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# Pseudo-naïve approaches to investment performance measurement

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# Pseudo-naïve approaches to investment performance measurement

**Carlo Alberto Magni***

**Abstract**

This paper makes use of Magni's (2013. *Insurance Mathematics and Economics*, 53, 747−756) Average Interest Rate (AIR) in order to find a performance index which does not depend on the valuation rate (i.e., benchmark return). To this end, we distort the AIR by dropping the discount factors in the formula. The resulting *modified* AIR (MAIR) is the ratio of overall (undiscounted) return to overall (undiscounted) capital. While seemingly a naïve metric, we show that it is a genuinely internal metric, capable of capturing an investment's economic profitability, as long as it is compared with an appropriate cutoff rate which adequately takes account of the opportunity cost of capital. The not-so naïve MAIR is then extended to several different capital bases; the result is that other well-known (allegedly naïve) metrics, such as cash multiple, undiscounted profitability, Modified Dietz and Simple Dietz return are given economic significance: each such metric is a (*pseudo-naïve*) performance index that correctly expresses the investment's amount of return per unit of a specific capital: overall capital, initial investment, total cash outflow, average cash outflow).

**Keywords**. Finance, investment, performance measurement, rate of return, average interest rate, naïve approach, Modified Dietz.

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# 1 Introduction

In a recent paper, Magni (2013) generalizes Makeham's formula, which is usually applied to financial transactions (e.g. bonds, loans) with constant interest rates (Makeham 1874; Glen 1893; Hossack and Taylor 1975; Ramlau-Hansen 1988; Astrup Jensen 1999a,b). The author supplies a generalized Makeham's formula where interest rates are allowed to vary over time. The formula lends itself to working as a valuation tool and decision criterion and easily reconciles with the Net Present Value (NPV). In particular, the value of interest of an asset is a function of an Average Interest Rate (AIR) which is a capital-weighted mean of the asset's interest rates or, equivalently, the ratio of the discounted value of returns to the discounted value of the outstanding balances. The AIR captures the investment's economic profitability, as long as it is compared with the valuation rate, which represents a benchmark return which discriminate between value creation and value destruction. The approach can be used for both *ex ante* and *ex post* assessment of an investment's performance, and it has been most recently applied to the problem of assessing fund managers' performance (Magni 2014).

A relevant feature of the AIR is that it is a function of the valuation rate. In this respect, this paper has two aims: (i) to derive a metric from the AIR which is unaffected by the valuation rate and which is NPV-consistent, (ii) to use the approach for giving economic significance to naïve approaches which are often used in corporate and financial practice, such as the *undiscounted profitability index*, the *cash multiple*, the *Modified Dietz* return (and some of its variants).

To pursue the first objective, we make use of a replicating strategy whereby an investor periodically deposits or withdraws from the benchmark a monetary amount which is equal to the cash flow generated by the investment under scrutiny. By comparing the returns of the investment and the foregone returns of the replicating asset, a modified AIR (MAIR) is derived, where returns and capital values are not discounted. This means that the metric is not affected by the valuation rate. But it also seems that the time value of money is not taken into account, for values are not discounted. While this seems to make MAIR a naïve metric, the replicating approach we use produces a modified benchmark return (a multiple of the benchmark return), which properly adjusts for the time value of money. We show that the product of the (undiscounted) total capital invested and the active return (difference between MAIR and modified benchmark return) coincides with the *value added* (accumulated NPV). Therefore, the MAIR correctly captures the investment's economic profitability as long as it is compared with the modified benchmark return.

A second aim is to use the MAIR approach for investigating the validity of some metrics which are often used in the corporate and financial practice and are considered naïve and, as such, are shunned (or, at best, only accepted as proxies of other indexes) just because they do not discount cash flows. We will deal with the *undiscounted profitability index*, the *cash multiple*, the *Modified Dietz* return and two variants of it. In particular, we show that all these metrics are variants of the MAIR. And while the latter measures the amount of return per unit of *total invested capital*, the undiscounted profitability index represents an amount of return per unit of *initial investment* and the Modified Dietz return is an amount of return per unit of average cash outflow. If, in a Modified Dietz return, the average cash outflow is replaced by the total cash outflow, the resulting metric represents the amount of return per unit of *total cash outflow*. The cash multiple is shown to express economic profitability per unit of *total cash outflow* as well, albeit in the form of (1+ rate of return).

In essence, all these metrics differ only by the capital base selected, so providing different but consistent pieces of information about the same investment, namely, a return on different capital bases: total capital, average cash outflow, initial investment etc. Each of them is easily reconciled with the investment's value added in the same way as the MAIR: one only needs associates each metric with a modified benchmark return, one which takes into account the capital base it refers to. It is even possible to derive a variant of the MAIR which can be directly compared with the benchmark return, as long as the capital base is the total capital invested in the replicating asset.

As a result, these naïve approaches are, in actual fact, *pseudo*-naïve approaches and, as such, can be fruitfully employed for *ex ante* or *ex post* performance measurement.

