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Monotone Trajectories of Differential Inclusions in Banach Spaces[1]

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Two existence results for monotone trajectories of differential inclusions $x'(t) \in F\big(t, x(t)\big)$ in a separable Banach space are obtained; they extend in two directions previous ones due to *Aubin-Cellina, Deimling* and *Haddad*.

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1. Introduction

Let X be a given nonempty compact subset of a separable Banach space E. A preorder $\succeq$ on X, that is a reflexive and transitive relation, is defined by a set-valued map P which to any $x \in X$ associates

$$P(x) = \{y \in X : y \succeq x\};$$

we recall that $\succeq$ is said to be a *continuous preorder* (see e.g. [9]) whenever P is a lower semicontinuous correspondence with a closed graph.

Let $F : [0, T] \times X \to E$ be a nonempty convex weakly compact set-valued map. Given x_0 in X we look for Lipschitz solutions of the differential inclusion:

$$w'(t) \in F(t, w(t)) \qquad w(0) = x_0 \tag{1.1}$$

which are viable, i.e. $w(t) \in X$ for all $t \in [0, T]$ and *monotone* with respect to the preorder P, that is

$$\text{for any } s, t \in [0, T],\ s < t \text{ implies } w(t) \in P(w(s)).$$

The same problem has been investigated by Haddad [9] (see also [1]) when E is a finite dimensional space. In [9] F is assumed to be globally upper semicontinuous and satisfies a tangential condition involving Bouligand's contingent cone $T_{P(x)}(x)$, where

$$T_{P(x)}(x) = \Big\{ v \in E : \liminf_{\gamma \to 0^+} \frac{d_{P(x)}(x + \gamma v)}{\gamma} = 0 \Big\};$$

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It is well known (see e.g. [1]), that $T_{P(x)}(x)$ is a nonempty closed cone which is larger than the tangent cone introduced by Clarke [7] which is given by

$$C_{P(x)}(x) = \Big\{ v \in E : \limsup_{\gamma \to 0^+,\, y \to x} \frac{d_{P(x)}(y + \gamma v) - d_{P(x)}(y)}{\gamma} = 0 \Big\}.$$

Tallos [10] studied the existence of viable trajectories for (1.1) in a finite dimensional space, when F is integrably bounded, measurable in t upper semicontinuous in x and satisfies a tangential condition involving Clarke's cone.

Existence theorems of viable solutions in Banach spaces where proven by Benabdellah-Castaing-Gamal Ibrahim [2] and by Castaing-Moussaoui-Syam [5] when F is measurable in t and upper semicontinuous in x. We also refer to [1] and [8] for a wide bibliography on the subject.

The present paper extends in several directions the results obtained by Aubin-Cellina [1], Deimling [8] and Haddad [9]. Even in the particular case when F is scalarly globally upper semicontinuous, we need a careful proof of the convergence of the approximated solutions via new compactness results in L^1_E (see e.g. [2], Thm.5.4).

The second difficulty is due to the various weak measurability assumptions on F. The above mentioned difficulties are solved by approximations techniques involving a careful proof of the convergence of approximated solutions via a result of convergence due to Castaing-Moussaoui-Syam [5], Lemma 6.5 (see also Lemma 2.1 below), a multivalued version of Scorza-Dragoni Theorem [4], Thm.2.2 (see also Theorem 2.3) and a multivalued version of Dugundji's "single-valued" extension Theorem [4], Thm.2.3 (see also Theorem 2.4). So Theorem 3.3 and Theorem 3.4 are new achievements on this subject.

In the sequel we shall denote by $cwk(E)$ the set of all nonempty convex weakly compact subsets of E and by B the closed unit ball of E; further we shall put $|A| = \sup\{||x|| : x \in A\}$ for any subset $A \subset E$ and refer to λ as the Lebesgue measure on the real line R.

Finally we remind that a multifunction F from a measurable space (S, Σ) to bounded subsets of a Banach space E is said to be *scalarly* Σ*-measurable*, see e.g. [5], if for any e' in the dual E' of E the scalar function

$$\delta^*(e', F(x)) = \sup_{v \in F(x)} \langle e', v \rangle$$

is $\Sigma-$measurable; analogously, when S is also a topological space, F is said to be *scalarly upper semicontinuous*, see e.g. [5], whenever the scalar function $\delta^*(e', F(x))$ is upper semicontinuous for each $e' \in E'$.

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2. Preliminary Results

In this section we summarize some fundamental results useful later on.

We begin with a lemma due to Castaing-Moussaoui-Syam [5]; we shall apply it to show the convergence of a sequence of approximated solutions.

Lemma 2.1. ([5], Lemma 6.5.) *Let (S, d) be a Souslin metrizable space. Let $F : [0, T] \times S \to cwk(E)$ satisfying:*

(i) *F is scalarly $\tau_\lambda([0,T]) \otimes \mathcal{B}(S)$–measurable;*
(ii) *for any t in $[0,T]$, $F(t,\cdot)$ is scalarly upper semicontinuous;*
(iii) $\sup_{(t,x)\in[0,T]\times S} |F(t,x)| < \infty$.

