Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 20, NUMBER 6 (2017) (Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

RESEARCH PAPER

ON GENERALIZED BOUNDARY VALUE PROBLEMS FOR A CLASS OF FRACTIONAL DIFFERENTIAL INCLUSIONS

Irene Benedetti [1], **Valeri Obukhovskii** [2], **Valentina Taddei** [3]

Abstract

We prove existence of mild solutions to a class of semilinear fractional differential inclusions with non local conditions in a reflexive Banach space. We are able to avoid any kind of compactness assumptions both on the nonlinear term and on the semigroup generated by the linear part. We apply the obtained theoretical results to two diffusion models described by parabolic partial integro-differential inclusions.

MSC 2010: Primary 34A08, 34G20, 34B10; Secondary 54C60, 35K91, 54C08

Key Words and Phrases: nonlocal conditions, fixed point theorem, fractional derivative

1. **Introduction**

Due to the more flexibility given by the non-integer derivatives, fractional calculus is an excellent tool for the description of memory and hereditary properties of various materials and processes. For instance, fractional derivatives find interesting applications in variational principles, control theory as well as in fractional Lagrangian and Hamiltonian dynamics. Among several different definitions we consider the Caputo fractional derivative. It is especially suitable for physical applications. Unlike the

pp. 1424–1446 , DOI: 10.1515/fca-2017-0075 DE GRUYTER

Riemann-Liouville fractional derivative, the Caputo derivative of a constant is zero and it allows a physical interpretation of the initial conditions as well as of boundary conditions. For a survey on the subject see e.g. [22, 25, 28]. We consider ultraslow processes, i.e. when the derivation order $\alpha \in (0,1)$. For results on intermediated processes, i.e. when $\alpha \in (1,2)$, see, e.g., [11].

Since the pioneering work of Byszewsky [10], nonlocal problems have been extensively studied for their interest in several contexts. For instance, Deng in [13] showed that the so called multipoint boundary value problem, which allows measurements at $t = t_i \in [0,b]$, $i = 1,\dots,n$ rather than just at the initial time $t = 0$, gives better results in the description of the diffusion phenomenon of a small amount of gas in a transparent tube. More recent results in this topic are due to Benedetti, Malaguti and Taddei [6], Benedetti, Taddei and Väth [8], García-Falset and Reich [17] and Paicu and Vrabie [26].

In this paper we give existence results for the solutions of two diffusion models driven by fractional parabolic differential equations with the nonlinearity depending on an integral term, precisely, for $t \in [0,b]$ and $x \in \Omega$, a bounded domain in $\mathbb{R}^n$ with a sufficiently regular boundary:

$$ {}^C D_t^\alpha z = \Delta z + f\left(t, x, \int_\Omega k(x,\xi) z(t,\xi)\, d\xi\right) \tag{1.1} $$

and

$$ \begin{aligned} {}^C D_t^\alpha z(t,x) &\in \gamma z(t,x) \\ &+ \left[f_1\left(t, x, \int_\Omega k(x,\xi) z(t,\xi)\, d\xi\right), f_2\left(t, x, \int_\Omega k(x,\xi) z(t,\xi)\, d\xi\right)\right]. \end{aligned} \tag{1.2} $$

We consider equations (1.1) and (1.2) associated with several nonlocal conditions, see Section 4 for the detailed problems description. These kind of problems, coming from applied sciences, describe anomalous diffusion in disordered materials or with memory effects. For instance, the first equation is a perturbation by means of a nonlocal forcing term of the diffusion of particles verifying a generalized Fick's second law, for other kind of perturbation we refer to [1] and [32]. Important applications include viscoelasticity and seismic-wave theory, diffusion in turbulent plasma, fractal media and porous media (see, e.g., [19] and the references therein). The second equation arises in population dynamics theory, it is a nonlinear perturbation of fractional epidemic models in which contacts between individuals are spatially distributed, for fractional epidemic, predator-pray, or birth-processing models see [3, 4, 14] and the references therein.

We transform equations (1.1) and (1.2) into the following class of fractional semilinear differential inclusions with non local conditions in abstract space:

$$\begin{cases} {}^{C}D^{\alpha}y(t) \in Ay(t) + F(t, y(t)), & \text{for a.e. } t \in [0, b], \\ Ly \in M(y) \\ 0 < \alpha < 1, \end{cases} \tag{1.3}$$

where y is a function with values in a reflexive Banach space E, ${}^{C}D^{\alpha}$ means the Caputo fractional derivative, A is the generator of a bounded C_0−semigroup $\{T(t)\}_{t\geq 0}$; $F : [0, b] \times E \multimap E$ is a multivalued map; $L : C([0, b]; E) \to E$ is a linear operator and $M : C([0, b]; E) \multimap E$ is a non necessarily linear multioperator.

The boundary condition considered is fairly general and obviously includes the initial valued problem, the periodic and anti-periodic problem and more general two-point problems as well as several nonlocal conditions. For instance, the following particular cases are covered by our general approach:

(i) $M(y) = \frac{1}{b}\int_0^b p(t)y(t)\,dt$ with $p \in L^1([0, b], \mathbb{R})$.

(ii) $M(y) = \sum_{i=1}^{n} \alpha_i y(s_i) + y_0$, with $y_0 \in E$, $\alpha_i \neq 0$, $s_i \in [0, b]$, $i = 1, \ldots, n$.

(iii) $M(y) \equiv B$, with $B \subset E$ a prescribed set.

In this paper we prove the existence of mild solutions to problem (1.3), obtaining the corresponding existence of solutions to (1.1) and (1.2) with $z \in C([0, b], L^2(\Omega, \mathbb{R}))$. We extend to semilinear differential inclusions a recent result obtained in [7] given for fully nonlinear inclusions. It is worth noting that considering a semilinear inclusion instead of a fully nonlinear one it is not a trivial generalization. In literature there exist several definitions of mild solutions to fractional semilinear differential inclusions. We consider the one introduced in [15, 34], since it is satisfied by a possible strong solution (for details see Section 2 and 3). For a different definition see the survey [30].

Contrary to the case of fully nonlinear fractional differential equations (or inclusions), see e.g. [2], few results are known for semilinear fractional equations or inclusions with nonlocal conditions. Some papers concern non local conditions for an equation (see [23]), others deal with a generalized Cauchy condition for inclusions (see [12, 24, 31]). However, in all quoted results and usually in literature in order to solve fractional differential problems of type (1.3) in an infinite dimensional framework some compactness assumptions are required on the semigroup generated by the nonlinear part, or on the nonlinear term. For instance, a regularity assumption in terms of measures of non compactness is required on the non linear term or the linear part is assumed to generate a compact semigroup (or a compact evolution operator).

Unlike all those results, by means of a technique based on weak topology and developed in [6], we are able to prove the existence of at least a solution of problem (1.3) avoiding any kind of compactness hypotheses both on the nonlinear term F and on semigroup generated by the linear part.

2. **Preliminaries**

Let $(E, \|\cdot\|)$ be a reflexive Banach space. We denote by E_w the space E endowed with the weak topology and by B the closed unit ball in E. For a set $\mathcal{M} \subset E$, the symbol $\overline{\mathcal{M}}^w$ means the weak closure of $\mathcal{M}$ while

$$\|\mathcal{M}\| := \sup\{\|y\| : y \in \mathcal{M}\}, \tag{2.1}$$

denotes the norm of a bounded set $\mathcal{M}$.

In the whole paper we denote by $\|\cdot\|_0$ and $\|\cdot\|_p$ the $C[0,b];E)$-norm and the $L^p([0,b];\mathbb{R})$-norm ($p \leq \infty$) of a function respectively. We recall (see [9, Theorem 4.3]) that a sequence $\{y_n\} \subset C([0,b];E)$ weakly converges to an element $y \in C([0,b];E)$ if and only if

1. $\{y_n\}$ is uniformly bounded, i.e., there exists a constant $\|y_n(t)\| \leq N$, for each $n \in \mathbb{N}$ and for each $t \in [0,b]$;
2. $y_n(t) \rightharpoonup y(t)$ for every $t \in [0,b]$.

