www.cefin.unimore.it

CEFIN Working Papers
No 21

Average Internal Rate of Return and investment decisions: A new perspective

by Carlo Alberto Magni

February 2010

CEFIN – Centro Studi di Banca e Finanza
Dipartimento di Economia Aziendale – Università di Modena e Reggio Emilia
Viale Jacopo Berengario 51, 41100 MODENA (Italy)
tel. 39-059.2056711 (Centralino) fax 059 205 6927

Average Internal Rate of Return and investment decisions: A new perspective

Carlo Alberto Magni
Department of Economics, University of Modena and Reggio Emilia
CEFIN – Center for Research in Banking and Finance
magni@unimo.it

Abstract. The internal rate of return (IRR) is often used by managers and practitioners for investment decisions. Unfortunately, it has serious flaws: (i) multiple real-valued IRRs may arise, (ii) complex-valued IRRs may arise, (iii) the IRR is, in general, incompatible with the net present value (NPV) in accept/reject decisions (iv) the IRR ranking is, in general, different from the NPV ranking, (v) the IRR criterion is not applicable with variable costs of capital. The efforts of economists and management scientists in providing a reliable project rate of return have generated over the decades an immense bulk of contributions aiming to solve these shortcomings. This paper offers a complete solution to this long-standing issue by changing the usual perspective: the IRR equation is dismissed and the evaluator is allowed to describe the project as an investment or a borrowing at his discretion. This permits to show that *any* arithmetic mean of the one-period return rates implicit in a project reliably informs about a project's profitability and correctly ranks competing projects. With such a measure, which we name "Average Internal Rate of Return", complex-valued numbers disappear and all the above mentioned problems are wiped out. The economic meaning is compelling: it is the project return rate implicitly determined by the market. The traditional IRR notion may be found back as a particular case.

Keywords. Decision analysis, investment criteria, capital budgeting, internal rate of return, investment stream, market rate, mean.

Introduction

The inception of the internal rate of return (IRR) traces back to Keynes (1936) and Boulding (1935, 1936a,b). This index is massively used as a tool for decision-making by scholars, managers, analysts, practitioners, and is taught to every student of any business and management school. The IRR decision criterion suggests to accept a project if and only if the IRR is greater than the cost of capital (usually, the market rate) and to rank competing projects via their IRRs: the higher a project IRR the higher its rank. Unfortunately, the IRR gives rise to serious conceptual and technical problems: (i) a real-valued IRR may not exist, so that the comparison with the cost of capital is not possible; (ii) multiple IRRs may arise, in which case the above mentioned comparison is problematic; (iii) compatibility with the Net Present Value (NPV) is not guaranteed, not even if the IRR is unique;[1] (iv) the IRR ranking is not equivalent to the NPV ranking; (v) the IRR may not be used if the cost of capital is variable over time. The economic and managerial literature has thoroughly investigated the IRR shortcomings and a huge amount of contributions in the past eighty years have been devoted to searching for corrective procedures capable of healing its flaws (e.g. Boulding 1935, 1936b, Samuelson 1946; Lorie and Savage 1955; Solomon 1956; Hirshleifer 1958; Pitchford and Hagger 1958; Bailey 1959; Karmel 1959; Soper 1959; Wright 1959; Kaplan 1965, 1967; Jean 1968; Arrow and Levhari 1969; Adler 1970; Ramsey 1970; Norstrøm 1967, 1972; Flemming and Wright 1971; Fairley and Jacoby 1975; Aucamp and Eckardt 1976; Bernhard 1967, 1977, 1979, 1980; De Faro 1978; Herbst 1978; Ross, Spatt and Dybvig 1980; Dorfman 1981; Cannaday, Colwell and Paley 1986; Gronchi 1986; 1987; Hajdasinski 1987, 2004; Promislow and Spring 1996; Tang and Tang 2003; Rocabert, Tarrío and Pérez 2005; Zhang, 2005; Kierulff 2008; Simerská 2008).

In particular, Pitchford and Hagger (1958), Soper (1959), Kaplan (1965, 1967), Gronchi (1986) individuate classes of projects having a unique real-valued IRR in the interval $(-1, +\infty)$. Jean (1968), Norstrøm (1972), Aucamp and Eckardt (1976), De Faro (1978), Bernhard (1979, 1980) individuate classes of projects with a unique IRR in the interval $(0, +\infty)$. Karmel (1959), Arrow and Levhari (1969), Flemming and Wright (1971), Ross, Spatt and Dybvig (1980) use the

[1] For example, the cash flow stream $(-4, 12, -9)$ has a unique IRR equal to 50%. According to the IRR criterion, the project must be accepted if the market rate is smaller than 50%, but the NPV is negative for any rate different from 50%, so the project is not worth undertaking. (Note that this example implicitly introduces a further class of problems: if a project is not unambiguously individuated as either an investment or a borrowing, the IRR profitability rule introduced in section 1 below is ambiguous.)

assumption of project truncability in order to make the IRR unique. Teichroew, Robichek and Montalbano (1965a,b) and Gronchi (1987) circumvent the IRR problems by using a pair of different return rates applied to the project balance depending on its sign (one of which is the market rate itself). The notion of *relevant* internal rate of return has been studied by Cannaday, Colwell and Paley (1986), Hajdasinski (1987) Hartman and Schafrick (2004). Issues related to the reinvestment assumptions in the IRR criterion and the adoption of the Modified Internal Rate of Return have been analyzed in several contributions, among which Lorie and Savage (1955), Lin (1976), Lohmann (1988), Hajdasinski (2004), Kierulff (2008) (see also the historical perspective of Biondi 2006).

No complete solution to the issue has so far appeared in the literature. Among the proposals, Hazen's (2003, 2009) approach stands out for the insights it gives on the problem. The author makes use of the notion of 'investment stream', which is the sequence of capitals periodically invested in the project. He shows that the problems of uniqueness and nonexistence of the IRR are overcome by considering that any IRR is univocally associated with its corresponding investment stream. One just has to compare the real part of the (possibly complex-valued) IRR with the market rate, and a positive sign signals profitability if the project is a net investment or value destruction if the project is a net borrowing.

However, three issues remain unsolved:

- complex-valued return rates and complex-valued capitals are devoid of economic meaning
- project ranking with the IRR is not compatible with NPV ranking
- the IRR may be applied only if the cost of capital is constant.

This paper offers a complete solution to the long-standing issue by providing a generalization of the IRR notion. The basic idea is that the capital invested in a project is arbitrarily selected by the evaluator, which means that the evaluator has complete freedom as for the financial interpretation of the project: any project may be seen, at the same time, as a net investment or as a net borrowing of any monetary amount. Any sequence of capitals (investment stream) univocally determines a sequence of one-period IRRs (internal return vector). The corresponding arithmetic mean is shown to represent an unfailing economic yield, here named 'Average Internal Rate of Return' (AIRR). This return rate correctly signals desirability of a project and correctly ranks any bundle of competing projects. The traditional IRR is obtained back as a particular case corresponding to a well-determined class of investment streams.

The approach purported in this work is computationally very simple and gets rid of complex-valued roots of polynomials; it admits of a straightforward economic interpretation as the project is reduced to its basic ingredients: (i) capital invested, (ii) average internal rate of return, (iii) cost of capital. The AIRR may then be interpreted as the unique real-valued rate of return on the capital invested in the project.

Three fundamental notions are brought into play to achieve the results:

I. the notion of *Chisini mean* (Chisini 1929; Graziani and Veronese 2009)

II. the notion of *return* (or *income*) alongside its relation with capital and cash flows (e.g. Teichroew, Robichek and Myers 1965a,b; Lee 1985; Brief and Peasnell 1996)

III. the notion of *residual income* (Preinreich 1936, 1937, 1938; Edwards and Bell 1961; Solomons 1965; Peasnell, 1982; Ohlson 1995; Egginton 1995; Magni 2009a).

The paper is structured as follows. Section 1 presents the mathematical notation and provides the notions of *investment stream* and *internal return vector* along with the notion of *return* as well as the recurrence equation for capital. Section 2 summarizes the approach of Hazen (2003) which essentially consists of deriving investment streams from the project's IRRs. Section 3 deals with accept/reject decisions: the IRR equation is dismissed and complex-valued numbers are swept away: the AIRR is derived as a real-valued rate of return from a linear equation. Hazen's decision criterion follows as a particular case of the AIRR criterion. Section 4 underlines that even a simple arithmetic mean of the period rates suffices to produce the correct result to the accept/reject problem. Section 5 deals with the economic interpretation of the AIRR: it is shown that it may be interpreted as the rate of return of the project obtained from the original one by reframing it as a one-period project and, in particular, as the rate of return of the capital actually invested in the project. Section 6 shows that the AIRR correctly ranks a bundle of projects: the AIRR ranking is the same as the NPV ranking. Some concluding remarks end the paper.

1. Mathematical notation and preliminary results

A *project* or *cash flow stream* is a sequence $\boldsymbol{x} = (x_0, x_1, x_2, \dots, x_T) \in \mathbb{R}^{T+1}$ of cash flows (monetary values). The net present value (NPV) of project $\boldsymbol{x}$ is

$$PV(\boldsymbol{x} \mid r) = \sum_{t=0}^{T} x_t \cdot (1+r)^{-t}$$

where $r > -1$ is the market rate.[2] The net future value (NFV) is the future value of $PV(\boldsymbol{x} \mid r)$ at some future date:

$$PV_t(\boldsymbol{x} \mid r) := PV(\boldsymbol{x} \mid r)(1+r)^t \qquad\qquad t \geq 1.$$

We say that a project is profitable (or is worth undertaking) if and only if $PV(\boldsymbol{x} \mid r) > 0$. Evidently, this is equivalent to $PV_t(\boldsymbol{x} \mid r) > 0$ for every t.

An internal rate of return for project x is a constant rate $k \neq -1$ such that $PV(\boldsymbol{x} \mid k) = 0$ or, which is the same, $PV_T(\boldsymbol{x} \mid k) = \sum_0^T x_t \cdot (1+r)^{T-t} = 0$. The IRR profitability rule may be stated as follows:

> *If the project is an investment, it is profitable if and only if $k > r$; if the project is a borrowing, it is profitable if and only if $k < r$.*

Let $c_t \in \mathbb{R}$, $t = 0,1,\dots,T$, and let

$$R_t := c_t - c_{t-1} + x_t, \qquad t = 1,\dots,T, \qquad \text{with} \quad c_0 = -x_0, \qquad c_T = 0. \tag{1a}$$

The term c_{t-1} represents the capital invested (or borrowed) in the period $[t-1, t]$, so the term R_t is the *return* generated by the project in that period. If $c_{t-1} \neq 0$, eq. (1a) may be framed as

$$c_t = c_{t-1}(1+k_t) - x_t, \qquad t = 1,\dots,T, \qquad \text{with} \quad c_0 = -x_0, \qquad c_T = 0 \tag{1b}$$

where $k_t := R_t / c_{t-1}$ is the period rate of return. Equation (1b) may be economically interpreted in the following way: at the beginning of every period, the capital c_{t-1} is invested (or borrowed) at the return rate k_t. The capital increases by the return generated in the period but decreases by the amount cash flow x_t, which is paid off to (or by) the investor. The return is often called *income* in business economics and accounting (Lee 1985; Penman 2010); the capital c_t, $t = 0,1,\dots,T$, is also

[2] The approach is compatible with a bounded-rationality perspective: in this case r is a subjective threshold (Magni 2009b).

known as project balance (Teichroew, Robichek and Montalbano 1965a,b), outstanding capital (Lohmann 1988; Peccati 1989; Gallo and Peccati 1993), unrecovered capital (Lohmann, 1988), unrecovered balance (Bernhard, 1962; Hajdasinski, 2004), and may also be interpreted as the *book value* of the project (therefore, the return rate k_t is interpretable as an accounting rate of return). Equation (1) is called 'clean surplus relation' in accounting (see Brief and Peasnell 1996).

