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DEMB Working Paper Series

N. 134

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September 2018

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ISSN: 2281-440X online



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New representations of sets of lower bounds and of sets with supremum in Archimedean order-unit vector spaces

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Abstract

Sets of lower bounds (also known as lower cuts), and sets with supremum, are characterized in the setting of Archimedean order-unit vector spaces (typically a Banach space ordered by means of a closed cone with non-empty interior). In this framework, our study proves that a set admits a supremum if and only if all the members of a newly defined family of supporting hyperplanes pass through a same point. This result is used to prove our second result. It states that a set is a lower cut if and only if it is bounded from above, downward, and contains the existing supremums of any of its subsets. As a consequence, we prove the following hidden convexity result: any set fulfilling the three above-mentioned conditions is necessarily a convex set.

Keywords: Partially ordered vector spaces, Archimedean spaces, order unit, set of lower bounds, supremum

2000 MSC: 46A40, 06F20, 52B11

1. Introduction

Recent developments in the field of multicriteria optimization renewed the interest in studying order-related properties for subsets of a vector space ordered by a convex cone. In this regard, the present article tackles two important types of

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subsets of a partially ordered vector space: the sets of lower bounds, also known as lower cuts, on one hand, and the sets which admit a supremum, on the other.

To be more specific, let us consider (X, \leq_X) , a partially ordered set; as customary, for any subset S of X , we denote by $L(S)$ the set of lower bounds of S , $L(S) := \{x \in X : x \leq_X y \ \forall y \in S\}$, and by $U(S)$ the set of upper bounds of S , $U(S) := \{x \in X : y \leq_X x \ \forall y \in S\}$. When $U(S)$ coincides with $U(\{x\})$ for some singleton $\{x\}$, we say that x is the supremum of the set S ; when S itself coincides with $L(A)$ for some subset A of X , we call S a *lower cut*¹ of X (see for instance [1, Definition 1]).

While the importance of the class of sets with supremum is broadly acknowledged, we need to make clear the relevance of the more technical notion of lower cut. To this respect, we address MacNeille's concept (see [16, Definition 20]) of a *Dedekind cut* of X , as being a pair (D_l, D_u) of subsets of X such that the relations

$$D_l = \{x \in X : x \leq_X y \ \forall y \in D_u\}, \quad D_u = \{x \in X : y \leq_X x \ \forall y \in D_l\}$$

hold true simultaneously. Obviously, (\emptyset, X) and (X, \emptyset) are two (trivial) cuts. By endowing the set of all the Dedekind cuts with the natural ordering $(D_l, D_u) \leq (E_l, E_u)$ iff $D_l \subset E_l$, MacNeille's completion theorem ([16, Theorem 25]) proves that the family of all the Dedekind cuts is a complete lattice, namely the smallest such lattice with the poset (X, \leq_X) embedded in it. It is now easy to see that the class of lower cuts of X and that of Dedekind cuts of X are in a one-to-one correspondence. Namely, given (D_l, D_u) a Dedekind cut, then D_l is the set of lower bounds of D_u , so it is a lower cut. On the other hand, given S a lower cut, the pair $(S, U(S))$ is a Dedekind cut.

Due to their interest, both sets with supremum and lower cuts have been extensively studied. In particular, several complete descriptions of lower cuts are available in the mathematical literature, for various classes of lattices (complete lattices, Heyting or Boolean algebras), as well as for partially ordered vectors spaces (for exact definitions of the notions used in this introduction, the reader is referred to subsection 2.1). However, the terminology under which the same mathematical object is identified varies from an author to another; in order to give a brief unitary description of the existing results, we have gathered the most significant theorems on this topic in the subsections 2.2 and 2.3.

¹ In [4, footnote 12, page 59], Birkhoff advocates in favor of naming this kind of sets *closed order ideals*, but this notation seems rarely - if ever - used.

Our first objective is to characterize sets possessing a supremum in partially ordered vector spaces. Let X be a vector space ordered by means of a pointed convex cone K . We define K^+ , the cone of all the real-valued linear functions defined on X which are non-negative over K . Given S , a subset of X , we call an *extreme hyperplane* of S any supporting hyperplane of S generated by one of the extreme directions of K^+ .

The case when X is finite dimensional and K is polyhedral is well understood. More precisely, it is known (see for instance [10, Proposition 4]) that a subset S of X possesses a supremum if and only if all of its extreme hyperplanes pass through a same point. Moreover, when the set S does have a supremum, then the common point of all its extreme hyperplanes coincides with the supremum of S .

However, this result fails to be true when the cone K is no longer polyhedral; the construction at [10, Section 4.2] illustrates the case of a three-dimensional vector space X ordered by means of a circular cone K (the so-called "ice-cream cone"), and provides an example of a subset of X which possesses a supremum, but for which the intersection of all of its extreme hyperplanes is empty.

In other words, given S a subset of X possessing a supremum, there may exist extreme hyperplanes of S which miss the supremum of S . Maitland Wright (see [17, Corollary 2.3]) noticed that the set of extreme hyperplanes missing the supremum of S is small - more precisely that it is a meagre subset of the set of all the extreme hyperplanes endowed with the duality topology. However, in order to state a pertinent characterization of sets with supremum, we need to accurately determine which of the extreme hyperplanes passes through the supremum of the point S , in the case when this point exists. Let us define a *c-extreme hyperplane* as being a supporting hyperplane of S generated by an extreme direction of K^+ at which the support function of S is continuous with respect to the topology of the duality between K^+ and X .

The first of our main results, Theorem 2, Section 4, proves that a subset S of an Archimedean order-unit vector space possesses a supremum if and only if all its c-extreme hyperplanes pass through the same point. In addition, if the set S has a supremum, then this supremum amounts to the intersection of all the c-extreme hyperplanes.