Let $(r_n)_n$ be a sequence of strictly positive numbers with $\lim_{n\to\infty} r_n = 0$. Let $(X_n)_{n\geq 1}$ be a sequence of λ-measurable mappings from $[0,T]$ to S which converges pointwisely to a λ-measurable mapping X, $(Y_n)_{n\geq 1}$ be a sequence in $L^1_E([0,T],\lambda)$ which $\sigma(L^1,L^\infty)$ converges to Y in $L^1_E([0,T],\lambda)$ and such that
$Y_n(t) \in \frac{1}{r_n}\int_{I_{t,r_n}} F(s,X_n(t))\,ds$ a.e. with $I_{t,r_n} = [0,T]\cap[t,t+r_n]$. Then

$$Y(t) \in F(t,X(t)) \quad a.e.$$

Next lemma is due to Haddad [9]; more precisely he gave his result in a finite dimensional space, but the same proof holds in an arbitrary Banach space; we shall apply it to construct a sequence of approximated solutions.

Lemma 2.2. *Let X be a locally compact subset of a Banach space E, $P: X \to X$ a given continuous preorder and $F: X \to cwk(E)$ a scalarly upper semicontinuous multifunction satisfying the following tangential condition:*

$$F(x) \cap T_{P(x)}(x) \neq \emptyset \qquad \text{for all}\, x \in X.$$

Given $x_0 \in X$, let $R > 0$ be such that $X_0 = X \cap (x_0 + RB)$ is compact and let $\alpha > 0$ be such that $\alpha \geq \mathrm{Sup}_{x\in X_0}|F(x)|$.
Then for every $\beta > 0$ it is possible to find two finite sequences
$0 = t_0 < t_1 < \ldots < t_{m-1} < \frac{R}{\alpha+\beta} \leq t_m$ and $\{x_0, x_1, \ldots, x_m\}$ such that for each $k = 0,1,\ldots,m-1$ we have $t_{k+1} - t_k < \beta$, $x_k \in X_0$, $x_{k+1} \in P(x_k)$ and the existence of $y_k \in X_0$ depending on x_k and $v_k \in F(y_k)$ satisfying

$$||x_k - y_k|| < \beta, \qquad ||\frac{x_{k+1} - x_k}{t_{k+1} - t_k} - v_k|| < \beta. \tag{2.1}$$

Proof. We recall that, for convex weakly compact nonempty set valued maps, scalar upper semicontinuity is equivalent to weak upper semicontinuity, that is upper semicontinuity when E is endowed with weak topology (see [6], Thm.II.20); therefore $F(X_0)$ is weakly compact. The result then follows by the same reasoning given in [9], Lemma I-1. □

The following result, due to Castaing-Monteiro Marques [4], is a multivalued version of Scorza-Dragoni Theorem.

Theorem 2.3. *Let X be a Polish space and Y be a convex compact metrizable subset of a Hausdorff locally convex space.*
Let $F: [0,T]\times X \to ck(Y)$ (nonempty convex compact subsets of Y) be a multifunction satisfying:
(i) *for all $t \in [0,T]$, graph $F_t = \{(x,y) \in X \times Y : y \in F(t,x)\}$ is closed in $X \times Y$;*

(ii) *for any* $x \in X$, $F(\cdot, x)$ *admits a* $(\tau_\lambda([0,T]), \mathcal{B}(Y))$*-measurable selection.*

Then, there exists a measurable multifunction $F_0 : [0,T] \times X \to ck(Y) \cup \{\emptyset\}$ *which has the following properties:*

(1) *there is a* λ*-null set* M*, independent of* (t,x)*, such that*

$$F_0(t,x) \subset F(t,x), \text{ for all} t \notin M \text{ and} x \in X;$$

(2) *if* $u : [0,T] \to X$ *and* $v : [0,T] \to Y$ *are* $\tau_\lambda([0,T])$*-measurable functions with* $v(t) \in F(t, u(t))$ *a.e., then* $v(t) \in F_0(t, u(t))$ *a.e.;*

(3) *for every* $\epsilon > 0$*, there is a compact subset* $I_\epsilon \subset [0,T]$ *such that* $\lambda([0,T] \setminus I_\epsilon) < \epsilon$*, the graph of the restriction* $F_{0 \setminus I_\epsilon \times X}$ *is closed and* $\emptyset \neq F_0(t,x) \subset F(t,x)$*, for all* $(t,x) \in I_\epsilon \times X$.

Reference. Castaing-Monteiro Marques ([4], Thm.2.2).

We also need the following multivalued version of Dugundji's extension Theorem.

Theorem 2.4. *Let* X *and* E *be Banach spaces and* $I \subset X$, $D \subset E$ *be nonempty closed sets. Let* E_σ *be the vector space* E *endowed with the* $\sigma(E, E')$*-topology. Let* $F : I \times D \to E_\sigma$ *be an upper semicontinuous multifunction with nonempty convex compact values in* E_σ *such that*

$$F(t,x) \subset c(t)(1 + \|x\|)B \quad \text{for all } (t,x) \in I \times D,$$

and some positive function c *defined on* I.