For a function $f : [0,b] \to E$, the definition of the Riemann-Liouville fractional derivative with $0 < \alpha < 1$ is the following:

$$[D^\alpha f](t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^\alpha}\, ds,$$

where Γ is the Euler function:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx.$$

The Caputo fractional derivative is defined through the Riemann-Liouville fractional derivative as

$$[{}^C D^\alpha f](t) = D^\alpha[f(\cdot) - f(0)](t).$$

Let us briefly recall that a multivalued map (multimap) $\Phi\colon X \multimap Y$ of topological spaces X and Y is a relation that assigns to every point $x \in X$ a nonempty set $\Phi(x) \subset Y$. A multimap Φ of Banach spaces is called weakly sequentially closed, provided the conditions $x_n \rightharpoonup x_0$, $y_n \rightharpoonup y_0$, and $y_n \in \Phi(x_n)$, imply $y_0 \in \Phi(x_0)$. It is clear that this condition is equivalent to the hypothesis that Φ has a weakly sequentially closed graph. A multimap

$\Phi\colon X \multimap Y$ is said to be upper semicontinuous (u.s.c. for short), if the set $\Phi^{-1}(V) := \{x \in X : \Phi(x) \subset V\}$ is open for every open subset $V \subseteq Y$.

Finally, for sake of completeness, we recall some results that we will need in the sequel. Firstly we state the Glicksberg-Ky Fan fixed point theorem ([16], [18]).

THEOREM 2.1. *Let $\mathcal{K}$ a non-empty compact convex subset of a locally convex topological vector space and $G : \mathcal{K} \multimap \mathcal{K}$ a u.s.c. multimap with closed, convex values. Then G has a fixed point $x_* \in \mathcal{K} : x_* \in G(x_*)$.*

We mention also a result contained in the so-called Eberlein-Smulian theory.

THEOREM 2.2. [21, Theorem 1, p. 219] *Let Ω be a subset of a Banach space X. The following statements are equivalent:*

1. *Ω is relatively weakly compact;*
2. *Ω is relatively weakly sequentially compact.*

We assume that $A : D(A) \subset E \to E$ is a linear, not necessarily bounded operator generating a bounded C_0-semigroup $T : \mathbb{R}_+ \to \mathcal{L}(E)$, i.e., a family of bounded linear operators $T(t) : E \to E$, for $t \in \mathbb{R}_+$ such that

(a) $T(0) = I$;
(b) $T(t + r) = T(t)T(r) = T(r)T(t)$ for every $t, r \in \mathbb{R}_+$;
(c) the function $t \in \mathbb{R}_+ \to T(t)x \in E$ is continuous for every $x \in E$.

Define the families of operators $\{S_\alpha(t)\}_{t\in[0,\infty)}$ and $\{T_\alpha(t)\}_{t\in[0,\infty)}$ in E by the formulas

$$S_\alpha(t)x = \int_0^\infty \phi_\alpha(s)T(t^\alpha s)x\, ds,$$

where ϕ_α is the probability density function

$$\phi_\alpha(s) = \frac{1}{\alpha} s^{-\frac{\alpha+1}{\alpha}} \psi_\alpha(s^{-1/\alpha}),$$

$$\psi_\alpha(s) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1} s^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha),$$

and

$$T_\alpha(t)x = \alpha \int_0^\infty s\phi_\alpha(s)T\left(t^\alpha s\right)x\, ds.$$

Remark 2.1. (See, e.g., [33]).

$$\int_0^\infty s\phi_\alpha(s)\,ds = \frac{1}{\Gamma(\alpha+1)};\ \int_0^\infty \phi_\alpha(s)\,ds = 1.$$

By Lemma 3.2 and 3.3 in [33] the next regularity result holds.

Lemma 2.1. *The operator functions S_α and T_α possess the following properties:*

a) *for every $t \in [0,\infty)$, $S_\alpha(t)$ and $T_\alpha(t)$ are linear and bounded operators. More precisely*

$$\|S_\alpha(t)\| \leq D$$

and

$$\|T_\alpha(t)\| \leq \frac{D\alpha}{\Gamma(1+\alpha)}$$

where

$$D = \sup_{t\in[0,\infty)} \|T(t)\|;$$

b) *the operator functions S_α and T_α are strongly continuous, i.e., for each $x \in E$, functions $t \in [0,\infty) \to S_\alpha(t)x \in E$ and $t \in [0,\infty) \to T_\alpha x \in E$ are continuous.*

According to [5], [33], [34] a function $y \in C([0,b];E)$ is a mild solution of the Cauchy problem

$$\begin{cases} {}^C D^\alpha y(t) \in Ay(t) + f(t), \quad \text{for a.e. } t \in [0,b], \\ y(0) = y_0 \\ 0 < \alpha < 1 \end{cases} \tag{2.2}$$

with $f \in L^p([0,b];E)$, $p > \frac{1}{\alpha}$, if it satisfies the integral formula

$$y(t) = S_\alpha(t)y_0 + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s)\,ds.$$

Therefore, defining $\mathcal{S}\colon L^p([0,b];E) \to C([0,b];E)$ as

$$\mathcal{S}(f)(t) = \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s)\,ds,\ t \in [0,b], \tag{2.3}$$

we see that a continuous function y is a mild solution of (2.2) if

$$y(t) = S_\alpha(t)y_0 + \mathcal{S}(f)(t),\ t \in [0,b]. \tag{2.4}$$

Hence, we can define the mild solution of (1.3) as follows.

DEFINITION 2.1. A continuous function $y : [0, b] \to E$ is a *mild solution* to problem (1.3) if and only if there exists a map $f \in L^p([0, b]; E), p > \frac{1}{\alpha}$, with $f(t) \in F(t, y(t))$ for a.e. $t \in [0, b]$ such that

$$y(t) = S_\alpha(t)y(0) + \mathcal{S}(f)(t),\ t \in [0, b]$$

and

$$Ly \in M(y).$$

3. Existence result

We will study problem (1.3) under the following assumptions.

(A) A is the generator of a bounded C_0-semigroup $\{T(t)\}_{t\geq 0}$.

Concerning the multivalued nonlinearity $F : [0, b] \times E \multimap E$ we will suppose that it has closed bounded and convex values and, moreover, the following conditions hold true:

($F1$) the multifunction $F(\cdot, c) : [0, b] \multimap E$ has a measurable selection for every $c \in E$, i.e., there exists a measurable function $f : [0, b] \to E$ such that $f(t) \in F(t, c)$ for a.e. $t \in [0, b]$;

($F2$) the multimap $F(t, \cdot) : E \multimap E$ is weakly sequentially closed for a.e. $t \in [0, b]$;

($F3$) condition of local integral boundedness: for every $r > 0$ there exists a function $\mu_r \in L^p([0, b]; \mathbb{R}_+)$ with $p > \frac{1}{\alpha}$ such that for each $c \in E, \|c\| \leq r$:

$$\|F(t, c)\| \leq \mu_r(t) \text{ for a.e. } t \in [0, b].$$

Notice that under conditions ($F1$) - ($F3$), by Proposition 3.1 in [7] the superposition multioperator $\mathcal{P}_F : C([0, b]; E) \multimap L^p([0, b]; E),\ p > \frac{1}{\alpha}$, given as

$$\mathcal{P}_F(y) = \{f \in L^p([0, b]; E) \ :\ f(t) \in F(t, y(t)) \text{ a.e. } t \in [0, b]\}$$

is well defined.

