k – Pyramidal One-Factorizations

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Abstract. We consider one-factorizations of complete graphs which possess an automorphism group fixing $k \geq 0$ vertices and acting regularly (i.e., sharply transitively) on the others. Since the cases $k = 0$ and $k = 1$ are well known in literature, we study the case $k \geq 2$ in some detail. We prove that both k and the order of the group are even and the group necessarily contains $k - 1$ involutions. Constructions for some classes of groups are given. In particular we extend the result of [7]: let G be an abelian group of even order and with $k - 1$ involutions, a one-factorization of a complete graph admitting G as an automorphism group fixing k vertices and acting regularly on the others can be constructed.

Key words. One-factorization, Sharply transitive permutation group, Starter.

1. Introduction

A *one-factor* in a graph is a set of pairwise disjoint edges that partition the set of vertices and a *one-factorization* in a graph is a partition of the set of edges into one-factors.

One-factorizations of complete graphs have been studied from different point of view, we refer to [14] and to the monograph [18] for a survey and for the general notions that will not be explicitly defined here. A complete graph admits a one-factorization if and only if it has an even number of vertices. For this reason, we will only be concerned with the complete graph K_v , v an even integer.

As the number of non-isomorphic one-factorizations of K_v rapidly explodes as v increases, [9], a general classification seems to be not possible. In an attempt to describe one-factorizations which have some degree of symmetry, one can impose conditions on the automorphism group. We recall that an automorphism group is a group of bijections on the vertex-set preserving the one-factorization.

Let k be an integer, with $0 \leq k \leq v$, in this paper we deal with the following

Questions. Does there exist a one-factorization of K_v admitting an automorphism group G fixing k vertices and acting regularly (i.e. sharply transitively) on the others? What can we say about the one-factorization? What can we say about G ?

The case $k = 1$ is completely settled. The existence is well known for any group G of odd order, [8], and the one-factorization is said to be 1-*rotational*, or *pyramidal* under G (see [14] for the terminology).

When $k = 0$, there exists a one-factorization of K_v admitting a cyclic sharply-vertex-transitive automorphism group except when $v = 2^n$, $n \geq 3$, [11]. This result was extended to all abelian groups in [7], and to many other classes of groups, see [3], [4], [5], [6], [16], giving ground to conjecture that for each group G of even order (except for the cyclic groups of order 2^n , $n \geq 3$) a one-factorization of a complete graph admitting G as a sharply-vertex-transitive automorphism group always exists. Further results are also in [13].

When $k = 2$, a one-factorization of K_v admitting a cyclic automorphism group fixing 2 vertices and acting regularly on the others is constructed in [14] for all even v 's. Other examples were constructed in [2] in case the automorphism group G is symmetrically sequenceable.

Paralleling the notations introduced in [14], a one-factorization with an automorphism group G which acts regularly on all but k pointwise fixed vertices, will be said *k-pyramidal* under G (when $k = 2$ the term *bipyramidal* is used in [14]). We will also say that G realizes a *k-pyramidal one-factorization*.

When $k \geq 2$ we prove that necessarily k is even and G contains exactly $k - 1$ involutions.

Hence, a group G of even order with $k - 1$ involutions stands as a candidate to realize a sharply-vertex-transitive one-factorization as well as a *k-pyramidal one-factorization* of a complete graph of suitable order.

In [7], Buratti proved that each abelian group of even order (except for the cyclic group of order 2^n , $n \geq 3$) realizes a sharply-vertex-transitive one-factorization. In [2], Anderson proved that each abelian group of even order with a unique involution realizes a bipyramidal one-factorization, see also Theorem 1 below.

In this note we extend these results and we prove that each abelian group of even order and with $k - 1$ ($k \geq 2$) involutions realizes a *k-pyramidal one-factorization* of a complete graph.

In addition we also examine some other classes of groups and we obtain an analogous result for the class of dihedral groups (thus extending a result of [3]), and for the class of Hamiltonian groups.

2. Preliminaries and Construction

Consider the complete graph K_v and denote respectively by V and E its vertex-set and its edge-set.

Proposition 1. *Let $\mathcal{F}$ be a one-factorization of K_v which is *k-pyramidal* under the action of a group G . If $k \geq 2$ then k is even and G has even order.*

Proof. Suppose k to be odd, $k \neq 1$. Let $\infty_1, \dots, \infty_k$ be the fixed vertices and let F_1 be the one-factor containing $[\infty_1, \infty_2]$. F_1 is fixed by G , as well as all its edges containing $\infty_3, \dots, \infty_k$. This implies the existence of at least one more vertex which

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is fixed by G : a contradiction. The group G acts regularly on an even number of vertices, therefore it has even order. $\square$

In the rest of the paper set $k = 2t > 0$, $v = 2n + 2t$ and denote by $\mathcal{F}$ a $2t$ -pyramidal one-factorization of K_v under the action of a group G . Obviously $2n$ is precisely the order of G in this case. Denote by $X = \{\infty_1, \dots, \infty_{2t}\}$ the set of fixed vertices and identify $V - X$ with the elements of G . The action of G on V can be assumed to occur by right multiplication: an element $g \in G$ fixes each element of X (i.e., $\infty_i g = \infty_i$, $i = 1, \dots, 2t$) and g maps a vertex $v \in V - X$ onto vg . This action extends to edges and one-factors. Hence if R is any subset of V we write: $Rg = \{xg \mid x \in R\}$, in particular if $S = [x, y]$ is an edge then $[x, y]g = [xg, yg]$. Furthermore, if U is a collection of subsets of V , then we write $Ug = \{Sg \mid S \in U\}$. In particular, if U is a collection of edges of K_v then $Ug = \{[xg, yg] \mid [x, y] \in U\}$.

