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MATERIAL AND DIGITAL TOOLS FOR GEOMETRY IN MATHEMATICS LABORATORY

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In this paper, we present two examples of activities carried out with material and digital tools following the methodology of mathematics laboratory. The aim is to show how activities with the two kinds of tools can enrich students' experience and allow them to construct mathematical meanings. These two examples contain the same artifact used with different didactical functionalities at different school grades.

INTRODUCTION

As reported in the Discussion Document of this ICMI study, the use of material manipulatives and digital tools is widespread in teaching and learning geometry. Our perspective is the use of both material manipulatives and digital tools within the same educational context, thinking about their intertwined use and basing on their respective potential (Maschietto, 2018). Two examples are discussed in this paper. The first example concerns the content of measurement of 2D figures (area and perimeter) at primary and low secondary school; it deals with Tangram puzzles and other artifacts, in their material and digital versions (including GeoGebra); the second example focuses on conic sections at high secondary school through the use of particular artifacts for geometry, the mathematical machines (Bartolini Bussi & Maschietto, 2011). All the activities are carried out within the methodology of the mathematics laboratory (Maschietto & Trouche, 2010).

With these two examples, this paper aims to contribute to the discussion on resources (topic C) and to provide elements for answering the following questions (Discussion document, p.12):

- What is the role of visual tools and manipulatives on geometrical processes?
- How can bridges be created between manipulation and visualization and mathematical thinking in geometry?
- How can students be helped to abstract from perceptive and concrete properties of physical objects?
- What is the place of language, signs, gestures in activities based on manipulatives?

USE OF ARTIFACTS IN MATHEMATICS LABORATORY

The history of mathematics laboratory in the European culture is long and interesting (Maschietto, 2015). In Italy, at the beginning of the 21st century, the UMI-CIIM (Italian Mathematical Union, Italian Commission on Mathematical Instruction) prepared a Curriculum for Mathematics for primary and secondary schools (Anichini et al., 2004), which gave a special emphasis to the mathematics laboratory in its methodological part. This emphasis is maintained in the Italian National Guidelines for Primary and Secondary Instruction, approved respectively in 2012 and 2010. Therefore, in the mathematics laboratory, relevant components are the use of tools and the interactions among people working together for the construction of mathematical meanings. These elements are central in the

Lowrie, T., Gutiérrez, A., & Emprin, F. (Eds.), *Pre-Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. 415-422). ICMI.

Theory of Semiotic Mediation (TSM, Bartolini Bussi & Mariotti, 2008), within which the mathematics laboratory activities reported in this paper are designed and analyzed. The main components of this theoretical framework are:

- The teacher uses an artifact to mediate mathematical meanings.
- An artifact is analyzed in terms of semiotic potential, defined as the double link between the artifact and students' personal senses and the artifact and mathematical knowledge; the conception of tasks for the students is based on this analysis.
- The didactical cycle of activities (working with artifacts, individual work, and collective discussion) always begins with the exploration of the artifact, according to the questions *How is the machine made?* and *What does the machine make?* (they support students' instrumental genesis; Vérillon & Rabardel, 1995); then, it focuses on the emergence of mathematical meanings embedded in the artifact by questions such as *Why does it make it?* and *What could happen if...?* (Bartolini et al., 2011).

The artifacts present in laboratory activities, as already written, can be both material and digital.

REFERENCES FOR LEARNING GEOMETRY

The Discussion Document highlights several components that characterize the teaching and learning of geometry. In this section, we refer to some of those components to support the analysis of the examples that will be presented in the following sections.

The first reference concerns the visualization of geometric figures. Duval (2005) distinguishes between iconic visualization and non-iconic visualization. The first is the most spontaneous approach to a drawing: a subject recognizes an object because its shape is similar to an already known object. In this sense, the shape resembles the typical shape of the real object represented and the contour of the object is mainly considered. Duval claims that this is the usual way of visualizing and pupils tend to deal with drawings through an iconic visualization. Under this visualization, the shapes appear stable: it is difficult to transform them into other similar shapes (for example, a square seen as a rhombus) or different ones (for example, non-congruent figures with the same area). According to a non-iconic visualization, a drawing is one of the representations of a geometrical object; the subject passes from the visual similarity of an object to a familiar shape to its geometric properties. This kind of visualization develops against the spontaneous way of seeing the shape by iconic visualization; passing from iconic visualization to non-iconic visualization is neither easy nor natural, but this transition is necessary to the learning of mathematical proof in geometry. Duval explains that using drawings to solve geometry problems involves three distinct operations: mereological deconstruction, instrumental deconstruction, and dimensional deconstruction. Mereological deconstruction consists of seeing the drawing as a union of superpositions or juxtapositions of figural units of the same dimension. Under an instrumental deconstruction, a drawing is considered the result of a construction process associated with a geometrical object, with the use of tools. Dimensional deconstruction is linked to a mathematical way of seeing drawings: a geometrical figure is considered as a set of figural units related to each other by geometric properties, and the drawing is analyzed accordingly. Non-iconic visualization is a necessary condition for dimensional deconstruction to be operational.

