Study of the Bidirectional Efficiency of Linear and Nonlinear Physical Systems

1st Roberto Zanasi
DIEF - University of Modena and Reggio Emilia
Via Pietro Vivarelli 10 Int. 1, 41100, Modena, Italy
roberto.zanasi@unimore.it

2nd Davide Tebaldi
DIEF - University of Modena and Reggio Emilia
Via Pietro Vivarelli 10 Int. 1, 41100, Modena, Italy
davide.tebaldi@unimore.it

***Abstract*—In this paper, a study of the bidirectional efficiency of linear and nonlinear physical systems is performed. The methodology to compute the bidirectional efficiency map of the system is described, highlighting which is the power flow orientation giving the maximum system efficiency. The designer can therefore easily evaluate whether a physical system, such as for instance an electric machine, is more suitable for being used in forward motor mode rather than in reverse generator mode or viceversa. Three different types of physical systems are modeled and simulated in Matlab/Simulink, and the different characteristics they exhibit in terms of efficiency are highlighted.**

Finally, the properties that the efficiency maps exhibit if the linear system is affected by symmetric or nonsymmetric nonlinearities are studied and commented in detail.

I. Introduction

The problem of addressing the physical systems efficiency evaluation is of primary importance in all industrial application segments, including smart-grid applications, automotive and agricultural industries and all companies making mechatronic systems. For all these industries, the common trend is to move towards the energy efficiency optimization, in order to gain benefits both in terms of improvement of the environmental sustainability and in terms of economic return.

The problem of systems efficiency evaluation is considered in [1]. In this case, the author provides a metric allowing to perform the evaluation of the energy system efficiency. In [2], the authors propose an energy efficiency evaluation system, while in [3] two types of electric machines are considered in order to make a comparison between the efficiency maps.

The problem of the power flow-based efficiency evaluation of physical systems based on the systems model has been addressed by the authors in [4]. The work presented in this paper represents an extent of the power flow-based efficiency evaluation from the unidirectional case in [4] to the bidirectional case, allowing to characterize the physical systems in all the possible operating modes. The analysis that is presented starts from the modeling of the physical elements whose efficiency is to be evaluated, which is performed by using the Power-Oriented Graphs (POG) modeling technique, see [5].

The potentialities of the study proposed in this paper are several. The designer can use this analysis to compute the bidirectional efficiency maps of the system both on the plane of the input power variables and on the plane of the output power variables, so as to identify the power flow orientation maximizing the area of the high-efficiency operating regions and to understand whether the system is more suitable for being used in *forward* power flow or in *reverse* power flow mode. Additionally, the condition upon the system parameters ensuring that the peak efficiency is the same with both orientations of the power flow is derived, and the different characteristics of the efficiency map in presence of symmetric and nonsymmetric nonlinearities are studied.

Three actual physical systems are considered as cases study: a DC electric motor, a mechanical gear transmission and an electric circuit. The bidirectional efficiency analysis is applied to these considered physical systems and the different characteristics that they exhibit in terms of efficiency are analyzed and commented in detail with the support of some simulation results.

This paper is organized as follows: Sec. II provides the definition of unidirectional efficiency of a linear system and shows some features that such efficiency exhibits on the plane of the output power variables (Sec. II-A) and on the plane of the input power variables (Sec. II-B). Sec. III provides the extension of the efficiency definition to the bidirectional power-flow case, both on the planes of the input and output power variables, by also extending the analysis to the case of nonlinear systems (Sec. III-A). In Sec. IV, some simulation results applied to the three considered actual cases study are shown, and the different characteristics they exhibit in terms of efficiency are highlighted and commented in detail. Finally, Sec. V reports the conclusions of the work presented in this paper.

II. Efficiency of linear systems

Let us consider a linear dissipative system $\mathbf{H}(s)$ characterized by two power sections, see Fig. 1, and let (y_1, u_1) and (y_2, u_2) denote the power variables characterizing the input and output power sections, respectively. Fig. 1 puts in evidence the chosen positive directions for the powers $P_1 = u_1 y_1$ and $P_2 = u_2 y_2$: P_1 is positive if it is entering the system, whereas P_2 is positive if it is exiting the system. Let $\mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ be the state space model of system $\mathbf{H}(s)$. The input output matrix $\mathbf{H}(s)$ can be expressed as

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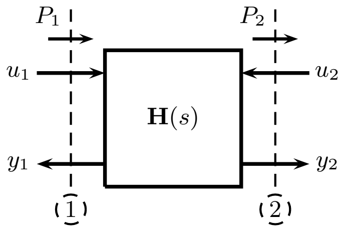

Fig. 1. A linear system $\mathbf{H}(s)$ characterized by two power sections.

follows: $\mathbf{H}(s) = \mathbf{C}(\mathbf{L}\, s - \mathbf{A})^{-1} + \mathbf{D}$. The steady-state matrix $\mathbf{H}_0 = \mathbf{H}(s)|_{s=0}$ can be denoted as follows, see [4]-(2):

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} a & b \\ c & -d \end{bmatrix}}_{\mathbf{H}_0} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\mathbf{u}} \tag{1}$$

The steady-state parameters of the system b and c can also be replaced by parameters f and g as:

$$\begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} g+f \\ g-f \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \frac{b-c}{2} \\ \frac{b+c}{2} \end{bmatrix} \tag{2}$$

By using (2), the strictly dissipative condition for the linear system $\mathbf{H}(s)$ introduced in [4] can also be expressed as:

$$a > 0, \quad d > 0, \quad ad > \frac{(b-c)^2}{4} = f^2 \tag{3}$$

that is $-\sqrt{ad} \le f \le \sqrt{ad}$ or, equivalently, $-1 \le f/\sqrt{ad} \le 1$.

Definition. *In steady-state condition, the "unidirectional power flow efficiency" $E(t)$ of a linear system having the structure shown in Fig. 1 is defined as follows, see [4]:*

$$E(t) = \begin{cases} \dfrac{P_2(t)}{P_1(t)} = \dfrac{u_2(t)\, y_2(t)}{u_1(t)\, y_1(t)} & \text{if} \quad (P_1 > 0) \bigwedge (P_2 > 0) \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

The unidirectional efficiency of a linear system can be shown and analyzed both on the output plane (y_2, u_2) and on the input plane (y_1, u_1): these two cases will be analyzed separately.