The remainder of the paper is structured as follows. In section 2 we summarise Magni's (2013, 2014) AIR approach; in section 3 we present the replicating strategy, and introduce the MAIR and the modified benchmark return, whose relations are stated in Proposition 1, which is the building block of any other result in the paper. Section 4 shows that the naïve (i.e., undiscounted) profitability index is a variant of the MAIR, which refers to a capital base equal to the initial invested capital. Section 5 shows that the Modified Dietz return and its variants are variants of the MAIR as well, either referred to total cash ouflow or average cash outflow. Section 6 shows that the cash multiple provides the same piece of information as that variant of the MD which makes use of the total cash outflow. Section 7 illustrates, by means of an example, how an investment is associated with a return function, which identifies, for each capital base, a performance index. The (not-so) naïve metrics previously presented lie on the graph on the function. Also, a further variant of the MAIR is identified which can be directly compared with the benchmark return for capturing economic profitability. Some concluding remarks end the paper.

## 2 The AIR approach

This section summarises the AIR approach Let $t = 0$ be the current date and $T_0 = \{0, 1, 2, \ldots, n\}$, $T_1 = \{1, 2, \ldots, n\}$. Let $\vec{f} = (f_0, f_1, \ldots, f_n)$ be any investment, where $f_t, t > 1$ represents the cash flows received (if positive) or injected (if negative) in the investment at time $t$. Without loss of generality, we assume $f_0 < 0$, so that $-f_0$ represents the initial investment. For any such investment, the following recursive equation holds:

$$P_t = P_{t-1} + I_t - f_t \tag{1}$$

where $P_t$ represents the capital which remains invested at time $t$ (outstanding capital or outstanding balance), $I_t$ is the return (interest) generated by the investment and $f_t$ is the cash flow distributed to (or contributed by) the investor. The initial and terminal condition of (1) are $P_0 = -f_0$ and $P_n = 0$ respectively. The initial condition may also be written as $K_1 + K_2 + \ldots + K_n = -f_0$ where $K_t = P_{t-1} - P_t$ is the capital repayment.

The economic value of the investment is $\mathcal{V} = \sum_{t \in T_0} f_t(1+r)^{-t}$ where $r$ is the valuation rate, which acts as a minimum required rate of return. It is also known as *cost of capital* or *benchmark return*. We will henceforth use the latter term throughout the paper, for the sake of consistency.[1] The Net Present Value (NPV) is the difference between economic value and investment's cost: $NPV = \mathcal{V} - P_0 = \sum_{t \in T_0} f_t(1+r)^{-t}$. For ex post valuation, the *value added* ($VA$) is often used, which is the accumulated value of $NPV$: $VA = NPV(1+r)^n = \sum_{t \in T_0} f_t(1+r)^{n-t}$.

An investment is economically profitable (i.e. it adds value for the investor) if and only if $NPV > 0$ ($VA > 0$). Let $v_t := (1+r)^{-1}$ be the current value of a monetary unit available at time $t$. Thus, the value of $I_t$ is $\mathcal{I}_t = I_t \cdot v_t$, the value of $K_t$ is $\mathcal{K}_t = K_t \cdot v_t$. As $f_t = I_t + K_t$, the value of any asset can be decomposed into an interest component and a capital component:

$$\mathcal{V} = \mathcal{I} + \mathcal{K}$$

[1] Note that, as is usual in the financial literature, by "benchmark return" we refer to a *rate* of return,.

where $\mathcal{I} := \sum_{t\in T_1} \mathcal{I}_t$ denotes the overall interest and $\mathcal{K} := \sum_{t\in T_1} \mathcal{K}_t$ is the value of the capital repayments.

Makeham's formula deals with loans and bonds: letting $i$ be the (assumed constant) interest rate, the value of interest can be written as

$$\mathcal{I} = \frac{i}{r}(P_0 - \mathcal{K}) \tag{2}$$

so that

$$\mathcal{V} = \frac{i}{r}(P_0 - \mathcal{K}) + \mathcal{K}. \tag{3}$$

Magni (2013) generalizes Makeham's formula by showing that, in case of varying interest rate $i_t, t \in T_1$, (2) is replaced by

$$\mathcal{I} = \frac{\bar{\imath}}{r}(P_0 - \mathcal{K}) \tag{4}$$

where $\bar{\imath}$ is the capital-weighted mean of the holding period rates $i_t$:

$$\bar{\imath} := \alpha_1 i_1 + \alpha_2 i_2 + \ldots \alpha_n i_n \qquad \alpha_t := \mathcal{P}_{t-1}/\mathcal{P} \tag{5}$$

with $\mathcal{P}_{t-1} = P_{t-1} \cdot v_t$ and $\mathcal{P} := \sum_{t\in T_0} \mathcal{P}_t$ (Magni 2013, Proposition 2.1). In other words, the AIR is the weighted mean of the holding period rates, where the weights are given by the respective invested capital.

It can be shown that be $P_0 - \mathcal{K} = r \cdot \mathcal{P}$ so that

$$NPV = (P_0 - \mathcal{K})\big(\frac{\bar{\imath}}{r} - 1\big) \tag{6}$$

(Magni 2013, eq. 13). Hence, any investment is economically profitable (i.e. it increases the investor's wealth) if and only if $\bar{\imath} > r$ (Magni 2013, Proposition 2.2). Note that. owing to the equality $i_t = I_t/P_{t-1}$, the AIR can also be written as a ratio of overall return to overall invested capital:

$$\bar{\imath} = \frac{\mathcal{I}}{\mathcal{P}} = \frac{\mathcal{I}_1 + \mathcal{I}_2 + \ldots + \mathcal{I}_n}{\mathcal{P}_0 + \mathcal{P}_1 + \ldots + \mathcal{P}_{n-1}} \tag{7}$$

which is especially useful whenever return from an investment is nonzero while the invested capital is zero.[2] The AIR does not suffer the difficulties incurred by the Internal Rate of Return (IRR). The latter is an interest rate $x$ such that $\sum_{t\in T_0} f_t(1+x)^{-t} = 0$. As such, multiple roots may arise, whereas the AIR is unique. Also, the

[2]If $P_{t-1} = 0$, then $i_t = I_t/P_{t-1}$ is not defined. However, it may well occur that $I_t \neq 0$ while $P_{t-1} = 0$. This may occur in financial portfolios where long positions and (i.e., investments) and short positions (i.e., borrowings) offset one another (and the return from the net position is nonzero).