In addition, we consider three further aspects. First, we suppose that an iconic visualization supports a global viewpoint on a drawing, while a non-iconic visualization supports a local/punctual viewpoint. In this sense, relationships between global/local viewpoints should be considered in teaching and learning geometry. Then, another aspect linked to visualization is the relationship between static and dynamic aspects, well highlighted by research in dynamic geometry environment. The variation in geometric drawings allows the identification of invariant properties (the "invariants") and relations between the properties of the represented figures; for example, they are useful in the formulation of theorem statements (Baccaglini-Frank et al., 2009). Finally, the way of seeing the figures and the language to describe them are closely linked. In constructing a geometric vocabulary, the definitions and relationships between the shapes recognized by the students come into play. For example, the type of visualization and the consequent interpretation of the drawings can support exclusive or inclusive classifications of geometric figures (for example, they concern triangles and quadrilaterals, mostly studied in primary and lower secondary schools). Several research studies have been interested in the role of definitions and the implications of these definitions in the conceptualization of geometric figures (2D and 3D).

THE FIRST EXAMPLE: PERIMETER AND AREA AT GRADES 4-5 AND 7

The first example is taken from a collaborative project on mathematics laboratory (https://sites.google.com/view/diffusioneculturamodena2020/home-page) between teachers and researchers. Two teaching experiments were designed and carried out on the topic of area and perimeter, one in primary school (grade 4-5, 4 classes) and one in low secondary school (grade 7, 10 classes). The choice of the topic is based on the results of research (Fandiño Pinilla & D'Amore, 2006) which highlight how these two concepts are kept separate in teaching practice (at least, in Italy) and how consequently students (but also teachers) have difficulty relating them. In particular, the tasks for students often deal with the conservation of areas and perimeter or contextual increase/decrease of measurements without discussing the variation of the area of isoperimetric figures or the variation of the perimeter of equivalent figures. The manipulatives (in their material and digital versions) used in these experiments were the Tangram puzzle, the geoboard (grade 4-5 only), and the mathematical machine for tracing the ellipse by the gardener's way (grade 7 only).

The didactical intervention for grade 7 is composed of three phases. In the first phase, the Tangram puzzle was initially explored in its material version (Figure 1, left), with the objectives of identifying the geometrical figures that constitute it (isosceles right triangles, square, rhombus) and the property (the invariance of the area) common to all the combinations of the seven pieces of the puzzle. The square configuration was represented on paper and pencil and/or by paper folding, favoring the transition from 3D to 2D representations and implementing instrumental and dimensional deconstructions. To support these two processes, the students were asked to write the procedure for drawing the square in small groups (Figure 1) and to exchange the texts with other groups for reviewing and obtaining a correct final version. A Geogebra book with incorrect Tangram squares (Figure 1, right) was assigned to consolidate the construction procedure and the properties of the figures. The digital tool is used to propose tasks that cannot performed with the material Tangram (GeoGebra figures offer strategy feedback by dragging some vertices) and to collect and assess qualitatively students' answers (they had to upload their answers). In this way, it is possible to assign an individual work, as set by the TSM. After this task, the students played the puzzle game in small groups; there the reproduction of the Tangram combinations is based on mereological deconstruction by juxtaposition accompanied by the gestures of moving pieces for composing new combinations.

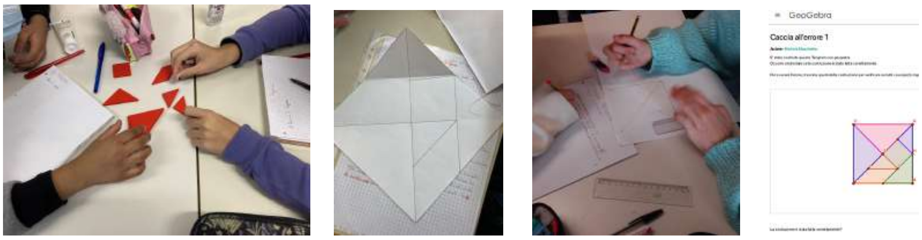
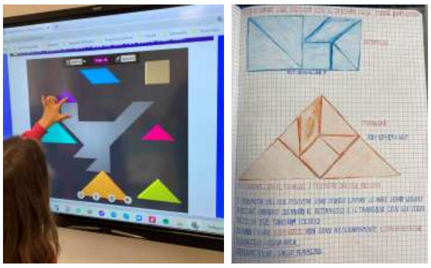


