

Dispersion of elastic waves in a layer interacting with a Winkler foundation

B. Erbaş,¹ J. Kaplunov,² A. Nobili,³ and G. Kılıç¹

¹*Anadolu University, Department of Mathematics, Yunus Emre Campus, Eskişehir, Turkey*

²*School of Computing and Mathematics, Keele, Staffordshire, ST5 5BG,*

UK^a

³*Dipartimento di Ingegneria Enzo Ferrari, via Vignolese 905,*

41122 Modena, Università degli Studi di Modena e Reggio Emilia,

Italy

1 Dispersion of plane harmonic waves in an elastic layer interacting with one or two-
2 sided Winkler foundation is analysed. The long-wave low frequency polynomial ap-
3 proximations of the full transcendental dispersion relations are derived for a relatively
4 soft foundation. The validity of the conventional engineering formulation of a Kirch-
5 hoff plate resting on an elastic foundation is investigated. It is shown that this
6 formulation has to be refined near the cut-off frequency of bending waves. The as-
7 sociated near cut-off expansion is obtained for both cases. A simple explicit formula
8 demonstrating veering of bending and extensional waves is presented for a one-sided
9 foundation.

^{a)}j.kaplunov@keele.ac.uk

10 I. INTRODUCTION

11 Dispersion of elastic waves in a free layer was investigated in great detail beginning with
12 the seminal contribution of Lamb ([Lamb, 1917](#)). The symmetry of the problem significantly
13 simplifies analysis enabling separation of all the quantities of interest into even and odd
14 parts with respect to the mid-plane. Violation of the symmetry, due to interaction with the
15 environment along one of the faces, leads to more sophisticated dispersion relations which
16 cannot be generally reduced to simpler ones for symmetric and antisymmetric waves. As an
17 example, we mention ([Kaplunov and Markushevich, 1993](#)) investigating vibration of elastic
18 layer resting on an acoustic half-space. A layer supported by a Winkler foundation is another
19 important asymmetric setup.

20 Bending of elastic structures on a Winkler foundation is usually treated in the framework
21 of approximate engineering theories beginning with static analysis of an Euler-Bernoulli
22 beam, see ([Frýba, 2013](#)) and references therein. Associated dynamic formulations are always
23 obtained just by incorporating an extra inertial term. However, such elementary trick does
24 not take into consideration the peculiarities of the bending wave propagation in the vicinity
25 of the cut-off frequency arising from the effect of the foundation. Formally, it corresponds
26 to the absence of the leading order term in the Taylor near cut-off expansion expressed in
27 terms of squared wavenumber. This phenomenon is also characteristic of formal dynamic
28 generalization of the original static semi-membrane equations for a thin cylindrical shell, see
29 ([Kaplunov *et al.*, 2016a](#); [Kaplunov and Nobili, 2017b](#); [Strozzi *et al.*, 2014](#)).

30 The current paper is aimed at revising the traditional 1D bending problem for a Kirchhoff
31 plate on a Winkler foundation starting from the exact solution of the plane strain problem
32 in linear elasticity with the main focus on a near cut-off behaviour. The fundamental
33 extensional mode with a zero cut-off frequency arising in the 2D formulation, is also tackled
34 in the paper.

35 The studied Winkler foundation may be treated as the leading order approximation
36 of the related problem for an elastic laminate subject to appropriate boundary condi-
37 tions along the faces, see (Aghalovyan, 2015). It still finds numerous applications in-
38 cluding modelling of transit and edge bending waves, (Brun *et al.*, 2013), (Kaplunov *et al.*,
39 2016b), (Kaplunov *et al.*, 2014), see also (Kaplunov and Nobili, 2017a) using refined Paster-
40 nak model. Among the publications on the subject we also mention (Elishakoff *et al.*, 2018;
41 Li *et al.*, 2009; Ponnusamy and Selvamani, 2012).

42 The paper is organized as follows. The dynamic equations in plane elasticity written in
43 terms of wave potentials are presented in Section II. Along with the boundary conditions
44 corresponding to the traditional ‘one-sided’ Winkler foundation, a symmetric ‘two-sided’
45 foundation is also considered. Various dispersion relations are derived in Section III. As
46 might be expected, the dispersion relations for symmetric and antisymmetric waves may
47 be separated from each other only for a two-sided foundation. The asymptotic expansions
48 corresponding to the bending cut-off frequencies are obtained for a relatively soft Winkler
49 foundation. Sections IV and V deal with long wave low frequency polynomial approxima-
50 tions of the transcendental dispersion relations derived in Section III for two and one-sided
51 foundations, respectively. The leading order polynomial approximations of the antisymmet-

52 ric dispersion relation in Section IV appears to contain a few extra terms in comparison
 53 with the conventional dynamic model of a plate on a Winkler foundation. The associated
 54 near cut-off expansion is also presented. The leading order approximation of the symmetric
 55 dispersion relation appears to be valid not only for a soft foundation and corresponds to the
 56 longitudinal extensional wave in a plate subject to a transverse compression. Asymptotic
 57 considerations in the last section lead to similar conclusions as in previous section. However,
 58 the related leading order polynomial dispersion relation and near cut-off asymptotic expan-
 59 sion take a more sophisticated form due to interaction between extensional and bending
 60 waves. Veering of the associated dispersion curves is specially emphasized.