A. Efficiency of linear systems on the output plane (y_2, u_2)

An example of the efficiency map $E(t) = E(y_2, u_2)$ on the plane of the output variables $(y_2,\, u_2)$ for a linear system $\mathbf{H}(s)$ characterized by the following parameters:

$$a = 0.2, \quad b = 0.6, \quad c = 0.8, \quad d = 0.1, \quad P_{max} = 100\, W \tag{5}$$

is shown in Fig. 2. The green area in Fig. 2 describes the set of points where the positive powers $P_1 > 0$ and $P_2 > 0$ flow from Section 1 to Section 2, thus the unidirectional efficiency $E(t)$ is positive and less than 1: $0 < E(t) < 1$. The red area in Fig. 2 describes the set of points where the negative powers $P_1 < 0$ and $P_2 < 0$ flow from Section 2 to Section 1, thus the unidirectional efficiency $E(t)$ is greater than 1: $E(t) > 1$. The magenta area in Fig. 2 describes the set of points where power $P_1 > 0$ is positive, power $P_2 < 0$ is negative and the unidirectional efficiency $E(t)$ is negative: $E(t) < 0$. These operating regions will be accounted for by the bidirectional efficiency, whose definition will be provided in Sec. III.

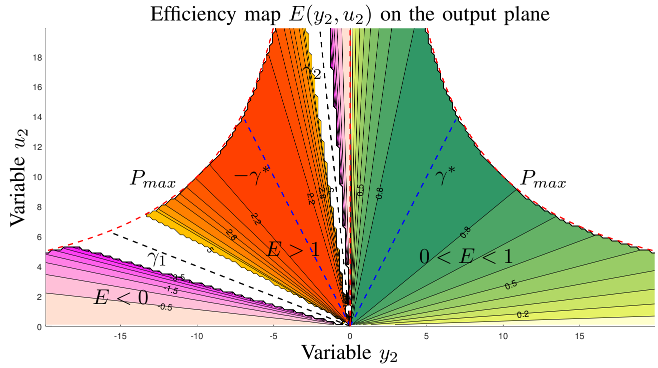

Fig. 2. Efficiency map $E(y_2, u_2)$ shown on the output plane $(y_2,\, u_2)$.

In [4] it has been shown that the efficiency $E(t)$ of a linear system $\mathbf{H}(s)$ is constant along straight lines of slope γ exiting from the origin of the output plane (y_2, u_2), see [4]-(9):

$$u_2 = \gamma\, y_2, \qquad \gamma \in [0,\ +\infty]. \tag{6}$$

and that the efficiency $E(\gamma)$ on the output plane $(y_2,\, u_2)$ can also be expressed as follows:

$$E(\gamma) = \frac{c^2 \gamma}{(d\,\gamma + 1)\,[(a\,d + b\,c)\,\gamma + a]} = \frac{c^2\gamma}{a + \beta\,\gamma + \delta\,\gamma^2} \tag{7}$$

where $\beta = 2\,a\,d + c\,b$ and $\delta = d\,(a\,d + c\,b)$, see [4]-(12). One can easily prove that if system $\mathbf{H}(s)$ is strictly dissipative, the two solutions $\gamma_{1,2}$ of the denominator $a + \beta\,\gamma + \delta\,\gamma^2$ of function $E(\gamma)$ in (7) are real and negative:

$$\gamma_1 = -\frac{a}{(a\,d + c\,b)} = -\frac{1}{d(1 + \bar{g}^2 - \bar{f}^2)}, \qquad \gamma_2 = -\frac{1}{d} \tag{8}$$

where

$$\bar{f} = \frac{f}{\sqrt{a\,d}}, \quad \bar{f} \in [-1, 1], \quad \bar{g} = \frac{g}{\sqrt{a\,d}}, \quad \bar{g} \in [-\infty, +\infty]. \tag{9}$$

The global behavior of efficiency $E(\gamma)$ in (7) as a function of parameter γ is shown in Fig. 3. One can easily show that the maximum value $E^* = E(\gamma^*) = E^*(\bar{f}, \bar{g})$ of function $E(\gamma)$ when $\gamma = \gamma^*$ given in [4]-(13), can also be written as follows:

$$E^* = \frac{c(\sqrt{ad + bc} - \sqrt{ad})}{b(\sqrt{ad + bc} + \sqrt{ad})} = \frac{(\bar{g} - \bar{f})^2}{(\sqrt{1 + \bar{g}^2 - \bar{f}^2} + 1)^2} \tag{10}$$

where $\bar{f}$ and $\bar{g}$ are defined in (9) and γ^* is:

$$\gamma^* = \sqrt{\frac{a}{d\,(a\,d + c\,b)}} = \frac{1}{d}\sqrt{\frac{1}{1 + \bar{g}^2 - \bar{f}^2}} \tag{11}$$

Moreover, one can easily prove that the minimum $\bar{E}^* = E(-\gamma^*) = \bar{E}^*(\bar{f}, \bar{g})$ of function $E(\gamma)$ when $\gamma = -\gamma^*$ is:

$$\bar{E}^* = \frac{c(\sqrt{ad + bc} + \sqrt{ad})}{b(\sqrt{ad + bc} - \sqrt{ad})} = \frac{(\bar{g} - \bar{f})^2}{(\sqrt{1 + \bar{g}^2 - \bar{f}^2} - 1)^2} \tag{12}$$

The global behavior of the maximum efficiency $E^*(\bar{f}, \bar{g})$ in (10) when $\bar{g} \in [-4,\ 4]$ and $\bar{f} \in [-1,\ 1]$ is shown in Fig. 4.

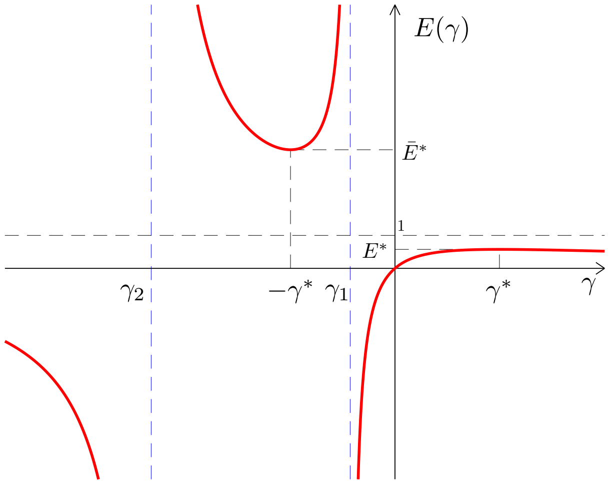

Fig. 3. Efficiency $E(\gamma)$ as a function of parameter γ.

