COMPUTING MATVEEV'S COMPLEXITY VIA CRYSTALLIZATION THEORY: THE ORIENTABLE CASE *

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7th June 2006

Abstract

By means of a slight modification of the notion of *GM-complexity* introduced in [8], the present paper performs a graph-theoretical approach to the computation of *(Matveev's) complexity* for closed orientable 3-manifolds. In particular, the existing crystallization catalogue $\mathcal{C}^{28}$ available in [18] is used to obtain upper bounds for the complexity of closed orientable 3-manifolds triangulated by at most 28 tetrahedra. The experimental results actually coincide with the exact values of complexity, for all but three elements. Moreover, in the case of at most 26 tetrahedra, the exact value of the complexity is shown to be always directly computable via crystallization theory.

Mathematics Subject Classification 2000: 57N10 - 57M15 - 57M20 - 57M50.
Key words: orientable 3-manifold; complexity; crystallization; spine; Heegaard diagram.

1. Introduction

As it is well-known, Matveev's notion of *complexity* is based on the existence, for each compact 3-manifold M^3, of a *simple spine*[1]: in fact, if M^3 is a compact 3-manifold, *(Matveev's)*

*Work performed under the auspicies of the G.N.S.A.G.A. of the C.N.R. (National Research Council of Italy) and financially supported by M.U.R.S.T. of Italy (project "Strutture geometriche delle varietà reali e complesse") and by Università degli Studi di Modena e Reggio Emilia (project "Strutture finite e modelli discreti di strutture geometriche continue").

[1]According to [21], a subpolyhedron $P \subset Int(M^3)$ is said to be a *simple spine* of M^3 if the link of each of its points can be embedded in Δ (the 1-skeleton of the 3-simplex) and M^3 - or M^3 minus an open 3-ball, in case $\partial M^3 = \emptyset$ - collapses to P.

complexity of M^3 is defined as

$$c(M^3) = min_P c(P),$$

where the minimum is taken over all simple spines P of M^3 and $c(P)$ denotes the number of true vertices[2] of the simple spine P.

In [8] (which is devoted only to the non-orientable case), a graph-theoretical approach to the computation of complexity is performed, via another combinatorial theory to represent 3-manifolds, which makes use of particular edge-coloured graphs, called *crystallizations* (see [14] or [4] for a survey on this representation theory, for PL-manifolds of arbitrary dimension): the existence of the crystallization catalogue $\tilde{\mathcal{C}}^{26}$ (due to [6]) for closed non-orientable 3-manifolds triangulated by at most 26 tetrahedra has allowed to complete the existing classification (due to [1]) of closed non-orientable 3-manifolds up to complexity six.

On the other hand, as already pointed out in [8], any crystallization catalogue obviously yields - via the notion of Gem-Matveev complexity, or GM-complexity, for short, - upper bounds for the complexity of any involved manifold. Since complexity and GM-complexity actually turn out to coincide for each manifold represented by catalogue $\tilde{\mathcal{C}}^{26}$, it appears to be an interesting problem to search for classes of 3-manifolds whose complexity can be directly computed via GM-complexity or, better, to give a characterization of the classes of 3-manifolds satisfying this property: see [8] (paragraph 1 - Open Problem).

The aim of the present paper is to face the above problem in the *orientable* case, by making use of the existing crystallization catalogue $\mathcal{C}^{28}$ (due to [18]) for closed orientable 3-manifolds triangulated by at most 28 tetrahedra.

For this purpose, a slight modification of the notion of GM-complexity, involving also non minimal crystallizations, is taken into account.

Algorithmic computation (easily implemented on computer) directly yields that, for all but three 3-manifolds involved in $\mathcal{C}^{28}$, GM-complexity and complexity coincide: see Proposition 7; moreover, if the attention is restricted to orientable 3-manifolds triangulated by at most 26 tetrahedra, then the exact value of the complexity turns out to be always directly computable via crystallizations.