Let $(U_\lambda)_{\lambda \in \Lambda}$ *be a locally finite open covering of* $X \setminus I$ *such that, for all* $\lambda \in \Lambda$, $0 < diam\, U_\lambda \leq \mathrm{d}\,(U_\lambda, I)$*, where*

$$d(U_\lambda, I) = \inf\{\|t_\lambda - s\| : t_\lambda \in U_\lambda, s \in I\}.$$

Let $(\psi_\lambda)_{\lambda \in \Lambda}$ *be a continuous partition of unity of* $X \setminus I$ *associated to the covering* $(U_\lambda)_{\lambda \in \Lambda}$*. For every* $\lambda \in \Lambda$*, choose* $t_\lambda \in I$ *such that* $d_{U_\lambda}(t_\lambda) < 2d(U_\lambda, I)$ *where*

$$d_{U_\lambda}(t_\lambda) = \inf\{\|t_\lambda - s\| : s \in U_\lambda\}.$$

Then the multifunction $\tilde{F}$ *defined on* $X \times D$ *by*

$$\tilde{F}(t,x) = \begin{cases} F(t,x), & \text{if } t \in I, x \in D \\ \sum\limits_{\lambda \in \Lambda} \psi_\lambda(t) F(t_\lambda, x), & \text{if } t \in X \setminus I, x \in D \end{cases}$$

is an upper semicontinuous extension of F *from* $X \times D$ *to* E_σ *with convex compact values in* E_σ*. Moreover, we have* $\tilde{F}(X \times D) \subset coF(I \times D)$ *(convex hull of the set* $F(I \times D)$*) and, if* c *is constant,* $\tilde{F}(t,x) \subset c(1 + \|x\|)B$*. In particular, if* $F(t,x) \subset K$ *for all* (t,x)*, where* K *is a convex set, then* $\tilde{F}(t,x) \subset K$.

Reference. Proof is a trivial adaptation of ([4], Thm.2.3) and it is omitted.

3. Existence results

We first give a local existence result (Theorem 3.1) for an autonomous r.h.s. $F = F(x)$, followed by a global existence one (Theorem 3.2) for $F = F(t, x)$, both of them under tangential conditions for the preorder P involving Bouligand's cone; in such results F is assumed to be scalarly upper semicontinuous.
In the sequel, under stronger tangential conditions involving Clarke's cone, we deal with the cases when $F = F(t, x)$ is scalarly measurable in (t, x) and scalarly upper semicontinuous in x (Theorem 3.3) and when $F = F(t, x)$ is separately measurable in t and scalarly upper semicontinuous in x (Theorem 3.4).

Theorem 3.1. *Let X be a locally compact subset of a separable Banach space E, $P : X \to X$ a given continuous preorder and $F : X \to cwk(E)$ a scalarly upper semicontinuous multifunction. Assume the following tangential condition:*

$$F(x) \cap T_{P(x)}(x) \neq \emptyset \qquad \text{for all}\, x \in X. \tag{3.1}$$

Then, for every $x_0 \in X$, there exist $T_0 > 0$ and a Lipschitz function $w : [0, T_0] \to X$ satisfying

$$\begin{aligned} &w(0) = x_0, \qquad w'(t) \in F(w(t)) \;\; \text{for almost all}\, t \in [0, T_0] \\ &w \text{ is monotone with respect to the preorder } P. \end{aligned} \tag{3.2}$$

Proof. Given $x_0 \in X$, let $R > 0$ be such that $X_0 = X \cap (x_0 + RB)$ is compact and let $\alpha > 0$ be such that $\alpha \geq \operatorname{Sup}_{x \in X_0} |F(x)|$.

Fix a strictly positive integer n; applying Lemma 2.2 with $\beta = \frac{1}{n}$, we can associate three finite sequences $\{t_k^{(n)}\}_{k=0,1,\ldots,m}$, $\{x_k^{(n)}\}_{k=0,1,\ldots,m}$ and $\{y_k^{(n)}\}_{k=0,\ldots m-1}$ with the properties given by the lemma; notice that it is not restrictive to put $t_m^{(n)} = \frac{R}{\alpha + \frac{1}{n}}$. Let $\frac{R}{\alpha} = T_0$ and define the function $w_n : [0, T_0] \to E$ as follows

$$w_n(t) = \begin{cases} x_k^{(n)} + (t - t_k^{(n)}) \frac{x_{k+1}^{(n)} - x_k^{(n)}}{t_{k+1}^{(n)} - t_k^{(n)}} & \text{for } t \in [t_k^{(n)}, t_{k+1}^{(n)}) \text{ and } k = 0, 1, \ldots, m-1 \\ x_m^{(n)} & \text{for } t \in [t_m^{(n)}, T_0]. \end{cases}$$

More precisely, on every interval $[t_k^{(n)}, t_{k+1}^{(n)}]$, with $k = 0, 1, \ldots, m-1$, w_n is the linear function interpolating $x_k^{(n)}$ and $x_{k+1}^{(n)}$, while w_n is extended with continuity on $[t_m^{(n)}, T_0]$.

