Controllability in dynamics of diffusion processes with nonlocal conditions

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Abstract

The paper deals with semilinear evolution equations in Banach spaces. By means of linear control terms, the controllability problem is investigated and the solutions satisfy suitable nonlocal conditions. The Cauchy multi-point condition and the mean value condition are included in the present discussion. The final configuration is always achieved with a control with minimum norm. The results make use of fixed point techniques; two different approaches are proposed, depending on the use of norm or weak topology in the state space. The discussion is completed with some applications to dynamics of diffusion processes.

AMS Subject Classification: Primary: 93B05, 34A60 Secondary: 34B15, 47H04, 47H09, 47H10

Keywords: Controllability, Semilinear differential inclusion, Nonlocal solution, Mild solution, Fixed point

1 Introduction

This paper concerns the second order integro-differential equation

$$z_{tt}(t,x) = \Delta z(t,x) + f\left(t, x, z(t,x), \int_D h(x,\xi) z(t,\xi)\, d\xi\right) + b(x)v(t,x), \qquad (1)$$

for $x \in D \subset \mathbb{R}^n$ bounded and with a sufficiently regular boundary and $t \in [0,T] := \mathcal{J}$ which is a model for the description of diffusive behaviours such as the propagation of electro-magnetic waves, the motion of a string or a membrane with external damping, the evolution of visco-elastic fluids and the heat propagation (see e.g. [15, 19, 20] and the references therein). The space domain D is a bounded subset of the euclidean space $\mathbb{R}^n, n \geq 1$. The nonlocal term in integral form appearing in (1) accounts of long distance interactions into the process (see e.g. [14], [16]). As usual, the Laplace operator stands for a diffusion behaviour of punctual type. The nonlinear part $f\left(t, x, z, \int_\Omega h(x,\xi) z(t,\xi)\, d\xi\right)$ can also be replaced by an interval such as, for instance,

$$\left[f_1\left(t, x, z(t,x), \int_D h(x,\xi) z(t,\xi)\, d\xi\right), f_2\left(t, x, z(t,x), \int_D h(x,\xi) z(t,\xi)\, d\xi\right)\right]. \quad (2)$$

This case occurs, for instance, when the function f in (1) displays some jump discontinuity or it is known up to some degree on uncertainty; in this case equation (1) turns to a multivalued dynamic.

With $v(t,x)$ given, by a solution of (1) we mean a function $z : \mathcal{J} \times D \to \mathbb{R}$ such that $z_t(t,x)$ exists, for $t \in \mathcal{J}$ and a.a. $x \in D$, $z(t,\cdot)$, $z_t(t,\cdot) \in L^2(D,\mathbb{R})$, $t \in \mathcal{J}$ and the map $y : \mathcal{J} \to L^2(D,\mathbb{R})$ defined by $y(t) = z(t,\cdot)$ belongs to $\mathbb{C}^1(\mathcal{J}; L^2(D,\mathbb{R}))$. The symbol $\mathcal{A}$ stands for the set of functions $z(t,x)$ with all previous properties. Equation (1) is then satisfied in integral form (see Section 6 for details).

We are interested in solutions of (1) which satisfy some nonlocal conditions. Consider, for instance, the *Cauchy multi-point condition*

$$z(0,x) = \sum_{i=1}^{p} \alpha_i\, z(t_i,x) + z_0(x), \quad z_t(0,x) = \sum_{j=1}^{q} \beta_j\, z_t(\tau_j,x) + \overline{z}_0(x), \quad x \in D \quad (3)$$

where

$$t_i, \tau_j \in [0,T],\ \alpha_i, \beta_j \in \mathbb{R},\ i = 1,...,p, j = 1,...,q \quad (4)$$

and $z_0(\cdot), \overline{z}_0(\cdot)$ are suitable real valued functions. A second important example of nonlocal condition is the *weighted mean value condition*

$$z(0,x) = z_0(x) + \int_0^T k_1(t) z(t,x)\, dt, \quad z_t(0,x) = \overline{z}_0(x) + \int_0^T k_2(t) z_t(t,x)\, dt, \quad x \in D, \quad (5)$$

with $z_0(\cdot), \overline{z}_0(\cdot)$ as in (3) and $k_i(\cdot)$, $i = 1,2$ suitable real valued functions. The initial value problem associated to (1), i.e. $z(0,\cdot) = z_0(\cdot)$ and $z_t(0,\cdot) = \overline{z}_0(\cdot)$ is clearly included, in both case. The class of nonlocal conditions that we are able to manage is, indeed, quite wide (see Definition 1.1).

The additional term $v(t,x)$ appearing in (1) accounts of external forces acting into the model, i.e. it is a control of linear type. The main aim of this paper is to investigate the possibility to act by $v(t,x)$ for obtaining a solution of (1) which satisfies some given nonlocal condition (for instance (3) or (5)) and reaches a prescribed configuration at time $t = T$, i.e. $z(T, \cdot) = z_1(\cdot)$, where $z_1(\cdot)$ stands for a suitable real valued function. This is known as the *exact controllability* problem associated to (1) (for short controllability problem in the sequel). The possibility to lead a system in finite time into a desired configuration is of great interest in several physical applications.

A *nonlocal solution* of (1) is a solution $z(t, x)$ of equation (1) with $v(t, x)$ given, satisfying

$$(z(0, \cdot), z_t(0, \cdot)) = g(z) + (z_0, \overline{z}_0) \tag{6}$$

where $g\colon \mathcal{A} \to L^2(D, \mathbb{R})$, $z_0(\cdot), \overline{z}_0(\cdot) \in L^2(D, \mathbb{R})$.

Definition 1.1 *We say that problem* (1)-(6) *is controllable if, for every* $z_0(\cdot), \overline{z}_0(\cdot), z_1(\cdot) \in L^2(D, \mathbb{R})$*, there is a control function* $v(t, x)$ *and a corresponding solution* $z(t, x)$ *of* (1) *satisfying both* (6) *and*

$$z(T, x) = z_1(x), \ \ x \in D. \tag{7}$$

As usual in this framework we transform problem (1)-(6) into its abstract setting. When assuming that the nonlinearity in (1) takes the form as in (2) and the linear term also depends on t, we arrive to the multivalued nonlocal problem

$$\dot{y}(t) \in A(t)y(t) + F\big(t, y(t)\big) + Bu(t), \quad \text{for a.e. } t \in \mathcal{J} := [0, T], \ \ T > 0 \tag{8}$$

$$y(0) = M(y) + y_0 \tag{9}$$

in some infinite dimensional Banach space $\mathbb{E}$; for instance $\mathbb{E} = W^{1,2}(D, \mathbb{R}) \times L^2(D, \mathbb{R})$. The multivalued map $F\colon \mathcal{J} \times \mathbb{E} \multimap \mathbb{E}$; $\{A(t)\}_{t \in \mathcal{J}}$ is a family of linear (not necessarily bounded) operators with same domain $D(A)$ dense in $\mathbb{E}$ and generating an evolution operator on $\mathbb{E}$ (see Definition 2.3), the control $u \in L^2(\mathcal{J}, \mathbb{U})$, where $\mathbb{U}$ is a reflexive Banach space with norm $\| \cdot \|_{\mathbb{U}}$, $B\colon \mathbb{U} \to \mathbb{E}$ and $M\colon C(\mathcal{J}, \mathbb{E}) \to \mathbb{E}$.

The discussion about problem (8)-(9) is presented in Section 3. We give there the notion of solution for (8)-(9) (Definition 3.1) and define its controllability (Definition 3.2). The main results of the paper are Theorems 3.1 and 3.2; they provide sufficient conditions for the controllability of (8)-(9). It was pointed out by Triggiani [25] that, in infinite dimensional Banach spaces, the compactness of the associated evolution operator is in contradiction with the controllability of a linear system while using locally L^p- controls, for $p > 1$. We overcome this lack of compactness by means of two different strategies; precisely by introducing suitable measures of noncompactness in Theorems 3.1 and by making use of the weak topology of $\mathbb{E}$ in Theorem 3.2. Fixed point techniques are used in both cases in the proofs. The final configuration is always achieved with a control with minimum norm in $L^2(\mathcal{J}, \mathbb{U})$. The proofs of Theorems 3.1 and 3.2 are, respectively, contained in Section 4 and Section 5.

The controllability in infinite dimensional setting with $M \equiv 0$ was recently treated in [21] and [5]; previous contributions are reported in [9] and in the survey [2]. A Cauchy multipoint condition

$$M(y) = y_0 + \sum_{k=1}^{p} c_k y(t_k)$$

with t_k fixed in $\mathcal{J}$ and given real values c_k, $k = 1, \dots, p$ was introduced in [4] (see also [3]); the linear part there takes the form $A = A(t, y)$, but $A(t, y)$ is a bounded operator.

The notion of *appoximate controllability* (see e.g. [24]) seems the most appropriate in the case of a compact evolution operator.

Section 6 deals with the controllability problem associated to equation (1); for the sake of simplicity we restrict there to the case when x is a real variable and D an interval and we consider the multivalued Cauchy condition (3). This section also contains a discussion concerning the cases when the use of Theorem 3.1 is preferable than Theorem 3.2 and vice-versa. The methods used in Section 3 in abstract setting is quite general and hence it can be used also for the study of some first order integro-differential models such as

$$z_t(t,x) \in \int_\Omega k(x,y)z(t,y)\, dy + f(t,x,z) + b(x)v(t,x), \quad \text{for } x \in \Omega, \text{ a.e. } t \in \mathcal{J}. \tag{10}$$

In Section 47 we discuss the controllability of (10). At last, Section 2 contains some basic notation and results concerning multivalued analysis and fixed point theory and some relevant example of measures of noncompactness.

2 Preliminaries

In this section we briefly introduce the theory of multivalued analysis, show some relevant examples of measures of noncompactness (*m.n.c.* for short) and discuss their main properties, recall some useful function spaces and the fixed point results used in the following. We denote by X or Y a topological space and by $\mathbb{E}$ or $\mathbb{F}$ an arbitrary Banach space with norm $\|\cdot\|$.

We start with the introduction of some definitions, notation and preliminary items from multivalued analysis and linear operators.

A *multivalued mapping* (*multimap* for short) $\phi\colon X \multimap Y$ is a relation that assigns to any point $x \in X$ a nonempty closed set $\phi(x) \subset Y$. Its *graph* is the set $Gr(\phi) := \{(x,y) \in X \times Y \mid y \in \phi(x)\}$. If $A \subseteq X$, then $\phi(A) := \bigcup_{a \in A} \phi(a)$ is called the *image* of *A through* ϕ. The multimap ϕ is *closed* if $Gr(\phi)$ is closed in $X \times Y$; ϕ is *sequentially closed* when the conditions $\lim_{n\to+\infty} x_n = x_0$, $\lim_{n\to+\infty} y_n = y_0$, and $y_n \in \phi(x_n)$, $n \in \mathbb{N}$, imply that $y_0 \in \phi(x_0)$. The single valued function $f\colon X \to Y$ is a *selector* of ϕ if $Gr(f) \subset Gr(\phi)$, i.e. $f(x) \in \phi(x)$, for every $x \in X$.

The multimap ϕ is *upper semi-continuous* (*u.s.c.* for short) if the set $\phi^{-1}(C) := \{x \in X \mid \phi(x) \cap C \neq \varnothing\}$ is closed, for every closed set $C \subseteq Y$; ϕ is *quasicompact* if $\phi(A)$ is relatively compact for every compact set $A \subset X$; ϕ is *locally compact* if, for every $x \in X$, there is a neighborhood $V(x)$ such that $\phi(V(x))$ is relatively compact. If ϕ is closed, locally compact and compact valued, then it is u.s.c. ([17, Theorem

1.1.5]). If, moreover, X and Y are metric spaces and ϕ is closed, quasicompact and compact valued, then it is again u.s.c. ([17, Theorem 1.1.12]).

We say that ϕ *has a fixed point* if $X \subseteq Y$ and there exists $x \in X$ such that $x \in \phi(x)$.

Given a sequence $\{x_n\} \subset \mathbb{E}$ we write $x_n \to x_0$ and $x_n \rightharpoonup x_0$, with $x_0 \in \mathbb{E}$, for denoting, respectively, the strong and weak convergence in $\mathbb{E}$.