IRR cannot generalize Makeham's formula: (4) does not hold if $\bar{\imath}$ is replaced by $x$; that is,

$$\mathcal{I} \neq \frac{x}{r}(P_0 - \mathcal{K}).$$

Furthermore, if the benchmark return is not constant, then IRR is of no use for capturing economic profitability. In contrast, the AIR easily copes with such as situation: $r$ is replaced by

$$\bar{r} = \alpha_1 r_1 + \alpha_2 r_2 + \ldots \alpha_n \tag{8}$$

where $\alpha_t$ is generalized as $\alpha_t := P_{t-1}v_{t,0} / \sum_{t \in T_1} P_{t-1}v_{t,0}$ with $v_{t,0} = (1+r_1)^{-1}(1+r_2)^{-1} \cdot \ldots (1+r_t)^{-t}$, so Makeham's formula is further generalized as

$$\mathcal{I} = \frac{\bar{\imath}}{\bar{r}}(P_0 - \mathcal{K}) \tag{9}$$

(Magni 2013, Proposition 3.1).

It is worth noting that $\bar{\imath} = \bar{\imath}(r)$ is a function of $r$, which is rather natural, given that, in order to aggregate the period returns $I_t$ and the capital values $P_t$, the time value of money must be taken into account. In the following section, we present a modified AIR which does not depend on $r$ but only on the period returns $I_t$ and on the interim values $P_t$.

## 3 The Modified Average Interest Rate

As seen, the AIR takes account of the time value of money via the discount factor $v_t$. In this section, we modify the AIR in such a way that the resulting value does not depend on $r$. The easiest way of getting rid of $r$ is to compute AIR without discounting the returns $I_t$'s and the interim values $P_t$'s. Let $\bar{\jmath}$ be such a *modified* AIR. This means that (7) becomes

$$\bar{\jmath} = \bar{\imath}(0) = \frac{I_1 + I_2 + \ldots + I_n}{P_0 + P_1 + \ldots + P_{n-1}} = \frac{i_1 P_0 + i_2 P_1 + \ldots + i_n P_{n-1}}{P_0 + P_1 + \ldots + P_{n-1}}. \tag{10}$$

The modified AIR (MAIR) only depends on the invested capitals $P_{t-1}$ and the corresponding period returns $I_t$.

At a first sight, this index seems to be a naïve one, just because it does not take into account the time value of money and, for this reason, it seems to be incapable of providing information on the investment's economic profitability. We now show that, contrary to appearances, (10) can be fruitfully used as a performance index.

Consider an investment strategy which replicates $\vec{f}$ from time 0 to time $n-1$. This consists of investing $P_0 = -f_0$ at $r$ and withdrawing or injecting $f_t$ at every date $t = 0, 1, 2, \ldots, n-1$. Letting $P'_t$ be the market value of such a strategy, one finds

$$P'_t = P'_{t-1}(1+r) - f_t \qquad t = 1, 2, \ldots, n \tag{11}$$

which says that, at the beginning of the interval $[t-1, t]$, the interim value $P'_{t-1}$ is invested and generates an interest equal to $r \cdot P'_{t-1}$. The end-of-period market value is then $P'_{t-1}(1+r)$. Subtracting the interim cash flow $f_t$ one finds the market value at the beginning of the period $[t, t+1]$. Denoting as $I'_t := r \cdot P'_{t-1}$ the interest generated in the period $[t-1, t]$, the recursive equation (11) can be written as

$$P'_t = P'_{t-1} + I'_t - f_t \qquad t = 1, 2, \ldots, n. \tag{12}$$

Note that this equation is similar to (1), with $I'_t = r \cdot P'_t$ replacing $I_t = i_t \cdot P_{t-1}$ and $P_0 = P'_0$.

From a cash-flow perspective, the investors receive the same cash flows as $\vec{f}$, but are left, at time $n$, with the terminal value $P'_n$.[3] The cash-flow vector of the replicating strategy, inclusive of the terminal value, is then $\vec{f'} = (f_0, f_1, \ldots, f_n + P'_n)$. Therefore, if the investor undertakes $\vec{f}$, she foregoes $\vec{f'}$. Note that $\vec{f'} = \vec{f} + (0, 0, \ldots, 0, P'_n)$ and that, solving (11), $P'_n = -\sum_{t=0}^{n} f_t(1+r)^{n-t} = -NPV(1+r)^n = -VA$, so $\vec{f'}$ can just be viewed as a replica of the project itself, net of the project's value added. Therefore, the terminal value $P'_n$ measures the opportunity cost of investing in the project: if $P'_n > 0$ ($VA < 0$) the opportunity cost of investing in the project exceeds the benefits, so $\vec{f}$ subtracts value to the investors (economic value is destroyed); if $P'_n < 0$ ($VA > 0$), the benefits are greater than the opportunity cost of investing in the project, and therefore $\vec{f}$ adds value (economic value is created). Denoting as $P := P_0 + P_1 + \ldots + P_{n-1}$ and $P' = P'_0 + P'_1 + \ldots + P'_{n-1}$ the overall invested capital in $\vec{f}$ and in $\vec{f'}$, respectively, we can state the following result.