Any vector $\boldsymbol{c} = (c_0, c_1, \dots, c_{T-1}) \in \mathbb{R}^T$ satisfying (1) is here labeled 'investment stream'.

Consider one-period project $\boldsymbol{x^t} = (\boldsymbol{0_{t-1}}, -c_{t-1}, c_t + x_t, \boldsymbol{0_{T-t}}) \in \mathbb{R}^{T+1}$ where $\boldsymbol{0_k} \in \mathbb{R}^k$ is the null vector. Any such project represents an investment (or borrowing) of amount c_{t-1} at time $t-1$, which generates an end-of-period payoff equal to $(c_t + x_t)$, $t = 1, 2, \dots, T$. Given that $c_0 = -x_0$ and $c_T = 0$, the equality

$$\sum_{t=1}^{T} \boldsymbol{x^t} = \boldsymbol{x}$$

holds, irrespective of the investment stream $\boldsymbol{c} \in \mathbb{R}^T$. Any project may then be viewed as a portfolio of T one-period projects. The possibility of splitting up any multi-period project into T one-period projects is conspicuous: it opens up the opportunity of interpreting k_t as the unique IRR of project $\boldsymbol{x^t}$, for (1b) may be reframed as

$$-c_{t-1} + \frac{c_t + x_t}{1 + k_t} = 0 \tag{2a}$$

or, which is the same,

$$c_{t-1}(1 + k_t) = c_t + x_t \tag{2b}$$

(the latter allows one to accept $k_t = -1$). It is easy to see that that such an equation leads to $c_T = -\sum_{t=0}^{T} x_t(1 + k_{t+1})(1 + k_{t+2}) \cdots (1 + k_T)$. Letting $PV_T(\boldsymbol{x}, \boldsymbol{k}) := \sum_{t=0}^{T} x_t(1 + k_{t+1})(1 + k_{t+2}) \cdots (1 + k_T)$ and using the terminal condition $c_T = 0$, one gets $PV_T(\boldsymbol{x}, \boldsymbol{k}) = 0$, which means that the sequence $\boldsymbol{k} = (k_1, k_2, \dots, k_T) \in \mathbb{R}^T$ of one-period IRRs represents an *internal return vector* (see Weingartner 1966; Peasnell 1982; Peccati 1989; Magni 2009a). There are infinite sequences $\boldsymbol{k}$ of real-valued numbers that satisfy $PV_T(\boldsymbol{x}, \boldsymbol{k}) = 0$; an IRR (if it exists in the real interval) is only a particular case of internal return vector such that all components are constant: $\boldsymbol{k} = (k, k, \dots, k)$.

It is important to underline that the internal return vector $\boldsymbol{k}$ and the investment stream $\boldsymbol{c}$ are in a biunivocal relation: once the capitals are arbitrarily fixed first, the corresponding one-period IRRs are univocally determined. This means that the analyst is free to describe the project with his own preferred investment stream. (Equation (1) may also be read from the internal return vector to the investment stream, which implies that, once the one-period rates are arbitrarily fixed first, the capitals are then univocally derived.)

2. Hazen's (2003) criterion: from IRRs to investment streams

Hazen (2003) focuses on multiple roots drawn from the classic IRR equation. If a real-valued IRR exists, the author considers the investment stream $\boldsymbol{c}(k)=(c_0,c_1(k),c_2(k),\dots,c_{T-1}(k))$ derived from the IRR, so that $c_t(k)=c_{t-1}(k)(1+k)-x_t$. The decision criterion the author proposes may be summarized in the following

Theorem 2.1. Suppose $k\in\mathbb{R}-\{-1\}$ is an IRR of project $\boldsymbol{x}$. Then,

(i) if $PV(\boldsymbol{c}(k)\mid r)>0$, project $\boldsymbol{x}$ is profitable if and only if $k>r$

(ii) if $PV(\boldsymbol{c}\ (k)\mid r)<0$, project $\boldsymbol{x}$ is profitable if and only if $k<r$

where $PV(\boldsymbol{c}\mid r):=\sum_0^{T-1}c_t\,(1+r)^{-t}$ (Hazen 2003, Theorem 4. See also Hazen 2009). The project is a *net investment* if $PV(\boldsymbol{c}\mid r)>0$, a *net borrowing* if $PV(\boldsymbol{c}\mid r)<0$. In the former case, an IRR is a rate of return, whereas in the latter case an IRR is a rate of cost. Theorem 2.1 entails that the analyst should follow the following steps:

a. solve the IRR equation and pick any one of the IRRs
b. compute the corresponding investment stream $\boldsymbol{c}(k)$ and calculate its present value $PV(\boldsymbol{c}(k)\mid r)$ to ascertain its financial nature (investment or borrowing)
c. if the project is an investment (borrowing), accept the project if and only if the IRR is greater (smaller) than the market rate.

This criterion brilliantly solves the problem of multiple roots, because to every root k there corresponds a unique $PV(\boldsymbol{c}(k)\mid r)$. The choice of which root to use is immaterial, for the project NPV may be written as

$$\frac{(k-r)}{1+r}\cdot PV(\boldsymbol{c}(k)\mid r)=PV(\boldsymbol{x}\mid r) \tag{3}$$

(see Hazen, 2003, Theorem 1; Lohman, 1988, eq. (43)). Equation (3) shows that the NPV of the project is obtained as the product of two factors: (i) the (discounted) difference between a project IRR and the market rate, (ii) the present value of the IRR-derived investment stream. The left-hand side of (3) is invariant under changes in the IRR. That is, let k^j and k^p be any two real-valued IRRs and let $\boldsymbol{c}(k^j)$ and $\boldsymbol{c}(k^p)$ be the corresponding investment streams. Then,

$$\frac{k^j - r}{1+r} \cdot PV\big(\boldsymbol{c}(k^j) \mid r\big) = \frac{k^p - r}{1+r} \cdot PV(\boldsymbol{c}(k^p) \mid r) = PV(\boldsymbol{x}, r).$$

This unfolds the opportunity of depicting the project in different ways: for each internal rate of return k^j, $j = 1,2, \dots, T$, the project may be interpreted as a net investment (borrowing) of amount $PV\big(\boldsymbol{c}(k^j) \mid r\big)$ with rate of return (cost) equal to k^j. Far from generating ambiguity, this multiple description of a project is computationally unfailing and economically meaningful: the NPV does not change under changes in the project description.

In a similar vein, the author successfully deals with complex-valued IRRs as well (see his Theorem 5), but the economic significance of the result is obfuscated: "We are currently unaware of an economic interpretation of complex-valued rates of return and complex-valued investment streams, and without such an interpretation it would be hard to justify any economic recommendation without resort to other performance measures such as present value" (p. 44).

Hazen's approach is quite successful in accept/reject decisions, but does not allow for a sufficient degree of freedom, so that complex-valued numbers may not be dismissed and competing projects may not be correctly ranked. In the next sections we show that allowing flexibility on the investment stream solves all the problems: first, one freely chooses an investment stream, then one computes the corresponding one-period return rates k_t, and, finally, calculates the arithmetic mean. This wipes out complex-valued numbers, and Hazen's criterion is derived as a particular case.

3. The use of AIRR in accept/reject decisions

We first provide a generalization of eq. (3).

Lemma 3.1. Consider an arbitrary investment stream $\boldsymbol{c} = (c_0, c_1, c_2, \dots, c_{T-1}) \in \mathbb{R}^T$. Then, the following equality holds:

$$PV(\boldsymbol{x} \mid r) = \sum_{t=1}^{T} (R_t - rc_{t-1}) \cdot (1+r)^{-t} \tag{4}$$

Proof: By eq. (1), $x_t = R_t + c_{t-1} - c_t$ for $t \geq 1$. Also, $(1+r)^{-(t+1)} - (1+r)^{-t} = -r \cdot (1+r)^{-(t+1)}$. Reminding that $c_T = 0$,

$$PV(\boldsymbol{x}\mid r)=\sum\nolimits_1^T R_t(1+r)^{-t}+\sum\nolimits_1^T c_{t-1}\,(1+r)^{-t}-\sum\nolimits_1^T c_t\,(1+r)^{-t}+x_0$$

$$=\sum\nolimits_1^T R_t(1+r)^{-t}+c_0(1+r)^{-1}+\sum\nolimits_1^{T-1} c_t\,(1+r)^{-(t+1)}-\sum\nolimits_1^{T-1} c_t\,(1+r)^{-t}-c_0$$

$$=\sum\nolimits_1^T R_t(1+r)^{-t}-\sum\nolimits_1^{T-1} r\,c_t(1+r)^{-(t+1)}-rc_0(1+r)^{-1}$$

$$=\sum\nolimits_1^T (R_t-rc_{t-1})(1+r)^{-t}.$$

(QED)

The term (R_t-rc_{t-1}) in (4) represents a "residual income", that is, it measures the return in excess of what could be earned by investing the capital c_{t-1} at the market rate r. The notion of residual income is well-known in managerial accounting and value-based management (Edwards and Bell 1961; Peasnell 1982; Egginton 1995; Martin and Petty 2000; Young and O'Byrne 2001; Martin, Petty and Rich 2003; Pfeiffer 2004. Baldenius and Reichelstein 2005; Pfeiffer and Schneider, 2007. See Magni, 2009a, for a review). If $c_{t-1}\neq 0$ for every $t=1,2,\dots,T$, k_t is defined so that (4) may be framed as $PV(\boldsymbol{x}\mid r)=\sum_1^T c_{t-1}(k_t-r)(1+r)^{-t}$. The margin (k_t-r) measures the residual income per unit of capital invested, so we henceforth call it "residual rate of return" (RRR). By replacing k_t with an internal rate of return k, the residual income becomes $c_{t-1}(k-r)$, which Lohmann (1988) labels "marginal return". Lohmannn's marginal return is then a particular case of residual income, and the IRR-determined investment stream $\boldsymbol{c}(k)$ is just one choice of an investment stream $\boldsymbol{c}\in\mathbb{R}^{\boldsymbol{T}}$ among infinite possible ones. In other words, eq. (4) holds whatever the choice of $\boldsymbol{c}\in\mathbb{R}^{\boldsymbol{T}}$ and eq. (3) is only a particular case of it.

Now we search for a *Chisini mean* (Chisini 1929; Graziani and Veronese 2009) of the one-period IRRs. That is, we search for that constant return rate $\bar{k}$ which, replaced to each one-period rates k_t in the residual-income expression, generates the project NPV: from

$$\sum\nolimits_{t=1}^T c_{t-1}(k_t-r)(1+r)^{-t}=\sum\nolimits_{t=1}^T c_{t-1}(\bar{k}-r)(1+r)^{-t}$$

one gets

$$\bar{k}=\frac{\sum_1^T k_t c_{t-1}(1+r)^{-t}}{\sum_1^T c_{t-1}(1+r)^{-t}}=\frac{\sum_1^T k_t c_{t-1}(1+r)^{-(t-1)}}{PV(\boldsymbol{c},r)}. \tag{5}$$

The mean $\bar{k}$ is an average of the one-period IRRs, and the weighs are given by the (discounted) capitals (see also Peccati 1989). We name this mean 'Average Internal Rate of Return' (AIRR).