As usual $\mathbb{C}([a,b],\mathbb{E})$ is the Banach space of continuous functions $f\colon [a,b] \to \mathbb{E}$ with norm $\|\cdot\|_{\mathbb{C}}$. The weak convergence in this space is discussed in the following lemma

Lemma 2.1 *(see [6, Theorem 4.3]) A sequence $\{x_n\} \subset \mathbb{C}([a,b],\mathbb{E})$ is weakly convergent to an element $x \in \mathbb{C}([a,b],\mathbb{E})$ if and only if*

1. *there exists $N > 0$ such that $\|x_n\|_{\mathbb{C}} \le N$, for each $n \in \mathbb{N}$;*
2. *$x_n(t) \rightharpoonup x(t)$ as $n \to \infty$, for every $t \in [a,b]$.*

A set $W \subset L^1([a,b],\mathbb{E})$ is called *integrably bounded* if there exists $\nu \in L^1([a,b],\mathbb{R}^+)$ such that $\|f(t)\| \le \nu(t)$ for a.a. $t \in [a,b]$ and all $f \in W$; *uniformly integrable* if, for any $\epsilon > 0$, there is $\delta > 0$ such that $\lambda(A) < \delta$ implies

$$\|\int_A f(t)\,dt\| < \epsilon \qquad \text{for every } f \in W,$$

where λ is the Lebesgue measure on $[a,b]$. The following result deals with the weak convergence in the usual Banach space $L^1([a,b],\mathbb{E})$ of integrable functions with norm $\|\cdot\|_1$.

Theorem 2.1 *(Dunford–Pettis)[10, Corollary 2.6] Let $W \subset L^1\big([a,b],\mathbb{E}\big)$ be uniformly integrable. If $\{w(t) : w \in W\}$ is relatively weakly compact for a.a. $t \in [a,b]$, then W is relatively weakly compact in $L^1\big([a,b],\mathbb{E}\big)$.*

A sequence $\{f_n\} \subset L^1([a,b],\mathbb{E})$ is called *semicompact,* if it is integrably bounded and the set $\{f_n(t)\}$ is relatively compact for a.a. $t \in [a,b]$. The following convergence result for semicompact sequences is useful in our discussion.

Theorem 2.2 *([17, Theorem 5.1.1]) Let $S\colon L^1([a,b],\mathbb{E}) \to \mathbb{C}([a,b],\mathbb{E})$ be an operator satisfying the following conditions*

(i) there is $L > 0$ such that $\|Sf - Sg\|_{\mathbb{C}} \le L\|f - g\|_1$ for all $f, g \in L^1([a,b],\mathbb{E})$;

(ii) for any compact $K \subset \mathbb{E}$ and sequence $\{f_n\} \subset L^1([a,b],\mathbb{E})$ such that $\{f_n(t)\} \subset K$ for a.a. $t \in [a,b]$ the weak convergence $f_n \rightharpoonup g$ implies $Sf_n \to Sg$.

Then for every semicompact sequence $\{f_n\} \subset L^1([a,b],\mathbb{E})$ the sequence $\{Sf_n\}$ is relatively compact in $\mathbb{C}([a,b],\mathbb{E})$ and, moreover, if $f_n \rightharpoonup f_0$ then $Sf_n \to Sf_0$.

By $\mathcal{L}(\mathbb{E},\mathbb{F})$ we denote the Banach space of *automorphism* (i.e. linear and continuous operators) $T\colon \mathbb{E}\to\mathbb{F}$ with norm $\|T\|_{\mathcal{L}} := \sup_{\|x\|\le 1}\|T(x)\|$. We say that the automorphism $T\in\mathcal{L}(\mathbb{E},\mathbb{F})$ has a *right inverse*, if there exists a linear continuous operator $\tilde{T}\colon \mathbb{F}\to\mathbb{E}$ such that $T\circ\tilde{T}$ is the identity on $\mathbb{F}$. The right inverse $\tilde{T}$ is called a *pseudoinverse* if $\tilde{T}(u)=x$ implies that $\|x\|=\min\{\|y\| \,:\, T(y)=u\}$

We recall now the definition of m.n.c., report its main properties and propose some relevant examples. Further information and all the proofs can be found, for instance, in [1] and [17].

Definition 2.1 *Let $\mathcal{P}$ be the family of all non-empty subsets of $\mathbb{E}$. A function $\beta\colon\mathcal{P}\to\mathcal{A}$ with values in a partially ordered set $(\mathcal{A},\ge)$ is called a m.n.c., provided that*

$$\beta(\overline{\operatorname{conv}}\,\Omega)=\beta(\Omega),\quad \textit{for every}\quad \Omega\in\mathcal{P}.$$

A m.n.c. β is said to be

(a) *monotone* if, for $\Omega_1,\Omega_2\in\mathcal{P},\Omega_1\subset\Omega_2$ we have $\beta(\Omega_1)\le\beta(\Omega_2)$,

(b) *nonsingular*, if $\beta\big(\{a\}\cup\Omega\big)=\beta(\Omega)$, for every $a\in\mathbb{E}$ and $\Omega\in\mathcal{P}$.

In addition, if $\mathcal{A}$ is a cone, a m.n.c. is:

(e) *algebraically semiadditive*, if $\beta(\Omega_1+\Omega_2)\le\beta(\Omega_1)+\beta(\Omega_2)$ for every $\Omega_1,\Omega_2\in\mathcal{P}$,

(e) *regular*, if $\beta(\Omega)=0$ is equivalent to Ω relatively compact,

(e) *real*, if $\mathcal{A}=[0,+\infty]$.

A relevant example of m.n.c. is the *Hausdorff m.n.c.* $\chi_{\mathbb{E}}$, given by

$$\chi_{\mathbb{E}}(\Omega):=\inf\{\epsilon>0 \mid \Omega \ \text{ has a finite } \epsilon\text{-net}\}; \tag{11}$$

$\chi_{\mathbb{E}}$ enjoys all the above properties. The following relation between the Hausdorff m.n.c. and a bounded linear operator can be established.

Lemma 2.2 *(see e.g. [1, 21]) Let $L:\mathbb{F}\to\mathbb{E}$ be a bounded linear operator. Then*

$$\chi_{\mathbb{E}}(L(\Omega))\le\|L\|_{\mathcal{L}}\chi_{\mathbb{F}}(\Omega),$$

for any $\Omega\subset\mathbb{F}$ nonempty and bounded.

We introduce now some important m.n.c. in the space of continuous functions. Let $\Omega\subset\mathbb{C}([a,b],\mathbb{E})$, $\Omega\ne\emptyset$

1. $\gamma(\Omega):=\sup_{t\in[a,b]}\chi_{\mathbb{E}}\big(\Omega(t)\big)$ is called *modulus of fibre noncompactness*, where $\Omega(t):=\{v(t)|\ v\in\Omega\}$ is the fibre set.

2. $\mod_C(\Omega) := \lim_{\delta\to 0} \sup_{y\in\Omega} \max_{|t_1-t_0|<\delta} \|y(t_1)-y(t_0)\|$ is called *modulus of equicontinuity.*

If $\Omega \subset \mathbb{C}([a,b],\mathbb{E})$ is such that $\mod_C(\Omega) = 0$, then it is an equicontinuous family of functions. We remark that neither γ nor $\mod_C(\Omega)$ are regular m.n.c. The further m.n.c. in $\mathbb{C}([a,b],\mathbb{E})$ with values in $(\mathbb{R}^+)^2$:

$$\nu(\Omega) = \max_{D\in\Delta(\Omega)} \big(\gamma(D), \mathrm{mod}_C(D)\big) \tag{12}$$

is then frequently used, where the ordering is the natural one introduced by the positive cone of $\mathbb{R}^2$ and $\Delta(\Omega)$ stands for the collection of all countable subsets of Ω. As a consequence of the Arzelà-Ascoli Theorem, the m.n.c. ν turns out to be regular; it is also monotone and nonsingular. The following result is useful for the computation of ν.

Lemma 2.3 *[17, Collorary 4.2.5] Let $\{f_n\} \subset L^1([a,b],\mathbb{E})$ be integrably bounded. Suppose that there exists $q \in L^1([a,b],\mathbb{R})$ such that*

$$\chi_{\mathbb{E}}\big(\{f_n(t)\}\big) \le q(t), \quad \text{for a.a. } t \in [a,b].$$

Then, for any t_1, $t_2 \in [a,b]$, $t_1 < t_2$, we have that

$$\chi_{\mathbb{E}}\left(\left\{\int_{t_1}^{t_2} f_n(t)\,dt\right\}\right) \le 2\int_{t_1}^{t_2} q(t)\,dt, \tag{13}$$

in the general case and

$$\chi_{\mathbb{E}}\left(\left\{\int_{t_1}^{t_2} f_n(t)\,dt\right\}\right) \le \int_{t_1}^{t_2} q(t)\,dt \tag{14}$$

in the case of a separable Banach space $\mathbb{E}$.

Definition 2.2 *Given a m.n.c. β, we say that the multimap $F\colon X \subset \mathbb{E} \multimap \mathbb{E}$ with compact values is β-*condensing *if, for every nonempty $\Omega \subset \mathbb{E}$ that is not relatively compact, we have*

$$\beta\big(F(\Omega)\big) \not\geq \beta(\Omega).$$

We briefly introduce the notion of evolution system and evolution operator and refer to [22] for further details.

Definition 2.3 *Let $\Delta := \{(t,s) \in [0,T]\times[0,T] \,:\, 0 \le s \le t \le T\}$. A two parameter family $\{\mathcal{U}(t,s)\}_{(t,s)\in\Delta}$, where $\mathcal{U}(t,s) : \mathbb{E} \to \mathbb{E}$ is a bounded linear operator and $\mathbb{E}$ a Banach space is called an* evolution system *if the following conditions are satisfied:*

1. $\mathcal{U}(s,s) = I,\ 0 \le s \le T$; $\quad \mathcal{U}(t,r)\mathcal{U}(r,s) = \mathcal{U}(t,s),\ 0 \le s \le r \le t \le T$;

2. $(t, s) \mapsto \mathcal{U}(t, s)$ *is strongly continuous on* Δ, *i.e. the map* $(t, s) \to \mathcal{U}(t, s)x$ *is continuous on* Δ *for every* $x \in \mathbb{E}$.

Given an evolution system, we can consider the respective evolution operator $\mathcal{U} : \Delta \to \mathcal{L}(\mathbb{E})$.

Since the evolution operator $\mathcal{U}$ is strongly continuous on the compact set Δ, by the uniform boundedness theorem there exists a constant constant $D_{\mathcal{U}}$ such that

$$\|\mathcal{U}(t, s)\|_{\mathcal{L}} \leq D_{\mathcal{U}}, \quad \text{for all } (t, s) \in \Delta. \tag{15}$$

Now we recall the Pettis measurability criterion.

Theorem 2.3 *[23, p. 278] Let* (S, Σ) *be a measure space,* $\mathbb{E}$ *be a separable Banach space. Then* $f : S \to \mathbb{E}$ *is measurable if and only if for any* $e \in \mathbb{E}'$ *the function* $e \circ f : S \to \mathbb{R}$ *is measurable with respect to* Σ *and the Borel* σ*-algebra in* $\mathbb{R}$.

At last we report the fixed point results used in the sequel.

Theorem 2.4 *(Sadovskiĭ) [17, Collorary 3.3.1] Let* D *be a closed and convex subset of* $\mathbb{E}$. *If* $F : D \multimap D$ *is a compact and convex valued, closed and* β*-condensing multimap with respect to a nonsingular m.n.c.* β, *then* F *has a fixed point.*

Theorem 2.5 *(Ky Fan) [13, Theorem 1] Let* Z *be a Hausdorff locally convex topological vector space and* V *a compact convex subset of* Z. *If* $F : V \multimap V$ *is an u.s.c. multimap with closed, convex values, then* F *has a fixed point.*

3 Statement of the problem

In this section we introduce the notion of controllability for system (8)-(9) and state its validity in two different cases (see Theorems 3.1 and 3.2).

We start with the discussion about $A(t)$ and B and introduce the operator G.

(A) $\{A(t) : \mathcal{D}(A) \to \mathcal{L}(\mathbb{E})\}_{t \in \mathcal{J}}$, with $D(A)$ dense in $\mathbb{E}$, is a family of linear (not necessarily bounded) operators generating an evolution operator $\{\mathcal{U}(t, s)\}_{(t,s) \in \Delta}$ (see Definition 2.3 and also [22]).

Remark 3.1 *(i) When* $A(t) \equiv A$, *then* $e^{At} := U(t, 0)$, $t \geq 0$ *is a strongly continuous semigroup (see [17] and [22]).*

(ii) Let $S : L^1(\mathcal{J}, \mathbb{E}) \to \mathbb{C}(\mathcal{J}, \mathbb{E})$ *be the generalized Cauchy operator defined by*

$$Sf(t) := \int_0^t \mathcal{U}(t, s) f(s)\, ds.$$

It is proved in [7, Theorem 2] (see also [8, Theorem 2]) that S *satisfies the assumptions of Theorem 2.2.*

We assume that

(B) $B\colon \mathbb{U} \to \mathbb{E}$ is linear and bounded with $D_B := \|B\|_{\mathcal{L}}$.