In Rockafellar's terminology ([19, page 162]), Theorem 2 may be seen as an *external representation* of subsets possessing a supremum in an Archimedean order-unit vector space.

The second aim of our article is to give an intrinsic description of lower cuts, i.e. a characterization in terms of algebraic properties fulfilled by their elements.

Appealing once again to Rockafellar’s terminology, such a result provides an *internal representation* of lower cuts.

The starting point of our analysis is the obvious fact that any lower cut S from a given poset X has three simple properties: S is bounded from above, S contains all the elements from X smaller than one of its own elements, and S contains all the existing suprema of its subsets. In the notation of [12, page 51], a set fulfilling the last two of the above-mentioned properties is called a *complete ideal*; we may thus rephrase our remark, and say that, in any poset (X, \leq_X) , any lower cut is a bounded from above complete ideal.

Several examples from the mathematical literature prove that a bounded from above complete ideal is not always a lower cut, not even when the poset (X, \leq_X) is a distributive lattice (see [13, Remark, page 939]), or a finite-dimensional vector space ordered by means of a convex cone (see [10, Section 4.3]). If however the poset (X, \leq) belongs to a special class of distributive lattices, namely the Heyting algebras, or if it is a finite-dimensional space ordered by a polyhedral cone, then it is known (see [13, Lemma 2.1] for the former setting, and [10, Theorem 1] for the latter) that any bounded from above complete ideal is automatically a lower cut.

Our second result, Theorem 4, proves that a similar characterization also holds for a fairly more general class of partially ordered vector spaces. Namely a subset of an *Archimedean order-unit vector space* is a lower cut if and only if it is a bounded from above complete ideal.

Since a lower cut of X is nothing but the intersection of an arbitrary family of translates of $-K$, the opposite of the ordering cone, we deduce, as an obvious corollary of Theorem 4, the following ”hidden convexity” result: in an Archimedean order-unit setting, any bounded from above complete ideal is convex.

Several open questions are gathered in Section 6, the concluding part of our article.

2. Notation, definitions, and known results

2.1. Definitions and notations

In the sequel, a *poset* (X, \leq_X) is a nonempty set X endowed with a reflexive, antisymmetric and transitive relation denoted \leq_X . Let S be a subset of X ; we call *infimum* of S the greatest element of $L(S)$, if such element exists. Similarly, the

supremum of S is the least element in $U(S)$, provided that $U(S)$ admits a least element.

By applying L and U in sequence we obtain two different closure operators on X :

$$LU(S) := L(U(S)) \quad \text{and} \quad UL(S) := U(L(S))$$

which are called the lower and upper *MacNeille closure* of S and are the smallest lower cut, and respectively the smallest upper cut, containing S .

Given (X, \leq_X) a poset, a central notion for our study is that of an *ideal* of X . As defined by Doyle in [9, Definition 1.3], an ideal² of the poset (X, \leq_X) , is a subset F of X which is downward (that is F contains any element of X which is smaller than some element of F), and closed under existing finite joins (meaning that F contains the supremum of any of its finite subsets, provided that the said supremum exists). An obvious example of ideals are the *principal ideals*, that is subsets F_x of X of the form $F_x := \{y \in X; y \leq_X x\}$, for some fixed $x \in X$.

When the ideal F is closed not only under existing finite joins, but under existing arbitrary joins, in other words, when F contains the supremum of any of its subsets possessing a supremum, then F is called a *complete ideal* of X . Let us remark that a principal ideal is always complete.

The poset (X, \leq_X) is called a *lattice* if every pair of elements $x, y \in X$ (and hence every finite subset) possesses an infimum and a supremum, indicated $x \wedge y$ and $x \vee y$ respectively. Two classes of lattices are of particular interest for this study. First, we define a *complete lattice*, as being a lattice such that any of its subsets (including the lattice itself) possesses an infimum and a supremum. A lattice is called *Dedekind complete* if every order bounded subset possesses an infimum and a supremum.

This article also addresses the class of the *distributive lattices*, that is those for which the relation $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ holds for arbitrary $x, y, z \in X$. Two remarkable sub-classes of distributive lattices are mentioned in the sequel. Let us first define, for every two elements a and b from X , the set $S_{a,b} = \{x \in X : a \wedge x \leq b\}$. A distributive lattice possessing greatest and least elements, and with the additional property that any of the sets $S_{a,b}$ admits a greatest element, is called a *Heyting algebra*.

² In order to avoid confusions, let us mention that several more restrictive definitions for an order ideal exist in the mathematical literature - for instance Frink's ideals (see [11, Definition, page 227]), or Niederle's S-ideals (see [18, Definition, page 288]); throughout this article, we will systematically consider ideals in Doyle's sense.

Furthermore, a Heyting algebra in which, for any element $x \in X$, there is $y \in X$ such that the infimum between x and y coincides with the least element of X , while the supremum between x and y is the greatest element of X (the element y is called the *complement of x*), is called a *Boolean algebra*. Heyting algebras were introduced in order to formalize intuitionistic logic, in the same way as Boolean algebras describe classical logic.

Of special concern for our article is a different family of posets, obtained by ordering a vector space X by means of a convex pointed cone $K \subset X$:

$$[x \leq_K y] \Leftrightarrow [y - x \in K] \quad \forall x, y \in X.$$

If the following two properties hold :

- given two vectors $x, y \in X$, the relation $n x \leq_K y$ for any $n \in \mathbb{N}$ implies that $x \leq_K 0$,
- there is a vector $e \in X$ such that, for any $x \in X$, $-n e \leq_K x \leq_K n e$ for large enough $n \in \mathbb{N}$,

we call the partially ordered vector space (X, \leq_K, e) an *Archimedean order-unit vector space*.

