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NONLOCAL PROBLEMS IN HILBERT SPACES

Irene Benedetti

Department of Mathematics and Informatics
University of Perugia, Italy

Luisa Malaguti

Department of Sciences and Methods for Engineering
University of Modena and Reggio Emilia, Italy

Valentina Taddei

Department of Physics, Informatics and Mathematics
University of Modena and Reggio Emilia, Italy

Abstract. An existence result for differential inclusions in a separable Hilbert space is furnished. A wide family of nonlocal boundary value problems is treated, including periodic, anti-periodic, mean value and multipoint conditions. The study is based on an approximation solvability method. Advanced topological methods are used as well as a Scorza Dragoni-type result for multivalued maps. The conclusions are original also in the single-valued setting. An application to a nonlocal dispersal model is given.

1. **Introduction.**

The paper deals with the nonlocal problem

$$\begin{cases} x'(t) \in F(t, x(t)) \text{ for a.e. } t \in [0, T] \\ x(0) = Mx \end{cases} \tag{1}$$

in a separable Hilbert space H where $F\colon [0,T] \times H \multimap H$ is a multivalued map (multimap) and $M\colon C([0,T], H) \to H$ is a bounded linear operator. The investigation of periodic, anti-periodic, mean value and multi-point solutions is included. The multivalued framework can be motivated by the introduction of control terms into the process, by the appearance of jump discontinuities or by an incomplete knowledge of the model as in Section 3. Advanced topological methods were recently used for the study of (1), based on suitable topological degrees (see e.g. [1], [8] and [10]). Recent results in this context can be found in [3], [4], [6], [7], [8] and [12]. In particular, in [4] and [6] the assumptions involve the weak topology in the state space; an abstract homotopy invariant is introduced in [8], in order to detect steady-states solutions of (1) with an additional m-accreative term appearing also in [12]. A new approach was proposed in [3], based on the approximation solvability method and it was showed there that a quite general family of nonlinear terms can be considered. While the transversality condition in [3] (see [3, (3.1)]) is taken on all an open set, we refine here that technique by assuming a strictly located condition (see condition (3) below), i.e. only on a suitable boundary. This change is not marginal since it requires the use of a Scorza Dragoni type result for multivalued functions (see e.g. [2] and [9, Proposition 5.1]) and the introduction of a sequence of auxiliary problems (see (6)). The main result is Theorem 1.1 below; we point out that it is new also in a single-valued framework.

We denote by H^ω the topological space H equipped with its weak topology and assume the following conditions

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(H) $(H, \|\cdot\|_H)$ is a separable Hilbert space which is *compactly embedded* in a Banach space $(E, \|\cdot\|_E)$ with the relation of norms:

$$\|w\|_E \leq q\|w\|_H \text{ for all } w \in H; \tag{2}$$

for some $q > 0$;

$(F1)$ F takes nonempty, convex, closed and bounded values and for every $w \in H$ the multifunction $F(\cdot, w)\colon [0,T] \multimap H$ is measurable;

$(F2)$ for a.e. $t \in [0,T]$ the multimap $F(t,\cdot)\colon H \multimap H$ is closed from H to H^ω;

$(F3)$ for a.e. $t \in I$ the multimap $F(t,\cdot)\colon H \multimap H$ is $E-E$ u.s.c.;

$(F4)$ for every bounded subset $\Omega \subset H$ there exists $v_\Omega \in L^1_+[0,T]$ such that for each $\omega \in \Omega$ we have

$$\|F(t,w)\|_H = \sup\{\|z\|_H \colon z \in F(t,w)\} \leq v_\Omega(t)$$

for a.e. $t \in [0,T]$.

(M) $M\colon C([0,T],H) \multimap H$ is a linear bounded operator satisfying $\|M\| \leq 1$.

Theorem 1.1. *Assume $(H), (F1)-(F4)$ and (M). In addition, suppose that there exists $R_0 > 0$ such that for every $w \in H$ with $\|w\|_H = R_0$, for a.e. $t \in [0,T]$ and $z \in F(t,w)$, we have*

$$\langle w, z\rangle \leq 0. \tag{3}$$

Then problem (1) admits a solution $x \in W^{1,1}([0,T],H)$ with $\|x(t)\|_H \leq R_0$, for a.e. $t \in [0,T]$.

Its proof appears in Section 2. Section 3 contains an application to a nonlocal dispersal model given by an integro-differential equation; its multivalued nature is given by the possible uncertainty of the integral kernel which is not determined, but belongs to a prescribed family of functions.

2. **Proof of Theorem 1.1.** Since H is separable, according to (F1) and the Kuratowski-Ryll-Nardzewski Theorem (see [11]), a measurable selection of $F(\cdot, y)$ exists for every $y \in H$. Thus we get that, (F2), (F4) and [3, Proposition 4] yield, for each $q \in C([0,T],H)$,

$$\mathcal{S}_q = \{f \in L^1([0,T],H) : f(t) \in F(t,q(t)) \text{ for a.e. } t \in [0,T]\} \neq \emptyset$$

(see [5, Proposition 2.2]). The proof splits into five steps.

