COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **9**, Number **4**, July **2010**

doi:10.3934/cpaa.2010.9.1083

pp. **1083–1098**

CONTINUOUS DEPENDENCE IN FRONT PROPAGATION OF CONVECTIVE REACTION-DIFFUSION EQUATIONS

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(Communicated by Shoji Yotsutani)

Abstract. Continuous dependence of the threshold wave speed and of the travelling wave profiles for reaction-diffusion-convection equations

$$u_t + h(u)u_x = \Big(d(u)u_x\Big)_x + f(u)$$

is here studied with respect to the diffusion, reaction and convection terms.

1. Introduction.

We consider a scalar parabolic reaction-diffusion-convection equation

$$u_t + h(u)u_x = \Big(d(u)u_x\Big)_x + f(u), \qquad \text{with } t \geq 0,\ x \in \mathbb{R}, \text{ and } u(x,t) \in [0,1] \tag{1}$$

where $h \in C[0,1]$ is an arbitrary nonlinear convective term, $d \in C^1[0,1]$ stands for a diffusivity coefficient, i.e. $d(u) > 0$ for all $u \in (0,1)$, and $f \in C[0,1]$ is a Fisher-type reaction term, that is

$$f(u) > 0 \quad \text{for every } u \in (0,1) \quad \text{and} \quad f(0) = f(1) = 0. \tag{2}$$

This equation maintains a constant interest in mathematical literature since it is a model for the investigation of several problems in population dynamics, chemical processes, epidemiology, cancer growth, nerve pulses and ecology (see, e.g., [8] and [16]). In some processes, in addition to diffusion and reaction, motion is also due to convection forces. The monograph [16] contains several models concerning ecological control strategies, predator-prey pursuit and evasion, ion-exchange columns, chromatography etc. These models include, as a fundamental ingredient, a convective flux. Equation (1) is also used for the study of dispersion due to population pressure (see [17]) and for the study of chemotaxis behavior under some simplifying

2000 *Mathematics Subject Classification.* Primary: 35K57, 92D25; Secondary: 34B40, 34B16.
Key words and phrases. Reaction-diffusion-convection equations, travelling wave solutions, threshold wave speed, continuous dependence.
This research is partially supported by the M.U.R.S.T. (Italy) as the PRIN Project "Equazioni differenziali ordinarie e applicazioni".

assumptions, and it appears when modeling the Gunn effect in semiconductors (see e.g., [4, 9]). Special cases of (1) occur also in the investigation of the heat transfer with convective transport (see [5] and references therein). In all these areas, a particular relevance is held by the so called *travelling wave solutions* of equation (1). These solutions u satisfy $u(x,t) = U(x-ct)$, for some sufficiently regular one-variable function U (the wave profile) and constant $c \in \mathbb{R}$ (the wave speed), and they connect the stationary states 0 and 1, that is satisfy the boundary conditions $U(-\infty) = 1, U(+\infty) = 0$. Put $\xi = x - ct$ (the wave coordinate), $U(\xi)$ is a solution of the following ordinary differential equation

$$-cU' + h(U)U' = (d(U)U')' + f(U). \tag{3}$$

The case $d(u) > 0$ for $u \in [0,1]$ (non-degenerate case), has been investigated in [14]. Under the following assumption

$$D^+(fd)(0) := \limsup_{u\to 0^+} \frac{f(u)d(u)}{u} < +\infty, \tag{4}$$

it is proved that there exists a threshold value c^* satisfying

$$2\sqrt{D_+(fd)(0)} + h(0) \le c^* \le 2\sqrt{\sup_{u\in(0,1]} \frac{f(u)d(u)}{u}} + \max_{u\in[0,1]} h(u),$$

with $D_+(fd)(0) := \liminf_{u\to 0^+} \frac{f(u)d(u)}{u}$, such that (1) admits travelling wave solutions with speed c if and only if $c \ge c^*$. Moreover, any travelling wave solution is decreasing, hence $0 \le U(\xi) \le 1$ for every $\xi \in \mathbb{R}$, and for every admissible speed c the travelling wave solution is unique (up to a variable shift). The value c^* is usually called *minimal (or threshold) wave speed*.

Travelling wave solutions play an important role in the investigation of (1). Indeed, it was proved (see [3] and [10]) that, for special cases of (1) and a wide class of initial conditions, any solution of (1) approaches the travelling wave solution having speed c^* when $t \to \infty$.

Nevertheless, the previous setting in which the travelling waves are actually defined and regular on the whole real line, is not satisfactory in various concrete situations. Indeed, for instance, when (1) models the spatial spreading of a population initially located in a bounded environment, since individuals diffuse with a finite speed, then equation (1) realistically should have the property of finite speed of propagation (see [7]), that is any solution satisfying a compactly supported initial condition, maintains a compact support in any time. This occurs if and only if the travelling wave solution having the threshold speed c^* vanishes at a finite value of the wave variable (see [7] and [12]). As it is well known, when the diffusion coefficient is positive (as in the heat equation), in general the dynamics does not exhibit such a behavior, contrary to the degenerate parabolic equations, occurring when $d(0) = 0$. We also refer to [18] for some concrete models where the diffusion coefficient vanishes at both the equilibria 0 and 1 (doubly-degenerate case).

A prototype of equation (1) in the degenerate case is the porous media equation, with reaction and convection terms,

$$u_t + bu^k u_x = \left(au^k u_x\right)_x + cu(1-u^k) \tag{5}$$

where $a, c, k > 0$ and $b \in \mathbb{R}$. The exact value of the threshold speed of its travelling wave solutions was recently obtained in [9].

In some models (see, e.g., [18]) the diffusion coefficient vanishes at both the equilibria 0 and 1 (doubly-degenerate case). In the degenerate case [doubly-degenerate case] (1) can support travelling wave solutions attaining the value 0 [both the values 0 and 1] at a finite value of the wave variable.

Only recently a detailed discussion and a sharp classification of the qualitative properties of the solutions have been carried on for such a type of equations. In particular, it was shown that if $d(0) = 0$ (see [9] and [12]) and/or $d(1) = 0$ (see [12]), then the travelling wave having speed c^* attains the equilibria 0 and/or 1 at finite values and the set $J := \{\xi \in R : 0 < U(\xi) < 1\}$ is a halfline or a bounded interval. In this case, $U, d(U)U' \in C^1(J)$, U is a solution of (3) in the open interval J, and satisfies the boundary conditions

$$\lim_{\xi \to (\inf J)^+} U(\xi) = 1, \qquad \lim_{\xi \to (\sup J)^-} U(\xi) = 0,$$

together with the following ones:

$$\lim_{\xi \to (\inf J)^+} d(U(\xi))U'(\xi) = \lim_{\xi \to (\sup J)^-} d(U(\xi))U'(\xi) = 0. \tag{6}$$

The previous conditions can be adopted as a unifying definition of travelling wave, both for the degenerate and the non-degenerate case, since (6) is trivially satisfied if $J = \mathbb{R}$ (see [12]).

For the special case $d(u) = u^k$ with $k > 0$, the relevant interpretation of c^* as the asymptotic speed of propagation of any solution $u(x,t)$ with a compactly supported initial condition was obtained in [10] for $h \equiv 0$, and in [15] for a wider family of models including convective terms.