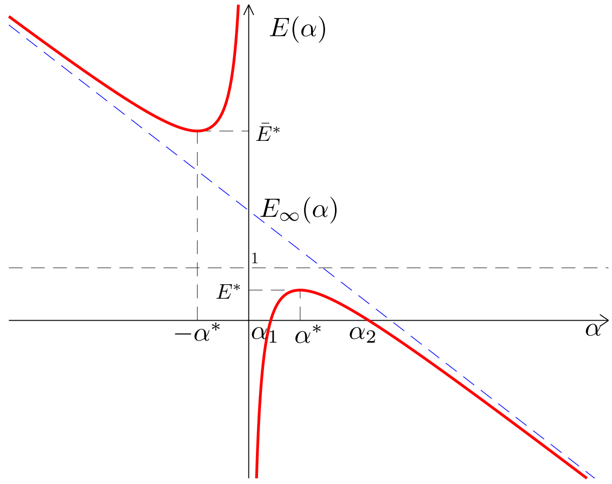

Fig. 6. Efficiency $E(\alpha)$ as a function of parameter α.

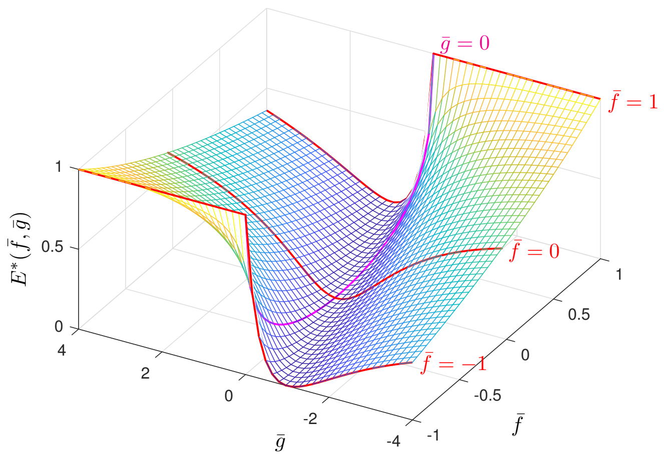

Fig. 4. Global behavior of the maximum efficiency $E^*(\bar{f}, \bar{g})$ when $\bar{g} \in [-4, \ 4]$ and $\bar{f} \in [-1, \ 1]$.

B. Efficiency of linear systems on the input plane (y_1, u_1)

An example of the efficiency map $E(t) = E(y_1, u_1)$ on the input plane (y_1, u_1) for the parameters given in (5) is shown in Fig. 5. The colored areas in Fig. 5 have the same meaning as the colored areas described for Fig. 2: green if $0 < E < 1$, red if $E > 1$ and magenta if $E < 0$.

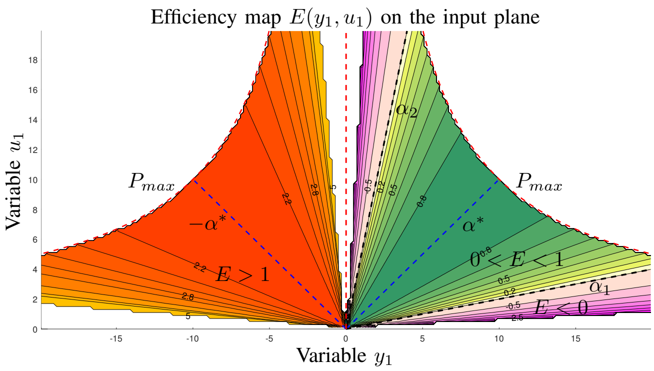

Fig. 5. Efficiency map $E(y_1, u_1)$ shown on the input plane (y_1, u_1).

In [4] it has been shown that, on the input plane (y_1, u_1), the efficiency $E(t)$ of a linear system $\mathbf{H}(s)$ is constant along straight lines of slope α exiting from the origin, see [4]-(14):

$$u_1 = \alpha \, y_1, \qquad\qquad \alpha \in [0, \ +\infty]. \tag{13}$$

and that the efficiency $E(\alpha)$ on the input plane (y_1, u_1) can also be written as follows:

$$E(\alpha) = \frac{(1-a\,\alpha)\,[(a\,d+b\,c)\,\alpha - d]}{b^2\alpha} = \frac{-d+\beta\,\alpha-\eta\,\alpha^2}{b^2\alpha} \tag{14}$$

where $\beta = 2\,a\,d + c\,b$ and $\eta = a(a\,d + b\,c)$, see [4]-(17). Moreover, the two solutions α_1 and α_2 of the numerator $-d + \beta\,\alpha - \eta\,\alpha^2$ of function $E(\alpha)$ in (14) are real and positive:

$$\alpha_1 = \frac{d}{(a\,d + c\,b)} = -\frac{1}{a(1 + \bar{g}^2 - \bar{f}^2)}, \qquad \alpha_2 = \frac{1}{a}. \tag{15}$$

where $\bar{f}$ and $\bar{g}$ are defined in (9). The global behavior of efficiency $E(\alpha)$ in (14) as a function of parameter α is shown in Fig. 6. The maximum value $E^* = E(\alpha^*) = E^*(\bar{f}, \bar{g})$ of function $E(\alpha)$ when $\alpha = \alpha^*$, see Fig. 6, is equal to the value E^* given in (10). The minimum value $\bar{E}^* = E(-\alpha^*) = \bar{E}^*(\bar{f}, \bar{g})$ of function $E(\alpha)$ when $\alpha = -\alpha^*$, see Fig. 6, is equal to the value $\bar{E}^*$ given in (12). The slope α^*, see [4]-(15), can be expressed as follows:

$$\alpha^* = \sqrt{\frac{d}{a(a\,d + b\,c)}} = \frac{1}{a}\sqrt{\frac{1}{(1 + \bar{g}^2 - \bar{f}^2)}} \tag{16}$$