We are now able to prove the following

Theorem 3.1. For any investment stream $\boldsymbol{c} \in \mathbb{R}^T$,

(i) if $PV(\boldsymbol{c} \mid r) > 0$, project $\boldsymbol{x}$ is profitable if and only if $\bar{k} > r$

(ii) if $PV(\boldsymbol{c} \mid r) < 0$, project $\boldsymbol{x}$ is profitable if and only if $\bar{k} < r$

(iii) project $\boldsymbol{x}$ is value-neutral (i.e. NPV = 0) if and only if $\bar{k} = r$

Proof: Owing to Lemma 3.1 and eq. (5), the equality

$$PV(\boldsymbol{x} \mid r) = \left(\bar{k} - r\right) \sum_{1}^{T} c_{t-1} \cdot (1+r)^{-t} = \frac{\bar{k} - r}{1+r} PV(\boldsymbol{c} \mid r) \qquad (6)$$

holds for any arbitrary investment stream. Hence, the thesis follows immediately. (QED)

Contrasting Theorem 2.1 and Theorem 3.1 from a formal point of view, we note that the margin $(k - r)$ is replaced by the residual rate of return $(\bar{k} - r)$; in other terms, the AIRR replaces the IRR. From a computational and conceptual point of view, a radical departure from Theorem 2.1 is consummated: the latter presupposes that the decision maker solves a T-degree equation in order to find a (real-valued or complex-valued) IRR; hence, the investor univocally determines the investment stream $\boldsymbol{c}(k)$ (and, therefore, $PV(\boldsymbol{c}(k) \mid r)$). In contrast, Theorem 3.1 leaves the decision maker free to choose any desired investment stream $\boldsymbol{c}$, whence an internal return vector $\boldsymbol{k}$ is univocally individuated, and the AIRR is consequently computed.

The average rate $\bar{k}$ is a reliable return rate because Theorem 3.1 just says that the product of $PV(\boldsymbol{c} \mid r)\left(\bar{k} - r\right)$ is invariant under changes in $\boldsymbol{c}$. It is important to stress that the AIRR itself is invariant under changes in $\boldsymbol{c}$, as long as $PV(\boldsymbol{c} \mid r)$ is unvaried. To see it, just consider that (6) implies

$$\bar{k} = \bar{k}\left(PV(\boldsymbol{c} \mid r)\right) = r + \frac{PV_1(\boldsymbol{x} \mid r)}{PV(\boldsymbol{c} \mid r)}$$

which means that the AIRR is a (hyperbolic) function of $PV(\boldsymbol{c} \mid r)$. For any fixed $P \in \mathbb{R}$ the equation $P = \sum_1^T c_{t-1} \cdot (1+r)^{-t}$ has infinite solutions, so any given $PV(\boldsymbol{c} \mid r) \in \mathbb{R}$ is associated with infinitely many investment streams which give rise to the same AIRR. Figure 1 illustrates the graph of the AIRR function for a positive-NPV project. The AIRR is greater (smaller) than the market rate for every positive (negative) $PV(\boldsymbol{c} \mid r)$. The triplet $(PV(\boldsymbol{c} \mid r)$, $\bar{k}$, $r)$ univocally determines the NPV: precisely, $PV(\boldsymbol{c} \mid r)\left(\bar{k} - r\right) = PV(\boldsymbol{x} \mid r)(1+r) = PV_1(\boldsymbol{x} \mid r)$. Graphically, $PV_1(\boldsymbol{x} \mid r)$ is the area of any rectangle with base $\overline{0P}$ and height $\left|\bar{k}(P) - r\right|$. Therefore, a project may always be viewed as either a net investment or a net borrowing of any amount at the decision maker's discretion: the overall economic analysis does not change, given that the NPV does not change: this is actually the reason why the choice of the investment stream is irrelevant. Using the notion of internal return vector and computing the Chisini mean of the one-period rates k_t, no complex-valued roots ever appear: only real numbers come into play, with the precise meaning of return rates. In other words, complex-valued numbers are removed *a priori* so that economic intuition is always preserved.

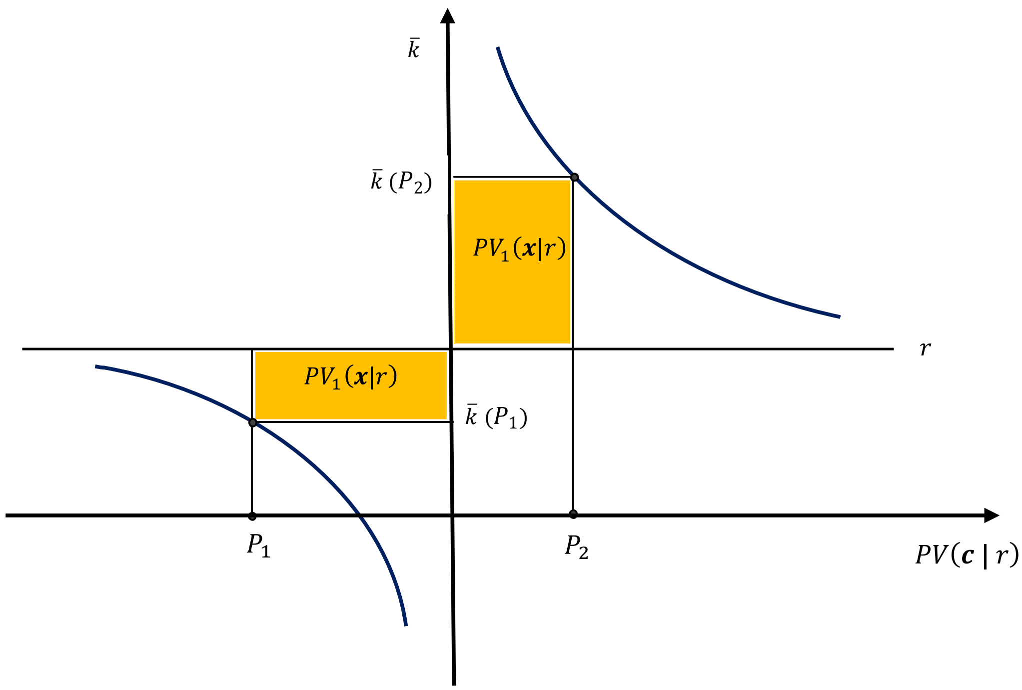

Figure 1. The graph of the AIRR function for a positive-NPV project. No matter which investment stream one chooses, the AIRR is always greater than the market rate for positive $PV(\boldsymbol{c}\mid r)$ and smaller than the market rate for negative $PV(\boldsymbol{c}\mid r)$ (i.e., the project is worth undertaking).

Computationally, the steps an analyst should follow are:

a. arbitrarily pick an investment stream $\boldsymbol{c}$ so that the project is described as either an investment or a borrowing
b. compute the corresponding one-period return rates and their average
c. if the project is an investment (borrowing), accept the project if and only if the AIRR is greater (smaller) than the market rate.

Remark 3.1. While $k_t = \frac{c_t + x_t}{c_{t-1}} - 1$ is not defined if $c_{t-1} = 0$, the AIRR is nonetheless defined, for the return R_t is well-defined for every $c_{t-1} \in \mathbb{R}$. Owing to Lemma 3.1 and the notion of Chisini mean, we may write

$$\bar{k} = \frac{\sum_1^T R_t (1+r)^{-(t-1)}}{PV(\boldsymbol{c} \mid r)} \tag{7}$$

so the AIRR is well-defined even if some capital is equal to zero, as long as the denominator is nonzero. (Obviously, given that the analyst arbitrarily chooses the investment stream, he will always pick one such that $PV(\boldsymbol{c}, r) \neq 0$). In such a way, in order to calculate the AIRR one does not even need compute rates of return, but only returns.

EXAMPLE

Consider the cash flow stream $\boldsymbol{x} = (-10, 30, -25)$ studied by Hazen (2003, p. 44), where a market rate equal to 10% is assumed. The project NPV is $PV(\boldsymbol{x} \mid r) = -3.39$, so the project is not profitable. No real IRR exists, but two complex-valued IRRs exist: $k^1 = 0.5 + 0.5i$ and $k^2 = 0.5 - 0.5i$. Instead of focusing on the complex-valued IRRs and calculating the complex-valued investment streams (whose economic meaning is obscure), one may more conveniently choose, at discretion, an investment stream and then compute the corresponding (real-valued) AIRR. For illustrative purposes, Table 1 collects four arbitrary investment streams. Any of the corresponding AIRRs provides correct information: for example, the first pattern is such that $PV(\boldsymbol{c^1} \mid 10\%) = 4.55 > 0$; this means that the project is framed as a net investment. The AIRR is $\bar{k} = -72\%$, which is smaller than the market rate 10%. Hence, by Theorem 3.1, the project is not worth undertaking. As for the second choice, we find $PV(\boldsymbol{c^2} \mid 10\%) = -8.18 < 0$ so the project is depicted as a net borrowing; by Theorem 3.1, the project is not worth undertaking, because the AIRR (now interpreted as a rate of cost) is $\bar{k} = 55.56\%$, which is greater than the market rate 10%. As for the third pattern, $PV(\boldsymbol{c^3} \mid 10\%) = -15.45 < 0$ so the project is seen as a net borrowing at a rate of cost of $\bar{k} = 34.12\%$, which is greater than the market rate 10%. Again, the project is deemed unprofitable. Analogously for the fourth case, where eq. (7) is used for computing the AIRR. Note that in any possible case the product of the RRR and the present value of the investment stream is invariant under changes in vector $\boldsymbol{c}$: for example, in the first case the RRR is -82%, which, applied to the amount invested 4.55, leads to the time-1 NFV which, discounted by one period, leads back to the NPV: $-82\% \cdot 4.55 \cdot 0.91 = -3.39$. Analogously in any other case.

Table 1. Complex-valued IRRs, real-valued AIRRs

Time	0	1	2	$PV(\boldsymbol{c}\vert\ 10\%)$	AIRR (%)	Market rate (%)
Cash Flows	−10	30	−25			
NPV	−3.39					
$\boldsymbol{c}^1$	10	−6	0			
Period rate		140%	316.7%	4.55	−72.00	10
$\boldsymbol{c}^2$	10	−20	0			
Period rate		0%	25%	−8.18	55.56	10
$\boldsymbol{c}^3$	10	−28	0			
Period rate (%)		−80%	−10.7%	−15.45	34.12	10
$\boldsymbol{c}^4$	10	0	0			
Period rate (%)		200%	undefined	10	−27.27	10

We now show that the IRR is just an AIRR associated with a specific class of investment streams. We first need the notion of PV-equivalent investment streams.

Definition. Two or more investment streams are said to be *PV-equivalent* if they have equal $PV(\boldsymbol{c} \mid r)$.

Consider the class of those investment streams $\boldsymbol{c}$ which are PV-equivalent to $\boldsymbol{c}(k)$; that is, $PV(\boldsymbol{c} \mid r) = PV(\boldsymbol{c}\ (k)\vert\ r)$. We call this class 'Hotelling class' (after Hotelling 1925). This class contains infinite elements, because there exist many infinite vectors $\boldsymbol{c} \in \mathbb{R}^T$ that fulfill the equation $\sum_0^{T-1} c_t(1+r)^{-t} = PV(\boldsymbol{c}\ (k)\vert\ r)$. Now, it is obvious that the AIRR generated by $\boldsymbol{c}(k)$ is k itself, for $\sum_0^{T-1} kc_t(k)(1+r)^{-t} / PV(\boldsymbol{c}(k), r) = k$. But any investment stream contained in the same class generates the same AIRR, since, as seen above, the AIRR does not depend on $\boldsymbol{c}$ as long as $PV(\boldsymbol{c} \mid r)$ is unvaried. That is, $\bar{k}\big(PV(\boldsymbol{c}\vert r)\big) = k$ for any $\boldsymbol{c}$ contained in the Hotelling class. For such investment streams, eq. (6) is identical to eq. (3). We have then proved the following

Theorem 3.2. A (real-valued) IRR is a particular case of AIRR generated by a Hotelling class of investment streams. The class contains infinite elements, so there exist infinite investment streams which give rise to that IRR as the AIRR of the class.