61 II. STATEMENT OF THE PROBLEM

62 Consider the plane strain problem for an elastic plate of thickness $2h$ either embedded in
 63 a Winkler elastic foundation with stiffness θ , Fig. 1(a), or supported by the latter along the
 64 lower face with traction free upper face, Fig. 1(b). We adapt the Lamé decomposition

$$\mathbf{u} = \text{grad } \phi + \text{curl } \boldsymbol{\psi}, \quad (1)$$

65 of the two-dimensional displacement vector $\mathbf{u} = (u_1, u_2)$ through the scalar potential
 66 $\phi(x, y, t)$ and the only nonzero component of the vector potential $\psi_z(x, y, t)$ both satis-
 67 fying the wave equation in two dimensions ($-\infty < x < \infty$, $-h \leq y \leq h$)

$$\Delta_2 \phi - \frac{1}{c_1^2} \phi_{,tt} = 0, \quad (2)$$

68 and

$$\Delta_2 \psi_z - \frac{1}{c_2^2} \psi_{z,tt} = 0, \quad (3)$$

with longitudinal and transverse wave speeds given by

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad \text{and} \quad c_2 = \sqrt{\frac{\mu}{\rho}},$$

69 where λ and μ are material constants, ρ is mass density, and $\Delta_2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the

70 2D Laplace operator, with comma in suffix denoting partial differentiation.

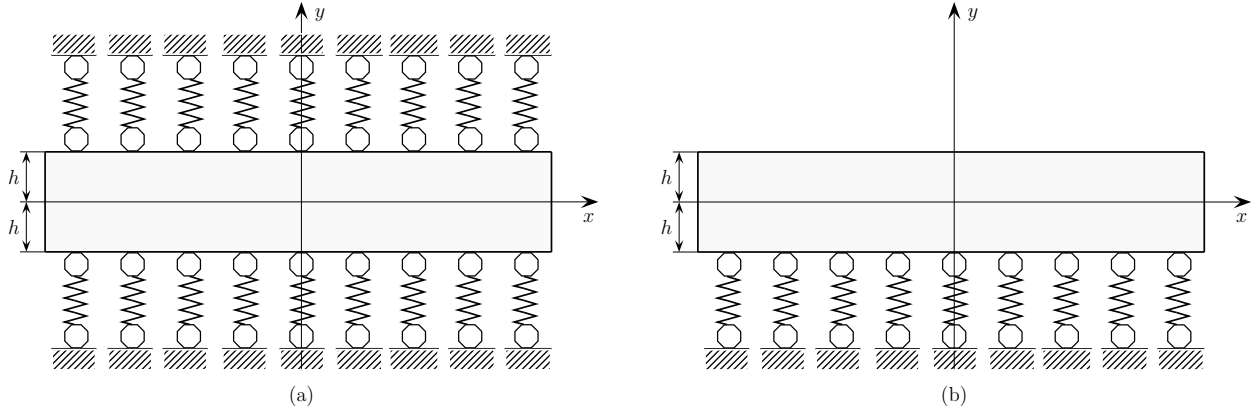


FIG. 1. Thin elastic layer embedded in a Winkler elastic foundation

71 The boundary conditions along the faces of the one-sided and two-sided foundations, see

72 Fig. 1, respectively, take the forms,

$$\sigma_{22} = \mp\theta u_2 \quad \text{and} \quad \sigma_{12} = 0, \quad \text{at } y = \pm h. \quad (4)$$

73 and

$$\sigma_{22} = 0 \quad \text{at } y = h; \quad \sigma_{22} = \theta u_2 \quad \text{at } y = -h; \quad \sigma_{12} = 0 \quad \text{at } y = \pm h. \quad (5)$$

74 The components of the displacement vector and stresses entering the boundary conditions

75 (4) and (5) are expressed through the potentials as, e.g. see (Achenbach, 2012),

$$u_1 = \phi_{,x} + \psi_{z,y}, \quad u_2 = \phi_{,y} - \psi_{z,x}. \quad (6)$$

76 and

$$\sigma_{22} = 2\mu(\phi_{,yy} - \psi_{z,xy}) + \lambda\Delta_2\phi, \quad (7)$$

$$\sigma_{12} = \mu(2\phi_{,xy} + \psi_{z,yy} - \psi_{z,xx}).$$

77 The main goal of the paper is to justify and refine the well-known low frequency engi-
78 neering model for bending vibration of a plate supported by a Winkler foundation. For the
79 configuration in Fig 1 (b), it is given by, e.g. see (Achenbach, 2012),

$$\frac{2Eh^3}{3(1-\nu^2)}\frac{\partial^4 w}{\partial x^4} + 2h\rho\frac{\partial^2 w}{\partial t^2} + \theta w = 0 \quad (8)$$

80 where $w \approx u_2(x, 0)$, $E = 2(1 + \nu)\mu$ is Young's modulus, and ν is Poisson's ratio.