**Proposition 1.** *The economic value created by $\vec{f}$ is given by*

$$VA = P \cdot (\bar{\jmath} - \bar{\rho}) \tag{13}$$

*where $\bar{\rho} := r \cdot (P'/P)$ is a modified benchmark return which adjusts for investment scale. Therefore, $\vec{f}$ is economically profitable if and only if*

$$\bar{\jmath} > \bar{\rho}. \tag{14}$$

[3] It is worth noting that, in financial economics, a replicating asset's cash flows are equal to the investment's cash flows from time 1 to time $n$, whereas our replicating asset's cash flows are equal to the investment's cash flows from time 0 to time $n-1$.

*Proof.*

$$\begin{aligned}
VA &= -P'_n \\
&= P'_0 - P_0 + P_n - P'_n \\
&= \sum_{t\in T_1}(P_{t-1} - P_t + f_t) - \sum_{t\in T_1}(P'_{t-1} - P'_t + f_t) \\
&= \sum_{t\in T_1} I_t - \sum_{t\in T_1} I'_t \\
&= \bar{\jmath}\cdot P - \sum_{t\in T_1} r\cdot P'_{t-1} \\
&= P(\bar{\jmath} - \bar{\rho}).
\end{aligned} \tag{15}$$

The second part of the proposition follows immediately. □

*Remark* 1. Proposition 1 shows that a seemingly naïve index such as $\bar{\jmath}$ is economically significant, as long as it is contrasted with an adjusted cost of capital which allows for investment scale. The latter is a multiple of the benchmark return, which derives from the idea of comparing two alternative scenarios: in the first one the investor invests in $\vec{f}$, in the second one the investor invests in the replicating asset $\vec{f}'$. The two scenarios not only imply different period rates of return ($i_t$ versus $r$) but also different amounts invested ($P_{t-1}$ versus $P'_{t-1}$). In this framework, the active return in the interval $[t-1, t]$ is $i_t \cdot P_{t-1} - r \cdot P'_{t-1}$.[4] In other words, if one invests in the replicating asset, one invests $P'_{t-1}$ at time $t-1$, not $P_{t-1}$ (see (1) and (11)). So, the interest generated by the replicating asset is $r \cdot P'_{t-1}$. Therefore, the overall interest generated is $r \cdot P'$. However, to earn $r$ on a capital base of $P'$ is equivalent to earn $\bar{\rho}$ on capital base of $P$. Thus, comparing the alternative overall returns, $\sum_{t\in T_1} I_t = \bar{\jmath}\cdot P$ and $\sum_{t\in T_1} I'_t = \bar{\rho}\cdot P$, boils down to comparing $\bar{\jmath}$ and $\bar{\rho}$.

*Remark* 2. It is worth noting that one can define modified benchmark returns period by period as $\rho_t = r\cdot(P'_{t-1}/P_{t-1})$ so that $I'_t = \rho_t \cdot P'_{t-1}$. This implies that that $\bar{\rho}$ is a weighted mean of the $\rho_t$'s, with the same weights as $\bar{\jmath}$ (so justifying the overbar in the symbol):

$$\bar{\rho} = \frac{\rho_1 P_0 + \rho_2 P_1 + \ldots + \rho_n P_{n-1}}{P_0 + P_1 + \ldots + P_{n-1}}. \tag{16}$$

Essentially, the modified benchmark return $\bar{\rho}$ is, formally, equivalent to the MAIR, with $\rho_t$ replacing $i_t$. The meaning should now be evident: in any given period, the investor earns $i_t$ while renouncing to earn $\rho_t$ on the same invested capital. Therefore, economic profitability is intuitively captured by the comparison of the weighted mean

[4] Under the AIR framework, the active return is $i_t \cdot P_{t-1} - r \cdot P_{t-1}$.

of the $i_t$'s and the weighted mean of $\rho_t$'s. Furthermore, reframing the modified benchmark returns as a weighted arithmetic mean makes it very easy to supply a generalization to the case of time-variant benchmark returns $r_t$: one can keep on using (16) with $\rho_t$ redefined as $\rho_t = r_t \cdot (P'_{t-1}/P_{t-1})$. Equivalently, it can be easily checked that $\bar{\rho} = \bar{r} \cdot P'/P$.

In the following sections we show that the MAIR approach can be applied to other seemingly naïve metrics: undiscounted profitability index, Modified Dietz return and variants, and the cash multiple. Reinterpreted within the MAIR framework, they will turn into helpful metrics, each providing a different (but consistent) piece of information on an investment's economic value created.

# 4 Undiscounted Profitability Index

Profitability Index (PI) is a metric which expresses the NPV as per unit of initial investment $P_0$:

$$PI = \frac{NPV}{P_0} = \frac{\sum_{t \in T_0} f_t \cdot (1+r)^{-t}}{P_0}. \tag{17}$$

In the financial and corporate practices, some decision makers use a variant of this index where cash flows are not discounted; we call it *undiscounted PI* ($uPI$):[5]

$$uPI = \frac{\sum_{t \in T_0} f_t}{P_0}. \tag{18}$$

This variant aims at capturing the performance per unit of initial investment but, as it does not take the time value of money into account, it is considered incorrect:

> "A few companies do not discount the benefits or costs before calculating the profitability index. The less said about these companies the better" (Brealey, Myers and Allen 2011, p. 143).

