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Research paper

A random elastic traffic equilibrium problem via stochastic quasi-variational inequalities

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ABSTRACT

The aim of the paper is to introduce a random elastic traffic equilibrium problem in a Hilbert space setting. The equilibrium condition is expressed by a random extension of the elastic Wardrop principle. Its characterization with a stochastic quasi-variational inequality is proved. Under suitable assumptions, the existence of a random equilibrium distribution is established. Furthermore, a numerical scheme to compute the random elastic traffic equilibrium distribution is presented. Finally a numerical example is discussed.

1. Introduction

The paper deals with a random elastic traffic equilibrium problem, namely a traffic problem where not only the perception of the expected equilibrium solution influences the choice of the network users but also the data are affected by a certain degree of uncertainty. In addition, it studies the stochastic quasi-variational inequality which equivalently expresses the random elastic Wardrop equilibrium condition. In particular, the existence of stochastic elastic equilibrium distributions is ensured under suitable assumptions. The stochastic quasi-variational characterization allows us to propose a numerical scheme to compute equilibrium distributions. Thanks to the equivalence of a stochastic quasi-variational inequality with a fixed point problem of a projection operator, the extragradient method has been extended to solve stochastic quasi-variational inequalities. Further, by means of an example connected to significant traffic elastic network models, the behavior of the random traffic elastic equilibrium problem is illustrated.

The random formulation of the traffic elastic networks is essential because the path flows as well as the travel demand often vary in a non-regular and unpredictable manner. Such an uncertainty can be not only caused by several factors such as the particular hour of the day, the particular day of the week, the particular week of the year, but also by a sudden accident or a traffic network maintenance work. We highlight that a Hilbert space setting is considered for the model, since it allows us to reach existence results. Indeed, an existence result is obtained applying the general theory of quasi-variational inequalities in infinite dimensional spaces.

It is worth mentioning that stochastic variational inequalities have been studied in several papers, see for example [1–5], where existence and uniqueness results are proved and approximation procedures are analyzed. Furthermore, several scholars have been

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studied the applications of stochastic variational inequalities to random equilibrium problems on networks (see for instance [6–10]). In [11], the random time-dependent oligopolistic market equilibrium problem from the policymaker's point of view is studied.

Quasi-variational inequalities, which are a special case of the quasi-equilibrium problems, have been considered in different settings under various assumptions (see, for example, [12–14] and the references therein for theoretical existence results). Such inequalities have an important role in the study of many problems come from optimization, game theory and transport planning. For this reasons, many recent articles concern stability, regularity and sensitivity analysis, such as [15–17] where several types of approximations and regularizations are discussed. A notable generalization is to examine the stochastic framework, namely to analyze the stochastic quasi-variational inequality problem. Many important and useful applications of these mathematical tools are known, from Nash games to transportation network equilibria. From the point of view of computational methods, quasi-variational inequalities do not have an extensive literature: we may address the reader only to few papers cited in [18]. Therefore, one of our aims is to propose a numerical scheme to solve stochastic quasi-variational inequalities. Up to our knowledge, there are not such numerical schemes available in the literature. Thus, our procedure is a preliminary approach to the development of numerical methods to compute solutions to stochastic quasi-variational inequalities. We underline that to overcome the implicit formulation of stochastic quasi-variational inequalities, we proceed by approximation updating at each iteration the evaluation of the elastic traffic demand in the approximated solution obtained at previous iteration. At every step a stochastic variational inequality has to be solved. We refer to the procedures similar to the ones in [19,20].

The contents of this paper are organized as follows. In Section 2, some preliminary notations are presented. In Section 3 the detailed random elastic traffic equilibrium model is introduced. A random elastic Wardrop equilibrium principle is given. Moreover, a stochastic quasi-variational characterization of the equilibrium is obtained. In Section 4 the existence of a random equilibrium flow is proved under suitable assumptions. In Section 5 a numerical method to solve the stochastic quasi-variational inequality which expresses the random elastic Wardrop equilibrium condition is proposed. In addition, a numerical example of random elastic traffic equilibrium network is studied and its random equilibrium distributions is determined by using the algorithm discussed. Finally, Section 6 summarizes our results and future works.

2. Notations

Let $(\Omega, A, \mathbb{P})$ be a probability space. Let $L^2(\Omega, \mathbb{R}^k, \mathbb{P})$ be the Hilbert space of random vectors $v : \Omega \to \mathbb{R}^k$ such that the expectation

$$\mathbb{E}\|v\|^2 = \int_\Omega \|v(\omega)\|^2 d\mathbb{P}(\omega)$$

is finite. We introduce the bilinear form on $(L^2(\Omega, \mathbb{R}^k, \mathbb{P}))^* \times L^2(\Omega, \mathbb{R}^k, \mathbb{P})$ as

$$\langle\langle \phi, w \rangle\rangle_{\mathbb{E}} = \int_\Omega \langle \phi(\omega), w(\omega) \rangle \, d\mathbb{P}(\omega),$$

where $\phi \in (L^2(\Omega, \mathbb{R}^k, \mathbb{P}))^* = L^2(\Omega, \mathbb{R}^k, \mathbb{P})$, $w \in L^2(\Omega, \mathbb{R}^k, \mathbb{P})$ and

$$\langle \phi(\omega), w(\omega) \rangle = \sum_{l=1}^{k} \phi_l(\omega) w_l(\omega).$$

Given a measurable set $\Omega \subset \mathbb{R}^n$, we denote by $|\Omega|$ its Lebesgue measure. If $\Omega \subset \mathbb{R}^n$ is a measurable set with strictly positive and finite measure and $f : \Omega \to \mathbb{R}$ is a measurable map, we shall denote the average integral by

$$⨍_\Omega f(x)\, dx = \frac{1}{|\Omega|} \int_\Omega f(x)\, dx.$$

Given a finite set E, the cardinality of E is denoted by $\sharp E$.