81 Originally the Winkler model was adapted for static equilibrium of a beam (Winkler,
82 1870). The inertial term was formally added at the latest stage as a D'Alembert force.
83 Let us show that such approach does not seem to be always justified. On substituting the
84 travelling wave solution $w = \exp i(kx - \omega t)$ where ω is frequency and k is wave number, into
85 equation (8), we get

$$\omega^2 - \frac{\theta}{2h\rho} = \frac{Eh^2k^4}{3\rho(1-\nu^2)}. \quad (9)$$

86 This relation may be treated as a Taylor expansion of a more general dispersion relation
87 near the cut-off frequency $\omega = (\theta/2h\rho)^{1/2}$. Then, it is not very clear, why this expansion
88 does not contain a term with k^2 , but only with k^4 ! It is interesting that such issue also
89 arises in the dynamic version of the semi-membrane theory for thin cylindrical shell, see
90 (Kaplunov and Nobili, 2017b).

91 In addition, we note that apart from aforementioned bending vibrations, the formulated
92 2D plane strain problem also support extensional vibrations usually neglected in the con-
93 siderations on the subject.

94 **III. DISPERSION RELATIONS**

We shall look for the solutions in the form of travelling waves

$$\phi(x, y, t) = \Phi(y) \exp i(kx - \omega t) \quad \text{and} \quad \psi_z(x, y, t) = \Psi_z(y) \exp i(kx - \omega t).$$

Then, we have from (2) and (3)

$$\Phi(\eta) = e_1 \cos(\alpha\eta) + o_1 \sin(\alpha\eta), \quad (10)$$

$$\Psi_z(\eta) = o_2 \cos(\beta\eta) + e_2 \sin(\beta\eta), \quad (11)$$

95 where $\eta = y/h$ and

$$\alpha^2 = \chi^2 \Omega^2 - K^2, \quad \beta^2 = \Omega^2 - K^2, \quad (12)$$

96 with $\chi = c_2/c_1 = \sqrt{(1-2\nu)/2(1-\nu)}$ and

$$K = kh \quad \text{and} \quad \Omega = \frac{\omega h}{c_2} \quad (13)$$

98 Consider first a two-sided foundation, see Fig. 1a. For symmetric waves when $o_i = 0$, $i = 1, 2$
 99 in (10) and (11), we substitute the latter into boundary conditions (4) taking into account the
 100 formulas (7). As a result, we arrive at the dispersion relation (cf. (Graff, 1975, Eq.(8.1.54)))

$$\frac{4\alpha\beta K^2}{(\beta^2 - K^2)^2} + \frac{\tan \beta}{\tan \alpha} + G \frac{\alpha \Omega^2}{(\beta^2 - K^2)^2} \tan \beta = 0, \quad (14)$$

101 where $G = \theta h/\mu$ is the dimensionless stiffness of the Winkler foundation. Numerical illus-
 102 tration of the dispersion curve (14) is shown in Fig. 2. Here and henceforth, if not stated
 103 explicitly we take $G = 0.01$ and $\nu = 0.25$.

104 Similarly, for antisymmetric waves ($e_i = 0$, $i = 1, 2$ in equations (10) and (11)) we
 105 have (cf. (Graff, 1975, Eq.(8.1.59)))

$$\frac{(\beta^2 - K^2)^2}{4\alpha\beta K^2} + \frac{\tan \beta}{\tan \alpha} - G \frac{\Omega^2}{4\beta K^2} \cot \alpha = 0. \quad (15)$$

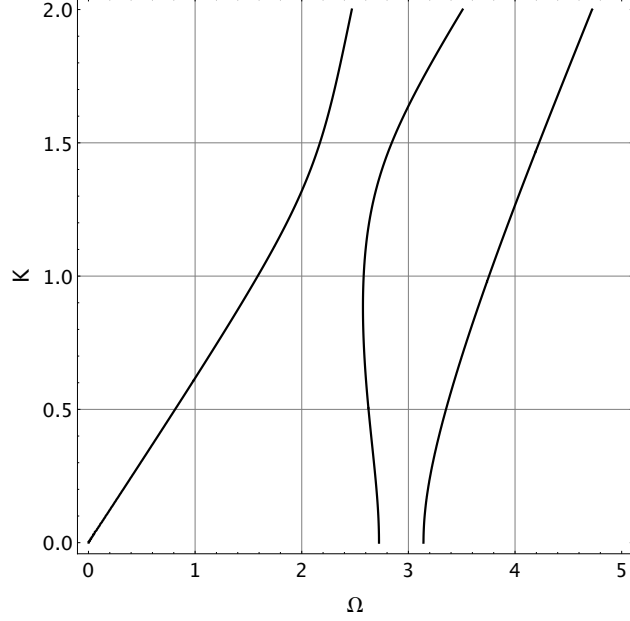


FIG. 2. Symmetric waves governed by the dispersion relation (14) for $G = 0.01$ and $\nu = 0.25$.