Denote by 1_G the identity of G and set $G = \{g_1 = 1_G, g_2, \dots, g_{2n}\}$.

Proposition 2. *The group G contains exactly $2t - 1$ involutions, it fixes $2t - 1$ one-factors of $\mathcal{F}$ and acts regularly on the others.*

Proof. The edge-set of the complete subgraph K_X is pointwise fixed by G , then the group G fixes at least $2t - 1$ one-factors of $\mathcal{F}$. Consider the set of edges of K_v of type $[\infty_1, g_i]$, as g_i varies in G . They belong to different factors and form an orbit of length $2n$ under the action of G . We conclude that G has exactly $2t - 1$ fixed one-factors and acts regularly on the others. Consider an edge $e = [g_i, g_j]$, $g_i, g_j \in G$. The stabilizer G_e has either cardinality 2 or it is trivial according to whether $g_j^{-1}g_i$ is an involution or not. If $G_e = \{1_G\}$ then $|Orb_G(e)| = 2n$ and each of these $2n$ edges belongs to a different one-factor. In fact if F denotes the one-factor containing e , the existence of $g \in G - \{1_G\}$ mapping e onto an edge of F forces F to be fixed by G and to contain $t + 2n$ distinct edges: a contradiction. If $|G_e| = 2$ then $|Orb_G(e)| = n$. The involution $g_j^{-1}g_i$ fixes e together with the one-factor, say F , to which e belongs. Therefore the one-factor F is fixed by G , it contains t edges with both vertices in X and $Orb_G(e)$ yields the other edges. In particular the element $g_j g_i^{-1}$ is an involution itself, say σ , and $Orb_G(e) = \{[g, \sigma g] \mid g \in T\}$ where T denotes a set of distinct representatives for the right cosets in G of the subgroup $\{1_G, \sigma\}$. We conclude that each involution $\tau \in G$ corresponds to a fixed one-factor, namely the one containing the orbit of the edge $[1_G, \tau]$, and viceversa. We conclude that the number of involutions in G is $2t - 1$. $\square$

Given a group G of even order $2n$ possessing $2t - 1$ involutions, $t \geq 1$, we want to test the existence of a one-factorization which is $2t$ -pyramidal under G . Moreover, we want the minimum amount of information which is necessary to reconstruct the one-factorization from G . Use previous notations and set $X = \{\infty_1, \dots, \infty_{2t}\}$, with $G \cap X = \emptyset$ and construct the complete graph with vertex-set $V = G \cup X$. Consider the natural action of G on vertices, edges and factors as before and let J_G be the set of involutions of G . An edge $e = [g_i, g_j]$ with both vertices in G will be called a *proper edge* and there are two possibilities:

- It has trivial stabilizer in G and $|Orb_G(e)| = 2n$. In this case $g_j g_i^{-1}$ is not an involution (otherwise $g_i^{-1} g_j$ fixes e), e is called a *long* edge and we set $\Phi(e) = \{g_i, g_j\}$ and $\partial e = \{g_j g_i^{-1}, g_i g_j^{-1}\}$.
- The element $g_j^{-1} g_i$ is an involution fixing e . The edge is called *short* in this case, $|Orb_G(e)| = n$, the element $g_j g_i^{-1}$ is an involution itself and we set $\partial e = \{g_j g_i^{-1}\}$.

If S is a set of long proper edges, we define $\partial S = \bigcup_{e \in S} \partial e$ and $\Phi(S) = \bigcup_{e \in S} \Phi(e)$. Obviously these unions can contain repeated elements and so, in general, will return a multiset.

Definition 1. Let $S_G = \{e_1, \dots, e_{n-t}\}$ be a set of $n-t$ distinct long and proper edges. We say that S_G is a weak-starter in G if the following conditions are satisfied:

- $\partial S_G = G - (J_G \cup \{1_G\})$
- $\Phi(e_i) \cap \Phi(e_j) = \emptyset$, for every $i, j \in \{1, \dots, n-t\}$.

When $|J_G| = 1$, this definition coincides with the definition of *right even starter* introduced in [2].

Furthermore, if G is an abelian group with an elementary abelian 2-Sylow subgroup P , then a weak-starter in G is the patterned frame starter in $G - P$, see [10] page 473. If G itself is elementary abelian, then a set S_G is a weak-starter in G if and only if S_G is the empty set.

Proposition 3. Let G be a group of order $2n$ which contains exactly $2t-1$ involutions. The existence of a weak-starter in G is equivalent to the existence of a one-factorization of K_{2n+2t} which is $2t$ -pyramidal under G .