Initially we assume that operators L and M satisfy the following conditions.

(L) $L : C([0, b]; E) \to E$ is a bounded linear operator;

(M1) $M : C([0, b]; E) \multimap E$ is a weakly sequentially closed multioperator, with convex, closed and bounded values, mapping bounded sets into bounded ones.

To formulate the next conditions, consider the subspace $C_0 \subset C([0, b]; E)$ consisting of functions $y(\cdot)$ having the form

$$y(t) = S_\alpha(t)y(0)$$

and denote $L_0 = L|_{C_0}$. We will assume that

(Λ) there exists a bounded linear operator $\Lambda \colon E \to C_0$ such that for every $y \in C(0,b], E)$, $w \in M(y)$, and $f \in P_F(y)$,

$$(I - L_0\Lambda)(w - L\mathcal{S}f) = 0,$$

where the operator $\mathcal{S}$ is defined by (2.3).

Remark 3.1. To present an example when condition (Λ) is fulfilled, consider the linear operator $K \colon E \to C_0$ defined as

$$K(y)(t) = S_\alpha(t)y$$

and define the linear operator $\widetilde{L} \colon E \to E$, $\widetilde{L}y = L_0(K(y))$. It is easy to see that condition (Λ) holds true if $\widetilde{L}$ has a bounded inverse $\widetilde{L}^{-1}$. Indeed, in this case we may take $\Lambda = K\widetilde{L}^{-1}$. In the particular case of $Ly = y(0)$ the last condition is trivially satisfied being $\widetilde{L} = I$. Moreover, in the case of a periodic problem ($Ly = y(0) - y(b)$; $My \equiv 0$) the last condition takes the form of the existence of a bounded inverse $(I - S_\alpha(b))^{-1}$ what is similar to an usual condition used in the search of periodic solutions for semilinear inclusions with classical derivative (see [20], Section 6.1, condition (A')).

Now, consider the multioperator $\mathcal{T} \colon C([0,b];E) \multimap C([0,b];E)$ defined as

$$\mathcal{T}(y) = \Lambda M(y) + (I - \Lambda L)\mathcal{S}\mathcal{P}_F(y). \tag{3.1}$$

Let $y \in C([0,b];E)$, be a fixed point of $\mathcal{T}$, i.e $y \in \mathcal{T}(y)$. Hence, there exist $w \in M(y)$ and $f \in \mathcal{P}_F(y)$, such that

$$y = \Lambda w + (I - \Lambda L)\mathcal{S}(f) = \Lambda(w - L\mathcal{S}(f)) + \mathcal{S}(f).$$

By condition (Λ) it follows

$$\begin{aligned} Ly &= L(\Lambda w + (I - \Lambda L)\mathcal{S}(f)) = L_0\Lambda w + L(I - \Lambda L)\mathcal{S}(f) \\ &= w - (w - L_0\Lambda w) + L\mathcal{S}(f) - L_0\Lambda L\mathcal{S}(f) \\ &= w - (I - L_0\Lambda)w + (I - L_0\Lambda)L\mathcal{S}(f) \\ &= w - (I - L_0\Lambda)(w - L\mathcal{S}(f)) = w \in M(y). \end{aligned}$$

Moreover, denote $z := w - L\mathcal{S}(f)$. By the definition of the operator Λ we have that $\Lambda z = \varphi \in C([0,b];E)$ defined as $\varphi(t) = S_\alpha(t)\varphi(0)$. Hence

$$y(0) = [\Lambda z](0) = \varphi(0),$$

and

$$y(t) = [\Lambda z](t) + [\mathcal{S}(f)](t) = S_\alpha(t)y(0) + [\mathcal{S}(f)](t), \ t \in [0,b],$$

yielding that y satisfies (2.4). Therefore, we have proven that any function $y \in C([0,b];E)$ which is a fixed point of the multioperator $\mathcal{T}$ is a mild solution of problem (1.3).

We will assume, additionally, the following condition posed on the multioperator M

(M2) M satisfies the following asymptotic estimate

$$\limsup_{\|u\|_0\to\infty} \frac{\|M(u)\|}{\|u\|_0} = l \ \text{ with } l < \frac{1}{\|\Lambda\|}. \tag{3.2}$$

It is usual in literature to assume that the limit in (3.2) is equal to zero. This is the case when the multimap M is bounded. On the contrary, we are able to consider operator M satisfying a linear growth condition. For our main result (see Theorem 3.1), instead of condition (F3), we need the stronger assumption below:

$(F3')$ $\sup_{\|x\|\le n} \|F(t,x)\| \le \varphi_n(t)$, for a.a. $t \in [0,b]$, with $\varphi_n \in L^p([0,b];\mathbb{R})$, $p > \frac{1}{\alpha}$ and such that

$$\liminf_{n\to\infty} \frac{1}{n} \left\{ \int_0^b |\varphi_n(s)|^p \, ds \right\}^{\frac{1}{p}} = 0. \tag{3.3}$$

In order to prove the existence of a fixed point of $\mathcal{T}$, let us study its properties.

Lemma 3.1. *The operator $\mathcal{S}$ is linear and bounded.*

Proof. The linearity follows from the linearity of the integral operator. We now prove that $\mathcal{S}$ is bounded. For every $\tau_1, \tau_2 \in [0,b]$,

$$\left(\int_{\tau_1}^{\tau_2} \left((\tau_2 - s)^{\alpha-1}\right)^{\frac{p}{p-1}} \, ds \right)^{\frac{p-1}{p}} \le \left[\frac{p-1}{\alpha p - 1} \right]^{\frac{p-1}{p}} (b)^{\alpha - \frac{1}{p}} =: C.$$

Thus, denoted $H := \frac{D\alpha}{\Gamma(1+\alpha)} C$, using Hölder inequality, we get for any $f \in L^p([0,b];E)$ and $t \in [0,b]$

$$\begin{aligned} \|\mathcal{S}(f)(t)\| &\le \int_0^t (t-s)^{\alpha-1} \|T_\alpha(t-s)\| \|f(s)\| \, ds \\ &\le \frac{D\alpha}{\Gamma(1+\alpha)} C \|f\|_p = H\|f\|_p. \end{aligned} \tag{3.4}$$

□

Proposition 3.1. *The multioperator $\mathcal{T}$ has a weakly sequentially closed graph.*

P r o o f. Let $\{x_m\} \subset C([0,b];E)$ and $\{y_m\} \subset C([0,b];E)$ satisfying $y_m \in \mathcal{T}(x_m)$ for all m and $x_m \rightharpoonup x$, $y_m \rightharpoonup y$ in $C([0,b];E)$; we will prove that $y \in \mathcal{T}(x)$.

By the weak convergence of the sequence $\{x_m\}$ in $C([0,b];E)$, it follows that there exists a constant $r > 0$ such that $\|x_m\|_0 < r$ for every $m \in \mathbb{N}$ and $x_m(t) \rightharpoonup x(t)$ for every $t \in [0,b]$. Therefore, it follows that $\|x(t)\| \leq \liminf_{m\to\infty} \|x_m(t)\| \leq r$ for all t. The fact that $y_m \in \mathcal{T}(x_m)$ means that there exist a sequence $\{f_m\}$, $f_m \in \mathcal{P}_F(x_m)$ and a sequence $w_m \in M(x_m)$ such that

$$y_m = \Lambda(w_m - L\mathcal{S}f_m) + \mathcal{S}f_m.$$

We observe that, according to (F3), $\|f_m(t)\| \leq \eta_r(t)$ for a.a. t and every m, i.e. $\{f_m\}$ is uniformly bounded and by the reflexivity of the space $L^p([0,b];E)$, we have the existence of a subsequence, denoted as the sequence, and a function g such that $f_m \rightharpoonup g$ in $L^p([0,b];E)$.