106 This dispersion relation is shown in Fig. 3. For a one-sided foundation, see Fig. 1(b),
 108 symmetric and antisymmetric waves are coupled with each other, resulting in four non-
 109 zero constants e_i and o_i , $i = 1, 2$ in (10) and (11). Then, we insert the latter into the
 110 ‘nonsymmetric’ boundary conditions (5) using formulae (7). The derived dispersion relation
 111 takes a more sophisticated form than above, and can be written as

$$\begin{aligned} & \frac{G\Omega^2}{16} \left(\gamma^4 \cos 2\alpha \frac{\sin 2\beta}{\beta} + K^2 \alpha^2 \cos 2\beta \frac{\sin 2\alpha}{\alpha} \right) - \\ & - \left[\left(\frac{\gamma^8}{4\alpha\beta} + \frac{K^4 \alpha\beta}{4} \right) \sin 2\beta \sin 2\alpha - \frac{\gamma^4 K^2}{2} \cos 2\beta \cos 2\alpha + \frac{\gamma^4 K^2}{2} \right] = 0 \end{aligned} \quad (16)$$

112 where $\gamma^2 = K^2 - \Omega^2/2$.

113 We remark that it is similar in a sense to the dispersion relation for an elastic plate lying
 114 on an acoustic half-space, e.g. (Kaplunov and Markushevich, 1993).

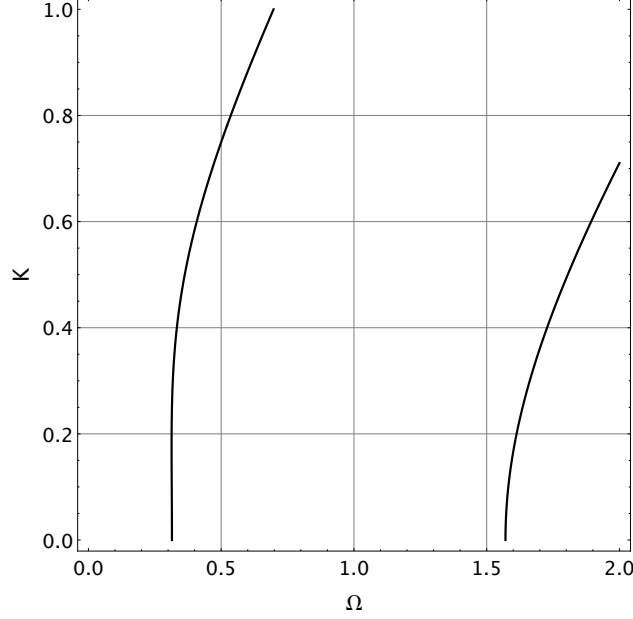


FIG. 3. Antisymmetric waves governed by the dispersion relation (15) for $G = 0.01$ and $\nu = 0.25$.

115 The cut-off frequencies ($K = 0$) of (16) affected by the Winkler foundation are given by

$$\tan 2\chi\Omega - \frac{G\chi}{\Omega} = 0 \quad (17)$$

116 For a relatively soft foundation ($G \ll 1$) the lowest cut-off frequency $\Omega = \Omega_*$ is expanded as

$$\Omega_* = \sqrt{\frac{G}{2}} \left(1 - \frac{G\chi^2}{3} + \dots \right). \quad (18)$$

117 As might be expected, the leading order term in (18) coincides with the value of the cut-off
 118 frequency (9) originated from the approximate model (8).

119 Dispersion relation (16) possesses also zero cut-off frequency $\Omega = 0$, similarly to the
 120 dispersion relation (14), related to symmetric waves in a plate supported by a two-sided
 121 foundation. In the latter case, we obtain, at $G \ll 1$, from the dispersion equation for
 122 antisymmetric waves (15) the counterparts of the formulae (17) and (18). These take the

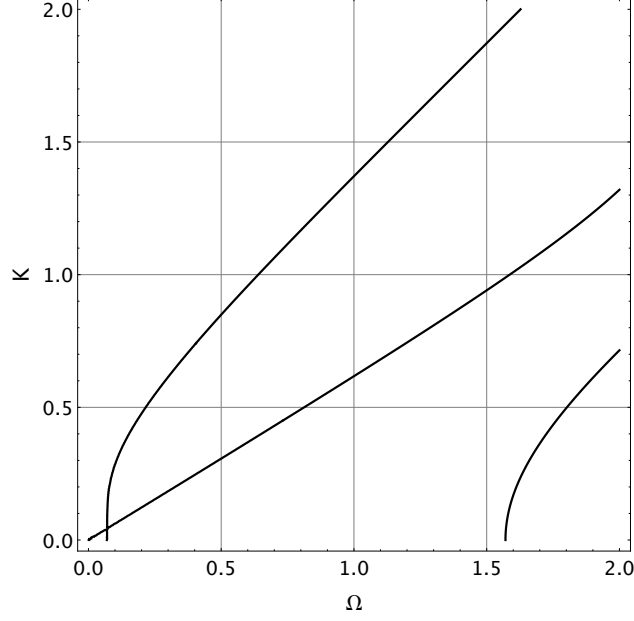


FIG. 4. Harmonic waves governed by the dispersion relation (16) for $G = 0.01$ and $\nu = 0.25$.

