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Photon-Graviton Amplitudes

N. Ahmadinia^{*}, F. Bastianelli[†], O. Corradini^{**}, José M. Dávila[‡] and C. Schubert^{*}

^{*}*Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Edificio C-3, Apdo. Postal 2-82, C.P. 58040, Morelia, Michoacán, México.*

[†]*Dipartimento di Fisica ed Astronomia, Università di Bologna and INFN, Sezione di Bologna, Via Irnerio 46, I-40126 Bologna, Italy.*

^{**}*Centro de Estudios en Física y Matemáticas Básicas y Aplicadas, Universidad Autónoma de Chiapas, C.P. 29000, Tuxtla Gutiérrez, México, and Dipartimento di Scienze Fisiche, Informatiche e Matematiche, Università di Modena e Reggio Emilia, Via Campi 213/A, I-41125 Modena, Italy.*

[‡]*Facultad de Ciencias, Universidad Autónoma del Estado de México, Instituto Literario 100, Toluca 50000, México.*

Abstract. We report on an ongoing study of the one-loop photon-graviton amplitudes, using both effective action and worldline techniques. The emphasis is on Kawai-Lewellen-Tye-like relations.

Keywords: Photon-graviton, KLT, worldline formalism

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MOTIVATIONS

In 1986 Kawai, Lewellen and Tye [1] found a relation between tree-level amplitudes in open and closed string theory which in the field theory limit also implies relations between amplitudes in gauge theory and in gravity. Schematically, such relations are of the form

$$(\text{gravity amplitude}) \sim (\text{gauge amplitude})^2 .$$

Their existence is related to the factorization of vertex operators for open and closed strings

$$V^{\text{closed}} = V_{\text{left}}^{\text{open}} \bar{V}_{\text{right}}^{\text{open}} . \quad (1)$$

For example, at the four and five point level one has [2]

$$\begin{aligned} M_4(1, 2, 3, 4) &= -is_{12}A_4(1, 2, 3, 4)A_4(1, 2, 4, 3) , \\ M_5(1, 2, 3, 4, 5) &= is_{12}s_{34}A_5(1, 2, 3, 4, 5)A_5(2, 1, 4, 3, 5) + is_{13}s_{24}A_5(1, 3, 2, 4, 5)A_5(3, 1, 4, 2, 5) , \end{aligned} \quad (2)$$

where

$$\begin{aligned} M_n &= \text{tree - level graviton amplitudes} , \\ A_n &= (\text{colour stripped}) \text{ tree - level gauge theory amplitudes} , \\ s_{ij} &= (k_i + k_j)^2 . \end{aligned} \quad (3)$$

Usually such relations are studied contrasting purely gravitational amplitudes with pure gauge theory amplitudes (see, however, [3]). In the line of work presented here we more generally study mixed amplitudes involving both gravitons and photons.

WORLDLINE FORMALISM IN FLAT SPACE-TIME

The worldline formalism is an alternative to Feynman diagrams that was initially also developed by Feynman [4, 5]. It allows one to write the full S-matrix for Scalar or Spinor QED in terms of first-quantized path integrals. Let us write it down here for the case of the quenched effective action in Scalar QED:

$$\Gamma(A) = \int d^4x \mathcal{L}(A) = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_{x(T)=x(0)} Dx(\tau) e^{-S[x(\tau)]}. \quad (4)$$

Here T is the proper time of the scalar loop, m the scalar mass, and the path integral is over closed loops in (euclidean) spacetime with the periodicity given by T . The worldline action $S[x(\tau)]$ has three parts:

$$S[x(\tau)] = S_0 + S_{\text{ext}} + S_{\text{int}} \quad (5)$$

where

$$S_0 = \int_0^T d\tau \frac{\dot{x}^2}{4}, \quad S_{\text{ext}} = ie \int_0^T d\tau \dot{x}^\mu A_\mu(x(\tau)), \quad S_{\text{int}} = -\frac{e^2}{8\pi^2} \int_0^T d\tau_1 \int_0^T d\tau_2 \frac{\dot{x}(\tau_1) \cdot \dot{x}(\tau_2)}{(x(\tau_1) - x(\tau_2))^2}. \quad (6)$$

Of those, S_0 describes the free propagation, S_{ext} the interaction with the external field, and S_{int} the exchange of virtual photons in the loop. The expansion of those interaction exponentials has the usual diagrammatic meaning (fig. 1).

FIGURE 1. Expansion of the interaction exponentials.

