EXACT CONTROLLABILITY OF INFINITE DIMENSIONAL SYSTEMS WITH CONTROLS OF MINIMAL NORM

L. MALAGUTI, S. PERROTTA, AND V. TADDEI

Dedicated to the memory of Professor Ioan I. Vrabie

Abstract. The paper deals with the exact controllability of a semilinear system in a separable Hilbert space. A bounded linear part is considered and a linear control introduced. The state space is compactly embedded in a Banach space and the nonlinear term is continuous in its state variable in the norm of the Banach space. An infinite sequence of finite dimensional controllability problems is introduced and the solution is obtained by a limiting procedure. To the best of our knowledge, the method is new in controllability theory. An application to an integro-differential system in euclidean spaces completes the discussion.

1. Introduction

This paper deals with the exact controllability in infinite dimensional spaces, by means of linear controls. We consider the semilinear equation

$$y'(t) = Ay(t) + f(t\,, y(t)) + Bu(t), \quad t \in [0, T],\ y(t) \in H, \tag{1.1}$$

with $0 < T < +\infty$, in the separable Hilbert space H and assume that the control term u belongs to $L^2([0, T], U)$ where U is a Hilbert space. The operators $A : H \to H$ and $B : U \to H$ are linear and bounded. We refer to Section 3 for the exact properties of $f\colon [0\,, T] \times H \to H$. System (1.1) is said to be controllable if every initial condition $y_0 \in H$ can be steered at time T to any $y_1 \in H$, i.e. if $y(0) = y_0$ and $y(T) = y_1$, by some admissible control u (see Definition 3.3).
Let $\{e^{At}\}_{t\in\mathbb{R}}$ be the group of continuous operators generated by A. The linear, bounded operator $G : L^2([0\,, T]\,, U) \to H$ defined by

$$G(u) = \int_0^T e^{A(T-s)}Bu(s)\,ds$$

is important in the study of the controllability of (1.1). In particular, when (1.1) is linear, that is $f(t\,, y) = f(t)$, the controllability of (1.1) is equivalent to the surjectivity of G (see e.g. [8, Sect. 4.1]). We refer to [17] for a wide discussion about controllability in the linear case.
In general it is hard to prove that the nonlinear system (1.1) is controllable. The study is usually carried out by means of a fixed point technique and the following equation is introduced

$$y'(t) = Ay(t) + f(t\,, q(t)) + Bu(t), \quad t \in [0, T], \tag{1.2}$$

where $q : [0\,, T] \to H$ is any continuous function. Since (1.2) is clearly linear with respect to y, the operator G is important in this discussion. In particular,

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when equation (1.2) is controllable, for every $y_0, y_1 \in H$ there exists a control $u \in L^2([0,T],U)$ such that

$$G(u) = y_1 - e^{AT}y_0 - \int_0^T e^{A(T-s)} f(s, q(s))\, ds. \tag{1.3}$$

The usual space of equivalence classes $L^2([0,T],U)/\ker G$ is then naturally involved. Let $\bar{G}$ be the operator induced by G in $L^2([0,T],U)/\ker G$; when G is onto, then $\bar{G}$ has an inverse which is again linear and bounded; the property is true in the more general setting of H and U arbitrary Banach spaces (see e.g. [11] and [15, Sect. II.5]). However, the knowledge of $\bar{G}$ and its inverse is not sufficient, in general, for the correct implementation of a topological method. It is also necessary to select a control function, in each equivalence class of $L^2([0,T],U)/\ker G$, and some properties of this selection map are needed. If the control space U is a Hilbert space, a unique control function with minimal norm, say $\bar{u}$, exists in each equivalence class, and the selection map $[u] \longmapsto \bar{u}$ is linear and bounded (see e.g. Proposition 2.2). A linearization method can then be correctly implemented and this explains why we set out the present discussion in Hilbert spaces. Moreover, the controllability of (1.1) can be obtained by the minimal norm control with respect to the family of controls that satisfy the necessary condition (1.3). The existence of a unique minimal norm element in each equivalence class of $L^2([0,T],U)/\ker G$ is true also when U is a uniformly convex Banach space but the selection map need not be linear in general; this topic is discussed in Remark 2.3. An alternative approach could be to make use of a multivalued analysis in order to account of all the controls in each equivalence class as suggested in [11].

Controllability can be investigated for a more general system obtained when A in (1.1) is replaced by a densely defined operator which generates a strongly continuous semigroup; nevertheless, such a system is never controllable when the associated semigroup is compact (see [16]). Controllability results can also be obtained for multivalued systems, occurring when f is a multivalued map, by some additional techniques proper of the multivalued analysis. Many contributions can be found about the controllability of semilinear systems in infinite spaces; we refer, in particular, to [1], [6], [11], [13] and [19]. The topological method is clearly showed in the first paper published by Magnusson-Pritchard-Quinn [11]; some preliminary results are also obtained there by the contraction principle or the Schauder fixed point theorem with a Lipschitzian nonlinear part. The controllability in the multivalued case is discussed in [6] and [13]. The Lipschitzianity condition is replaced in [13] by a weaker regularity involving the Hausdorff measure of noncompactness and the fixed point result for condensing maps (see [10, Chap. 2 and Corollary 3.3.1]) is used; the use of the weak topology in the state space is exploited in [6]. The discussion in [19] is based on the fixed point-type application of the Schmidt existence theorem and it makes use of the Kuratowski measure of noncompactness; one-sided Lipschitz conditions and some convexity are further assumed; the nonlinearity is autonomous. Several additional papers can be found where the investigation is based on contradictory assumptions that G is onto and A generates a compact semigroup. The discussion in other papers is set in some Banach space, but the properties of the selection map of one control in each equivalence class are not clarified, in some cases the selection is neither introduced.

As it is known, every separable Hilbert space has an orthonormal basis $(e_n)_{n\in\mathbb{N}}$. Starting from this property we define infinitely many simpler controllability problems settled in the finite dimensional spaces $H_n := \text{span}\{e_1, ..., e_n\}$ (see $(\mathcal{P}_n)$ in Section 3). Then we assume that the Hilbert space H is compactly embedded in a

Banach space $(E\,,\|\cdot\|_E)$ and obtain the controllability of (1.1) by a limiting procedure. This *approximation solvability method* was recently pointed out in [4] for the study of boundary value problems and extended in [5] to second order equations. To the best of our knowledge, the technique is new in controllability theory. We assume that the nonlinear term $f(t,\cdot)$ is continuous with respect to $\|\cdot\|_E$, for almost every t, and we do not need to introduce any measure of noncompactness, the only amount of compactness required being the mentioned compact embedding. Our main results are Theorem 3.4 and Theorem 3.5. The former requires a strict sublinear restriction on f, while the latter deals with a case where f is sublinear and it is small with respect to the linear part. This discussion is in Section 3. In Section 4 we apply our abstract results for the controllability of an integro-differential equation in $\mathbb{R}^n$. Notations and preliminary results appear in Section 2.
There is an increasing interest in the study of solutions of infinite dimensional systems that satisfy some nonlocal condition; the nonlinear term in these models may also include some delay. These topics are widely investigated in the recent book by Burlică-Necula-Roşu-Vrabie [7]. The present controllability discussion can be extended, with suitable changes, to these models.

2. Notations and preliminary results

Let H be an infinite dimensional separable real Hilbert space with a scalar product $(\cdot,\cdot)$. Denote by $\|\cdot\|$ its norm and by $\{e_n\}_{n\in\mathbb{N}}$ an *orthonormal basis*, whose existence is granted by the separability of H. Let $H_n = \operatorname{span}\{e_1,\ldots,e_n\}$ denote the n-dimensional Hilbert space generated by the first n vectors of the basis, and $P_n : H \to H_n$ the *natural projection*, $P_n(x) = \sum_{k=1}^{n}(x\,,e_n)e_n$.

Proposition 2.1. *Let H be a separable Hilbert space and P_n be the natural projections. Then*

(i) $\|P_n(x)\| \le \|x\| \quad \forall n \in \mathbb{N},\ \forall x \in H$;
(ii) *if $x_j \rightharpoonup x$ then $P_n x_j \to P_n x$ for every $n \in \mathbb{N}$;*
(iii) *if $x_n \rightharpoonup x$ then $P_n(x_n) \rightharpoonup x$.*

Proof. (i) and (ii) are well known results, (iii) is proved, for example, in [5, Lemma 6]. □

The following proposition shows that every surjective, bounded, linear operator defined in a Hilbert space admits a right inverse of minimal norm.