An obvious example of an Archimedean order-unit space is provided by a Banach space X ordered by a convex pointed cone K which is also closed and possesses a non-empty interior; the role of the order-unit e is played by any of the vectors from the interior of K .

The case of Archimedean order-unit vector spaces was first considered through a very important example : the poset $(C_b(S), \leq, 1)$ of all the bounded and continuous real functions defined on some topological space, endowed with the pointwise ordering, the constant function 1 being its order unit. In this setting, Dilworth proved (see [7, Theorem 4.1] - equivalent formulations may be found in [20] and [6]), that (A, B) is a Dedekind cut in $(C_b(S), \leq, 1)$ if and only if

$$A = \{f \in C_b(S) : f(x) \leq g(x) \quad \forall x \in S\}$$

and

$$B = \{f \in C_b(S) : f(x) \geq g(x) \quad \forall x \in S\},$$

where $g : S \rightarrow \mathbb{R}$ is a *normal* lower semicontinuous function, that is a lower semicontinuous function which can be obtained as the lower semicontinuous envelope of some upper semi-continuous bounded function.

In order to generalize Dilworth's description to an abstract Archimedean order-unit vector space (X, \leq_K, e) , we need the following concepts. Following Kadison (see [14]), let us define

$$\|\cdot\|_e : X \rightarrow \mathbb{R}_+, \quad \|x\|_e := \{\inf a > 0 : -ae \leq x \leq ae\};$$

obviously, this function is a norm on X . Let us denote by X^* the topological dual of the normed space $(X, \|\cdot\|_e)$, and by $\langle \cdot, \cdot \rangle : X^* \times X \rightarrow \mathbb{R}$, the duality application between X^* and X .

The positive dual cone K^+ is the set of all the linear functions on X which are non-negative on K . Clearly, every element in K^+ is continuous with respect to the Kadison norm. Thus, it holds that

$$K^+ = \{x^* \in X^* : \langle x^*, x \rangle \geq 0 \ \forall x \in K\},$$

and the topology of the duality between K^+ and K is nothing but the trace on K^+ of the standard w^* -topology on X^* .

It is furthermore easy to see that K^+ possesses a w^* -compact basis (called the *state space* of X)

$$B := \{x^* \in K^+ : \langle x^*, e \rangle = 1\},$$

whose set of extreme points (visibly non-empty, as a consequence of Krein-Milman theorem) will be denoted by $\text{ext}(B)$.

Throughout this article, all the topological notions - closures, continuities etc - on X and on X^* , will always be considered with respect to the norm and, respectively, the w^* -topology.

2.2. Internal representations

One of the main aims of this article is to prove that the lower cuts of an Archimedean order-unit vector space (X, \leq_K, e) may be characterized as being the subsets S of X fulfilling a list of three properties :

- S is bounded from below,
- it is *downward*, i.e. if $x \in S$, and $y \leq_K x$, then $y \in S$,
- it is *sup-containing*, i.e. if $T \subset S$ and the supremum of T exists, then it belongs to S .

Following Rockafellar's terminology, we may rephrase our main result, by saying that this list of properties is an *internal* representations of lower cuts in an Archimedean order-unit space.

The purpose of this subsection is to collect the examples known to us of internal representations of lower cuts scattered through the mathematical literature.

2.2.1. An internal representation of lower cuts in complete lattices

Let us first consider the case when the lattice (X, \leq_X) is complete. In this setting, it is easy to prove (see [4, Ex 2, page 59]) that the lower cuts, the complete ideals and the principal ideals are the same thing. Consequently, the following result is established.

Theorem A. *Let (X, \leq_X) be a complete lattice, and S be a subset of X . Then S is a lower cut of X if and only if S is downward and has a greatest element.*

Accordingly, the two above properties constitute an internal representation of lower cuts in the case of a complete lattice.

2.2.2. An internal representation of lower cuts in Heyting algebras

In the setting of Heyting algebras, the classes of lower cuts and complete ideals are larger than the class of principal ideals. However a well-known result [13, Lemma 2.1] proves that the lower cuts and the complete ideals coincide, and implies the following statement.

Theorem B. *Let (X, \leq_X) be a Heyting algebra, and S be a subset of X . Then S is a lower cut of X if and only if S is a complete ideal, that is if and only if S is downward and sup-containing.*

It is worth noticing that Theorem B does not hold for the larger class of distributive lattices, as shown by the example in [13, Remark, page 939].

2.2.3. An internal representations of lower cuts for finite dimensional vector spaces ordered by a polyhedral cone

A different class of posets for which an internal representation of lower cuts is available is the family of finite dimensional vector spaces for which the order is given by a polyhedral cone. In the case when K is the intersection of a finite number of closed half-spaces, [10, Theorem 1] establishes the following internal representation for lower cuts.

Theorem C. *Let (X, \leq_K) be a finite dimensional vector space ordered by means of a polyhedral cone K , and S be a subset of X . Then S is a lower cut of X if and only if it is a bounded from above complete ideal, that is if and only if it is bounded from above, downward and sup-containing.*

A standard corollary of Theorem C proves a hidden convexity result. Indeed S is a lower cut of X if there exists $A \subset X$ such that

$$S = L(A) = \{x \in X : x \leq_K a, \forall a \in A\} = \bigcap_{a \in A} a - K, \quad (1)$$

which is a convex set. Hence any bounded from above complete ideal of a finite dimensional vector space with a polyhedral ordering cone, being a lower cut, is automatically convex. As proved by the analysis in [10, Section 4.3], this results, as well as Theorem C, are no longer true if the polyhedrality assumption is dropped.