The extension to Spinor QED involves the insertion of a “spin-factor” into the path integral [5, 6].

WORLDLINE FORMALISM IN CURVED SPACE-TIME

To include background gravity, one starts with the obvious generalization of S_0 to a curved space,

$$S_0 = \frac{1}{4} \int_0^T d\tau \dot{x}^2 \rightarrow \frac{1}{4} \int_0^T d\tau \dot{x}^\mu g_{\mu\nu}(x(\tau)) \dot{x}^\nu. \quad (7)$$

We will assume that the geometry is such that the metric can be treated as a perturbation of flat space-time:

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + \kappa h_{\mu\nu}(x). \quad (8)$$

Unlike the flat space path integral in (4), the construction of the curved space path integral leads to mathematical subtleties that have been resolved only quite recently (see [7] and refs therein).

First, in curved space, the path integral measure is nontrivial. Covariance requires that the naive infinite-dimensional measure Dx be supplied with a factor $\prod_{0 \leq \tau < T} \sqrt{\det g_{\mu\nu}(x(\tau))}$. At least if an analytic calculation of the path integral is intended, this factor cannot be used as it stands. We exponentiate it as follows [8]:

$$\mathcal{D}x \equiv Dx \prod_{0 \leq \tau < T} \sqrt{\det g_{\mu\nu}(x(\tau))} = Dx \int_{\text{PBC}} DaDbDc e^{-S_{\text{gh}}[x,a,b,c]} \quad (9)$$

with the ghost action

$$S_{\text{gh}}[x, a, b, c] = \int_0^T d\tau \frac{1}{4} g_{\mu\nu}(x(\tau)) [a^\mu(\tau) a^\nu(\tau) + b^\mu(\tau) c^\nu(\tau)]. \quad (10)$$

One-loop graviton amplitudes can then be computed by choosing the perturbation $h_{\mu\nu}(x)$ as a plane-wave background. Each graviton then gets represented by a vertex operator

$$V_{\text{scal}}^h[k, \varepsilon] = \varepsilon_{\mu\nu} \int_0^T d\tau [\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu] e^{ik \cdot x(\tau)}. \quad (11)$$

The path integral becomes gaussian, and can be calculated using the two-point worldline correlators

$$\begin{aligned} \langle x^\mu(\tau_1) x^\nu(\tau_2) \rangle &= -G_B(\tau_1, \tau_2) \delta^{\mu\nu}, \quad G_B(\tau_1, \tau_2) = |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{T} - \frac{T}{6}, \\ \langle a^\mu(\tau_1) a^\nu(\tau_2) \rangle &= 2\delta(\tau_1 - \tau_2) \delta^{\mu\nu}, \quad \langle b^\mu(\tau_1) c^\nu(\tau_2) \rangle = -4\delta(\tau_1 - \tau_2) \delta^{\mu\nu}. \end{aligned} \quad (12)$$

After a suitable regularization, the ghost field contributions will cancel all UV - divergent terms [7].

Second, these cancellations leave finite ambiguities, and finite worldline counterterms are needed to get the correct amplitudes. The curved space σ model behaves effectively like a UV divergent but super-renormalizable 1-D QFT, requiring only a small number of counterterms needed. Those are regularization dependent and in general noncovariant:

$$\Delta S_{CT} = \int_0^T d\tau V_{CT}. \quad (13)$$

So far they have been determined for three regularization schemes [7]:

- Time slicing: $V_{CT} = -\frac{1}{4}R - \frac{1}{12}g^{\mu\nu}g^{\alpha\beta}g_{\lambda\rho}\Gamma_{\mu\alpha}^\lambda\Gamma_{\nu\beta}^\rho$.
- Mode regularization: $V_{CT} = -\frac{1}{4}R + \frac{1}{4}g^{\mu\nu}\Gamma_{\mu\alpha}^\beta\Gamma_{\nu\beta}^\alpha$.
- 1-D dimensional regularization: $V_{CT} = -\frac{1}{4}R$. This is the only known covariant regularization.

Once a regularization scheme has been fixed and the counterterms been determined, there are no further ambiguities.

GRAVITON-PHOTON-PHOTON AMPLITUDE

The simplest nontrivial case of a mixed graviton-photon amplitude has one graviton and two photons (fig. 2).

FIGURE 2. Graviton-photon-photon amplitude.