Proposition 2.2. *Let H be a Hilbert space, W a Banach space and $G : H \to W$ be a surjective, bounded, linear operator. Then there exists a bounded linear operator $\tilde{G}^{-1} : W \to H$ such that, for every $w \in W$, $G \circ \tilde{G}^{-1}(w) = w$ and*

$$\left\|\tilde{G}^{-1}(w)\right\| = \min\left\{\|u\| : G(u) = w\right\}. \tag{2.1}$$

Proof. Since G is bounded, $K = \ker G$ is a closed subspace of H and the quotient space H/K is a Banach space with the norm

$$|||[u]||| = \inf_{v\in[u]} \|v\|, \qquad [u] = \{v \in H : G(v) = G(u)\} = u + K \tag{2.2}$$

(see [15, Theorem 5.1]). The bounded linear operator $\bar{G} : H/K \to W$ defined by $\bar{G}([u]) = G(u)$ is one to one and onto, then there exists $\bar{G}^{-1} : W \to H/K$ linear and bounded.
The weak lower semicontinuity of the norm and the weak compactness of the closed balls in H imply that the infimum in (2.2) is reached. Moreover, the strict convexity

of the norm in a Hilbert space implies the uniqueness of the minimum, that is for every $u \in H$ there exist one and only one $\bar{u} \in [u]$ such that

$$\|\|[u]\|\| = \min_{v\in[u]} \|v\| = \|\bar{u}\|. \tag{2.3}$$

Denote by $K^\perp$ the orthogonal complement of K, then $H = K \oplus K^\perp$, that is for every $u \in H$ there exist only one $k_u \in K$ and only one $h_u \in K^\perp$ such that $u = h_u + k_u$ and $K \cap K^\perp = \{0\}$. Moreover $\|u\|^2 = \|h_u\|^2 + \|k_u\|^2 \geq \|h_u\|^2$. For every $u \in H$

$$G(u) = G(h_u) + G(k_u) = G(h_u),$$

then $h_u \in [u]$. Moreover, for every $v \in [u]$

$$v - u = k \in K \quad \Rightarrow \quad h_v - h_u = k - k_v + k_u \in K \cap K^\perp \quad \Rightarrow \quad h_v = h_u.$$

Therefore $\|\|[u]\|\| = \|h_u\|$.
The function $\pi : H/K \to K^\perp$, $\pi[u] = h_u$, is an isometry and $\tilde{G}^{-1} : W \to K^\perp$ defined by $\tilde{G}^{-1} = \pi \circ \bar{G}^{-1}$ has the following properties: for every $w \in W$

(a) for every $u \in G^{-1}(w)$

$$G(\tilde{G}^{-1}(w)) = G(\pi(\bar{G}^{-1}(w))) = G(\pi([u])) = G(h_u) = w;$$

(b) since π is an isometry, by (2.3)

$$\|\tilde{G}^{-1}(w)\| = \left\|\pi(\bar{G}^{-1}(w))\right\| = \left\|\left\|\bar{G}^{-1}(w)\right\|\right\| = \min_{v\in\bar{G}^{-1}(w)} \|v\|,$$

but $v \in \bar{G}^{-1}(w)$ if and only if $G(v) = w$, then

$$\|\tilde{G}^{-1}(w)\| = \min\left\{\|v\| : G(v) = w\right\}.$$

□

Remark 2.3. Notice that, if H is simply a uniformly convex Banach space, for every $G : H \to W$ linear, bounded and onto it is possible to define a function $\tilde{G}^{-1} : W \to H$ such that (2.1) holds, but in general this function is not linear. In fact, in general, the map $\pi : H/K \to H$ such that, for every $u \in H$, $\pi([u]) \in [u]$ and

$$\|\pi([u])\| = \min_{v\in[u]} \|v\|,$$

is homogeneous but not additive. For example[1] consider $H = \mathbb{R}^3$ with the norm $\|(x\,,y\,,z)\|_4 = \sqrt[4]{x^4 + y^4 + z^4}$, $W = \mathbb{R}^2$ with the euclidian norm and the linear map $G : H \to W$ defined by $G(x\,,y\,,z) = (x - y\,,y - z)$. G is onto and $K = \ker G = \{(t\,,t\,,t) : t \in \mathbb{R}\}$. Then $[(x\,,y\,,z)] = \{(x + t\,,y + t\,,z + t) : t \in \mathbb{R}\}$,

$$\|\|[(x\,,y\,,z)]\|\| = \|(x + \bar{t}\,,y + \bar{t}\,,z + \bar{t})\|_4 = \min_{t\in\mathbb{R}} \|(x + t\,,y + t\,,z + t)\|_4$$

and $\pi[(x\,,y\,,z)] = (x + \bar{t}\,,y + \bar{t}\,,z + \bar{t})$. Simple computations lead to

$$\pi[(1, 0, 0)] = \left(\frac{\sqrt[3]{2}}{1 + \sqrt[3]{2}}\,, \frac{-1}{1 + \sqrt[3]{2}}\,, \frac{-1}{1 + \sqrt[3]{2}}\right)$$

$$\pi[(0, 1, 0)] = \left(\frac{-1}{1 + \sqrt[3]{2}}\,, \frac{\sqrt[3]{2}}{1 + \sqrt[3]{2}}\,, \frac{-1}{1 + \sqrt[3]{2}}\right)$$

$$\pi[(1, 1, 0)] = \left(\frac{1}{1 + \sqrt[3]{2}}\,, \frac{1}{1 + \sqrt[3]{2}}\,, \frac{-\sqrt[3]{2}}{1 + \sqrt[3]{2}}\right)$$

then $\pi[(1, 0, 0)] + \pi[(0, 1, 0)] \neq \pi[(1, 1, 0)]$.

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[1] J. Rosenberg, Lectures notes for the course of Functional Analysis, University of Maryland

In the sequel, for vector valued functions we will consider strong measurability (or simply *measurability*) and *Bochner integrability.* Since H is separable, for functions with values in H measurability is indifferently strong and weak measurability and the integrals are indifferently Bochner or Pettis integrals (see e.g. [18, Theorem 1.1.3]).

Remark 2.4. If a function $\varphi : [0\,,T] \to H$ is measurable, respectively integrable, in H and H is continuously embedded in a Banach space E, trivially φ is measurable, respectively integrable, in E too and the integral in E is equal to the integral in H.

For $1 \le p \le \infty$, $L^p([0\,,T]\,,H)$, $L^p(0\,,T)$ when $H = \mathbb{R}$, denotes the Banach space of equivalence classes of functions $y : [0\,,T] \to H$ such that y is measurable in $[0\,,T]$ and $\|y\|_p < +\infty$, where $\|\cdot\|_p$ is the usual norm defined by

$$\|y\|_p = \left(\int_0^T \|y(t)\|^p \, dt\right)^{\frac{1}{p}}$$

when $1 \le p < \infty$ and

$$\|y\|_\infty = \operatorname*{ess\,sup}_{t\in[0\,,T]} \|y(t)\|$$

when $p = \infty$. Since H is a Hilbert space, also $L^2([0\,,T]\,,H)$ is a Hilbert space with the inner product

$$(y_1\,,y_2)_2 = \int_0^T (y_1(t)\,,y_2(t))\,dt.$$

$\mathcal{C}([0\,,T]\,,H)$, the space of continuous functions $y : [0\,,T] \to H$, is a Banach space with the norm

$$\|y\|_\infty = \max\{\|y(t)\| : \ t \in [0\,,T]\}.$$

In the last section we will consider the Sobolev spaces $W^{1,p}(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$, with the norms

$$\|u\|_{1,p} = \left(\int_\Omega |u(x)|^p \, dx + \sum_{i=1}^n \int_\Omega |u_{x_i}(x)|^p \, dx\right)^{\frac{1}{p}}$$

when $1 \le p < \infty$ and

$$\|u\|_{1,\infty} = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| + \sum_{i=1}^n \operatorname*{ess\,sup}_{x\in\Omega} |u_{x_i}(x)|$$

when $p = \infty$. If $p = 2$, $W^{1,2}(\Omega)$ is a Hilbert space endowed with the inner product

$$\langle u, v\rangle_{1,2} = \int_\Omega u(x)v(x)\,dx + \sum_{i=1}^n \int_\Omega u_{x_i}(x)v_{x_i}(x)\,dx.$$

$\mathcal{AC}([0\,,T]\,,H)$ denotes the space of absolutely continuous functions $y : [0\,,T] \to H$. Recall that a function $y : [0\,,T] \to H$ is absolutely continuous if and only if there exists $g \in L^1([0\,,T]\,,H)$ such that

$$y(t) = y(0) + \int_0^t g(s)\,ds, \quad t \in [0\,,T].$$

Moreover y is a.e. differentiable on $[0\,,T]$ and $y'(t) = g(t)$ for a.e. $t \in [0\,,T]$ (see [2, Chap. I, Theorem 2.1]).