The slopes α and γ are related as follows, see [4]-(19):

$$\alpha = \frac{d\gamma + 1}{(a\,d + b\,c)\gamma + a}, \qquad \gamma = \frac{a\alpha - 1}{d - \alpha(a\,d + b\,c)}. \tag{17}$$

From (17) it follows that:

$$\gamma \in [0, \infty] \leftrightarrow \alpha \in [\alpha_1, \alpha_2], \quad \alpha \in [-\infty, 0] \leftrightarrow \gamma \in [\gamma_2, \gamma_1]. \tag{18}$$

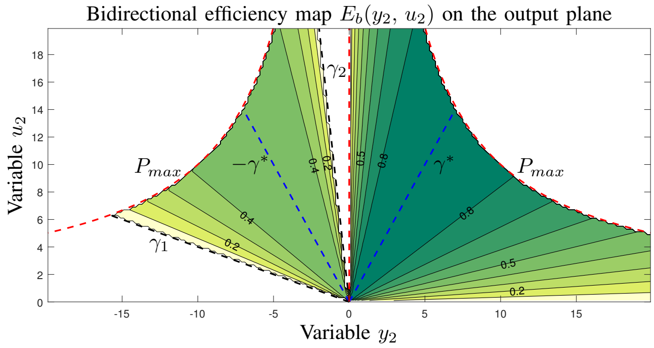

Fig. 7. Bidirectional efficiency map E_b shown on the output plane (y_2, u_2).

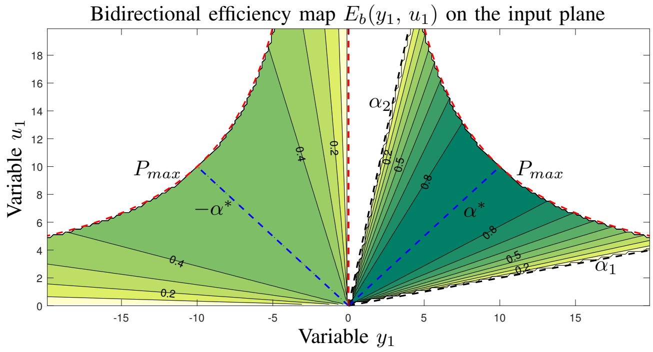

Fig. 8. Bidirectional efficiency map E_b shown on the input plane (y_1, u_1).

III. Bidirectional Efficiency of Linear Systems

Let us consider again the linear dissipative system $\mathbf{H}(s)$ shown in Fig. 1. From (7), (14) and (18), and from Fig. 3 and Fig. 6, it follows that the unidirectional efficiency $E(t)$ defined in (4) satisfies the following implications:

$$\begin{aligned} & 0<E(t)<1 \Leftrightarrow P_2(t)>0 \Leftrightarrow \gamma\in[0, +\infty] \Rightarrow P_1(t)>0 \\ & E(t)>1 \Leftrightarrow P_1(t)<0 \Leftrightarrow \alpha\in[-\infty, 0] \Rightarrow P_2(t)<0 \\ & E(t)<0 \Leftrightarrow \begin{cases} \gamma\in(-\infty, \gamma_2)\bigcup(\gamma_1, 0) \Rightarrow P_2(t)<0 \\ \alpha\in(0, \alpha_1)\bigcup(\alpha_2, +\infty) \Rightarrow P_1(t)>0 \end{cases} \end{aligned} \tag{19}$$

Property 1: If system $\mathbf{H}(s)$ is dissipative, then in (19) the case $(P_2(t) > 0) \bigwedge (P_1(t) < 0)$ can never happen.

Proof. In fact from (19), if $P_2(t) > 0$ and $P_1(t) < 0$, two positive power flows oriented towards the external of the system would be present, which is not possible since the system $\mathbf{H}(s)$ is supposed to be dissipative. Indeed, this case does not appear in (19). □

From the previous considerations, the definition of unidirectional efficiency $E(t)$ can be extended as follows.

Definition. *The "bidirectional power flow efficiency" $E_b(t)$ of the linear system $\mathbf{H}(s)$ shown in Fig. 1 is defined as follows:*

$$E_b(t) = \begin{cases} E(t) & \text{if} \quad (P_2(t) > 0) \bigwedge (P_1(t) > 0) \\ \dfrac{1}{E(t)} & \text{if} \quad (P_2(t) < 0) \bigwedge (P_1(t) < 0) \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

where $E(t)$ is the unidirectional efficiency defined in (4).

The two bidirectional efficiency maps $E_b(y_2, u_2)$ on the output plane (y_2, u_2) and $E_b(y_1, u_1)$ on the input plane (y_1, u_1), obtained using the parameters given in (5), are shown in Fig. 7 and Fig. 8, respectively. These two efficiency maps have been plotted only for $u_2 > 0$ and for $u_1 > 0$ respectively because, as regards linear systems, it can be proved that the bidirectional efficiency maps are symmetric with respect to the origin. The global behavior of the bidirectional efficiencies $E_b(\gamma)$ and $E_b(\alpha)$ as a function of parameters γ and α are shown in Fig. 9 and in Fig. 10, respectively. The parameters γ_1, γ_2, γ^*, α_1, α_2 and α^* present in Fig. 7, Fig. 8, Fig. 9 and Fig. 10 have been defined in (8), (11), (15) and (16), respectively. The maximum value $\tilde{E}^*$ present in Fig. 9 and Fig. 10 is defined as follows: $\tilde{E}^* = 1/\bar{E}^*$. One can easily show that E^* is the *maximum forward efficiency* that can be obtained from the system when the power flows from section 1 to section 2, see Fig. 1, while $\tilde{E}^*$ is the *maximum reverse efficiency* that can be obtained from the system when the power flows from section 2 to section 1.

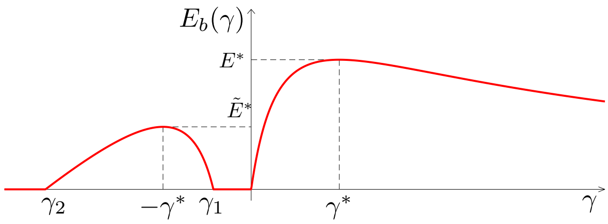

Fig. 9. Bidirectional efficiency $E_b(\gamma)$ as a function of parameter γ.