Lemma 3.1 and (L1) imply that $\mathcal{S}f_m \rightharpoonup \mathcal{S}g$ in $C([0,b];E)$ and $L\mathcal{S}f_m \rightharpoonup L\mathcal{S}g$ in E. The operator M maps bounded sets in bounded sets and it is weakly sequentially closed, hence, up to subsequence, $w_m \rightharpoonup w$ in E, with $w \in M(x)$. In conclusion, we have

$$y_m \rightharpoonup \Lambda(w - L\mathcal{S}g) + \mathcal{S}g = y_0,$$

thus, by the uniqueness of the weak limit in E, we obtain that $y_0 \equiv y$.

Reasoning as [7, Proposition 4.1] it is possible to prove that $g(t) \in F(t, x(t))$ for a.a. $t \in [0,b]$, i.e. that $y \in \mathcal{T}(x)$. □

PROPOSITION 3.2. *The multioperator $\mathcal{T}$ is weakly compact.*

P r o o f. By Theorem 2.2 it is sufficient to prove that $\mathcal{T}$ is weakly relatively sequentially compact.

Let $\{x_m\} \subset C([0,b];E)$ be a bounded sequence and $\{y_m\} \subset C([0,b];E)$ satisfying $y_m \in \mathcal{T}(x_m)$ for all m. By the definition of the multioperator $\mathcal{T}$, there exist a sequence $\{f_m\}$, $f_m \in \mathcal{P}_F(x_m)$, and a sequence $w_m \in M(x_m)$ such that

$$y_m = \Lambda(w_m - L\mathcal{S}f_m) + \mathcal{S}f_m.$$

Reasoning as in Proposition 3.1, we have that there exists a subsequence, denoted as the sequence, and a function g such that $f_m \rightharpoonup g$ in $L^p([0,b];E)$. Moreover, since the multioperator M maps bounded sets into bounded sets and $\{x_m\}$ is bounded, we obtain that, up to subsequence, $w_m \rightharpoonup \overline{w} \in E$ as $m \to \infty$. Therefore

$$y_m \rightharpoonup \Lambda(\overline{w} - L\mathcal{S}g) + \mathcal{S}g$$

in $C([0,b];E)$. □

Proposition 3.3. *The multioperator $\mathcal{T}$ has convex and weakly compact values.*

P r o o f. Fix $x \in C([0,b];E)$ since F and M are convex valued, the set $\mathcal{T}(x)$ is convex from the linearity of the involved operators. The weak compactness of $\mathcal{T}(x)$ follows from Propositions 3.2 and 3.1. □

Theorem 3.1. *Under assumptions* $(A),(F1),(F2),(F3'),(L),(\Lambda)$, $(M1)$ *and* $(M2)$, *problem (1.3) has at least a mild solution.*

P r o o f. Fix $n \in \mathbb{N}$, consider Q_n the closed ball of radius n of $C([0,b];E)$. We show that there exists $n \in \mathbb{N}$ such that the operator $\mathcal{T}$ maps the ball Q_n into itself.

Assume to the contrary, that there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \in Q_n$, $y_n \in \mathcal{T}(x_n)$ and $y_n \notin Q_n$ for all $n \in \mathbb{N}$. By the definition of $\mathcal{T}$, there exist a sequence $\{f_n\} \subset \mathcal{P}_F(x_n)$ and a sequence $w_n \in M(x_n)$ such that

$$y_n = \Lambda(w_n - L\mathcal{S}f_n) + \mathcal{S}f_n.$$

From the assumption $y_n \notin Q_n$ we must have, for any n,

$$n < \|y_n\|_0 \leq \|\Lambda\|\{\|w_n\|_0 + \|L\|H\|f_n\|_p\} + H\|f_n\|_p, \tag{3.5}$$

where H is defined in (3.4). Moreover $x_n \in Q_n$ implies, by (F3$'$), that $\|f_n(t)\| \leq \varphi_n(t)$ for a.a. $t \in [0,b]$, hence $\|f_n\|_p \leq \|\varphi_n\|_p$. Consequently

$$n < \|\Lambda\|\{\|M(x_n)\|_0 + \|L\|H\|\varphi_n\|_p\} + H\|\varphi_n\|_p.$$

Therefore

$$\frac{1}{\|\Lambda\|} < \frac{\|M(x_n)\|_0}{n} + \|L\|H\frac{\|\varphi_n\|_p}{n} + \frac{H}{\|\Lambda\|}\frac{\|\varphi_n\|_p}{n}.$$

Notice that if $\|x_n\|_0 \leq H_1 < +\infty$ for any $n \in \mathbb{N}$ then

$$\lim_{n\to\infty} \frac{\|M(x_n)\|}{n} = 0,$$

because M maps bounded sets into bounded sets.

If $\lim\limits_{n\to\infty} \|x_n\|_0 = +\infty$ by hypothesis (M2) we have

$$\lim_{n\to\infty} \frac{\|M(x_n)\|}{n} \leq \limsup_{n\to\infty} \frac{\|M(x_n)\|}{\|x_n\|_0} \leq \limsup_{\|u\|_0\to\infty} \frac{\|M(u)\|}{\|u\|_0} = l < \frac{1}{\|\Lambda\|}.$$

In both cases

$$\lim_{n\to\infty} \frac{\|M(x_n)\|}{n} < \frac{1}{\|\Lambda\|}.$$

Moreover, according to (3.3), there exists a subsequence, still denoted as the sequence, such that

$$\lim_{n\to\infty} \frac{1}{n}\left\{\int_0^b |\varphi_n(s)|^p\, ds\right\}^{\frac{1}{p}} = 0. \tag{3.6}$$

Hence

$$\frac{1}{\|\Lambda\|} \leq \quad \limsup_{n\to\infty}\left\{\frac{\|M(x_n)\|}{n} + \|L\|H\frac{\|\varphi_n\|_p}{n} + \frac{H}{\|\Lambda\|}\frac{\|\varphi_n\|_p}{n}\right\} < \frac{1}{\|\Lambda\|}$$

giving the contradiction.

The conclusion then follows from Theorem 2.1 as in [7, Theorem 4.1]. □

When considering the periodic problem, Theorem 3.1 turns into the following assertion.

Corollary 3.1. *Assume* $(A), (F1), (F2), (F3')$ *and suppose that the operator* $I - S_\alpha(b)$ *has a bounded inverse. Then the periodic problem*

$$\begin{cases} {}^C D^\alpha y(t) \in Ay(t) + F(t, y(t)), & \text{for a.e. } t \in [0, b] \\ y(0) = y(b) \\ 0 < \alpha < 1, \end{cases}$$

has at least a mild solution.

Remark 3.2. For a periodic problem, conditions (M1) and (M2) are trivially fulfilled with $l = 0$.