We shall show that $(w_n)_n$ admits a subsequence which pointwisely converges in norm topology on $[0, T_0]$, to a Lipschitz function w satisfying both (3.2) and the requested monotonicity.

First of all it is convenient to introduce, for every $n \in N$, the following piecewise constant function $y_n : [0, T_0] \to X_0$ given by

$$y_n(t) = \begin{cases} y_k^{(n)} & \text{for } t \in [t_k^{(n)}, t_{k+1}^{(n)}) \text{ and } k = 0, 1, \ldots, m-1 \\ y_{m-1}^{(n)} & \text{for } t \in [t_m^{(n)}, T_0]. \end{cases}$$

Since $w_n'(t)$ is piecewise constant with

$$w_n'(t) = \begin{cases} \frac{x_{k+1}^{(n)} - x_k^{(n)}}{t_{k+1}^{(n)} - t_k^{(n)}} & \text{for } t \in (t_k^{(n)}, t_{k+1}^{(n)}) \text{ and } k = 0, 1, ..., m-1 \\ 0 & \text{for } t \in (t_m^{(n)}, T_0), \end{cases}$$

from (2.1) it yields

$$w_n'(t) \in F(y_n(t)) \cup \{0\} + \frac{1}{n} B \quad \text{for almost all } t \in [0, T_0], \tag{3.3}$$

and this implies

$$||w_n'(t)|| \le \alpha + \frac{1}{n} \le \alpha + 1 \quad \text{a.e.} \tag{3.4}$$

Moreover, for $t \in [t_k^{(n)}, t_{k+1}^{(n)}]$ and $k = 0, 1, ..., m-1$, since $t_{k+1}^{(n)} - t_k^{(n)} < \frac{1}{n}$, we have

$$\begin{aligned} &||w_n(t) - x_k^{(n)}|| = ||w_n(t) - w_n(t_k^{(n)})|| \le (t - t_k^{(n)})(\alpha + 1) \le \\ &\le (t_{k+1}^{(n)} - t_k^{(n)})(\alpha + 1) \le \frac{\alpha + 1}{n} \end{aligned}$$

and $w_n(t) \equiv x_m^{(n)}$ on $[t_m^{(n)}, T_0]$. Therefore we obtain

$$w_n(t) \in X_0 + \left(\frac{\alpha + 1}{n}\right) B \quad \text{for all } t \in [0, T_0] \text{ and } n \in N. \tag{3.5}$$

Let us consider now the sequence $(w_n')_n$; by (3.4) it is bounded and uniformly integrable in $L_E^1([0, T_0], \lambda)$; by (3.3) it holds

$$w_n'(t) \in F(X_0) \cup \{0\} + \frac{1}{n} B \quad \text{a.e. in } [0, T_0];$$

denote with Φ the balanced convex hull of $F(X_0)$, then Φ is convex weakly compact and we have

$$w_n'(t) \subset \Phi + \frac{1}{n} B \quad \text{a.e. in } [0, T_0].$$

Using measurable selections, $w_n'(t)$ can be expressed as $w_n'(t) = v_n(t) + h_n(t)$, for all $t \in [0, T_0]$, where v_n belongs to the set S_Φ of all measurable selections of Φ and h_n is measurable with $h_n(t) \in \frac{1}{n} B$, $\forall t$. Since S_Φ is weakly compact, $(v_n)_n$ is relatively weakly compact. Since $h_n \to 0$ in L^1, $(w_n')_n$ is relatively $\sigma(L^1, L^\infty)$ compact. So we can extract a subsequence, again denoted $(w_n')_n$ converging to $w' \in L_E^1([0, T_0], \lambda)$ in $\sigma(L^1, L^\infty)$-topology.

Let $w : [0, T_0] \to E$ be given by $w(t) = x_0 + \int_0^t w'(s)\, ds$; by (3.4) w is an $\alpha-$Lipschitz function satisfying $w_n(t) \rightharpoonup w(t)$ in the $\sigma(E, E')$-topology, for all $t \in [0, T_0]$.

Notice that, by (3.5) the following set $W = \{w_n(t) : t \in [0, T_0] \text{ and } n \in N\}$ is relatively compact for the norm topology, hence we have $w_n(t) \to w(t)$ in norm for all $t \in [0, T_0]$.