Property 2: For the linear dissipative system $\mathbf{H}(s)$, the maximum forward and reverse efficiencies E^* and $\tilde{E}^*$ are equal iff the condition $|b| = |c|$ is satisfied.

Proof. Imposing $E^* = \tilde{E}^* = 1/\bar{E}^*$ and using (10) and (12), one obtains the constraint $b^2 = c^2$, from which $|b| = |c|$. □

The global behavior of the bidirectional efficiency $E_b(\gamma)$ as a function of parameter γ when $a = 1$, $d = 0.02$ and $|b| = |c| = 0.2485$ is reported in Fig. 11. When $|b| = |c|$, the maximum efficiency $E^* = \tilde{E}^* = E^*(q)$ can be expressed as follows:

$$E^*(q) = \frac{|\sqrt{1+q}-1|}{\sqrt{1+q}+1} = \begin{cases} E_+^*(q) & \text{if} \quad q > 0 \\ E_-^*(q) & \text{if} \quad -1 \le q \le 0 \end{cases} \tag{21}$$

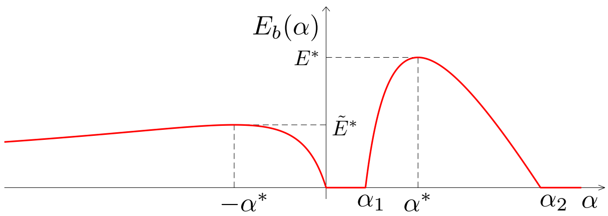

Fig. 10. Bidirectional efficiency $E_b(\alpha)$ as a function of parameter α.

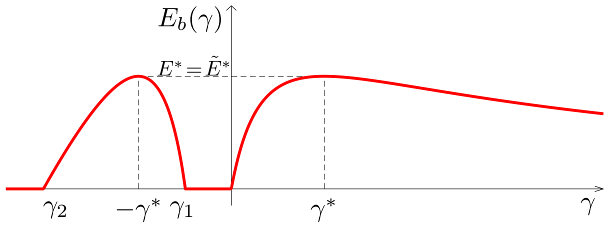

Fig. 11. Bidirectional efficiency $E_b(\gamma)$ when $|b| = |c|$.

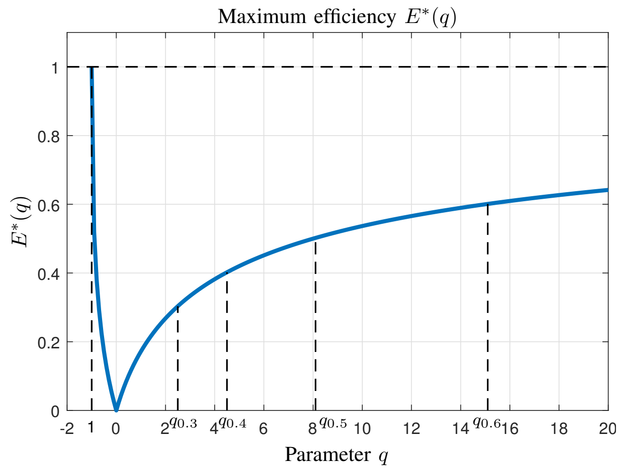

Fig. 12. Maximum efficiency $E^*(q)$ as a functions of parameter q.

where $E_+^*(q)$ refers to $b = c$, $E_-^*(q)$ refers to $b = -c$:

$$E_+^*(q) = \frac{\sqrt{1+q}-1}{\sqrt{1+q}+1}, \qquad E_-^*(q) = -\frac{\sqrt{1+q}-1}{\sqrt{1+q}+1},$$

and parameter q is defined as follows:

$$q = \frac{b\,c}{a\,d} = \bar{g}^2 - \bar{f}^2. \tag{22}$$

The global behavior of function $E^*(q)$ when $q \in [-1,\ +\infty]$ is shown in Fig. 12. One can easily prove that function $E^*(q)$ in (21) can be inverted as follows:

$$q(E^*) = \begin{cases} \left(\dfrac{E^*+1}{1-E^*}\right)^2 - 1 & \text{for} \quad q > 0 \\ \left(\dfrac{-E^*+1}{1+E^*}\right)^2 - 1 & \text{for} \quad -1 \leq q \leq 0 \end{cases} \tag{23}$$

From (21) and (23) it follows that: $E^*(q) > 0.8$ if $q > 81.6$, $E^*(q) > 0.85$ if $q > 152$, $E^*(q) > 0.9$ if $q > 361.6$ and $E^*(q) > 0.95$ if $q > 1522$.

A. Bidirectional efficiency of nonlinear systems

Let us consider the block scheme shown in Fig. 13 describing a linear system $\mathbf{H}(s)$ with additional nonlinear friction terms. In order to be dissipative, the nonlinear functions $\tilde{u}_1 = f_1(y_1)$ and $\tilde{u}_2 = f_2(y_2)$ are supposed to be defined only in the I and III quadrants of the input and output planes, see Fig. 14. The definition of unidirectional efficiency $E(t)$ for the nonlinear system shown Fig. 13 on the output and input planes can be given as follows, see [4]-Sec. III:

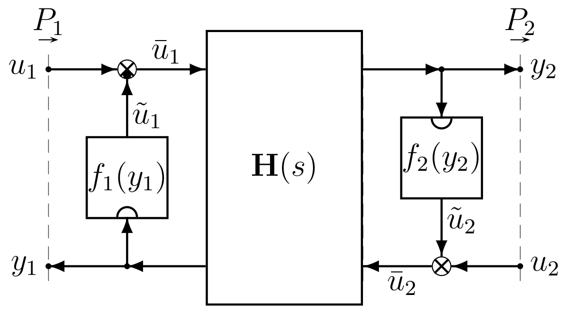

Fig. 13. Linear system $\mathbf{H}(s)$ with additional nonlinear friction terms.