EXAMPLE

Consider the cash flow stream $\boldsymbol{x} = (-10, 5, 8, 3)$ and assume the market rate is 5%. The project has a unique real-valued IRR equal to $k = 29.59\%$. This IRR is but the AIRR corresponding to the Hotelling class, i.e. the set of those investment streams such that $PV(\boldsymbol{c} \mid 5\%) = 19.68$. An element of this class is $\boldsymbol{c} = (10, 8, 2.27)$, as can be easily verified. Its associated internal return vector is $\boldsymbol{k} = (30\%,\ 28.4\%,\ 32.02\%)$ which leads to $\bar{k} = 29.59\%$. Another PV-equivalent investment stream is $\boldsymbol{c} = (10,\ 7,\ 3.32)$, which generates the internal return vector $\boldsymbol{k} = (20\%,\ 61.75\%, -9.7\%)$, whence $\bar{k} = 29.59\%$. There are infinitely many investment streams in the same class that lead to $\bar{k} = 29.59\%$. The IRR-determined vector is $\boldsymbol{c}(k) = (10, 7.96, 2.31)$, which is obviously associated with $\boldsymbol{k} = (29.59\%,\ 29.59\%,\ 29.59\%)$, so that $\bar{k} = 29.59\%$. We stress that $\boldsymbol{c}(k)$ is only one element of the class; any other PV-equivalent investment stream supplies the same AIRR and the same answer on desirability of the project: the project is worth undertaking, for $\bar{k} = 29.59\% > 5\% = r$.

Theorem 3.2 allows us to set aside the traditional interpretation of the IRR as that constant rate of return which is applied to the capital periodically invested in the project. The IRR is, more properly, an average AIRR corresponding to infinitely many PV-equivalent investment streams; the constant internal return vector $\boldsymbol{k} = (k\ \ k, \dots, k)$ is only one among other ones contained in the Hotelling class. And given that Theorem 3.1 tells us that the decision makers may freely choose their own preferred investment stream (and, therefore, their own preferred class of investment streams), the role of the IRR is diminished: only the AIRR notion counts and any AIRR correctly solves the accept/reject decision problem: the IRR is only one instance of it.

Remark 3.3. Evidently, Theorem 3.2 implies that Hazen's decision criterion is a particular case of the AIRR criterion. The former requires the solution of the IRR equation, but such a solution is just the AIRR corresponding to any investment stream belonging to a Hotelling class. And a Hotelling class is only one class among other infinitely many classes that the analyst may use.

EXAMPLE

Consider the following mineral-extraction project, first illustrated by Eschenbach (1995, Section 7.6) and, later, by Hazen (2003). The cash flow stream is $\boldsymbol{x} = (-4,\ 3,\ 2.25,\ 1.5,\ 0.75,\ 0, -0.75, -1.5, -2.25)$ and the real-valued IRRs are $k^1 = 10.43\%$ and

$k^2 = 26.31\%$. Assuming a market rate equal to $r = 5\%$, the NPV is $PV(\boldsymbol{x} \mid 5\%) = -0.338$. We compute the AIRRs associated with ten different investment streams, collected in Table 2. The first five investment streams depict the project as a net borrowing ($PV(\boldsymbol{c} \mid 5\%) < 0$), whereas the remaining five investment streams depict the project as a net investment ($PV(\boldsymbol{c} \mid 5\%) > 0$). As the reader may note, the AIRRs associated with the borrowing-type (investment-type) description are greater (smaller) than the market rate; no matter how the investment stream is chosen, the comparison between AIRR and market rate always supplies the correct answer: the project is not worth undertaking. In particular, the first three investment streams are PV-equivalent and belong to a Hotelling class: $PV(\boldsymbol{c^1} \mid 5\%) = PV(\boldsymbol{c^2} \mid 5\%) = PV(\boldsymbol{c^3} \mid 5\%) = -6.53$ so the AIRR is the same: $\bar{k} = 10.43\%$. The fourth investment stream belongs to another Hotelling class and is just the investment stream determined by k^2. The fifth one is such that $PV(\boldsymbol{c^5} \mid 5\%) = -7.195$ and $\bar{k} = 9.93\%$. Among the other five investment streams, $\boldsymbol{c^7}$ and $\boldsymbol{c^8}$ are PV-equivalent: $PV(\boldsymbol{c^7} \mid 5\%) = PV(\boldsymbol{c^8} \mid 5\%) = 27.145$, so they belong to the same class and therefore supply the same AIRR, which is equal to 3.69%. The last investment stream is $\boldsymbol{c^{10}} = (4, 0, 0, 0, 0, 0, 0, 0)$. In this case, eq. (7) is employed to compute the AIRR. Figure 2 depicts the graph of the AIRR function associated with this project. We stress again that the areas of the rectangles with base $\overline{0\, PV(\boldsymbol{c}|r)}$ and height $|\bar{k} - r|$ are equal and correspond to the project's time-1 NFV.

4. The *simple* arithmetic mean

The AIRR is a weighted average, the weighs being the capitals discounted at the market rate. This section shows that it is possible to rest on the simple arithmetic mean. For example, consider again the project described in Table 2 and focus on $\boldsymbol{c^9} = (4, 4.2, 4.41, 4.63, 4.862, 5.105, 5.36, 5.628)$. This choice implies $PV(\boldsymbol{c^9} \mid 5\%) = 32$. Rather than computing the weighted arithmetic mean of the period rates, let us compute the simple arithmetic mean of the period rates:

$$\frac{1}{8}(80\% + 58.57\% + 39.01\% + 21.19\% + 5\% - 9.69\% - 22.98\% - 139.97\%) = 3.89\%.$$

But $3.89\% = \bar{k}(32)$. That is, the weighted arithmetic mean is equal to the simple arithmetic mean. The reason is that the capitals in $\boldsymbol{c^9}$ grow at the market rate:

$$\boldsymbol{c^9} = (4, 4(1.05), 4(1.05)^2, 4(1.05)^3, 4(1.05)^4, 4(1.05)^5, 4(1.05)^6, 4(1.05)^7).$$

In general, suppose $\boldsymbol{c} = (-x_0, -x_0(1+r), \dots, -x_0(1+r)^{T-1})$. Then $c_{t-1} = -x_0(1+r)^{t-1}$ for $t = 1, \dots, T$, so that

$$\bar{k} = \frac{\sum_1^T k_t c_{t-1}(1+r)^{-(t-1)}}{PV(\boldsymbol{c}, r)} = \frac{-\sum_1^T k_t x_0}{-T x_0} = \frac{k_1 + k_2 + \cdots + k_T}{T}.$$

Table 2. A mineral extraction project (market rate= 5%)

Time	0	1	2	3	4	5	6	7	8	
Cash flows	−4	3	2.25	1.5	0.75	0	−0.75	−1.5	−2.25	
Net borrowing										**AIRR**
$\boldsymbol{c}^1$	4	1.417	−0.685	−2.256	−3.242	−3.58	−3.203	−2.037	0	
Period rate		10.43%	10.43%	10.43%	10.43%	10.43%	10.43%	10.43%	10.43%	**10.43%**
$\boldsymbol{c}^2$	4	2	1	−0.5	−1	−4	−2	−10.5	0	
		25%	62.5%	0%	−50%	300%	−31.25	499.9%	−78.57%	**10.43%**
$\boldsymbol{c}^3$	4	2.05	0.34	−1.068	−2	−4	−3	−6.826	0	
Period rate		26.25%	26.34%	27.06%	17.04%	100%	−6.25%	177.52%	−67.04%	**10.43%**
$PV(\boldsymbol{c}^1 \mid 5\%) = PV(\boldsymbol{c}^2 \mid 5\%) = PV(\boldsymbol{c}^3 \mid 5\%) = -6.53$										
$\boldsymbol{c}^4$	4	2.052	0.342	−1.068	−2.098	−2.65	−2.598	−1.781	0	
Period rate		26.31%	26.31%	26.31%	26.31%	26.31%	26.31%	26.31%	26.31%	**26.31%**
$PV(\boldsymbol{c}^4 \mid 5\%) = -1.665$										
$\boldsymbol{c}^5$	4	2	−2	−3	−4.08	−4	−2	−1	0	
Period rate		25%	−87.5%	−25%	11%	−1.96%	−31.25	25%	125%	**9.93%**
$PV(\boldsymbol{c}^5 \mid 5\%) = -7.195$										
Net investment										
$\boldsymbol{c}^6$	4	2	1	1	−0.1	−0.3	−2	−1	0	
Period rate		25%	62.5%	150%	−35%	200%	816.67	25%	125%	**−1.88%**
$PV(\boldsymbol{c}^6 \mid 5\%) = 5.155$										
$\boldsymbol{c}^7$	4	4	4	4	4	4	4	4	0	
Period rate		75%	56.25%	37.5%	18.75%	0%	−18.75	−37.5%	−156.25	**3.69%**
$\boldsymbol{c}^8$	4	3	5	6	1	8	3	1.745	0	
Period rate		50%	141.67	50%	−70.8%	700%	−71.87	−91.8%	−228.9%	**3.69%**
$PV(\boldsymbol{c}^7 \mid 5\%) = PV(\boldsymbol{c}^8 \mid 5\%) = 27.145$										
$\boldsymbol{c}^9$	4	4.2	4.41	4.63	4.862	5.105	5.36	5.628	0	
Period rate		80%	58.57%	39.01%	21.19%	5%	−9.69%	−22.98%	−139.97%	**3.89%**
$PV(\boldsymbol{c}^9 \mid 5\%) = 32$										
$\boldsymbol{c}^{10}$	4	0	0	0	0	0	0	0	0	
Period rate		−25%	undef.	undef.	undef.	undef.	undef.	undef.	undef.	**−3.87%**
$PV(\boldsymbol{c}^{10} \mid 5\%) = 4$										