Proposition 1, on which the MAIR approach relies, can be used to give a more robust theoretical status to the $uPI$. To this end, first consider that, for any given cash-flow vector $\vec{f}$, the following equalities hold:

$$\begin{aligned} f_t &= I_t + K_t \\ P_0 &= K_1 + K_2 + \ldots + K_n. \end{aligned} \tag{19}$$

[5] In the following, we will assume that $r$ is constant. All results easily extend to the case of time-variant benchmark returns by replacing $r$ with $\bar{r}$.

Therefore,

$$\sum_{t=1}^{n} I_t = \sum_{t=1}^{n} f_t - \sum_{t=1}^{n} K_t = -P_0 + \sum_{t=1}^{n} f_t = \sum_{t=0}^{n} f_t. \tag{20}$$

Equation (10) is then reframed as

$$\bar{\jmath} = \bar{\jmath}(P) = \frac{\sum_{t\in T_0} f_t}{P} \tag{21}$$

where the dependence on the capital base $P$ is highlighted. Note that the numerator is equal to the numerator of the undiscounted PI, so the latter indeed represents an overall return; the only difference between (18) and (21) lies in the fact that, in the latter, $P_0$ is used in place of $P$. Economically, both metrics express an overall return on a capital base, but the capital base is different: the MAIR considers the overall outstanding capital, whereas the $uPI$ considers the initial investment. It can be noted, from the proof of Proposition 1, that (13) does not depend on the interim values $P_t, t = 1, 2, \ldots, n-1$, so the following generalization holds for any $x \neq 0$:

$$\text{VA} = x \cdot (\bar{\jmath}(x) - \bar{\rho}(x)) \tag{22}$$

where

$$\bar{\jmath}(x) = \frac{\sum_{t\in T_0} f_t}{x} \tag{23}$$

is the investment rate of return as referred to a capital base of $x$, and $\bar{\rho}(x) = r\cdot(P'/x)$ is the corresponding modified benchmark return; $\bar{\jmath}(x)$ and $\bar{\rho}(x)$ represent the return functions as referred to $\vec{f}$ and $\vec{f}'$ respectively. If one picks $x = P_0$, one gets the $uPI$: $\bar{\jmath}(P_0) = \sum_{t\in T_1} f_t/P_0$. In other words, $uPI$ and MAIR are just different values taken on by the same function $\bar{\jmath}(x) = \sum_{t\in T_0} f_t/x$, which measures the investment total return (= net cash flow) for different capital bases. Proposition 1 can then be reformulated in terms of initial investment, so turning a naïve metric into an economically significant one.

**Proposition 2.** *The undiscounted PI is a variant of the MAIR, such that return is expressed as per unit of initial investment:*

$$uPI = \bar{\jmath}(P_0).$$

*A direct relation with VA is established:*

$$VA = P_0 \cdot (uPI - \bar{\rho}(P_0)),$$

*so the investment is economically profitable if and only if* $uPI > \rho(P_0)$, *where* $\bar{\rho}(P_0) = r \cdot \frac{P'}{P_0}$. *Also,*

$$PI(1+r)^n = uPI - \bar{\rho}(P_0)$$

# 5 Modified Dietz Return and its variants

In investment performance measurement, analysts often use the IRR for assessing fund or portfolio's performance. However, given that the computation of IRR usually requires an iterative trial-and-error procedure, they sometimes rely on an approximation of it, called *Modified Dietz* return (*MD*) (see Spaulding 2011; Fischer and Wermers 2013):

> the IRR is solved iteratively (i.e., by trial and error). As noted above, a rule-of-thumb is to start with the result you'd obtain by using Modified Dietz formula (which serves as the "first order" approximation to the IRR). (Spaulding 2011, pp. 98-99).[6]

Let $V_0$ be the amount initially invested by the client and $V_n$ be the terminal value of the fund; let $F_t > 0$ (<) represents a contribution to (withdrawal from) the fund. The MD return is calculated as

$$MD = \frac{V_n - V_0 - F}{V_0 + F^*} \tag{24}$$

where $F := \sum_{t=1}^{n-1} F_t$ and $F^* := \sum_{t \in T_1} \frac{n-t}{n} \cdot F_t$. Essentially, this index aims at computing the net cash flows per unit of *average cash outflow*, where the latter is obtained as the linearly time-weighted mean of the exogenous cash flows.[7]

For example, consider the investment of 100 in a fund, followed by a deposit of 20 after two periods and a withdrawal of 10 after six periods. Assume the investor liquidates the investment after nine periods and the ending value is 120. Thus, $n = 9$, $V_0 = 100, V_9 = 120$, $F_2 = 20, F_6 = -10$, $F_t = 0$ for $t \in T_0 - \{2, 6\}$, so that

$$MD = \frac{120 - 100 - 20 + 10}{100 + \left(20 \cdot \frac{7}{9} - 10 \cdot \frac{3}{9}\right)} = 0.891.$$

In percentage terms, $MD = 8.91\%$.

It is easy to see that *MD* is just the value taken on by $\bar{\jmath}(x)$ at $x = V_0 + F^*$. To this end, just consider that deposits and withdrawals are equal to the investment's interim cash flows changed in sign, that is, $F_t = -f_t$, $t = 1, 2, \ldots, n-1$. Also, the first cash flow is $f_0 = -V_0$ and the last cash flow is $V_n = f_n$. Therefore, $V_n - V_0 - F = \sum_{\in T_0} f_t$. As $P_0 = -f_0$ and, remembering (22), the following result holds.