Given a bounded and linear operator $A : H \to H$, the semigroup generated by A is defined by

$$e^{At} = \sum_{k=0}^\infty \frac{A^k t^k}{k!}, \quad t \ge 0.$$

Similarly, $-A$ is a bounded and linear operator generating the semigroup $\{e^{-At}\}_{t\geq 0}$. It is then well known (see [14, p. 22]) that A generates the group defined as

$$e^{At} = \begin{cases} e^{At} & \text{if } t \geq 0 \\ e^{-A(-t)} & \text{if } t < 0. \end{cases}$$

We have that

$$\frac{d}{dt}e^{At} = Ae^{At}, \quad \text{for every } t \in \mathbb{R},$$

then $\left\{e^{At}\right\}_{t\in\mathbb{R}}$ is the $\mathcal{C}^0$ group generated by A.
It is well known (see e.g. [14, p. 2]) that

$$\|e^{At}\| \leq e^{\|A\||t|} \quad \text{for all } t \in \mathbb{R}$$

and, given $g \in L^1([0,T],H), y_0 \in H$, the initial value problem

$$\begin{cases} y'(t) = Ay(t) + g(t) \\ y(0) = y_0 \end{cases}$$

has a unique absolutely continuous solution satisfying the constant variation formula

$$y(t) = e^{At}y_0 + \int_0^t e^{A(t-s)}g(s)ds, \qquad t \in [0\,,T].$$

3. The abstract problem

In this section we consider the control problem

$$(\mathcal{P}) \qquad \begin{cases} y'(t) = Ay(t) + f(t\,,y(t)) + Bu(t) \\ y(0) = y_0 \end{cases}$$

where $A : H \to H$ is a bounded linear operator, $f : [0\,,T] \times H \to H$, H is a separable Hilbert space, and $B : U \to H$ is a bounded linear operator defined in a Hilbert space U. Moreover, suppose that H is compactly embedded in a Banach space $(E\,,\|\cdot\|_E)$, therefore there exists a costant $\lambda > 0$ such that $\|\cdot\|_E \leq \lambda\|\cdot\|$. In the sequel we will consider the following assumptions on f and B:

- (f_1) for every $y \in H$ the function $f(\cdot\,,y) : [0\,,T] \to H$ is measurable with respect to the Lebesgue measure on $[0\,,T]$ and the Borel measure on H;
- (f_2) for almost every $t \in [0\,,T]$ the function $f(t\,,\cdot) : H \to H$ is continuous with respect to the norm in the Banach space E;
- (f_3) for every bounded set $D \subset H$ there exists $\nu_D \in L^1(0\,,T)$ such that
$$\|f(t\,,y)\| \leq \nu_D(t) \quad \text{ for each } y \in D \text{ and for a.e. } t \in [0\,,T];$$
- (B) the linear operator $G : L^2([0\,,T]\,,U) \to H$ defined by
$$G(u) = \int_0^T e^{A(T-s)}Bu(s)\,ds$$
is onto.

Now we need to show the measurability of $f(\cdot\,,q(\cdot))\colon [0\,,T] \to H$ when $q \in \mathcal{C}([0,T]\,,H)$. This property is trivially satisfied if we replace assumption (f_2) with

> for almost every $t \in [0,T]$ the function $f(t\,,\cdot)\colon H \to H$ is continuous with respect to the norm in H.

The following proposition contains the proof of such mesaurability under condition (f_2).

Proposition 3.1. *Let conditions (f_1), (f_2), (f_3) hold and let $q \in \mathcal{C}([0,T],H)$. Then the function $f(\cdot,q(\cdot)) : [0,T] \to H$ is measurable.*

Proof. For every $n \in \mathbb{N}$ consider the piecewise constant functions

$$q_n(t) = \begin{cases} q(\frac{T}{n}(i-1)) & \text{if } \frac{T}{n}(i-1) \le t < \frac{T}{n}i,\ i = 1, ..., n \\ q(T) & \text{if } t = T \end{cases}$$

and

$$f_n(t) = f(t, q_n(t)),\ \ t \in [0, T],\quad n \in \mathbb{N}.$$

From condition (f_1) we get that f_n is measurable in H, hence in E (Remark 2.4). Since q is continuous, $q_n(t) \overset{\|\cdot\|}{\to} q(t)$, therefore the compact embedding and (f_2) imply that $f_n(t) \overset{\|\cdot\|_E}{\to} f(t, q(t))$ for every $t \in [0, T]$, i.e. $f(\cdot, q(\cdot))$ is measurable as function from $[0, T]$ to E.
Moreover, for every $n \in \mathbb{N}$ and $t \in [0, T]$, $\|q_n(t)\| \le \|q\|_\infty$. From condition (f_3) we get that $\{f_n\}_n$ is weakly relatively compact in $L^1([0, T), H)$, hence there exists a subsequence, still denoted as the sequence, weakly converging to $g \in L^1([0, T], H)$. Mazur's lemma then implies that there exists a sequence of convex combinations $\{\tilde{f}_n\}_n$ converging to g in $L^1([0, T), H)$. We can then conclude that, eventually passing to a subsequence, $\tilde{f}_n(t) \overset{\|\cdot\|}{\to} g(t)$ and $\tilde{f}_n(t) \overset{\|\cdot\|_E}{\to} g(t)$ for a.e. $t \in [0, T]$ and the uniqueness of the limit implies that $f(\cdot, q(\cdot)) = g \in L^1([0, T), H)$, hence it is measurable.

□

By Proposition 2.2 the linear operator G admits a right inverse of minimal norm.

Proposition 3.2. *If G is the linear operator defined in (B), then there exists a bounded linear operator $\tilde{G}^{-1} : H \to L^2([0\,,T]\,,U)$, $G \circ \tilde{G}^{-1}(w) = w$ for all $w \in H$, with the property that*

$$\left\|\tilde{G}^{-1}(w)\right\|_2 = \min\left\{\|u\|_2 :\, G(u) = w\right\}, \quad \forall w \in H. \tag{3.1}$$

Recall that, by Remark 2.3, this result in general does not hold if U (and then $L^2([0\,,T]\,,U)$) is not a Hilbert space, even if U is a uniformly convex Banach space.

Definition 3.3. We say that problem $(\mathcal{P})$ is *controllable* on $[0\,,T]$ if for all y_0, y_1 in H there exist $u \in L^2([0\,,T]\,,U)$ and a solution $y \in \mathcal{AC}([0\,,T]\,,H)$ to $(\mathcal{P})$ such that $y(T) = y_1$.

We will give two controllability results by assuming more restrictive growth assumptions instead of (f_3). In both proofs the idea is to approximate the original problem by a family of problems in finite dimensional spaces: for every n we obtain the finite dimensional control problem

$$\begin{cases} y'(t) = P_nAy(t) + P_nf(t\,,y(t)) + P_nBu(t) \\ y(0) = P_ny_0 \end{cases} \tag{$\mathcal{P}_n$}$$

where $P_n : H \to H_n$ is the natural projection.

Theorem 3.4. *Let conditions (f_1), (f_2) and (B) hold. In addition suppose that for every $N \in \mathbb{N}$ there exists $\varphi_N \in L^1([0\,,T]\,,[0\,,+\infty))$ such that*

$$\begin{cases} \liminf\limits_{N\to\infty} \dfrac{1}{N}\displaystyle\int_0^T \varphi_N(s)\,ds = 0, \\ \sup\limits_{\|x\|\le N} \|f(t\,,x)\| \le \varphi_N(t) \quad \text{for a.e. } t \in [0\,,T]. \end{cases} \tag{f_3^1}$$

Then problem $(\mathcal{P})$ is controllable.