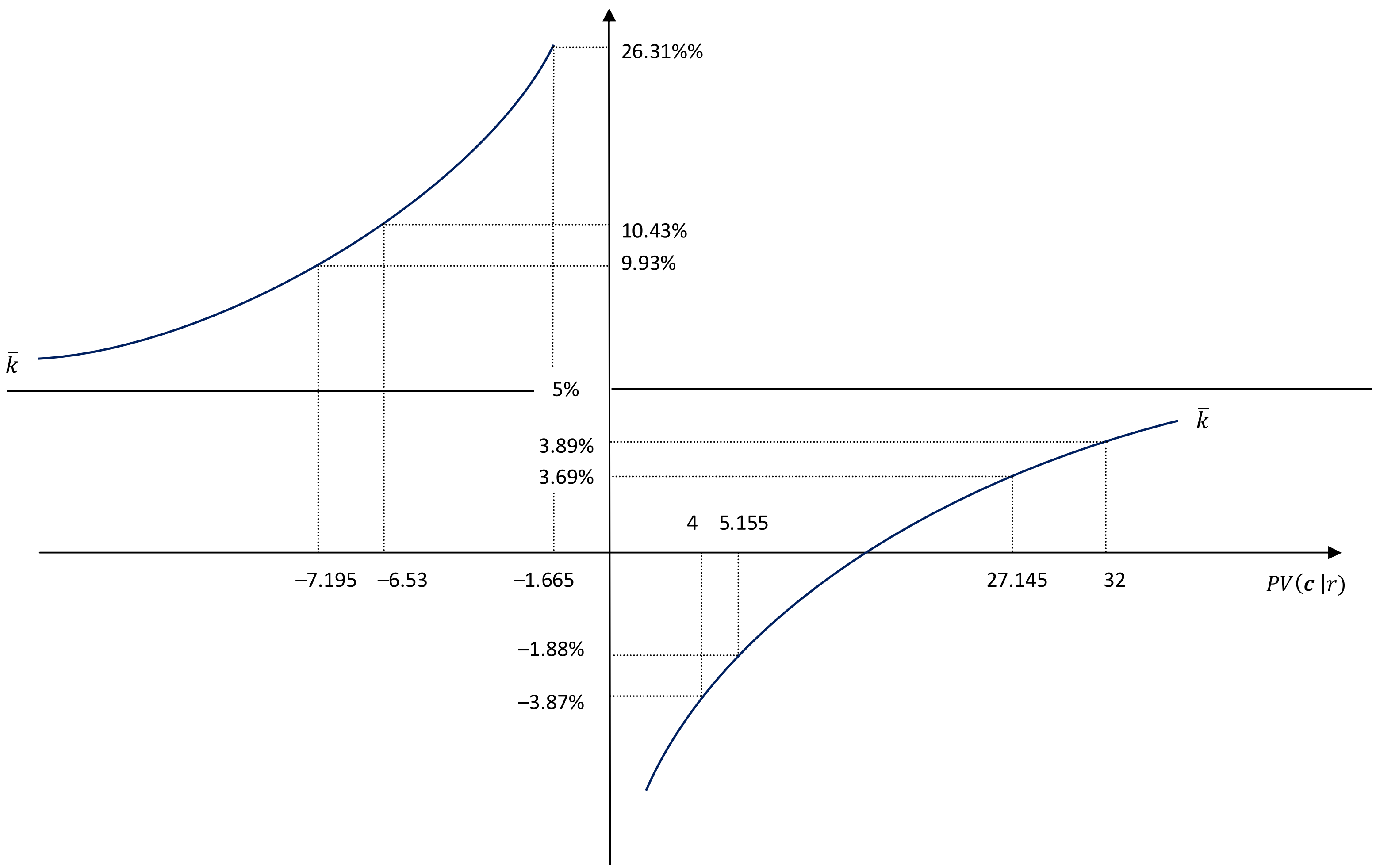

Figure 2. Mineral extraction example (see Table 2)— any AIRR is a reliable return rate associated with a class of PV-equivalent investment streams: contrasted with the market rate, it signals that the project is not worth undertaking. The project's IRRs (10.43% and 26.31%) are but two different values taken on by the AIRR function corresponding to two different Hotelling classes.

The same result applies if the investment stream is PV-equivalent to $\boldsymbol{c} = (-x_0, -x_0(1+r), \dots, -x_0(1+r)^{T-1})$, because the AIRR does not depend on $\boldsymbol{c}$ as long as $PV(\boldsymbol{c} \mid r)$ is unvaried. Then, from Theorem 3.1, the following result holds.

Theorem 4.1. Suppose the investment stream is $\boldsymbol{c} = (-x_0, -x_0(1+r), \dots, -x_0(1+r)^{T-1})$ or PV-equivalent to it. If $x_0 < 0$ (respectively, $x_0 > 0$) a project is profitable if and only if the simple arithmetic mean of its period rates is greater (respectively, smaller) than the market rate:

$$\frac{\sum_{t=1}^{T} k_t}{T} > r \ \ \text{(respectively, } \frac{\sum_{t=1}^{T} k_t}{T} < r\text{)}.$$

Theorem 4.1 is intriguing in that the simple arithmetic mean is the easiest one to use and understand, and the financial nature of the project is unambiguously revealed by the sign of the first cash flow (the project is a net investment if $x_0 < 0$, a net borrowing if $x_0 > 0$).

EXAMPLE

Consider the cash flow stream $\boldsymbol{x} = (-10,\ 4, 5, 6)$; the market rate is $r = 10\%$. If one chooses $\boldsymbol{c} = (10, 11, 12.1)$, the assumption of Theorem 4.1 holds. The internal return vector is $\boldsymbol{k} = (50\%, 55.45\%, -50.41\%)$ and the simple arithmetic mean of the period rates is $\frac{50\%+55.45\%-50.41\%}{3} = 18.35\% > 10\%$. Therefore, the project is profitable. This is confirmed by the NPV, which is equal to $PV(\boldsymbol{x} \mid 10\%)$ =2.28. The latter may be found by applying the RRR (=8.35%) to $PV(\boldsymbol{c} \mid 10\%) = 30$ and discounting back by one period.

5. Economic interpretation of AIRR.

We have shown that the economic analysis of a project depends on the fundamental triplet $(PV(\boldsymbol{c} \mid r),\ \bar{k},\ r)$. While the third component is exogenously given, the first one and the second one depend on a choice upon the decision maker. The latter may choose any investment stream, and the fundamental triplet determines the project NPV:

$$PV(\boldsymbol{x} \mid r) = \frac{PV(\boldsymbol{c} \mid r)(\bar{k} - r)}{1+r}.$$

To economically interpret the above equality, suppose a decision maker has the opportunity of investing in a one-period project $\boldsymbol{y} = (y_0, y_1)$, with $y_0 = -PV(\boldsymbol{c} \mid r)$ and $y_1 = PV(\boldsymbol{c} \mid r)\big(1 + \bar{k}\big)$. The NPV of $\boldsymbol{y}$ is

$$PV(\boldsymbol{y}\,|\,r) = -PV(\boldsymbol{c}\,|\,r) + \frac{PV(\boldsymbol{c}\,|\,r)\big(1+\bar{k}\big)}{1+r}$$

which evidently coincides with $PV(\boldsymbol{x}\,|\,r)$. This means that the use of AIRR enables the decision maker to transform project $\boldsymbol{x}$ into an economically equivalent one-period project. The IRR of $\boldsymbol{y}$ is the solution of $PV_T(\boldsymbol{y}\,|\,r) = 0$, which is just $\bar{k}$. We then maintain that the correct economic yield is just $\bar{k}$, bearing the unambiguous meaning of internal rate of return.

One may wonder whether, among the infinite values the AIRR may take on, there is any one which is particularly significant from an economic point of view. While we stress that the choice of $\boldsymbol{c}$ is subjective (and irrelevant to the decision-making process), it actually seems "natural" in many cases to consider c_0 as the total capital invested. After all, if one invests 10 dollars at time 0, it is natural to search for the rate of return of those 10 dollars. That is, one may choose $\boldsymbol{c}$ such that $PV(\boldsymbol{c}\,|\,r) = 10$. In general, one may pick $\boldsymbol{c}$ such that $PV(\boldsymbol{c}\,|\,r) = -x_0$ (there are infinite investment streams that fulfil the requirement, the simplest of all being $\boldsymbol{c} = (-x_0\ , 0, 0\ , \dots\ , 0)$; so doing, project $\boldsymbol{x}$ is turned into an equivalent one-period project $\boldsymbol{y} = \Big(\, x_0, -x_0(1+\bar{k}(x_0)\Big)$ whose NPV is

$$PV(\boldsymbol{y}\,|\,r) = x_0 + \frac{-x_0\big(1+\bar{k}(x_0)\big)}{1+r} = \ PV(\boldsymbol{x}\,|\,r).$$

Note that this implies

$$-x_0\left(1+\bar{k}(x_0)\right) = \sum_{t=1}^{T} x_t(1+r)^{-(t-1)}$$

so that the cash flows which will be generated from time 1 to time T are all compressed back to time 1. The interpretation is economically interesting: reminding that $V_k := \sum_{t=k+1}^{T} x_t(1+r)^{-(t-k)}$ is the so-called *market value* of the project as of time k, $k = 0, 1, \dots, T-1$, we have $\sum_{t=1}^{T} x_t(1+r)^{-(t-1)} = x_1 + \sum_{t=2}^{T} x_t(1+r)^{-(t-1)} = x_1 + V_1$. Therefore, if the investor invests $-x_0$ in project $\boldsymbol{x}$, it is as if he invested $-x_0$ in a one-period project generating a terminal payoff consisting of the cash flow x_1 and the end-of-period market value V_1. That is, $-x_0\left(1+\bar{k}(x_0)\right) = x_1 + V_1$. Therefore, $\bar{k}(x_0)$ represents the rate of return on the c_0 dollars invested. Such a return rate, depending on V_1, is implicitly determined by the market. Note that, in such a way, the project NPV is reduced to the economically evident relation "value minus cost":

$$PV(\boldsymbol{x}\,|\,r) = \frac{x_1+V_1}{1+r} + x_0 = \underbrace{V_0}_{\text{value}} - \underbrace{c_0}_{\text{cost}}.$$

Should other outlays occur after the initial one, the investor may well consider, more generally, the sum of the outlays as the total capital invested, so that $\boldsymbol{c}$ is selected such that $PV(\boldsymbol{c}\,|\,r) = -\sum_{t:x_t<0} x_t$ (or $PV(\boldsymbol{c}\,|\,r) = -\sum_{t:x_t>0} x_t$ if the initial cash flow is an inflow).

Remark 5.1. An interpretation of project $\boldsymbol{x}$ as a one-period project is provided in Hazen (2009, eqs. (1)-(2)) as well, but the author's interpretation is bounded by the use of the IRR, which univocally determines $PV(\boldsymbol{c}(k)\,|\,r)$, so making it impossible to consider $PV(\boldsymbol{c}\,|\,r) = -x_0$ (let alone $PV(\boldsymbol{c}\,|\,r) = -\sum_{t:x_t<0} x_t$), which is the only way to compute the rate of return on the capital actually invested.

EXAMPLE

Consider a cash flow stream $\boldsymbol{x} = (-10,\ 2,\ 8,\ 3,\ 1)$. The market rate is 3% so that $PV(\boldsymbol{x}\,|\,3\%) = 3.116$. Consider now project $\boldsymbol{y} = (-10,\ 13.51)$. Its unique real-valued IRR is $k^y = 35.1\%$, which is equal to project $\boldsymbol{x}$'s AIRR $\bar{k}(c_0) = \bar{k}(10)$. The NPV of $\boldsymbol{y}$ is $PV(\boldsymbol{y}\,|\,3\%) = -10 + 13.51(1 + 0.03)^{-1} = 3.116 = PV(\boldsymbol{x}\,|\,3\%)$. Note that 11.51 represents the market value of project $\boldsymbol{x}$ as of time 1. Therefore, $\boldsymbol{y}$ is just the very project $\boldsymbol{x}$ disguised as a one-period project: the investor invests 10 and receives the time-1 cash flow along with the market value of project $\boldsymbol{x}$: $2 + 11.51 = 10 \cdot (1 + 0.1351)$: we may say that the investor invests his 10 dollars in a project whose economic yield, implicitly determined by the market, is 35.1%.

Consider the project described in Table 1, which entails an investment of 10 dollars. As seen, the traditional IRR does not exist. This is irrelevant to the analyst, for the rate of return of those 10 dollars does exist: it is -27.27%, the AIRR associated with the fourth investment stream.

Consider the project described in Table 2, which entails an investment of 4 dollars. The rate of return of those 4 dollars is -3.87%, corresponding to $\boldsymbol{c}^{\boldsymbol{10}}$. However, should the analyst consider all the negative outflows as investments, then it means that the overall investment is equal to 8.5 dollars. In this case, one may choose, for example, $\boldsymbol{c} = (4,\ 4.725,\ 0,0,0,0,0,0)$ so that $PV(\boldsymbol{c}\,|\,r) = 8.5$ and the rate of return of those 8.5 dollars invested is 0.827%, as may be easily checked.