Proof. Suppose the existence in G of a weak-starter $S_G = \{[g_i, h_i], i = 1, \dots, n-t\}$. We construct a one-factorization $\mathcal{F}$ of K_{2n+2t} . Let $\mathcal{H} = \{H_1, \dots, H_{2t-1}\}$ be a one-factorization of K_X . Pair each factor H_i with an involution $\sigma_i \in G$ in such a way that distinct factors of $\mathcal{H}$ are paired with distinct involutions. Complete each one-factor H_i to a one-factor R_i of K_{2n+2t} adding all proper short edges $[g, \sigma_i g]$, $g \in T_i$, denoting by T_i a set of distinct representatives for the right cosets of $\{1_G, \sigma_i\}$ in G . In this way, we obtain $2t-1$ distinct one-factors for $\mathcal{F}$ and each of them is fixed by G . Observe also that each proper edge of R_i is short and $\partial e = \{\sigma_i\}$. Let $\{a_1, \dots, a_{2t}\} = G - \Phi(S_G)$. Construct a one-factor F containing all edges $[\infty_i, a_i]$, $i = 1, \dots, 2t$, together with the edges of S_G . Set $\mathcal{F} = \{R_1, \dots, R_{2t-1}\} \cup \{Fg \mid g \in G\}$, $\mathcal{F}$ is a set of one-factors and contains at most $(n+t)(2n+2t-1)$ edges. To prove that $\mathcal{F}$ is a one-factorization of K_{2n+2t} , it is sufficient to prove that each edge e of the complete graph with vertex-set $G \cup X$ appears in at least one one-factor of $\mathcal{F}$. If e has both its vertices in X , then it is an edge of K_X and appears in exactly one one-factor $R_i \in \mathcal{F}$. If $e = [\infty_i, h]$, $h \in G$, then $e \in F a_i^{-1} h$. If e is a short proper edge, then $\partial e = \{\sigma_i\}$ and e lies in R_i , the one-factor associated to the involution σ_i . If $e = [x, y]$ is a long proper edge, let $[g_i, h_i] \in S_G$ such that $\partial e = \partial[g_i, h_i]$. If $yx^{-1} = h_i g_i^{-1}$, then $[x, y] \in Fg$ with $g = g_i^{-1} x$; otherwise if $yx^{-1} = g_i h_i^{-1}$ then

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$[x, y] \in Fg$ with $g = g_i^{-1}y$. Obviously $\mathcal{F}$ is $2t$ -pyramidal under G . For the converse follow the proof of Proposition 2: a weak-starter S_G is the set $\{e_1, \dots, e_{n-t}\}$ of proper edges of a non-fixed one-factor F . In fact each e_i is long and $\Phi(e_i) \cap \Phi(e_j) = \emptyset$, $i \neq j$. Moreover if $e_i = [g_i, h_i]$ and $e_j = [g_j, h_j]$, $i \neq j$, then it is $\partial e_i \cap \partial e_j = \emptyset$ otherwise either $h_j^{-1}h_i$ or $g_j^{-1}h_i$ maps e_j onto e_i and fixes the one-factor F which is a contradiction. $\square$

In general the one-factorization constructed from a weak-starter is not unique. In the above construction different choices of $\mathcal{H}$ as well as different choices of the edges with a vertex in X and the other in G can lead to non isomorphic one-factorizations.

Proposition 4. *A group G of order $2n$ with $2t - 1$ involutions and such that $2t > n$, realizes a $2t$ -pyramidal one-factorization of a complete graph.*

Proof. To prove the statement it is sufficient to prove the existence of a weak-starter in G . If $t = n$ (that is G is an elementary abelian 2-group), then the empty set is a weak-starter in G . Suppose $t < n$ and suppose a weak-starter does not exist in G . Let S be a maximum cardinality set of long edges such that ∂S and $\Phi(S)$ do not contain repeated elements, and denote by m its cardinality. Since S is not a weak-starter, we have $m < n - t$ and there exists a non-identity element $g \in G - \partial S$ such that g is not an involution of G . We also have $|\Phi(S)| = 2m < 2n - 2t$, this implies $|\Phi(S)| < n$ by the hypothesis on $2t$. Consider the edge set $E = \{[x, y] : \partial([x, y]) = \{g, g^{-1}\}\}$. Through each vertex $a \in G$ there are exactly two edges of E , namely $[a, ga]$ and $[a, g^{-1}a]$, so that $|E| = 2n$. Furthermore at most $2|\Phi(S)| < 2n$ edges of E have at least one vertex in $\Phi(S)$ and then there exists at least one edge $e \in E$ with both the vertices in $G - \Phi(S)$. The set $\tilde{S} = S \cup \{e\}$ is a set of long edges such that $\partial \tilde{S}$ and $\Phi(\tilde{S})$ do not contain repeated elements: a contradiction. $\square$

In the next sections we will examine some classes of groups. We will make use of the notions of *sequenceability* and *R-sequenceability* in a finite group G . These notions, together with some relevant results, are briefly summarized below.