Let $G\colon L^2(\mathfrak{J}, \mathbb{U}) \to \mathbb{E}$ be the linear bounded operator defined by

$$Gu := \int_0^T \mathfrak{U}(T,s)Bu(s)\, ds. \tag{16}$$

When $L^2(\mathfrak{J}, \mathbb{U})$ is reflexive, which is the case here because $\mathbb{U}$ is reflexive, the operator G has a right inverse if and only if it has a pseudoinverse (see Section 2 and [2, page 9]); we denote it G^{-1}. We assume that

(G1) G has a pseudoinverse G^{-1} and put $D_G := \left\|G^{-1}\right\|_{\mathcal{L}}$.

(F1) $F\colon \mathfrak{J} \times \mathbb{E} \multimap \mathbb{E}$ has nonempty, bounded, closed, convex values;

(F2) $F(\cdot, x)\colon \mathbb{E} \multimap \mathbb{E}$ has a measurable selection for every $x \in \mathbb{E}$.

Conditions (A), (B), (G1), (F1) and (F2) will always be assumed. For the sake of simplicity, we will no more mention them.

The solution of problem (8)-(9) with $u \in L^2(\mathfrak{J}, \mathbb{U})$ given, is intended in integral form, precisely

Definition 3.1 *Let $u \in L^2(\mathfrak{J}, \mathbb{U})$ be given. A function $y(\cdot) \in \mathbb{C}(\mathfrak{J}, \mathbb{E})$ is called a* mild solution *of problem* (8)-(9) *if the multimap $t \multimap F\big(t, y(t)\big)$ has a selection $f \in L^1(\mathfrak{J}, \mathbb{E})$ satisfying*

$$y(t) = \mathfrak{U}(t,0)[M(y) + y_0] + \int_0^t \mathfrak{U}(t,s)f(s)\, ds + \int_0^t \mathfrak{U}(t,s)Bu(s)\, ds, \quad t \in \mathfrak{J}.$$

Definition 3.2 *Problem* (8)-(9) *is called* controllable *on $\mathfrak{J}$ if, for any $y_0, y_1 \in \mathbb{E}$, there is a control $u \in L^2(\mathfrak{J}, \mathbb{U})$ such that the corresponding mild solution $y(\cdot)$ satisfies $y(T) = y_1$.*

As pointed out in Introduction, the notion of controllability is classical when $M \equiv 0$.

Example 3.1 *(i) In the linear Cauchy case*

$$\dot{y}(t) = A(t)y(t) + Bu(t), \quad \textit{for a.e. } t \in \mathfrak{J}, \tag{17}$$

i.e. when $F \equiv 0$ and $M \equiv 0$, the controllability is equivalent to the existence of a pseudoinverse of G (see e.g. [9]); this is the motivation for assumption (G1).

(ii) If $y(\cdot)$ is a controllable solution of (8)-(9) *with control $u \in L^2(\mathfrak{J}, \mathbb{U})$ (see Definition 3.2), then*

$$Gu = y_1 - \mathfrak{U}(T,0)[M(y) + y_0] - \int_0^T \mathfrak{U}(T,s)f(s)\, ds,$$

with $f \in L^1(\mathfrak{J}, \mathbb{E})$ satisfying $f(t) \in F(t, y(t))$ for a.a. $t \in \mathfrak{J}$.

In several preliminary results (see Section 4) we also need the following, quite general, growth condition on F.

(F3) for each bounded set $\Omega \subset \mathbb{E}$, there exists $\mu_\Omega \in L^1(\mathfrak{J}, \mathbb{R})$ satisfying

$$\|F(t,x)\| \leq \mu_\Omega(t), \quad \text{for all } x \in \Omega \text{ and a.e. } t \in \mathfrak{J}.$$

We establish the controllability of problem (8)-(9) in two different sets of regularity and growth conditions on F and M, which cause the use of different techniques. In particular, when the regularities involve the norm topology in $\mathbb{E}$, we assume

(Es) $\mathbb{E}$ is a separable Banach space;

(F4s) $F(t,\cdot)\colon \mathbb{E} \multimap \mathbb{E}$ is closed for a.e. $t \in \mathfrak{J}$;

(F5s) there exists a function $k \in L^1(J, \mathbb{R}^+)$ such that, for every bounded set $\Omega \subset \mathbb{E}$,

$$\chi_{\mathbb{E}}\big(F(t,\Omega)\big) \leq k(t)\chi_{\mathbb{E}}(\Omega), \text{ for a.e. } t \in \mathfrak{J}.$$

(M1s) $M\colon \mathbb{C}(\mathfrak{J},\mathbb{E}) \to \mathbb{E}$ is continuous;

(M2s) $\chi_{\mathbb{E}}\big(M(Q)\big) \leq m\gamma(Q)$ for some constant $m > 0$ and every $Q \subset \mathbb{C}(\mathfrak{J},\mathbb{E})$ bounded.

(G2s) there exists a function $g \in L^1(J, \mathbb{R}^+)$ such that, for every bounded set $\Omega \subset \mathbb{E}$,

$$\chi_{\mathbb{U}}\big(G^{-1}(\Omega)(t)\big) \leq g(t)\chi_{\mathbb{E}}(\Omega), \text{ for a.e. } t \in \mathfrak{J}.$$

In order to simplify notation, we denote in the following by (Hs) this group of assumptions, i.e.

(Hs) the assumptions: (Es), (F4s), (F5s), (M1s), (M2s), (G2s) are satisfied.

Remark 3.2 *Let $F\colon \mathfrak{J} \times \mathbb{E} \mapsto \mathbb{E}$ be a multimap satisfying (F5s). If Ω is compact in $\mathbb{E}$, from the regularity of the Hausdorff m.n.c. we have that $\chi_{\mathbb{E}}\left(F(t,\Omega)\right) = 0$ for a.a. $t \in \mathfrak{J}$. Hence $F(t,\cdot)$ is quasi-compact, for a.a. $t \in \mathfrak{J}$. If, moreover, also (F1) and (F4s) are satisfied then F is compact valued and closed and $F(t,\cdot)$ is u.s.c. for a.a. $t \in \mathfrak{J}$ (see Section 2).*

When the regularities are given by means of weak topology in $\mathbb{E}$ we assume

(Ew) $\mathbb{E}$ is a reflexive Banach space;

(F4w) $F(t,\cdot)\colon \mathbb{E} \multimap \mathbb{E}$ is weakly sequentially closed for a.e. $t \in \mathfrak{J}$, i.e. if $x_n \rightharpoonup x_0$, $y_n \rightharpoonup y_0$ and $y_n \in F(t,x_n)$ for all $n \in \mathbb{N}$, then $y_0 \in F(t,x_0)$;

(M1w) $M\colon \mathbb{C}(\mathfrak{J},\mathbb{E}) \to \mathbb{E}$ is weakly sequentially continuous;

(M2w) $M\colon \mathbb{C}(\mathfrak{J},\mathbb{E}) \to \mathbb{E}$ is bounded on bounded sets.

As before, we introduce the notation

(Hw) the assumptions: (Ew), (F4w), (M1w) and (M2w) are satisfied.

Example 3.2 *Let* $M(q) := Lq$, *where* $L \in \mathcal{L}\big(\mathbb{C}(\mathfrak{J},\mathbb{E}),\mathbb{E}\big)$; *then* M *satisfies (M1s), (M1w) and (M2w). Consider now* $L(q) = \sum_{i=1}^{n} \alpha_i q(t_i)$, *with* $t_1 < t_2 < \ldots < t_n$ *and* $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, *or* $L(q) = \int_0^T \alpha(s)q(s)ds$ *with* $\alpha \in L^1(\mathfrak{J},\mathbb{R})$; *let* $Q \subset C(\mathfrak{J},\mathbb{E})$ *be bounded. By the algebraic semiadditivity of the Hausdorff measure of noncompactness and Lemma 2.3, we respectively have*

$$\chi_{\mathbb{E}}(M(Q)) \leq \sum_{i=1}^{n} |\alpha_i| \chi_{\mathbb{E}}(Q(t_i)) \leq \sum_{i=1}^{n} |\alpha_i| \gamma(Q)$$

and

$$\chi_{\mathbb{E}}(M(Q)) \leq 2 \int_0^T |\alpha(s)| \chi_{\mathbb{E}}(Q(s)) ds \leq 2\|\alpha\|_1 \gamma(Q),$$

implying that they both satisfies (M2s).

We need in the following to consider the *Nemytskiĭ operator* $P_F : \mathbb{C}(\mathfrak{J},\mathbb{E}) \multimap L^1(\mathfrak{J},\mathbb{E})$ *associated to* F which is defined by

$$P_F(y) := \big\{ f \in L^1(\mathfrak{J},\mathbb{E}) \mid f(t) \in F\big(t, y(t)\big) \text{ a.e. on } \mathfrak{J} \big\}, \text{ for all } y \in \mathbb{C}(\mathfrak{J},\mathbb{E}). \qquad (18)$$

P_F is well-defined, in our settings, as showed by the following lemmas.

Lemma 3.1 *[17, Theorem 1.3.5] Under conditions (Es), (F4s) and (F5s), the set* $P_F(y)$ *is nonempty, for all* $y \in \mathbb{C}(\mathfrak{J},\mathbb{E})$.

Lemma 3.2 *[5, Proposition 4.1] Under assumptions (Ew), (F3) and (F4w) the set* $P_F(y)$ *is nonempty, for all* $y \in \mathbb{C}(\mathfrak{J},\mathbb{E})$.

Theorem 3.1 and Theorem 3.2 contains our main results about the controllability of problem (8)-(9). They treat both the case when $F(t,\cdot)$ is strictly sublinear in its variable x (see condition *(s1)(i)*) and when $F(t,\cdot)$ is possibly linear (see condition *(s2)(i)*). Their proofs appear, respectively, in Section 4 and Section 5. They exploit fixed point arguments and, hence, require the introduction of a solution operator H, which will be defined in (24). The estimates on H involve, in particular, the value $D_B D_G D_{\mathfrak{U}} \sqrt{T}$; for simplifying notation, we put in the following

$$C := D_B D_G D_{\mathfrak{U}} \sqrt{T}. \qquad (19)$$

Additional restrictions on M and C are required, in the case when $F(t,\cdot)$ is possibly linear.

Theorem 3.1 *Assume* (Hs). *Then problem* (8)-(9) *is controllable if one of the following conditions* (s1), (s2) *or* (s3) *is satisfied.*

(s1) *(i)* *There exists* $\psi_n \in L^1(\mathfrak{J}, \mathbb{R}^+)$, $n \in \mathbb{N}$ *such that* $\sup_{\|x\| \leq n} \|F(t,x)\| \leq \psi_n(t)$, *for a.e.* $t \in \mathfrak{J}$ *and* $\liminf_{n\to\infty} \frac{1}{n} \int_0^T \psi_n(s)\, ds = 0$;

(ii) $\limsup_{\|q\|_{\mathbb{C}} \to +\infty} \frac{\|M(q)\|}{\|q\|_{\mathbb{C}}} < \frac{1}{D_{\mathfrak{U}}(1+C)}$;

(iii) $D_{\mathfrak{U}}(1 + D_{\mathfrak{U}} D_B \|g\|_1)(m + \|k\|_1) < 1.$

(s2) *(i)* $\|F(t,x)\| \leq \beta(t)(1 + \|x\|)$, $x \in \mathbb{E}$, *a.e.* $t \in \mathfrak{J}$ *with* $\beta \in L^1(\mathfrak{J}, \mathbb{R}^+)$;

(ii) $\lim_{\|q\|_{\mathbb{C}} \to +\infty} \frac{\|M(q)\|}{\|q\|} = 0$;

(iii) *the estimate* (s1)(iii) *is satisfied and* $D_{\mathfrak{U}}(1+C)\|\beta\|_1 < 1$.