2.3. External representations

Given \mathcal{S} , a class of subsets of some set X , we say that a pair $(\mathcal{F}, \mathcal{I})$ provides an *external representation* of \mathcal{S} , when :

- \mathcal{F} is a family of subsets of X , $\mathcal{F} \subset 2^X$,
- \mathcal{I} is a family of subsets of \mathcal{F} , $\mathcal{I} \subset 2^{\mathcal{F}}$,
- the transfer function $T : \mathcal{I} \rightarrow 2^X$, defined as $T(J) := \bigcap_{F \in J} F$, for any $J \in \mathcal{I}$, satisfies $T(\mathcal{I}) = \mathcal{S}$.

When the transfer function is one-to-one, the representation is said to be *exact*.

A well-known example of an external representation is given by the construction of closed convex sets in a locally convex space X as intersections of closed half-spaces. In this case, \mathcal{F} is the family of all the closed half-spaces of X , while $\mathcal{I} = 2^{\mathcal{F}}$, the family of all the subsets of \mathcal{F} . Indeed, any intersection of closed half-spaces is a closed convex set, and any closed convex set may be expressed as the intersection of at least one family of closed half-spaces, so the previously defined pair $(\mathcal{F}, 2^{\mathcal{F}})$ constitutes an external representation of the class of closed convex sets. However, since the same closed convex set may be obtained as the intersection of several families of closed half-spaces, this representation is not exact.

Exact external representations for the family of lower cuts are available for two very important classes of posets: Boolean algebras, and Archimedean order-unit vector spaces.

2.3.1. External representation of lower cuts in Boolean algebras

In the case when (X, \leq_X) is a Boolean algebra, we define the family \mathcal{F} as containing all the maximal proper ideals of (X, \leq_X) , that is all the non-empty ideals F such that X is the only ideal strictly larger than F . The Stone topology on the set \mathcal{F} is defined by endowing \mathcal{F} with the neighborhood basis $(G_x)_{x \in X}$, where G_x is the subset of \mathcal{F} composed by all the maximal ideals containing the element x .

An important feature of the Stone topology is that it contains a large number of regular sets, that is open sets coinciding with the interior of their closure.

For instance, all the neighborhoods of the form G_x are regular. The family \mathcal{I} is composed by all regular subsets of \mathcal{F} in the Stone topology.

Combining two classical results, that is Stone's representation theorem ([12, Theorem 6, page 78]) on one hand, and the folk theorem [12, Theorem 11, page 93] (Dilworth attributes this result to Birkhoff) on the other, we deduce the following theorem.

Theorem D. *Let (X, \leq_X) be a Boolean algebra, and \mathcal{S} be the class of lower cuts of (X, \leq_X) . The pair $(\mathcal{F}, \mathcal{I})$, where \mathcal{F} is the family of all the maximal proper ideals of (X, \leq) , and \mathcal{I} is the family of all the regular subsets of \mathcal{F} in the Stone topology, is an external representation of \mathcal{S} .*

Moreover, the transfer function $T : \mathcal{I} \rightarrow \mathcal{S}$ is a lattice isomorphism between the lattice of regular sets in the Stone topology of \mathcal{F} , and the lattice of lower cuts of (X, \leq_X) .

Let us remark that intersecting all the maximal ideals from a set of the form G_x gives us the principal ideal F_x of all the elements in X smaller than x . More generally, Theorem D says that intersecting all the maximal ideals which are elements of a regular set in the Stone topology, always amounts to a lower cut, not necessarily to a principal ideal. Reciprocally, every lower cut can be obtained in this fashion.

2.3.2. External representation of lower cuts in Archimedean order-unit spaces

Relation (1) shows that the lower cuts in an ordered vector space can be represented as intersections of translates of $-K$: any such intersection forms a lower cut, and every lower cut can be written as in (1) taking for instance $A = U(S)$.

When the poset (X, \leq_X) is an Archimedean order-unit vector space an external representation in terms of halfspaces is also possible.

The result presented below is implicit in the embedding theorem by Maitland Wright [17, Theorem 2.9]. We refer to the reformulation [2, Theorem 2.9] given by Becker as the starting point of a further extension to spaces without order unit.

To any element x in X there corresponds the function $\phi_x : \text{ext}(B) \rightarrow \mathbb{R}$, defined by the relation

$$\phi_x(x^*) = \langle x^*, x \rangle \quad \forall x^* \in \text{ext}(B) \rightarrow \mathbb{R}.$$

Obviously, ϕ_x is a bounded w^* -continuous function, and the operator mapping x into ϕ_x is a linear one-to-one Archimedean order-unit vector spaces morphism from (X, K, e) to $(C_b(\text{ext}(B)), \leq, 1)$.

Maitland Wright's theorem states that (A, B) is a Dedekind cut in (X, \leq_K, e) if and only if there is some normal w^* -lower semicontinuous function $g : \text{ext}(B) \rightarrow \mathbb{R}$ such that

$$A = \{x \in X : \langle x^*, x \rangle \leq g(x^*) \quad \forall x^* \in \text{ext}(B)\}$$

and

$$B = \{x \in X : \langle x^*, x \rangle \geq g(x^*) \quad \forall x^* \in \text{ext}(B)\}.$$

Moreover, the above-depicted correspondence between a Dedekind cut (A, B) and the normal w^* -upper semicontinuous function g is a one-to-one and onto Dedekinding complete lattices morphism between the MacNeille completions of (X, \leq_K, e) and $(C_b(\text{ext}(B)), \leq, 1)$.