Theorem 3.5. *Let conditions (f_1), (f_2), and (B) hold. In addition suppose that there exists $\alpha \in L^1([0\,,T]\,,[0\,,+\infty))$ such that*

$$(f_3^2) \qquad \begin{cases} \|f(t\,,x)\| \le \alpha(t)(1+\|x\|) & \text{for a.e. } t\in[0\,,T], \text{ for every } x\in H,\\ e^{\|A\|T}\|\alpha\|_1\left(1+e^{\|A\|T}\sqrt{T}\|B\|\|\tilde{G}^{-1}\|\right)<1. \end{cases}$$

Then problem ($\mathcal{P}$) is controllable.

Remark 3.6. Both (f_3^1) and (f_3^2) imply (f_3).

Proof of Theorem 3.4. We have to prove that for all $y_0, y_1 \in H$ there exists $u \in L^2([0\,,T]\,,U)$ and a solution $y(\cdot)\in\mathcal{AC}([0\,,T]\,,H)$ to ($\mathcal{P}$) such that $y(T)=y_1$.

Step 1. For every $n\in\mathbb{N}$ we will prove that there exists a solution $y_n:[0\,,T]\to H_n$ to ($\mathcal{P}_n$) such that $y_n(T)=P_ny_1$.
For $N>0$ consider the subset of $\mathcal{C}([0\,,T]\,,H_n)$ defined by

$$Q_N^n=\{q\in\mathcal{C}([0\,,T]\,,H_n):\ \|q(t)\|\le N,\ \forall t\in[0\,,T]\}$$

and the integral operator $\mathcal{T}_n: Q_N^n\to\mathcal{C}([0\,,T]\,,H_n)$ defined by

$$\begin{aligned}\mathcal{T}_n(q)(t)=P_ne^{At}y_0+&\int_0^t P_ne^{A(t-s)}f(s\,,q(s))\,ds\\ &+\int_0^t P_ne^{A(t-s)}B\left(\tilde{G}^{-1}(p_q)(s)\right)ds\quad t\in[0\,,T],\end{aligned}$$

where $\tilde{G}^{-1}$ is the right-inverse of G defined in Proposition 3.2 and

$$p_q=y_1-e^{AT}y_0-\int_0^T e^{A(T-s)}f(s\,,q(s))\,ds\in H.$$

In order to apply Schauder's fixed point theorem, we have to prove that for every $n\in\mathbb{N}$

Claim 1: for every $N>0$, $\mathcal{T}_n(Q_N^n)$ is relatively compact;
Claim 2: for every $N>0$, $\mathcal{T}_n: Q_N^n\to\mathcal{C}([0\,,T]\,,H_n)$ is continuous;
Claim 3: there exists $\bar{N}>0$, which does not depend on n, such that $\mathcal{T}_n(Q_{\bar{N}}^n)\subseteq Q_{\bar{N}}^n$.

Proof of Claim 1. Since H_n is finite dimensional, by Ascoli-Arzelà theorem we have to prove that continuous functions in $\{\mathcal{T}_n(q):\ q\in Q_N^n\}$ are bounded and equicontinuous. For every $q\in Q_N^n$ and $t\in[0\,,T]$, consider $x(t)=\mathcal{T}_n(q)(t)$. Note that

$$\begin{aligned}\|p_q\|&\le\|y_1\|+\|e^{AT}y_0\|+\left\|\int_0^T e^{A(T-s)}f(s\,,q(s))\,ds\right\|\\ &\le\|y_1\|+\ e^{\|A\|T}\left[\|y_0\|+\int_0^T\varphi_N(t)\,dt\right]\end{aligned}$$

and set $C_1=\|y_1\|+e^{\|A\|T}[\|y_0\|+\|\varphi_N\|_1]$. Analogously, by (i) of Proposition 2.1,

$$\left\|P_ne^{At}y_0+\int_0^t P_ne^{A(t-s)}f(s\,,q(s))\,ds\right\|\le C_1,$$

and, by Cauchy-Schwarz inequality in $L^2(0\,,T)$,

$$\left\|\int_0^t P_n e^{As} B\left(\tilde{G}^{-1}(p_q)(s)\right) ds\right\| \leq \int_0^T e^{\|A\|s}\left\|B\left(\tilde{G}^{-1}(p_q)(s)\right)\right\| ds$$
$$\leq e^{\|A\|T}\|B\|\int_0^T \left\|\tilde{G}^{-1}(p_q)(s)\right\|_U ds \leq e^{\|A\|T}\|B\|\sqrt{T}\left\|\tilde{G}^{-1}(p_q)\right\|_2$$
$$\leq e^{\|A\|T}\|B\|\sqrt{T}\left\|\tilde{G}^{-1}\right\|\|p_q\| \leq C_1 e^{\|A\|T}\|B\|\sqrt{T}\left\|\tilde{G}^{-1}\right\|.$$

Finally, by the very definition of $\mathcal{T}_n$,

$$\|x(t)\| \leq C_1 + C_1 e^{\|A\|T}\|B\|\sqrt{T}\left\|\tilde{G}^{-1}\right\|, \tag{3.2}$$

therefore $\mathcal{T}_n(Q_N^n)$ is bounded.
By the same arguments, and recalling

$$y(t) = e^{At}y_0 + \int_0^t e^{A(t-s)} f(s\,,q(s))\,ds + \int_0^t e^{A(t-s)} B\left(\tilde{G}^{-1}(p_q)(s)\right) ds$$

is the absolutely continuous solution of $y'(t) = Ay(t) + f(t,q(t)) + B\tilde{G}^{-1}(p_q)(t)$ and P_n is linear and bounded, for every $0 \leq t_1 < t_2 \leq T$

$$\|x(t_1) - x(t_2)\| = \|P_n y(t_1) - P_n y(t_2)\| \leq \|y(t_1) - y(t_2)\| \leq \int_{t_1}^{t_2} \|y'(s)\| ds$$
$$\leq \|A\|\left[C_1 + C_1 e^{\|A\|T}\|B\|\sqrt{T}\left\|\tilde{G}^{-1}\right\|\right](t_2 - t_1)$$
$$+ \int_{t_1}^{t_2} \varphi_N(s)\,ds + C_1\|B\|\sqrt{t_2 - t_1}\left\|\tilde{G}^{-1}\right\|,$$

then the functions in $\mathcal{T}_n(Q_N^n)$ are equicontinuous.

Proof of Claim 2. Let $\{q_j\}_j$ be a sequence in Q_N^n uniformly convergent to $q \in Q_N^n$. We have to prove that $\mathcal{T}_n(q_j) \to \mathcal{T}_n(q)$ as $j \to \infty$ uniformly on $[0\,,T]$. From Claim 1 we know $\{\mathcal{T}_n(q_j)\}_j$ is equicontinuous. To conclude it is sufficient to prove that $\mathcal{T}_n(q_j)(t) \to \mathcal{T}_n(q)(t)$ as $j \to \infty$ for every $t \in [0\,,T]$.
Fix $t \in [0, T]$ and $s \in [0\,,t]$ such that, by (f_3^1),

$$\|f(s\,,q_j(s))\| \leq \varphi_N(s).$$

Then, for every subsequence $\{f(s, q_{j_k}(s))\}_k$ there exists $w(s) \in H$ and a subsequence $\{f(s, q_{j_{k_h}}(s))\}_h$ such that

$$f(s, q_{j_{k_h}}(s)) \rightharpoonup w(s) \quad \text{ in } H$$

which implies that

$$f(s, q_{j_{k_h}}(s)) \xrightarrow{\|\cdot\|_E} w(s).$$

Since $q_j \to q$ in $\mathcal{C}([0,T], H)$, (f_2) implies that

$$f(s, q_j(s)) \xrightarrow{\|\cdot\|_E} f(s, q(s))$$

i.e. $w(s) = f(s, q(s))$. Thus each subsequence of $\{f(s, q_j(s))\}_j$ admits a subsequence weakly converging to $f(s, q(s))$. We get that