123 form

$$\tan(\chi\Omega) - \frac{G\chi}{\Omega} = 0. \quad (19)$$

124 and

$$\Omega_* = \sqrt{G} \left(1 - \frac{G\chi^2}{6} + \dots \right). \quad (20)$$

125 **IV. LONG-WAVE LOW FREQUENCY APPROXIMATION FOR TWO-SIDED**
 126 **FOUNDATION**

127 **A. Antisymmetric Motion**

128 At long wave, low frequency limit

$$\Omega \ll 1 \quad \text{and} \quad K \ll 1 \quad (21)$$

129 transcendental dispersion relation (15) under the assumption of a relatively weak Winkler
 130 foundation $G \ll 1$ has a polynomial expansion. At leading order it takes the form

$$K^4 - \frac{(1 + \nu)}{2} K^2 \Omega^2 - \frac{3(1 - \nu)}{2} \Omega^2 - \frac{(1 - 2\nu)}{4} \Omega^4 + \frac{3(1 - \nu)}{2} G = 0 \quad (22)$$

131 At $K = 0$, it supports two-term asymptotic formula (20) for cut-off frequency Ω_* .

132 Over the vicinity of the cut-off frequency $\delta = \Omega^2 - G \ll G$, the polynomial equation (22)

133 to within asymptotically small terms may be rewritten as

$$\delta = \frac{G^2(1 - 2\nu)}{6(1 - \nu)} + \frac{2}{3(1 - \nu)} K^4 - \frac{1 + \nu}{3(1 - \nu)} K^2 G + \dots \quad (23)$$

134 Note that at $\delta \gg G^2$, when $K \gg G^{1/2}$, this expansion reduces to

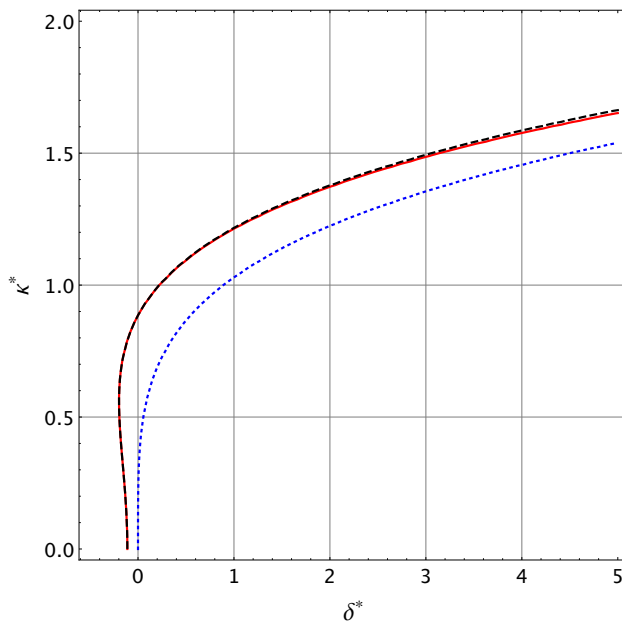


FIG. 5. Antisymmetric waves governed by (16) (black, solid), (23) (red, dashed), and (24) (blue dotted) for $G = 0.01$ and $\nu = 0.25$ in the scaled variables $\delta^* = \delta/G^2$ and $\kappa^* = K/\sqrt{G}$.

135

136

$$K^4 - \frac{3(1 - \nu)}{2} \Omega^2 + \frac{3(1 - \nu)}{2} G = 0, \quad (24)$$

137 corresponding to the conventional engineering model of a Kirchhoff plate resting on a Winkler
 138 foundation, see Section II.

139 It is clear that the inertial term in equation (24) may be neglected at $\Omega \ll G^{1/2}$ resulting
 140 in quasi-static behaviour

$$K^4 + \frac{3(1-\nu)}{2}G = 0. \quad (25)$$

141 On the other hand, at $\Omega \gg G^{1/2}$ we arrive at the dispersion relation for a free plate given
 142 by

$$K^4 - \frac{3(1-\nu)}{2}\Omega^2 = 0. \quad (26)$$

143

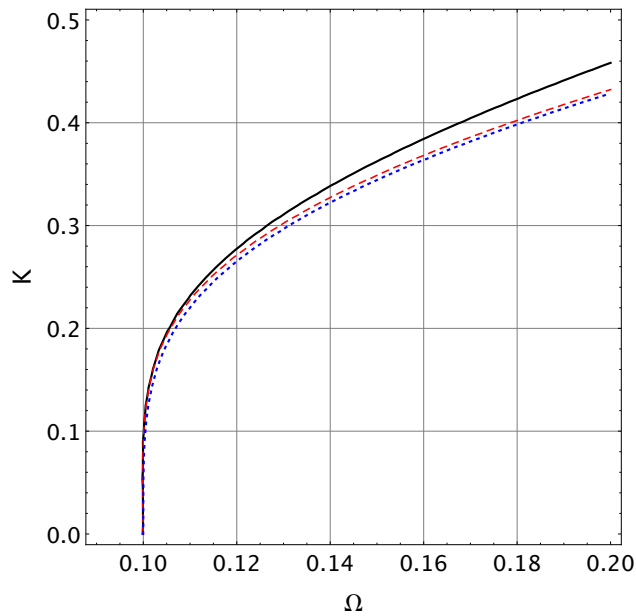


FIG. 6. Antisymmetric waves governed by (16) (black, solid), (23) (red, dashed), and (24) (blue dotted) for $G = 0.01$ and $\nu = 0.25$.