To obtain an external representation of lower cuts, consider the family \mathcal{F} composed by all the closed half-spaces of the normed space $(X, \|\cdot\|_e)$ of the form $F_{f^*, a}$,

$$F_{f^*, a} := \{x \in X : \langle f^*, x \rangle \leq a\},$$

with $f^* \in \text{ext}(B)$ and $a \in \mathbb{R}$.

The family \mathcal{I} gathers all the subsets of \mathcal{F} of the form J_ϕ , for all the normal real-valued lower semi-continuous functions ϕ defined on $\text{ext}(B)$.

Maitland Wright's theorem may then be stated as follows.

Theorem E. *Let (X, \leq_K, e) be an Archimedean order-unit vector space, and \mathcal{S} be the class of lower cuts of (X, \leq_X) . The pair $(\mathcal{F}, \mathcal{I})$, where \mathcal{F} is the family of all the closed half-spaces of $(X, \|\cdot\|_e)$ of the form $F_{f^*, a}$ with $f^* \in \text{ext}(B)$ and $a \in \mathbb{R}$, and \mathcal{I} is the family of all the subsets of \mathcal{F} of the form J_ϕ , for all the normal real-valued lower semi-continuous functions ϕ defined on $\text{ext}(B)$, is an exact external representation of \mathcal{S} .*

Moreover, the transfer function $T : \mathcal{I} \rightarrow \mathcal{S}$ is a lattice isomorphism between the lattice of normal lower semi-continuous functions on $\text{ext}(B)$ and the lattice of lower cuts of (X, \leq_X) .

Theorem E describes thus a systematic manner to generate all the lower cuts from a given Archimedean order-unit vector space. To this end, it suffices to take ϕ , a normal real-valued lower semi-continuous function defined on $\text{ext}(B)$, and to intersect all the closed half-spaces of the form $(H_{f^*, \phi(f^*)})_{f^* \in \text{ext}(B)}$. One is then assured, on one hand, to obtain a lower cut, and on the other, that any lower cut of X can be obtained via this construction.

3. An intermediate result: an external representation of the lower closure

Let S be a nonempty and bounded above subset of the Archimedean order-unit vector space X . A key step in achieving both our main results, is to establish an external representation for the set $LU(S)$.

Since S is a non-empty set, its support function,

$$\sigma_S : X^* \rightarrow \mathbb{R} \cup \{+\infty\}, \quad \sigma_S(x^*) = \sup\{\langle x^*, x \rangle : x \in S\}, \quad \forall x^* \in X^*,$$

is an extended-real-valued convex lower semi-continuous function. Hence it is lower semi-continuous on $\text{ext}(B)$, but not necessarily normal l.s.c. An important role in representing $LU(S)$ as the intersection of a family of closed half-spaces of X will be played by the set B_S , of all the points of $\text{ext}(B)$ at which the function σ_S is continuous.

Lemma 1. *Let S be a non-empty bounded from above subset of the Archimedean order-unit vector space (X, \leq_K, e) . Then the set B_S is dense in $\text{ext}(B)$.*

Proof of Lemma 1: A topological space T is called a *Baire space* provided that the intersection of any sequence of dense open subsets of T is dense in T . A celebrated result of Choquet ([8, Page 355]) states that the set of all the extreme points of a compact convex subset of a locally convex space is a Baire space in the induced topology. We may thus conclude that $\text{ext}(B)$ is a Baire space in the w^* -topology.

Consider now $x \in S$ and $y \in M(S)$ (the existence of these two vectors is ensured by the fact that S is both non-empty and bounded from above); the definition of the Kadison norm $\|\cdot\|_e$ easily implies that

$$-\|x\|_e \leq \sigma_S(x^*) \leq \|y\|_e \quad \forall x^* \in B.$$

Accordingly, the restriction of σ_S on $\text{ext}(B)$ is a bounded function. A classical result, (see [5, Partie 5, Ex 20]), states that the set of points of continuity of a real-valued lower semi-continuous function over a Baire space is a dense G_δ set. By applying this result to the bounded (thus real-valued) function σ_S , we deduce that B_S is a dense subset of $\text{ext}(B)$. \square

As already stated, normal lower semi-continuous functions play a key role in characterizing lower sets in Archimedean order-unit vector spaces. The following lemma states a basic property of this family of lower semi-continuous functions.

Lemma 2. *Let U be a dense subset of the topological space T , and let $w : U \rightarrow \mathbb{R}$ be a continuous bounded function. Then, among the family of lower semi-continuous functions defined on T which coincide on U with w , there is exactly one normal function.*

Proof of Lemma 2: It is easy to see that the function

$$w_l : T \rightarrow \mathbb{R}, \quad w_l(x) := \liminf_{y \rightarrow x, y \in U} w(y) \quad \forall x \in T$$

is the largest among all the lower semi-continuous functions defined on T which agree with w on U . Similarly,

$$w_u : T \rightarrow \mathbb{R}, \quad w_u(x) := \limsup_{y \rightarrow x, y \in U} w(y) \quad \forall x \in T$$

is the smallest among all the upper semi-continuous functions defined on T and agreeing on U with w .

Let us first prove that w_l is normal. To this respect, consider ϕ , the lower semi-continuous envelope of w_u . As w_u is continuous (at least) at any point of U , it follows that ϕ agrees with w_u , and thus with w , on U . Since w_l is the largest lower semi-continuous functions agreeing with w , it results that $\phi \leq w_l$. As obviously $w_l \leq w_u$, we deduce that the lower semi-continuous functions w_l is sandwiched between w_u and its lower semi-continuous envelope. Accordingly, w_l is the lower semi-continuous envelope of the upper semi-continuous function w_u , and thus w_l is normal.