Definition 2. *A non-trivial finite group G of order n , with identity 1_G , is said to be sequenceable if its elements can be listed in a sequence $g_1, g_2, \dots, g_n$ in such a way that the quotients $g_2g_1^{-1}, g_3g_2^{-1}, \dots, g_ng_{n-1}^{-1}$ are distinct.*

Definition 3. *A non-trivial finite group G of even order $2n$ with identity 1_G and with a unique involution j is said to be symmetrically sequenceable if its elements can be listed in a sequence (symmetric sequence) $g_1, g_2, \dots, g_{2n}$ in such a way that the quotients $g_2g_1^{-1}, g_3g_2^{-1}, \dots, g_{2n}g_{2n-1}^{-1}$ are distinct and $g_{n+i} = g_{n-i+1}j$, $i = 1, \dots, n$.*

Definition 4. *A group G of order n with identity 1_G is said to be R-sequenceable if the elements of $G - \{1_G\}$ can be listed in an R-sequence $g_1, g_2, \dots, g_{n-1}$ such that the quotients $g_2g_1^{-1}, g_3g_2^{-1}, \dots, g_{n-1}g_{n-2}^{-1}, g_1g_{n-1}^{-1}$ are distinct.*

For a recent survey on this topic we refer to [15]. Usually the quotients on a sequence (either symmetric or not) or on an R -sequence $g_1, \dots, g_t$ are defined by $g_i^{-1}g_{i+1}, i = 1, \dots, t$, nevertheless our definitions 2, 3 and 4 are equivalent to those given in [15] and are more efficient in our context. Observe also that if $g_1, \dots, g_n$ is a sequence then for each $h \in G$, $g_1h, \dots, g_nh$ is a sequence itself. Moreover, it is a symmetric sequence if and only if the previous one is symmetric.

We simply recall that each solvable group with a unique involution, except for the quaternion group Q_8 , is symmetrically sequenceable, [1].

Each abelian 2-group is R -sequenceable if and only if it is not cyclic, [12].

The only groups known to be non-sequenceable are the abelian groups with more than one involution, the quaternion group Q_8 and the dihedral groups D_6 and D_8 . Each dihedral group of order at least 10 is sequenceable.

It was proved in [2] that each symmetrically sequenceable group, together with Q_8 , possesses a right even starter (i.e., a weak-starter in our terminology) this, together with Proposition 3, proves the following:

Theorem 1 [2]. *Each symmetrically sequenceable group, together with the quaternion group Q_8 , realizes a bipyramidal one-factorization of a complete graph.*

3. Abelian k –Pyramidal One-Factorizations

In this section we prove some preliminary Lemmas and Propositions which will lead to the main Theorem 2.

We will denote by Z_n the cyclic group of order n in additive notation.

Lemma 1. *Let G be a sequenceable group then $G \times Z_2$ admits a weak-starter.*

Proof. Let $g_1, g_2, \dots, g_n$ be a sequence for G . Consider the set of edges $S' = \{e_1, e_2, \dots, e_{n-1}\}$ where $e_i = [(g_i, x_i), (g_{i+1}, y_i)], i = 1, \dots, n-1$ and $x_1 = y_1 = 0$, i.e., $e_1 = [(g_1, 0), (g_2, 0)]$, and each edge e_{i+1} is defined from the previous edge e_i setting $e_{i+1} = \Gamma(e_i)$ where:

$$\Gamma(e_i) = \begin{cases} [(g_{i+1}, y_i + 1), (g_{i+2}, y_i + 1)] & \text{if } g_{i+2}g_{i+1}^{-1} = (g_{r+2}g_{r+1}^{-1})^{-1} \Rightarrow r \geq i \\ [(g_{i+1}, y_i + 1), (g_{i+2}, y_i)] & \text{if } g_{i+2}g_{i+1}^{-1} = (g_{r+2}g_{r+1}^{-1})^{-1} \Rightarrow r < i \end{cases}$$

Denote by I the set of all short edges in S' . The set $S = S' - I$ is a weak-starter in $G \times Z_2$. In fact it is obvious that $\Phi(e_i) \cap \Phi(e_j) = \emptyset$ when $i \neq j$. Moreover, let (g, x) be an element of order greater than 2 in $G \times Z_2$. Let (g_i, g_{i+1}) and (g_r, g_{r+1}) be the two pairs of elements of the sequence for G such that $g_{i+1}g_i^{-1} = g$ and $g_{r+1}g_r^{-1} = g^{-1}$. Denote by m and $\bar{m}$ the minimum and the maximum between i and r , respectively. If $x = 0$ then $(g, x) = \partial([(g_m, y), (g_{m+1}, y)])$ and otherwise, if $x = 1$, then $(g, x) = \partial([(g_{\bar{m}}, y), (g_{\bar{m}+1}, y + 1)])$, where y is a suitable element in Z_2 . $\square$

Lemma 2. *Let G be an R -sequenceable group of even order then $G \times Z_2$ admits a weak-starter.*

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Proof. Let $g_1, \dots, g_{n-1}$ be an R -sequence of G . Without loss in generality we can suppose that $g_1 g_{n-1}^{-1}$ is an involution of G . Paralleling the construction of Lemma 1, consider the set $S' = \{e_1, \dots, e_{n-2}\}$ where $e_i = [(g_i, x_i), (g_{i+1}, y_i)]$, $i = 1, \dots, n-2$, $x_1 = y_1 = 0$, i.e., $e_1 = [(g_1, 0), (g_2, 0)]$, and each edge e_{i+1} is defined by the previous edge e_i setting $e_{i+1} = \Gamma(e_i)$, where Γ is defined as in Lemma 1. Denote by I the set of all short edges in S' . The set $S = S' - I$ is a weak-starter in $G \times Z_2$. $\square$