Clearly, equation (5) depends on the constants a, b, k, and more in general in (1) the diffusivity, the convection and reaction terms, can be viewed as parameters. Hence, the interest about the dependence on the parameters arises very naturally and the aim of this paper is just to investigate the continuous dependence of the threshold speed c^* and of the wave profiles $U(\xi)$ on the nonlinear terms appearing in (1).

Recently, some researchers started this study. In [6] the continuous dependence and further regularity of the minimal speed c^* was established in the particular non-degenerate case $h(u) \equiv 0$, $d(u) \equiv 1$ and $f(u) = u^m(1-u)$, $m \geq 1$. Subsequently, a general study of the continuous dependence of c^* and of the corresponding profile U^* was carried on in [1] recovering the degenerate equations, but again in absence of convection ($h(u) \equiv 0$). Such an investigation is based on a variational approach introduced in [2], where the following characterization of the minimal speed was obtained

$$\frac{1}{(c^*)^2} = \inf\left\{ \int_{-\infty}^{\infty} \frac{1}{2} e^t (u'(t))^2 \ \mathrm{d}t \ : \ u \in H^1(e^t), \ \int_{-\infty}^{\infty} e^t F(u(t)) \ \mathrm{d}t = 1 \right\}$$

where $F(u) := \int_0^u f(s) \ \mathrm{d}s$ and $H^1(e^t) := \{u \in H^1_{\mathrm{loc}}(\mathbb{R}) : \ e^t u(t) \in H^1(\mathbb{R})\}$. By using this approach, in [1] the continuous dependence of the minimal speed with respect on the diffusion and reaction terms, and the continuous dependence of the corresponding profiles U^*, have been proved. More in detail, given a sequence $(d_n)_n$ of positive diffusion terms, uniformly convergent to d_0, and given a sequence $(f_n)_n$ of Fisher-type reaction terms, such that $\frac{f_n(u)}{u}$ uniformly converges to $\frac{f_0(u)}{u}$ in $(0,1]$, then $c^*(d_n, f_n) \to c^*(d_0, f_0)$. Moreover, as for the corresponding profiles U_n^*, in [1] it was showed that, when $c_0^* > 2\sqrt{d_0(0) f_0'(0)}$, then $U_n^* \to U_0^*$ in $H^1(e^{\frac{c_0^*}{d_0(0)}} t)$ if $d_0(0) >$

0; while $U_n^* \to U_0^*$ in $H^1(e^{\alpha}t)$ for every $\alpha > 0$ if $d_0(0) = 0$ but $\inf_{n\geq 0} \dot{d}_n(0) > 0$. This approach also allowed to discuss the fastness of the rate of decay at 0 of the solutions u^*, both in the case of constant diffusion, and in the case of non-constant diffusion (degenerate or non-degenerate).

Due to the presence of a non-constant term multiplying the first derivative u_x, the variational technique used in [1] seems to be not appropriate for the study of the reaction-diffusion-convection equation (1). So, we introduce an alternative approach for this analysis based on differential inequalities applied to the following first order singular boundary value problem

$$\begin{cases} \dot{z}(u) = h(u) - c - \frac{f(u)d(u)}{z(u)}, \quad u \in (0,1) \\ z(u) < 0 \\ z(0^+) = z(1^-) = 0, \end{cases} \tag{7}$$

to which the investigation can be reduced, due to the monotonicity of the wave profiles (see, e.g., [12]). In Section 2 we discuss the main properties of (7) and the most important comparison techniques used for its investigation. The study of the behavior of the minimal speeds and of the fronts in the case of monotone convergence of the nonlinear terms is treated in Section 3. The discussion about the convergence of $c_n^*(h_n, d_n, f_n)$ to $c^*(h_0, d_0, f_0)$ in the general case is contained in Section 4. Firstly we prove that, under the sole conditions ensuring the existence of travelling wave solutions, the lower semi-continuity of c^* is guaranteed (see Theorem 4.1 and Example 1). As showed in [1] for the case of no convective effects, further regularities have to be assumed to achieve the continuous dependence of c^* on the nonlinear terms of the equation (see Theorem 4.2). The present investigation provides an extension to reaction-diffusion-convection equations of the study carried on in [1]. However, Theorem 4.2 improves the analogous convergence result in [1], even in the particular case of a null convective effect (see Remark 1 and Example 2). The problem of the convergence of the wave profiles is studied in Section 5. The main result is the following, whose proof is presented at the end of the section.

Theorem 1.1. *Let $(h_n)_n$, $(d_n)_n$ and $(f_n)_n$ be sequences of continuous functions uniformly convergent, respectively, to h_0, d_0 and f_0. Assume, for $n \geq 0$, that $d_n \in C^1[0,1]$ with $d_n(u) > 0$ on $(0,1)$ and (2), (4) are satisfied.*

Let $U_n(\xi)$ be the profile of the corresponding travelling wave solution with speed $c_n \geq c_n^*$ such that $U_n(0) = \frac{1}{2}$. If $c_n \to c_0$, then

(i) *$c_0 \geq c_0^*$ and $(U_n)_n$ converges to U_0 uniformly on all the real line;*
(ii) *$(U_n)_n$ converges to U_0 in $C^1_{\rm loc}(J)$ where J is the (maximal) open interval where $0 < U_0(\xi) < 1$.*

The choice of the sequential (discrete) point of view to study the continuous dependence on the parameters of the problem, is essentially due to a simpler notation requirement. Of course, all the results presented in this paper could be rewritten also in the setting of terms depending on a continuous parameter $k \in \mathbb{R}$, assuming that h, d, f are continuous functions of the two variables $(u,k) \in [0,1] \times \mathbb{R}$. In this framework, the continuous dependence of the minimal speed means the continuity of $c^*(k)$ as a function of the parameter k and the convergence of the profiles in $C^1_{\rm loc}(J)$ means that $U(t;k)$ and $U'(t;k)$ are continuous with respect to both the variables. This last statement is guaranteed by the uniform convergence on compact sets of $\mathbb{R}$.

2. **Notations and preliminary results.** This section is devoted to the statement of preliminary results which will be used in the following. Most of these results are generalizations to the present case of analogous ones proved by the authors in [13] and [14], see also the discussions before the statements.

Till Section 4 we focus our attention on the following singular first order boundary value problem

$$\begin{cases} \dot{z}(u) = h(u) - c - \dfrac{g(u)}{z(u)}, \quad u \in (0,1) \\ z(u) < 0 \\ z(0^+) = z(1^-) = 0, \end{cases} \tag{8}$$

where c is a given constant, and $h, g : [0,1] \to \mathbb{R}$ are continuous functions.

Comparing problem (8) with (7), notice that the function $g(u)$ replaces the product $f(u)g(u)$. Since f and d are positive in $(0,1)$ and f vanishes at 0 and 1, throughout Sections 2-4 we always assume that

$$g(0) = g(1) = 0, \quad g(u) > 0 \text{ in } (0,1).$$

The investigation of the solvability of problem (8) has been carried on in [14]. In particular, the following Proposition is consequence of [14, Theorem 1.4] (for $d(u) \equiv 1$ and h replaced by $-h$), combined to [14, Lemma 2.2].