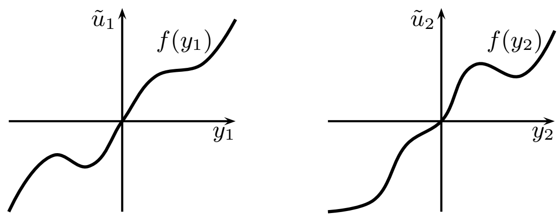

Fig. 14. Nonlinear functions $\tilde{u}_1 = f_1(y_1)$ and $\tilde{u}_2 = f_2(y_2)$.

1) Unidirectional efficiency $E(y_2, u_2)$ on the output plane:

$$E(y_2, u_2) = \frac{u_2\,y_2}{u_1\,y_1} = \frac{u_2\,y_2}{(\bar{u}_1 + f_1(y_1))\,y_1} \tag{24}$$

where

$$\begin{bmatrix} \bar{u}_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \frac{d}{c} & \frac{1}{c} \\ \frac{a\,d+b\,c}{c} & \frac{a}{c} \end{bmatrix} \begin{bmatrix} \bar{u}_2 \\ y_2 \end{bmatrix}, \quad \bar{u}_2 = u_2 + f_2(y_2). \tag{25}$$

2) Unidirectional efficiency $E(y_1, u_1)$ on the input plane:

$$E(y_1, u_1) = \frac{u_2\,y_2}{u_1\,y_1} = \frac{(\bar{u}_2 - f_2(y_2))\,y_2}{u_1\,y_1} \tag{26}$$

where

$$\begin{bmatrix} \bar{u}_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -\frac{a}{b} & \frac{1}{b} \\ \frac{a\,d+b\,c}{b} & -\frac{d}{b} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ y_1 \end{bmatrix}, \quad \bar{u}_1 = u_1 - f_1(y_1). \tag{27}$$

The bidirectional efficiency $E_b(y_2, u_2)$ on the output plane and the bidirectional efficiency $E_b(y_1, u_1)$ on the input plane can be directly obtained from (20) by using (24) and (26), respectively.

Property 3: Let us consider a nonlinear system having the structure described in Fig. 13 and Fig. 14. If both nonlinear functions $f_2(y_2)$ and $f_1(y_1)$ are symmetric with respect to the origin, then both bidirectional efficiency maps $E_b(y_2,\, u_2)$ and $E_b(y_1,\, u_1)$ of the considered system are symmetric with respect to the origin of the output and input planes, respectively.

Proof. Using (24) and (25), one can easily verify that $E(-y_2,\, -u_2) = E(y_2,\, u_2)$. Similarly, from (26) and (27), one can easily prove that $E(-y_1,\, -u_1) = E(y_1,\, u_1)$. □

Numerical example: considering the nonlinear system of Fig. 13, composed by a linear function $\mathbf{H}(s)$ with the parameters given in (5) and the following two *symmetric* nonlinear functions $f_1(y_1)$ and $f_2(y_2)$, see Fig. 15:

$$\begin{aligned} f_1(y_1) &= 0.6\,\mathrm{sgn}(y_1) + 0.02\,y_1|y_1| \\ f_2(y_2) &= 0.9\,\mathrm{sgn}(y_2) + 0.03\,y_2|y_2| \end{aligned} \tag{28}$$

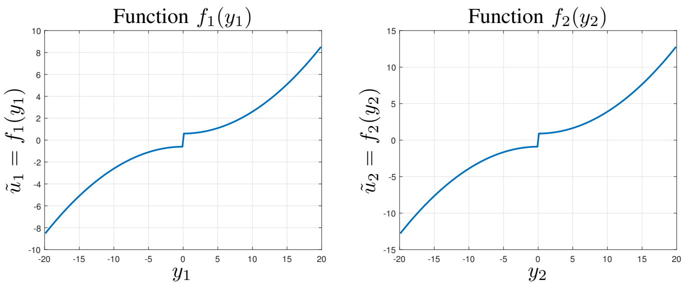

Fig. 15. Symmetric nonlinear functions $\tilde{u}_1 = f_1(y_1)$ and $\tilde{u}_2 = f_2(y_2)$.

one obtains the *symmetric* bidirectional efficiency map $E_b(y_2, u_2)$ on the output plane shown in Fig. 16. Additionally, one can easily show that the bidirectional efficiency map $E_b(y_1, u_1)$ on the input plane is *symmetric* with respect to the origin as well. If the *symmetric* nonlinear functions (28) are

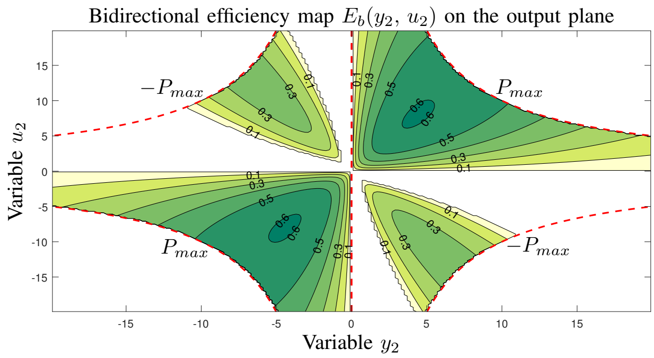

Fig. 16. Bidirectional efficiency map $E_b(y_2, u_2)$ on the output plane in presence of the two symmetric nonlinear functions $f_1(y_1)$ and $f_2(y_2)$.

substituted by the following *nonsymmetric* nonlinear functions $f_1(y_1)$ and $f_2(y_2)$, see Fig. (17):

$$f_1(y_1) = \begin{cases} \log(y_1+1) & \text{if } \; y_1 \geq 0 \\ -0.038\, y_1^2 & \text{if } \; y_1 < 0 \end{cases}$$
$$f_2(y_2) = \begin{cases} \exp(y_2/100) - 1 & \text{if } \; y_2 \geq 0 \\ -0.5\sqrt{|y_2|} & \text{if } \; y_2 < 0 \end{cases} \tag{29}$$

one obtains the *nonsymmetric* bidirectional efficiency map $E_b(y_2, u_2)$ on the output plane shown in Fig. 18.