$$f(s, q_j(s)) \rightharpoonup f(s, q(s)) \quad \text{ in } H$$

for a.e. $s \in [0, t]$. On the other hand, the boundedness of $e^{A(t-s)}$ then yields

$$e^{A(t-s)} f(s, q_j(s)) \rightharpoonup e^{A(t-s)} f(s, q(s)) \quad \text{ in } H, \tag{3.3}$$

therefore, by Proposition 2.1 (ii)

$$P_n e^{A(t-s)} f(s, q_j(s)) \to P_n e^{A(t-s)} f(s, q(s))$$

for a.e. $s \in [0,t]$. Since

$$\|P_n e^{A(t-s)} f(s, q_j(s))\| \le e^{\|A\|T} \varphi_N(s),$$

we can conclude that

$$P_n e^{A(t-\cdot)} f(\cdot, q_j(\cdot)) \to P_n e^{A(t-\cdot)} f(\cdot, q(\cdot))$$

in $L^1([0,t], H_n)$ by the dominated convergence theorem. In particular

$$\int_0^t P_n e^{A(t-s)} f(s, q_j(s)) ds \to \int_0^t P_n e^{A(t-s)} f(s, q(s)) ds. \tag{3.4}$$

Now we want to prove that $p_{q_j} \rightharpoonup p_q$ in H. Setting

$$g_j(s) = e^{A(T-s)} f(s, q_j(s)) \quad \text{ and } \quad g(s) = e^{A(T-s)} f(s, q(s)), \qquad s \in [0\,, T],$$

we have to prove that $\int_0^T g_j(s)\, ds \rightharpoonup \int_0^T g(s)\, ds$. By (3.3), for every $w \in H$ and for a.e. $s \in [0\,, T]$

$$(w\,, g_j(s)) \to (w\,, g(s)) \quad \text{ as } j \to \infty,$$

moreover

$$|(w\,, g_j(s))| \le \|w\| e^{\|A\|T} \varphi_N(s),$$

then, by dominated convergence theorem, for every $w \in H$

$$\left(w\,, \int_0^T g_j(s)\, ds\right) = \int_0^T (w\,, g_j(s))\, ds \to \int_0^T (w\,, g(s))\, ds = \left(w\,, \int_0^T g(s)\, ds\right)$$

as $j \to \infty$, proving that $p_{q_j} \rightharpoonup p_q$.

Let us show that, for every $t \in [0\,, T]$,

$$\int_0^t e^{A(t-s)} B\left(\tilde{G}^{-1}(p_{q_j})(s)\right) ds \rightharpoonup \int_0^t e^{A(t-s)} B\left(\tilde{G}^{-1}(p_q)(s)\right) ds \tag{3.5}$$

in H. Consider $\phi \in H$ and define the linear functional $\tilde{\phi} : H \to \mathbb{R}$ as

$$\tilde{\phi}(w) = \left(\phi, \int_0^t e^{A(t-s)} B\left(\tilde{G}^{-1}(w)(s)\right) ds\right), \qquad w \in H.$$

By continuity of B and $\tilde{G}^{-1}$ and by Cauchy-Schwarz inequality in $L^2(0\,, T)$, for every $w \in H$ we have

$$\left|\tilde{\phi}(w)\right| \le \|\phi\| \left\| \int_0^t e^{A(t-s)} B\left(\tilde{G}^{-1}(w)(s)\right) ds \right\| \le \|\phi\| e^{\|A\|T} \|B\| \sqrt{T} \left\|\tilde{G}^{-1}\right\| \|w\|.$$

Therefore $\tilde{\phi}$ is bounded and $\tilde{\phi}(p_{q_j}) \to \tilde{\phi}(p_q)$, then (3.5) follows. As before we can show that

$$\int_0^t P_n e^{A(t-s)} B\left(\tilde{G}^{-1}(p_{q_j})(s)\right) ds \to \int_0^t P_n e^{A(t-s)} B\left(\tilde{G}^{-1}(p_q)(s)\right) ds. \tag{3.6}$$

Then (3.4) and (3.6) imply that $\mathcal{T}_n(q_j)(t) \to \mathcal{T}_n(q)(t)$ for every $t \in [0\,, T]$ and the claim is proved.

Proof of Claim 3. By (3.2), for every $n \in \mathbb{N}$

$$\sup_{q \in Q_N^n} \|\mathcal{T}_n(q)\|_\infty \le C_2 \left(\|y_1\| + e^{\|A\|T}[\|y_0\| + \|\varphi_N\|_1]\right), \tag{3.7}$$

with $C_2 = 1 + e^{\|A\|T} \|B\| \sqrt{T} \left\|\tilde{G}^{-1}\right\|$. Recall that, by (f_3^1),

$$\liminf_{N \to \infty} \frac{\|\varphi_N\|_1}{N} = \liminf_{N \to \infty} \frac{1}{N} \int_0^T \varphi_N(s)\, ds = 0,$$

then

$$\liminf_{N\to\infty}\left(\frac{\|y_1\|+e^{\|A\|T}\|y_0\|}{N}+e^{\|A\|T}\frac{\|\varphi_N\|_1}{N}\right)=0.$$

Therefore, there exists $\bar{N}>0$ (indipendent of n) such that

$$\frac{1}{\bar{N}}\sup_{q\in Q_{\bar{N}}}\|\mathcal{T}_n(q)\|_\infty\le C_2\left(\frac{\|y_1\|+e^{\|A\|T}\|y_0\|}{\bar{N}}+e^{\|A\|T}\frac{\|\varphi_{\bar{N}}\|_1}{\bar{N}}\right)<1$$

and the claim is proved.

Finally, applying Schauder's fixed point theorem we prove that for every $n\in\mathbb{N}$ there exists a fixed point y_n of $\mathcal{T}_n$. The function $y_n:[0\,,T]\to H_n$ with the control $u=\tilde{G}^{-1}\left(p_{y_n}\right)$ is a solution to $(\mathcal{P}_n)$ such that $y_n(T)=P_ny_1$.

Step 2. We shall prove that the sequence $\{y_n\}_n$ found in the previous step, verifying

$$\begin{aligned}y_n(t)=P_ne^{At}y_0+&\int_0^t P_ne^{A(t-s)}f(s\,,y_n(s))\,ds\\ &+\int_0^t P_ne^{A(t-s)}B\left(\tilde{G}^{-1}(p_{y_n})(s)\right)ds,\end{aligned}\tag{3.8}$$

admits a subsequence converging to a function $y:[0\,,T]\to H$ such that

$$y(t)=e^{At}y_0+\int_0^t e^{A(t-s)}f(s\,,y(s))\,ds+\int_0^t e^{A(t-s)}B\left(\tilde{G}^{-1}(p_y)(s)\right)ds\tag{3.9}$$

and $y(T)=y_1$. Then y is a solution to $(\mathcal{P})$ and $u=\tilde{G}^{-1}(p_y)$ is the associated control.

For every $N>0$, let Q_N be the subset of $\mathcal{C}([0\,,T]\,,H)$ defined by

$$Q_N=\{q\in\mathcal{C}([0\,,T]\,,H):\ \|q(t)\|\le N,\ \forall t\in[0\,,T]\}\,.$$

By *Claim 3*, for every $n\in\mathbb{N}$, $y_n\in Q_{\bar{N}}$ then, setting $g_n(t)=f(t\,,y_n(t))$,

$$\|g_n(s)\|\le\varphi_{\bar{N}}(s)\qquad\text{for a.e. } s\in[0\,,T].$$

Hence $\{g_n:\ n\in\mathbb{N}\}$ is relatively weakly compact in $L^1([0\,,T]\,,H)$ (see [9] Theorem 1, p. 101) and there exists a subsequence $\{g_{n_k}\}_k$ and $g\in L^1([0\,,T]\,,H)$ such that $g_{n_k}\rightharpoonup g$ in $L^1([0\,,T]\,,H)$. Fix $t\in[0,T]$. A fortiori, $g_{n_k}\rightharpoonup g$ in $L^1([0\,,t]\,,H)$ and, as in the proof of (3.5) , it follows that

$$\int_0^t e^{A(t-s)}g_{n_k}(s)\,ds\rightharpoonup\int_0^t e^{A(t-s)}g(s)\,ds\quad\text{ in } H.$$