6. Ranking projects

It is well-known in the economic and managerial literature as well as in real-life applications that ranking competing projects by comparing their IRRs clashes with the NPV ranking. The economic and managerial literature have strived to overcome the IRR faults, but project ranking with the IRR is so far an unsolved problem. The reason is that the use of a traditional IRR determines the present value of investment stream *univocally*. More precisely, suppose that competing cash flows $\boldsymbol{x}$ and $\boldsymbol{y}$ are under consideration and let $\boldsymbol{c}(k^x)$ and $\boldsymbol{c}(k^y)$ be the investment stream associated with the IRRs, k^x and k^y, respectively. We have

$$PV(\boldsymbol{x} \mid r) = PV(\boldsymbol{c}(k^x) \mid r) \cdot \frac{k^x - r}{1+r} \qquad PV(\boldsymbol{y} \mid r) = PV(\boldsymbol{c}(k^y) \mid r) \cdot \frac{k^y - r}{1+r}.$$

According to the IRR decision criterion, the higher a project IRR, the higher its rank. But for consistency with NPV to hold, $PV(\boldsymbol{c}(k^x)| \, r)$ and $PV(\boldsymbol{c}(k^y)| \, r)$ must be equal: "if the net investments … are very different, then comparing the internal rates … will tell us little about the relative desirability of $\boldsymbol{x}$ and $\boldsymbol{y}$ in present value terms." (Hazen, 2003, p. 42). The conceptual and formal shift accomplished by the AIRR approach (let the investment stream be arbitrarily chosen) unlocks the bounds on the investment stream (and, therefore, on its present value) so that the analyst may soundly rank competing projects via their AIRRs. This is substantiated in the following

Theorem 6.1. Consider competing projects $\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n}$. The ranking of the projects via their AIRRs is equal to the NPV ranking, provided that the investment streams of the projects are chosen so as to be PV-equivalent.

Proof: First of all, we note that there are infinitely many ways to get the same $PV(\boldsymbol{c}| \, r)$ for the n projects at hand, so the requirement of PV-equivalence gives no problem. Let $\boldsymbol{c}_j$ be the investment stream selected for project $\boldsymbol{x_j}$, $j = 1,2, \dots n$ such that

$$PV(\boldsymbol{c}_1 \mid r) = PV(\boldsymbol{c}_2 \mid r) = \cdots = PV(\boldsymbol{c}_n \mid r).$$

Let $\bar{k}_j$ be the AIRR associated to $\boldsymbol{c}_j$. From eq. (6), it follows that $\max_{1 \le j \le n} PV(\boldsymbol{x_j} \mid r)$ is equivalent to $\max\limits_{1 \le j \le n} \bar{k}_j$ if $PV(\boldsymbol{c}_j \mid r) > 0$ and to $\min_{1 \le j \le n} \bar{k}_j$ if $PV(\boldsymbol{c}_j \mid r) < 0$. (QED)

EXAMPLE

Consider the following projects: $\boldsymbol{x_1} = (-100, 10, 10, 110)$, $\boldsymbol{x_2} = (-90,\ 69, 10, 12, 20)$, $\boldsymbol{x_3} = (-35,\ 50,\ -18)$ and let $r = 5\%$ (see Table 3). Because $PV(\boldsymbol{x_1}|\ 5\%) = 13.6$, $PV(\boldsymbol{x_2}|\ 5\%) = 11.6$, $PV(\boldsymbol{x_3}|\ 5\%) = -3.7$, the NPV ranking is : $\boldsymbol{x_1} \succ \boldsymbol{x_2} \succ \boldsymbol{x_3}$. The IRR of the first project is 10%, the IRR of the second project is 12.61%, no real IRR exists for the third project. This means that the IRR ranking of the three projects is not even possible; also, the first two projects are incorrectly ranked, given that the second project's IRR is greater than the first one's. Let us choose three PV-equivalent investment streams. Among the infinite possible choices, we choose, for illustrative purposes,

$$\boldsymbol{c}_1 = (100,\ 20,\ 10)$$
$$\boldsymbol{c}_2 = (90,\ 10,\ 5,\ 27.85)$$
$$\boldsymbol{c}_3 = (35,\ 97.77).$$

Hence, $PV(\boldsymbol{c}_1|\ 5\%) = PV(\boldsymbol{c}_2)|\ 5\%) = PV(\boldsymbol{c}_3|\ 5\%) = 128.12$. After calculating the one-period rates for all projects, one finds the following AIRRs for the three projects: $\bar{k}_1 = 16.16\% > \bar{k}_2 = 14.51\% > \ \bar{k}_3 = 1.96\%$. The projects are described as net investments, so the higher the AIRR, the higher the rank. Therefore, $\boldsymbol{x_1} \succ \boldsymbol{x_2} \succ \boldsymbol{x_3}$, which is just the NPV ranking (see also Figure 3).

The ranking via the AIRR may be even more fruitfully reframed in terms of residual rate of return $\bar{k} - r$. The latter provides, at one time, information about profitability and information about rank. In the above example, the RRRs are

$$\bar{k}_1 - r = 11.16\%.$$
$$\bar{k}_2 - r = 9.51\%$$
$$\bar{k}_3 - r = -3.04\%.$$

Note that the third RRR is negative, so signaling that the project is not profitable. The second RRR is positive, so, while project 1 is preferred to project 2, the latter is worth undertaking. By discounting the RRR and multiplying it by the present value of the investment stream one gets back the NPV: $11.16\%(1.05)^{-1}\cdot128.12=13.6$ for the first one, $9.5\%(1.05)^{-1}\cdot128.12=11.6$ for the second one, $-3.04\%(1.05)^{-1}\cdot128.12=-3.7$ for the third one.

Either the AIRR or, equivalently, the RRR is a perfect substitute of the NPV, in both accept/reject decisions and project ranking.

Table 3. Project ranking

Time	0	1	2	3	4
Project x_1					
Cash flows	−100	10	10	110	
Inv. Stream	100	20	10	0	
Period rate		−70%	0%	1000%	
AIRR	**16.16%**				
RRR	11.16%				
Project x_2					
Cash flows	−90	69	10	12	20
Inv. stream	90	10	5	27.85	0
Period rate		−12.2%	50%	697%	−28.2%
AIRR	**14.51%**				
RRR	9.51%				
Project x_3					
Cash flows	−35	50	−18		
Inv. stream	35	97.77	0		
Period rate		322.2%	−118.4%		
AIRR	**1.96%**				
RRR	−3.04%				
ranking	$\boldsymbol{x_1 \succ x_2 \succ x_3}$				

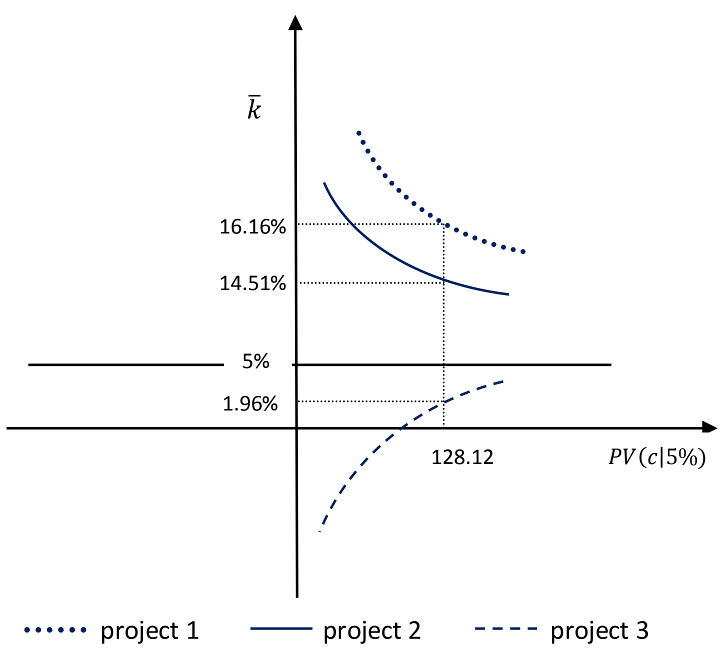

Figure 3. Project ranking with the AIRR (see Table 3). The greater the AIRR, the higher the project rank. Evidently, the result holds for every $PV(\boldsymbol{c}\mid 5\%)$. (We omit the graph for $PV(c\mid 5\%) < 0$).

The simple arithmetic mean may be employed for ranking projects as well, if proper care is taken. Let us begin with projects with equal initial outflow (or inflow), which is the case of a decision maker who is endowed with a capital to be invested in some alternative.

Theorem 6.2. Consider competing projects $\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n}$ with respective length $T_1, T_2, \dots, T_n$ and equal initial cash flow x_0. Suppose that the investment stream for each project is PV-equivalent to $\boldsymbol{c} = (-x_0, -x_0(1+r), \dots, -x_0(1+r)^{T-1})$, where $T = \max(T_1, T_2, \dots, T_n)$. Then, the ranking of the projects via the arithmetic mean of the period rates is equivalent to the NPV ranking.

Proof: If $\boldsymbol{c}_j$ is PV-equivalent to $\boldsymbol{c} = (-x_0, -x_0(1+r), \dots, -x_0(1+r)^{T-1})$ for all $j = 1,2, \dots, n$, then, owing to Theorem 4.1, the AIRR is equal to the simple arithmetic mean. The thesis follows from Theorem 6.1. (QED)

EXAMPLE

Suppose the manager of a firm is endowed by the shareholders with additional equity to be invested in some business. Suppose he has the opportunity of employing the capital in three economic activities: $\boldsymbol{x_1} = (-100, 40, 0, 80, 0)$, $\boldsymbol{x_2} = (-100, \ 60, 10, 10, 20)$, $\boldsymbol{x_3} = (-100, \ 113, 10, 0, 0)$. The market rate is 5% so that

$$PV(\boldsymbol{x_1} |\, 5\%) = 7.2$$
$$PV(\boldsymbol{x_2} |\, 5\%) = -8.69$$
$$PV(\boldsymbol{x_3} |\, 5\%) = 16.69.$$

For simplicity, we pick the same investment stream for all projects:

$$\boldsymbol{c}_1 = \boldsymbol{c}_2 = \boldsymbol{c}_3 = (100, 105, 110.25, 115.76)$$

which is just $\boldsymbol{c} = (-x_0, -x_0(1+r), \dots, -x_0(1+r)^{T-1})$, with $x_0 = -100$, $r = 5\%$, $T = 4$. The internal return vectors are, respectively,

$$\boldsymbol{k}_1 = (45\%, 5\%, 77.56\%, -100\%),$$
$$\boldsymbol{k}_2 = (65\%, 14.5\%, 14.1\%, -82.7\%)$$
$$\boldsymbol{k}_3 = (118\%, 14.5\%, \ 5\%, -100\%).$$

The simple arithmetic means are:

project $\boldsymbol{x_1}$	6.89%
project $\boldsymbol{x_2}$	2.72%
project $\boldsymbol{x_3}$	9.38%.

The ranking is then $\boldsymbol{x_3} \succ \boldsymbol{x_1} \succ \boldsymbol{x_2}$, which is the same as the NPV ranking.