3. The model

Let us introduce a traffic network $\mathcal{N}$, which is represented by a graph $G = [N, L]$, where N is the set of nodes (i.e. cross-roads, airports, railway stations) and L is the set of directed links between the nodes. We denote by a a generic link of the network connecting a pair of nodes and by r a generic path consisting of a sequence of links which connect an origin/destination (O/D) pair of nodes. In the network there are n links and m paths. Moreover we introduce the following notations:

- $\mathcal{W}$ denotes the set of O/D pairs and $\sharp \mathcal{W} = l$;
- given an O/D pair w_j, $j = 1, \dots, l$, $\mathcal{R}_j$ denotes the set of paths connecting the pair w_j and $\sharp \mathcal{R}_j = m_j$;
- $\mathcal{R}$ denotes the set of all the paths in the network and $\sharp \mathcal{R} = m$.

Therefore, it results

$$m = \sum_{j=1}^{l} m_j.$$

and evidently $m > l$.

Table 1
List of symbols used in the random traffic equilibrium model.

Symbol	Description
f	Link flow function
F	Path flow function
$\lambda,\ \mu$	Lower and upper capacity constraints
ρ	Elastic travel demand
Δ	Link-path incidence matrix
Φ	Pair-path incidence matrix
c	Link cost function
C	Path cost function

Let $f \in L^2(\Omega, \mathbb{R}^n, \mathbb{P})$ be the random link flow vector-function and let $F \in L^2(\Omega, \mathbb{R}^m, \mathbb{P})$ denote the random path flow vector-function. The relationship between link and path flows is given by:

$$f_a(\omega) = \sum_{r=1}^{m} \delta_{ar} F_r(\omega), \quad a = 1, \dots, n, \mathbb{P} - \text{a.s.} \quad \text{or} \quad f(\omega) = \Delta F(\omega), \quad \mathbb{P} - \text{a.s.},$$

where Δ is the link-path incidence matrix, whose typical entry δ_{ar} is 1 if the link a is contained in the path r and 0 otherwise. Let $\lambda, \mu \in L^2(\Omega, \mathbb{R}^m, \mathbb{P})$ be the random capacity constraints functions, such that $0 \le \lambda_r(\omega) < \mu_r(\omega)$, for every $r = 1 \dots m$, $\mathbb{P}$-a.s. Let us assume $\lambda_r(\omega) \le F_r(\omega) \le \mu_r(\omega)$, for every $r = 1 \dots m$, $\mathbb{P}$-a.s. Let Φ be the O/D pair-path incidence matrix, whose typical entry φ_{jr} is 1 if path r connects the O/D pair w_j and 0 otherwise. Let $\rho \in L^2(\Omega \times \mathbb{R}^m, \mathbb{R}^l, \mathbb{P})$ represent the random elastic travel demand vector-function, depending on the expected equilibrium pattern. We summarize the symbols used in the model in Table 1.

Thus we consider

$$D = \left\{ F \in L^2(\Omega, \mathbb{R}^m, \mathbb{P}) : \ \lambda(\omega) \le F(\omega) \le \mu(\omega), \ \mathbb{P} - \text{a.s.} \right\}, \tag{1}$$

which is a bounded, closed and convex subset of $L^2(\Omega, \mathbb{R}^m, \mathbb{P})$.

We require that a random traffic conservation law is fulfilled, namely random flows and hence travelers are not lost or generated in the network. In the static case, this traffic conservation law is expressed saying that the sum of the path flows associated to each O/D pair must be equal to the travel demand for that O/D pair. A more general situation is the one in which the random travel demand is supposed to depend on the users' evaluation of flows and it is plausible to expect that travelers evaluate the network practicability by an average on the sample space. Therefore, we consider a formulation of the traffic equilibrium problem where the random travel demand ρ is not fixed but depends on the random equilibrium distribution H and, then, the random conservation law reads as

$$\sum_{r=1}^{m} \varphi_{jr} F_r(\omega) = \fint_{\Omega} \rho_j(\omega, H(\sigma)) d\mathbb{P}(\sigma), \quad j = 1, \dots, l, \ \mathbb{P} - \text{a.s.},$$

or

$$\Phi F(\omega) = \fint_{\Omega} \rho(\omega, H(\sigma)) d\mathbb{P}(\sigma), \quad \mathbb{P} - \text{a.s.}.$$

Hence, the set of random feasible flows depends also on the expected random equilibrium distribution. As a consequence, it is the value of the multifunction $\mathbb{K} : D \rightrightarrows D$ defined as follows:

$$\mathbb{K}(H) = \left\{ F \in D : \ \Phi F(\omega) = \fint_{\Omega} \rho(\omega, H(\sigma)) d\mathbb{P}(\sigma), \ \mathbb{P} - \text{a.s.} \right\}.$$

To ensure the nonemptyness of $\mathbb{K}(H)$, we assume that $\Phi\lambda(\omega) \le \Phi F(\omega) \le \Phi\mu(\omega)$, $\mathbb{P}$-a.s.

Let $c \in L^2(\Omega \times \mathbb{R}^n, \mathbb{R}^n, \mathbb{P})$ denote the random link cost vector-function. We highlight that we are concerning the general case, namely with the case of asymmetric costs, i.e. the cost on a link does not depend only on the flow on that link, but it is affected by the flows on all the links in the network. Let us denote by $C \in L^2(\Omega \times \mathbb{R}^m, \mathbb{R}^m, \mathbb{P})$ the random path cost vector-function. It results

$$C_r(\omega, F(\omega)) = \sum_{a=1}^{n} \delta_{ar} c_a(\omega, f(\omega)), \quad r = 1, \dots, m, \ \mathbb{P} - \text{a.s.},$$
$$\text{or} \quad C(\omega, F(\omega)) = \Delta^T c(\omega, f(\omega)) = \Delta^T c(\omega, \Delta F(\omega)), \quad \mathbb{P} - \text{a.s.}$$

Now we are able to introduce the random elastic Wardrop principle.