By Proposition 2.1 (iii)

$$P_{n_k}e^{At}y_0\overset{\|\cdot\|}{\to}e^{At}y_0\tag{3.10}$$

and

$$P_{n_k}\int_0^t e^{A(t-s)}g_{n_k}(s)\,ds\rightharpoonup\int_0^t e^{A(t-s)}g(s)\,ds\ \text{ in } H\text{, for every } t\in[0\,,T],\tag{3.11}$$

then

$$\begin{aligned}p_{y_{n_k}}=y_1-e^{AT}y_0-\int_0^T e^{A(T-s)}f(s\,,y_{n_k}(s))\,ds\rightharpoonup&\\ \rightharpoonup p=y_1-e^{AT}y_0-&\int_0^T e^{A(T-s)}g(s)\,ds.\end{aligned}$$

in H. Reasoning as in the proof of (3.5) we get that for every $t\in[0\,,T]$

$$\int_0^t e^{A(t-s)}B\left(\tilde{G}^{-1}(p_{y_{n_k}})(s)\right)ds\rightharpoonup\int_0^t e^{A(t-s)}B\left(\tilde{G}^{-1}(p)(s)\right)ds\quad\text{ in } H$$

and, by Proposition 2.1(iii),

$$\int_0^t P_{n_k} e^{A(t-s)} B\left(\tilde{G}^{-1}(p_{y_{n_k}})(s)\right) ds \rightharpoonup \int_0^t e^{A(t-s)} B\left(\tilde{G}^{-1}(p)(s)\right) ds \tag{3.12}$$

in H. Finally, by (3.10), (3.11) and (3.12) for every $t \in [0\,,T]$ we have

$$y_{n_k}(t) \rightharpoonup y(t) = e^{At} y_0 + \int_0^t e^{A(t-s)} g(s)\, ds + \int_0^t e^{A(t-s)} B\left(\tilde{G}^{-1}(p)(s)\right) ds$$

in H and, by compact embedding, $y_{n_k}(t) \xrightarrow{\|\cdot\|_E} y(t)$. Therefore, by (f_2), for almost every $s \in [0,T]$,

$$f(s\,, y(s)) = \lim_{k\to\infty} f(s, y_{n_k}(s))$$

and, by (f_3^1),

$$g_{n_k} = f(\cdot, y_{n_k}(\cdot)) \longrightarrow f(\cdot\,, y(\cdot)) \quad \text{ in } L^1([0\,,T]\,, E).$$

Since the dual space of $L^1([0\,,T]\,,E)$ is continuously embedded in the dual space of $L^1([0\,,T]\,,H)$, a fortiori

$$g_{n_k} = f(\cdot, y_{n_k}(\cdot)) \rightharpoonup f(\cdot\,, y(\cdot)) \quad \text{ in } L^1([0\,,T]\,, H).$$

Then, by the unicity of the weak limit of $\{g_{n_k}\}_k$, $g(t) = f(t\,, y(t))$ for almost every $t \in [0\,,T]$ and

$$p = y_1 - e^{AT} y_0 - \int_0^T e^{A(T-s)} f(s\,, y(s))\, ds.$$

Therefore (3.9) is verified, moreover

$$\begin{aligned} y(T) &= e^{AT} y_0 + \int_0^T e^{A(T-s)} f(s\,, y(s))\, ds + \int_0^T e^{A(T-s)} B\left(\tilde{G}^{-1}(p_y)(s)\right) ds \\ &= e^{AT} y_0 + \int_0^T e^{A(T-s)} f(s\,, y(s))\, ds + G\tilde{G}^{-1}(p_y) \\ &= e^{AT} y_0 + \int_0^T e^{A(T-s)} f(s\,, y(s))\, ds + p_y = y_1. \end{aligned}$$

□

Proof of Theorem 3.5. In the proof of previous theorem the strictly sublinearity condition

$$\liminf_{N\to\infty} \frac{1}{N} \int_0^T \varphi_N(s)\, ds = 0$$

plays a role only in the proof of Claim 3. Then we can repeat all the proof (except Claim 3) with $\varphi_N(t) = \alpha(t)(1+N)$.

It remains to prove that there exists $\bar{N} > 0$ such that $\mathcal{T}_n(Q^n_{\bar{N}}) \subseteq Q^n_{\bar{N}}$, for every $n \in \mathbb{N}$.

Suppose, by contradiction, that for every $N \in \mathbb{N}$ there exist $\bar{n} = \bar{n}(N) \in \mathbb{N}$ and $q_N \in Q^{\bar{n}}_N$ such that $\mathcal{T}_{\bar{n}}(q_N) \notin Q^{\bar{n}}_N$. By (3.7) and $\mathcal{T}_{\bar{n}}(q_N) \notin Q^{\bar{n}}_N$ we have that

$$N < \|\mathcal{T}_{\bar{n}}(q_N)\|_\infty \le C_2 \left(\|y_1\| + e^{\|A\|T}[\|y_0\| + (N+1)\|\alpha\|_1]\right),$$

with C_2 as in the proof of Theorem 3.4. Now dividing by N the first and the last term in the previous inequality and passing to the limit for $N \to \infty$ we obtain

$$1 < C_2 \left(\frac{\|y_1\| + e^{\|A\|T}\|y_0\|}{N} + e^{\|A\|T} \left(\frac{1}{N} + 1\right) \|\alpha\|_1 \right) \to C_2 e^{\|A\|T} \|\alpha\|_1.$$

From (f_3^2) and the definition of C_2 it follows

$$1 \le e^{\|A\|T} \|\alpha\|_1 \left(1 + e^{\|A\|T} \sqrt{T} \|B\| \left\|\tilde{G}^{-1}\right\|\right) < 1,$$

a contradiction. □

4. Applications

As an application of the previous results we will prove the controllability for a control problem of the form

$$z_t(t\,,x) = \int_\Omega k(x,\xi)z(t,\xi)d\xi + g(t\,,x\,,z(t\,,x)) + b(x)u(t\,,x), \tag{4.1}$$

$t \in [0\,,T]$, $x \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is open, bounded with Lipschitz boundary. It is not restrictive to suppose that Ω is connected. The nonlinear evolution equation obtained by (4.1) in the absence of control term, is frequently used for describing the spatial dispersal of organisms (see e.g. [3] and references therein) where z is the density of a single species which is considered in an n-dimensional habitat.

Using the previous notations, the Hilbert space $U = H = W^{1,2}(\Omega)$ is compactly embedded in the Hilbert space $E = L^2(\Omega)$. In order to rewrite problem (4.1) in abstract form, we identify z and u respectively with functions $t \mapsto z(t,\cdot)$ and $t \mapsto u(t,\cdot)$. We look for solution $z \in \mathcal{AC}([0\,,T]\,,W^{1,2}(\Omega))$ associated to the control $u \in L^2([0\,,T]\,,W^{1,2}(\Omega))$.
Suppose that $b \in W^{1,\infty}(\Omega)$, $|b(x)| \geq b_0 > 0$ for every $x \in \Omega$, and consider the following assumptions on functions $k : \Omega\times\Omega \to \mathbb{R}$ and $g : [0\,,T]\times\Omega\times\mathbb{R} \to \mathbb{R}$:

- (k) $k \in C^1(\overline{\Omega}\times\overline{\Omega})$;
- (g_1) $g(\cdot\,,x\,,p) : [0\,,T] \to \mathbb{R}$ is measurable for a.e. $x \in \Omega$ and for every $p \in \mathbb{R}$;
- (g_2) $g(t\,,\cdot\,,p) : \Omega \to \mathbb{R}$ is weakly differentiable for a.e. $t \in [0\,,T]$ and for every $p \in \mathbb{R}$;

moreover there exist four non-negative functions $\beta, \gamma, \delta, \eta \in L^1(0\,,T)$ such that

- (g_3) $|g(t\,,x\,,0)| \leq \eta(t)$ for a.e. $t \in [0\,,T]$ and $x \in \Omega$;
- (g_4) $|g(t\,,x\,,p_1) - g(t\,,x\,,p_2)| \leq \delta(t)|p_1 - p_2|$ for a.e. $t \in [0\,,T]$, $x \in \Omega$ and for every $p_1, p_2 \in \mathbb{R}$;
- (g_5) $|g_{x_i}(t\,,x\,,p)| \leq \beta(t)|p| + \gamma(t)$ for a.e. $t \in [0\,,T]$, $x \in \Omega$ and for every $p \in \mathbb{R}$, $i = 1\ldots n$.