2. Crystallizations and GM-complexity

In this section, in order to introduce our graph-theoretical approach to the computation of complexity, we briefly recall few basic concepts of the representation theory of PL-manifolds by crystallizations. For general PL-topology, Heegaard splittings of 3-manifolds and elementary notions about graphs and embeddings, we refer to [17], [16] and [27] respectively.

Crystallization theory represents PL n-manifolds by means of $(n+1)$-*coloured graphs*, that is, it is a representation theory which can be used in any dimension. On the other hand, since this paper concerns only 3-manifolds, the following definitions and results will be given for $n = 3$, although they mostly hold for each $n \geq 0$.

[2]Recall that a point of the simple spine P is said to be a *true vertex* if its link is homeomorphic to Δ.

Moreover, throughout the paper all manifolds will be closed and connected.

Given a pseudocomplex K, triangulating a 3-manifold M, a *coloration* on K is a labelling of its vertices by $\Delta_3 = \{0, 1, 2, 3\}$, which is injective on each simplex of K.

The dual 1-skeleton of K is a (multi)graph Γ embedded in $|K| = M$; we can define on $\Gamma = (V(\Gamma), E(\Gamma))$ an *edge-coloration* i.e. a map $\gamma : E(\Gamma) \to \Delta_3$ in the following way: $\gamma(e) = c$ iff the vertices of the face dual to e are coloured by $\Delta_3 - \{c\}$[3].

The pair (Γ, γ) is called a *4-coloured graph representing* M or simply a *gem=graph encoded manifold* (see [18]).

In the following, to avoid long notations, we will often omit the edge-coloration, when it is not necessary, and we will simply write Γ instead of (Γ, γ).

It is easy to see that, starting from Γ, we can always reconstruct $K(\Gamma) = K$ and hence the manifold M (see [14] and [4] for more details).

Given $i, j \in \Delta_3$, we denote by $(\Gamma_{i,j}, \gamma_{i,j})$ the 3-coloured graph such that $\Gamma_{i,j} = (V(\Gamma), \gamma^{-1}(\{i, j\}))$ and $\gamma_{i,j} = \gamma_{|_{\gamma^{-1}(\{i,j\})}}$ i.e. it is obtained from Γ by deleting all edges which are not i- or j-coloured; the connected components of $\Gamma_{i,j}$ will be called $\{i, j\}$*-residues* of Γ and their number will be denoted by $g_{i,j}$.

As a consequence of the definition, a bijection is established between the set of $\{i, j\}$-residues of Γ and the set of 1-simplices of $K(\Gamma)$, whose endpoints are labelled by $\Delta_3 - \{i, j\}$.

Moreover, for each $c \in \Delta_3$, the connected components of the 3-coloured graph $\Gamma_{\hat{c}}$ obtained from Γ by deleting all c-coloured edges, are in bijective correspondence with the c-coloured vertices of $K(\Gamma)$; we will call Γ *contracted* iff $\Gamma_{\hat{c}}$ is connected for each $c \in \Delta_3$, i.e. if $K(\Gamma)$ has exactly four vertices.

A contracted 4-coloured graph representing a 3-manifold M is called a *crystallization* of M.

Several topological properties of M can be "read" as combinatorial properties of any crystallization (or more generally any gem) Γ of M: as an example, M is orientable iff Γ is bipartite.

Relations among crystallization theory and other classical representation methods for PL manifolds have been deeply analyzed (see [4]; sections 3, 6, 7). In particular, for our purposes, it is useful to recall the strong connection existing between crystallizations and Heegaard diagrams.

If Γ is a bipartite (resp. non bipartite) crystallization of a 3-manifold M, for each pair $\alpha, \beta \in \Delta_3$, let us set $\{\hat{\alpha}, \hat{\beta}\} = \Delta_3 - \{\alpha, \beta\}$ and let $F_{\alpha,\beta}$ be the orientable (resp. non orientable) surface of genus $g_{\alpha,\beta} - 1 = g_{\hat{\alpha},\hat{\beta}} - 1$, obtained from Γ by attaching a 2-cell to each $\{i, j\}$-residue such that $\{i, j\} \neq \{\alpha, \beta\}$ and $\{i, j\} \neq \{\hat{\alpha}, \hat{\beta}\}$.