144

145

146 We also remark that near cut-off approximation (23) corresponds to the equation of
 147 motion

$$\begin{aligned} \frac{2Eh^3}{3(1-\nu^2)} \left(\frac{\partial^2}{\partial x^2} + \frac{\theta(1+\nu)^2}{Eh^3} \right) \frac{\partial^2 w}{\partial x^2} + \\ + 2h\rho \frac{\partial^2 w}{\partial t^2} + \theta \left(1 - \frac{\theta(1-2\nu)(1+\nu)}{3Eh(1-\nu)} \right) w = 0 \end{aligned} \quad (27)$$

148 containing extra terms in comparison with the traditional formulation (8).

149 B. Symmetric Motion

150 Now, we have at leading order from transcendental dispersion relation (14) over the
 151 domain (21)

$$K^2 = \frac{(2-2\nu) + G(1-2\nu)}{2(2+G(1-\nu))} \Omega^2 \quad (28)$$

152 not making yet additional assumptions on the dimensionless parameter G . At $G \ll 1$, the
 153 latter can be expanded as

$$K^2 \left(1 + \frac{\nu^2 G}{2(1-\nu)} + \dots \right) = \frac{1-\nu}{2} \Omega^2 \quad (29)$$

154 corresponding to the approximate model of plate transverse compression, ([Kaplunov et al.,](#)
 155 [1998](#)), see also Fig 1, governed by the one dimensional equation

$$\frac{2Eh}{1-\nu^2} \frac{\partial^2 u_1}{\partial x^2} - 2\rho h \frac{\partial^2 u_1}{\partial t^2} = -\frac{2h\nu}{1-\nu} \frac{\partial Q}{\partial x} \quad (30)$$

156 where $Q = \mp \theta u_2$ and $u_2 = \mp \frac{\nu h}{1-\nu} \frac{\partial u_1}{\partial x}$.

157 As it might be expected, at $G = 0$, the dispersion relation (29) coincides with that in the
 158 classical plane stress theory given by

$$K^2 = \frac{1-\nu}{2} \Omega^2 \quad (31)$$

159 At $G \gg 1$ we have from (28), at leading order,

$$K^2 = \frac{1 - 2\nu}{2 - 2\nu} \Omega^2, \quad (32)$$

160 which also may easily be deduced from the 2D plane strain limiting problem, see Section 2,

161 with the following mixed boundary conditions along the faces

$$\sigma_{12} = 0, \quad u_2 = 0 \quad \text{at} \quad y = \pm h. \quad (33)$$

162

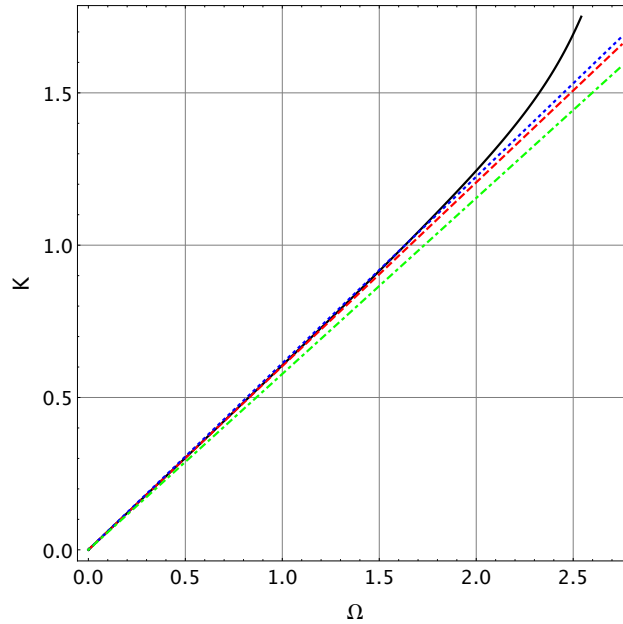


FIG. 7. Symmetric waves governed by the dispersion relations (16) (solid, black), (28) (red, dashed), (31) (blue dotted), and (32) (green, dot-dashed) for $G = 1$ and $\nu = 0.25$.

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164

165 **V. LONG-WAVE LOW FREQUENCY APPROXIMATION FOR ONE SIDED**
 166 **FOUNDATION**

167 In this case, symmetric and antisymmetric plate motions are coupled with each other
 168 resulting in a more sophisticated polynomial shortened form of the original transcendental
 169 dispersion relation (16). We have at leading order, assuming (21), and also $G \ll 1$,

$$\begin{aligned} & \frac{G\Omega^2}{32} - \frac{G}{16(1-\nu)}K^2 + \frac{1}{8} \left(\frac{1}{1-\nu} + \frac{G(7-12\nu+4\nu^2)}{6(1-\nu)^2} \right) K^2\Omega^2 - \\ & - \frac{1}{16} \left(1 + \frac{G(5-8\nu)}{6(1-\nu)} \right) \Omega^4 - \frac{G}{12(1-\nu)}K^4 + \frac{3-2\nu}{12(1-\nu)^2}\Omega^2K^4 - \\ & - \frac{(11-16\nu+4\nu^2)}{48(1-\nu)^2}\Omega^4K^2 - \frac{1}{12(1-\nu)^2}K^6 + \frac{3-4\nu}{48(1-\nu)}\Omega^6 = 0 \end{aligned} \quad (34)$$