[6] *MD* is also used as an approximation of the so-called Time-Weighted Rate of Return by chain-linking a series of Modified Dietz returns (Spaulding 2011, p. 92).

[7] Note that $V_0$ is the first term of the mean and its weight is 1.

**Proposition 3.** *The Modified Dietz return is a variant of the MAIR corresponding to the return expressed per unit of average cash outflow* $P^* = P_0 + F^*$*:*

$$MD = \bar{\jmath}(P^*).$$

*A direct relation with VA is established:*

$$VA = P^* \cdot \big(MD - \bar{\rho}(P^*)\big)$$

*where* $\bar{\rho}(P^*) = r \cdot \frac{P'}{P^*}$*. Hence, the investment is economically profitable if and only if* $MD > \bar{\rho}(P^*)$*.*

A variant of the MD return is sometimes used when the magnitude of the cash flows and the linear time-weighting makes the average cash outflow excessively low. In this case, time-weighting is dropped and all contributions are moved to the beginning of the interval and all withdrawals are moved to the end. In this case, the MD return is modified as

$$MD^m = \frac{V_n - V_0 - F}{F^+} \tag{25}$$

where $F^+ := V_0 + \sum_{t:F_t>0} F_t$. In this case, the MD return is the ratio of net cash flow to the total cash flow injected into the fund. Therefore, $MD^m$ represents the overall return referred to a capital base equal to the total cash flow investment. Proposition 3 can be employed with $F^+$ replacing $P^*$, producing the following corollary.

**Corollary 1.** *The variant of MD with no weighting (eq.* (25)*) is a variant of the MAIR expressed as per unit of total cash flow contributed:*

$$MD^m = \bar{\jmath}(F^+).$$

*A direct relation with VA is established:*

$$VA = F^+ \cdot \big(MD^m - \bar{\rho}(F^+)\big)$$

*and the investment is economically profitable if and only if* $MD^m > \bar{\rho}(F^+)$ *where* $\bar{\rho}(F^+) := rP'/F^+$*.*

Note that if $F_t \geq 0$ for every $0 < t < n$ (i.e., no contributions are made after the initial investment $V_0$), the undiscounted PI coincides with $MD^m$ and Proposition 2 conicides with Corollary 1.

Another variant of the MD is the Simple Dietz (SD) which is just a MD which assume that deposits and withdrawals occur in the middle of the interval $[0, n]$ so that the weight is 0.5 for all interim cash flows. Formally,

$$SD = \frac{V_n - V_0 - F}{P_0 + \frac{F}{2}}. \tag{26}$$

Therefore, $SD$ is but a variant of MAIR with a still different capital base.

**Corollary 2.** *SD is a variant of MAIR corresponding to the return expressed per unit of average cash outflow, under the assumption that interim cash flows occur in the middle of the period* $[0, n]$*. The same conclusions as made for Corollary 1 hold, with SD and* $P_0 + F/2$ *replacing* $MD^m$ *and* $F^+$*, respectively.*

## 6 Cash Multiple

Another rule of thumb sometimes employed by managers and practitioners is to divide inflows by outflows:

> "A naïve approach that is often used is to divide the inflows by the outflows [...] This formula does not work in a multiperiod setting. [...] The naïve approach – because it does not properly take into account the timing of the cash flows – is not a correct measure of return" (Rao, 1992, pp. 74–75)

This index is often called *Cash Multiple* (CM).

> "Real-world practitioners often use IRR and the *cash multiple* (or multiple of money) as alternative valuation metrics. [...] The cash multiple (also called the multiple of money or absolute return) is the ratio of the total cash received to the total cash invested. [...] The cash multiple is a common metric used by investors in transactions such as this one. It has an obvious weakness: The cash multiple does not depend on the amount of time it takes to receives the cash" (Berk and De Marzo, 2011, p. 663).

Formally, letting $f^- = -\sum_{t:f_t<0} f_t$ be the total cash invested (outflows) and $f^+ = \sum_{t:f_t>0} f_t$ be the total cash received (inflows), the CM is computed as

$$CM = \frac{f^+}{f^-}. \tag{27}$$

Consider, for example, $\vec{f} = (-40, 50, -60, 120)$. This means $f^+ = 170, f^- = 100$, so that

$$CM = \frac{170}{100} = 1.7.$$

Let us compute $MD^m$ for this investment as well. As $V_0 = 100$, $V_3 = 120$, $F_1 = -50$, $F_2 = 60$ (so that $F = 10$) and $F^+ = 100$, one gets

$$MD^m = \frac{120 - 40 - 10}{100} = 0.7.$$

Note that $CM = 1.7 = 1 + 0.7 = DM^m$. This is not an exception; the result holds in general, given that $V_n - V_0 - F = \sum_{t\in T_0} f_t = f^+ - f^-$ and $F^+ = f^-$, whence

$$1 + DM^m = 1 + \frac{V_n - V_0 - F}{F^+} = 1 + \frac{f^+ - f^-}{f^-} = \frac{f^+}{f^-} = CM.$$

Therefore, $CM$ essentially supplies the same piece of information as $DM^m$; both refer to the total cash outflow but the latter has the the form of a rate of return, the former has the form of (1+rate of return).