Proposition 1. *Assume that*

$$D^+g(0) := \limsup_{u \to 0^+} \frac{g(u)}{u} < +\infty. \tag{9}$$

Then, there exists a real value c^* such that (8) is solvable if and only if $c \geq c^*$, and the solution is unique. Moreover, c^* satisfies

$$2\sqrt{D_+g(0)} + h(0) \leq c^* \leq 2\sqrt{\sup_{u \in (0,1]} \frac{g(u)}{u}} + \max_{u \in [0,1]} h(u), \tag{10}$$

where $D_+g(0) := \liminf_{u \to 0^+} \frac{g(u)}{u}$.

In our study we will deal with equations having various nonlinear terms and speeds. So, from now on, we will use the notation $z(u; c, h, g)$ to denote the (unique) solution of (8), in order to avoid misunderstandings. Similarly, the notation $c^*(h, g)$ will stand for the minimal admissible speed.

Our approach for handling problem (8) is based on differential inequalities and upper and lower solutions techniques. The following Lemma is a key result in this matter, and it will be also used for proving the main results in the subsequent sections.

Lemma 2.1. *For a fixed constant $c \in \mathbb{R}$, assume that there exists $\zeta \in C^1(0,1)$ such that*

$$\dot{\zeta}(u) \geq h(u) - c - \frac{g(u)}{\zeta(u)}, \quad \text{for all } u \in (0,1) \tag{11}$$

and $\zeta(0^+) = 0$, $\zeta(u) < 0$ for all $u \in (0,1)$. Then, problem (8) has a solution $z(u)$ satisfying

$$\zeta(u) \leq z(u) < 0 \quad \text{for all } u \in (0,1). \tag{12}$$

Proof. The proof of this Lemma is based on a preliminary comparison result for strict inequalities, which was proved in [13] (see Lemma 8) in the special case $h(u) \equiv 0$. The same argument works also in the present general context. From [13, Lemma 8] it follows that, if (11) holds with strict inequality, then the statement of Lemma 2.1 holds with both the inequalities in (12) strict. Assume now that (11) holds. Then, for every $\epsilon > 0$ we have

$$\dot{\zeta}(u) > h(u) - c - \epsilon - \frac{g(u)}{\zeta(u)}, \qquad \text{for every } u \in (0,1).$$

Hence, the boundary value problem

$$\begin{cases} \dot{z}(u) = h(u) - c - \epsilon - \frac{g(u)}{z(u)}, & u \in (0,1) \\ z(u) < 0 \\ z(0^+) = z(1^-) = 0. \end{cases} \tag{13}$$

possesses a solution. If c^* denotes the threshold value for problem (8), we have $c^* \leq c + \epsilon$. By the arbitrariness of $\epsilon > 0$, we get $c^* \leq c$, that is problem (8) has a solution $z(u)$. Notice that if $z(u) < \zeta(u)$ for some $u \in (0,1)$ then

$$\dot{z}(u) = h(u) - c - \frac{g(u)}{z(u)} < h(u) - c - \frac{g(u)}{\zeta(u)} \leq \dot{\zeta}(u).$$

Hence $\dot{z}(u) < \dot{\zeta}(u)$ whenever $z(u) < \zeta(u)$. If $z(\bar{u}) < \zeta(\bar{u})$ for some $\bar{u} \in (0,1)$, then $z(u) < \zeta(u)$ and $\dot{z}(u) < \dot{\zeta}(u)$ for every $u \in [\bar{u}, 1)$, implying $z(1^-) < \zeta(1^-) \leq 0$, a contradiction. Therefore, $z(u) \geq \zeta(u)$, for every $u \in (0,1)$. □

As an immediate consequence, the following monotonicity result holds.

Lemma 2.2. *Let h_1, h_2, g_1, g_2 be continuous functions, with g_1, g_2 satisfying* (9). *Assume that $h_1(u) \leq h_2(u)$ and $g_1(u) \leq g_2(u)$ for every $u \in [0,1]$. Consider problem* (8) *for $h = h_i$, and $g = g_i$, $i = 1,2$ and let c_i^*, $i = 1,2$ be the threshold value of the wave speed. Then $c_1^* \leq c_2^*$. Moreover, let $c_1 \geq c_2 \geq c_2^*$, and denote by z_i the solution of problem* (8) *for $h = h_i$, $c = c_i$ and $g = g_i$, $i = 1,2$. Then we have*

$$z_1(u) \geq z_2(u) \qquad \text{for every } u \in [0,1].$$

Proof. Let z_i^* denote the (unique) solution of problem (8) for $h = h_i$, $c = c_i^*$ and $g = g_i$. Since

$$\dot{z}_2^*(u) \geq h_1(u) - c_2^* - \frac{g_1(u)}{z_2^*(u)} \qquad \text{for every } u \in (0,1),$$

by Lemma 2.1 we get that problem (8), for $h = h_1$ and $g = g_1$, is solvable for $c = c_2^*$, and this implies $c_1^* \leq c_2^*$.

Let us consider now $c_1 \geq c_2 \geq c_2^*$. We have

$$\dot{z}_2(u) \geq h_1(u) - c_1 - \frac{g_1(u)}{z_2(u)} \qquad \text{for every } u \in (0,1),$$

implying that there exists a solution ζ of problem (8) for $h = h_1$, $c = c_1$ and $g = g_1$, satisfying $z_2(u) \leq \zeta(u)$ in $[0,1]$. By the uniqueness of the solution, we conclude that $\zeta = z_1$. □

We will use also the following auxiliary result about the interval of existence of the solutions.

Lemma 2.3. *Each negative solution $z(u)$ of the equation*

$$\dot{z} = h(u) - c - \frac{g(u)}{z(u)}, \quad u \in [a,b] \subset (0,1) \tag{14}$$

can be extended on $(0,b]$.

Moreover, if there exists a negative strict upper-solution ζ of equation (14) in $(0, b]$, i.e. a C^1-function satisfying

$$\dot{\zeta}(u) > h(u) - c - \frac{g(u)}{\zeta(u)},\ u \in (0, b], \qquad \zeta(b) < z(b),$$

then $\zeta(u) < z(u)$ for every $u \in (0, b]$.

Proof. First observe that the solution $z(u)$ of equation (14) can not blow up at a finite value. Moreover, if there exists a value $u_0 \in (0, 1)$ such that $z(u) \to 0$ as $u \to u_0^+$, then since $g(u_0) > 0$, we deduce $\dot{z}(u) \to +\infty$ as $u \to u_0^+$, a contradiction. Therefore, $z(u)$ can be extended on $(0, b]$.

Let $\bar{u} := \inf\{0 < u \leq b : \zeta(s) < z(s) \quad \text{for every } s \in [u, b]\}$. If $\bar{u} > 0$, then $\zeta(\bar{u}) = z(\bar{u})$ and $\dot{\zeta}(\bar{u}) > h(\bar{u}) - c - \frac{g(\bar{u})}{z(\bar{u})} = \dot{z}(\bar{u})$, a contradiction. Hence, $\bar{u} = 0$. □

The last result of this section concerns the regularity of the solution of (8) at 0.