It is well-known (see [14] or [4], together with their references) that a regular embedding[4] $i_{\alpha,\beta} : \Gamma \to F_{\alpha,\beta}$ exists. Moreover, if $\mathcal{D}$ (resp. $\mathcal{D}'$) is an arbitrarily chosen $\{\alpha, \beta\}$-residue (resp. $\{\hat{\alpha}, \hat{\beta}\}$-residue) of Γ, the triple $\mathcal{H}_{\alpha,\beta,\mathcal{D},\mathcal{D}'} = (F_{\alpha,\beta}, \mathbf{x}, \mathbf{y})$, where $\mathbf{x}$ (resp. $\mathbf{y}$) is the set of the images of all $\{\alpha, \beta\}$-residues (resp. $\{\hat{\alpha}, \hat{\beta}\}$-residues) of Γ, but $\mathcal{D}$ (resp. $\mathcal{D}'$), is a Heegaard diagram of M.

[3] Note that an edge-coloration is characterized by being injective on each pair of adjacent edges of the graph.

[4] A cellular embedding i of a 4-coloured graph Γ into a surface is said to be *regular* if there exists a cyclic permutation ε of Δ_3 such that the regions of i are bounded by the images of $\{\varepsilon_j, \varepsilon_{j+1}\}$-residues of Γ ($j \in \mathbb{Z}_4$).

Conversely, given a Heegaard diagram $\mathcal{H} = (F, \mathbf{x}, \mathbf{y})$ of M and $\alpha, \beta \in \Delta_3$, there exists a construction which, starting from $\mathcal{H}$ yields a crystallization Γ of M such that $\mathcal{H} = \mathcal{H}_{\alpha,\beta,\mathcal{D},\mathcal{D}'}$ for a suitable choice of $\mathcal{D}$ and $\mathcal{D}'$ in Γ (see [15])[5].

Now, let us denote by $\mathcal{R}_{\mathcal{D},\mathcal{D}'}$ the set of regions of $F_{\alpha,\beta} - (\mathbf{x} \cup \mathbf{y}) = F_{\alpha,\beta} - i_{\alpha,\beta}((\Gamma_{\alpha,\beta} - \mathcal{D}) \cup (\Gamma_{\hat{\alpha},\hat{\beta}} - \mathcal{D}'))$.

Definition 3. Let M be a closed 3-manifold, and let Γ be a crystallization of M. With the above notations, *Gem-Matveev complexity* (or simply *GM-complexity*) of Γ is defined as the non-negative integer

$$c_{GM}(\Gamma) = min\left\{\#V(\Gamma) - \#(V(\mathcal{D}) \cup V(\mathcal{D}') \cup V(\Xi)) \ / \ \alpha, \beta \in \Delta_3, \mathcal{D} \in \Gamma_{\alpha,\beta}, \mathcal{D}' \in \Gamma_{\hat{\alpha},\hat{\beta}}, \Xi \in \mathcal{R}_{\mathcal{D},\mathcal{D}'}\right\}$$

while *(non-minimal) GM-complexity* of M is defined as the minimum value of GM-complexity, where the minimum is taken over all crystallizations of M :

$$c'_{GM}(M) = min\{c_{GM}(\Gamma) \ / \ \Gamma \text{ is a crystallization of } M\}$$

As a direct consequence of the definition, (non-minimal) GM-complexity turns out to be an upper estimation of the manifold complexity:

Proposition 1 [9] *For every closed 3-manifold M,*

$$c(M) \leq c'_{GM}(M)$$

□

Remark 1. Definition 3 is a slight modification (already suggested in [9]) of the previous definition of *GM-complexity* of a 3-manifold M denoted by $c_{GM}(M)$ and originally introduced in [8]. In fact, $c_{GM}(M)$ is defined as the minimum value of $c_{GM}(\Gamma)$, too, but the minimum is taken only over *minimal* crystallizations Γ of the manifold, i.e. crystallizations of M having minimal number of vertices.