Let us now focus on a bundle of projects $\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n}$ with different initial cash flows. We exploit the fact that the NPV of a project does not change if the project is virtually integrated with a value-neutral investment. For example, let $\boldsymbol{z} = (z_0,\ 0, 0, \dots, -z_0(1+r)^T)$. Project $\boldsymbol{z}$ is a *mute* operation: $PV(\boldsymbol{z}\,|\,r) = 0$ for any $z_0 \in \mathbb{R}$ Thus, $PV(\boldsymbol{x}+\boldsymbol{z}\,|\,r) = PV(\boldsymbol{x}|\,r)$, so one may always use the *integrated* project vector $\boldsymbol{x}+\boldsymbol{z}$ rather than $\boldsymbol{z}$ for economic analysis purposes. Pick any project $\boldsymbol{x_p}$ and consider the mute operation $\boldsymbol{z_j}$, $j = 1,2,, \dots, n$, such that the integrated project $\boldsymbol{x_j}+\boldsymbol{z_j}$ has the same initial cash flow as $\boldsymbol{x_p}$ for all j (obviously, if some projects have the same initial cash flow as $\boldsymbol{x_p}$, then $\boldsymbol{z_j}$ is the null vector). Then, Theorem 6.2 may be applied to $\boldsymbol{x_1}+\boldsymbol{z_1}, \boldsymbol{x_2}+\boldsymbol{z_2}, \dots, \boldsymbol{x_n}+\boldsymbol{z_n}$. But the latter are financially equivalent to $\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n}$. We have then proved the following

Theorem 6.3. Consider competing projects $\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n}$ with different initial cash flows. Using the appropriate (fictitious) mute operations in order to harmonize the initial cash flows, the ranking via the simple arithmetic means coincides with the NPV ranking.[3]

EXAMPLE

Consider the following three projects:

$$\boldsymbol{x_1} = (-100,\ 40, 0,\ 80, 0)$$
$$\boldsymbol{x_2} = (-100, 60, 10, 10, 20)$$
$$\boldsymbol{x_3} = (-10,\ 30, -25, 0, 0)$$

and let 5% be the market rate, which implies

$$PV(\boldsymbol{x_1}|\,5\%) = 7.2$$
$$PV(\boldsymbol{x_2}|\,5\%) = -8.69$$
$$PV(\boldsymbol{x_3}|\,5\%) = -4.1.$$

The initial outlay of $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ is 100, whereas the initial outlay of $\boldsymbol{x_3}$ is only 10, so we use the following mute operations: $\boldsymbol{z_1} = \boldsymbol{z_2} = (0, 0, 0, 0, 0)$, $\boldsymbol{z_3} = (-90, 0,\ 0,\ 0,\ 109.4)$. The latter implies $\boldsymbol{x_3}+\boldsymbol{z_3} = (-100,\ 30, -25,\ 0, 109.4)$. We then apply Theorem 6.3 to the integrated projects $\boldsymbol{x_1}$, $\boldsymbol{x_2}$, $\boldsymbol{x_3}+\boldsymbol{z_3}$. Consider, for example, $\boldsymbol{c} = (100,\ 105, 110.25, 115.76)$ for all the projects. It is straightforward to compute the economic yields (simple arithmetic means):

[3] Note that the theorem includes those cases where $x_0 = 0$ (i.e., the project starts at time $t > 0$.)

project $\boldsymbol{x_1}$	6.89%
project $\boldsymbol{x_2}$	2.72%
project $\boldsymbol{x_3} + \boldsymbol{z_3}$	3.92%.

Then, the project ranking is $\boldsymbol{x_1} \succ \boldsymbol{x_3} \succ \boldsymbol{x_2}$, the same as the NPV ranking.

Remark 6.1. As a final remark, we notice that we have implicitly assumed throughout the paper that the market rate is constant. While the IRR criterion is not capable of handling variable market rates, our approach is easily generalized. The NPV of a project will be $PV(\boldsymbol{x}|\, r) = \sum_1^T c_{t-1}(\bar{k} - r_t)\, v_t$, where r_t is the market rate holding in the period $[t-1, t]$ and $v_t := [(1+r_1)(1+r_2) \cdot \ldots \cdot (1+r_t)]^{-1}$ is the discount factor. Searching for a Chisini mean $\bar{r}$ of the market rates, one solves $\sum_1^T c_{t-1}(\bar{k} - r_t)\, v_t = \sum_1^T c_{t-1}(\bar{k} - \bar{r}) v_t$ getting to

$$\bar{r} = \frac{\sum_1^T c_{t-1} r_t v_t}{\sum_1^T c_{t-1} v_t}.$$

All results proved in the paper hold with $\bar{r}$ replacing r.

Concluding remarks

Scholars, managers, practitioners have long since recognized that the NPV criterion is a theoretically sound decision criterion for capital budgeting in most circumstances (e.g. Fisher 1930; Weston and Copeland 1988; Brealey and Myers 2000; MacMinn 2005), and even real options may be framed in terms of an 'expanded' NPV.[4] However, managers, analysts, practitioners often find it useful (or are explicitly required) to supply a performance measure in terms of rates rather than present values. Unfortunately, the venerable internal rate of return (IRR) is not a reliable profitability index because it may not exist, multiple roots may arise and, in general, is incompatible with the NPV. This paper presents the notion of Average Internal Rate of Return (AIRR), which generalizes the usual IRR notion. The AIRR is an average of one-period return rates derived from investment streams which are freely chosen by the analyst. An investment (borrowing) is worth undertaking if and only the AIRR is greater (smaller) than the market rate and the AIRR ranking is the same as the NPV ranking. One may also equivalently use the

[4] The real options approach is but a sophisticated version of the traditional NPV model, where the set of alternatives is inclusive of the options implicit in the project: ''one can always redefine NPV by subtracting from the conventional calculation the opportunity cost of exercising the option to invest, and then say that the rule 'invest if NPV is positive' holds once this correction has been made'' (Dixit and Pindyck 1994, p. 7. See also Smith and Nau 1995).

residual rate of return (RRR), which is defined as the difference between AIRR and market rate: this index signals, at one time, desirability of a project and its rank among other competing projects. The RRR is then a perfect substitute of the NPV and represents the excess return on one unit of invested capital.

The AIRR notion, which is defined even when the one-period rates are not, is a general notion encompassing the IRR notion: the latter is only one particular case of AIRR: it is the AIRR associated with a specific class of present-value-equivalent investment streams, here named "Hotelling class". *Any* one AIRR works well in association with its own investment stream, because any combination AIRR/investment stream univocally determines the NPV of the project. The analyst may emphasize any one combination to the neglect of the others. A natural choice is such that the present value of the investment stream is equal to the initial outlay of the project (or the sum of all outlays); in this case, the resulting AIRR is conveniently interpreted as the market-determined rate of return on the capital actually invested in the project. Among the various AIRRs, the simple arithmetic mean is particularly salient because it is easily understood by real-life decision makers and the financial nature of the project is unambiguously determined by the sign of the first cash flow.

The results obtained stem from a basic idea: the capital periodically invested (or borrowed) in the project is conventionally determined by the analyst. In other words, it is not a project that is intrinsically an investment or a borrowing, but it is the very evaluator who decides whether the project is an investment or a borrowing. The approach here endorsed is thus radically different from the usual approach centered on the IRR notion, because it dismisses the IRR equation, guarantees description flexibility and warrants correctness of decisions (i.e., equivalence with the NPV criterion). The AIRR approach solves the long-standing problem of finding an economically significant return rate capable of helping decision makers make correct decisions.

NPV and AIRR are perfectly equivalent performance measures: whether one will make use of the NPV criterion or the AIRR criterion is only a matter of practical issues, subjective tastes, purpose of the analysis or institutional commitment.

References

Adler, M. 1970. The true rate of return and the reinvestment rate. *The Engineering Economist*, 15, 185–187.

Arnold, G., Davies, M. (Eds.) 2000. *Value-based Management: Context and Application*. Chichester, UK: John Wiley & Sons.

Arrow, K.J., Levhari, D. 1969. Uniqueness of the internal rate of return with variable life of investment. *The Economic Journal*, 79 (September), 560–566.

Aucamp, D.L., Eckardt, W.L. 1976. A sufficient condition for unique non-negative internal rate of return: Comment. *Journal of Financial and Quantitative Analysis*, 11 (June), 329–332.

Bailey, M.J. 1959. Formal criteria for investment decisions. *Journal of Political Economy*, 67 (October), 476–488.

Baldenius, T., Reichelstein, S., 2005. Incentives for efficient inventory management: The role of historical cost. *Management Science*, 51 (7), 1032–1045.

Bernhard, R.H. 1962. Discount methods for expenditure evaluation—A clarification of their assumptions. *The Journal of Industrial Engineering*, 18(1), 19–27.

Bernhard, R.H. 1967. Communications to the Editor—Probability and rates of return: Some critical comments. *Management Science*, 13(7) (March), 598–600.

Bernhard, R.H. 1977. Unrecovered investment, uniqueness of the internal rate, and the question of project acceptability. *Journal of Financial and Quantitative Analysis*, 12 (March), 33–38.

Bernhard, R.H. 1979. A more general sufficient condition for a unique internal rate of return. *Journal of Financial and Quantitative Analysis*, 14 (June), 337–341.

Bernhard, R.H. 1980. A simplification and an extension of the Bernhard-De Faro sufficient condition for a unique non-negative internal rate of return. *Journal of Financial and Quantitative Analysis* 15(1), 201–209.

Biondi, Y. 2006. The double emergence of the modified internal rate of return: The neglected financial work of Duvillard (1755–1832) in a comparative perspective. *The European Journal of the History of Economic Thought*, 13(3), 311–335.

Boulding, K.E. 1935. The theory of the single investment. *Quarterly Journal of Economics*, 49 (May), 475–494.

Boulding, K.E. 1936a. Time and investment. *Economica*, 3 (May), 196–220.

Boulding, K.E. 1936b. Time and investment: A reply. *Economica*, 3 (November), 440–442.

Brealey, R., Myers, S.C. 2000. *Principles of Corporate Finance*. New York: McGraw-Hill.

Brief, R.P., Peasnell, K.V. (Eds.) 1996. *Clean Surplus: A Link Between Accounting and Finance*. New York and London: Garland Publishing.

Cannaday, R.E., Colwell, P.F., Paley, H. 1986. Relevant and irrelevant internal rates of return. *The Engineering Economist*, 32(1) (Fall), 17–38.

Chisini, O. 1929. Sul concetto di media [On the concept of mean], *Periodico di matematiche*, 4, 106–116.

De Faro, C. 1978. A sufficient condition for a unique non-negative internal rate of return: further comment. *Journal of Financial and Quantitative Analysis*, 13 (September), 577–584.

Dixit, A. and Pindyck, R. 1994. *Investment Under Uncertainty*. Princeton, NJ : Princeton University Press.

Dorfman, R. 1981. The meaning of internal rates of return. *The Journal of Finance*, 36(5) (December), 1011–1021.

Edwards, E., Bell, P. 1961. *The Theory and Measurement of Business Income*. Berkeley University of California Press.

Egginton, D. 1995. Divisional performance measurement: residual income and the asset base. *Management Accounting Research*, 6, 201–222.

Eschenbach, T.G. 1995. *Engineering Economy: Applying Theory to Practice*. Chicago: Irwin.

Fairley, W., Jacoby, H.D. 1975. Investment analysis using the probability distribution of the internal rate of return. *Management Science*, 21(12) (August), 1428–1437.

Fisher, I., 1930. *The Theory of Interest*. New York: MacMillan.(Reprinted by Augustum M. Kelley: Clifton, NJ, 1974.

Flemming, J.S., Wright, J.F. 1971. Uniqueness of the internal rate of return: A generalization. *The Economic Journal*, 81 (June), 256–262.

Gallo, P. Peccati, L. 1993. The appraisal of industrial investments: a new method and a case study. *International Journal of Production Economics*, 30-31, 465–476.