Corollary 1. *Let $\dot{g}(0)$ exist (finite). Then there exists $\dot{z}(0)$ and*

$$\dot{z}(0) = \frac{1}{2}\left(h(0) - c \pm \sqrt{(h(0) - c)^2 - 4\dot{g}(0)}\right). \tag{15}$$

Moreover, if $\dot{g}(0) = 0$ and z^* denotes the solution corresponding to the threshold value c^*, we have $\dot{z}^*(0) = h(0) - c^*$.

Proof. The proof of (15) can be easily obtained following the same argument used to prove [11, Lemma 1]. The second part of the assertion is a consequence of some results proved in [12]. More in detail, let $D(u) := u$ and $\tilde{g}(u) := \frac{g(u)}{u}$. Since $\dot{g}(0) = 0$, then $\tilde{g}$ admits a continuous extension in $[0, 1]$, vanishing at 0. Then, functions D and $\tilde{g}$ satisfy all the assumptions of Theorem 1.1 and Theorem 2.1 in [12]. Combining their statements we get $\dot{z}^*(0) = h(0) - c^* < 0$ if $c^* > h(0)$, while $\dot{z}^*(0) = 0$ if $c^* = h(0)$. □

3. **Monotone convergence.**

We begin the study of the continuous dependence from the parameters for problem (8) in the case of monotone sequences.

Theorem 3.1. *Let $(h_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ be increasing sequences of continuous functions defined in $[0, 1]$, convergent to continuous functions h_0 and g_0, respectively. Assume that condition (9) holds for $g = g_n$, $n \geq 0$. Then the sequence of the corresponding threshold values $(c_n^*)_{n\geq 1}$ given by Proposition 1 is increasing and converges to c_0^*.*

Moreover, let $(c_n)_{n\geq 1}$ be a decreasing sequence of real numbers converging to a value $c_0 \geq c_0^*$. Then, denoted by $z_n(u)$ the solution $z(u; c_n, h_n, g_n)$, the sequence $(z_n)_{n\geq 1}$ is decreasing and uniformly convergent to z_0.

Proof. By virtue of Lemma 2.2 we get $c_n^* \leq c_{n+1}^* \leq c_0^*$. Then,

$$\hat{c} := \sup_{n\geq 1} c_n^* \leq c_0^*. \tag{16}$$

Let us consider a decreasing sequence $(c_n)_n$ converging to a value $c_0 \geq c_0^* \geq \hat{c}$. By Lemma 2.2 we have $z_n(u) \geq z_{n+1}(u) \geq z_0(u)$ for every $u \in [0, 1]$. Consequently, the function $\tilde{z}(u) := \inf_{n\geq 1} z_n(u) = \lim_n z_n(u)$ is well defined in $[0, 1]$, and it holds $z_0(u) \leq \tilde{z}(u) \leq z_1(u)$. Notice that for a fixed closed interval $[a, b] \subset (0, 1)$ we have

$$0 < \frac{g_n(u)}{-z_n(u)} \leq \frac{\max_{u\in[a,b]} g_0(u)}{\min_{u\in[a,b]}(-z_1(u))} = -\frac{\max_{u\in[a,b]} g_0(u)}{\max_{u\in[a,b]}(z_1(u))}.$$

Moreover, being $h_1(u) \le h_n(u) \le h_0(u)$ for every $n \ge 1$ and $u \in [a,b]$, we can apply the dominated convergence theorem obtaining

$$\tilde{z}(u) - \tilde{z}(a) = \int_a^u h_0(s)\, \mathrm{d}s - c_0(u-a) - \int_a^u \frac{g_0(s)}{\tilde{z}(s)}\, \mathrm{d}s,$$

that is $\tilde{z}$ is a solution in $[a,b]$ of the differential equation in (8), for $h = h_0$, $c = c_0$, $g = g_0$. By the arbitrariness of the interval $[a,b]$, we get that $\tilde{z}$ solves the differential equation on the whole interval $(0,1)$. Finally, since $z_0(u) \le \tilde{z}(u) \le z_1(u)$ for every $u \in (0,1)$, we obtain that $\tilde{z}(0^+) = \tilde{z}(1^-) = 0$, and then $\tilde{z}$ is a solution of problem (8) for $h = h_0$, $c = c_0$, $g = g_0$. By the uniqueness of the solution, we conclude that $\tilde{z} = z_0$, i.e. $(z_n)_{n\ge 1}$ is a decreasing sequence convergent to z_0. The uniformity follows from the Dini's theorem.

Now consider the decreasing sequence $(\hat{c}+\frac{1}{n})_n$, where $\hat{c}$ is defined in (16). Observe that

$$z_n(u) \ge \int_0^u h_n(s)\,\mathrm{d}s - c_n u \ge \int_0^u h_1(s)\,\mathrm{d}s - c_1 u \quad \text{for all } u \in [0,1] \text{ and } n \ge 1.$$

The function $\tilde{z}(u) = \inf_n z_n(u)$ is therefore well defined and reasoning as before we get that it is a solution of the equation $\dot{z}(u) = h_0(u) - \hat{c} - \frac{g_0(u)}{z(u)}$ in $(0,1)$. Since $\tilde{z}(u) \ge \int_0^u h_1(s)\,\mathrm{d}s - c_1 u$ for every $u \in (0,1)$, we get $\tilde{z}(0^+) = 0$. Therefore, according to Lemma 2.1, the boundary value problem

$$\begin{cases} \dot{z}(u) = h_0(u) - \hat{c} - \frac{g_0(u)}{z(u)}, & u \in (0,1) \\ z(0^+) = z(1^-) = 0 \\ z(u) < 0, \quad \text{for } u \in (0,1) \end{cases}$$

is solvable. This implies that $\hat{c} \ge c_0^*$. Taking (16) into account, we conclude that $\hat{c} = c_0^*$, i.e. $c_n^* \to c_0^*$. □

An analogous result for the reversed monotonicity does not hold, as the following example shows.

Example 1. Consider a sequence $(h_n)_n$ of continuous functions on $[0,1]$ satisfying $\max_{u\in[0,1]} h_n(u) = h_n(0)$ for all $n \in \mathbb{N}$, and the sequence $(g_n)_n$ of continuous functions on $[0,1]$ defined by

$$g_n(u) := \begin{cases} u(2-u) & \text{for } u \in \left[0, \frac{1}{n}\right] \\ (1-u)\big(u + \frac{1}{n-1}\big), & \text{for } u \in \left(\frac{1}{n}, 1\right]. \end{cases}$$

As it is easy to check, the sequence (g_n) is decreasing and convergent to the function $g_0(u) := u(1-u)$ in $[0,1]$. Moreover, each function g_n, $n \ge 0$, is concave and satisfies condition (9). From (10) we have

$$c_n^* = h_n(0) + 2\sqrt{\dot{g}_n(0)}, \qquad \text{for every } n \ge 0.$$

Since $\dot{g}_n(0) = 2$ for $n \ge 1$, while $\dot{g}_0(0) = 1$, from the convergence of $(h_n)_n$ to h_0 we conclude that

$$c_n^* \to h_0(0) + 2\sqrt{2} > h_0(0) + 2 = c_0^*.$$

Nevertheless, a partial continuous dependence result for the reversed type of monotonicity holds.