4. **Applications**

4.1. **Time-fractional diffusion model.** Given a bounded domain $\Omega \subset \mathbb{R}^n$ with C^2-boundary, we consider the integro-differential equation

$$\begin{cases} {}^C D_t^\alpha z = \Delta z + f\left(t, x, \int_\Omega k(x,\xi) z(t,\xi)\, d\xi\right), t \in [0,b], & x \in \Omega \\ z(0,x) = z(b,x). & x \in \Omega \end{cases}. \tag{4.1}$$

The equation $D_t^\alpha z = z_{xx}$, known as time-fractional diffusion equation or generalized diffusion equation, is a mathematical model of wide application in science, due to anomalous diffusion effects in disordered materials where the environment is constrained and trapping and binding of particles can occur. It describes anomalous diffusion characterized by the mean square displacement of particles from the original starting site, verifying a generalized Fick's second law. Important applications include viscoelasticity and seismic-wave theory, diffusion in turbulent plasma, fractal media and porous media (see, e.g., [19] and the references therein). We consider a perturbed equation in the multidimensional case, like in [1] and [32], but in our case the forcing term is nonlocal. A periodic condition is associated to the system. We assume

(i) for all $r \in \mathbb{R}$, $f(\cdot,\cdot,r) : [0,b] \times \Omega \to \mathbb{R}$ is measurable;
(ii) for a.a. $t \in [0,b]$ and $x \in \Omega$, $f(t,x,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous;
(iii) there exist $\varphi \in L^p([0,b];\mathbb{R})$, with $p > \frac{1}{\alpha}$, and $\mu : [0,+\infty) \to [0,+\infty)$ increasing such that, for a.a. $x \in \Omega$ and every $t \in [0,b]$ and $r \in \mathbb{R}$, $|f(t,x,r)| \le \varphi(t)\mu(|r|)$ and

$$\lim_{r \to +\infty} \frac{\mu(r)}{r} = 0.$$

(iv) $k : \Omega \times \Omega \to \mathbb{R}$ is measurable with $k(x,\cdot) \in L^2(\Omega;\mathbb{R})$ and $\|k(x,\cdot)\|_2 \le 1$ for a.a. $x \in \Omega$.

Let $y : [0,b] \to L^2(\Omega;\mathbb{R})$ be the map defined by $y(t) = z(t,\cdot)$. We can write the periodic problem (4.1) as the fractional inclusion with impulses in the Hilbert space $E = L^2(\Omega;\mathbb{R})$

$$\begin{cases} {}^C D^\alpha y(t) \in Ay(t) + F(t,y(t)),\ t \in [0,b],\ y(t) \in E \\ y(b) - y(0) = 0 \end{cases}, \tag{4.2}$$

where $A : W^{2,2}(\Omega;\mathbb{R}) \cap W_0^{1,2}(\Omega;\mathbb{R}) \to L^2(\Omega;\mathbb{R})$ is the linear operator defined as $Ay = \Delta y$, $F : [0,b] \times E \to E$ is the single valued map

$$F(t,y)(x) = f\left(t, x, \int_\Omega k(x,\xi)y(\xi)\,d\xi\right)$$

and the maps $L : C([0,b];E) \to E$ and $M : C([0,b];E) \to E$ are respectively defined as $L = y(b) - y(0)$ and $M(y) \equiv 0$.

We show, now, that all the hypotheses of Corollary 3.1 are satisfied. It is known that A generates a strongly continuous semigroup of contractions $T(t)$ on E (see e.g. [29], Theorem 4.1.3). Hence hypothesis (A) is satisfied. Moreover, it is well known that

$$T(t)y = \sum_{i=1}^{\infty} e^{-\lambda_i t}(y,e_i)e_i,$$

where $\{e_i\}_i$ is an orthonormal basis of $L^2(\Omega;\mathbb{R})$ formed by eigenvectors of A corresponding to the eigenvalues λ_i.

Given $y \in L^2(\Omega;\mathbb{R})$ and denoted by $\mathcal{L}(g)(\lambda)$ the Laplace transform of the function g, we get that

$$\begin{aligned}(I-S_\alpha(b))y = & \; y-\int_0^\infty \Phi_\alpha(s)T(b^\alpha s)y ds \\ = & \sum_{i=1}^\infty\left[1-\int_0^\infty \Phi_\alpha(s)e^{-\lambda_i(b^\alpha s)}ds\right](y,e_i)e_i \\ = & \sum_{i=1}^\infty[1-\mathcal{L}(\Phi_\alpha)(\lambda_i b^\alpha)](y,e_i)e_i.\end{aligned}$$

Now, since $\lambda_i > 0$ for every i and Φ_α is a probability density function, it follows that $\mathcal{L}(\Phi_\alpha)(\lambda_i b^\alpha) \neq \mathcal{L}(\Phi_\alpha)(0) = 1$ for every i, i.e. that $I - S_\alpha(b)y = 0$ if and only if $y = 0$. Hence $I - S_\alpha(b)$ is injective. Moreover the operator $S_\alpha(b) : E \to E$ is compact. In fact, take $\{y_n\}_n \subset E$ bounded. Since E is reflexive, there exists a subsequence still denoted as the sequence such that y_n weakly converges to $y_0 \in E$. Denote now $\mathcal{L}(\Phi_\alpha)(\lambda_i b^\alpha) = c_i$. Then $S_\alpha(b)y_n = \sum_{i=1}^\infty c_i(y_n, e_i)e_i$. By the weak convergence, we get that $(y_n, e_i) \to (y_0, e_i)$ for every i. Since $\{e_i\}$ is an orthonormal basis for E, it follows that $\sum_{i=1}^\infty c_i(y_n, e_i)e_i \to \sum_{i=1}^\infty c_i(y_0, e_i)e_i$, hence the compactness of $S_\alpha(b)$. According to the Fredholm alternative it follows that $I - S_\alpha(b)$ is also surjective, thus invertible.

We prove, now, that the map F verifies condition (F1),(F2) and (F3′). Notice first of all that Pettis measurability theorem (see [27, p. 278]), the separability of $L^2([0,T];\mathbb{R})$ and conditions (i) and (ii) imply that F is globally measurable (see [20, Corollary 1.3.1]), hence, being single-valued, it satisfies condition (F1).

We now prove that $F(t,\cdot)$ is weakly sequentially continuous for a.a. $t \in [0,b]$. To this aim take $y_n \rightharpoonup y$ in $L^2(\Omega,\mathbb{R})$. From (iv) we get that

$$\int_\Omega k(x,\xi)y_n(\xi)d\xi \to \int_\Omega k(x,\xi)y(\xi)d\xi$$

for a.a. $x \in \Omega$, thus (ii) implies that

$$f(t,x,\int_\Omega k(x,\xi)y_n(\xi)d\xi) \to f(t,x,\int_\Omega k(x,\xi)y(\xi)d\xi)$$

for a.a. $x \in \Omega$. Moreover, according to (iv), we have, for a.a. $x \in \Omega$ and every $y \in L^2(\Omega;\mathbb{R})$,

$$\left|\int_\Omega k(x,\xi)y(\xi)d\xi\right| \le \int_\Omega |k(x,\xi)||y(\xi)|d\xi \le \|k(x,\cdot)\|_2\|y\|_2 \le \|y\|_2,$$

thus (iii) implies, for a.a. $t \in [0,b]$ and every $y \in L^2(\Omega;\mathbb{R})$,

$$\left| f\left(t, x, \int_\Omega k(x,\xi)y(\xi)d\xi\right)\right| \leq \varphi(t)\mu\left(\left|\int_\Omega k(x,\xi)y(\xi)d\xi\right|\right) \leq \varphi(t)\mu(\|y\|_2). \tag{4.3}$$

Since the weak convergence implies the boundedness in norm, (4.3) implies the existence of a positive constant L such that $|f\left(t,x,\int_\Omega k(x,\xi)y_n(\xi)d\xi\right)| \leq \varphi(t)\mu(L)$ for a.a. $x \in \Omega$ and every $n \in \mathbb{N}$, and (F2) follows from the Lebesgue's dominated convergence theorem.

Finally, (4.3) yields

$$\|F(t,y)\|_2 = \sqrt{\int_\Omega \left[f\left(t,x,\int_\Omega k(x,\xi)y(\xi)d\xi\right)\right]^2 dx} \leq \varphi(t)\mu(\|y\|_2)\sqrt{|\Omega|},$$

and so also the growth condition (F3′) is satisfied with $\varphi_n(t) = \varphi(t)\mu(n)\sqrt{|\Omega|}$, with

$$\liminf_{n\to\infty} \frac{1}{n}\int_0^b \varphi_n(t)dt = \liminf_{n\to\infty} \frac{\mu(n)}{n}\|\varphi\|_1\sqrt{|\Omega|} = 0.$$

The solvability of problem (4.1) then follows from Corollary 3.1.