(s3) *(i)* $\|F(t,x)\| \leq \beta(t)\rho\big(\|x\|\big)$ *and* $\|M(q)\| \leq \sigma(\|q\|)$ *for* $x \in \mathbb{E}$, $q \in \mathbb{C}(\mathfrak{J}, \mathbb{E})$ *and a.a.* $t \in \mathfrak{J}$ *where* $\rho, \sigma \in L^1(\mathbb{R}^+, \mathbb{R}^+)$ *are increasing functions and* $\beta \in L^1(\mathfrak{J}, \mathbb{R}^+)$;

(ii) $\forall y_0, y_1 \in \mathbb{E}\, \exists L = L(y_0, y_1) : C\|y_1\| + D_{\mathfrak{U}}(1+C)\left(\sigma(L) + \|\beta\|_1 \rho(L) + \|y_0\|\right) < L$;

(iii) *the estimate* (s1)(iii) *is satisfied.*

Theorem 3.2 *Assume (Hw). Then problem* (8)-(9) *is controllable if one of the following conditions* (w1), (w2) *or* (w3) *is satisfied.*

(w1) The assumptions (s1)(i) *and* (s1)(ii) *are satisfied;*

(w2) the assumptions (s2)(i) *and* (s2)(ii) *are satisfied;*

(w3) The assumptions (s3)(i) *and* (s3)(ii) *are satisfied.*

Remark 3.3 *We remark that the growth condition* (F3) *on the nonlinear term* F *is always satisfied when* (s1)(i), (s2)(i) *or* (s3)(i) *are true.*

Given $q \in \mathbb{C}(\mathfrak{J}, \mathbb{E})$, $f \in P_F(q)$ and $y_0, y_1 \in \mathbb{E}$, we define

$$p_q(f) := y_1 - \mathfrak{U}(T,0)[M(q) + y_0] - \int_0^T \mathfrak{U}(T,s) f(s)\, ds \tag{20}$$

and

$$u_{f,q}(t) := G^{-1}\big(p_q(f)\big)(t), \quad \text{for a.e. } t \in \mathfrak{J}, \tag{21}$$

In addition to the generalized Cauchy operator Sf (see Remark 3.1*(ii)*), we need a further operator between the same spaces because of the presence of a control into the model. Precisely, let $S_1 : L^1(\mathcal{J}, \mathbb{E}) \to \mathbb{C}(\mathcal{J}, \mathbb{E})$ be defined by

$$S_1 f(t) := \int_0^t \mathcal{U}(t,s) B \overline{u}_f(s)\, ds \tag{22}$$

with $\overline{u}_f := G^{-1} \int_0^T \mathcal{U}(T,\sigma) f(\sigma)\, d\sigma$.

Lemma 3.3 *The operator S_1 satisfies the assumptions of Theorem 2.2.*

Proof. The case when $A(t) = A$ for all $t \in \mathcal{J}$ and A generates a C_0-semigroup is discussed in [21, Lemma 1]. We use a similar reasoning in this general framework. Let $f, g \in L^1(\mathcal{J}, \mathbb{E})$; for $t \in \mathcal{J}$ we have

$$\begin{aligned} \|(S_1 f)(t) - (S_1 g)(t)\| \leq\ & D_{\mathcal{U}} D_B \int_0^T \|(G^{-1} \textstyle\int_0^T \mathcal{U}(T,\sigma)(f(\sigma) - g(\sigma))\, d\sigma)(s)\|_{\mathbb{U}}\, ds \\ \leq\ & D_{\mathcal{U}} D_B \sqrt{T} \|G^{-1} \textstyle\int_0^T \mathcal{U}(T,\sigma)(f(\sigma) - g(\sigma))\, d\sigma\|_{L^2(\mathcal{J},\mathbb{U})} \\ \leq\ & C \| \textstyle\int_0^T \mathcal{U}(T,\sigma)(f(\sigma) - g(\sigma))\, d\sigma)\| \\ \leq\ & D_{\mathcal{U}} C \|f - g\|_{L^1(\mathcal{J},\mathbb{E})}. \end{aligned} \tag{23}$$

We proved that S_1 satisfies condition *(i)* in Theorem 2.2.

Let $\{f_n\} \subset L^1(\mathcal{J}, \mathbb{E})$ be such that $\{f_n(t)\} \subset K$ for a.a. $t \in \mathcal{J}$ and $f_n \rightharpoonup g$ in $L^1(\mathcal{J}, \mathbb{E})$ with K compact in $\mathbb{E}$. Since $\{f_n\}$ is semicompact, by applying Theorem 2.2 to the generalized Cauchy operator S, we have that $Sf_n(T) \to Sg(T)$ in $\mathbb{E}$. Hence, for every $t \in \mathcal{J}$, by (23) we have

$$\|S_1 f_n(t) - S_1 g(t)\| \leq C \|S f_n(T) - S g(T)\| \to 0$$

i.e. $S_1 f_n \to S_1 g$ in $C(\mathcal{J}, \mathbb{E})$. The proof is complete. ∎

Given $y_0, y_1 \in \mathbb{E}$, we introduce the solution operator $H \colon \mathbb{C}(\mathcal{J}, \mathbb{E}) \multimap \mathbb{C}(\mathcal{J}, \mathbb{E})$ defined by

$$H(q) := \{x_{f,q} \ :\ f \in P_F(q)\}, \quad \text{for all } q \in \mathbb{C}(\mathcal{J}, \mathbb{E}), \tag{24}$$

where

$$x_{f,q}(t) := \mathcal{U}(t,0)[M(q) + y_0] + \int_0^t \mathcal{U}(t,s) f(s)\, ds + \int_0^t \mathcal{U}(t,s) B u_{f,q}(s)\, ds, \text{ for all } t \in \mathcal{J},$$

with $u_{f,q}$ defined in (21). It is then straightforward to show the following estimate for the solution operator H

$$\|x_{f,q}(t)\| \leq C\|y_1\| + D_{\mathcal{U}}(1 + C)\left(\|M(q)\| + \|f\|_1 + \|y_0\|\right), \ t \in \mathcal{J}, \tag{25}$$

with $x_{f,q} \in H(q)$, $q \in \mathbb{C}(\mathcal{J}, \mathbb{E})$ and C defined in (19).

Remark 3.4 Let $y \in \mathbb{C}(\mathcal{J}, \mathbb{E})$ be a fixed point of H, i.e. $y = x_{f,y}$ for some $f \in P_F(y)$. It is easy to see that y is a solution to problem (8)-(9) with $u_{f,y}$ as in (21). Furthermore, since

$$\begin{aligned} Gu_{f,y} = &\ GG^{-1}\Big(y_1 - \mathcal{U}(T,0)[M(y)+y_0] - \textstyle\int_0^T \mathcal{U}(T,s)f(s)\,ds\Big) \\ = &\ y_1 - \mathcal{U}(T,0)[M(y)+y_0] - \textstyle\int_0^T \mathcal{U}(T,s)f(s)\,ds, \end{aligned}$$

we have that

$$y(T) = x_{f,y}(T) = \mathcal{U}(T,0)[M(y)+y_0] + \int_0^T \mathcal{U}(T,s)f(s)\,ds + Gu_{f,y} = y_1.$$

Hence, when H has a fixed point for every $y_0, y_1 \in \mathbb{E}$, then problem (8)-(9) is controllable (see Definition 3.2) and the control $u_{f,y}$ associated to the fixed point has minimal norm.

4 Proof of Theorem 3.1

In this part we investigate the controllability of problem (8)-(9) when the regularities in $\mathbb{E}$ are given by means of the norm topology. In order to guarantee that the associated Nemytskiĭ operator is well defined, we always assume that $\mathbb{E}$ is a separable Banach space. In some preliminary results we show that the solution operator H defined in (24) has convex and compact values (Lemma 4.1 and Lemma 4.3), its graph is closed in $\mathbb{C}(\mathcal{J}, \mathbb{E}) \times \mathbb{C}(\mathcal{J}, \mathbb{E})$ (Lemma 4.2) and it is $\nu-$condensing (Lemma 4.4) with respect to the vector-valued m.n.c. ν defined in (12) (see Section 2). The proof of Theorem 3.1 completes this part.

Lemma 4.1 *The multivalued operator H has convex values.*

Proof. Let $q \in \mathbb{C}(\mathcal{J}, \mathbb{E})$, x_1, $x_2 \in H(q)$ and $\lambda \in [0,1]$. There exist f_1, $f_2 \in P_F(q)$ such that $x_i = x_{f_i,q}$, $i = 1,2$. Notice that

$$\begin{aligned} \lambda u_{f_1,q} + (1-\lambda)u_{f_2,q} &= \lambda G^{-1}\left(p_q(f_1)\right) + (1-\lambda)G^{-1}\left(p_q(f_2)\right) = G^{-1}\big(p_q(\lambda f_1 + (1-\lambda)f_2)\big) \\ &= u_{\lambda f_1 + (1-\lambda) f_2}. \end{aligned}$$

Then

$$\begin{aligned} \lambda x_1(t) + (1-\lambda)x_2(t) = \mathcal{U}(t,0)[M(q)+y_0] &+ \int_0^t \mathcal{U}(t,s)\big(\lambda f_1(s) + (1-\lambda)f_2(s)\big)\,ds \\ &+ \int_0^t \mathcal{U}(t,s)Bu_{\lambda f_1+(1-\lambda)f_2}(s)\,ds, \quad t \in \mathcal{J}. \end{aligned}$$

Since F is convex-valued, we have that $\lambda f_1 + (1-\lambda)f_2 \in P_F(q)$ and hence that $\lambda x_1 + (1-\lambda)x_2 \in H(q)$. ■

Lemma 4.2 *Assume conditions* (F3)*,* (F4s)*,* (F5s) *and* (M1s)*. Then, the multivalued operator H is closed in $\mathbb{C}(\mathfrak{J},\mathbb{E})\times\mathbb{C}(\mathfrak{J},\mathbb{E})$.*

Proof. We prove in the following that H is sequentially closed, i.e. that given q_j, $q\in\mathbb{C}(\mathfrak{J},\mathbb{E}), y_j\in H(q_j), y\in\mathbb{C}(\mathfrak{J},\mathbb{E})$, with $j\in\mathbb{N}$, if

$$q_j\to q,\ y_j\to y\ \text{ in }\ \mathbb{C}(\mathfrak{J},\mathbb{E}),\ \text{ while }\ j\to\infty,$$

then $y\in H(q)$. Since $\mathbb{C}(\mathfrak{J},\mathbb{E})$ is a Banach space, it implies the closure of H. Let $f_j\in P_F(q_j)$ be such that

$$y_j(t)=\mathcal{U}(t,0)[M(q_j)+y_0]+\int_0^t\mathcal{U}(t,s)f_j(s)\,ds+\int_0^t\mathcal{U}(t,s)Bu_j(s)\,ds,\ \ t\in\mathfrak{J}\qquad(26)$$

where $u_j:=u_{f_j,q_j}$ with u_{f_j,q_j} defined in (21) and $p_{q_j}(f_j)$ in (20), for $j\in\mathbb{N}$. From *(M1s)* we have that $M(q_j)\to M(q)$ in $\mathbb{E}$ and hence that $\mathcal{U}(\cdot,0)M(q_j)\to\mathcal{U}(\cdot,0)M(q)$ in $\mathbb{C}(\mathfrak{J},\mathbb{E})$ as $j\to\infty$.
According to the convergence of $\{q_j\}$ we can also find a bounded set $\Omega\subset\mathbb{E}$ satisfying

$$q_j(t),q(t)\in\Omega,\ \text{ for all }\ t\in\mathfrak{J},\ j\in\mathbb{N}.\qquad(27)$$

So, by *(F3)*, there exists $\mu_\Omega\in L^1(\mathfrak{J},\mathbb{R}^+)$ such that $\|f_j(t)\|\le\mu_\Omega(t)$ for a.a. $t\in\mathfrak{J}$ and all $j\in\mathbb{N}$; hence the sequence $\{f_j\}$ is integrably bounded in $L^1(\mathfrak{J},\mathbb{E})$.
By *(F5s)* the set $\{f_j(t)\}$ is relatively compact, for a.a. $t\in\mathfrak{J}$ and then $\{f_j\}$ is semicompact; hence it is weakly relatively compact in $L^1(\mathfrak{J},\mathbb{E})$ by Theorem 2.1. By passing to a subsequence, denoted as usual as the sequence, we have that $f_j\rightharpoonup f\in L^1(\mathfrak{J},\mathbb{E})$. By applying Theorem 2.2 to the generalized Cauchy operator (Remark 3.1*(ii)*) we obtain that $Sf_j\to Sf$ in $\mathbb{C}(\mathfrak{J},\mathbb{E})$. In particular,

$$\int_0^T\mathcal{U}(T,s)f_j(s)ds\to\int_0^T\mathcal{U}(T,s)f(s)ds$$

implying that $p_{q_j}(f_j)\to p_q(f)$ and then $u_j\to u_{f,q}$ in $L^2(\mathfrak{J},\mathbb{U})$, as $j\to\infty$. Moreover, for $t\in\mathfrak{J}$,

$$\begin{aligned}\left\|\int_0^t\mathcal{U}(t,s)Bu_j(s)\,ds-\int_0^t\mathcal{U}(t,s)Bu_{f,q}(s)\,ds\right\|&\le D_{\mathcal{U}}D_B\int_0^T\|u_j(s)-u_{f,q}(s)\|ds\\&\le D_{\mathcal{U}}D_B\sqrt{T}\|u_j-u_{f,q}\|_{L^2(\mathfrak{J},\mathbb{U})}\to 0.\end{aligned}$$