Theorem 3.2. *Let $(h_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ be decreasing sequences of continuous functions defined in $[0,1]$, convergent to continuous functions h_0 and g_0, respectively. Assume that condition (9) holds for $g = g_n$, $n \geq 0$. Then the sequence of the corresponding threshold values $(c_n^*)_{n\geq 1}$ given by Proposition 1 is decreasing and*

$$\inf c_n^* \geq c_0^*.$$

Moreover, let $(c_n)_{n\geq 1}$ be an increasing sequence of real numbers, converging to c_0, and such that $c_1 \geq c_1^*$. Then, denoted by $z_n(u)$ the solution $z(u; c_n, h_n, g_n)$, $n \geq 0$, the sequence $(z_n)_{n\geq 1}$ is increasing and uniformly convergent to z_0.

Proof. From Lemma 2.2 we have that $c_n^* \geq c_{n+1}^* \geq c_0^*$ and $z_n(u) \leq z_{n+1}(u) \leq z_0(u)$ for every $u \in [0,1]$. Hence, the function $\tilde{z}(u) := \sup_{u\in[0,1]} z_n(u)$ is negative in $(0,1)$ and vanishes at $u = 0$ and $u = 1$. Moreover, for a fixed closed interval $[a,b] \subset (0,1)$ we have

$$0 < \frac{g_n(u)}{-z_n(u)} \leq \frac{\max\limits_{u\in[a,b]} g_1(u)}{\min\limits_{u\in[a,b]}(-z_0(u))} = -\frac{\max\limits_{u\in[a,b]} g_1(u)}{\max\limits_{u\in[a,b]}(z_0(u))}.$$

Hence, being $h_0(u) \leq h_n(u) \leq h_1(u)$ for every $n \geq 1$ and $u \in [a,b]$, we can apply the dominated convergence theorem obtaining, as in the proof of Theorem 3.1, that $\tilde{z}$ is a solution of problem (8) for $h = h_0$, $c = c_0$, $g = g_0$. By the uniqueness of the solution, we conclude that $\tilde{z} = z_0$. □

4. **Continuity of the threshold values c^*.** As we showed in Example 1, when dealing with a decreasing sequence of reaction terms $(g_n)_n$, one can not expect the convergence of the threshold values c_n^*, but at most a semicontinuity property:

$$\inf c_n^* \geq c_0^*.$$

The lower semicontinuity of c_n^* is a general property, as the following result shows.

Theorem 4.1. *Let $(h_n)_n$ and $(g_n)_n$ be sequences of continuous functions uniformly convergent to functions h_0 and g_0 respectively. Assume that condition (9) holds for every $n \geq 0$. Then*

$$\liminf_{n\to\infty} c_n^* \geq c_0^*. \tag{17}$$

Proof. For every $u \in [0,1]$ and $n \geq 1$, let

$$\hat{h}_n(u) := \min\{h_0(u), \inf_{k\geq n} h_k(u)\} \qquad \hat{g}_n(u) := \min\{g_0(u), \inf_{k\geq n} g_k(u)\}. \tag{18}$$

Due to the uniform convergence of the sequence $(h_n)_n$ to h_0, it is easy to check that the functions $\hat{h}_n$ are well defined and the sequence $(\hat{h}_n)_n$ is increasing and uniformly convergent to h_0 in $[0,1]$.

Let us now show that each function $\hat{h}_n$ is continuous. To this aim, let us fix $\bar{n} \in \mathbb{N}$. First of all, notice that $\hat{h}_{\bar{n}}$ is upper semicontinuous, since it is the infimum of a family of continuous functions. Let us assume, by contradiction, that

$$\hat{h}_{\bar{n}}(u_0) > \lim_{s\to+\infty} \hat{h}_{\bar{n}}(u_s) =: L \tag{19}$$

for some $u_0 \in [0,1]$ and some sequence $(u_s)_s$ converging to u_0 in $[0,1]$. Let $\epsilon > 0$ be such that $\hat{h}_{\bar{n}}(u_0) - 3\epsilon > L$. Then, by virtue of the definition of L, the continuity of h_0 and the uniform convergence of $(h_n)_n$ towards h_0, an integer $n^* \geq \bar{n}$ exists such that

$$h_n(u_s) > h_0(u_s) - \epsilon \geq h_0(u_0) - 2\epsilon \geq \hat{h}_{\bar{n}}(u_0) - 2\epsilon > L + \epsilon > \hat{h}_{\bar{n}}(u_s),$$

for every $n, s \geq n^*$. Hence, we deduce that

$$\hat{h}_{\bar{n}}(u_s) = \min\{h_0(u_s), \min_{\bar{n} \leq k \leq n^*} h_k(u_s)\} \quad \text{for every } s \geq n^*.$$

Since the function $u \mapsto \min\{h_0(u), \min_{\bar{n} \leq k \leq n^*} h_k(u)\}$ is continuous on all $[0,1]$, we get

$$\hat{h}_{\bar{n}}(u_s) \to \min\{h_0(u_0), \min_{\bar{n} \leq k \leq n^*} h_k(u_0)\} \geq \hat{h}_{\bar{n}}(u_0),$$

in contradiction with (19).

Similarly one can show that each function $\hat{g}_n$ is well defined and continuous, and that the sequence $(\hat{g}_n)_n$ is increasing and uniformly convergent to g_0. Moreover, being $g_n(0) = g_n(1) = 0$ for every $n \geq 0$, we get $\hat{g}_n(0) = \hat{g}_n(1) = 0$ for every $n \geq 1$. Similarly we get $g_n(u) > 0$ in $(0,1)$ for every $n \geq 1$. Finally, since $\hat{g}_n(u) \leq g_n(u)$ in $[0,1]$, we get $D^+\hat{g}_n(0) < +\infty$ for every $n \geq 1$. Therefore, according to Proposition 1, for every $n \geq 1$ there exists a real value σ_n^* such that the boundary value problem

$$\begin{cases} \dot{z}(u) = \hat{h}_n(u) - c - \dfrac{\hat{g}_n(u)}{z(u)}, \quad u \in (0,1) \\ z(0^+) = z(1^-) = 0 \\ z(u) < 0, \quad \text{for } u \in (0,1) \end{cases} \tag{20}$$

is solvable if and only if $c \geq \sigma_n^*$. Moreover, we can apply Theorem 3.1 to deduce that the sequence $(\sigma_n^*)_{n\geq 1}$ is increasing and convergent to c_0^*. Since $\hat{h}_n(u) \leq h_n(u)$ for every $n \geq 1$ and every $u \in [0,1]$, by virtue of Lemma 2.2, we get $\sigma_n^* \leq c_n^*$, which implies $\liminf c_n^* \geq c_0^*$. □

In view of Example 1, in order to obtain the continuity of the threshold value c^*, we need to add some further requirements, concerning the infinitesimal asymptotic of $g_n(u) - g_0(u)$ as $u \to 0$.