Remark 4.1. Condition (g_5) implies that for every ball $B_r \subset \Omega$

$$|g(t\,,x_1\,,p) - g(t\,,x_2\,,p)| \leq \sqrt{n}\,[\beta(t)|p| + \gamma(t)]\,\|x_1 - x_2\|, \tag{4.2}$$

for a.e. $t \in [0\,,T]$, for every $x_1, x_2 \in B_r$ and $p \in \mathbb{R}$.

In order to apply Theorem 3.5 we have to define A, f and B. The bounded and linear operator $A : W^{1,2}(\Omega) \to W^{1,2}(\Omega)$ is

$$Ay(x) = \int_\Omega k(x,\xi)y(\xi)\,d\xi, \qquad x \in \Omega,\ y \in W^{1,2}(\Omega),$$

the function $f : [0\,,T]\times W^{1,2}(\Omega) \to W^{1,2}(\Omega)$ is

$$f(t,y)(x) = g(t\,,x\,,y(x)), \qquad t \in [0\,,T],\ x \in \Omega,\ y \in W^{1,2}(\Omega)$$

and the linear operator $B : W^{1,2}(\Omega) \to W^{1,2}(\Omega)$ is

$$(By)(x) = b(x)y(x), \qquad x \in \Omega,\ y \in W^{1,2}(\Omega).$$

First of all we have to show that A, f and B are well defined.

The operator B is obviously linear and maps $W^{1,2}(\Omega)$ in itself since $b \in W^{1,\infty}(\Omega)$. Moreover B is bounded and $\|B\| \leq \sqrt{2}\|b\|_{1,\infty}$. In fact, for every $u \in W^{1,2}(\Omega)$

$$\begin{aligned}
\|Bu\|_{1,2} &= \left(\|bu\|_2^2 + \sum_{i=1}^n \|\partial_{x_i}(bu)\|_2^2 \right)^{\frac{1}{2}} \\
&\leq \left(\|b\|_\infty^2 \|u\|_2^2 + \sum_{i=1}^n \|b_{x_i} u + b u_{x_i}\|_2^2 \right)^{\frac{1}{2}} \\
&\leq \left(\|b\|_\infty^2 \|u\|_2^2 + 2\sum_{i=1}^n \left[\|b_{x_i} u\|_2^2 + \|b u_{x_i}\|_2^2 \right] \right)^{\frac{1}{2}} \\
&\leq \left(\|b\|_\infty^2 \|u\|_2^2 + 2\sum_{i=1}^n \left[\|b_{x_i}\|_\infty^2 \|u\|_2^2 + \|b\|_\infty^2 \|u_{x_i}\|_2^2 \right] \right)^{\frac{1}{2}} \\
&\leq \left(2\|b\|_\infty^2 \|u\|_{1,2}^2 + 2\sum_{i=1}^n \|b_{x_i}\|_\infty^2 \|u\|_2^2 \right)^{\frac{1}{2}} \\
&\leq \sqrt{2}\|b\|_{1,\infty}\|u\|_{1,2}.
\end{aligned}$$

Now we have to show that for every $t \in [0\,,T]$ and $y \in W^{1,2}(\Omega)$, $f(t\,,y(\cdot))$ is in $W^{1,2}(\Omega)$, hence that for almost every fixed $t \in [0\,,T]$:

(i) $f(t\,,y)(\cdot) = g(t\,,\cdot\,,y(\cdot)) \in L^2(\Omega)$;
(ii) for every $i = 1,...,n$ there exists $\partial_{x_i} g(t\,,\cdot\,,y(\cdot)) \in L^2(\Omega)$.

(i) By (4.2) and (g_4), for every ball $B_r \subset \Omega$, for a.e. $t \in [0\,,T]$, $x_1, x_2 \in B_r$ and for every $p_1, p_2 \in \mathbb{R}$

$$|g(t\,,x_1\,,p_1) - g(t\,,x_2\,,p_2)| \leq \delta(t)|p_1 - p_2| + \sqrt{n}\,[\beta(t)|p_1| + \gamma(t)]\,\|x_1 - x_2\|, \tag{4.3}$$

therefore $g(t\,,\cdot\,,\cdot)$ is continuous (locally Lipschitz) in $\Omega \times \mathbb{R}$, then $g(t\,,\cdot\,,y(\cdot))$ is measurable. Moreover (g_3) and (g_4) imply that

$$|g(t\,,\cdot\,,y(\cdot))| \leq \delta(t)|y(\cdot)| + \eta(t) \in L^2(\Omega). \tag{4.4}$$

(ii) $g(t\,,\cdot\,,\cdot)$ is locally Lipschitz, y is absolutely continuous on segments of almost all straight lines that are parallel to coordinate axes, then also $g(t\,,\cdot\,,y(\cdot))$ is absolutely continuous on the same segments and the weak derivatives coincide with the classical ones ([12, Theorem 1 p. 4 and Theorem 2 p. 6]). If $\{\mathrm{e}_i : 1 \leq i \leq n\}$ is the standard basis in $\mathbb{R}^n$, then by (g_2), (g_4) and (g_5), for a.e. $t \in [0\,,T]$, $x \in \Omega$ and every h sufficiently small

$$\begin{aligned}
&|g(t\,,x + h\mathrm{e}_i\,,y(x + h\mathrm{e}_i)) - g(t\,,x\,,y(x))| \\
&\quad \leq |g(t\,,x + h\mathrm{e}_i\,,y(x + h\mathrm{e}_i)) - g(t\,,x + h\mathrm{e}_i\,,y(x))| \\
&\qquad + |g(t\,,x + h\mathrm{e}_i\,,y(x)) - g(t\,,x\,,y(x))| \\
&\qquad\quad \leq \delta(t)|y(x + h\mathrm{e}_i) - y(x)| + |h|[\beta(t)|y(x)| + \gamma(t)]
\end{aligned}$$

for every $i = 1, \ldots n$. Therefore

$$|\partial_{x_i}[g(t\,,x\,,y(x))]| \leq \delta(t)|y_{x_i}(x)| + \beta(t)|y(x)| + \gamma(t) \in L^2(\Omega), \tag{4.5}$$

that is $\partial_{x_i} g(t\,,\cdot\,,y(\cdot)) \in L^2(\Omega)$ for a.e. $t \in [0\,,T]$ and every $y \in W^{1,2}(\Omega)$.

Finally we have to prove that $Ay \in W^{1,2}(\Omega)$, for every $y \in W^{1,2}(\Omega)$. From (k) we get that for every $i = 1,..,n$, there exists $\partial_{x_i} Ay(x) = \int_\Omega k_{x_i}(x,\xi)y(\xi)\,d\xi$ and, denoted by $K = \sqrt{\|k\|_\infty^2 + \sum_{i=1}^n \|k_{x_i}\|_\infty^2}$, it follows that

$$\begin{aligned}\|Ay\|_{1,2}^2 &= \int_\Omega \left[\int_\Omega k(x,\xi)y(\xi)\,d\xi\right]^2 dx + \sum_{i=1}^n \int_\Omega \left[\int_\Omega k_{x_i}(x,\xi)y(\xi)\,d\xi\right]^2 dx \\ &\le \|k\|_\infty^2 \|y\|_2^2 |\Omega|^2 + \sum_{i=1}^n \|k_{x_i}\|_\infty^2 \|y\|_2^2 |\Omega|^2\end{aligned}$$

Thus A is bounded and $\|A\| \le K|\Omega|$.

We shall show that hypothesis (f_1), (f_2), (f_3^2) and (B) hold.