**Corollary 3.** *The Cash Multiple is equal to* $1 + DM^m$. *Therefore, the investment is economically profitable if and only if* $CM > 1 + \bar{\rho}(F^+)$*, given that* $f^- = F^+$. *Evidently,*

$$VA = F^-\big(CM - \bar{\rho}(F^+)\big).$$

Note that, if $F^+ = P_0$ (i.e. the only contribution is the initial investment), then $CM = 1 + uPI$.

# 7 The pseudo-naïve metrics

Consider an investment of €100 in a portfolio, followed by withdrawals of €45, €75 after one periods and three periods, respectively, and by further injections of €40 and €25 after two periods and four periods. After five periods, the portfolio value is €60 and the investment is liquidated. We assume that the portfolio's holding period rates are $i_1 = 3.2\%, i_2 = -5.4\%, i_3 = 10.8\%, i_4 = -21.5\%, i_5 = 22.9\%$ and that the benchmark return is $r = 3\%$. Table 1 collects the investment's cash flows $f_t$ $(= -F_t)$, the investment's interim values $P_t$ and the replicating asset's values $P'_t$. The overall return, equal to the net cash flows, is €15. This means that the investment's return function is $\bar{\jmath}(x) = 15/x$. As $P' = 339$, the benchmark's return function is $\bar{\rho}(x) = 0.03 \cdot 339/x = 10.17/x$, where $x$ represents the capital base. The metrics studied in the previous sections are just different values taken on by the investment's return function for different choices of the capital base. Therefore, they are consistent pieces of information:

- MAIR is equal to $\bar{\jmath}(332) = 4.51\%$; it represents the return on the €332 total invested capital
- $MD^m$ is equal to $\bar{\jmath}(165) = 9.09\%$; it represents the return on the €165 total cash outflow

- $CM$ is equal to $1+\bar{\jmath}(165)=1.0909$; it has the same meaning as $MD^m$, but, instead of a percentage, it is expressed as an accumulation factor
- undiscounted $PI$ is $\bar{\jmath}(10)=15\%$ and represents the return on the €100 initial investment
- $MD$ is $\bar{\jmath}(63)=23\%$ and represents the return on the €63 average cash outflow
- $SD$ is $\bar{\jmath}(42.5)=35.29\%$ and represents the return on the €42.5 average cash outflow, under the assumption that cash flows occur in the middle of period $[0,n]$.

(see Table 2). Note that the value of the metrics decreases as the capital base $x$ increases, given the hyperbolic shape of the return function. The same occurs to the modified benchmark rates. Consistency is guaranteed by (22): for each of the metrics computed, the product of the active return and the capital base supplies the value added, which is equal to the terminal value of the replicating asset, changed in sign: $VA=-P'_5=4.83$.[8]

Table 1: Relevant data

| $t$ | $f_t$ | $i_t$ | $P_t$ | $P'_t$ |
|---|---|---|---|---|
| 0 | −100 | | 100 | 100 |
| 1 | 45 | 3.2% | 58.2 | 58 |
| 2 | −40 | −5.4% | 95.1 | 99.7 |
| 3 | 75 | 10.8% | 30.3 | 27.7 |
| 4 | −25 | −21.5% | 48.8 | 53.6 |
| 5 | 60 | 22.9% | 0 | −4.83 |
| TOTAL | 15 | | 332 | 339 |

It should now be evident that many other capital bases can be selected; any pair $(x,\bar{\jmath})$ lies on the graph of the return function $\bar{\jmath}(x)$, and any pair $(x,\bar{\rho})$ lies on the graph of $\bar{\rho}(x)$ (see Figure 1). They all have the common feature that the numerator is, invariably, equal to the net cash flow (inflows minus outflows). Only the capital

[8]This is obviously equal to the accumulated sum of the investment's cash flows:

$$4.83=-100(1.03)^5+45(1.03)^4-40(1.03)^2+75(1.03)^2-25(1.03)+60=NPV(1.03)^5.$$

Table 2: Rates of return and modified benchmark returns

| | MAIR | $MD^m$ | CM | $uPI$ | $MD$ | $SD$ |
|---|---|---|---|---|---|---|
| $x$ | 332 | 165 | 165 | 100 | 63 | 42.5 |
| $\bar{\jmath}(x)$ | 4.51% | 9.09% | 1.0909 | 15.00% | 23.81% | 35.29% |
| $\bar{\rho}(x)$ | 3.06% | 6.16% | 1.0616 | 10.17% | 16.14% | 23.93% |
| $\bar{\jmath}(x) - \bar{\rho}(x)$ | 1.45% | 2.93% | 2.93% | 4.83% | 7.66% | 11.36% |
| $VA$ | 4.83 | 4.83 | 4.83 | 4.83 | 4.83 | 4.83 |

base changes. All these metrics do not discount cash flows, and, for this reason, a comparison of any $\bar{\jmath}(x)$ with $r$ does not guarantee correct information on the increase in wealth. But the role of the replicating asset is just that of providing a tool for taking the time value of money into account (via the ratio of $P'$ to $x$) and enables the construction of the modified benchmark return $\bar{\rho}(x)$ which is homogenous (and, therefore, comparable) to $\bar{\jmath}(x)$. In such a way, there naïveté disappears and they are turned into economically significant indexes: any $\bar{\jmath}(x)$ correctly informs how much return has been generated per unit of capital $x$ and the active return $\bar{\jmath}(x)-\bar{\rho}(x)$ tells how much value has been added per unit of capital $x$. These *pseudo-naïve* metrics, have a simple, intuitive connection with the investment's value added, as this is just equal to the product of the metric's capital base and the active return.