170 Let us first transform this equation into two identical forms

$$\begin{aligned} & \frac{1}{8(1-\nu)} \left(\Omega^2 - \frac{G}{2} \right) \left(K^2 - \frac{1-\nu}{2}\Omega^2 \right) = -\frac{G(7-12\nu+4\nu^2)}{48(1-\nu)^2}K^2\Omega^2 + \\ & + \frac{G(5-8\nu)}{96(1-\nu)}\Omega^4 + \frac{G}{12(1-\nu)}K^4 - \frac{3-2\nu}{12(1-\nu)^2}\Omega^2K^4 + \\ & + \frac{4\nu^2-16\nu+11}{48(1-\nu)^2}\Omega^4K^2 - \frac{3-4\nu}{48(1-\nu)}\Omega^6 + \frac{1}{12(1-\nu)^2}K^6, \end{aligned} \quad (35)$$

171 and

$$\begin{aligned} & \frac{1}{12(1-\nu)^2}K^2 \left(K^4 - \frac{3(1-\nu)}{2}(\Omega^2 - G/2) \right) = \\ & = \frac{G}{32}\Omega^2 - \frac{1}{16} \left(1 + \frac{G(5-8\nu)}{6(1-\nu)} \right) \Omega^4 - \frac{G}{12(1-\nu)}K^4 + \\ & + \frac{3-2\nu}{12(1-\nu)^2}\Omega^2K^4 - \frac{11-16\nu+4\nu^2}{48(1-\nu)^2}\Omega^4K^2 + \frac{3-4\nu}{48(1-\nu)}\Omega^6. \end{aligned} \quad (36)$$

172 in order to get a better idea of the studied extension and bending waves, respectively. Next,
 173 we obtain the two-term asymptotic expansions of the latter

$$K^2 = \frac{1-\nu}{2}\Omega^2 \left(1 - \frac{G\nu^2}{3\delta(1-\nu)}\Omega^2 + \frac{\nu^2}{6\delta(1-\nu)}\Omega^4 + \dots \right), \quad (37)$$

174 and

$$K^4 = \frac{3(1-\nu)\delta}{2} \left(1 - \frac{1}{4\delta^{3/2}} \sqrt{\frac{2(1-\nu)}{3}} \left(4\delta G - \left(G + \frac{4(3-2\nu)}{1-\nu} \delta \right) \Omega^2 + 2\Omega^4 \right) + \dots \right) \quad (38)$$

175 where, now, $\delta = \Omega^2 - G/2$. They clarify the physical meaning of long wave low frequency ap-
 176 proximation (34), in particular, the leading order term in (38) corresponds to the dispersion
 177 relation for a beam resting on an elastic foundation, see (8).

178 The derived formulae (37) and (38) are valid outside the vicinity of the lowest bending
 179 cut-off frequency, see (18), namely at, $\delta \gg G^2$ and $\delta \gg G^{4/3}$, respectively. Instead, at
 180 $\delta \ll G$, we arrive at a near cut-off expansion of (34) given by

$$\delta \left(1 - \frac{4}{G(1-\nu)} K^2 + \frac{2\delta}{G} \right) = -\frac{G^2(1-2\nu)}{6(1-\nu)} + \frac{8}{3G(1-\nu)^2} \left(\frac{3-8\nu+4\nu^2}{16} K^2 + \frac{1}{2G} K^4 - \frac{1}{G^2} K^6 \right) \quad (39)$$

181 It is worth noting that at $\delta \gg G^2$ it coincides at leading order with (37) and (38) provided
 182 that $K \sim G^{1/2}$ and $K \gg G^{1/2}$, respectively. We also remark that near the point $\Omega^2 = G/2$
 183 and $K^2 = G(1-\nu)/4$ both expansions (37) and (39) at leading order may reduce to

$$\delta_\Omega \left(\frac{1-\nu}{2} \delta_\Omega - \delta_K \right) = \frac{\nu^2}{32}, \quad (40)$$

184 where

$$\delta_\Omega = \frac{\Omega^2 - G/2}{G^{3/2}} \quad \text{and} \quad \delta_K = \frac{K^2 - (1-\nu)G/4}{G^{3/2}}.$$

185 This explicit formula illustrates veering (Mace and Manconi, 2012) of the extensional and
 186 the bending dispersive curves first noted in Section 3.