Definition 3.1. A feasible flow $H \in \mathbb{K}(H)$ is a random elastic equilibrium flow if and only if for each O/D pair $w_j \in \mathcal{W}$, for each couple of paths $s, q \in \mathcal{R}_j$, and $\mathbb{P}$-a.s., it results

$$C_q(\omega, H(\omega)) < C_s(\omega, H(\omega)) \ \Rightarrow \ H_q(\omega) = \mu_q(\omega) \ \textit{or} \ H_s(\omega) = \lambda_s(\omega). \tag{2}$$

Definition 3.1 means that the users choose the less expensive routes. Let us show the characterization of the random elastic equilibrium condition by means of a stochastic quasi-variational inequality.

Theorem 3.1. *A feasible flow $H \in \mathbb{K}(H)$ is a random elastic equilibrium flow if and only if*

$$H \in \mathbb{K}(H): \quad \langle\langle C(H), F - H \rangle\rangle_{\mathbb{E}} \geq 0, \quad \forall F \in \mathbb{K}(H), \tag{3}$$

namely

$$H \in \mathbb{K}(H): \quad \int_{\Omega} \sum_{j=1}^{l} \sum_{r \in \mathcal{R}_j} C_r(\omega, H(\omega)) \left(F_r(\omega) - H_r(\omega)\right) d\mathbb{P}(\omega) \geq 0, \quad \forall F \in \mathbb{K}(H),$$

Proof. At first, let us suppose that condition (2) holds. For each O/D pair $w_j \in \mathcal{W}$, let us consider the following sets

$$\begin{aligned} A &= \{q \in \mathcal{R}_j : \ H_q(\omega) < \mu_q(\omega), \quad \mathbb{P}-\text{a.s.}\}, \\ B &= \{s \in \mathcal{R}_j : \ H_s(\omega) > \lambda_s(\omega), \quad \mathbb{P}-\text{a.s.}\} \end{aligned}$$

By virtue of (2), it follows

$$C_q(\omega, H(\omega)) \geq C_s(\omega, H(\omega)), \quad \forall q \in A, \ \forall s \in B, \ \mathbb{P}-\text{a.s.}$$

Hence, there exists a function $\gamma_{w_j} : \Omega \to \mathbb{R}$ such that

$$\inf_{q \in A} C_q(\omega, H(\omega)) \geq \gamma_{w_j}(\omega) \geq \sup_{s \in B} C_s(\omega, H(\omega)), \quad \mathbb{P}-\text{a.s.}$$

Let $F \in \mathbb{K}(H)$ be arbitrary. Thus, for every $r \in \mathcal{R}_j$ such that $C_r(\omega, H(\omega)) < \gamma_{w_j}(\omega)$, $\mathbb{P}$-a.s., we deduce $r \notin A$. As a consequence, we have $H_r(\omega) = \mu_r(\omega)$, $\mathbb{P}$-a.s., and $F_r(\omega) - H_r(\omega) \leq 0$, $\mathbb{P}$-a.s. Therefore, we obtain

$$(C_r(\omega, H(\omega)) - \gamma_{w_j}(\omega))(F_r(\omega) - H_r(\omega)) \geq 0, \quad \mathbb{P}-\text{a.s.}$$

With similar arguments, for each $r \in \mathcal{R}_j$ such that $C_r(\omega, H(\omega)) > \gamma_{w_j}(\omega)$, $\mathbb{P}$-a.s., it follows $r \notin B$ and

$$(C_r(\omega, H(\omega)) - \gamma_{w_j}(\omega))(F_r(\omega) - H_r(\omega)) \geq 0, \quad \mathbb{P}-\text{a.s.}$$

Then, it results

$$\sum_{r \in \mathcal{R}_j} C_r(\omega, H(\omega))(F_r(\omega) - H_r(\omega)) \geq \gamma_{w_j}(\omega) \sum_{r \in \mathcal{R}_j} (F_r(\omega) - H_r(\omega)) = \gamma_{w_j}(\omega)(\rho_{w_j}(\omega) - \rho_{w_j}(\omega)) = 0$$

and, summing up for each $w_j \in W$, $j = 1, \dots, l$, and integrating on Ω, the stochastic quasi-variational inequality (3) holds.

Conversely, we argue by contradiction. Let us assume that (2) is not verified, hence there exist an O/D pair $w_j \in \mathcal{W}$, two paths $q, s \in \mathcal{R}_j$ and a set $G \subseteq \Omega$ with positive measure such that

$$C_q(\omega, H(\omega)) < C_s(\omega, H(\omega)) \Longrightarrow H_q(\omega) < \mu_q(\omega) \text{ and } H_s(\omega) > \lambda_s(\omega), \quad \mathbb{P}-\text{a.s. in } G.$$

Let us consider

$$\delta(\omega) = \min\{\mu_q(\omega) - H_q(\omega), H_s(\omega) - \lambda_s(\omega)\}, \quad \mathbb{P}-\text{a.s.}$$

It results $\delta(\omega) > 0$, $\mathbb{P}$-a.s. in G. Let us fix $F \in \mathbb{K}(H)$ such that

$$\begin{aligned} F_q(\omega) &= \begin{cases} H_q(\omega) + \delta(\omega), & \mathbb{P}-\text{a.s. in } G, \\ H_q(\omega), & \mathbb{P}-\text{a.s. in } \Omega \setminus G, \end{cases} \\ F_s(\omega) &= \begin{cases} H_s(\omega) - \delta(\omega), & \mathbb{P}-\text{a.s. in } G, \\ H_s(\omega), & \mathbb{P}-\text{a.s. in } \Omega \setminus G, \end{cases} \\ F_r(\omega) &= H_r(\omega), \quad \text{for } r \neq q, s, \ \mathbb{P}-\text{a.s.} \end{aligned}$$