Lemma 3. *The group Z_{8n} admits a weak-starter S such that $0, 4n \notin \Phi(S)$.*

Proof. Consider the following sets:

$$\begin{aligned} S_1 &= \{[1, 2n+1], [5n, n+1]\} \\ S_2 &= \{[2n+1-i, 2n+1+i]/i = 1, \dots, n-1\} \\ S_3 &= \{[n+1-i, 3n+i]/i = 1, \dots, n-1\} \\ S_4 &= \{[-i, 4n+i]/i = 1, \dots, n-1\} \\ S_5 &= \{[5n+i, 7n+1-i]/i = 1, \dots, n\} \end{aligned}$$

we have:

$$\begin{aligned} \partial S_1 &= \{\pm 2n, \pm(4n-1)\}, \Phi(S_1) = \{1, n+1, 2n+1, 5n\}. \\ \partial S_2 &= \{\pm 2i, i = 1, \dots, n-1\}, \Phi(S_2) = \{n+2, n+3, \dots, 2n, 2n+2, 2n+3, \dots, 3n\}. \\ \partial S_3 &= \{\pm(2n+2i-1), i = 1, \dots, n-1\}, \\ \Phi(S_3) &= \{2, 3, \dots, n, 3n+1, 3n+2, \dots, 4n-1\}. \\ \partial S_4 &= \{\pm(4n+2i), i = 1, \dots, n-1\}, \\ \Phi(S_4) &= \{4n+1, 4n+2, \dots, 5n-1, 7n+1, 7n+2, \dots, 8n-1\}. \\ \partial S_5 &= \{\pm(2n-2i+1), i = 1, \dots, n\}, \Phi(S_5) = \{5n+1, 5n+2, \dots, 7n\}. \end{aligned}$$

The union $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$ is a weak-starter in Z_{8n} such that $0, 4n \notin \Phi(S)$. $\square$

Lemma 4. *The group $Z_{8m} \times Z_{8n}$ admits a weak-starter.*

Proof. Let $S_1 = \{[a_1, b_1], \dots, [a_{4m-1}, b_{4m-1}]\}$ be a weak-starter in Z_{8m} and let $S_2 = \{[x_1, y_1], \dots, [x_{4n-1}, y_{4n-1}]\}$ be a weak-starter in Z_{8n} . By Lemma 3, we can assume $0, 4m \notin \Phi(S_1)$ as well as $0, 4n \notin \Phi(S_2)$, so that $\Phi(S_1) = Z_{8m} - \{0, 4m\}$ and $\Phi(S_2) = Z_{8n} - \{0, 4n\}$. Let $c_1, \dots, c_{4m}, c_{4m+1}, \dots, c_{8m}$ be a symmetric sequence in Z_{8m} such that $c_{4m} = 0$ and $c_{4m+1} = 4m$. Similarly, let $d_1, \dots, d_{4n}, d_{4n+1}, \dots, d_{8n}$ be a symmetric sequence in Z_{8n} such that $d_{4n} = 4n$ and $d_{4n+1} = 0$.

Recall that both the sets $\pm\{c_{i+1} - c_i \mid i = 1, \dots, 4m-1\}$ and $\pm\{c_{i+1} - c_i \mid i = 4m+1, \dots, 8m-1\}$ cover the elements of $Z_{8m} - \{0, 4m\}$ exactly once.

In the same manner, both the sets $\pm\{d_{i+1} - d_i \mid i = 1, \dots, 4n-1\}$ and $\pm\{d_{i+1} - d_i \mid i = 4n+1, \dots, 8n-1\}$ cover the elements of $Z_{8n} - \{0, 4n\}$ exactly once.

Consider the following sets:

$$Q = \{(a_i, x_j), (b_i, y_j), [(a_i, y_j), (b_i, x_j)] \mid [a_i, b_i] \in S_1, [x_j, y_j] \in S_2\}$$

$$R = \{(c_i, 0), (c_{i+1}, 4n)\} \mid i = 1, \dots, 4m - 1\}$$

$$U_1 = \{(c_{4m+2i+1}, 0), (c_{4m+2i+2}, 0)\} \mid i = 0, \dots, 2m - 1\}$$

$$U_2 = \{(c_{4m+2i+2}, 4n), (c_{4m+2i+3}, 4n)\} \mid i = 0, \dots, 2m - 2\}$$

$$Z = \{(0, d_i), (4n, d_{i+1})\} \mid i = 1, \dots, 4n - 1\}$$

$$W_1 = \{(0, d_{4n+2i+1}), (0, d_{4n+2i+2})\} \mid i = 0, \dots, 2n - 1\}$$

$$W_2 = \{(4m, d_{4n+2i+2}), (4m, d_{4n+2i+3})\} \mid i = 0, \dots, 2n - 2\}$$

Set $U = U_1 \cup U_2$ and $W = W_1 \cup W_2$, we have:

$\partial Q = (Z_{8m} - \{0, 4m\}) \times (Z_{8n} - \{0, 4n\})$, $\Phi(Q) = \Phi(S_1) \times \Phi(S_2)$ and $\Phi(Q)$ is disjoint from both $\{0, 4m\} \times Z_{8n}$ and $Z_{8m} \times \{0, 4n\}$.

$$\partial R = (Z_{8m} - \{0, 4m\}) \times \{4n\},$$

$$\Phi(R) = \{(c_j, 0), j = 1, \dots, 4m - 1\} \cup \{(c_j, 4n), j = 2, \dots, 4m\}.$$

$$\partial U = (Z_{8m} - \{0, 4m\}) \times \{0\},$$

$$\Phi(U) = \{(c_j, 0), j = 4m + 1, \dots, 8m\} \cup \{(c_j, 4n), j = 4m + 2, \dots, 8m - 1\}.$$

$$\partial Z = \{4m\} \times (Z_{8n} - \{0, 4n\}),$$

$$\Phi(Z) = \{(0, d_j), j = 1, \dots, 4n - 1\} \cup \{(4m, d_j), j = 2, \dots, 4n\}.$$

$$\partial W = \{0\} \times (Z_{8n} - \{0, 4n\}),$$

$$\Phi(W) = \{(0, d_j), j = 4n + 1, \dots, 8n\} \cup \{(4m, d_j), j = 4n + 2, \dots, 8n - 1\}.$$

The set $Q \cup R \cup U \cup Z \cup W$ is a weak-starter in $Z_{8m} \times Z_{8n}$. $\square$

Lemma 5. *Let G be an R -sequenceable group of even order then $G \times Z_{8m}$ admits a weak-starter.*

Proof. Let $g_1, \dots, g_{n-1}$ be an R -sequence of G . Let $S' = \{[x_j, y_j] \mid j = 1, \dots, 4n - 1\}$ be a starter of Z_{8m} such that $0, 4m \notin \Phi(S')$, take for example the weak-starter described in lemma 3. Set $g_n = g_1$ and consider the following sets:

$$R = \{[(g_i, x_j), (g_{i+1}, y_j)] \mid i = 1, \dots, n - 1, j = 1, \dots, 4m - 1\}$$

$$T = \{[(1_G, x_j), (1_G, y_j)] \mid j = 1, \dots, 4m - 1\}$$

The set $G \times Z_{8m} - (\Phi(R) \cup \Phi(T))$ is exactly $G \times \{0, 4m\}$, and $\partial(R \cup T) = G \times (Z_{8m} - \{0, 4m\})$. The set $G \times \{0, 4m\}$ is a subgroup of $G \times Z_{8m}$ and it is isomorphic to $G \times Z_2$, then apply Lemma 2 and construct a weak-starter U in $G \times \{0, 4m\}$. The set $S = R \cup T \cup U$ is a weak-starter in $G \times Z_{8m}$. $\square$

Lemma 6. *Let G be a sequenceable group then $G \times Z_4$ admits a weak-starter.*

Proof. Let $g_1, \dots, g_n$ be a sequence of G . Consider the set

$$S' = \{e_1, \dots, e_{n-1}, f_1, \dots, f_{n-1}\}$$

where

$$e_i = [(g_i, x_i), (g_{i+1}, y_i)], \quad f_i = [(g_i, s_i), (g_{i+1}, t_i)], \quad i = 1, \dots, n-1$$

with $x_1 = t_1 = 0$ and $y_1 = s_1 = 1$, i.e., $e_1 = [(g_1, 0), (g_2, 1)]$, $f_1 = [(g_1, 1), (g_2, 0)]$, and each pair (e_{i+1}, f_{i+1}) is defined from the previous pair (e_i, f_i) setting $(e_{i+1}, f_{i+1}) = \mu((e_i, f_i))$ where:

$$\mu((e_i, f_i)) = \begin{cases} ([(g_{i+1}, y_i + 2), (g_{i+2}, t_i + 2)], [(g_{i+1}, t_i + 2), (g_{i+2}, y_i + 2)]) \\ \quad \text{if } g_{i+2}g_{i+1}^{-1} = (g_{r+2}g_{r+1}^{-1})^{-1} \text{ implies } r \geq i \\ ([(g_{i+1}, y_i + 2), (g_{i+2}, y_i + 2)], [(g_{i+1}, t_i + 2), (g_{i+2}, t_i)]) \\ \quad \text{if } g_{i+2}g_{i+1}^{-1} = (g_{r+2}g_{r+1}^{-1})^{-1} \text{ implies } r < i \end{cases}$$