Remark 4.1. Similarly as before it is possible to show that also the multipoint boundary value problem

$$\begin{cases} {}^C D_t^\alpha z = \Delta z + f\left(t,x,\int_\Omega k(x,\xi)z(t,\xi)\,d\xi\right), & t \in [0,b], x \in \Omega \\ z(0,x) = \sum_{i=1}^n \alpha_i z(s_i,x) + z_0(x) & x \in \Omega \end{cases}$$

with $z_0 \in L^2(\Omega;\mathbb{R}), \alpha_i \neq 0,\ s_i \in [0,b],\ i = 1,\ldots,n$, or the weighted boundary value problem

$$\begin{cases} {}^C D_t^\alpha z = \Delta z + f\left(t,x,\int_\Omega k(x,\xi)z(t,\xi)\,d\xi\right), & t \in [0,b], x \in \Omega \\ z(0,x) = \dfrac{1}{b}\displaystyle\int_0^b p(t)z(t,x)\,dt & x \in \Omega \end{cases}$$

with $p \in L^1([0,b],\mathbb{R})$, are solvable, provided respectively that $\sum_{i=1}^n |\alpha_i|$ and $\frac{\|p\|_1}{b}$ are sufficiently small.

Indeed, introducing as before the auxiliary map $y : [0,b] \to L^2(\Omega;\mathbb{R})$ defined as $y(t) = z(t,\cdot)$, in both cases $Ly = y(0)$ and condition (Λ) is trivially satisfied, see Remark 3.1. Moreover, in the first case $M(y) = \sum_{i=1}^n \alpha_i y(t_i) + y_0$, with $y_0 = z_0(\cdot)$, is the translation of a linear and bounded single valued operator, hence it is a weakly sequentially closed multioperator. Furthermore

$$\begin{aligned}\frac{\|\sum_{i=1}^n \alpha_i y(s_i)+y_0\|}{\|y\|_0} &\le \frac{\sum_{i=1}^n |\alpha_i||y(s_i)|+\|y_0\|}{\|y\|_0}\\ &\le \frac{\|y\|_0\sum_{i=1}^n |\alpha_i|+\|y_0\|}{\|y\|_0} = \sum_{i=1}^n|\alpha_i| + \frac{\|y_0\|}{\|y\|_0}.\end{aligned}$$

Hence

$$\lim_{\|y\|_0\to\infty} \frac{\|\sum_{i=1}^n \alpha_i y(s_i)+y_0\|}{\|y\|_0} \le \sum_{i=1}^n |\alpha_i|.$$

In the second case $M(y)=\frac{1}{b}\int_0^b p(s)y(s)ds$ is a weakly continuous single valued operator, thus it is a weakly sequentially continuous multioperator. Moreover we have

$$\frac{\|\frac{1}{b}\int_0^b p(t)y(t)\,dt\|}{\|y\|_0} \le \frac{\|p\|_1}{b}.$$

Hence

$$\lim_{\|y\|_0\to\infty}\frac{\|\frac{1}{b}\int_0^b p(t)y(t)\,dt\|}{\|y\|_0} \le \frac{\|p\|_1}{b}.$$

Notice that by Lemma 2.1$\|\Lambda\|\le 1$. So if we respectively assume that

$$\sum_{i=1}^n |\alpha_i| < 1$$

and

$$\frac{\|p\|_1}{b} < 1,$$

we get that also condition (M2) holds and all the hypotheses of Theorem 3.1 are satisfied also by the multipoint and the weighted boundary value problems.

4.2. **Fractional integro-differential model.** This application concerns the integro-differential equation

$$\begin{cases} {}^C D_t^{\alpha} z(t,x) \in \gamma z(t,x) + \\ \qquad \left[f_1\left(t,x,\int_\Omega k(x,\xi)z(t,\xi)\,d\xi\right), f_2\left(t,x,\int_\Omega k(x,\xi)z(t,\xi)\,d\xi\right)\right], \\ \qquad\qquad t\in[0,b], x\in\Omega, \\ z(b,x)\in B(z(0,x)) \quad x\in\Omega, \end{cases} \tag{4.4}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a sufficiently regular boundary. We assume the following hypotheses:

(i) for all $r\in\mathbb{R}, i=1,2,\ f_i(\cdot,\cdot,r):[0,b]\times\Omega\to\mathbb{R}$ is measurable;

(ii) for a.a. $t \in [0,b]$ and $x \in \Omega$, $f_1(t,x,\cdot) : \mathbb{R} \to \mathbb{R}$ is lower semicontinuous and $f_2(t,x,\cdot) : \mathbb{R} \to \mathbb{R}$ is upper semicontinuous;
(iii) $f_1(t,x,r) \leq f_2(t,x,r)$ in $[0,b] \times \Omega \times \mathbb{R}$;
(iv) there exist $\varphi \in L^p([0,b];\mathbb{R})$, with $p > \frac{1}{\alpha}$, and a non decreasing function $\mu : [0,\infty) \to [0,\infty)$ such that, for a.a. $x \in \Omega$ and every $t \in [0,b], r \in \mathbb{R}$ and $i = 1,2$, we have $|f_i(t,x,r)| \leq \varphi(t)\mu(|r|)$ with

$$\liminf_{r\to\infty} \frac{\mu(r)}{r} = 0; \tag{4.5}$$

(v) $k\colon \Omega\times\Omega \to \mathbb{R}$ is measurable with $k(x,\cdot) \in L^2(\Omega;\mathbb{R})$ and $\|k(x,\cdot)\|_2 \leq 1$ for all $x \in \Omega$;
(vi) $\gamma < 0$;
(vii) $B : \mathbb{R} \multimap \mathbb{R}$ is bounded, convex, i.e. for any $x, y \in \mathbb{R}$ and any $\lambda \in [0,1]$ it holds

$$\lambda B(x) + (1-\lambda)B(y) \subset B(\lambda x + (1-\lambda y),$$

upper semicontinuous and with closed values.

Problem (4.4) can be represented in the form of the following abstract system in the Hilbert space $E = L^2(\Omega;\mathbb{R})$

$$\begin{cases} {}^C D^\alpha y(t) \in Ay(t) + F(t,y(t)) \\ y(b) \in \widetilde{M}(y(0)), \end{cases} \tag{4.6}$$

where $y : [0,b] \to E$ is defined as $y(t) = z(t,\cdot), F : [0,b] \times E \multimap E$ is the multimap

$$F(t,y)(x) = \left[f_1\left(t,x,\int_\Omega k(x,\xi)y(\xi)d\xi\right), f_2\left(t,x,\int_\Omega k(x,\xi)y(\xi)d\xi\right)\right]$$

and A is a bounded linear operator in E generating the non-compact semigroup of contractions

$$T(t)y(x) = e^{\gamma t}y(x).$$

System (4.6) reads as an attainability problem, i.e. the study of the existence of at least a trajectory of the system reaching a given set at the final time. The attainability problem (4.6) can be written as (1.3) with $L(y) = y(b)$ and $M(y) = \widetilde{M} \circ \theta$, where θ is the evaluation operator, i.e. the linear and bounded operator $\theta\colon C([0,b];E) \to E, \theta(y) = y(0)$, and $\widetilde{M} : E \to E$ is the multimap defined as $\widetilde{M}(y)(x) = B(y(x))$ for a.a. $x \in \Omega$.