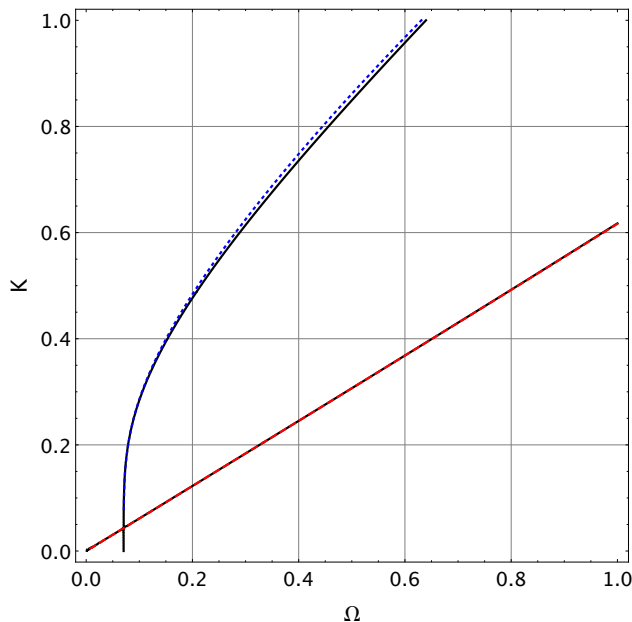


FIG. 8. Comparison of the dispersion curves corresponding to (34) (black solid), (37) (red, dashed), and (38) (blue dotted), for $G = 0.01$ and $\nu = 0.25$.

190 VI. CONCLUSIONS

191 The long wave low-frequency shortened polynomial forms of the ‘exact’ transcendental
 192 dispersion relations for an elastic layer interacting with two and one-sided Winkler founda-
 193 tions are analysed. It is shown that the traditional engineering approximation within the
 194 framework of the classical theory of plate bending is not uniformly valid. It fails near the
 195 cut-off frequencies characteristic of elastically supported structures. The near cut-off asymp-
 196 totic expansions, see (23) and (39), are derived; in doing so, the expansion for a one-sided
 197 foundation takes a pretty sophisticated form, due to interaction of bending and extensional
 198 waves. At the same time, the simple explicit formula (40) visualising veering of two types
 199 of waves is obtained.

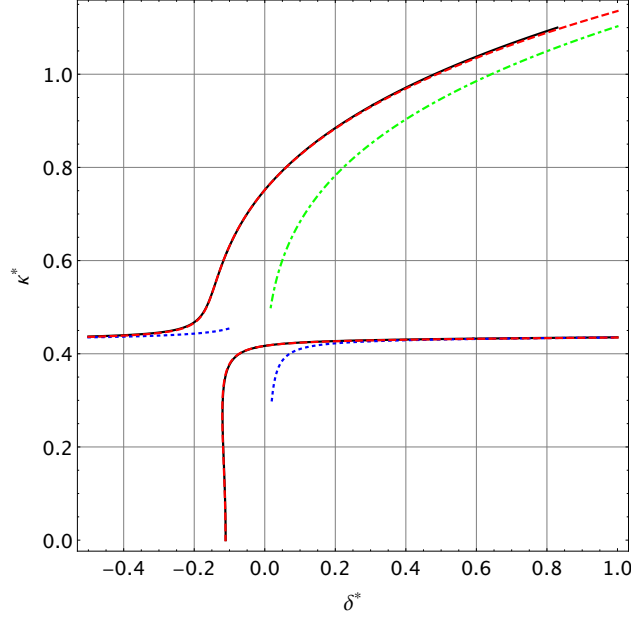


FIG. 9. Comparison of the dispersion curves corresponding to (34) (black solid), (39) (red, dashed), (37) (blue, dotted), and (38) (green, dotdashed), for $G = 0.01$ and $\nu = 0.25$ in the scaled variables $\delta^* = \delta/G^2$ and $\kappa^* = K/\sqrt{G}$.

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 202 for Visiting Scientists and Scientists on Sabbatical Leave for his visit to Anadolu University,
 203 Turkey.

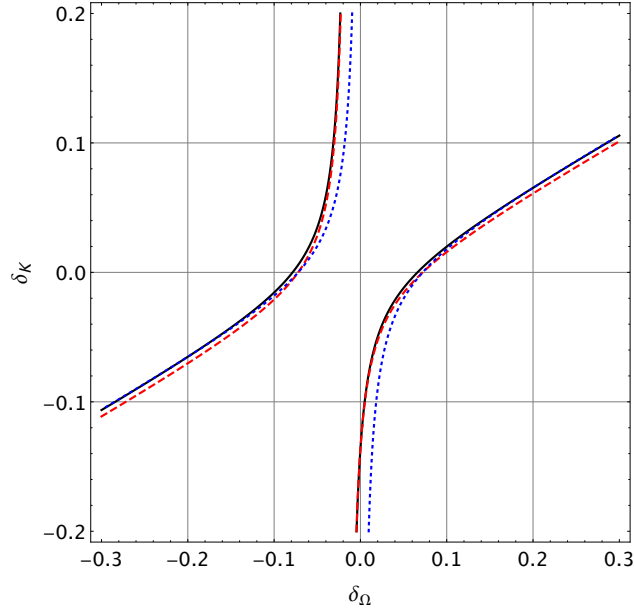


FIG. 10. Veering of the dispersion curves according to (34) (black, solid), (39) (red, dashed), and (40) (blue, dotted) for $G = 0.01$, $\nu = 0.25$ in the scaled variables δ_Ω and δ_K .

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