Thus $y_j\to z$ in $\mathbb{C}(\mathfrak{J},\mathbb{E})$, where

$$z(t):=\mathcal{U}(t,0)[M(q)+y_0]+\int_0^t\mathcal{U}(t,s)f(s)\,ds+\int_0^t\mathcal{U}(t,s)Bu_{f,q}(s)\,ds,\ \ t\in\mathfrak{J}.$$

By the uniqueness of the limit we obtain that $z=y$. Now we prove that $f\in P_F(q)$. In fact, by Mazur's convexity Theorem we obtain a sequence

$$\tilde{f}_j=\sum_{i=0}^{k_j}\lambda_{j,i}f_{j+i},\qquad\lambda_{j,i}\ge 0,\ \ \sum_{i=0}^{k_j}\lambda_{j,i}=1$$

such that $\tilde{f}_j \to f$ in $L^1(\mathcal{J},\mathbb{E})$ and then, up to a subsequence, $\tilde{f}_j(t) \to f(t)$ for a.a. $t \in \mathcal{J}$. By Remark 3.2, $F(t,\cdot)$ is u.s.c. for a.a. $t \in \mathcal{J}$. Let $t \in \mathcal{J}$ be such that $\tilde{f}_j(t) \to f(t)$ as $j \to \infty$, $f_j(t) \in F(t, q_j(t))$ and $F(t,\cdot)$ is u.s.c. Given $\epsilon > 0$, since $q_j(t) \to q(t)$ as $j \to \infty$, there is $j_0 \in \mathbb{N}$ such that $f_j(t) \in F(t, q_j(t)) \subset F(t, q(t)) + \epsilon B_0$ for $j \geq j_0$, where B_0 is the ball of radius 1 centered at 0 in $\mathbb{E}$. Since F is compact and convex valued, this implies that $f(t) \in F(t, q(t))$. Hence $f \in P_F(q)$ so that the proof is complete. ■

Lemma 4.3 *In the same assumptions of Lemma 4.2 the multivalued operator H has compact values and it is u.s.c.*

Proof. Let $Q \subset \mathbb{C}(\mathcal{J},\mathbb{E})$ be compact and consider $\{y_j\} \subset H(Q)$. The function y_j satisfies (26) with $f_j \in P_F(q_j)$ for $j \in \mathbb{N}$.

The compactness of Q implies the existence of a subsequence, still denoted as the sequence, such that $q_j \to q \in \mathbb{C}(\mathcal{J},\mathbb{E})$. By the same reasoning as in Lemma 4.2 we can prove that $\{y_j\}$, and hence $H(Q)$, is relatively compact in $\mathbb{C}(\mathcal{J},\mathbb{E})$. It implies that H is quasi-compact. Since H is closed (see Lemma 4.2) we have, in particular, that H is compact valued. At last, by [17, Theorem 1.1.12] (see Section 2), H is u.s.c. ■

Lemma 4.4 *If the assumptions of Lemma 4.2 and conditions* (M2s)*,* (G2s) *and* (s1)(iii) *are satisfied, then H is ν-condensing (ν is defined in* (12)*).*

Proof. Let $\Theta \subset \mathbb{C}(\mathcal{J},\mathbb{E})$ be bounded and such that

$$\nu\big(H(\Theta)\big) \geq \nu(\Theta) \tag{28}$$

in the sense of the order generated by the cone $(\mathbb{R}^+)^2$.
Consider the sequence $\{y_j\} \subset H(\Theta)$ such that

$$\nu\big(H(\Theta)\big) = \big(\gamma(\{y_j\}), \quad \operatorname{mod}_{\mathbb{C}}(\{y_j\})\big). \tag{29}$$

Then, we can find sequences $\{q_j\} \subset \Theta$, $\{f_j\} \subset L^1(\mathcal{J},\mathbb{E})$ such that $f_j \in P_F(q_j)$ and y_j satisfies (26) for $j \in \mathbb{N}$. By means of conditions *(M2s)* and *(F5s)* and applying Lemma 2.2, we have

$$\chi_{\mathbb{E}}(\{\mathcal{U}(t,0)[M(q_j)+y_0]\}) \leq D_{\mathcal{U}}[\chi_{\mathbb{E}}(M(\{q_j\}))+\chi_{\mathbb{E}}(\{y_0\})] \leq D_{\mathcal{U}} m \gamma(\{q_j\}), \;\; t \in \mathcal{J} \tag{30}$$

and

$$\begin{aligned} \chi_{\mathbb{E}}\big(\{\mathcal{U}(t,s)f_j(s)\}\big) \; &\leq D_{\mathcal{U}}\chi_{\mathbb{E}}\big(\{f_j(s)\}\big) \leq D_{\mathcal{U}}\chi_{\mathbb{E}}\big(F(s,\{q_j(s)\})\big) \\ &\leq D_{\mathcal{U}}k(s)\chi_{\mathbb{E}}\left(\{q_j(s)\}\right) \leq D_{\mathcal{U}}k(s)\gamma(\{q_j\}), \;\; (t,s) \in \Delta. \end{aligned}$$

Since $\{q_j\}$ is bounded, we can find a bounded set $\Omega \subset \mathbb{E}$ such that $q_j(t) \in \Omega$ for $t \in \mathcal{J}$ and $j \in \mathbb{N}$. So, by *(F3)* and with a similar reasoning as in the proof of Lemma 4.2, we obtain that $\{\mathcal{U}(t,\cdot)f_j(\cdot)\}$ is integrably bounded in $L^1([0,t],\mathbb{E})$ for $t \in \mathcal{J}$. By (14) (see Lemma 2.3), we obtain

$$\chi_{\mathbb{E}}\left(\left\{\int_0^t \mathcal{U}(t,s)f_j(s)\,ds\right\}\right) \leq D_{\mathcal{U}}\|k\|_1\gamma(\{q_j\}), \;\; t \in \mathcal{J}. \tag{31}$$

By Lemma 2.3, the semiadditivity of the Hausdorff m.n.c. and assumption *(G2s)*, we can estimate

$$\begin{aligned}\chi_{\mathbb{E}}\left(\{\mathcal{U}(t,s)Bu_j(s)\}\right) &\le D_{\mathcal{U}}D_B\chi_{\mathbb{U}}(\{u_j(s)\})\\ &\le D_{\mathcal{U}}D_B g(s)\chi_{\mathbb{E}}\left(\{p_{q_j}(f_j)\}\right)\\ &\le D_{\mathcal{U}}D_B g(s)[D_{\mathcal{U}}m\gamma(\{q_j\})+D_{\mathcal{U}}\|k\|_1\gamma(\{q_j\})]\\ &= D_{\mathcal{U}}^2 D_B g(s)(m+\|k\|_1)\gamma(\{q_j\}),\ \ (t,s)\in\Delta.\end{aligned}$$

It implies, for $t\in\mathcal{J}$,

$$\chi_{\mathbb{E}}\left(\left\{\int_0^t \mathcal{U}(t,s)Bu_j(s)\,ds\right\}\right)\le D_{\mathcal{U}}^2 D_B\|g\|_1(m+\|k\|_1)\gamma(\{q_j\}). \tag{32}$$

Therefore, by conditions (30), (31) and (32) and according to (26), we have

$$\gamma(\{y_j\})\le D_{\mathcal{U}}(1+D_{\mathcal{U}}D_B\|g\|_1)(m+\|k\|_1)\gamma(\{q_j\}).$$

Owing to*(s1)(iii)* and (28) we can conclude that $\gamma(\{q_j\})=\gamma(\{y_j\})=0$. We claim that the set $\{y_j\}$ is relatively compact in $\mathbb{C}(\mathcal{J},\mathbb{E})$. In this case $\{y_j\}$ is equicontinuous and then $\mod_C(\{y_j\})=0$; hence $\nu\left(H(\Theta)\right)=(0,0)$; by (28) also $\nu\left(\Theta\right)=(0,0)$. Θ is then relatively compact by the regularity of the m.n.c. ν and hence H is ν-condensing. In order to complete the proof, it remains to show the relative compactness of $\{y_j\}$. Let $z_j:=y_1-\mathcal{U}(T,0)[M(q_j)+y_0]$, $j\in\mathbb{N}$ and consider the sequence $\{\omega_j\}$ defined by

$$\omega_j(t):=Bv_j(t)\quad\text{and}\quad v_j:=G^{-1}(z_j),\ \ j\in\mathbb{N}\text{ and a.a. }t\in\mathcal{J}.$$

Notice that y_j, $j\in\mathbb{N}$ satisfies the following estimate

$$y_j(t)=\mathcal{U}(t,0)[M(q_j)+y_0]+Sf_j(t)+\int_0^t\mathcal{U}(t,s)\omega_j(s)\,ds-S_1f_j(t),\ \ t\in\mathcal{J}. \tag{33}$$

By $\gamma(\{q_j\})\le\gamma(\Theta)\le\gamma\left(H(\Theta)\right)=\gamma(\{y_j\})=0$ and *(M2s)* we have that $\chi_{\mathbb{E}}\left(\{M(q_j)\}\right)\le m\gamma(\{q_j\})=0$. Hence, by the regularity of the Hausdorff m.n.c. we obtain that $\{M(q_j)\}$ is relatively compact in $\mathbb{E}$, implying that $\{\mathcal{U}(\cdot,0)M(q_j)\}$ is a relatively compact subset of $\mathbb{C}(\mathcal{J},\mathbb{E})$. Since Θ is bounded, by applying condition (F3) we get that $\{f_j\}$ is integrably bounded. Moreover $\gamma(\{q_j\})=0$ implies $\chi_E(\{q_j(t)\})=0$ for every t, hence by (F5s) we have that $\{f_j\}$ is semicompact. Since both S and S_1 satisfy the assumptions of Theorem 2.2 (see Remark 3.1 and Lemma 3.3, respectively) we have that also $\{Sf_j\}$ and $\{S_1f_j\}$ are relatively compact in $\mathbb{C}(\mathcal{J},\mathbb{E})$.
The sequence $\{\omega_j\}\subset L^2(\mathcal{J},\mathbb{E})$ and then $\{\omega_j\}\subset L^1(\mathcal{J},\mathbb{E})$. Again by the relative compactness of $\{M(q_j)\}$ we get that also the set $\{z_j\}$ is relatively compact in $\mathbb{E}$. By the continuity of G^{-1} the set $\{G^{-1}z_j\}$ is relatively compact in $L^2(\mathcal{J},\mathbb{U})$ and then also in $L^1(\mathcal{J},\mathbb{U})$. This implies that $\{\omega_j\}$ is relatively compact in $L^1(\mathcal{J},\mathbb{E})$. At last it is easy to show that $\{S\omega_j\}$ is relatively compact in $\mathbb{C}(\mathcal{J},\mathbb{E})$. The claim is proved and the proof is complete. ■

Proof of Theorem 3.1 Fix $y_0, y_1 \in \mathbb{E}$ and consider the solution operator H defined in (24); let $Q_r \subset \mathbb{C}(\mathcal{J}, \mathbb{E})$, $r \in \mathbb{N}$ be the closed ball with radius $r > 0$ and center in 0. Assume that

$$H(Q_{r_0}) \subseteq Q_{r_0} \tag{34}$$

for some $r_0 > 0$. By means of Lemmas 4.1, 4.2, 4.3 and 4.4 we obtain that the multimap $H\colon Q_{r_0} \multimap Q_{r_0}$ satisfies all the assumptions of Theorem 2.4 and then H has a fixed point $y \in Q_{r_0}$, i.e. $y \in H(y)$. Since the conclusion is valid for all $y_0, y_1 \in \mathbb{E}$, then problem (8)-(9) is controllable (see Remark 3.4).
The proof of (34) is based on the estimate (25) and differs according to the assumption *(s1)*, *(s2)* or *(s3)*.