Theorem 4.2. *Let $(h_n)_n$ and $(g_n)_n$ be sequences of continuous functions uniformly convergent to functions h_0 and g_0 respectively. Assume that condition* (9) *holds for every $n \geq 0$ and let $\dot{g}_0(0)$ exists finite.*

Suppose that

$$\limsup_{u\to 0,\ n\to\infty} \frac{g_n(u) - g_0(u)}{u} \leq 0. \tag{21}$$

Then $c_n^* \to c_0^*$.

Proof. Let $z_0(u)$ denote the solution $z(u; c_0^*, h_0, g_0)$. From Corollary 1 we get that $\dot{z}_0(0)$ exists and satisfies (15).

First assume that $\dot{z}_0(0) < 0$. In this case, by assumption (21) and the uniform convergence of $(h_n)_n$, we have that for every $\epsilon > 0$ there exists an integer $\bar{n}$ and a positive number $\delta > 0$ such that

$$\frac{g_n(u)}{z_0(u)} - \frac{g_0(u)}{z_0(u)} > -\frac{\epsilon}{2} \quad \text{for all } n \geq \bar{n} \text{ and } 0 < u \leq \delta; \tag{22}$$

$$|h_n(u) - h_0(u)| < \frac{\epsilon}{2} \quad \text{for all } n \geq \bar{n} \text{ and } u \in [0,1]. \tag{23}$$

Therefore,

$$\dot{z}_0(u) > h_n(u) - c_0^* - \epsilon - \frac{g_n(u)}{z_0(u)} \quad \text{for every } u \in (0,\delta], n \geq \bar{n}. \tag{24}$$

Let $\zeta(u) := z(u; c_0^* + \epsilon, h_0, g_0)$. From Lemma 2.2 we have $\zeta(u) > z_0(u)$ for every $u \in (0,1)$. Let $\alpha \in (z_0(\delta), \zeta(\delta))$ be fixed, and let w be the solution of the initial value problem

$$\begin{cases} \dot{w} = h_0(u) - c_0^* - \epsilon - \dfrac{g_0(u)}{w(u)} \\ w(\delta) = \alpha. \end{cases} \tag{25}$$

As a consequence of Lemma 2.3, w is defined on all $(0,\delta]$ and $w(u) > z_0(u)$ for every $u \in (0,\delta]$. Then, $w(0^+) = 0$. Moreover, since w and ζ solve the same differential equation, we get $w(u) < \zeta(u)$ whenever w is defined. Therefore, w is defined on the whole interval $(0,1)$. Since ζ is the unique solution of problem (8) for $c = c_0^* + \epsilon$, $h = h_0$ and $g = g_0$, we get that $w(1^-) < 0$.

Let us now consider the initial value problem

$$\begin{cases} \dot{\psi} = h_n(u) - c_0^* - \epsilon - \dfrac{g_n(u)}{\psi(u)} \\ \psi(\delta) = \alpha \end{cases} \tag{26}$$

and let ψ_n denote its unique solution. By Lemma 2.3 we get that $\psi_n(u)$ is defined on $(0,\delta]$, and taking (24) into account, $z_0(u) < \psi_n(u)$ for every $n \geq \bar{n}$ and $u \in (0,\delta]$. Thus, $\psi_n(0^+) = 0$ for every $n \geq \bar{n}$. From the continuous dependence on the data for problem (26), we get the existence of an integer $\tilde{n} \geq \bar{n}$ such that $\psi_n(1) < 0$ for every $n \geq \tilde{n}$. Then, by applying Lemma 2.1, we deduce that problem (8) is solvable for $c = c_0^* + \epsilon$, $h = h_n$ and $g = g_n$. This yields $c_n^* \leq c_0^* + \epsilon$ for every $n \geq \tilde{n}$ and the assertion follows from Theorem 4.1.

Assume now $\dot{z}_0(0) = 0$. Observe that from (15) we get $\dot{g}_0(0) = 0$; moreover by Corollary 1 we deduce $c_0^* = h_0(0)$. If the strict inequality holds in formula (21), then we get $g_n(u) < g_0(u)$ for n large enough and u in a right neighborhood of 0. Hence, in this case (22) holds and the proof proceeds as above. So, let us now consider the case

$$\limsup_{u\to 0,\, n\to\infty} \frac{g_n(u) - g_0(u)}{u} = 0.$$

Since $\dot{g}_0(0) = 0$, we have

$$\limsup_{u\to 0,\, n\to\infty} \frac{g_n(u)}{u} = \limsup_{u\to 0,\, n\to\infty} \left(\frac{g_0(u)}{u} + \frac{g_n(u) - g_0(u)}{u} \right) = 0. \tag{27}$$

Let $\epsilon > 0$ be fixed. Taking (27) and the uniform convergence of $(h_n)_n$ into account, since $c_0^* = h_0(0)$, we get the existence of an integer $\bar{n}$ and a real $\delta > 0$ such that

$$\frac{g_n(u)}{u} < \frac{\epsilon^2}{8} \quad \text{and} \quad h_n(u) - c_0^* < \frac{\epsilon}{4} \quad \text{for every } n \geq \bar{n},\ u \in (0,\delta). \tag{28}$$

Put, as before, $\zeta(u) := z(u; c_0^* + \epsilon, h_0, g_0)$, from Lemma 2.2 we obtain $\zeta(u) > z_0(u)$ for every $u \in (0,1)$. Let $\alpha \in (z_0(\delta), \zeta(\delta))$ be fixed, and denote again by w and ψ_n the solutions of the initial value problem (25) and (26), respectively. As above, we get that w is defined on the whole interval $(0,1)$ with $w(1^-) < 0$, and ψ_n is defined on $(0,\delta]$.

Let us now consider the straightline $\gamma(u) = -\frac{\epsilon}{2}u$. From (28) we have

$$h_n(u) - c_0^* - \epsilon - \frac{g_n(u)}{\gamma(u)} < \frac{\epsilon}{4} - \epsilon + \frac{\epsilon}{4} < -\frac{\epsilon}{2} = \dot{\gamma}(u), \tag{29}$$

i.e. γ is a strict upper-solution of equation (14) for $h = h_n$, $g = g_n$ and $c = c_0^* + \epsilon$. Hence, by Lemma 2.3, we obtain that $\gamma(u) < \psi_n(u)$ for every $u \in (0,\delta)$, implying that $\psi_n(0^+) = 0$. Moreover, by the continuous dependence on the data for problem (26), since $w(1^-) < 0$, we get the existence of an integer $\tilde{n} \geq \bar{n}$ such that $\psi_n(1) < 0$ for every $n \geq \tilde{n}$. Then, by applying again Lemma 2.1, we deduce that problem (8) is solvable for $c = c_0^* + \epsilon$, $h = h_n$ and $g = g_n$, which implies $c_n^* \leq c_0^* + \epsilon$ for every $n \geq \tilde{n}$. □

Remark 1. Theorem 4.2 improves the analogous convergence result proved in [1], even in the case of absence of convective effects. Indeed, assumption (21) is weaker than the uniform convergence of the sequence $(\frac{g_n(u)}{u})_n$ in $(0,1)$, as the following example shows.