In the present paper we will always refer to (non-minimal) GM-complexity $c'_{GM}(M)$ but, for sake of conciseness, we will simply write GM-complexity.

It is well known that complexity is additive with respect to the connected sum of manifolds; GM-complexity can be easily proved to be subadditive as shown in the following

Proposition 2 *For each pair of closed 3-manifolds M_1, M_2, the following inequality holds:*

$$c'_{GM}(M_1 \# M_2) \leq c'_{GM}(M_1) + c'_{GM}(M_2).$$

[5]This correspondence between Heegaard diagrams and crystallizations allows to prove the coincidence between the Heegaard genus of M and its *regular genus*, a combinatorial PL-manifold invariant, based on regular embeddings, which is defined in arbitrary dimension. Interesting results about classification of PL-manifolds via regular genus may be found, for example, in [13], [11], [5], [12]

Hint of the proof. The proof consists essentially of two steps.
Step 1. Let $\Gamma^{(1)}, \Gamma^{(2)}$ be crystallizations of M_1 and M_2 respectively, such that $c'_{GM}(M_k) = c_{GM}(\Gamma^{(k)})$ $(k = 1, 2)$. With the notations of Definition 3, for each $k = 1, 2$, let $\mathcal{H}_k$ be the Heegaard diagram associated to $\Gamma^{(k)}$ and Ξ_k the region of $\mathcal{H}_k$ realizing $c'_{GM}(\Gamma^{(k)})$; if we denote by n_k the number of vertices of $\mathcal{H}_k$ and by m_k the number of vertices of Ξ_k, then $c_{GM}(M_k) = n_k - m_k$. We perform the connected sum of M_1 and M_2 with respect to two 3-balls B_1 and B_2 such that, for each $k = 1, 2$, B_k is contained in one of the two handlebodies defined by $\mathcal{H}_k$ and intersects the Heegaard surface of $\mathcal{H}_k$ in a 2-disc D_k contained in *int* Ξ_k. In this way we obtain a Heegaard diagram $\mathcal{H}$ of $M_1\#M_2$ having $n_1 + n_2$ vertices and containing a region $\Xi = \Xi_1\#\Xi_2$ with $m_1 + m_2$ vertices.
Step 2. By applying to the diagram $\mathcal{H}$ the construction of [15], a crystallization Γ of $M_1\#M_2$ is obtained, with the property that $c'_{GM}(\Gamma) \leq n_1 + n_2 - (m_1 + m_2) = c'_{GM}(\Gamma^{(1)}) + c'_{GM}(\Gamma^{(2)})$.
□

Remark 2. The additivity of c'_{GM} can be proved for the restricted class of manifolds having GM-complexity coinciding with the complexity. In fact, in this case, the additivity of the complexity and the above Proposition yield

$$c'_{GM}(M_1\#M_2) \leq c'_{GM}(M_1) + c'_{GM}(M_2) = c(M_1) + c(M_2) = c(M_1\#M_2) \leq c'_{GM}(M_1\#M_2).$$

This result ensures that, as far as we are interested in the coincidence of GM- and Matveev's complexity, we can restrict our attention to prime manifolds.

Actually, direct computation proves that the additive property holds for all manifolds represented by catalogue $\mathcal{C}^{28}$ (as it is already known for catalogue $\tilde{\mathcal{C}}^{26}$).