The definition of bidirectional efficiency given in (20) can also be extended, in a similar way, to the full nonlinear

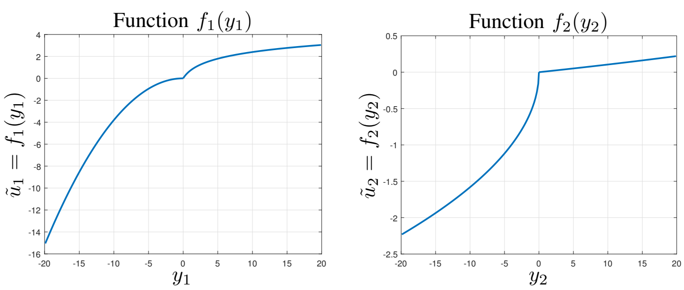

Fig. 17. Nonsymmetric nonlinear functions $\tilde{u}_1 = f_1(y_1)$ and $\tilde{u}_2 = f_2(y_2)$.

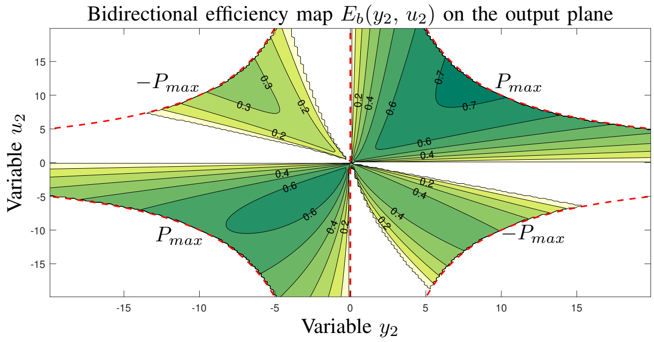

Fig. 18. Bidirectional efficiency map $E_b(y_2, u_2)$ on the output plane in presence of the two nonsymmetric nonlinear functions $f_1(y_1)$ and $f_2(y_2)$.

systems $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ defined in [4]-Fig. 8 and [4]-Sec. III.B. In this case, the bidirectional efficiency maps are usually nonsymmetric.

IV. Bidirectional Efficiency: Simulation Results

A. *Efficiency of a DC electric motor.*

Let us consider the following state space model $\mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ of a DC electric motor, see [4]-Fig. (9):

$$\underbrace{\begin{bmatrix} L & \\ & J \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{I} \\ \dot{\omega} \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -R & -K \\ K & -B \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I \\ \omega \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} V \\ \tau \end{bmatrix}}_{\mathbf{u}}$$

with $\mathbf{C} = \mathbf{I}_2$ and $\mathbf{D} = \mathbf{0}_2$. The corresponding steady-state matrix $\mathbf{H}_0$, see (1), is characterized by the parameters:

$$a = \frac{B}{R\,B + K^2}, \quad b = \frac{K}{R\,B + K^2} = c, \quad d = \frac{R}{R\,B + K^2}.$$

The DC motor satisfies the condition $|b| = |c|$, see Prop. 2, and therefore its maximum efficiency $E^* = \bar{E}^* = E^*(q)$ is expressed by (21) when $q > 0$ (since b and c have the same sign). The maximum efficiency of the DC motor remains the same both when it works *forward* as a motor and when it works *reverse* as a generator. In this case, the parameter q and the slope γ^* have the following form:

$$q = \frac{K^2}{R\,B} > 0, \qquad \gamma^* = B\sqrt{1 + \frac{K^2}{R\,B}}.$$

The maximum efficiency of a DC motor increases if parameter K increases or if parameters R and B decrease. Additionally, for constant q, the optimal slope γ^* is proportional to coefficient B: the DC motor is more suitable for working in the high torque-low speed region if B is large, and in the low torque-high speed region if B is small. The bidirectional efficiency map of a DC motor on the output plane (ω, τ), when $R = 1\,[\Omega]$, $B = 9.5{\cdot}10^{-3}\,[Nm/(rad/s)]$ and $K = 2\,[Nm/A]$, is shown in Fig. 19.

B. *Efficiency of a mechanical gear transmission.*

Let us consider the mechanical gear transmission shown in Fig. 20. The corresponding state space equations are given

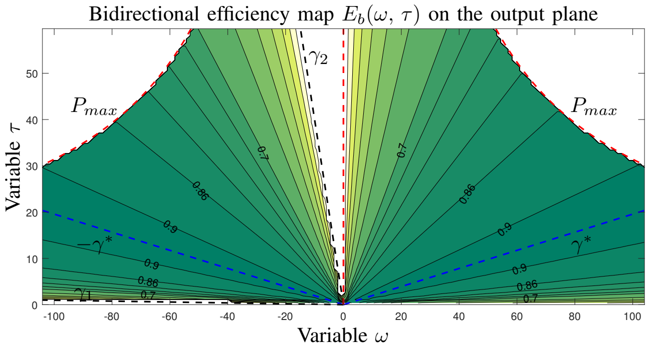

Fig. 19. Bidirectional efficiency map $E_b(\omega,\,\tau)$ of the DC motor on the output plane.

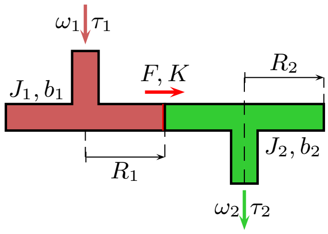

Fig. 20. Mechanical gear transmission.

in [4]-(23). In this case, the steady-state matrix $\mathbf{H}_0$ has the following form, see [4]-(24):

$$\mathbf{H}_0=\begin{bmatrix} \frac{1}{b_1+b_2 r^2} & \frac{-r}{b_1+b_2 r^2} \\ \frac{r}{b_1+b_2 r^2} & \frac{-r^2}{b_1+b_2 r^2} \end{bmatrix}=\begin{bmatrix} a & b \\ c & -d \end{bmatrix} \quad \text{where} \quad r=\frac{R_1}{R_2}. \quad (30)$$

The gear transmission satisfies the condition $|b|=|c|$ and its maximum efficiency $E^*=\tilde{E}^*=E^*(q)$ is expressed by (21) when $-1\leq q\leq 0$ (since b and c have opposite sign). Its maximum efficiency E^* remains the same when it works in the *forward* and in the *reverse* direction. For this system, the parameter q and the slope γ^* have the following values:

$$q=\frac{b\,c}{a\,d}=-1, \qquad \gamma^*=\sqrt{\frac{a}{d\,(a\,d+c\,b)}}=\infty.$$

From (21), it follows that the maximum efficiency of the system is $E^*(q)=1$, when $q=-1$. The bidirectional efficiency map of the gear transmission system on the output plane $(\omega_2,\,\tau_2)$, when $r=0.36$, $b_1=23.4\cdot 10^{-3}\,[Nm/(rad/s)]$ and $b_2=43\cdot 10^{-3}\,[Nm/(rad/s)]$, is shown in Fig. 21.