We have accordingly proved that the class of all normal lower semi-continuous functions defined on T and coinciding on U with w , is non empty. Let us now prove that this class contains exactly one element. To this respect, let us consider ψ , a normal lower semi-continuous functions defined on T which coincide on U with w , and ζ , an upper semi-continuous function whose lower envelope is ψ .

The definition of w_l implies that

$$\psi(x) \leq w_l(x) \quad \forall x \in T. \quad (2)$$

On the other hand, ζ dominates ψ , so for any $x \in U$ it holds that $\zeta(x) \geq \psi(x) = w(x)$. But ζ is upper semi-continuous, so

$$\zeta(x) \geq \limsup_{y \rightarrow x, y \in U} \zeta(y) \geq \limsup_{y \rightarrow x, y \in U} w(y) = w_u(x) \quad \forall x \in T. \quad (3)$$

From relations (2) and (3) we deduce that both the lower semi-continuous function w_l , and the upper semi-continuous function w_u are sandwiched between ψ and ζ ; as ψ is ζ 's lower semi-continuous envelope, it follow that $\psi = w_l$.

The class of all the normal lower semi-continuous functions defined on T and coinciding on U with w reduces accordingly to w_l , and Lemma 2 is completely proved. \square

We are now in a position to state and prove the announced representation of the set $LU(S)$ as the intersection of a family of closed half-spaces supporting S . Given the halfspace $H_{f^*,a} := \{x \in X : \langle f^*, x \rangle \leq a\}$, we say that H is a supporting halfspace for the set $S \subset X$ if $a = \sigma_S(f^*) = \sup_{s \in S} \langle f^*, s \rangle$ (without the need for such supremum to be attained in S). The term supporting hyperplane is used in the sequel with analogous meaning.

Theorem 1. *Let S be a non-empty bounded from above subset of the Archimedean order-unit vector space (X, \leq_K, e) . Then*

$$LU(S) = \bigcap_{f^* \in B_S} H_{f^*, \sigma_S(f^*)}, \quad (4)$$

where B_S is the subset of $\text{ext}(B)$ at which σ_S is continuous.

Proof of Theorem 1: We know from Lemma 1 that B_S is dense in $\text{ext}(B)$, and since σ_S is continuous bounded on B_S , it is possible to define the function

$$\phi : \text{ext}(B) \rightarrow \mathbb{R}, \quad \phi(f^*) = \liminf_{g^* \rightarrow f^*, g^* \in B_S} \sigma_S(g^*).$$

From Lemma 2, we know that ϕ is the unique normal lower semi-continuous function defined on $\text{ext}(B)$ which agrees with σ_S on B_S .

Let us first prove that

$$LU(S) = \bigcap_{f^* \in \text{ext}(B)} H_{f^*, \phi(f^*)}. \quad (5)$$

As a first consequence of Theorem E, it follows that the set $\bigcap_{f^* \in \text{ext}(B)} H_{f^*, \phi(f^*)}$ is one of the lower cuts of X . From the definition of ϕ , it follows that this function dominates σ_S at any point of $\text{ext}(B)$; so

$$S \subset \bigcap_{x^* \in X^*} H_{x^*, \sigma_S(x^*)} \subset \bigcap_{f^* \in \text{ext}(B)} H_{f^*, \sigma_S(f^*)} \subset \bigcap_{f^* \in \text{ext}(B)} H_{f^*, \phi(f^*)}.$$

The set $\bigcap_{f^* \in \text{ext}(B)} H_{f^*, \phi(f^*)}$ is thus one of the lower cuts containing the set S . But $LU(S)$ is the smallest such lower cut; it results that

$$LU(S) \subset \bigcap_{f^* \in \text{ext}(B)} H_{f^*, \phi(f^*)}. \quad (6)$$

In order to prove that the inclusion in (6) is in fact an equality, we can exploit once again Theorem E, in order to deduce that

$$LU(S) = \bigcap_{f^* \in \text{ext}(B)} H_{f^*, \zeta(f^*)} \quad (7)$$

for some normal lower semi-continuous function ζ defined on $\text{ext}(B)$. Again from Theorem E, we know that the transfer function is a lattice isomorphism, so that it is monotone. From relation (6) we have that ζ is dominated by ϕ , while relation (7) yields that ζ dominates σ_S .

Accordingly, ζ is sandwiched between σ_S and ϕ ; as σ_S and ϕ agree on B_S , we deduce that ζ also agrees on B_S with σ_S . We have thus proved that ζ is a normal lower semi-continuous function amounting to σ_S on B_S ; but Lemma 2 says that ϕ is the unique such function, so $\zeta = \phi$, and our claim (5) follows from (7).

All what remains to be proved in order to demonstrate relation (4), is that

$$\bigcap_{f^* \in \text{ext}(B)} H_{f^*, \phi(f^*)} = \bigcap_{f^* \in B_S} H_{f^*, \phi(f^*)}. \quad (8)$$

To prove relation (8), let us pick a function $f^* \in \text{ext}(B)$ and a vector $x \in \bigcap_{g^* \in B_S} H_{g^*, \phi(g^*)}$; obviously,

$$\langle g^*, x \rangle \leq \phi(g^*) \quad \forall g^* \in B_S. \quad (9)$$

Moreover, B_S is dense in $\text{ext}(B)$, and $f^* \in \text{ext}(B)$; relation (9) implies that

$$\langle f^*, x \rangle = \lim_{g^* \rightarrow f^*, g^* \in B_S} \langle g^*, x \rangle = \liminf_{g^* \rightarrow f^*, g^* \in B_S} \langle g^*, x \rangle \leq \liminf_{g^* \rightarrow f^*, g^* \in B_S} \phi(g^*) = \phi(f^*).$$

In other words,

$$\bigcap_{g^* \in B_S} H_{g^*, \phi(g^*)} \subset H_{f^*, \phi(f^*)} \quad \forall f^* \in \text{ext}(B);$$

consequently,

$$\bigcap_{g^* \in B_S} H_{g^*, \phi(g^*)} \subset \bigcap_{g^* \in \text{ext}(B)} H_{g^*, \phi(g^*)}.$$

As the reverse inclusion obviously holds, relation (8) is established.