Graziani, R., Veronese, P. 2009. How to compute a mean? The Chisini approach and its applications. *The American Statistician*, 63(1) (February), 33–36.

Gronchi, S. 1986. On investment criteria based on the internal rate of return. *Oxford Economic Papers*, 1 (March), 174–180.

Gronchi, S. 1987. *Tasso di rendimento e valutazione dei progetti* [Rate of return and project evaluation]. Milano: Franco Angeli.

Hajdasinski, M.M. 1987. On relevant and irrelevant rates of return. *The Engineering Economist*, 32(4) (Summer), 347–353.

Hajdasinski, M.M. 2004. Technical note—the internal rate of return (IRR) as a financial indicator. *The Engineering Economist*, 49, 185–197.

Hartman, J.C., Schafrick, I.C. 2004. The relevant internal rate of return. *The Engineering Economist*, 49, 139–158.

Hazen, G.B. 2003. A new perspective on multiple internal rates of return. *The Engineering Economist*, 48(1), 31–51.

Hazen, G.B. 2009. An extension of the internal rate of return to stochastic cash flows. *Management Science*, 55(6) (June), 1030–1034

Herbst, A. 1978. The unique, real internal rate of return: Caveat emptor. *Journal of Financial and Quantitative Analysis*, 13 (June), 363–370.

Hirshleifer, J. 1958. On the theory of optimal investment decision. *Journal of Political Economy*, 66 (August), 329–352.

Hotelling, H., 1925. A general mathematical theory of depreciation. *Journal of the American Statistical Association*, September, 27–38. Reprinted in Brief R.P. (Ed.), 1984. *Depreciation and Capital Maintenance*. New York, Garland Publishing, pp. 15–29.

Jean, W.H. 1968. On multiple rates of return. *The Journal of Finance*, 23 (March), 187–191.

Kaplan, S. 1965. A note on a method for precisely determining the uniqueness or non-uniqueness of the internal rate of return for a proposed investment. *The Journal of Industrial Engineering*, 16 (January/February), 70–71.

Kaplan, S. 1967. Computer algorithms for finding exact rates of return. *Journal of Business*, 40 (October), 389–392.

Karmel, P.H. 1959. The marginal efficiency of capital. *Economic Record*, 35 (December), 429–434.

Keynes, J.M. 1936. *The General Theory of Employment, Interest and Money*. London: MacMillan.

Kierulff, H. 2008. MIRR: A better measure. *Business Horizons*. 51, 321–329.

Lee, T. 1985. *Income and Value Measurement. Theory and Practice*, third edition. Worcester, UK: Billing & Sons.

Lin, A.Y.S. 1976. The modified internal rate of return and investment criterion. *The Engineering Economist*, 21(4), 237–247.

Lohmann, J.R. 1988. The IRR, NPV and the fallacy of the reinvestment rate assumption, *The Engineering Economist*, 33(4) (Summer), 303–330.

Lorie, J., Savage, L.J. 1955. Three problems in capital budgeting. *Journal of Business*, 28 (October), 229–239.

MacMinn, R.D. 2005. *The Fisher Model and Financial Markets*. Singapore: World Scientific.

Magni, C.A. 2009a. Splitting up value: A critical review of residual income theories. *European Journal of Operational Research*, 198(1) (October), 1–22.

Magni, C.A. 2009b. Investment decisions, net present value, and bounded rationality. *Quantitative Finance*, 9(8) (December), 967–979.

Martin, J.D., Petty, J.W. 2000. *Value Based Management. The Corporate Response to Shareholder Revolution*. Boston, MA: Harvard Business School.

Martin, J.D., Petty, J.W., Rich, S.P. 2003. An analysis of EVA and other measures of firm performance based on residual income. Hankamer School of Business working paper. Available at SSRN:<http://papers.ssrn.com/sol3/papers.cfm?abstract id=412122>.

Norstrøm, C.J. 1967. Communication to the Editor — A note on "Mathematical analysis of rates of return under certainty". *Management Science*, 13(5) (June), 429.

Norstrøm, C.J. 1972. A sufficient condition for a unique non-negative internal rate of return. *Journal of Financial and Quantitative Analysis*, 7(3) (June), 1835–1839.

Ohlson. J.A. 1995. Earnings, book values, and dividends in equity valuation. *Contemporary Accounting Research*, 11(2) (Spring), 661–687.

Peasnell, K.V. 1982. Some formal connections between economic values and yields and accounting numbers. *Journal of Business Finance & Accounting*, 9(3), 361–381.

Peccati, L. 1989. Multiperiod analysis of a levered portfolio. *Decisions in Economics and Finance*, 12(1) (March), 157–166. Reprinted in J. Spronk & B. Matarazzo (Eds.), 1992, *Modelling for Financial Decisions*. Berlin: Springer-Verlag.

Penman D. 2010. *Financial Statement Analysis and Security Valuation*, fourth edition. New York NY: McGraw Hill Irwin.

Pfeiffer, T. 2004. Net present value-consistent investment criteria based on accruals: A generalisation of the residual income-identity. *Journal of Business Finance & Accounting*, 31(7/8) (September), 905–926.

Pfeiffer, T., Schneider, G. 2007. Residual income-based compensation plans for controlling investment decisions under sequential private information. *Management Science*, 53(3), 495-507.

Pitchford, J.D. Hagger, A.J. 1958. A note on the marginal efficiency of capital. *The Economic Journal*, 68 (December), 597–600.

Preinreich, G. 1936. The fair value and yield of common stock. *The Accounting Review*. Reprinted in R. Brief & K.V. Peasnell (Eds.) 1996. *Clean Surplus: A Link Between Accounting and Finance*. New York and London: Garland Publishing.

Preinreich, G. 1937. Valuation and amortization. *The Accounting Review*, 12(3) (September), 209–226.

Preinreich, G. 1938. Annual survey of economic theory: the theory of depreciation. *Econometrica*, 6(1) (January), 219–241.
Promislow S.D., Spring D. 1996. Postulates for the internal rate of return of an investment project. *Journal of Mathematical Economics*, 26(3), 325–361.
Ramsey, J.B. 1970. The marginal efficiency of capital, the internal rate of return, and net present value: An analysis of investment criteria. *Journal of Political Economy*, 78(5) (September/October), 1017–1027.
Rocabert, J.P., Tarrío, J.A., Pérez, M.J. 2000. Anomalies in net present value calculations. A solution. *Revista de Economía Pública,* 173(2), 47–60.
Ross, S.A., Spatt, C.S., Dybvig P.H. 1980. Present values and internal rates of return. *Journal of Economic Theory*, 23 (August), 66–81.
Samuelson, P. 1964. Tax deductibility of economic depreciation to insure invariant valuations. *Journal of Political Economy*, 72, 604–606.
Simerská, C. 2008. Remarks on the economic criterion — the internal rate of return. In J. Chleboun, P. Přikryl, K. Segeth, T. Vejchodskỳ (Eds.) , *Programs and Algorithms of Numerical Mathematics 14*, p.170–176. Prague: Institute of Mathematics Academy of Sciences of the Czech Republic. Available at: <www.math.cas.cz/~panm/proceedings/PANM14proc.pdf>.
Smith, J.E. Nau, R.F. 1995. Valuing risky options: option pricing theory and decision analysis. *Management Science*, 41(5) (May), 795–816.
Solomon, E. 1956. The arithmetic of capital budgeting decisions. *Journal of Business*, 29 (April), 124–129.
Solomons, D. 1965. *Divisional Performance: Measurement and Control*. Richard D. Irwin, Homewood, IL.
Soper, C.S. 1959. The marginal efficiency of capital: A further note. *The Economic Journal*, 69 (March), 174–177.
Tang, S.L., Tang, H.J. 2003. The variable financial indicator IRR and the constant economic indicator NPV. *The Engineering Economist*, 48(1), 69–78.
Teichroew, D., Robichek, A.A., Montalbano, M. 1965a. Mathematical analysis of rates under certainty. *Management Science,* 11(3) (January), 395–403.
Teichroew, D., Robichek, A.A., Montalbano, M. 1965b. An analysis of criteria for investment and financing decisions under certainty. *Management Science,* 12(3) (November), 151–179.
Weingartner, H.M. 1966. The generalized rate of return. *The Journal of Financial and Quantitative Analysis*, 1(3) (September), 1–29.
Weston, J.F., Copeland, T.E. 1988. *Managerial Finance*, second British ed. Cassell . Educational Limited, London, UK.
Wright, J.F. 1959. The marginal efficiency of capital. *The Economic Journal*, 69 (December), 813–816.
Young, S.D., O'Byrne, S.F. 2001. *EVA and Value-Based Management*. McGraw-Hill.
Zhang, D. 2005. A different perspective on using multiple internal rates of return: the IRR parity technique. *The Engineering Economist*, 50, 327–335.

www.cefin.unimore.it

Previously published “CEFIN Working Papers”

20 *The skew pattern of implied volatility in the DAX index options market,* by Muzzioli S. (December 2009)

19 *Accounting and economic measures: An integrated theory of capital budgeting,* by Magni C.A. (December 2009)

18 *Exclusions of US-holders in cross-border takeover bids and the principle of equality in tender offers*, by Mucciarelli F. (May 2009).

17 *Models for household portfolios and life-cycle allocations in the presence of labour income and longevity risk*, by Torricelli C. (March 2009)

16 *Differential evolution of combinatorial search for constrained index tracking*, by Paterlini S, Krink T, Mittnik S. (March 2009)

15 *Optimization heuristics for determining internal rating grading scales*, by Paterlini S, Lyraa M, Pahaa J, Winker P. (March 2009)

14 *The impact of bank concentration on financial distress: the case of the European banking system*, by Fiordelisi F, Cipollini A. (February 2009)

13 *Financial crisis and new dimensions of liquidity risk: rethinking prudential regulation and supervision*, by Landi A, Gualandri E, Venturelli V. (January 2009)

12 *Lending interest rate pass-through in the euro area: a data-driven tale*, by Marotta G. (October 2008)

11 *Option based forecast of volatility: an empirical study in the Dax index options market*, Muzzioli S. (May 2008)

10 *Lending interest rate pass-through in the euro area*, by Marotta G. (March 2008)

9 *Indebtedness, macroeconomic conditions and banks’ losses: evidence from Italy*, by Torricelli C, Castellani S, Pederzoli C. (January 2008)

8 *Is public information really public? The role of newspapers*, Ferretti R, Pattarin F. (January 2008)

7 *Differential evolution of multi-objective portfolio optimization*, by Paterlini S, Krink T. (January 2008)

6 *Assessing and measuring the equity gap and the equity*, by Gualandri E, Venturelli V. (January 2008)

5 *Model risk e tecniche per il controllo dei market parameter*, Torricelli C, Bonollo M, Morandi D, Pederzoli C. (October 2007)

4 *The relations between implied and realised volatility, are call options more informative than put options? Evidence from the Dax index options market*, by Muzzioli S. (October 2007)

3 *The maximum LG-likelihood method: an application to extreme quantile estimation in finance*, by Ferrari D., Paterlini S. (June 2007)

2 *Default risk: Poisson mixture and the business cycle*, by Pederzoli C. (June 2007)

1 *Population ageing, household portfolios and financial asset returns: a survey of the literature*, by Brunetti M. (May 2007)

CEFIN – Centro Studi di Banca e Finanza
Dipartimento di Economia Aziendale – Università di Modena e Reggio Emilia
Viale Jacopo Berengario 51, 41100 MODENA (Italy)
tel. 39-059.2056711 (Centralino) fax 059 205 6927