It follows that $F \in \mathbb{K}(H)$ and

$$\begin{aligned} \langle\langle C(H), F - H \rangle\rangle_{\mathbb{E}} &= \int_{\Omega} \sum_{j=1}^{l} \sum_{r \in \mathcal{R}_j} C_r(\omega, H(\omega))(F_r(\omega) - H_r(\omega)) d\mathbb{P}(\omega) \\ &= \int_{E} \delta(\omega)[C_q(\omega, H(\omega)) - C_m(\omega, H(\omega))] d\mathbb{P}(\omega) < 0, \end{aligned}$$

which is a contradiction. Therefore, the proof is completed. □

4. An existence result for equilibrium distributions

Making use of a general existence theorem proved in [21], we can obtain that there exists a solution to the stochastic quasi-variational inequality (3), under suitable assumptions on the path cost vector-function C and on the elastic travel demand vector-function ρ.

Theorem 4.1. *Let $C \in L^2(\Omega \times \mathbb{R}^m, \mathbb{R}^m, \mathbb{P})$ and $\rho \in L^2(\Omega \times \mathbb{R}^m, \mathbb{R}^l, \mathbb{P})$ be two functions such that*

(i) $C(\omega, F)$ is measurable in ω, for every $F \in L^2(\Omega, \mathbb{R}^m, \mathbb{P})$, continuous in F, $\mathbb{P}$-a.s., and

$$\exists \gamma \in L^2(\Omega): \ \|C(\omega, F)\| \leq \gamma(\omega) + \|F\|, \quad \forall F \in L^2(\Omega, \mathbb{R}^m, \mathbb{P}),\ \mathbb{P} - \text{a.s.};$$

(ii) $\rho(\omega, F)$ is measurable in ω, for every $F \in L^2(\Omega, \mathbb{R}^m, \mathbb{P})$, continuous in F, $\mathbb{P}$-a.s., and

$$\exists \psi \in L^1(\Omega): \ \|\rho(\omega, F)\| \leq \psi(\omega) + \|F\|^2, \quad \forall F \in L^2(\Omega, \mathbb{R}^m, \mathbb{P}),\ \mathbb{P} - \text{a.s.};$$

(iii) $\exists \nu(\omega) \geq 0$, $\mathbb{P}$-a.s., $\nu \in L^2(\Omega)$ such that

$$\|\rho(\omega, F) - \rho(\omega, G)\| \leq \nu(\omega)\|F - G\|, \quad \forall F, G \in L^2(\Omega, \mathbb{R}^m, \mathbb{P}),\ \mathbb{P} - \text{a.s.}$$

Then, the stochastic quasi-variational inequality (3) admits a solution.

Proof. Taking into account assumptions (i) and (ii) and if $H \in L^2(\Omega, \mathbb{R}^m, \mathbb{P})$, it follows

$$\omega \mapsto C(\omega, H(\omega)) \in L^2(\Omega, \mathbb{R}^m, \mathbb{P})$$

and

$$\omega \mapsto \rho(\omega, H(\omega)) \in L^1(\Omega, \mathbb{R}^l, \mathbb{P}).$$

Furthermore, by assumptions (i) and (ii), C and ρ are Nemytskii operators for assumptions (i) and (ii). Thus, if $\{H^n\}$ is a sequence such that $H^n \to H$, in $L^2(\Omega, \mathbb{R}^m, \mathbb{P})$, we have

$$\|C(H^n) - C(H)\| \to 0$$

and

$$\|\rho(H^n) - \rho(H)\| \to 0.$$

Therefore, the functions C and ρ are continuous in $L^2(\Omega, \mathbb{R}^m, \mathbb{P})$ and $L^1(\Omega, \mathbb{R}^l, \mathbb{P})$, respectively.

We claim that $\mathbb{K}(H)$ is a closed multifunction, namely if for every two arbitrary sequences $\{H^n\}$ and $\{F^n\}$ such that $H^n \to H$ and $F^n \to F$ in $L^2(\Omega, \mathbb{R}^m, \mathbb{P})$, with $F^n \in \mathbb{K}(H^n)$, for every $n \in \mathbb{N}$, then $F \in \mathbb{K}(H)$. Indeed, we fix two arbitrary convergent sequences $\{H^n\}$ and $\{F^n\}$ in $L^2(\Omega, \mathbb{R}^m, \mathbb{P})$. Since $F^n \in \mathbb{K}(H^n)$, we have that $\lambda_r(\omega) \leq F_r^n(\omega) \leq \mu_r(\omega)$, for every $r = 1, \dots, m$, $\mathbb{P}$-a.s. By virtue of the convergence of the sequence $\{F^n\}$ in $L^2(\Omega, \mathbb{R}^m, \mathbb{P})$, we obtain that F satisfies capacity constraints. In addition, for $j = 1, \dots, l$, it results

$$\begin{aligned}
&\left\| \int_\Omega \rho(\omega, H^n(\sigma)) d\mathbb{P}(\sigma) - \int_\Omega \rho(\omega, H(\sigma)) d\mathbb{P}(\sigma) \right\| \\
&\qquad \leq \int_\Omega \|\rho(\omega, H^n(\sigma)) - \rho(\omega, H(\sigma))\| d\mathbb{P}(\sigma) \\
&\qquad \leq \nu(\omega) \int_\Omega \|H^n(\sigma) - H(\sigma)\| d\mathbb{P}(\sigma) \\
&\qquad \leq \nu(\omega) \left(\int_\Omega \|H^n(\sigma) - H(\sigma)\|^2 d\mathbb{P}(\sigma) \right)^{\frac{1}{2}} |\Omega|^{\frac{1}{2}}.
\end{aligned}$$

Since $H^n \to H$, in $L^2(\Omega, \mathbb{R}_+^m, \mathbb{P})$, and by assumption (ii), we deduce

$$\int_\Omega \rho(\omega, H^n(\sigma)) d\mathbb{P}(\sigma) \to \int_\Omega \rho(\omega, H(\sigma)) d\mathbb{P}(\sigma).$$

Furthermore, since $F^n \to F$, in $L^2(\Omega, \mathbb{R}^m, \mathbb{P})$, we obtain that $\Phi F^n \to \Phi F$, in $L^1(\Omega, \mathbb{R}^l, \mathbb{P})$. As a consequence, we get that $F \in \mathbb{K}(H)$.