Let us show that Theorem 3.1 can be applied to the abstract formulation of the system (4.4). Given $y \in L^2(\Omega;\mathbb{R})$, recalling that Φ_α is a probability density function, we get that

$$LKy = S_\alpha(b)y = \int_0^\infty \Phi_\alpha(s)T(b^\alpha s)yds = y\int_0^\infty \Phi_\alpha(s)e^{\gamma b^\alpha s}ds,$$

i.e. $LK = \beta I$, where β denotes the real number $\int_0^\infty \Phi_\alpha(s)e^{\gamma b^\alpha s}ds \neq 0$. Hence we have that LK is invertible and so condition (Λ) is satisfied.

Reasoning as in Subsection 4.1 and using Pettis measurability Theorem, it is possible to show that the maps $t \mapsto f_i(t,\cdot,\int_\Omega y(s)ds)$, $i = 1,2$ are measurable selections of $F(\cdot,y)$ for every $y \in L^2(\Omega;\mathbb{R})$; hence condition (F1) is satisfied. Moreover, from (v), we easily get that for a.a. $x \in \Omega$ and $y \in L^2(\Omega,\mathbb{R})$

$$\left|\int_\Omega k(x,\xi)y(\xi)d\xi\right| \leq \|y\|_2$$

thus from (iv) that

$$\left|f_i\left(t,x,\int_\Omega k(x,\xi)y(\xi)d\xi\right)\right| \leq \varphi(t)\mu(\|y\|_2)$$

for $i = 1,2$. Therefore the growth condition (F3′) is fulfilled with $\varphi_n(t) = \varphi(t)\mu(n)\sqrt{|\Omega|}$.

Now we verify condition (F2). Fix $t \in [0,b]$ and consider the sequences $\{y_n\},\{\beta_n\} \subset L^2(\Omega;\mathbb{R})$ satisfying $y_n \rightharpoonup y$, $\beta_n \rightharpoonup \beta$ in $L^2(\Omega;\mathbb{R})$ and $\beta_n \in F(t,y_n)$ for all $n \in \mathbb{N}$. Since $\beta_n \rightharpoonup \beta$, applying Mazur's convexity lemma, we have the existence of a sequence

$$\tilde{\beta}_n = \sum_{i=0}^{k_n} \delta_{n,i}\beta_{n+i} \quad \delta_{n,i} \geq 0, \sum_{i=0}^{k_n} \delta_{n,i} = 1$$

such that $\tilde{\beta}_n \to \beta$ in $L^2(\Omega;\mathbb{R})$ and up to a subsequence denoted as the sequence $\tilde{\beta}_n(x) \to \beta(x)$ for a.a. $x \in \Omega$. By definition we have, for a.a. $x \in \Omega$,

$$\begin{aligned}\textstyle\sum_{i=0}^{k_n} \delta_{n,i} f_1\left(t,x,\int_\Omega k(x,\xi)y_{n+i}(\xi)d\xi\right) &\leq \tilde{\beta}_n(x)\\ &\leq \textstyle\sum_{i=0}^{k_n} \delta_{n,i} f_2\left(t,x,\int_\Omega k(x,\xi)y_{n+i}(\xi)d\xi\right).\end{aligned}$$

Reasoning again as in Subsection 4.1 and taking the limit as $n \to \infty$, according to (ii), we obtain that

$$f_1\left(t,x,\int_\Omega k(x,\xi)y(\xi)d\xi\right) \leq \beta(x) \leq f_2\left(t,x,\int_\Omega k(x,\xi)y(\xi)d\xi\right),$$

i.e. that $\beta \in F(t,y)$. We have showed that $F(t,\cdot)$ has weakly sequentially closed graph.

It remains to prove conditions (M1) and (M2). From (vii) we obtain that M has convex and closed values and it is bounded. Moreover, since θ is linear and bounded, it is sufficient to prove that $\widetilde{M}$ is weakly sequentially closed. Consider the sequences $\{y_n\},\{\zeta_n\} \subset L^2(\Omega;\mathbb{R})$ satisfying $y_n \rightharpoonup$

y, $\zeta_n \rightharpoonup \zeta$ in $L^2(\Omega;\mathbb{R})$ and $\zeta_n \in \widetilde{M}(y_n)$ for all $n \in \mathbb{N}$. Since $\zeta_n \rightharpoonup \zeta$, applying Mazur's convexity lemma, we have the existence of a sequence

$$\tilde{\zeta}_n = \sum_{i=0}^{k_n} \delta_{n,i}\zeta_{n+i} \quad \delta_{n,i} \geq 0, \sum_{i=0}^{k_n} \delta_{n,i} = 1$$

such that $\tilde{\zeta}_n \to \zeta$ in $L^2(\Omega;\mathbb{R})$ and up to a subsequence denoted as the sequence $\tilde{\zeta}_n(x) \to \zeta(x)$ for a.a. $x \in \Omega$. By definition, $\zeta_n(x) \in B(y_n(x))$ for a.a. $x \in \Omega$, thus the convexity of B yields $\tilde{\zeta}_n(x) \in B(\tilde{y}_n(x))$ for a.a. $x \in \Omega$, with $\tilde{y}_n = \sum_{i=0}^{k_n} \delta_{n,i} y_{n+i}$. Taking the limit as $n \to \infty$, from the upper semicontinuity of B we obtain $\zeta(x) \in B(y(x))$ for a.a. $x \in \Omega$, i.e. $\zeta \in \widetilde{M}(y)$. We have proved the weakly sequentially closedness of $\widetilde{M}$, i.e. condition (M1).

Finally, since B is bounded, there exists a constant $L > 0$ such that $|\zeta(x)| \leq L$ for every $y \in E, \zeta \in \widetilde{M}(y)$ and a.a. $x \in \Omega$. Therefore, we have

$$\|\zeta\|_2 = \sqrt{\int_\Omega \zeta(x)^2 dx} \leq L\sqrt{|\Omega|}$$

which implies

$$\limsup_{\|y\|_2\to\infty} \frac{\|\widetilde{M}(y)\|_2}{\|y\|_2} \leq \limsup_{\|y\|_2\to\infty} \frac{L\sqrt{|\Omega|}}{\|y\|_2} = 0.$$

We conclude that (M2) is satisfied with $l = 0$, thus all the assumptions of Theorem (3.1) are satisfied and the existence of a solution of (4.4) is proved.

Remark 4.2. Typical examples of multimap B trivially satisfying condition (vii) of the application above are $B(x) = [0, L]$ for every $x \in \mathbb{R}$ or $B(x) = [0, \min\{x, L\}]$, for every $x \in \mathbb{R}$.

Acknowledgements

The paper is partially supported by the project “Problemi non locali di evoluzione: teoria e applicazioni” of the University of Modena and Reggio Emilia and by the Ministry of Education and Science of the Russian Federation (the Agreement number 02.a03.21.0008 of 24 June 2016).

Benedetti and Taddei are partially supported by Gruppo Nazionale di Analisi Matematica, Probabilitá e le loro Applicazioni of Istituto Nazionale di Alta Matematica.

Benedetti is partially supported by the project “Fondi ricerca di base-2015: Inclusioni differenziali e disuguaglianze variazionali in spazi astratti” of the Department of Mathematics and Computer Science of the University of Perugia.

Obukhovskii is supported by the Ministry of Education and Science of the Russian Federation in the frameworks of the project part of the state work quota (Project No 1.3464.2017/4.6).

References

[1] M. Abu Hamed, A.A. Nepomnyashchy, Domain coarsening in a subdiffusive Allen-Cahn equation. *Phys. D* **308** (2015), 52–58; DOI:10.1016/j.physd.2015.06.007.

[2] B. Amhad, J. Nieto, Existence of solutions for nonlocal boundary value problems of higher-order nonlinear fractional differential equations. *Abstr. Appl. Anal.* **2009** (2009), 9 pp.; DOI:10.1155/2009/494720.

[3] E. Ahmed, A.M.A. El-Sayed, H.A.A. El-Saka, Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models. *J. Math. Anal. Appl.* **325**, No 1 (2007), 542–553; DOI:10.1016/j.jmaa.2006.01.087.