Therefore, by the inequalities after (3.4) and by Lemma 2.2, also the sequence $(y_n)_n$ converges pointwisely to the function w in $[0, T_0]$, since, for every $t \in [t_k^{(n)}, t_{k+1}^{(n)})$ with $k = 0, 1, ..., m-1$, it holds

$$||y_n(t) - w(t)|| \leq ||y_k^{(n)} - x_k^{(n)}|| + ||w_n(t_k^{(n)}) - w_n(t)|| + ||w_n(t) - w(t)||.$$

We shall prove now that w satisfies the differential inclusion (3.2) for almost all $t \in [0, T_0]$. Let $(e'_k)_k$ be a dense sequence in E' for the Mackey topology. Let $k, n \in N$ and $A \in \tau_\lambda([0, T_0])$; by (2.1) we get, in particular

$$w'_n(t) \in F(y_n(t)) + \frac{1}{n}B \quad \text{for almost every } t \in [0, t_m^{(n)}],$$

we also recall that $w'_n(t) \equiv 0$ on $(t_m^{(n)}, T_0)$ with $T_0 - t_m^{(n)} \to 0$ when $n \to +\infty$; therefore we have

$$\int_A \langle e'_k, w'_n(t)\rangle \, dt \leq$$
$$\leq \int_A \delta^*\Big(e'_k, F(y_n(t))\Big) \, dt + \frac{\lambda(A)}{n}||e'_k|| - \lambda(A \cap [t_m^{(n)}, T_0])\delta^*(e'_k, F(y_{m-1}^{(n)}));$$

since w'_n converges to w' in $\sigma(L^1, L^\infty)$, y_n converges pointwisely to w and F is scalarly upper semicontinuous, applying Fatou's Lemma we obtain

$$\int_A \langle e'_k, w'(t)\rangle \, dt \leq \int_A \delta^*\Big(e'_k, F(w(t))\Big) \, dt.$$

Hence we can find a negligible set M in $[0, T_0]$ such that

$$\langle e'_k, w'(t)\rangle \leq \delta^*\Big(e'_k, F(w(t))\Big) \quad \text{for all } t \in [0, T_0]\backslash M \text{ and } k \in N;$$

this implies (see [6], Lemma III.34)

$$w'(t) \in F(w(t)) \quad \text{for almost all } t \in [0, T_0]$$

and (3.2) holds.

To complete the proof it remains to show that w is a monotone function with respect to the preorder P. To this end take $s, t \in [0, T_0]$ with $s < t$.

First suppose $t < T_0$; in this case, for n large enough, we have $s \in [t_h^{(n)}, t_{h+1}^{(n)})$ and $t \in [t_k^{(n)}, t_{k+1}^{(n)})$ with $h + 1 \leq k$ and $h, k \in \{1, ..., m\}$. Then by the transitivity of P we deduce from Lemma 2.2 $x_k^{(n)} \in P(x_{h+1}^{(n)})$. By pointwise convergence of $(w_n)_n$ to w and Lemma 2.2 it is easy to prove that $(t_h^{(n)}, x_h^{(n)}) \to (s, w(s))$ and $(t_k^{(n)}, x_k^{(n)}) \to (t, w(t))$ as $n \to +\infty$; as the graph of P is closed we then get $w(t) \in P(w(s))$.

When $t = T_0$, we obtain $w(T_0) \in P(w(s))$ by the continuity of w.

The proof is then complete. □

Theorem 3.2. *Let X be a compact subset of a separable Banach space E; $P : X \to X$ a given continuous preorder and $F : [0,T] \times X \to cwk(E)$ a scalarly upper semicontinuous correspondence satisfying the following tangential condition*

$$F(t,x) \cap T_{P(x)}(x) \neq \emptyset \quad \text{for all } (t,x) \in [0,T] \times X. \tag{3.6}$$

Then, for every $x_0 \in X$, there exists a Lipschitz function $w : [0,T] \to X$ such that

$$w(0) = x_0, \qquad w'(t) \in F(t, w(t)) \text{ for almost all } t \in [0,T]$$
$$w \quad \text{is monotone with respect to the preorder } P.$$

Proof. Consider the following multifunctions

$$H : [0,+\infty) \times X \to R \times E \quad \text{defined by}$$
$$H(t,x) = \begin{cases} \{1\} \times F(t,x) & \text{when } t \leq T \\ \{1\} \times F(T,x) & \text{when } t > T \end{cases}$$

and

$$\hat{P} : [0,+\infty) \times X \to [0,+\infty) \times X \quad \text{given by}$$
$$\hat{P}(t,x) = [t,+\infty) \times P(x).$$

It is easy to show that H is a nonempty convex weakly compact set-valued map; moreover, for any η' in $(R \times E)'$ one has $\delta^*(\eta', H(t,x)) = \eta'((1,0_E)) + \delta^*(e', F(t,x))$ for all $t \in [0,T]$ and $x \in X$, where e' denotes the restriction of η' to $\{0_R\} \times E$; hence H is also scalarly upper semicontinuous. Finally $\hat{P}$ is a continuous preorder on $[0,+\infty) \times X$.
We recall (see e.g. [1]) that an element v of E belongs to the cone $T_{P(x)}(x)$ if and only if there exists a sequence $(\gamma_n)_n$ of positive numbers converging to zero and a sequence $(x_n)_n$ in $P(x)$ such that $\frac{x_n - x}{\gamma_n} \to v$ as $n \to +\infty$, hence

$$\{1\} \times T_{P(x)}(x) \subset T_{\hat{P}(t,x)}(t,x), \quad \text{for every } (t,x) \in [0,+\infty) \times X;$$

thus condition (3.6) implies the following tangential condition on H

$$H(t,x) \cap T_{\hat{P}(t,x)}(t,x) \neq \emptyset \quad \text{for all } (t,x) \in [0,+\infty) \times X$$

and we have verified that all the assumptions of Theorem 3.1 hold.
Therefore, given $x_0 \in X$, it is possible to find a positive constant μ and a Lipschitz function $u : [0,\mu] \to [0,+\infty) \times X$ satisfying

$$u(0) = (0, x_0), \qquad u'(\xi) \in H(u(\xi)) \text{ for almost every } \xi \in [0,\mu]$$
$$u \text{ is monotone with respect to the preorder } \hat{P}.$$