Now we show that the multifunction $\mathbb{K}$ is lower semicontinuous, namely if for every $\{H^n\}$ such that $H^n \to H$, in $L^2(\Omega, \mathbb{R}^m, \mathbb{P})$, and for every $F \in \mathbb{K}(H)$, then there exists a sequence $\{F^n\}$ such that $F^n \to F$, in $L^2(\Omega, \mathbb{R}^m, \mathbb{P})$, with $F^n \in \mathbb{K}(H^n)$, for every $n \in \mathbb{N}$. Thus, let us consider an arbitrary sequence $\{H^n\}$ such that $H^n \to H$, in $L^2(\Omega, \mathbb{R}^m, \mathbb{P})$, and $F \in \mathbb{K}(H)$. Let us fix $n \in \mathbb{N}$ and $\omega \in \Omega$. Let us set:

$$\begin{aligned}
A_j &= \{r \in \{1, \dots, m\} : \ \varphi_{jr} = 1\}, \\
B_j(n, \omega) &= \{r \in A_j : \ \hat{\rho}_j(\omega) - \hat{\rho}_j^n(\omega) \leq 0\},
\end{aligned}$$

$$C_j(n,\omega) = \{r \in A_j : 0 < \hat{\rho}_j(\omega) - \hat{\rho}_j^n(\omega) \le F_r(\omega) - \lambda_r(\omega)\},$$
$$D_j(n,\omega) = \{r \in A_j : F_r(\omega) - \lambda_r(\omega) < \hat{\rho}_j(\omega) - \hat{\rho}_j^n(\omega)\},$$

where $j \in \{1,\dots,l\}$ and

$$\hat{\rho}_j(\omega) = \fint_\Omega \rho_j(\omega, H(\sigma))d\mathbb{P}(\sigma),$$
$$\hat{\rho}_j^n(\omega) = \fint_\Omega \rho_j(\omega, H^n(\sigma))d\mathbb{P}(\sigma).$$

Hence, we consider the sequence $\{F^n\}$ as follows

$$F_r^n(\omega) = \begin{cases} F_r(\omega), & \text{if } r \in B_j \cup D_j, \ \mathbb{P}-\text{a.s.}, \\ F_r(\omega) - \dfrac{\hat{\rho}_j(\omega) - \hat{\rho}_j^n(\omega)}{\sum_{s\in C_j} \varphi_{js}}, & \text{if } r \in C_j, \ \mathbb{P}-\text{a.s.} \end{cases}$$

If $r \in B_j \cup D_j$, then $F_r^n(\omega) = F_r(\omega)$, $\mathbb{P}$-a.s. and, since $F \in \mathbb{K}(H)$, we have

$$\lambda_r(\omega) \le F_r^n(\omega) \le \mu_r(\omega), \quad \mathbb{P}-\text{a.s.}$$

On the other hand, if $r \in C_j$, it results

$$\lambda_r(\omega) < F_r^n(\omega) = F_r(\omega) - \frac{\hat{\rho}_j(\omega) - \hat{\rho}_j^n(\omega)}{\sum_{s\in C_j} \varphi_{js}} \le \mu_r(\omega), \quad \mathbb{P}-\text{a.s.}$$

As a consequence, F^n satisfies the capacity constraints, for every $n \in \mathbb{N}$. Furthermore, we get, for every $j = 1,\dots,l$,

$$\begin{aligned}\sum_{r=1}^m \varphi_{jr} F_r^n(\omega) &= \sum_{r\in A_j} \varphi_{jr} F_r^n(\omega) \\ &= \sum_{r\in B_j\cup D_j} \varphi_{jr} F_r^n(\omega) + \sum_{r\in C_j} \varphi_{jr}\left(F_r(\omega) - \frac{\hat{\rho}_j(\omega) - \hat{\rho}_j^n(\omega)}{\sum_{s\in C_j} \varphi_{js}}\right) \\ &= \sum_{r\in A_j} \varphi_{jr} F_r(\omega) - (\hat{\rho}_j(\omega) - \hat{\rho}_j^n(\omega)) = \hat{\rho}_j^n(\omega).\end{aligned}$$

Therefore, $F^n \in \mathbb{K}(H)$, for every $n \in \mathbb{N}$. It remains to prove that $F^n \to F$, in $L^2(\Omega, \mathbb{R}^m, \mathbb{P})$. For every $j = 1,\dots,l$, one has

$$\begin{aligned}\int_\Omega \left|\sum_{r=1}^m \varphi_{jr} \ (F_r^n(\omega) - F_r(\omega))\right|^2 d\mathbb{P}(\omega) &= \int_\Omega \left|\hat{\rho}_j(\omega) - \hat{\rho}_j^n(\omega)\right|^2 \ d\mathbb{P}(\omega) \\ &\le \fint_\Omega \left|\rho_j(\omega, H(\omega)) - \rho_j^n(\omega, H^n(\omega))\right| d\mathbb{P}(\omega) \\ &\le \fint_\Omega \nu(\omega)\|H - H^n\| d\mathbb{P}(\omega) \\ &\le \left(\fint_\Omega \nu^2(\omega) d\mathbb{P}(\omega)\right)^{\frac{1}{2}} \|H - H^n\|.\end{aligned}$$