Example 2. Let us consider $h_n(u) :\equiv 0$ and $g_n(u) := \min\{nu^2, u(1-u)\}$, $n \geq 1$. Clearly, $(g_n)_n$ uniformly converges to $g_0(u) = u(1-u)$ in $[0,1]$. Moreover, we have $g_n(u) - g_0(u) = \min\{nu^2 - u(1-u), 0\}$ and $\frac{g_n(u)-g_0(u)}{u} = \min\{nu - (1-u), 0\} \leq 0$. Condition (21) therefore holds, implying $c_n^* \to c_0^*$. However, $\frac{g_n(u)}{u} = \min\{nu, 1-u\}$ does not uniformly converges to $\frac{g_0(u)}{u} = 1 - u$ in $(0,1)$.

Example 3. Let us consider the following porous media equation with reaction and convection terms

$$u_t + h(u)u_x = (u^m)_{xx} + f(u)$$

with $m > 1$ and $h(u) \geq 0$ for every $u \in [0,1]$. Taking $d(u) := mu^{m-1}$, this equation can be seen as a particular case of (1). Supposing that f is differentiable at 0 when $1 < m < 2$, according to Theorem 4.2 we are able to state that the minimal admissible speed $c^*(m)$ is a continuous function of the parameter m (see also the discussion in Introduction about the extension from the discrete to the continuous point of view). Observe that in [9] the exact value of $c^*(m)$ was determined in the particular case $h(u) = bu^{m-1}$, $f(u) = cu(1-u^{m-1})$, with $b \in \mathbb{R}$, $c > 0$.

5. **Convergence of the profiles.**

In this section we will prove Theorem 1.1 about the convergence of the wave profiles. According to the approach used in this paper, we first prove the continuous dependence of the solutions of the singular problem (8).

Theorem 5.1. *Let $(h_n)_n$ and $(g_n)_n$ be sequences of continuous functions uniformly convergent to functions h_0 and g_0 respectively, and let $(c_n)_n$ be a sequence in $\mathbb{R}$ convergent to c_0, satisfying $c_n \geq c_n^*$ for every $n \in \mathbb{N}$. Assume that condition (9) holds for every $n \geq 0$, and let $z_n(u) := z(u; c_n, h_n, g_n)$, $n \geq 0$. Then the sequence $(z_n)_n$ converges to z_0, uniformly in each compact interval $[a,b] \subset (0,1)$.*

Proof. By virtue of Theorem 4.1 we have $c_0 \geq c_0^*$, so z_0 is well defined. According to Theorem 3.1, the sequence $z(u; c_0 + \frac{1}{n}, h_0, g_0)_n$ is decreasing and uniformly converges to z_0. Let $\epsilon > 0$ be fixed. Then there exists a real number $\sigma_0 > 0$ such that

$$z(u; c_0 + \sigma_0, h_0, g_0) < z_0(u) + \frac{\epsilon}{2} \quad \text{for every } u \in (0,1). \tag{30}$$

For every $n \geq 1$, let us consider problem (20), with $\hat{h}_n$ and $\hat{g}_n$ defined as in (18), and let σ_n^* be the corresponding threshold value. Since the sequence $(\sigma_n^*)_n$ increases and converges to $c_0^* \leq c_0$, we have $\sigma_n^* < c_0 + \sigma_0 + \frac{1}{n}$ for every $n \geq 1$. Let $w_n(u) := z(u; c_0 + \sigma_0, \hat{h}_n, \hat{g}_n)$. By virtue of Theorem 3.1, we get that $(w_n)_n$ decreases and uniformly converges to $z(u; c_0+\sigma_0, h_0, g_0)$. Therefore, fixed $\epsilon > 0$, there exists an integer $\bar{n}$ such that for every $n \geq \bar{n}$ we have

$$w_n(u) \leq z(u; c_0+\sigma_0, h_0, g_0) + \frac{\epsilon}{2} \quad \text{for every } u \in (0,1). \tag{31}$$

On the other hand, we can assume without restriction that $c_n \leq c_0 + \sigma_0$ for every $n \geq \bar{n}$. According to Lemma 2.2 we have

$$z_n(u) \leq w_n(u), \quad \text{for all } u \in [0,1] \text{ and } n \geq \bar{n}. \tag{32}$$

Combining (30), (31) and (32), for all $n \geq \bar{n}$ we have that

$$z_n(u) \leq w_n(u) \leq z_0(u) + \epsilon, \quad \text{for every } u \in (0,1). \tag{33}$$

Let $u_0 \in (0,1)$ be fixed, and assume by contradiction that for a subsequence, again labelled $(z_n)_n$, we have

$$\lim_{n\to+\infty} z_n(u_0) =: L < z_0(u_0). \tag{34}$$

Let $\ell \in (L, z_0(u_0))$. For a fixed $\eta \geq 0$, let ω_η denote the unique solution of the initial value problem

$$\begin{cases} \dot{z}(u) = h_0(u) - c_0 + \eta - \dfrac{g_0(u)}{z(u)} \\ z(u_0) = \ell. \end{cases} \tag{35}$$

Since $z_0(u_0) > \omega_0(u_0)$, and z_0 and ω_0 solve the same differential equation, we get $z_0(u) > \omega_0(u)$ whenever ω_0 is defined. As it is easy to check from the differential equation in (35), ω_0 can not blow up at any finite value, and so it is defined on the whole interval $[u_0, 1]$, with $\omega_0(u) < z_0(u)$ for every $u \in [u_0, 1)$. Moreover, since

$$\dot{z}_0(u) = h_0(u) - c_0 - \frac{g_0(u)}{z_0(u)} > h_0(u) - c_0 - \frac{g_0(u)}{\omega_0(u)} = \dot{\omega}_0(u) \quad \text{for every } u \in [u_0, 1),$$

we deduce that $\omega_0(1) < 0$.

According to the continuous dependence of the solution of problem (35) on the parameter η, there exists a value $\eta_0 > 0$ such that the solution ω_{η_0} exists in $[u_0, 1]$, with $\omega_{\eta_0}(1) < 0$. Therefore,

$$h_n(u) - c_n - \frac{g_n(u)}{\omega_{\eta_0}(u)} \;\to\; h_0(u) - c_0 - \frac{g_0(u)}{\omega_{\eta_0}(u)} \quad \text{uniformly in } [u_0, 1],$$

and there exists an integer $\bar{n}$ such that

$$h_n(u) - c_n - \frac{g_n(u)}{\omega_{\eta_0}(u)} < h_0(u) - c_0 - \frac{g_0(u)}{\omega_{\eta_0}(u)} + \eta_0 = \dot{\omega}_{\eta_0}(u) \tag{36}$$

for all $n \geq \overline{n}$ and $u \in [u_0, 1]$. On the other hand, since $L < \ell$, by (34) we can also find an integer $m \geq \bar{n}$ such that $z_m(u_0) < \ell$. Let us consider the initial value problem

$$\begin{cases} \dot{z}(u) = h_m(u) - c_m - \dfrac{g_m(u)}{z(u)} \\ z(u_0) = \ell \end{cases} \tag{37}$$

and let $\zeta(u)$ denote its solution. Since $\zeta(u_0) = \ell = \omega_{\eta_0}(u_0)$, by (36) we get $\dot{\zeta}(u_0) < \dot{\omega}_{\eta_0}(u_0)$. Moreover, for any $\bar{u} \in (u_0, 1)$ such that $\zeta(\overline{u}) < \omega_{\eta_0}(\overline{u})$, by (36) we have

$$\dot{\zeta}(\overline{u}) = h_m(\overline{u}) - c_m - \frac{g_m(\overline{u})}{\zeta(\overline{u})} < h_m(\overline{u}) - c_m - \frac{g_m(\overline{u})}{\omega_{\eta_0}(\overline{u})} < \dot{\omega}_{\eta_0}(\bar{u}).$$