Let $T_0 = \min\{\mu, T\}$; the previous condition then implies the existence of a Lipschitz function $w : [0,T_0] \to X$ such that

$$w(0) = x_0, \qquad w'(t) \in F(t, w(t)) \text{ for almost all } t \in [0,T_0]$$
$$w \text{ is monotone with respect to the preorder } P;$$

in fact it is enough to put $u(\xi) = (t(\xi), w(\xi))$ and notice that $t'(\xi) \equiv 1$.

Observe that H is bounded on $[0, +\infty) \times X$, and let $\alpha = \text{Sup}_{(t,x)\in[0,+\infty)\times X} |H(t, x)|$; if $T_0 < T$, Theorem 3.1 can be applied again as from the initial condition $(T_0, w(T_0))$, then w can be extended to $[0, T]$ in such a way that it remains α-Lipschitz and monotone with respect to P and this completes the proof. □

Theorem 3.3. *Let X be a compact subset of a separable Banach space E, $P : X \to X$ a given continuous preorder and $F : [0, T] \times X \to cwk(E)$ a correspondence satisfying:*

(i) *F is scalarly $\tau_\lambda([0, T]) \otimes \mathcal{B}(X)$–measurable;*
(ii) *$F(t, \cdot)$ is scalarly upper semicontinuous on X, for any $t \in [0, T]$;*
(iii) *there exists a balanced convex weakly compact set K in E such that $F(t, x) \subset K$ for all $(t, x) \in [0, T] \times X$;*
(iv) *$F(t, x) \cap C_{P(x)}(x) \neq \emptyset$ for all $(t, x) \in [0, T] \times X$.*

Then, for every $x_0 \in X$, there exists a Lipschitz function $w : [0, T] \to X$ satisfying

$$w(0) = x_0, \quad w'(t) \in F(t, w(t)) \text{ for almost all } t \in [0, T]$$

w is monotone with respect to the preorder P.

Proof. The following method will be used: for $h > 0$ we shall define an approximation F_h of the set-valued map F which enjoys more regularity than F, in fact F_h is globally scalarly upper semicontinuous and apply to F_h Theorem 3.2; thanks to new results due to Castaing-Moussaoui-Syam [5] (see also Lemma 2.1) we then pass to the limit when $h \to 0$ and obtain a monotone trajectory satisfying the original inclusion.

First notice that conditions (i) and (iii) imply τ_λ–measurability of $F(\cdot, x)$ on $[0, T]$ for every $x \in X$; indeed, since $\tau_\lambda([0, T])$ is a complete tribe, the class of universally measurable sets originated from $\tau_\lambda([0, T])$ coincides with $\tau_\lambda([0, T])$ (see [6], page 73), hence the assertion follows from [6], Thm.III-37.

Let $(r_n)_n$ be a sequence of strictly positive numbers converging to zero.

For every $n \in N$ and $t \in [0, T]$, put $I_{t,r_n} = [0, T] \cap [t, t+r_n]$ and consider the correspondence $F_n : [0, T] \times X \longmapsto E$ given by

$$F_n(t, x) = \frac{1}{r_n} \int_{I_{t,r_n}} F(s, x)\, ds,$$

where $\int_{I_{t,r_n}} F(s, x)\, ds$ is the Aumann integral of $F(\cdot, x)$ on I_{t,r_n}.

Since F is convex and weakly compact, by [5], Thm. 3.2 we derive that the set S_x of all integrable selections of $F(\cdot, x)$ is convex weakly compact in $L^1_E([0, T], \lambda)$; hence $F_n(t, x)$ is a convex weakly compact subset of E, for all $n \in N$ and $(t, x) \in [0, T] \times X$; moreover, reasoning as in [5], Prop.5.3, we obtain that F_n is scalarly upper semicontinuous.

Given $(t, x) \in [0, T] \times X$, consider the measurable multifunction $F(\cdot, x) \cap C_{P(x)}(x)$ defined on $[0, T]$; in consequence of (iv) it is nonempty valued; let $v : [0, T] \to E$ be an integrable selection of $F(\cdot, x) \cap C_{P(x)}(x)$; we have $\frac{1}{r_n} \int_{I_{t,r_n}} v(s)\, ds \in F_n(t, x)$ and since $C_{P(x)}(x)$ is a closed convex cone we also get $\frac{1}{r_n}\int_{I_{t,r_n}} v(s)\,ds \in C_{P(x)}(x)$, hence tangential condition (3.6) holds for each F_n.