In [9] we computed this amplitude in the low-energy limit, for both the scalar and spinor loop cases. In this case it is actually simplest to use the corresponding low-energy effective Lagrangians [10, 11]:

$$\begin{aligned} \mathcal{L}_{\text{scal}}^{h\gamma\gamma} &= \frac{e^2}{180(4\pi)^2 m^2} \left[15\left(\xi - \frac{1}{6}\right) R F_{\mu\nu}^2 - 2R_{\mu\nu} F^{\mu\alpha} F^\nu{}_\alpha - R_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta} + 3(\nabla^\alpha F_{\alpha\mu})^2 \right], \\ \mathcal{L}_{\text{spin}}^{h\gamma\gamma} &= \frac{e^2}{180(4\pi)^2 m^2} \left[5R F_{\mu\nu}^2 - 26R_{\mu\nu} F^{\mu\alpha} F^\nu{}_\alpha + 2R_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta} + 24(\nabla^\alpha F_{\alpha\mu})^2 \right], \end{aligned} \quad (14)$$

where ξ represents a non-minimal coupling to gravity. Using, as is usual, gravitons that are helicity eigenstates and factorize into vector polarizations,

$$\varepsilon_{0\mu\nu}^{++}(k_0) = \varepsilon_{0\mu}^+(k_0)\varepsilon_{0\nu}^+(k_0), \quad \varepsilon_{0\mu\nu}^{--}(k_0) = \varepsilon_{0\mu}^-(k_0)\varepsilon_{0\nu}^-(k_0), \quad (15)$$

one finds that the only non-vanishing components of the on-shell amplitude are the ones where all helicities are equal,

$$\begin{aligned} A_{\text{spin}}^{(++++)} &= \frac{\kappa e^2}{90(4\pi)^2 m^2} [01]^2 [02]^2, & A_{\text{spin}}^{(----)} &= \frac{\kappa e^2}{90(4\pi)^2 m^2} \langle 01 \rangle^2 \langle 02 \rangle^2, \\ A_{\text{spin}}^{(+-+-)} &= -2A_{\text{scal}}^{(+-+-)}, & A_{\text{spin}}^{(----)} &= -2A_{\text{scal}}^{(----)}. \end{aligned} \quad (16)$$

Here we have used the standard spinor helicity notation. Comparison with the low-energy limits of the four-photon amplitudes [12] one indeed finds a Kawai-Lewellen-Tye-like relation, that is, effectively the graviton has factored into two photons.

ONE-GRAVITON FOUR-PHOTON AMPLITUDE

We are now working on the one-graviton four-photon amplitude (the one-graviton three-photon one vanishes for parity reasons). Although the corresponding effective Lagrangians are also available [13], the expressions are already quite cumbersome, and we prefer here a direct calculation starting from the worldline expression,

$$\Gamma_{\text{scal}}^{h,4\gamma} = \int_0^\infty \frac{dT}{T} \frac{e^{-m^2 T}}{(4\pi T)^2} \left(-\frac{\kappa}{4T}\right) (-ie)^4 \langle V_{\text{scal}}^h(k_0) V_{\text{scal}}^\gamma(k_1) V_{\text{scal}}^\gamma(k_2) V_{\text{scal}}^\gamma(k_3) V_{\text{scal}}^\gamma(k_4) \rangle. \quad (17)$$

Here the graviton vertex operator has been given in (11) and the photon one is

$$V_{\text{scal}}^\gamma[k, \varepsilon] = ie \int_0^T d\tau \varepsilon \cdot \dot{x} e^{ik \cdot x} \quad (18)$$

(see [14] for the corresponding vertex operators for a spinor loop).

However, (17) comes from the effective action and thus represents only the one-particle-irreducible part of the amplitude. Contrary to the above three-point case, the five-point amplitude has already reducible contributions (fig. 3).

FIGURE 3. One-graviton four-photon amplitude.

Thus it will be important to find out how to combine the irreducible and reducible diagrams in an algebraically nice way. This problem has been studied at tree-level by various authors [15, 16], and remarkable simplifications were found for the sum of diagrams in the case of gravitational Compton scattering [17, 18]. However, we are not aware of similar studies at the one-loop level. To address these issues in the worldline formalism, we first should learn more about how to compare Feynman diagram and worldline calculations in quantum gravity. We are in the process of doing this.

FIGURE 4. Scalar-scalar-graviton vertex.