Figure 1: group work; construction of the tangram in paper and pencil; GeoGebra book

The activities with the Tangram puzzle allow students to pay attention to the invariance of the area for each combination of the 7 pieces, and to the measurement of this area and the area of the components using the smallest triangle taken as the unit. Furthermore, this game represents a good context to ask questions about the perimeter of the combinations and to look for combinations with minimum/maximum perimeter, already in primary school (Fig. 2). By the comparison of different combinations, the non-preservation of the perimeter emerges (Fig. 2, center and right).



The perimeters of the two polygons are different, even if the areas are the same since we have built the rectangle and the triangle with the same pieces of two identical tangram. Therefore, equivalent figures are not necessarily isoperimetric.

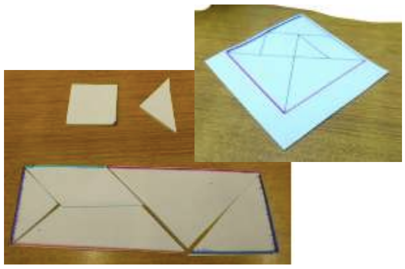
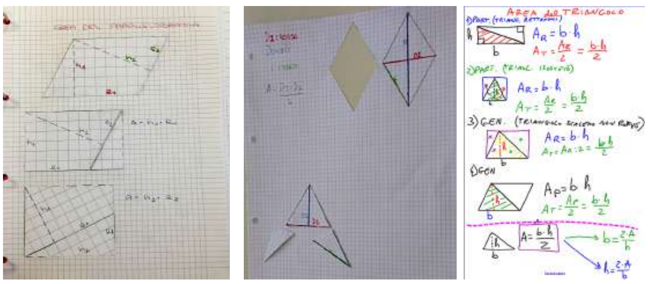


Figure 2: Tangram at IWB; notebook from grade 4-5; comparison of perimeter

The gestures performed by the Tangram puzzle related to the juxtaposition of pieces and the manipulation of equivalent combinations by moving pieces were recalled in studying the areas of quadrilaterals and obtaining their corresponding formulas (Figures 3, left and center). In this case, a digital tool is used to support collective discussion and mereological deconstruction. This work allows students to understand those formulas (Figure 3, right), which are often studied by heart.



S1: We obtained the formulas ourselves and I memorized the formulas better.
S2: Finding the formulas was an extra stimulus and then we remember them better.

Figure 3. Puzzle of quadrilaterals and students' opinions on these activities

In the second phase, the construction of the mathematical machine tracing the ellipse ("ellipsograph") following the gardener's way was proposed to grade 7 classes for asking questions on non-equivalent isoperimetric figures. It was built by students with recycled material (cardboard, pins, and string; Figure 4, left and center). The exploration followed the structure of the questions referring to the TSM and it was guided by a worksheet (Figure 4, center). The ellipse was also traced in the GeoGebra environment (Figure 4, right); students were invited to explore it by varying the parameters (length of the string and focal distance) as homework. The construction in GeoGebra allows students to go beyond the physical constraints of the mathematical machine: by varying the parameters, the students can see different drawings of the ellipse which cannot all be drawn with the ellipsograph. This task aims to support iconic visualization for those students who meet the curve for the first time.

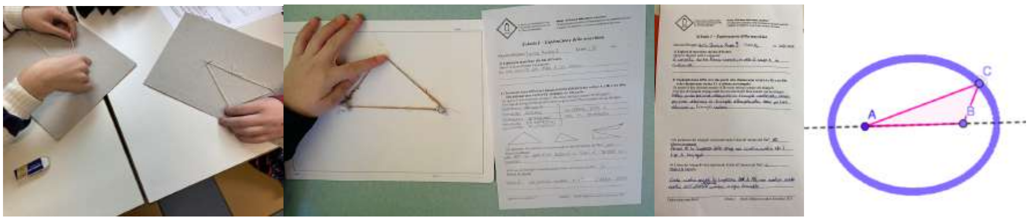


Figure 4. Ellipsograph, worksheets, and GeoGebra figure

The teaching experiment ends with the proposal of a challenge (third phase): "Build a machine that does the opposite of the ellipsograph: equivalent non-isoperimetric triangles". This assignment requires identifying a change in perimeter and ensuring the invariance of the area; that is another variation to study, and it is a real problem to solve for the students (referring to the question *What could happen if...*). Different machines were proposed (Figure 5, left and center): some of them worked, and others did not. During the classwork, the teacher discussed the solutions, also within the GeoGebra environment, and proposed other problems on equivalent figures. In this case, too, new tasks are proposed by GeoGebra books (Figure 4, right), aiming to support a non-iconic visualization.