3. Experimental data from catalogue $\mathcal{C}^{28}$

In the literature, a lot of subsequent cataloguing results for closed orientable irreducible 3-manifolds according to their complexity exist: in [21] Matveev himself lists all such manifolds with complexity $c \leq 6$; in [25] Ovchinnikov obtains a table for $c = 7$ (see also [22] - Appendix 9.3, where part of Ovchinnikov's table is reproduced); in [19] Martelli and Petronio re-obtain via bricks decomposition the previous results and extend the catalogue up to complexity 9 (see also http://www.dm.unipi.it/pages/petronio/public_html/files/3D/c9/c9_census.html for explicit censuses); finally in [23] (see also [24]) Matveev improves the above classifications by solving the cases $c = 10$ and $c = 11$.

Note that closed (orientable and non-orientable[6]) 3-manifolds up to complexity 6 are also classified by Burton's PhD thesis, which contains a catalogue of their minimal triangulations, obtained by face-pairing graphs: see [2].

[6]Burton's approach allows to include also the case of irreducible and $\mathbb{P}^2$-irreducible non-orientable 3-manifolds, which are further classified up to complexity 7 in [3].

The aim of the present section is to compare, by means of experimental results, complexity and GM-complexity of orientable 3-manifolds with "small" coloured decompositions; for this purpose, the existing catalogue $\mathcal{C}^{28}$ of rigid and bipartite crystallizations with at most 28 vertices ([18]) is a basic tool.

In fact, $\mathcal{C}^{28}$ yields a catalogue of closed orientable 3-manifolds, ordered by the minimal number of tetrahedra in their coloured triangulations:

Proposition 3 [18] *There exist exactly sixty-nine closed connected prime orientable 3-manifolds, which admit a coloured triangulation consisting of at most 28 tetrahedra. They are: the sphere $\mathbb{S}^3$; the orientable $\mathbb{S}^2$-bundle over $\mathbb{S}^1$ (i.e. $\mathbb{S}^2 \times \mathbb{S}^1$); the six Euclidean orientable 3-manifolds; twenty-three lens spaces; twenty-one quotients of $\mathbb{S}^3$ by the action of their finite (non-cyclic) fundamental groups; further seventeen topologically undetected orientable 3-manifolds.*[7]

A direct estimation for GM-complexity can be performed for all manifolds represented in $\mathcal{C}^{28}$ by means of an easily implemented computer program[8], which works as follows:

- given a crystallization (Γ, γ), let us fix
  - a partition $\{\{\varepsilon_0, \varepsilon_1\}, \{\varepsilon_2, \varepsilon_3\}\}$ of Δ_3;
  - an $\{\varepsilon_0, \varepsilon_1\}$-residue $\mathcal{D}$ of Γ;
  - an $\{\varepsilon_2, \varepsilon_3\}$-residue $\mathcal{D}'$ of Γ;
  - a pair of integers $i, j \in \{0, 1\}$;
  - an $\{\varepsilon_i, \varepsilon_{j+2}\}$-residue Ξ_0 of Γ;
- consider the following subgraph of Γ,

$$\Xi_1 = \left(\bigcup_{\substack{e \in E(\Xi_0) \cap E(\mathcal{D}) \\ \gamma(e) = \varepsilon_i}} \Gamma_{\varepsilon_i, \varepsilon_{2+(j+1) mod\, 2}}(e) \right) \cup \left(\bigcup_{\substack{f \in E(\Xi_0) \cap E(\mathcal{D}') \\ \gamma(f) = \varepsilon_{j+2}}} \Gamma_{\varepsilon_{(i+1) mod\, 2}, \varepsilon_{j+2}}(f) \right)$$

  where $\Gamma_{a,b}(c)$ denotes the connected component of $\Gamma_{a,b}$ containing the edge c.