(p1) *Assume conditions (s1)* We claim that $H(Q_{n_0}) \subseteq Q_{n_0}$ for some $n_0 \in \mathbb{N}$. We reason by contradiction and hence assume the existence of two sequences $\{q_n\}$, $\{x_n\} \subset \mathbb{C}(\mathcal{J}, \mathbb{E})$ such that $q_n \in Q_n$, $x_n = x_{f_n, q_n}$, for some $f_n \in P_F(q_n)$ and $x_n \notin Q_n$ for all $n \in \mathbb{N}$. Hence there exists $\{t_n\} \subset \mathcal{J}$ satisfying $\|x_n(t_n)\| > n$ and then, by (25),

$$n < \|x_n(t_n)\| \leq C\|y_1\| + D_{\mathcal{U}}(1+C)\left(\|M(q_n)\| + \|f_n\|_1 + \|y_0\|\right), \quad n \in \mathbb{N}. \tag{35}$$

By condition *(s1)(i)*, we have

$$n < \|x_n(t_n)\| \leq C\|y_1\| + D_{\mathcal{U}}(1+C)\left(\|M(q_n)\| + \|\psi_n\|_1 + \|y_0\|\right), \quad n \in \mathbb{N}.$$

When dividing by n and computing $\liminf$, by *(s1)(i)-(ii)* we arrive to the following contradictory conclusion

$$\begin{aligned} 1 \leq & \liminf_{n\to\infty} \left[\frac{C\|y_1\|}{n} + D_{\mathcal{U}}(1+C)\left(\frac{\|M(q_n)\|}{n} + \frac{\|\psi_n\|_1}{n} + \frac{\|y_0\|}{n}\right)\right] \\ \leq & \liminf_{n\to\infty} \left[\frac{C\|y_1\|}{n} + D_{\mathcal{U}}(1+C)\left(\frac{\|M(q_n)\|}{\|q_n\|_{\mathbb{C}}} + \frac{\|\psi_n\|_1}{n} + \frac{\|y_0\|}{n}\right)\right] < 1. \end{aligned}$$

Assumption (34) is then satisfied, in this case, for every $y_0, y_1 \in \mathbb{E}$.

(p2) *Assume conditions (s2)* Again we claim that $H(Q_{n_0}) \subseteq Q_{n_0}$ for some $n_0 \in \mathbb{N}$ and we reason by contradiction. We consider, in particular, the sequences $\{q_n\}$, $\{x_n\}$, $\{f_n\}$ and $\{t_n\}$ introduced in *(p1)*. By (35) and *(s2)(i)* we obtain

$$n < \|x_n(t_n)\| \leq C\|y_1\| + D_{\mathcal{U}}(1+C)\left(\|M(q_n)\| + \|y_0\| + \int_0^T \beta(s)(1 + \|q_n(s)\|)\, ds\right).$$

Dividing previous inequality by n and passing to the limit, by *(s2)(ii)* and *(s2)(iii)* we arrive to the contradictory conclusion

$$
\begin{aligned}
1 \leq & \lim_{n\to\infty} \left[\frac{C\|y_1\|}{n} + D_{\mathfrak{U}}(1+C)\left(\frac{\|M(q_n)\|}{n} + \frac{\|\beta\|_1(1+n)}{n} + \frac{\|y_0\|}{n} \right)\right] \\
\leq & \lim_{n\to\infty} \left[\frac{C\|y_1\|}{n} + D_{\mathfrak{U}}(1+C)\left(\frac{\|M(q_n)\|}{\|q_n\|_{\mathbb{C}}} + \frac{\|\beta\|_1(1+n)}{n} + \frac{\|y_0\|}{n} \right)\right] \\
= & D_{\mathfrak{U}}(1+C)\|\beta\|_1 < 1.
\end{aligned}
$$

Assumption (34) is then satisfied, in this case, for every $y_0, y_1 \in \mathbb{E}$.

(p3) *Assume conditions (s3)* Given $y_0, y_1 \in \mathbb{E}$, consider Q_L with $L = L(y_0, y_1)$ as in *(s3)(ii)*. Take $q \in Q_L$ and $y \in H(q)$; then $y = x_{f,q}$ for some $f \in P_F(q)$. Notice that, by *(s3)(i)*,

$$\|f\|_1 \leq \int_0^T \beta(t)\rho(\|q(t)\|)\, dt \leq \|\beta\|_1\rho(\|q\|_{\mathbb{C}}).$$

Therefore, by (25), (s3)(i)-(ii), we obtain

$$
\begin{aligned}
\|x_n(t)\| \leq & \ C\|y_1\| + D_{\mathfrak{U}}(1+C)\left(\sigma(\|q\|_{\mathbb{C}}) + \|\beta\|_1\rho(\|q\|_{\mathbb{C}}) + \|y_0\|\right) \\
\leq & \ C\|y_1\| + D_{\mathfrak{U}}(1+C)\left(\sigma(L) + \|\beta\|_1\rho(L) + \|y_0\|\right) < L,
\end{aligned}
$$

implying that $H(Q_L) \subseteq Q_L$ and (34) is true also in this case for any choice of $y_0, y_1 \in \mathbb{E}$. The proof is complete.

5 Proof of Theorem 3.2

In this part we investigate the controllability of problem (8)-(9) when the regularities in $\mathbb{E}$ are given by means of the weak topology. We always assume that the Banach space $\mathbb{E}$ is reflexive. The discussion, as in Section 4, exploits a fixed point technique and involves the solution multioperator H. We already know (Lemma 4.1) that H has convex values. In some preliminary results we show that H has a weakly sequentially closed graph in $\mathbb{C}(\mathfrak{J}, \mathbb{E}) \times \mathbb{C}(\mathfrak{J}, \mathbb{E})$ (Lemma 5.1) and that H has closed values and it is weakly compact when restricted to bounded sets (Lemma 5.2). We need, in the following, the Eberlein-Šmulian theory (see e.g. [18]); it states that, in $\mathbb{E}$, the relative sequential weak compactness and the sequential weak compactness are, respectively, equivalent to the relative weak compactness and the weak compactness. The proof of Theorem 3.2 completes this part.

Lemma 5.1 *Assume conditions* (F3)*,* (F4w) *and* (M1w)*. Then the multioperator H is weakly sequentially closed in $\mathbb{C}(\mathfrak{J}, \mathbb{E}) \times \mathbb{C}(\mathfrak{J}, \mathbb{E})$.*

Proof. Let q_j, $q \in \mathbb{C}(\mathfrak{J}, \mathbb{E})$, $y_j \in H(q_j)$, $y \in \mathbb{C}(\mathfrak{J}, \mathbb{E})$, with $j \in \mathbb{N}$, be such that

$$q_j \rightharpoonup q, \ y_j \rightharpoonup y \ \text{ in } \ \mathbb{C}(\mathfrak{J}, \mathbb{E}), \ \text{ while } \ j \to \infty. \tag{36}$$

The result is proved if $y \in H(q)$. Notice that y_j satisfies (26) for some $f_j \in P_F(q_j)$, $j \in \mathbb{N}$. By the characterization of the weak convergence in $\mathbb{C}(\mathcal{J}, \mathbb{E})$ (see Lemma 2.1), condition (27) is satisfied, for some bounded $\Omega \subset \mathbb{E}$. According to *(F3)* there exists $\mu_\Omega \in L^1(\mathcal{J}, \mathbb{R})$ such that $\|f_j(t)\| \leq \mu_\Omega(t)$ for a.a. $t \in \mathcal{J}$ and $j \in \mathbb{N}$; hence, by the reflexivity of $\mathbb{E}$ and Theorem 2.1, there is a subsequence, still denoted as the sequence, satisfying $f_j \rightharpoonup f \in L^1(\mathcal{J}, \mathbb{E})$. Given $\phi \colon \mathbb{E} \to \mathbb{R}$, linear and bounded and $t \in \mathcal{J}$, consider the operator $\Phi \colon L^1([0,t], \mathbb{E}) \to \mathbb{R}$ defined by

$$\Phi(h) := \phi\left(\int_0^t \mathcal{U}(t,s)h(s)\, ds\right).$$

Since Φ is clearly linear and bounded and the weak convergence: $f_j \rightharpoonup f$ is true also in $L^1([0,t], \mathbb{E})$, we have that

$$\Phi(f_j) = \phi\left(\int_0^t \mathcal{U}(t,s)f_j(s)\, ds\right) \rightharpoonup \phi\left(\int_0^t \mathcal{U}(t,s)f(s)\, ds\right) = \Phi(f).$$

By the arbitrariness of ϕ we conclude that

$$\int_0^t \mathcal{U}(t,s)f_j(s)\, ds \rightharpoonup \int_0^t \mathcal{U}(t,s)f(s)\, ds.$$

Consequently, since $M(q_j) \rightharpoonup M(q)$ by *(M1w)*, we obtain that $p_{q_j}(f_j) \rightharpoonup p_q(f)$ as $j \to \infty$; hence, the linearity and boundedness of G^{-1} imply that $u_j := u_{f_j,q_j} \rightharpoonup u_{f,q}$ in $L^2(\mathcal{J}, \mathbb{U})$. Since, for $t \in \mathcal{J}$, also the weak convergence $u_j \rightharpoonup u_{f,q}$ in $L^1([0,t], \mathbb{U})$ is satisfied, with a similar reasoning as before we have

$$\int_0^t \mathcal{U}(t,s)Bu_j(s)\, ds \rightharpoonup \int_0^t \mathcal{U}(t,s)Bu(s)\, ds, \;\; t \in \mathcal{J} \text{ in } \mathbb{E}.$$

By (26), we obtained that $y_j(t) \rightharpoonup z(t)$ in $\mathbb{E}$, where

$$z(t) := \mathcal{U}(t,0)[M(q) + y_0] + \int_0^t \mathcal{U}(t,s)f(s)\, ds + \int_0^t \mathcal{U}(t,s)Bu_{f,q}(s)\, ds, \;\; t \in \mathcal{J}$$

and then, by the uniqueness of the weak limit $z = y$.

What is left to show is that $f \in P_F(q)$. By *(F3)*, *(F4w)* and the definition of f_j, there is $N_0 \subset \mathcal{J}$ with Lebesgue measure $\lambda(N_0) = 0$ such that $\|F(t, q_j(t))\| \leq \mu_\Omega(t)$ with Ω as in (27), $F(t,\cdot) \colon \mathbb{E} \multimap \mathbb{E}$ is weakly sequentially closed and $f_j(t) \in F(t, q_j(t))$, for all $t \notin N_0$ and $j \in \mathbb{N}$.

Due to the Mazur's convexity Theorem, for each $j \in \mathbb{N}$, there exists $k_j \in \mathbb{N}$ and positive numbers $\beta_{j,i}$, $i = 0, \ldots, k_j$, such that $\sum_{i=0}^{k_j} \beta_{j,i} = 1$ and $g_j := \sum_{i=0}^{k_j} \beta_{j,i} f_{j+i} \to f$ in $L^1(\mathcal{J}, \mathbb{E})$. From the sequence $\{g_j\}$ we extract a subsequence, as usual denoted as the sequence, such that $g_j(t) \to f(t)$, for all $t \in \mathcal{J} \setminus N_1$, with $\lambda(N_1) = 0$.

Put $N := N_0 \cup N_1$ and assume, by a contradiction, that there exists $t_0 \in \mathcal{J} \setminus N$ such that $f(t_0) \notin F(t_0, q(t_0))$. Since $F(t_0, q(t_0))$ is closed and convex, by the Hahn–Banach

Theorem we obtain a weakly open and convex set V such that $F(t_0, q(t_0)) \subset V$ and $f(t_0) \notin \overline{V}^w = \overline{V}$.
Let F_Ω be the restriction of the multimap $F(t_0, \cdot)$ to the bounded set Ω introduced in (27). We show that F_Ω is weakly u.s.c. We know that $\|F_\Omega(x)\| = \|F(t_0, x)\| \leq \mu_\Omega(t_0)$ for every $x \in \Omega$. Therefore, F_Ω is weakly relatively compact, by the reflexivity of the space $\mathbb{E}$. Since $F(t_0, \cdot)$ is weakly sequentially closed, F_Ω is weakly compact by the Eberlein-Šmulyan theory; in particular it is compact valued. With no loss of generality we can assume Ω weakly closed implying that F_Ω is also closed. Hence it is weakly u.s.c. by [17, Theorem 1.1.5] (see Section 2).
Therefore, we can find a weakly open set V_1 of $q(t_0)$ satisfying $F_\Omega(V_1 \cap \Omega) \subset V$. Since $q_j(t_0) \rightharpoonup q(t_0)$, then there exists $j_0 \in \mathbb{N}$ with the property that $q_j(t_0) \in V_1 \cap \Omega$, for all $j > j_0$. Thus $f_j(t_0) \in F(t_0, q_j(t_0)) \subset V$, for $j > j_0$, but V is convex, and so $g_j(t_0) \in V$, which leads to the contradiction $f(t_0) \in \overline{V}$. The proof is complete.
■

Lemma 5.2 *If the assumptions of Lemma 5.1 and (M2w) are satisfied, then H maps bounded sets into weakly relatively compact sets.*