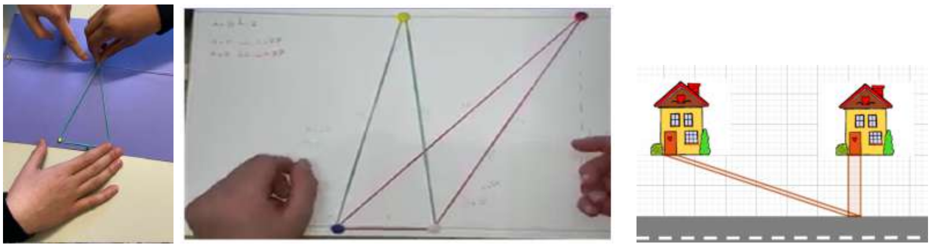


Figure 5. The new machine and new problems within GeoGebra books

THE SECOND EXAMPLE: CONIC SECTIONS AT GRADE 11

The study of conic sections is a classic mathematical content in Italian high secondary schools for grade 11 students. The National Guidelines for High Secondary School contain a precise reference to the approach to be followed, in which the conic sections must be studied from both a synthetic and analytical geometric point of view. The mathematical machines (they belong to the Museum System and Botanical Garden of the University of Modena e Reggio Emilia, www.mmlab.unimore.it) are suitable for proposing a synthetic approach (i.e., Maschietto & Bartolini Bussi, 2011). Two types of conicographs were considered in the didactical cycles (Dondi & Maschietto, 2022): conicographs for ellipse, parabola, and hyperbola functioning by taut string, and crossed-parallelogram conicographs for ellipse and hyperbola. They all embed the metric definitions (i.e., with foci and directrix).

The activities are structured in three phases: in the initial phase, all those elements considered necessary for the exploration of the mathematical machines are presented to students; the central phase is dedicated to the exploration of the conicographs; in the final phase, the theory of conic sections is presented through historical references. The mathematical machines are proposed with two different didactical functionalities (Maschietto, 2015): the taut string conicographs are used to introduce the new mathematical curves, their definitions, and properties; while the crossed-parallelogram conicographs are used to propose a unitary vision of conic sections and to approach the meanings of tangent and normal straight lines to a curve at a point of the curve itself. In particular, for taut string conicographs, the definition of the geometric locus generated by the machine is required, while for crossed-parallelogram machines the justification of the type of curve drawn is solicited (question *Why does it make it?*). Students' work is guided by worksheets, constructed consistently with the framework of the TSM.

We report an excerpt from the exploration of the gardener's ellipsograph by a group of grade 11 students (called A1, A2, A3, and A4). They generally recognize the drawn curve according to an iconic and global visualization related to the curve seen at low secondary school (above all in astronomy), but the analysis of the trace and the identification of parameters, variables and invariants for the definition of the geometric locus is not trivial for them, because it requires a non-iconic visualization and a local viewpoint on the curve. In this excerpt, first, variables and fixed lengths are identified (#3, #4), then a variable triangle (#8) as well as its isoperimetry property (#9, #14) even if it is not explicitly expressed in these terms and does not concern the vertex P (until #15). Nevertheless, the definition of the geometric locus (local viewpoint) needs a formulation involving the point P. We note the use of the terms "the same" (#9, #12, #14) and "equal" (#15) instead of "constant".

2	A2:	So, what does it vary [], [*he forms a triangle with the thread, Figure 6 left*]?
3	A4:	the distance between the two pins […]
4	A4:	this does not vary [*he points to the focal distance*] and this varies [*he points to the other sides of the triangle while A2 moves the thread*] […]
5	A3:	then it varies the distance [*he tightens the string and moves it around the pins*]
6	A*:	the distance of P from F1 and F2 varies [*the students write on their worksheet*]

The students then read the notes and wondered if there were other variables. […]

7	A2:	Beyond the point [*P*], the length of the two sides also varies. […]
8	A2:	Let's look at it as a triangle.
9	A3:	This [*side of the triangle*] plus this [*other side of the triangle*] is always the same.
10	A2:	Let's see more triangles. The two angles vary, and the lengths of the sides vary... up to approximately this limit... [*Figure 6, center*].
11	A4:	They always vary.