- construct inductively the sequence $\{\Xi_k\}_{k=1,\ldots,m}$, where

$$\Xi_k = \left(\bigcup_{\substack{e \in E(\Xi_{k-1}) \cap E(\mathcal{D}) \\ \gamma(e) = \varepsilon_i}} \Gamma_{\varepsilon_i, \varepsilon_{2+(j+1) mod\, 2}}(e) \right) \cup \left(\bigcup_{\substack{f \in E(\Xi_{k-1}) \cap E(\mathcal{D}') \\ \gamma(f) = \varepsilon_{j+2}}} \Gamma_{\varepsilon_{(i+1) mod\, 2}, \varepsilon_{j+2}}(f) \right)$$

  and $m \in \mathbb{Z}$ is such that $E(\Xi_m) \cap (E(\mathcal{D}) \cup E(\mathcal{D}')) = \emptyset$;

[7]Lins's classification simply identifies these seventeen 3-manifolds by means of their fundamental groups. However, five of these groups are given in [18] as semidirect products of $\mathbb{Z}$ by $\mathbb{Z} \times \mathbb{Z}$ induced by matrices of $GL(2; \mathbb{Z})$, and in [7] the corresponding 3-manifolds are actually proved to be torus bundles over $\mathbb{S}^1$.

[8]The C++ program for GM-complexity computation is available on the Web: http://cdm.unimo.it/home/matematica/casali.mariarita/DukeIII.htm

- compute the number $\bar{c} = \#V(\Gamma) - \#(V(\mathcal{D}) \cup V(\mathcal{D}') \cup V(\Xi_m))$ for all possible choices of $\Xi_0, i, j, \mathcal{D}', \mathcal{D}$ and for all possible partitions $\{\{\varepsilon_0, \varepsilon_1\}, \{\varepsilon_2, \varepsilon_3\}\}$ of Δ_3; it is very easy to check that the minimal value assumed by variable $\bar{c}$ exactly coincides with $c_{GM}(\Gamma)$;

- for each 3-manifold M^3 represented in $\mathcal{C}^{28}$, the above algorithm is applied to the first crystallization $\bar{\Gamma}$ of M^3 listed in the catalogue, and then to every crystallization $\Gamma \in \mathcal{C}^{28}$ with $K(\Gamma) = M^3$; the minimal value of their GM-complexities obviously yields an upper estimation for $c'_{GM}(M^3)$.

The obtained results are shown in details in Table 1 of [10]. That Table also contains, for each prime 3-manifold involved in $\mathcal{C}^{28}$, the corresponding Matveev's description (see [22] - Appendix 9.1 and Appendix 9.3) and/or the associated Seifert structure: in fact, for each element of Lins's catalogue, we have also performed the "translation" into Matveev's notation, which allows a more efficient topological identification and a direct knowledge of the complexity. The identifications were usually carried out through the computation of GM-complexity and the comparison of homology groups, and in some cases with the aid of a powerful computer program for 3-manifold recognition elaborated by Matveev and his research group and written by V.Tarkaev.[9]

As a consequence, the following improvement of Lins's classification is obtained, with unambiguous identification of the encoded 3-manifolds, via JSJ decompositions and fibering structures: [10]

Proposition 4 *The sixty-nine closed connected prime orientable 3-manifolds which admit a coloured triangulation consisting of at most 28 tetrahedra are:*

- $\mathbb{S}^3$*;*

- $\mathbb{S}^2 \times \mathbb{S}^1$*;*

- *the six Euclidean orientable 3-manifolds;*

- *twenty-three lens spaces;*

- *twenty-one quotients of* $\mathbb{S}^3$ *by the action of their finite (non-cyclic) fundamental groups;*

[9] Computer program *"Three-manifold Recognizer"* is available on the Web: http://www.csu.ac.ru/~trk/

[10] In the statement of Proposition 4, the following conventions are assumed:

- for each matrix $A \in GL(2;\mathbb{Z})$ with $\det(A) = +1$, $TB(A) = T \times I/A$ is the orientable torus bundle over $\mathcal{S}^1$ with monodromy induced by A;

- for each matrix $A \in GL(2;\mathbb{Z})$ with $\det(A) = -1$, $(K\tilde{\times}I) \cup (K\tilde{\times}I)/A$ is the orientable 3-manifold obtained by pasting together, according to A, two copies of the orientable $I-$bundle over the Klein bottle K;

- $(F, (p_1, q_1), , \dots, (p_k, q_k))$ is the Seifert fibered manifold with base surface F and k disjoint fibres, having (p_i, q_i), $i = 1, \dots, k$ as non-normalized parameters;

Moreover, the geometric structures are given according to [26].