In addition, we deduce, for every $j = 1,\dots,l$,

$$\begin{aligned}\left(\sum_{r=1}^m \varphi_{jr}(F_r^n(\omega) - F_r(\omega))\right)^2 &= \left(\sum_{r\in A_j}^m \varphi_{jr}(F_r^n(\omega) - F_r(\omega))\right)^2 \\ &= \left(\sum_{r\in B_j\cup D_j}^m \varphi_{jr}(F_r^n(\omega) - F_r(\omega)) + \sum_{r\in C_j}^m \varphi_{jr}(F_r^n(\omega) - F_r(\omega))\right)^2 \\ &= \left(\sum_{r\in C_j}\left(-\frac{\hat{\rho}_j(\omega) - \hat{\rho}_j^n(\omega)}{\sum_{s\in C_j} \varphi_{js}}\right)\right)^2 = \left(\sum_{r\in C_j}\left(\frac{\hat{\rho}_j(\omega) - \hat{\rho}_j^n(\omega)}{\sum_{s\in C_j} \varphi_{js}}\right)\right)^2 \\ &\ge \sum_{r\in C_j}\left(\frac{\hat{\rho}_j(\omega) - \hat{\rho}_j^n(\omega)}{\sum_{s\in C_j} \varphi_{js}}\right)^2 \ge \frac{1}{m^2}\sum_{r\in C_j}(F_r^n(\omega) - F_r(\omega))^2 \\ &= \frac{1}{m^2}\sum_{r\in A_j}(F_r^n(\omega) - F_r(\omega))^2 = \frac{1}{m^2}|F^n(\omega) - F(\omega)|^2.\end{aligned}$$

Hence, we obtain, for every $j = 1,\dots,l$,

$$0 \le \frac{1}{m^2}\int_\Omega |F^n(\omega) - F(\omega)|^2 d\mathbb{P}(\omega)$$

$$\leq \int_{\Omega}\Big(\sum_{r=1}^{m}\varphi_{jr}(F_r^n(\omega)-F_r(\omega))\Big)^2 d\mathbb{P}(\omega)$$
$$\leq \left(\fint_{\Omega} \nu^2(\omega)d\mathbb{P}(\omega)\right)^{\frac{1}{2}}\|H-H^n\|.$$

Since $H^n \to H$, in $L^2(\Omega,\mathbb{R}^m,\mathbb{P})$, the sequence $\{F^n\}$ converges to F, in $L^2(\Omega,\mathbb{R}^m,\mathbb{P})$. Finally it is easy to verify that $\mathbb{K}(H)$ is a closed, bounded and convex subset of D, for each $H \in D$, and, being D compact, $\mathbb{K}(H)$ is compact too, for each $H \in D$. Consequently, by virtue of the Tan theorem in [21], there exists at least one solution to the stochastic quasi-variational inequality (3). □

5. Numerical procedures and examples

In this section, we analyze a numerical procedure to solve a stochastic quasi-variational inequality of type (3). In the last decades, many numerical methods have been developed to solve the following class of variational inequalities (see for instance [22,23])

$$\langle C(x), y-x\rangle \geq 0, \quad \forall y \in K,$$

where K is a nonempty, convex and compact subset of $\mathbb{R}^n$ and $C : \mathbb{R}^n \to \mathbb{R}^n$. Moreover, some of these numerical methods have found applications also to a more general class of variational inequalities such as tensor variational inequalities, see [24]. For the reader's convenience, we briefly summarize some numerical methods and the ideas behind them.

- **Projection method:** given an initial point x_0 and a nonnegative suitable steplength α, the numerical procedure updates iteratively x_k through the projection on K of the quantity $x_k - \alpha C(x_k)$ and observing that the projection $Pr_K(x_k - \alpha C(x_k))$ is the solution of the quadratic programming problem

  $$\min_{y\in K} \frac{1}{2}\langle y, y\rangle - \langle x_k - \alpha C(x_k), y\rangle.$$

  Here the key observation is the fact that x is a solution of a variational inequality if and only if $x = Pr_k(x - \alpha C(x))$. To ensure the convergence of this method strong hypothesis on C and on α are required as well as the Lipschitz constant whose estimate is in general nontrivial.
- **Extragradient method:** this method refines the projection one. More precisely, given an initial point x_0 and a nonnegative suitable steplength α, the numerical scheme is based on a double projection according to the formulas:

  $$\begin{aligned} \overline{x_k} &= Pr_K(x_k - \alpha C(x_k)) \\ x_{k+1} &= Pr_K(x_k - \alpha C(\overline{x_k})). \end{aligned} \tag{4}$$

  This method overcomes the difficulty of having restrictive assumptions on C for the projection method. Here the Lipschitz constant is still necessary for choosing α. The convergence rate is at best linear (see [25]).
- **Extragradient method with adaptive steplength:** this algorithm is a modified version of the extragradient method. Here at each iteration, which follows the double projection as in (4), the parameter α is also updated as follows:

  $$\alpha_k = \min\left\{\frac{\alpha_{k-1}}{2}, \frac{\|x_k - \overline{x_k}\|}{\sqrt{2}(C(x_k) - C(\overline{x_k}))}\right\}$$

  The advantages of this method (whose rate of convergence is also linear see [25]) rely on the fact that weaker assumptions to ensure convergence are requested. In particular, the Lipschitz constant here is not needed in the choice of the steplength sequence.

We describe here the computational procedure for solving stochastic quasi-variational inequalities

$$\langle\langle C(x), y-x\rangle\rangle_{\mathbb{E}} \geq 0, \quad \forall y \in K(x),$$

where $K : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a multifunction with nonempty, convex and compact values and $C : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$. The procedure consists to compute iteratively the solution to a stochastic variational inequality problem (using one of the method described above). More precisely, choosing an initial feasible solution x_0, the setting parameters (such as the steplength α, the threshold ε, capacity constraints etc.) and the sample sequence $\{\omega^k\}$, it solves the stochastic variational inequality associated with the feasible set $K(x_0)$

whose solution determines a new set of feasible solutions and then a new stochastic variational inequality problem. The scheme of the procedure is described in Algorithm 1.