Therefore F_n satisfies all the assumptions of Theorem 3.2 and for every $x_0 \in X$ it is possible to find a Lipschitz function $w_n : [0,T] \to X$ such that

$$\begin{aligned} &w_n(0) = x_0, \quad w_n'(t) \in F_n(t, w_n(t)) \text{ a.e.} \\ &w_n \text{ is monotone with respect to } P. \end{aligned} \tag{3.7}$$

For each $n \in N$, by (iii) and the definition of F_n, we obtain $F_n(t,x) \subset K$ for all $(t,x) \in [0,T] \times X$, so given $\alpha > 0$ with $|K| \le \alpha$, the sequence $(w_n)_n$ turns out to be equi–α–Lipschitz.

Now recall that the set S_K of all measurable selections of K is convex and $\sigma(L^1, L^\infty)$ compact (see [6], Corollary V.4); by (3.7) $w_n'(t) \in K$ a.e., we can then apply Eberlein-Šmulian's Theorem to extract a subsequence, again denoted $(w_n')_n$, converging, for the $\sigma(L^1, L^\infty)$–topology, to a function $w' \in S_K$, hence $(w_n(t))_n$ weakly converges to $w(t) = x_0 + \int_0^t w'(s)\,ds$, for every $t \in [0,T]$.

Notice now that, for all $n \in N$ and $t \in [0,T]$, $w_n(t)$ belongs to the compact set X, therefore the sequence pointwisely $(w_n(t))_n$ converges to $w(t)$ in norm topology.

By (3.7) all trajectories w_n are monotone with respect to the preorder P, since P has closed graph also w is monotone with respect to P.

To complete the proof it remains thus to show that $w'(t) \in F(t, w(t))$ for almost every $t \in [0,T]$. According to (3.7) we then apply Lemma 2.1 to $(w_n')_n$, $(w_n)_n$ and to F so that we obtain $w'(t) \in F(t, w(t))$ a.e. as desired. □

Let us mention that when F is only scalarly measurable on $[0,T]$ for each fixed $x \in X$ and scalarly upper semicontinuous on X for each fixed $t \in [0,T]$, the conclusion of Theorem 3.3 does not hold (see e.g. Bothe [3], Example 2). We deal now with a stronger tangential condition and a weaker measurable assumption. Namely we have the following version of Theorem 3.3.

Theorem 3.4. *Let X be a compact subset of a separable Banach space E, $P : X \to X$ a given continuous preorder and $F : [0,T] \times X \to cwk(E)$ a correspondence satisfying*

(i) *$F(\cdot, x)$ admits a $\tau_\lambda([0,T])$-measurable selection, for all $x \in X$;*
(ii) *$F(t, \cdot)$ is scalarly upper semicontinuous on X, for any $t \in [0,T]$;*
(iii) *there is a balanced convex weakly compact subset K of E such that $F(t,x) \subset K \cap C_{P(x)}(x)$ for each $(t,x) \in [0,T] \times X$.*

Then, for every $x_0 \in X$, there exists a Lipschitz function $w : [0,T] \to X$ satisfying

$$\begin{aligned} &w(0) = x_0, \quad w'(t) \in F(t, w(t)) \textit{ for almost all } t \in [0,T] \\ &w \textit{ is monotone with respect to the preorder } P. \end{aligned}$$

Proof. In consequence of (ii), $F(t, \cdot)$ is for each t upper semicontinuous from X to E_σ, where E_σ denotes the vector space E endowed with the weak $\sigma(E, E')$-topology (see [6], Thm.II.20); hence it is obvious that graph $F_t = \{(x,y) \in X \times K : y \in F(t,x)\}$ is compact in the compact metrizable space $X \times K_\sigma$, where K_σ denotes K with the metric associated to the weak topology, for which it is a compact convex set.

By virtue of Theorem 2.3, there exists a multifunction $F_0 : [0,T] \times X \to cwk(K) \cup \{\emptyset\}$ which has the properties (1)-(3) of Theorem 2.3, that is

(1) there is a λ-null set M, independent of (t,x), such that

$$F_0(t,x) \subset F(t,x), \text{ for all } t \notin M \text{ and } x \in X;$$

(2) if $u : [0,T] \to X$ and $v : [0,T] \to K$ are $\tau_\lambda([0,T])$-measurable functions with $v(t) \in F(t,u(t))$ a.e., then $v(t) \in F_0(t,u(t))$ a.e.;

(3) for every $\epsilon > 0$, there is a compact subset $I_\epsilon \subset [0,T]$ such that $\lambda([0,T] \setminus I_\epsilon) < \epsilon$, the graph of the restriction $F_{0 \setminus I_\epsilon \times X}$ is closed in $I_\epsilon \times X \times K_\sigma$ and $\emptyset \neq F_0(t,x) \subset F(t,x)$, for all $(t,x) \in I_\epsilon \times X$.