WORLDLINE REPRESENTATION FOR TREE-LEVEL DIAGRAMS

Let us show here only the simplest example, namely how to reproduce in the worldline formalism the basic graviton-scalar-scalar vertex (fig. 4). This vertex would normally be derived from the space-time action

$$S = \int d^D x \sqrt{g} \frac{1}{2} \left[g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 + \xi R \phi^2 \right]. \quad (19)$$

Here we start with the worldline representation of the full scalar propagator in the background:

$$\langle 0 | T \phi(x_1) \phi(x_2) | 0 \rangle = \frac{1}{-\square_g + m^2 + \xi R} = \int_0^\infty dT \int Dx Da Db Dc e^{-\int_0^1 d\tau [\frac{1}{4T} g_{\mu\nu}(x) (\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu) + T m^2 + T \xi R]}. \quad (20)$$

As before we set $h_{\mu\nu}(x) \equiv g_{\mu\nu}(x) - \delta_{\mu\nu} = \kappa \epsilon_{\mu\nu} e^{ik \cdot x}$ and separate out the term linear in the polarization $\epsilon_{\mu\nu}$. The x -path integral now must be taken with boundary conditions $x(0) = x_2, x(T) = x_1$. We introduce the fluctuation variable $z(\tau)$ by $x(\tau) = x_2 + \frac{\tau}{T}(x_1 - x_2) + z(\tau)$, with the Wick contraction rule [7]

$$\langle z^\mu(\tau_1) z^\nu(\tau_2) \rangle = -2T \delta^{\mu\nu} \Delta(\tau_1, \tau_2), \quad \Delta(\tau_1, \tau_2) = \tau_1 \tau_2 - \tau_1 \theta(\tau_2 - \tau_1). \quad (21)$$

Fourier transforming in x_1, x_2 , we get (now setting $\xi = 0$)

$$\begin{aligned} \langle \phi(p_1) \phi(p_2) \rangle_{(1)} &= (2\pi)^D \delta^D(p_1 + p_2 + k) \int dx_1 \int_0^\infty \frac{dT}{(4\pi T)^{D/2}} e^{-\frac{k^2}{4T} - T m^2} e^{ip_1 x_1} \\ &\times \left(\frac{-\kappa}{4T} \epsilon_{\mu\nu} \right) \int_0^1 d\tau \langle (x_1^\mu x_1^\nu + 2x_1^\mu z^\nu + \dot{z}^\mu \dot{z}^\nu + a^\mu a^\nu + b^\mu c^\nu) \rangle \end{aligned} \quad (22)$$

where the subscript ‘(1)’ refers to a single interaction with the background, and

$$\begin{aligned} \langle (\dot{z}^\mu \dot{z}^\nu + a^\mu a^\nu + b^\mu c^\nu) e^{ik \cdot z(\tau)} \rangle &= \left[-2T \delta^{\mu\nu} (\bullet\bullet\Delta + \bullet\Delta\bullet) \Big|_\tau - 4T^2 k^\mu k^\nu (\bullet\Delta|_\tau)^2 \right] e^{T k^2 (\tau^2 - \tau)} \\ &= \left[-2T \delta^{\mu\nu} \partial_\tau (\bullet\Delta|_\tau) - 4T^2 k^\mu k^\nu \left(\tau - \frac{1}{2} \right)^2 \right] e^{T k^2 (\tau^2 - \tau)} \end{aligned} \quad (23)$$

(here a ‘dot’ before (after) a $\Delta(\tau_1, \tau_2)$ denotes a derivative with respect to τ_1 (τ_2)).

After some simple mathematical manipulations one gets:

$$\begin{aligned} \langle \phi(p_1) \phi(p_2) \rangle_{(1)} &= (2\pi)^D \delta^D(p_1 + p_2 + k) \left(-\frac{\kappa}{4} \epsilon_{\mu\nu} \right) (-\partial_{p_{1\mu}} \partial_{p_{1\nu}}) \int_0^1 d\tau \int_0^\infty \frac{dT}{T} e^{-T[p_1^2 + m^2 + (k^2 + 2k \cdot p_1)\tau]} \\ &= (2\pi)^D \delta^D(p_1 + p_2 + k) \left(\frac{\kappa}{4} \epsilon_{\mu\nu} \right) \int_0^1 d\tau \int_0^\infty \frac{dT}{T} 4p_1^\mu p_1^\nu T^2 e^{-T[p_1^2 + m^2 + (k^2 + 2k \cdot p_1)\tau]} \\ &= (2\pi)^D \delta^D(p_1 + p_2 + k) \frac{1}{p_1^2 + m^2} (\epsilon_{\mu\nu} \kappa p_1^\mu p_1^\nu) \frac{1}{p_2^2 + m^2}. \end{aligned} \quad (24)$$

This is indeed the correct expression for the vertex.

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