Let I be the set of edges f_i such that $g_{i+1}g_i^{-1}$ is an involution (if G has odd order then I is the empty set), in this case by definition of μ we have $\partial f_i = \partial e_i$, therefore if $\bar{S} = S' - I$, then it is $\partial \bar{S} = \partial S'$. Let $e_0 = [(g_1, 2), (g_1, 3)]$, we prove that the set $S = \bar{S} \cup \{e_0\}$ is a weak-starter in $G \times Z_4$. In fact $\Phi(S')$ does not contain repeated vertices and the same holds for $\Phi(S)$. Moreover, let $u \geq 0$ be the number of involutions in G , therefore I contains u elements, $G \times Z_4$ has $2u + 1$ involutions and the number of elements $(g, x) \in G \times Z_4$ such that $(g^2, 2x) \neq (1_G, 0)$ is $4n - 2u - 2$. Observe that this number is exactly $|\partial S|$. In fact $|\bar{S}| = 2n - u - 2$, each edge of $\bar{S}$ is long as well as e_0 and this implies $|\partial S| = 4n - 2u - 2$. Now, to prove that S is a weak-starter, it is sufficient to prove that each element (g, x) with $(g^2, 2x) \neq (1_G, 0)$ is in ∂S . If g is an involution of G , there is exactly one index $i \in \{1, \dots, n-1\}$ such that $g = g_{i+1}g_i^{-1}$, in this case $(g, x) \in \partial e_i$. Otherwise, if $g^2 \neq 1_G$ there is a pair of indices $1 \leq i < j \leq n-1$ such that $\{g, g^{-1}\} = \{g_{i+1}g_i^{-1}, g_{j+1}g_j^{-1}\}$. We have either $(g, x) \in \partial e_i \cup \partial f_i$ or $(g, x) \in \partial e_j \cup \partial f_j$ according to whether $x \in \{1, -1\}$ or $x \in \{0, 2\}$. Finally, if $g = 1_G$ we have $(g, x) \in \partial e_0 = \{(1_G, \pm 1)\}$. $\square$

Lemma 7. *Let G be an R -sequenceable group of even order then $G \times Z_4$ admits a weak-starter.*

Proof. Let $g_1, \dots, g_{n-1}$ be an R -sequence of G . Without loss of generality we can suppose $g_1g_{n-1}^{-1}$ is an involution of G . We repeat the construction of Lemma 6 and we consider the set

$$S' = \{e_1, \dots, e_{n-2}, f_1, \dots, f_{n-2}\}$$

where

$$e_i = [(g_i, x_i), (g_{i+1}, y_i)], \quad f_i = [(g_i, s_i), (g_{i+1}, t_i)], \quad i = 1, \dots, n-2$$

with $x_1 = t_1 = 0$ and $y_1 = s_1 = 1$, i.e., $e_1 = [(g_1, 0), (g_2, 1)]$, $f_1 = [(g_1, 1), (g_2, 0)]$, and each pair (e_{i+1}, f_{i+1}) is defined from the previous pair (e_i, f_i) setting $(e_{i+1}, f_{i+1}) = \mu((e_i, f_i))$ where μ is defined as in Lemma 6.

Let I be the set of edges f_i such that $g_{i+1}g_i^{-1}$ is an involution (if $g_1g_{n-1}^{-1}$ is the unique involution in G , then I is the empty set), in this case by definition of μ we have $\partial f_i = \partial e_i$, therefore if $\bar{S} = S' - I$ then $\partial \bar{S} = \partial S'$. Moreover, the edges of $\bar{S}$ are long and $\Phi(\bar{S})$ does not contain repeated elements. Proceeding as in the previous Lemma 6, we can observe that the elements in $\partial \bar{S}$ are distinct and cover all the elements of $G \times Z_4$ which are different from the identity and the involutions, except for $(g_1g_{n-1}^{-1}, \pm 1)$ and $(1_G, \pm 1)$. Let $(g_{n-1}, x) \notin \Phi(\bar{S})$, i.e., $(g_{n-1}, x) \notin \Phi(\{e_{n-2}, f_{n-2}\})$, if $x = \pm 1$ let $e_{n-1} = [(g_{n-1}, x), (g_1, 2)]$, otherwise let $e_{n-1} = [(g_{n-1}, x), (g_1, 3)]$. Observe that $(g_1, 2)$ and $(g_1, 3)$ are not in $\Phi(\bar{S})$ and $\partial e_{n-1} = \{(g_1g_{n-1}^{-1}, \pm 1)\}$. Finally, let $e_0 = [(1_G, 0), (1_G, 1)]$, the set $S = \bar{S} \cup \{e_0\} \cup \{e_{n-1}\}$ is a weak-starter in $G \times Z_4$. $\square$

Proposition 5. *Let G be an R -sequenceable group containing $2t - 1$ involutions and admitting a weak-starter S_G . Let K be a group of odd order. The group $G \times K$ realizes a $2t$ -pyramidal one-factorization of a complete graph.*

Proof. To prove the statement it is sufficient to construct a weak-starter in $G \times K$. Denote by $g_1, g_2, \dots, g_{n-1}$ an R -sequence of G and by $S_G = \{[x_i, y_i] / i = 1, \dots, n - t\}$ a weak-starter in G ; we express the group K as union of the disjoint sets K_1, K_2 and $\{1_K\}$, defined in such a way that $k \in K_1$ iff $k \neq 1_K$ and $k^{-1} \in K_2$. Consider the sets of edges:

$$\begin{aligned} Q &= \{[(1_G, k), (1_G, k^{-1})] | k \in K_1\} \\ S &= \{[(x_i, 1_K), (y_i, 1_K)] | i = 1, \dots, n - t\} \\ R &= \{[(g_{i+1}, k), (g_i, k^{-1})] | k \in K_1, i = 1, \dots, n - 1\} \quad (\text{where } g_n = g_1) \end{aligned}$$