Proof. Let $Q \subset \mathbb{C}(\mathcal{J}, \mathbb{E})$ be bounded and denote by Ω the bounded subset of $\mathbb{E}$ such that

$$q(t) \in \Omega, \quad q \in Q, \, t \in \mathcal{J}. \tag{37}$$

According to the Eberlein-Šmulyan theory, the weak relative compactness of $H(Q)$ is equivalent to its weak sequential relative compactness, so we prove in the following the latter property.
Let $\{q_j\} \subset Q$ and $\{y_j\} \subset \mathbb{C}(\mathcal{J}, \mathbb{E})$ with $y_j \in H(q_j), j \in \mathcal{J}$; hence y_j satisfies (26) for some $f_j \in P_F(q_j), j \in \mathcal{J}$. Reasoning as in the proof of Lemma 5.1 but with Ω defined as in (37), we can find a subsequence, still denoted as the sequence, such that $f_j \rightharpoonup f$, with $f \in L^1(\mathcal{J}, \mathbb{E})$. By the reflexivity of $\mathbb{E}$ and *(M2w)*, there is a subsequence of $\{M(q_j)\}$, again denoted as the sequence, such that $M(q_j) \rightharpoonup w \in \mathbb{E}$. Therefore

$$y_j(t) \rightharpoonup z(t) := \mathcal{U}(t, 0)[w + y_0] + \int_0^t \mathcal{U}(t, s) f(s) \, ds + \int_0^t \mathcal{U}(t, s) B u(s) \, ds, \quad t \in \mathcal{J} \tag{38}$$

where $u := G^{-1}\big(p\big)$ with $p := y_1 - \mathcal{U}(T, 0)[w + y_0] - \int_0^T \mathcal{U}(T, s) f(s) \, ds$.
Notice, moreover, that $\{y_j\}$ is bounded in $\mathbb{C}(\mathcal{J}, \mathbb{E})$. In fact, condition *(M2w)* implies the existence of a positive constant $\overline{M}$ satisfying $\|M(q)\| \leq \overline{M}$, $q \in Q$ and then, by (25),

$$\|y_j(t)\| = \|x_{f_j, q_j}(t)\| \leq C\|y_1\| + D_{\mathcal{U}}(1 + C)(\overline{M} + \|\mu_\Omega\|_1 + \|y_0\|), \quad t \in \mathcal{J}, \tag{39}$$

again with Ω as in (37) and C defined in (19). By the characterization of the weak convergence in $\mathbb{C}(\mathcal{J}, \mathbb{E})$ (see Lemma 2.1) we conclude that $y_j \rightharpoonup z$ in $\mathbb{C}(\mathcal{J}, \mathbb{E})$ and the proof is complete. ■

Remark 5.1 *Let the assumptions of Lemma 5.2 be satisfied. By Lemma 5.1 and the Eberlein-Šmulyan theory, it is easy to show that H is weakly compact valued.*

Proof of Theorem 3.2 Let Q_r be as in the proof of Theorem 3.1. With a similar reasoning as there it is possible to show, in all the cases *(w1)*, *(w2)* and *(w3)*, that $H(Q_{r_0}) \subset Q_{r_0}$ for some $r_0 > 0$. Let H_0 be the restriction of H to Q_{r_0}. Then H_0 is convex and weakly compact valued, by Lemma 4.1 and Remark 5.1, respectively. Moreover, by Lemmas 5.1 and the Eberlein-Šmulyan theory, H_0 is weakly closed. At last, the estimate $H_0(Q_{r_0}) \subseteq Q_{r_0}$ implies that H_0 is weakly compact. Therefore, by applying [17, Theorem 1.1.5] (see Section 2), we conclude that H_0 is weakly u.s.c. Hence H_0 has a fixed point $y \in Q_{r_0}$, by Theorem 2.5. The property is true for every $y_0, y_1 \in \mathbb{E}$ and then problem (8)-(9) is controllable (see Remark 3.4).

6 About the controllability of problem (1)-(3)

This part deals with the exact controllability of problem (1)-(3). We make use of the results and techniques discussed in Sections 4 and 5. For the sake of simplicity we restrict to a one-dimensional state space, i.e. we assume that $x \in K := [0, L]$ and we further require that the solution satisfies the Dirichlet condition. We consider, precisely

$$\begin{cases} z_{tt} = z_{xx} + f\left(t, x, \int_0^L h(x,\xi) z(t,\xi) d\xi\right) + b(x)v(t,x), \quad x \in K, \quad t \in \mathfrak{J} \\ z(t,0) = z(t,L) = 0 \quad t \in \mathfrak{J} \\ z(0,x) = \sum_{i=1}^{p} \alpha_i z(t_i, x) + z_0(x), \; z_t(0,x) = \sum_{j=1}^{q} \beta_j z_t(\tau_j, x) + z_1(x), \quad x \in K, \end{cases} \tag{40}$$

with t_i, τ_j, α_i and β_j as in (4). Both the electric voltage and the current in a double conductor satisfy equation (40). The interacting quantity f contains an integral term which takes into account the effects of finite velocity to standard heat or mass transport equation. We assume

(a) $f(\cdot, \cdot, c) : \mathfrak{J} \times K \to \mathbb{R}$ is measurable, for all $c \in \mathbb{R}$;

(b) $f(t, x, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous, for a.a. $(t, x) \in \mathfrak{J} \times K$;

(c) there exist $\eta \in L^1(\mathfrak{J}; \mathbb{R}^+)$ and $\lambda : [0, \infty) \to [0, \infty)$ increasing such that, for a.a. $t \in \mathfrak{J}, x \in K$ and every $c \in \mathbb{R}$,

$$|f(t, x, c)| \leq \eta(t)\lambda(|c|)$$

and $\liminf\limits_{c \to \infty} \dfrac{\lambda(c)}{c} = 0$;

(d) $h\colon K\times K\to\mathbb{R}$ is measurable with $h(x,\cdot)\in L^2(K;\mathbb{R})$ and $\|h(x,\cdot)\|_2\le 1$ for a.a. $x\in K$;

(e) $b\in L^\infty(K,\mathbb{R})$ is such that $\|b\|_\infty>0$; $z_0,\ z_1\in L^2(K;\mathbb{R})$.

Take the Hilbert space $\mathbb{E}=\mathbb{U}:=L^2(K;\mathbb{R})$. Problem (40) can then be written, in abstract setting, in the form

$$\begin{cases} r''(t)=Ar(t)+F(t,r(t))+Bw(t),\ t\in\mathcal{J},\\ r(0)=\sum_{i=1}^{p}\alpha_i r(t_i)+\overline{y}_0;\ r'(0)=\sum_{j=1}^{q}r'(\tau_j)+\overline{y}_1\end{cases}\tag{41}$$

where $r(t):=z(t,\cdot)$, $w(t):=v(t,\cdot)$, $\overline{y}_0:=z_0(\cdot)$, $\overline{y}_1:=z_1(\cdot)$. The functions $F\colon\mathcal{J}\times\mathbb{E}\to\mathbb{E}$ and $B\colon\mathbb{E}\to\mathbb{E}$ are defined by

$$F(t,r)(x)=f\left(t,x,\int_0^L h(x,\xi)r(\xi)d\xi\right),\ \ Bw(x)=b(x)w(x)\ \text{ for a.a. } x\in K.$$

The problem is well-posed by assumptions *(a)-(e)*. Let $A\colon D(A)=\{r\in W^{2,2}(K;\mathbb{R}): r(0)=r(L)=0\}\to L^2(K;\mathbb{R})$ be the Laplace operator $Ar=r''$.
Observe that $-A$ is a self-adjoint and positive definite operator on $L^2(K;\mathbb{R})$ with a compact inverse, hence there exists a unique positive definite square root $(-A)^{1/2}$ with domain $D((-A)^{1/2})=\{y\in W^{1,2}(K;\mathbb{R}): y(0)=y(L)=0\}$ (see, e.g. [21]). Denoting by $\mathcal{E}$ the Hilbert space $W_0^{1,2}(K;\mathbb{R})\times L^2(K;\mathbb{R})$ with norm $\|y\|_{\mathcal{E}}=\|y_1\|_{W_0^{1,2}}+\|y_2\|_2$, the linear operator $\mathcal{A}: W_0^{2,2}(K;\mathbb{R})\times L^2(K;\mathbb{R})\to\mathcal{E}$ defined by

$$\mathcal{A}=\begin{pmatrix}0 & I\\ A & 0\end{pmatrix},$$

generates a strongly continuous semigroup of contractions $\mathcal{U}(t)$ (see, e.g., [12]). Hence (41) can be transformed into the new problem

$$\begin{cases} y'(t)=\mathcal{A}y(t)+\mathcal{F}(t,y(t))+\mathcal{B}u(t),\ t\in\mathcal{J}\\ y(0)=\mathcal{M}(y)\end{cases}\tag{42}$$

in $\mathcal{E}$ which involves a first order equation. The function $\mathcal{F}:\mathcal{J}\times\mathcal{E}\to\mathcal{E}$ is defined by

$$\mathcal{F}(t,y)=\mathcal{F}(t,(y_1,y_2))=(0,F(t,y_1)),$$

$\mathcal{B}:\mathcal{E}\to\mathcal{E}$ can be expressed as $\mathcal{B}(u_1,u_2)=(0,Bu_1)$ and $\mathcal{M}\colon\mathbb{C}(\mathcal{J},\mathcal{E})\to\mathcal{E}$ is given by

$$\mathcal{M}(y)=\mathcal{M}(y_1,y_2)=\left(\sum_{i=1}^{p}\alpha_i y_1(t_i),\sum_{j=1}^{q}\beta_j y_2(\tau_j)\right).$$

We introduce the notation $\varphi_y\colon\mathcal{J}\times K\to\mathbb{R}$ with

$$\varphi_y(t,x):=f\left(t,x,\int_0^L h(x,\xi)y_1(\xi)d\xi\right),\tag{43}$$

so that $\mathcal{F}(t,y) = (0, \varphi_y(t,\cdot))$, $t \in \mathcal{J}$, $y \in \mathcal{E}$. By applying Theorem 3.2 we prove the controllability of problem (42) and, hence, of (40).

Condition *(A)* is satisfied and

$$\|\mathcal{U}(t)\|_{\mathcal{L}} \leq 1, \;\; t \in \mathcal{J}.$$

It is easy to show that $\mathcal{B}\colon \mathbb{E} \times \mathbb{E} \to \mathbb{E}$ is linear and bounded and then *(B)* is satisfied with $\|\mathcal{B}\| = \|b\|_\infty$. Since the equation in (42) is single-valued, also *(F1)* is trivially satisfied.

Now we prove *(F2)*, by means of Theorem 2.3. Fix $y = (y_1, y_2) \in \mathcal{E}$ and let $e\colon \mathcal{E} \to \mathbb{R}$ be linear and bounded. Hence there is $\psi \in L^2(K,\mathbb{R})$ satisfying

$$e \circ \mathcal{F}(t,y) = e\,(0, F(t,y_1)) = \int_0^L \psi(x)\varphi_y(t,x)\,dx, \;\; t \in \mathcal{J},$$

with φ_y as in (43). By *(d)* and the properties of y_1, it is clear that $\int_0^L h(\cdot,\xi)y_1(\xi)d\xi$ is a Borel-measurable function in K. Hence, by *(a)-(c)*, also the map $(t,x) \longmapsto \psi(x)\varphi_y(t,x)$ is Borel measurable in $\mathcal{J} \times K$. It implies that $e \circ \mathcal{F}(\cdot, y)$ is measurable in $\mathcal{J}$ and then, by Theorem 2.3, condition *(F2)* is satisfied.

Now we prove that $\mathcal{F}(t,\cdot)$ is weakly continous for a.a. $t \in \mathcal{J}$. In fact, let $\{y_n\} \subset \mathcal{E}$ with $y_n = (y_{n,1}, y_{n,2}) \rightharpoonup y = (y_1,y_2) \in \mathcal{E}$. Then $y_{n,1} \rightharpoonup y_1$ in $W_0^{1,2}(K,\mathbb{R})$ and then $y_{n,1} \rightharpoonup y_1$ in $L^2(K,\mathbb{R})$; by *(d)* we obtain,

$$\int_0^L h(x,\xi)y_{n,1}(\xi)d\xi \to \int_0^L h(x,\xi)y_1(\xi)d\xi$$

for a.a. $x \in K$. Let $c_0 > 0$ be such that

$$\left|\int_0^L h(x,\xi)y_{n,1}(\xi)d\xi\right| \leq c_0, \;\; x \in K, \, n \in \mathbb{N}.$$

Therefore, we have by *(b)* that

$$\varphi_{y_n}(t,x) \to \varphi_y(t,x), \;\; a.a.(t,x) \in \mathcal{J} \times K$$

and the convergence is dominated since, by *(c)*, $|\varphi_{y_n}(t,x)| \leq \eta(t)\lambda(c_0)$. Hence $\varphi_{y_n}(t,\cdot) \to \varphi_y(t,\cdot)$ in $L^2(K,\mathbb{R})$ and property *(F4w)* is satisfied.