[4] M. Alipour, L. Beghin, D. Rostamy, Generalized fractional nonlinear birth processes. *Methodol. Comput. Appl. Probab.* **17**, No 3 (2015), 525–540; DOI:10.1007/s11009-013-9369-0.

[5] C.T. Anh, T.D. Ke, On nonlocal problems for retarded fractional differential equations in Banach spaces. *Fixed Point Theory* **15**, No 2 (2014), 373–392.

[6] I. Benedetti, L. Malaguti, V. Taddei, Nonlocal semilinear evolution equations without strong compactness: theory and applications. *Bound. Value Probl.* **60** (2013), 18 pp.; DOI:10.1186/1687-2770-2013-60.

[7] I. Benedetti, V. Obukovskii, V. Taddei, On noncompact fractional order differential inclusions with generalized boundary condition and impulses in a Banach space. *J. Funct. Sp.* **2015** (2015), 1–10; DOI:10.1155/2015/651359.

[8] I. Benedetti, V. Taddei, M. Väth, Evolution problems with nonlinear nonlocal boundary conditions. *J. Dyn. Diff. Equat.* **25**, No 2 (2013), 477–503; DOI:10.1007/s10884-013-9303-8.

[9] S. Bochner, A.E. Taylor, Linear functionals on certain spaces of abstractly-valued functions. *Ann. Math.* **39**, No 4 (1938), 913–944; DOI:10.2307/1968472.

[10] L. Byszewski, Theorems about the existence and uniquencess of a solutions of a semilinear evolution nonlocal Cauchy problem. *J. Math. Anal. Appl.* **162**, No 2 (1991), 494–505; DOI:10.1016/0022-247X(91)90164-U.

[11] A. Cernea, A note on the existence of solutions for some boundary value problems of fractional differential inclusions. *Fract. Calc. Appl. Anal.* **15**, No 2 (2012), 183–194; DOI:10.2478/s13540-012-0013-4; https://www.degruyter.com/view/j/fca.2012.15.issue-2/ s13540-012-0013-4/s13540-012-0013-4.xml.
[12] A. Cernea, A note on mild solutions for nonconvex fractional semilinear differential inclusion. *Ann. Acad. Rom. Sci. Ser. Math. Appl.* **5**, No 1-2 (2013), 35–45.
[13] K. Deng, Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions. *J. Math. Anal. Appl.* **179** (1993), 630–637.
[14] Y. Ding, H. Ye, A fractional-order differential equation model of HIV infection of CD4C T-cells. *Mathematical and Computer Modelling* **50**, No 3-4 (2009), 386–392; DOI:10.1016/j.mcm.2009.04.019.
[15] M.M. El-Borai, Some probability densities and fundamental solutions of fractional evolution equations. *Chaos Solit. Fract.* **14**, No 3 (2002), 433–440; DOI:10.1016/S0960-0779(01)00208-9.
[16] K. Fan, Fixed point and minimax theorems in locally convex topological linear spaces. *Proc. Nat. Acad. Sci. U.S.A.* **38**, No 2 (1952), 121–126.
[17] J. García-Falset, S. Reich, Integral solutions to a class of nonlocal evolution equations. *Comm. Contemp. Math.* **12**, No 6 (2010), 1032–1054; DOI:10.1142/S021919971000410X.
[18] L.I. Glicksberg, A further generalization of the Kakutani fixed theorem with application to Nash equilibrium points. *Proc. Amer. Math. Soc.* **3** (1952), 170–174; DOI:10.2307/2032478.
[19] A. Hanyga, Multidimensional solutions of space-fractional diffusion equations. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **457**, No 2016 (2001), 2993–3005; DOI:10.1098/rspa.2001.0849.
[20] M. Kamenskii, V. Obukhovskii, P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces.* de Gruyter Ser. in Nonlinear Analysis and Applications # 7, Walter de Gruyter, Berlin - New York (2001).
[21] L.V. Kantorovich, G.P. Akilov, *Functional Analysis.* Pergamon Press, Oxford (1982).
[22] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North Holland Mathematics studies # 204, Elsevier (2006).
[23] K. Li, J. Peng, J. Gao, Existence results for semilinear fractional differential equations via Kuratowski measure of noncompactness, *Fract.*

Calc. Appl. Anal. **15**, No 4 (2012), 591–610; DOI:10.2478/s13540-012-0041-0; https://www.degruyter.com/view/j/fca.2012.15.issue-4/s13540-012-0041-0/s13540-012-0041-0.xml.

[24] X. Liu, Z. Liu, On the bang-bang principle for a class of fractional semilinear evolution inclusions. *Proc. Roy. Soc. Ed. A* **144**, No 2 (2014), 333–349; DOI: https://doi.org/10.1017/S030821051200128X.

[25] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations.* A Wiley-Interscience Publ., John Wiley and Sons, Inc., New York (1993).

[26] A. Paicu, I.I. Vrabie, A class of nonlinear evolution equations subjected to nonlocal initial conditions. *Nonl. Anal.* **72**, No 11 (2010), 4091–4100; DOI:10.1016/j.na.2010.01.041.

[27] B.J. Pettis, On the integration in vector spaces. *Trans. Amer. Math. Soc.* **44**, No 2 (1938), 277–304.

[28] I. Podlubny, *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications.* Mathematics in Science and Engineering # 198, Academic Press, Inc., San Diego (1999).

[29] I.I. Vrabie, *C_0-Semigroups and Applications* North-Holland Math. Studies # 191, North-Holland Publishing Co., Amsterdam (2003).

[30] J. Wang, M. Feckan, Y. Zhou, A survey on impulsive fractional differential equations. *Fract. Calc. Appl. Anal.* **19**, No 4 (2016), 806–831; DOI:10.1515/fca-2016-0044; https://www.degruyter.com/view/j/fca.2016.19.issue-4/fca-2016-0044/fca-2016-0044.xml.

[31] J. Wang, A.G. Ibrahim, M. Feckan, Nonlocal impulsive fractional differential inclusions with fractional sectorial operators on Banach spaces. *Appl. Math. Comp.* **257** (2015), 103–118; DOI:10.1016/j.amc.2014.04.093.

[32] J.Y. Yang, J.F. Huang, D.M. Liang, Y.F. Tang, Numerical solution of fractional diffusion-wave equation based on fractional multistep method. *Appl. Math. Model.* **38**, No 14 (2014), 3652–3661; DOI:10.1016/j.apm.2013.11.069.

[33] Z. Zhang, B. Liu, Existence of mild solutions for fractional evolution equations. *Fixed Point Theory* **15**, No 1 (2014), 325–334.

[34] Y. Zhou, F. Jiao, Existence of mild solutions for fractional neutral evolution equations. *Comp. Math. Appl.* **59**, No 3 (2010), 1063–1077; DOI:10.1016/j.camwa.2009.06.026.

[1] *Dipartimento di Matematica e Informatica*
Università degli Studi di Perugia
I–06123 Perugia, ITALY

e-mail: irene.benedetti@dmi.unipg.it

[2] *Faculty of Physics and Mathematics*
Voronezh State Pedagogical University
394043 Voronezh, RUSSIA
and the RUDN University
117198 Moscow, RUSSIA

e-mail: valerio-ob2000@mail.ru

[3] *Dipartimento di Scienze e Metodi per l'Ingegneria*
Università di Modena e Reggio Emilia
I–42122 Reggio Emilia, ITALY

e-mail: valentina.taddei@unimore.it

Received: April 30, 2017

Please cite to this paper as published in:

Fract. Calc. Appl. Anal., Vol. **20**, No 6 (2017), pp. 1424–1446,
DOI: 10.1515/fca-2017-0075