A2 moves the string.

13	A3:	That is, if we measure this [*he points to the distance between the foci and P*]
15	A2:	The sum of these two [*sides of the triangle between the focus and point P*] is equal.

Question 3 of the worksheet asked to draw three different configurations of the machine. The objective was to stimulate students to provide a graphical representation of the parameters and the invariant (i.e., by measuring the length of the string and the sides of triangles). However, all answers contain iconic and global representations (Figure 6, right), even if sometimes the invariant is written.

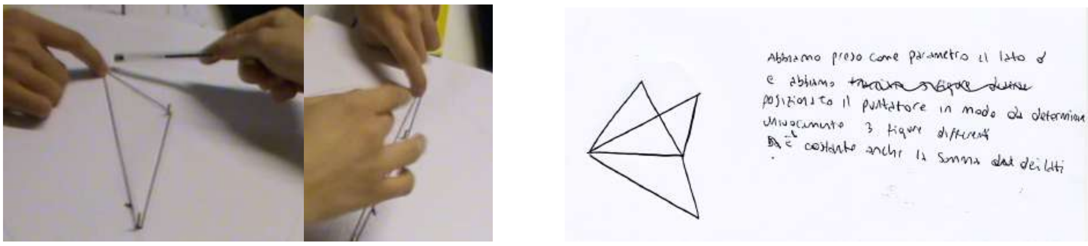


Figure 6. Triangle configuration and representation of these configurations (Question 3)

Contrary to a classical lesson in which the definitions of conic sections are stated (and the students often learn them by heart), working with the mathematical machines allows them to discover and build the definitions and to identify properties, by exploring and exploiting the machines. In the identification of parameters, variables and invariants, the analysis shows some difficulties in referring to the meanings of terms they have already encountered in another context before the study of conic sections. The use of different machines allows students' conceptualization to be tested: for example, each conicograph offers different material constraints and different representations for the parameters of the curves. The construction of the ellipsograph in the GeoGebra environment (as in Figure 4, right), asked after the work with the material machine, highlights the need to work more on parameters, variables, and invariants: for instance, in that construction, the possibility of choosing foci as free points hides the choose of that parameter of the curve. Compared with the first example discussed in this paper, GeoGebra is used with different didactical functionalities: there the students were supported to observe the shape of the ellipse depends on the variation of its parameters, while here the students were asked to make explicit their conceptualization of variables and parameters by constructing the curve.

CONCLUDING REMARKS

This paper presents two examples of didactical activities to contribute to the discussion about the use of material and digital tools in learning and teaching geometry. Through them, we aim to show how a bridge between the manipulation of artifacts and mathematical thinking in geometry can be created, proposing tasks by which the students describe procedures and drawings, represent the mathematical objects in different environments, and look for regularities and invariants. Concerning the use of only material artifacts, the use of digital tools allows students to perform tasks and problems that can not be proposed through the formesr; for teachers, it can be possible to structure sub-tasks and implement evaluation and strategy feedback, supporting in this way students learning by collective and individual tasks. On the other hand, the use of material artifacts encourages collaborative group work and students' exploration processes within a rich semiotic activity. In our perspective, each kind of tool offer complementary representations and approaches to mathematical objects. In the two examples, the activities carried out by the students through the digital tool often corresponded to individual activities, completing the didactical cycles of the TSM.

The artifacts are chosen following their preliminary analysis (semiotic potential in the TSM), which suggests which characteristics can be exploited with students at a certain grade and how to construct appropriate tasks. For instance, the use of the gardeners' ellipsograph at different grades is based on this kind of analysis. Furthermore, several tasks can involve the same artifact, because the emergence and the construction of mathematical meanings requires time (also for instrumental genesis), and it is not always possible to activate and end these processes in only one or two lessons. Another common element between the two examples is that the gestures made during a task can be performed in other tasks or with other artifacts. For instance, the gesture of moving pieces in the Tangram puzzle for composing new configurations is used then for quadrilaterals, and in general for transforming figures into equivalent ones that are not congruent. For conic section tracers, the gesture of usage of tightening a string is common to all three conicographs; in addition, it corresponds to creating a segment (cognitive root for the straight line) and gives a strong experience of a straight line. The analyses show that students very often have an iconic visualization, which can be questioned by tasks involving mereological deconstruction and the comparison between the variation of geometrical quantities and the invariant properties. Thus, the transition to a non-iconic visualization is supported.

In our classes, the manipulation of material artifacts, intertwined with digital tools, supported students' engagement in geometry activities. And this is not a secondary aspect for student' learning.

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