- *six (non euclidean) torus bundles $TB(A)$:*
  - *the Nil ones, with complexity 6, associated to matrices* $\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$*;*
  - *the Nil ones, with complexity 7, associated to matrices* $\begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$*;*
  - *the Sol ones, with complexity 7, associated to matrices* $\begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix}$*;*
- *two Nil 3-manifolds of type* $(K\tilde{\times}I)\cup(K\tilde{\times}I)/A$*, with complexity 6, i.e. the ones associated to matrices* $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$*;*
- *another Nil 3-manifold with complexity 6, i.e. the manifold with Seifert structure* $(\mathbb{S}^2,(3,2),(3,1),(3,-2))$*;*
- *eight Seifert 3-manifolds with complexity 7:*
  - *the Nil 3-manifold* $(\mathbb{RP}^2,(2,1),(2,3))$;
  - *the* $SL_2\mathbb{R}$ *3-manifolds* $(\mathbb{RP}^2,(2,1),(3,-1))$, $(\mathbb{S}^2,(2,1),(2,1),(2,1),(3,-4))$, $(\mathbb{S}^2,(2,1),(3,1),(7,-6))$, $(\mathbb{S}^2,(2,1),(4,1),(5,-4))$, $(\mathbb{S}^2,(3,1),(3,1),(5,-3))$, $(\mathbb{S}^2,(3,1),(3,1),(4,-3))$ *and* $(\mathbb{S}^2,(2,1),(3,1),(7,-5))$.

$\square$

Experimental data from catalogue $\mathcal{C}^{28}$ yield interesting information in order to compare different complexity notions.

First of all, we can consider, together with the complexity, the so called *gem-complexity* of M^3, i.e. the non-negative integer $k(M^3)=\frac{p}{2}-1$, p being the minimum order of a crystallization of M^3: see, for example, [6] - paragraph 5 or [8] - Remark 1, where the problem of possible relations between the complexity $c(M^3)$ and gem-complexity $k(M^3)$ is pointed out.

On one hand, catalogues $\mathcal{C}^{28}$ and $\tilde{\mathcal{C}}^{26}$ allow us to check that, for the first segments of 3-manifold censuses, "restricted" gem-complexity implies "restricted" complexity:

Proposition 5 *Let M^3 be a closed 3-manifold.*

(a) $k(M^3)\le 12 \quad\Longrightarrow\quad c(M^3)\le 6$;

(b) If M^3 is assumed to be orientable, then $k(M^3)\le 13 \quad\Longrightarrow\quad c(M^3)\le 7$.

Proof. Statement (b) is a direct consequence of Proposition 4, since $c(M^3)\le 7$ holds for every manifold M^3 encoded by elements of $\mathcal{C}^{28}$.

Statement (a) concerns both orientable and non-orientable 3-manifolds. In the orientable case, it follows from identification results contained in the first part of Table 1 of [10]: in fact, $k(M^3) \leq 12$ implies the existence of a rigid cristallization $\bar{\Gamma} \in \mathcal{C}^{28}$, with $\#V(\bar{\Gamma}) \leq 26$, representing M^3, and this immediately yields $c(M^3) \leq 6$ (as it may be seen in the last column of Table 1 itself). On the other hand, in the non-orientable case, statement (a) is a direct consequence of results contained in [6] and [8] (see also [8] - Remark 1, where the set of irreducible and $\mathcal{P}^2$-irreducible non-orientable 3-manifolds up to complexity $c = 6$ is proved to coincide exactly with the set of such manifolds up to gem-complexity $k = 12$).