Algorithm 1

Step 1 (initialization):
Set $i = 0$, initialize the parameters, the sample $\{\omega^k\}$ and choose an initial feasible set K_0;
Select an initial point $x_0 \in K_0$;
Step 2 (iteration):
Solve the variational inequality $\langle C(\omega^k, x_i), x - x_i \rangle \geq 0$, $\forall x \in K_i$;
Denote by x_{i+1} the found solution;
Update the feasible set K_{i+1} according to x_{i+1};
Set $i = i + 1$;
Step 3 (stop criteria):
Check the stop criteria (for example $|x_i - x_{i-1}| < \varepsilon$): end the procedure if it is fulfilled, otherwise go to **Step 2**.

The procedure described above is extremely general and adaptable to different choices, for example, the extragradient method to solve the stochastic variational inequality at Step 2 or the stop criteria at Step 3.

In the literature, numerical schemes to solve variational inequalities in the deterministic framework have been intensively studied (see for example [22]). More recently, these methods, such as the projection method or the extragradient method with its variants, have been extended also for stochastic variational inequalities. For example, in [19,20], the convergence and complexity analysis are studied in different settings, i.e. unbounded feasible set, unbounded operator etc. We refer to the recent work [26] to describe the assumptions necessary to ensure the convergence of an extragradient scheme for stochastic variational inequalities. Precisely, considering the following assumptions:

(a) the conditional first moments are zero;
(b) the conditional second moments are bounded $\mathbb{P}-$ a.s.;
(c) $C(x)$ is Lipschitz continuous and bounded;

the convergence of the extragradient method is guarantees by Proposition 2 in [26]. We recall here the statement.

Theorem 5.1. *Let us suppose that assumptions (a), (b), (c) hold true and C is a strongly pseudomonotone map. Then, the extragradient scheme generates a sequence $\{x_k\}$ such that $\{x_k\}$ is bounded $\mathbb{P}-$ a.s. and any limit point of $\{x_k\}$ is a solution of the stochastic variational inequality*

$$\langle\langle C(x), y - x \rangle\rangle_{\mathbb{E}} \geq 0, \quad \forall y \in K$$

in an a.s. sense.

On the other hand, the convergence of the few methods available for quasi-variational inequalities are based on the existence of a fixed point for the multifunction involved. One of the best results is proved in [27] and states as follows.

Theorem 5.2. *Let H be an Hilbert space. If the map $C : H \to H$ is Lipschitz continuous and strongly monotone with positive constants L and μ, respectively, and $K : H \rightrightarrows H$ is a multifunction with nonempty closed and convex values such that*

$$\| Pr_{K(x)}(z) - Pr_{K(y)}(z) \| \leq l \|x - y\|, \quad \forall x, y, z \in H, \tag{5}$$

with $l + \sqrt{1 - \mu^2/L^2} < 1$, then the quasi-variational inequality

$$\langle C(x^*), y - x^* \rangle \geq 0, \quad \forall y \in K(x^*), \tag{6}$$

has a unique solution.

It is also worth mentioning that in [18] the authors proved that it is sufficient to require $l < \mu/L$ in Theorem 5.2. The assumption (5) is a kind of strengthening of the contraction property for multifunction $K(x)$. The convergence analysis for the extragradient method for solving quasi-variational inequalities is presented in [28]. Precisely, the extragradient procedure converges to the unique solution of the quasi-variational inequality (6) under the following assumptions:

(a) K_0 is a nonempty closed convex subset of the Hilbert space H, $k : H \to H$ is a Lipschitz continuous function with Lipschitz constant $l > 0$, and $K : H \rightrightarrows H$ is a multifunction of the form $K(x) = k(x) + K_0$, for every $x \in H$;
(b) the map $C : H \to H$ is Lipschitz continuous and strongly monotone with positive constants L and μ, respectively;
(c) the parameter α and constants l, L and μ satisfy the following conditions

$$0 < \alpha < \frac{1}{L}, \quad 0 < l < \frac{\mu}{L} \sqrt{\frac{\alpha(1 - \alpha^2 L^2)}{(\alpha + 1)(1 - \alpha^2 L^2 + \alpha\mu)}};$$

for every initial approximation $x_0 \in K_0$.

5.1. The algorithm for the random elastic equilibrium problem

In order to compute numerically a random elastic equilibrium flow according to Definition 3.1, we use Algorithm 1. Precisely, we first fix the capacity constraints functions $\lambda(\omega)$ and $\mu(\omega)$ and then the domain D as in (1). Given an initial elastic travel demand ρ_0 and a flow $H_0 \in D$, the initial feasible set is given by

$$\mathbb{K}_0(H_0) = \left\{ F \in D : \ \Phi F(\omega) = \fint_\Omega \rho_0(\omega^k, H_0(\sigma))d\mathbb{P}(\sigma), \ \mathbb{P} - \text{a.s.} \right\}$$

where Φ is the incidence matrix and $\{\omega^k\}$ is the sample sequence.