Recalling Example 3.2, also *(M1w)-(M2w)* are true and

$$\lim_{\|y\| \to \infty} \frac{\|\mathcal{M}(y)\|}{\|y\|} = \sum_{i=1}^p |\alpha_i| + \sum_{j=1}^q |\beta_j|.$$

Concerning assumption *(G1)* we remind to [12, Example VI.8.10].
Finally, from (c) and (d) we get that

$$\begin{aligned}
\|\mathcal{F}(t,y)\| &= \sqrt{\int_0^L (f(t,x,\int_0^L h(x,\xi)y_1(\xi)d\xi))^2 dx} \\
&\le \eta(t)\sqrt{\int_0^L (\lambda(\int_0^L |h(x,\xi)y_1(\xi)|d\xi))^2 dx} \\
&\le \eta(t)\sqrt{\int_0^L (\lambda(\|h(x,\cdot)\|_2\|y_1\|_2))^2 dx} \\
&\le \eta(t)\sqrt{\int_0^L (\lambda(\|y\|_{\mathcal{E}}))^2 dx} = \eta(t)\lambda(\|y\|_{\mathcal{E}})\sqrt{L},
\end{aligned}$$

hence *(s1i)* holds with $\psi_n(t) = \eta(t)\lambda(n)|L|^{\frac{1}{2}}$.

Notice in particular that, since $y(t) = (r(t), r'(t))$, $t \in \mathcal{J}$ with $y \in C(\mathcal{J}; L^2(K;\mathbb{R}) \times L^2(K;\mathbb{R}))$ we obtain that the map $r\colon \mathcal{J} \to L^2(K;\mathbb{R})$, $t \longmapsto z(t,\cdot)$ belongs to $C^1(\mathcal{J}; L^2(K;\mathbb{R}))$.
By means of Theorem 3.2*(w1)* we arrive to the following result

Theorem 6.1 *Consider problem* (40)*, assume conditions* (a)-(e) *and let*

$$\sum_{i=1}^{p} |\alpha_i| + \sum_{j=1}^{q} |\beta_j| < \frac{1}{1 + \|b\|_\infty \|G^{-1}\| \sqrt{T}}, \tag{44}$$

then problem (40) *is controllable.*

Let us now consider equation

$$\begin{cases} z_{tt} = z_{xx} + f(t,x,z) + b(x)v(t,x), \quad \text{for } x \in K, \ t \in \mathcal{J} \\ z(t,0) = z(t,L) = 0 \quad t \in \mathcal{J} \\ z(0,x) = \sum_{i=1}^{p} \alpha_i z(t_i,x) + z_0(x), \ z_t(0,x) = \sum_{j=1}^{q} \beta_j z_t(\tau_j,x) + z_1(x) \quad x \in K, \end{cases} \tag{45}$$

under conditions (a), (e) and

f) $f(t,x,\cdot)$ is Lipschitzian for a.e. $(t,x) \in \mathcal{J} \times K$ with constant $k(t)$ for some $k \in L^1(\mathcal{J},\mathbb{R})$;

g) $|f(t,x,c)| \le h(t)p(x)$, for a.a. $t \in \mathcal{J}, x \in K$ and every $c \in \mathbb{R}$, with $h \in L^1(\mathcal{J},\mathbb{R})$ and $p \in L^2(K,\mathbb{R})$.

As in the previous example, system (45) can be written in the abstract form (42) , with $F : \mathcal{J} \times \mathbb{E} \to \mathbb{E}$ defined by

$$F(t,r)(x) = f(t,x,r(x)).$$

To prove the controllability of (45) by applying Theorem 3.1*(s1)*, we check assumptions *(F4s), (F5s)* and *(s1iii)*.
If $y_n = (y_{n,1}, y_{n,2}) \to y = (y_1, y_2) \in \mathbb{E}$, then $y_{n,1} \to y_1 \in L^2(H, \mathbb{R})$. From (f) we then get that, for a.a. $t \in \mathfrak{J}$,

$$\|F(t, y_{n,1}) - F(t, y_1)\|_2^2 = \int_0^L |f(t, x, y_{n,1}(x)) - f(t, x, y_1(x))|^2 dx \leq k^2(t)\|y_{n,1} - y_1\|_2^2 \to 0,$$

i.e. that $\mathfrak{F}(t, \cdot)$ is continuous, hence closed. On the other hand, from (g) we get that $\mathfrak{F}(t, \cdot)$ is bounded for a.a $t \in \mathfrak{J}$, which implies *(F5s)*.
By means of Theorem 3.1*(s1)* we arrive to the following result

Theorem 6.2 *Consider problem* (45)*, assume conditions* (a),(e)-(g) *and* (44)*. Suppose that* (G2s) *holds as well as*

$$(1 + \|b\|_\infty \|g\|_1) \left(\sum_{i=1}^{p} |\alpha_i| + \sum_{j=1}^{q} |\beta_j| + \|k\|_1 \right) < 1, \tag{46}$$

then problem (45) *is controllable.*

Remark 6.1 *Now we compare Theorem 6.1, i.e. the usage of the weak topology for the solvability of the associated abstract problem, and Theorem 6.2, i.e. the usage of the strong topology. The weak topology involves a lower number of conditions, namely only (a)-(e) and* (44)*. On the other hand, the strong topology, imposing stronger conditions, allows to consider more general nonlinear terms which may depend explicitly on the state term z and not only on its weighted mean value.*

7 Controllability of first order integro-differential dynamics

Let $\mathfrak{J} := [0, T]$, Ω be a bounded domain in $\mathbb{R}^n$ with sufficiently regular boundary. We will consider problem

$$\begin{cases} z_t(t, x) \in \int_\Omega k(x, y) z(t, y)\, dy + f(t, x, z) + b(x) v(t, x), & \text{for } x \in \Omega, \text{ a.e. } t \in \mathfrak{J}, \\ z(0, x) = \sum_{i=1}^{n} \alpha_i z(t_i, x) + z_0(x), & x \in \Omega, \\ z(T, x) = z_1(x), & x \in \Omega, \end{cases} \tag{47}$$

where

$$f(t, x, z) := \left[f_1\left(t, x, \int_\Omega h(x, \xi) z(t, \xi)\, d\xi\right), f_2\left(t, x, \int_\Omega h(x, \xi) z(t, \xi)\, d\xi\right) \right]. \tag{48}$$

Such integro-differential inclusion describes population dispersal. Diffusion operators such as the integral contained therein introduce a long distance dispersal effect in the

equation, hence are frequently preferable than the classical punctual diffusion terms such as the Laplace operator. The multivalued term represents the external influence on the process. It takes into account the long-distance dispersal and describes the dispersion via a kernel, which specifies the probability that an individual moves from one location to another.
We assume that

a) $f_i(\cdot,\cdot,p)\colon \mathcal{J}\times\Omega\to\mathbb{R}$ is measurable, for all $p\in\mathbb{R}$, $i=1,2$;

b) $f_1(t,x,\cdot)\colon \mathbb{R}\to\mathbb{R}$ is lower semicontinuous, while $f_2(t,x,\cdot)\colon \mathbb{R}\to\mathbb{R}$ is upper semicontinuous, for a.e. $t\in\mathcal{J}$, $x\in\Omega$, with $f_1(t,x,p)\le f_2(t,x,p)$;

c) there exist $\eta\in L^1(\mathcal{J},\mathbb{R}^+)$ and a non-decreasing function $\lambda\colon\mathbb{R}^+\to\mathbb{R}^+$ such that

$$\|f_i(t,x,p)\|\le\eta(t)\lambda(|p|),\quad \text{for } x\in\Omega,\ \text{a.e. } t\in\mathcal{J}, \tag{49}$$

and $\displaystyle\lim_{p\to+\infty}\frac{\lambda(p)}{p}=0$;

d) $h\colon\Omega\times\Omega\to\mathbb{R}$ is measurable and $h(x,\cdot)\in L^2(\Omega,\mathbb{R})$ with $\|h(x,\cdot)\|_2\le 1$ for all $x\in\Omega$;

e) $k\in L^2(\Omega\times\Omega,\mathbb{R})$;

f) $b\in L^\infty(\overline{\Omega},\mathbb{R})$ is such that $|b(x)|\ge c>0$ for a.a. $x\in\Omega$; $z_0\in L^2(\Omega,\mathbb{R})$.

Take the Hilbert space $\mathbb{E}=\mathbb{U}:=L^2(\Omega,\mathbb{R})$. Let $A\colon\mathbb{E}\to\mathbb{E}$ be defined as

$$A\colon w\mapsto Aw\colon x\mapsto\int_\Omega k(x,y)w(y)\,dy. \tag{50}$$

It is well-defined, bounded and linear. Therefore, it generates a strongly continuous semigroup $e^{At}:=\sum_{n=0}^\infty\frac{t^n}{n!}A^n$. This shows that Assumption *(A)* is satisfied. Moreover, we can estimate that

$$\|e^{At}\|\le e^{\|A\|T}\le e^{\|k\|_2T}. \tag{51}$$

Define $F\colon\mathcal{J}\times\mathbb{E}\multimap\mathbb{E}$ to be $F(t,w)(x):=f(t,x,w)$.
Denoting $w:=z(t,\cdot)$ we can rewrite the above system in a form

$$\begin{cases} w'(t)\in Aw(t)+F\big(t,w(t)\big)+Bu(t), & \text{a.e. } t\in\mathcal{J},\\ w(0)=\sum_{i=1}^n\alpha_iw(t_i)+w_0,\\ w(T)=w_1,\end{cases} \tag{52}$$

where $w_i:=z_i(\cdot)$, $i=0,1$, $u(t)=v(t,\cdot)$.
Now we will check if all assumptions in Theorem 3.2(w1) are satisfied.

Trivially *(F1)* is fulfilled. For every $x \in \mathbb{E}$ function $t \mapsto f_1(t, \cdot, \int_\Omega k(\cdot, y)x(y)\ dy)$ is a measurable selection of $F(\cdot, x)$, which is Assumption *(F2)*. Verification of conditions *(F4w)* and *(s1)(i)* was done in [5]. Furthermore, $B\colon \mathbb{U} \to \mathbb{E}$ defined as $w \mapsto b(\cdot)w(\cdot)$, is linear and bounded, hence Assumption *(B)* is fulfilled with $\|B\| = \|b\|_\infty$. Let us define $G\colon L^2(\mathcal{J}, \mathbb{U}) \to \mathbb{E}$ as follows

$$u \mapsto \int_0^T e^{A(T-s)} Bu(s)\ ds. \tag{53}$$

We will prove that G is surjective. Observe that $-A$ is a bounded and linear operator and it generates a strongly continuous semigroup e^{-At}. Let $z \in \mathbb{E}$ and define $u_z(s)(x) := \frac{1}{Tb(x)} e^{-A(T-s)} z(x)$. Then it is easy to check that $u_z \in \mathbb{U}$. Furthermore, for every $x \in \Omega$,

$$Bu_z(s)(x) = \frac{1}{T} e^{-A(T-s)} z(x)$$

thus

$$e^{A(T-s)} Bu_z(s) = \frac{1}{T} e^{A(T-s)} e^{-A(T-s)} z = \frac{1}{T} z,$$

so

$$\int_0^T e^{A(T-s)} Bu_z(s)\ ds = \int_0^T \frac{1}{T} z\ ds = z. \tag{54}$$

Therefore, $G(u_z) = z$. Hence, there exists pseudoinverse (see the discussion below formula (16)) $G^{-1}\colon \mathbb{E} \to L^2(\mathcal{J}, \mathbb{U})$ and Assumption *(G1)* is satisfied. We can estimate

$$\|G^{-1}\| \le \sup_{\|z\|_{\mathbb{E}}=1} \|u_z\|_{L^2(\mathcal{J},\mathbb{U})} \le \sup_{\|z\|_{\mathbb{E}}=1} \sqrt{\int_{\mathcal{J}} \left(\int_\Omega \left| \frac{1}{b(x)T} e^{-A(T-s)} z(x) \right|^2 dx \right) ds} \tag{55}$$

$$\le \frac{1}{c\sqrt{T}} e^{\|k\|_2 T} \tag{56}$$

Assumptions (M1w) and (M2w) follow from Example 3.2.

Thus, by means of Theorem 3.2*(w1)* we arrive to the following result

Theorem 7.1 *Consider problem* (47)*, assume conditions* (a)-(f) *and let*

$$\sum_{i=1}^n |\alpha_i| \le \frac{c}{e^{\|A\|T}(c + \|b\|_\infty e^{2T\|k\|_2})},$$

then problem (47) *is controllable.*

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