Theorem 1 is completely proved by combining relations (5) and (8). \square

4. A characterization of sets with supremum

The following standard result provides a sufficient condition for the existence of a supremum.

Proposition 1. *Let (X, \leq_K, e) be an Archimedean order-unit vector space, S be a subset of X , and assume that there is U , a dense subset of $\text{ext}(B)$ such that all the supporting hyperplanes of S corresponding to elements form U pass through a same point \bar{x} :*

$$\sigma_S(f^*) = \langle f^*, \bar{x} \rangle \quad \forall f^* \in U. \quad (10)$$

Then, \bar{x} is the supremum of S .

Proof of Proposition 1 : We need to prove that

$$\bar{x} \in U(S), \quad (11)$$

and that

$$[y \in U(S)] \Rightarrow [\bar{x} \leq_K y]. \quad (12)$$

In establishing both these results, we will use the following obvious statement :

$$[x \leq_K y] \Leftrightarrow [\langle f^*, x \rangle \leq \langle f^*, y \rangle \quad \forall f^* \in U]. \quad (13)$$

Indeed, let us pick $y \in S$. Relation (10) implies that

$$\langle f^*, y \rangle \leq \sigma_S(f^*) = \langle f^*, \bar{x} \rangle \quad \forall f^* \in U, \quad (14)$$

and by combining relations (13) and (14), we deduce that $y \leq_K \bar{x}$ for any $y \in S$. We have thus proved relation (11).

Let us now consider $y \in U(S)$; hence, $z \leq_K y$ for any $z \in S$. In other words,

$$\langle f^*, z \rangle \leq \langle f^*, y \rangle \quad \forall z \in S, f^* \in K^+. \quad (15)$$

By taking the supremum over $z \in S$, we infer from (15) that

$$\sigma_S(f^*) \leq \langle f^*, y \rangle \quad \forall f^* \in K^+. \quad (16)$$

Relations (10) and (16) prove that

$$\langle f^*, \bar{x} \rangle \leq \langle f^*, y \rangle \quad \forall f^* \in U, \quad (17)$$

and relation (12) follows by combining relations (13) and (17). \square

As a consequence of Theorem 1, the following result establishes a necessary condition for the existence of the supremum of a set.

Proposition 2. *Let (X, \leq_K, e) be an Archimedean order-unit space, \bar{x} be a vector from X , and S be a set whose supremum is \bar{x} . Then, all the supporting hyperplanes of S corresponding to elements from $\text{ext}(B)$ at which σ_S is continuous pass through the point \bar{x} :*

$$\sigma_S(f^*) = \langle f^*, \bar{x} \rangle \quad \forall f^* \in B_S. \quad (18)$$

Proof of Proposition 2 : Being the supremum of S , \bar{x} is one of the upper bounds of S : $\bar{x} \in U(S)$. We have already remarked (see relation (16)) that in this case it follows that

$$\sigma_S(f^*) \leq \langle f^*, \bar{x} \rangle \quad \forall f^* \in K^+. \quad (19)$$

But \bar{x} is precisely the least element of $U(S)$; accordingly, $U(S) = \bar{x} + K$, and it easily results that $LU(S) = \bar{x} - K$. Consequently, $\bar{x} \in LU(S)$.

The set S is bounded from above (by \bar{x}) and non-empty (indeed, $U(\emptyset) = X \neq \bar{x} + K = U(S)$). It is thus possible to apply to S the conclusion of Theorem 1, and deduce from relation (4) that $\bar{x} \in \bigcap_{f^* \in B_S} P_{f^*, \sigma_S(f^*)}$, where $P_{f^*, a} = \{x \in X : \langle f^*, x \rangle = a\}$. Equivalently,

$$\sigma_S(f^*) \geq \langle f^*, \bar{x} \rangle \quad \forall f^* \in B_S. \quad (20)$$

Relation (18) follows by combining relations (19) and (20). \square

Theorem 2 establishes the desired characterization of sets with supremum. This result is a direct consequence of Lemma 1 and Propositions 1 and 2, and does not need a proof.

Theorem 2. *Let (X, \leq_K, e) be an Archimedean order-unit vector space, and S be a non-empty bounded from above subset of X . Then S admits a supremum if and only if all the supporting hyperplanes of S corresponding to elements from $\text{ext}(B)$ at which σ_S is continuous pass through a same point \bar{x} :*

$$\sigma_S(f^*) = \langle f^*, \bar{x} \rangle \quad \forall f^* \in B_S. \quad (21)$$

When relation (21) holds, the element \bar{x} is the supremum of S .

5. A characterization of lower cuts

The main objective of this section is to prove an internal representation of lower cuts in Archimedean order-unit vector spaces.

Theorem 3 plays an instrumental role in proving our main result. It is customarily called *Hayes' criterion for extremality* (for a sketch of a proof, see [15, Theorem 1.8.1]; a similar result has been proved by Benoist et al. ([3, Lemma 2.1]) by using convex processes on Banach spaces).

Theorem 3. *Let (X, \leq_K) be a partially ordered vector space, such that $K - K = X$, and f^* be a non-null element from K^+ . The two following properties are equivalent :*

- i) f^* is an extreme direction of K^+ ,*
- ii) $\sup_{x \leq x_1, x \leq x_2} \langle f^*, x \rangle = \min(\langle f^*, x_1 \rangle, \langle f^*, x_2 \rangle)$ for any pairs of vectors x_1, x_2 in X .*

The desired internal representation of lower cuts reads as follows.