It is easy to verify that $Q \cup S \cup R$ is a weak-starter in $G \times K$. $\square$

The above Lemmas 2, 5 and 7 lead to the following:

Proposition 6. *Let G be an R -sequenceable group of even order. For each $t \geq 1$ the group $G \times Z_{2^t}$ admits a weak-starter.*

We are now able to prove the following:

Theorem 2. *Each abelian group G of even order $2n$ and with $2t - 1$ involutions, realizes a $2t$ -pyramidal one-factorization of the complete graph K_{2n+2t} .*

Proof. Let P be the 2-Sylow subgroup of G , then $G = P \times K$, where K is an odd order abelian group. From the fundamental theorem on the structure of finite abelian groups, $P = Z_{2^{n_1}} \times Z_{2^{n_2}} \times \dots \times Z_{2^{n_s}}, s \geq 1$. If $s = 1$ then G is symmetrically sequenceable, [1], and the assertion follows from Theorem 1. If $s \geq 2$, the group P is a non cyclic abelian 2-group and then it is R -sequenceable, [12]. Therefore, the proof follows from Proposition 5 as soon as we construct a weak-starter in P . If $P = Z_{2^{n_1}} \times Z_{2^{n_2}}$ (i.e., $s = 2$), there are two possibilities: either $n_1, n_2 \geq 3$ or at least

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one of them, say n_2 , is less than 3. In the former case we construct a weak-starter in P using Lemma 4, in the latter case, since $Z_{2^{n_1}}$ is sequenceable, a weak-starter is constructed applying Lemma 1 or Lemma 6. Now suppose $P = Z_{2^{n_1}} \times Z_{2^{n_2}} \times \dots \times Z_{2^{n_s}}$, with $s > 2$. Since $Z_{2^{n_1}} \times \dots \times Z_{2^{n_{s-1}}}$ is an even order R –sequenceable group, we apply proposition 6 to construct a weak-starter in P . $\square$

4. Dihedral and Hamiltonian k –pyramidal One–Factorizations

Non-abelian k –pyramidal one–factorizations are obtained in the previous chapters in some special cases (see for examples Theorem 1, Lemmas 1, 2, 6, 7 and Proposition 6, when G is not abelian). In this section we look at two more classes of groups, namely dihedral and Hamiltonian groups.

As a consequence of Proposition 4 we immediately obtain the following statement:

Theorem 3. *Each dihedral group D_{2n} realizes either a $(n + 1)$ –pyramidal or a $(n + 2)$ –pyramidal one–factorization of a complete graph, according to whether n is odd or even.*

We recall that each dihedral group realizes a sharply-vertex transitive one–factorization, see [3], hence the previous Theorem 3 completes this result.

Proposition 7. *Let G be a group of even order admitting a weak-starter. The group $G \times (Z_2)^m$, $m \geq 2$, admits a weak-starter itself.*

Proof. Let $S = \{[u_1, v_1] \dots, [u_s, v_s]\}$ be a weak-starter in G and let $a_1, \dots, a_{2^m-1}$ be an R –sequence of the elementary abelian 2–group $(Z_2)^m$. Set $a_{2^m} = a_1$ and for each $j \in \{1, \dots, 2^m - 1\}$ let $A_j = \{[(u_i, a_j), (v_i, a_{j+1})] \mid i = 1, \dots, s\}$ and let $A_0 = \{[(u_i, 0), (v_i, 0)], i = 1, \dots, s\}$. It is easy to check that $A_0 \cup A_1 \cup \dots \cup A_{2^m-1}$ is a weak-starter in $G \times (Z_2)^m$. $\square$

Recall that a *Hamiltonian group* is defined to be a group in which every subgroup is normal. Apart from the abelian groups, each Hamiltonian group is the direct product of the quaternion group Q_8 , together with an elementary abelian 2–group and an odd order group (see [17, p.253]).

Theorem 4. *Each Hamiltonian group G of even order $2n$ and with $2t - 1$ involutions, realizes a $2t$ –pyramidal one–factorization of the complete graph K_{2n+2t} .*

Proof. If G is abelian, the assertion follows from Theorem 2. Suppose G is not abelian, then $G = Q_8 \times K \times (Z_2)^m$, with K of odd order. From Theorem 1 we know that $Q_8 \times K$ admits a weak-starter, in fact, if K is not trivial, then the group $Q_8 \times K$ is solvable and with a unique involution and by [1] it is symmetrically sequenceable. If $m \geq 2$, the existence of a weak-starter in G follows from Proposition 7. Suppose $m = 1$. If K is not trivial, then the assertion follows from Lemma

1. If K is trivial, we conclude the proof by exhibiting a weak-starter in $Q_8 \times Z_2$. The group Q_8 can be presented as follows: $Q_8 = \langle a, b : a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle$. The set $\{(a, 0), (1, 0)\}, \{(a^{-1}, 0), (ab, 1)\}, \{(b^{-1}, 0), (b^2, 0)\}, \{(ab, 0), (b^2, 1)\}, \{(a, 1), (b, 1)\}, \{(b, 0), (ab, 1)\}$ is a weak-starter in $Q_8 \times Z_2$. $\square$

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