(f_1). We have to prove that, for every $y \in W^{1,2}(\Omega)$, the map $f(\cdot\,,y) : [0\,,T] \to W^{1,2}(\Omega)$ is (weakly) measurable, that is that for every $w \in W^{1,2}(\Omega)$ the function $L(f(\cdot\,,y)) : [0\,,T] \to \mathbb{R}$ defined by

$$L(f(t\,,y)) = \int_\Omega g(t\,,x\,,y(x))w(x)\,dx + \sum_{i=1}^n \int_\Omega \partial_{x_i}[g(t\,,x\,,y(x))]w_{x_i}(x)\,dx$$

is measurable.
By (4.3) and (g_1), g is a Carathéodory function, then it is globally measurable in the set $[0\,,T] \times \Omega \times \mathbb{R}$. Moreover, for every measurable $y : \Omega \to \mathbb{R}$, $g(t\,,x\,,y(x))$ is measurable too. Since $\partial_{x_i}[g(t\,,x\,,y(x))]$ is a.e. limit of measurable functions, it is measurable. Then the funcion $h : [0\,,T] \times \Omega \to \mathbb{R}$ defined by

$$h(t\,,x) = g(t\,,x\,,y(x))w(x) + \sum_{i=1}^n \partial_{x_i}[g(t\,,x\,,y(x))]w_{x_i}(x)$$

is globally measurable, hence, by Fubini's theorem, also $L(f(\cdot\,,y))$ is measurable in $[0\,,T]$.

(f_2). For a.e. $t \in [0\,,T]$, the function $f(t\,,\cdot)$ is continuous with respect to the L^2 norm. Indeed, from (g_4) it follows that, for every $y_1, y_2 \in W^{1,2}(\Omega)$

$$\begin{aligned}\|f(t\,,y_1) - f(t\,,y_2)\|_2{}^2 &= \int_\Omega |g(t\,,x\,,y_1(x)) - g(t\,,x\,,y_2(x))|^2\,dx \\ &\le \delta(t)^2 \int_\Omega |y_1(x) - y_2(x)|^2\,dx = \delta(t)^2 \|y_1 - y_2\|_2{}^2.\end{aligned}$$

(f_3^2). In order to verify condition (f_3^2) we have to prove that, in our hypothesis, there exists $\alpha \in L^1(0\,,T)$ such that $\|f(t\,,y)\|_{1,2} \le \alpha(t)(1 + \|y\|_{1,2})$ for a.e. $t \in [0\,,T]$ and for every $y \in W^{1,2}(\Omega)$. By (4.4)

$$\|f(t\,,y)\|_2 \le \delta(t)\|y\|_2 + \sqrt{|\Omega|}\eta(t),$$

whereas by (4.5)

$$\|\partial_{x_i} f(t\,,y)\|_2 \le \beta(t)\|y\|_2 + \gamma(t)\sqrt{|\Omega|} + \delta(t)\|y_{x_i}\|_2, \qquad i = 1, \dots n.$$

Therefore, for every $y \in W^{1,2}(\Omega)$,

$$\begin{aligned}\|f(t\,,y)\|_{1,2}^2 \le &[\delta(t)\|y\|_2 + \sqrt{|\Omega|}\eta(t)]^2 + \sum_{i=1}^n [\beta(t)\|y\|_2 + \gamma(t)\sqrt{|\Omega|} + \delta(t)\|y_{x_i}\|_2]^2 \\ \le &[2\delta(t)^2 + 3n\beta(t)^2]\|y\|_2^2 + 3\delta(t)^2 \sum_{i=1}^n \|y_{x_i}\|_2^2 + [2|\Omega|\eta(t)^2 + 3n\gamma(t)^2|\Omega|] \\ \le &[3\delta(t)^2 + 3n\beta(t)^2]\|y\|_{1,2}^2 + |\Omega|[2\eta(t)^2 + 3n\gamma(t)^2] \le \alpha(t)^2(\|y\|_{1,2} + 1)^2\end{aligned}$$

where $\alpha(t) = \max\left\{\sqrt{3\delta(t)^2 + 3n\beta(t)^2}\,, \sqrt{|\Omega|[2\eta(t)^2 + 3n\gamma(t)^2]}\right\}$.

(B). The linear operator $G : L^2([0\,,T]\,,W^{1,2}(\Omega)) \to W^{1,2}(\Omega)$ is defined by

$$Gu = \int_0^T e^{A(T-s)}[bu(s)]\,ds.$$

Since $b \in W^{1,\infty}(\Omega)$, $|b(x)| \geq b_0 > 0$ and $\{e^{At}\}_{t\in\mathbb{R}}$ is a group, we have that $e^{A(s-T)}w \in W^{1,2}(\Omega)$ for every $w \in W^{1,2}(\Omega)$ and $s \in [0,T]$ and the function $s \to u(s) = \frac{e^{A(s-T)}w}{Tb} \in L^2([0\,,T]\,,W^{1,2}(\Omega))$. In fact, similarly as for $\|B\|$, it follows that, for every $s \in [0,T]$,

$$\begin{aligned}\|u(s)\|_{1,2} \leq & \frac{\sqrt{2}}{T}\left\|\frac{1}{b}\right\|_{1,\infty}\left\|e^{A(s-T)}w\right\|_{1,2} \leq \frac{\sqrt{2}}{T}\left\|\frac{1}{b}\right\|_{1,\infty}\left\|e^{A(s-T)}\right\|\|w\|_{1,2}\\ \leq & \frac{\sqrt{2}}{T}\left\|\frac{1}{b}\right\|_{1,\infty} e^{\|A\|(T-s)}\|w\|_{1,2}\end{aligned}$$

then u is in $L^2([0\,,T]\,,W^{1,2}(\Omega))$. Morever, since A generates a group,

$$G(u) = \int_0^T e^{A(T-s)}[bu(s)]\,ds = \frac{1}{T}\int_0^T e^{A(T-s)}e^{A(s-T)}w\,ds = \frac{1}{T}\int_0^T w\,ds = w,$$

then G is onto.
As to $\|\tilde{G}^{-1}\|$, for every $w \in W^{1,2}(\Omega)$ the very same calculations yield

$$\begin{aligned}\|\tilde{G}^{-1}(w)\|_2 \leq \|u\|_2 &= \left(\int_0^T \|u(s)\|_{1,2}^2\,ds\right)^{\frac{1}{2}}\\ &\leq \frac{\sqrt{2}}{T}\left\|\frac{1}{b}\right\|_{1,\infty}\left(\int_0^T e^{2\|A\|(T-s)}ds\right)^{\frac{1}{2}}\|w\|_{1,2}\\ &= \frac{\sqrt{2}}{T}\left\|\frac{1}{b}\right\|_{1,\infty}\frac{e^{2\|A\|T}-1}{2\|A\|}\|w\|_{1,2}.\end{aligned}$$

Finally we can apply Theorem 3.5.
Since $r \mapsto \frac{e^r-1}{r}$ is increasing and $\left\|\frac{1}{b}\right\|_{1,\infty} \leq \frac{\|b\|_{1,\infty}}{b_0^2}$, if

$$e^{K|\Omega|T}\|\alpha\|_1\left(1 + e^{K|\Omega|T}\,\frac{\|b\|_{1,\infty}^2}{b_0^2}\,\frac{e^{2K|\Omega|T}-1}{K|\Omega|\sqrt{T}}\right) < 1,$$

with α and K defined above, then (f_3^2) holds and we can conclude that for every $y_0, y_T \in W^{1,2}(\Omega)$ there exist $\bar{u} \in L^2([0\,,T]\,,W^{1,2}(\Omega))$ and $\bar{z} \in \mathcal{AC}([0\,,T]\,,W^{1,2}(\Omega))$ such that for $x \in \Omega$

$$\begin{cases} \bar{z}_t(t\,,x) = \displaystyle\int_\Omega k(x,\xi)\bar{z}(t,\xi)d\xi + g(t\,,x\,,\bar{z}(t\,,x)) + b(x)\bar{u}(t\,,x), & t \in [0\,,T],\\ \bar{z}(0\,,x) = y_0(x),\\ \bar{z}(T\,,x) = y_T(x).\end{cases} \tag{4.6}$$

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(L. Malaguti and V. Taddei) Department of Sciences and Methods for Engineering, University of Modena and Reggio Emilia, I-42122 Italy
E-mail address, L. Malaguti: `luisa.malaguti@unimore.it`

(S. Perrotta) Department of Physics Informatics and Mathematics, University of Modena and Reggio Emilia, I-41125 Italy
E-mail address, S. Perrotta: `stefania.perrotta@unimore.it`

E-mail address, V. Taddei: `valentina.taddei@unimore.it`