Theorem 4. *Let (X, \leq_K, e) be an Archimedean order-unit vector space, and S be a non-empty subset of X . Then, S is a lower cut if and only if the three following properties are fulfilled :*

- i) S is bounded from above,*
- ii) S is downward,*
- iii) S is sup-containing.*

Proof of Theorem 4 : As the ‘only if’ part is obvious, let us address the ‘if’ part of the proof.

Let x be a vector from $LU(S)$, the smallest lower cut containing S . Theorem 4 is completely proved if we show that $x \in S$.

The first step in proving our claim is to compute $\sigma_A(f^*)$, where $A := S \cap (x - K)$, and $f^* \in B_S$.

Let us first remark that, since the vector x lies in $LU(S)$, it results that $\langle f^*, x \rangle \leq \sigma_{LU(S)}(f^*)$. Theorem 1 implies that

$$LU(S) = \bigcap_{g^* \in B_S} H_{g^*, \sigma_S(g^*)},$$

thus

$$\sigma_{LU(S)}(g^*) \leq \sigma_S(g^*) \quad \forall g^* \in B_S;$$

in particular, $\sigma_{LU(S)}(f^*) \leq \sigma_S(f^*)$. We may thus conclude that

$$\langle f^*, x \rangle \leq \sigma_S(f^*). \quad (22)$$

Let us now pick ε , a positive number. By the definition of the support function it results that there exists $y \in S$ such that

$$\sigma_S(f^*) - \varepsilon \leq \langle f^*, y \rangle \leq \sigma_S(f^*); \quad (23)$$

combine relations (23) and (22) to deduce that

$$\langle f^*, x \rangle - \varepsilon \leq \min(\langle f^*, x \rangle, \langle f^*, y \rangle) \leq \langle f^*, x \rangle. \quad (24)$$

Hayes’ theorem says that

$$\min(\langle f^*, x \rangle, \langle f^*, y \rangle) = \sigma_B(f^*), \quad (25)$$

where $B := (x - K) \cap (y - K)$. From relations (25) and (24) we infer that

$$\langle f^*, x \rangle - \varepsilon \leq \sigma_B(f^*) \quad \forall \varepsilon > 0;$$

consequently,

$$\langle f^*, x \rangle \leq \sigma_B(f^*). \quad (26)$$

Since $y \in S$, from property *ii*) it follows that $y - K \subset S$; accordingly,

$$B = (y - K) \cap (x - K) \subset S \cap (x - K) = A,$$

so

$$\sigma_B(f^*) \leq \sigma_A(f^*). \quad (27)$$

From relations (26) and (27) we deduce that

$$\langle f^*, x \rangle \leq \sigma_A(f^*). \quad (28)$$

Let us now take into account the facts that the vector x is one of the upper bounds of the set A , and that $f^* \in K^+$, in order to deduce that

$$\sigma_A(f^*) \leq \langle f^*, x \rangle. \quad (29)$$

It suffices to combine relations (29) and (28) to obtain that

$$\sigma_A(f^*) = \langle f^*, x \rangle. \quad (30)$$

Let us now apply Lemma 1 to the non-empty and bounded from above set S , and infer that B_S is dense in $\text{ext}(B)$. By combining the conclusions of Proposition 1 with B_S standing for U , with the relation (30), we deduce that x is the supremum of the set A . But A is a subset of S ; property *iii*) implies thus that the supremum of A , that is x , is a member of S . The proof of Theorem 4 is thus complete. \square

6. Conclusions

In any poset, a lower cut is automatically a bounded from above complete ideal; the reciprocal is, in general, false, but there are several - apparently unrelated - exceptions.

Three examples of posets with the property that any bounded from above complete order ideal is a lower cut may be found in the mathematical literature: this is the case for complete lattices, for Heyting algebras, and for finite-dimensional vector spaces with a polyhedral ordering cone. This article add to this list a fourth setting in which bounded from above complete ideals and lower cuts are the same thing - the class of Archimedean order-unit vector spaces (see Theorem 4).

However, there are partially ordered vectors spaces with this property which neither are ordered by means of a polyhedral cone, nor admit an order unit. For instance, the space ℓ_2 , of all the real square-summable sequences, ordered by ℓ_2^+ , the cone of all the ℓ_2 -sequences with positive terms, has no order unit. As a consequence, the dual positive cone K^+ does not admit a w^* -compact basis; more generally, none of the conditions required by the theorem used in our article is

satisfied. And yet, the cone ℓ_2^+ is Choquet simplicial, so any bounded from above subset of ℓ_2 admits a supremum. Consequently, the bounded from above complete ideals, the lower cuts, and the principal ideals all coincide; Theorem 4 is thus true in a partially ordered vector space which does not admit an order unit.

A first question which, at the best of our knowledge, is open, is to characterize all the partially ordered vector spaces in which bounded from above complete ideals are always lower cuts. This class should contain all the Archimedean order-unit spaces, but also spaces such as $(\ell_2, \leq_{\ell_2^+})$. An important step in solving this problem would be to determine whether an Archimedean space, that is a vector space ordered by the means of a lineally closed convex cone, has this property or not.

A more general question is to determine what common feature, shared by Heyting algebras and by Archimedean order-unit vector spaces, yields the fact that bounded from above complete ideals are always lower cuts.

Funding

A. Zaffaroni gratefully acknowledges financial support from Università di Modena e Reggio Emilia under the grant FAR2016 (SigFriED) and from Fondazione Cassa di Risparmio di Modena, (project 2015.0333 EU-SKEW).

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