By property (3), one gets a sequence of compact sets $I_n \subset [0,T]$ with $\lambda([0,T] \setminus I_n) = \epsilon_n \to 0$ such that the restriction of F_0 to $I_n \times X$ has compact graph in $I_n \times X \times K_\sigma$; we may also assume that $(I_n)_n$ is increasing.

Since, for each n, F_0 is upper semicontinuous from $I_n \times X$ to E_σ, by Theorem 2.4 it admits an upper semicontinuous extension $\tilde{F}_n$ from $[0,T] \times X$ to E_σ with nonempty convex compact values in E_σ satisfying

$$\tilde{F}_n(t,x) \subset K \cap C_{P(x)}(x) \quad \text{for all } (t,x) \in [0,T] \times X.$$

Let $\alpha = \sup\{\|k\| : k \in K\}$; notice that each $\tilde{F}_n$ is also scalarly upper semicontinuous (see [6], Thm.II.20), hence, for every n, we can apply Theorem 3.2 in order to obtain an α-Lipschitz function $w_n : [0,T] \to X$ such that

$$\begin{aligned} &w_n(0) = x_0, \\ &w_n'(t) \in \tilde{F}_n(t, w_n(t)) \text{ a.e.} \\ &w_n \text{ is monotone with respect to the preorder } P. \end{aligned} \tag{3.8}$$

By the construction of $\tilde{F}_n$, (3.8) implies

$$w_n'(t) \in F_0(t, w_n(t)), \text{ for all } t \in I_n \setminus M_n \tag{3.9}$$

with $\lambda(M_n) = 0$.

Since K is convex weakly compact and $w_n'(t) \in K$ a.e., repeating the same arguments given in the proof of Theorem 3.3, we can extract a subsequence, again denoted $(w_n')_n$, such that w_n' converges for the $\sigma(L^1, L^\infty)$-topology to $w' \in S_K$ and for all t

$\lim_{n \to +\infty} w_n(t) = w(t) = x_0 + \int_0^t w'(s)\,ds$ with respect to the norm topology.

Let $M_0 = ([0,T] \setminus \cup_n I_n) \cup \cup_n M_n$, which is a set with zero measure. If $t \notin M_0$, then there is $p = p(t)$ such that $t \in I_n \setminus M_n$ for all $n \geq p$, so that by (3.9) $w_n'(t) \in F_0(t, w_n(t))$, hence follows $\langle e', w_n'(t) \rangle \leq \delta^*\Big(e', F_0(t, w_n(t))\Big)$ for all $e' \in E'$.

For $t \notin M_0$ and $e' \in E'$, since F_0 is scalarly upper semicontinuous in $I_p \times X$ and $w_n(t) \to w(t)$ in norm topology, we have

$$\limsup_{n\to+\infty} \delta^*\Big(e', F_0(t, w_n(t))\Big) \leq \delta^*\Big(e', F_0(t, w(t))\Big)$$

so that, for such t

$$\limsup_{n\to+\infty} \langle e', w_n'(t)\rangle \leq \delta^*\Big(e', F_0(t, w(t))\Big)$$

where the right-hand side is a measurable function.
For all measurable sets $A \subset [0, T]$ and every $e' \in E'$, by Fatou's Lemma, it follows

$$\int_A \langle e', w'(t)\rangle \, dt = \lim_{n\to+\infty} \int_A \langle e', w_n'(t)\rangle \, dt \leq \int_A \delta^*\Big(e', F_0(t, w(t))\Big) \, dt.$$

Since E is separable, this is known to imply that

$$w'(t) \in F_0(t, w(t)) \subset F(t, w(t)) \text{ a.e.}$$

Finally, we recall that P has closed graph, by (3.8) w is then monotone with respect to P and this completes the proof. □

4. Comments

It is worth to compare the results obtained here with those given by Aubin-Cellina [1], Deimling [8] and Haddad [9].

For dim E $< \infty$, Haddad [9] proved a necessary and sufficient condition for the existence of monotone solutions of (1.1).

Via a constructive algorithm based on Lemma 2.2, Theorem 3.3 extends in two directions the sufficient condition given by Haddad, since dim E $= \infty$ and F is globally scalarly measurable and scalarly upper semicontinuous in $x \in X$.

Theorem 3.3 extends even a result given by Aubin-Cellina ([1], Thm.4.2.3) dealing with the case when E is a Hilbert space and F is globally upper semicontinuous, with norm-compact convex values.

Deimling obtained an existence result ([8], Thm.4) in Banach spaces when X is only closed, P has closed graph, F is independent of t and satisfies the normal growth conditions; he also assumed the following compactness type condition $\alpha(F(B)) \leq k\alpha(B)$ for some $k \geq 0$ and all bounded $B \subset X$, where α denotes the Kuratowski's measure of noncompactness; such condition implies the compactness, with respect to the norm topology, of $F(x)$ for all $x \in X$ and consequently Deimling's Thm.4 [8] and Theorem 3.1 are not comparable. Finally Deimling proved his result with a different method based on the use of a measure of noncompactness and Zorn's Lemma.

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