□

On the other hand, for all manifolds in the catalogue $\mathcal{C}^{28}$ "restricted" complexity implies "restricted" gem-complexity, too. More precisely, we can state:

Proposition 6 *Let M^3 be a closed orientable 3-manifold with complexity $c(M^3) = c$. If $0 \leq c \leq 4$, then $k(M^3) \leq 5 + 2c$.*

Proof. It is well-known, within crystallization theory, that $\mathbb{S}^1 \times \mathbb{S}^2$ admits a (non-rigid) order eight crystallization; hence, $k(\mathbb{S}^1 \times \mathbb{S}^2) = 3$. This fact, together with a direct comparison between Table 1 of [10] and the tables of [22] - Appendix 9.1 allows to state that all closed orientable 3-manifolds with complexity 0 (resp. 1) (resp. 2) (resp. 3) (resp. 4) admit a gem with at most 12 (resp. 16) (resp. 20) (resp. 24) (resp. 28) vertices. Hence, the corresponding gem-complexities satisfy the claimed inequality.

□

The above results naturally suggest the following

Conjecture: $k(M^3) \leq 5 + 2c(M^3)$ for any closed orientable 3-manifold M^3.

Moreover, experimental data concerning GM-complexity estimation for closed orientable 3-manifolds represented by the crystallization catalogue $\mathcal{C}^{28}$ - appearing in the fifth column of Table 1 of [10], - allow us to prove directly the following properties, and therefore to establish a comparison between GM-complexity and complexity. Note that, for sake of notational simplicity, $M^3 \in \mathcal{C}^{2p}$ ($p \in \mathbb{Z}$) is written in order to indicate a manifold M^3 which admits a rigid crystallization belonging to the catalogue $\mathcal{C}^{2p}$ of all rigid bipartite crystallizations with order $\leq 2p$.

Proposition 7

(a) $c'_{GM}(M^3) \leq c(M^3) + 1 \qquad \forall M^3 \in \mathcal{C}^{28}$;

(b) $c'_{GM}(M^3) = c(M^3) \qquad \forall M^3 \in \mathcal{C}^{26}$;

(c) $c'_{GM}(M^3) = c(M^3) \qquad \forall M^3 \in \mathcal{C}^{28} - \Big\{(\mathbb{RP}^2, (2,1), (3,-1)), \ (\mathbb{S}^2, (3,1), (3,1), (5,-3)), \ (\mathbb{S}^2, (2,1), (2,1), (2,1), (3,-4))\Big\}$.

$\square$

Remark 3. It is an open problem to compute the values (belonging to the set $\{7, 8\}$) of $c'_{GM}((\mathbb{RP}^2, (2,1), (3,-1)))$, $c'_{GM}((\mathbb{S}^2, (3,1), (3,1), (5,-3)))$, $c'_{GM}((\mathbb{S}^2, (2,1), (2,1), (2,1), (3,-4)))$.

Remark 4. For all manifolds, but one, encoded in catalogue $\mathcal{C}^{28}$ and whose GM-complexity and complexity coincide, GM-complexity is realized by a minimal cristallization in the sense of gems (according to the original definition of GM-complexity $c_{GM}(M)$, introduced in [8]). More precisely, if $\mathbb{S}^3/G$ denotes the quotient space of $\mathbb{S}^3$ by the action of group G and $P_{24} =< x, y \mid x^2 = (xy)^3 = y^3, x^4 = 1 >$, the following result holds:
$\forall M^3 \in \mathcal{C}^{28}$ so that $c'_{GM}(M^3) = c(M^3)$, $M^3 \neq \mathbb{S}^3/(P_{24} \times \mathbb{Z}_5)$, then $c'_{GM}(M^3) = c_{GM}(\Gamma)$, with $\#V(\Gamma) \leq \#V(\Gamma')$, $\forall \Gamma'$ representing M^3.

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