Therefore, ζ is defined in the whole interval $(0,1)$, with $\zeta(u) < \omega_{\eta_0}(u)$ and $\dot\zeta(u) < \dot\omega_{\eta_0}(u)$ for all $u \in (u_0, 1)$. Hence, $\zeta(1^-) < 0$. By the uniqueness of the solution of any initial value problem associated to the equation in (37), since $z_m(u_0) < \ell = \zeta(u_0)$, we deduce that $z_m(u) < \zeta(u)$ for all $u \in (u_0, 1)$, in contradiction with $z_m(1^-) = 0 > \zeta(1^-)$. Consequently, (34) is false and

$$\liminf_{n\to+\infty} z_n(u_0) \ge z_0(u_0),$$

which joined to (33) leads to $\lim_{n\to+\infty} z_n(u_0) = z_0(u_0)$, i.e, the pointwise convergence of $(z_n)_n$ to z_0 is proved. The uniform convergence in each compact interval $[a,b] \subset (0,1)$ follows from the equiboundedness of the sequence $(z_n')_n$ in every compact interval contained in $(0,1)$. Indeed, since the sequences $(h_n)_n$ and $(g_n)_n$ are uniformly convergent, they are equibounded. Let $M_h \ge |h_n(u)|$, $M_g \ge g_n(u)$, for all $u \in [a,b]$, $n \in \mathbb{N}$, and let $C \ge |c_n|$ for all $n \in \mathbb{N}$. Let $-m_0 := \max_{[a,b]} z_0(u) < 0$, and let $\varepsilon = m_0/2$. From (33), it follows that $z_n(u) \le z_0(u) + m_0/2 \le -m_0/2 < 0$ for $u \in [a,b]$ and n sufficiently large. Taking into account that z_n is a solution of the equation (8) for $c = c_n$, $h = h_n$, $g = g_n$, we obtain

$$|z_n'(u)| \le M_h + C + \frac{2M_g}{m_0}$$

for every $u \in [a,b]$ and n sufficiently large, i.e. the sequence $(z_n')_n$ is equibounded. □

The uniform convergence of the solutions z_n allows us to prove the convergence of the profiles stated in Theorem 1.1. In order to give the proof of this result we also need the following Lemma proved in [1].

Lemma 5.2. ([1, Lemma 2.5]) *Let $(w_n)_{n\ge 0}$, $w_n : \mathbb{R} \to [0,1]$, be a sequence of continuous decreasing functions satisfying*

$$\lim_{t\to-\infty} w_n(t) = 1, \quad \textit{and} \quad \lim_{t\to+\infty} w_n(t) = 0, \quad n \ge 0.$$

Assume that $w_n(t) \to w_0(t)$ for every t in a dense subset of the interval $(\alpha_0, \beta_0) := \{t \in \mathbb{R} : 0 < w_0(t) < 1\}$. Then $w_n \to w_0$ uniformly on $\mathbb{R}$.

Proof of Theorem 1.1. According to Theorem 4.1, we have $c_0 \ge c_0^*$, hence the profile U_0 with speed c_0 is well defined. Let z_n denote the solution $z(u; c_n, h_n, g_n)$, for $n \ge 0$. It is possible to prove (see, e.g., [12, Theorem 2.1]) that U_n is the solution of problem

$$(P_n) \qquad \begin{cases} u' = \dfrac{z_n(u)}{d_n(u)}, & t \in (\tau_n, t_n) \\ u(0) = \dfrac{1}{2}, \end{cases}$$

on its maximal existence interval (τ_n, t_n), with $-\infty \le \tau_n < 0 < t_n \le +\infty$, i.e., $U_n(\tau_n^+) = 1$, $U_n(t_n^-) = 0$.

First we prove the pointwise convergence. Let $[a,b] \subset (\tau_0, t_0)$ be a fixed interval, with $a < 0 < b$, and let $U_0([a,b]) = [\alpha, \beta] \subset (0,1)$. According to Theorem 5.1 and the uniform convergence of $(d_n)_n$, we have that

$$\frac{z_n(u)}{d_n(u)} \to \frac{z_0(u)}{d_0(u)}$$

uniformly on $[\alpha, \beta]$. Therefore, according to the continuous dependence of the solution on a parameter, at least for a sufficiently large n, U_n is defined on $[a, b]$ and $(U_n)_n$ converges to U_0 uniformly on $[a, b]$.

Consider now the case when for some $\xi_0 < 0$ we have $U_0(\xi_0) = 1$. Assume by contradiction the existence of $\eta_0 \in (\frac{1}{2}, 1)$ and of a subsequence $(U_{n_k})_k$ such that

$$U_{n_k}(\xi_0) < \eta_0, \qquad \text{for every } k \in \mathbb{N}. \tag{38}$$

In the interval $[\xi_0, 0]$, $U_{n_k}(\xi)$ is the solution of (P_{n_k}) and according to the uniform convergence of $\frac{z_{n_k}(u)}{d_{n_k}(u)}$ to $\frac{z_0(u)}{d_0(u)}$ on $[\eta_0, \frac{1}{2}]$ we have that $U_{n_k} \to U_0$ uniformly on $[\xi_0, 0]$, in contradiction with (38). Thus $U_n(\xi_0) \to 1 = U_0(\xi_0)$. Using an analogous argument it is possible to prove that, if $U_0(\xi_1) = 0$ for some $\xi_1 > 0$, then $U_n(\xi_1) \to 0$, and the pointwise convergence of $(U_n)_n$ to U_0 on all the real line is proved.

The uniform convergence follows as an application of Lemma 5.2.

Finally, let $[a, b] \subset \mathbb{R}$ be such that $0 < U_0(\xi) < 1$ for all $\xi \in [a, b]$. From the uniform convergence, according to Theorem 5.1 and the convergence of $(d_n)_n$, we have that

$$U_n'(t) = \frac{z_n(U_n(t))}{d_n(U_n(t))} \to \frac{z_0(U_0(t))}{d_0(U_0(t))} = U_0'(t)$$

uniformly on $[a, b]$. □

Remark 2. If $c_n^* \to c_0^*$, from Theorem 1.1 it follows that the minimal speed profiles U_n^* converge to U_0^* uniformly on all the real line and in $C^1_{\text{loc}}(J^*)$, where $J^* \subset \mathbb{R}$ is the maximal open interval where $0 < U_0^*(\xi) < 1$.

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Received July 2009; revised January 2010.

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