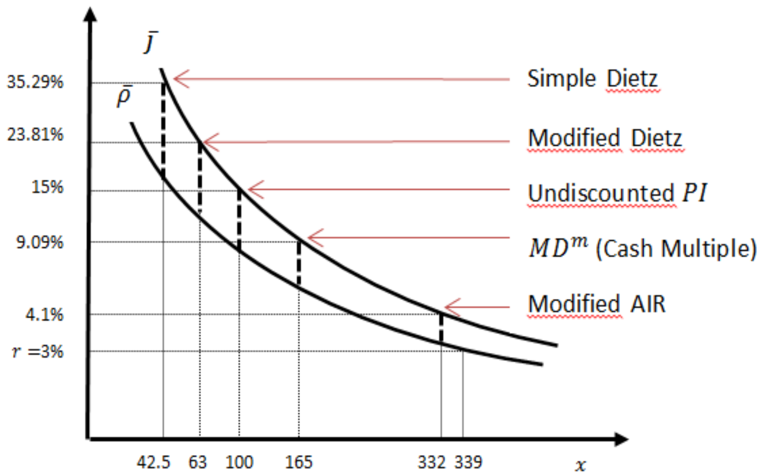


Figure 1: The return function and the pseudo-naïve metrics.

A nice byproduct of this approach is that, if the capital selected is $P'$, the modified benchmark return is just $r$: $\bar{\rho}(P') = rP'/P' = r$ (see also Figure 1). This leads to the following result.

**Corollary 4.** *The pseudo-naïve metric* $\bar{\jmath}(P')$ *is the only variant of the MAIR which captures value creation via a direct comparison with the benchmark return* $r$*. In particular,*

$$VA = P' \cdot \big(\bar{\jmath}(P') - r\big) \tag{28}$$

*and the investment is economically profitable if and only if* $\bar{\jmath}(P') > r$*.*

In the above example, $P' = 339$, so that $\bar{\jmath}(339) = 4.42\%$.

*Remark* 3. Note that, unlike the other metrics, $\bar{\jmath}(P')$ is a function of $r$, as $P'$ is itself a function of $r$. This makes it similar to the AIR. More precisely, one can appreciate the difference between $\bar{\jmath}(P')$ and $\bar{\imath}$ by rewriting them in a comparable form. From (6), remembering that $r\mathcal{P} = P_0 - \mathcal{K}$, one gets

$$\bar{\imath} = r + \frac{\sum_{t\in T_0} f_t v_t}{\mathcal{P}}.$$

In contrast,

$$\bar{\jmath}(P') = r + \frac{\sum_{t\in T_0} f_t - rP'}{P'}.$$

As $\sum_{t\in T_0} f_t - rP' = VA = NPV(1+r)^n$, one can write

$$\bar{\imath} = r + \frac{NPV}{\mathcal{P}}$$

and

$$\bar{\jmath}(P') = r + \frac{NPV}{v_n P'}.$$

The two indexes only differ by the capital base. Therefore, $\bar{\jmath}(P')$ is, at the same time, a variant of the AIR and a variant of the MAIR.

# 8 Concluding remarks

A generalization of Makeham's formula, based on an average-based approach (Magni 2013, 2014), leads to the *average interest rate* (AIR). Beside assessing the value of interest of an asset, the AIR is useful for performance measurement as well: compared with benchmark return, it correctly captures an investment's economic profitability. The AIR is computed as the weighted mean of the investment's discounted capital values and its value is affected by the benchmark return. In many circumstances, the analyst searches for an endogenous metric, that is, a metric which only depends on the investments made and the returns earned by the investments. In this paper, we just considered a modified AIR (MAIR) where capital values are not discounted, so that the resulting metrics only depend on the investment's incomes and capitals, not on the benchmark return. By using a replicating strategy, we showed that the seemingly naïve MAIR properly measures wealth creation or destruction, as long as it is compared with a modified benchmark return which takes account of the difference between the capital base of the investment under examination and the total capital that would be invested in the replicating strategy.

While the MAIR is a return on a total capital invested, we extend it to other capital bases, finding that it turns into metrics which are often used in corporate or financial practice, and which are considered rules of thumb or, at best, proxies of more appropriate indexes. In particular, we showed that the undiscounted profitability index is just a MAIR where the capital base is the initial investment; the Modified Dietz return is the MAIR that would result from considering the average cash outflow as the capital base; a widely used variant of MD considers total cash outflow as the capital base, so it is, again, a MAIR with a further different capital base; the Cash Multiple uses the same capital base as the variant of MD. The Simple Dietz method is another variant of MD, and, as such, a variant of the MAIR.

All these metrics are just different values taken on by the investment's return function, which is defined as the ratio of total return (or, equivalently, net cash flow) to any measure of capital. Depending on the notion of capital selected (total capital, average capital, total cash outflow, initial investment etc.), the various metrics are generated, which supply different but consistent pieces of information about an investment's economic profitability.

We showed that the naïveté of all such metrics is fictitious, as long as one supplies the appropriate cutoff rate (minimum required rate of return), which is able to discriminate between value creation and value destruction. They are all variants of the MAIR, lying on the graph of the same return function: associated with its proper modified benchmark return, each such metric becomes a pseudo-naïve metric.

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