At Step 2, we choose to apply the extragradient method for solving the associated stochastic variational inequality

$$\langle C(\omega^k, H_i(\omega^k)), F(\omega^k) - H_i(\omega^k) \rangle \geq 0, \quad \forall F \in \mathbb{K}_i,$$

obtaining the ith step solution H_i. Moreover, the update of the feasible set is made by upgrading the travel demand (we follow here some ideas in [29]):

$$\rho_{i+1} = \frac{1}{\log(\log(1+i))}\rho_0 + \left(1 - \frac{1}{\log(\log(1+i))}\right)\rho_i,$$

which implies

$$\mathbb{K}_{i+1} = \left\{ F \in D : \ \Phi F(\omega) = \fint_\Omega \rho_{i+1}(\omega^k, H_i(\sigma))d\mathbb{P}(\sigma), \ \mathbb{P} - \text{a.s.} \right\}.$$

Finally, the stop criteria is the following: given $\varepsilon > 0$, if

$$|\rho_i - \rho_{i-1}| < \varepsilon$$

the algorithm stops and $H^* = H_i$ is the desired solution.

The convergence analysis of the proposed algorithm is not one of the main purposes of this article. Anyway, we remark that combining the convergence results of numerical schemes for stochastic variational inequalities and the ones for quasi-variational inequalities recalled in the first part of this section, the convergence of the algorithm is ensured.

5.2. An example

We test now our algorithm on a simple example. We consider a transportation pattern for the network shown in Fig. 1 consists in six nodes and eight links. We assume that the O/D pairs are represented by $w_1 = (P_1, P_5)$ and $w_2 = (P_2, P_6)$, which are respectively connected by the following paths:

$$w_1 \ : \ \begin{cases} R_1 = (P_1, P_2) \cup (P_2, P_5) \\ R_2 = (P_1, P_4) \cup (P_4, P_5), \end{cases} \qquad w_2 \ : \ \begin{cases} R_3 = (P_2, P_3) \cup (P_3, P_6) \\ R_4 = (P_2, P_5) \cup (P_5, P_6) \\ R_5 = (P_2, P_5) \cup (P_5, P_3) \cup (P_3, P_6). \end{cases}$$

We fix the capacity constraints functions as $\lambda_i = 0$, $\mu_i = 25$, $i = 1, \dots, 5$, and consequently

$$D = \left\{ F \in L^2(\Omega, \mathbb{R}^m, \mathbb{P}) : \ 0 \leq F_i(\omega) \leq 25, \ i = 1, \dots, 5, \ \mathbb{P} - \text{a.s.} \right\}.$$

Choosing an initial elastic travel demand ρ_0 and an initial flow $H_0 \in D$ which are uniformly distributed random variables with supports in $[1, 20]$, the initial feasible set is given by

$$\mathbb{K}_0(H_0) = \left\{ F \in D : \ \Phi F(\omega) = \fint_\Omega \rho_0(\omega, H_0(\sigma))d\mathbb{P}(\sigma), \ \mathbb{P} - \text{a.s.} \right\}.$$

The path cost vector-function $C \in L^2(\Omega \times \mathbb{R}^5, \mathbb{R}^5, \mathbb{P})$ we consider has components

$$\begin{aligned}
&C_1(\omega, H(\omega)) = 2H_1(\omega) + H_4(\omega) + 1, \quad \mathbb{P} - \text{a.s.}, \\
&C_2(\omega, H(\omega)) = 5H_2(\omega) + 3H_5(\omega) + 1, \quad \mathbb{P} - \text{a.s.}, \\
&C_3(\omega, H(\omega)) = 4H_3(\omega) + 3, \quad \mathbb{P} - \text{a.s.}, \\
&C_4(\omega, H(\omega)) = 3H_1(\omega) + 2H_4(\omega) + 3, \quad \mathbb{P} - \text{a.s.}, \\
&C_5(\omega, H(\omega)) = 2H_2(\omega) + 5H_5(\omega) + 2, \quad \mathbb{P} - \text{a.s.}
\end{aligned}$$

It can easily verify that C is a strongly monotone function.

We implement the algorithm described before making use of Matlab and run on a PC with 32 GB RAM, HP EliteBook 830 G7. We obtain the equilibrium distributions of the numerical example for some different samples of the random variables. We collect them in Table 2 together with the computational time required.

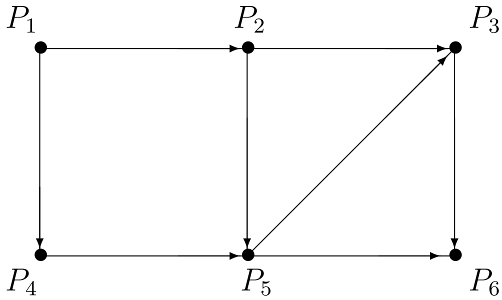

Fig. 1. A network model.

Table 2
Numerical results.

Sample n.	tcomp	H_1	H_2	H_3	H_4	H_5
1	0.3700	0.0000	0.0000	0.8611	12.1328	5.0524
2	0.3800	1.9126	0.0000	3.0411	10.3384	5.4826
3	0.3400	1.4929	0.0000	1.4145	4.6719	2.9639
4	0.3000	0.5486	0.0000	0.3125	2.1534	1.3898
5	0.3200	2.7876	0.0000	2.1661	5.0911	3.9089
6	0.3600	2.4373	0.0000	2.5164	7.1903	4.5381
7	0.3300	2.7876	0.0000	2.1661	5.0911	3.9089

6. Conclusions

In this paper, starting from the elastic Wardrop equilibrium principle governing the traffic networks, we presented a model which includes uncertainty on the data, specifically on the path flows as well as on the elastic travel demand. Thus we considered a random elastic traffic equilibrium problem, we introduced a random elastic traffic equilibrium condition and we obtained the equivalence between this equilibrium condition and a stochastic quasi-variational inequality. In addition, we established an existence theorem. Finally, we proposed a numerical scheme to compute random elastic traffic equilibrium distributions.

Further work is to extend the random approach to other models such as the case of mergers/acquisitions of companies and financial problems.

CRediT authorship contribution statement

Annamaria Barbagallo: Conceptualization, Data curation, Formal analysis, Funding acquisition, Methodology, Supervision. **Serena Guarino Lo Bianco:** Conceptualization, Data curation, Formal analysis, Software.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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