C. Asymmetric efficiency of a static electric system.

Very often, physical systems satisfy the condition $|b|=|c|$, meaning that their maximum forward and reverse efficiencies are the same. A physical system that does not satisfy the condition $|b|=|c|$ is, for example, the electric system shown in Fig. 22, where the physical element within the dotted line is a gyrator characterized by the following equations: $I_b=K\,V_1$ and $I_a=K\,V_2$. This system is described by the following

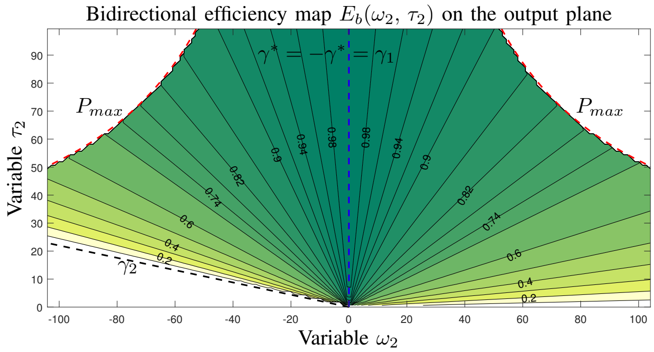

Fig. 21. Bidirectional efficiency map $E_b(\omega_2,\,\tau_2)$ of the gear transmission system on the output plane.

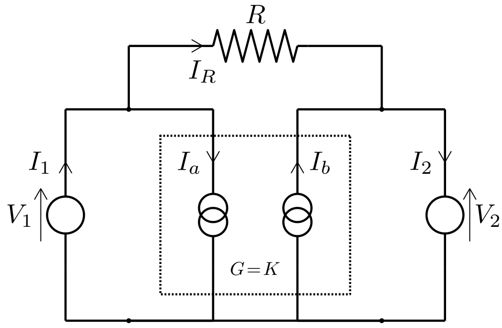

Fig. 22. Static electric system.

state space equations:

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}}=\underbrace{\begin{bmatrix} \frac{1}{R} & K-\frac{1}{R} \\ K+\frac{1}{R} & -\frac{1}{R} \end{bmatrix}}_{\mathbf{H}_0}\underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\mathbf{u}}=\underbrace{\begin{bmatrix} a & b \\ c & -d \end{bmatrix}}_{\mathbf{H}_0}\mathbf{u}$$

where $y_1=I_1$, $y_2=I_2$, $u_1=V_1$ and $u_2=V_2$. The bidirectional efficiency map $E_b(I_2,\,V_2)$ of the considered electric system, obtained when $R=4$ and $K=2$, is shown in Fig. 23. The global behavior of the corresponding bidirectional efficiency $E_b(\gamma)$ as a function of parameter γ is shown in Fig. 24. For this system, the maximum forward efficiency E^*, the maximum reverse efficiency $\tilde{E}^*=1/\bar{E}^*$ and the slope γ^*,

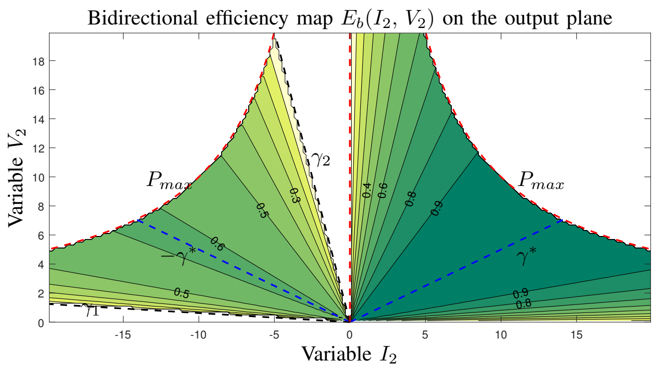

Fig. 23. Bidirectional efficiency map $E_b(I_2,\,V_2)$ of the considered static electric system on the output plane.

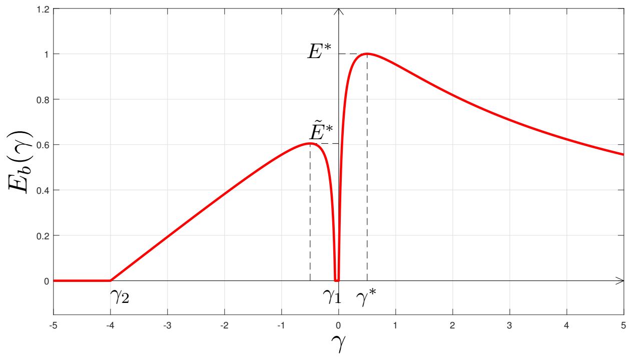

Fig. 24. Bidirectional efficiency $E_b(\gamma)$ of the considered electric system.

see (10), (12) and (11) respectively, have the following values:

$$E^* = 1, \quad \tilde{E}^* = \frac{(K - \frac{1}{R})^2}{(K + \frac{1}{R})^2} = 0.605, \quad \gamma^* = \frac{1}{K} = 0.5.$$

In this case, the maximum forward efficiency E^* is equal to one, because for this system the relation $a\,d = (b-c)^2/4$ holds, meaning that the system is not strictly dissipative, see (3).

V. Conclusions

In this paper, a study of the bidirectional efficiency of linear and nonlinear physical systems has been done. This study shows how the bidirectional efficiency maps of the considered system can be computed both on the plane of the input and of the output power variables, thus characterizing the system performance in all the operating conditions. The analysis also highlights the relation among the system parameters ensuring the same peak efficiency for the considered system both in *forward* and in *reverse* mode and how the efficiency map changes when the system is affected by symmetric or non-symmetric nonlinearities. The results presented in this paper are then applied to some physical systems of interest in order to perform the computation of their bidirectional efficiency maps, showing the effectiveness of the presented analysis in several industrial applications.

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