Step 1. *Introduction of a sequence of problems in a finite dimensional space.* Denote by $\{e_n\}_{n=1}^\infty$ an orthonormal basis of H and for every $n \in \mathbb{N}$, let H_n be an $n-$dimensional subspace of H with the basis $\{e_k\}_{k=1}^n$ and P_n be the projection of H onto H_n. For the sake of simplicity, we denote by F and M also their restrictions $F/_{[0,T]\times H_n}$ and $M/_{[0,T]\times H_n}$. We first prove that, for every fixed $n \in \mathbb{N}$, the problem

$$\begin{cases} x'(t) \in P_n F(t,x(t)), \text{ for a.e. } t \in [0,T], \\ x(0) = P_n M x, \end{cases} \tag{4}$$

has a solution. By a solution to (4) we mean a function $x \in W^{1,1}([0,T],H_n)$ such that $x(0) = P_n M x$ and there exists $f \in \mathcal{S}_x$ such that $x'(t) = P_n f(t)$ for a.e. $t \in [0,T]$.

Let us denote by $K = \{x \in H : \|x\|_H < R_0\}$ and by Q_n the closed and convex set $C([0,T], \overline{K} \cap H_n)$.

Step 2. *Introduction of a sequence of approximating problems.* To prove that problems (4) are solvable we use an approximation result of Scorza-Dragoni type (see e.g. [9, Proposition 5.1]). According to Urisohn lemma, given $\varepsilon \in (0, R_0)$, there exists a continuous function $\mu : H \to [0,1]$ such that $\mu \equiv 0$ on $H \setminus \{x \in H : R_0 - \varepsilon < \|x\|_H < R_0 + \varepsilon\}$ and $\mu \equiv 1$ on $\{x \in H : R_0 - \frac{\varepsilon}{2} \leq \|x\|_H \leq R_0 + \frac{\varepsilon}{2}\}$. Trivially the function $\phi : H \to \mathbb{R}$ defined by

$$\phi(x) = \begin{cases} \mu(x)\dfrac{x}{\|x\|_H} & R_0 - \varepsilon \leq \|x\|_H \leq R_0 + \varepsilon \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

is well-defined, continuous and bounded in H.

Since P_n is a linear operator, from (F1) it easily follows that $P_nF\colon [0,T]\times(\overline{K}\cap H_n) \multimap H_n$ has nonempty, convex and bounded values. We prove now that the map P_nF has closed values too. Given $(t,x)\in[0,T]\times(\overline{K}\cap H_n)$, take $\{w_m\}\subset P_nF(t,x)$ converging to w and $\overline{w}_m\in F(t,x)$ with $w_m = P_n\overline{w}_m$ for all $m\in\mathbb{N}$. From (F1), without loss of generality, we get the existence of $v\in F(t,x)$ such that $\overline{w}_m \rightharpoonup v$, hence that $w_m \rightharpoonup P_nv$, for the linearity and continuity of P_n. From the uniqueness of the weak limit we then obtain $w = P_nv$, i.e. that P_nF has closed values. Since P_n is continuous, (F1) also implies that $P_nF(\cdot,x)$ is measurable for every $x\in\overline{K}\cap H_n$. For a.e. $t\in[0,T]$, (F4) implies that $P_nF(t,\cdot)$ is a compact multimap. Finally, given $x_m\to x\in\overline{K}\cap H_n, w_m\to w$ with $w_m\in P_nF(t,x_m)$ and $\overline{w}_m\in F(t,x_m)$ with $w_m = P_n\overline{w}_m$, the convergence of $\{x_m\}$, conditions (F2) and (F4) yield the existence of a subsequence, denoted as the sequence, such that $\overline{w}_m\rightharpoonup v\in F(t,x)$. According to the linearity and continuity of P_n, we then get $w = P_nv$, i.e. that $P_nF(t,\cdot)$ is closed, hence u.s.c.. Let τ be the Lebesgue measure in $\mathbb{R}$. Applying [9, Proposition 5.1] to $P_nF\colon[0,T]\times(\overline{K}\cap H_n)\multimap H_n$ we obtain a multimap $G_n : [0,T]\times(\overline{K}\cap H_n)\multimap H_n$ with closed, bounded, convex and possibly empty values satisfying $G_n(t,q)\subset P_nF(t,q)$ for every (t,q) and a monotone decreasing sequence $\{\theta_m\}_m$ of subsets of $[0,T]$ such that $[0,T]\setminus\theta_m$ is compact, $\tau(\theta_m)<\frac{1}{m}$ for every $m\in\mathbb{N}$ and G_n is nonempty valued and u.s.c. in $([0,T]\setminus\theta_m)\times(\overline{K}\cap H_n)$. Obviously $\tau\left(\cap_{m=1}^{\infty}\theta_m\right)=0$ and $\lim_{m\to\infty}\chi_{\theta_m}(t)=0$, for every $t\notin\cap_{m=1}^{\infty}\theta_m$ where χ_A is the characteristic function of a set $A\subset\mathbb{R}$.

Now we introduce the initial problem

$$\begin{cases} x'(t)\in G_n(t,x(t))-\phi(x(t))\left(v_{\overline{K}}(t)\chi_{\theta_m}(t)+\frac{1}{m}\right), \text{ for a.e. } t\in[0,T]\\ x(0)=P_nMx. \end{cases} \tag{6}$$

Step 3. *Solvability of problem* (6). To prove that problem (6) has a solution, we shall apply a classical continuation principle (see, e.g. [2, Proposition 2]). Fix $q\in Q_n$; according to the properties of G_n and since $S_q\neq\emptyset$ we have that

$$\mathcal{R}_{nq}=\{f\in L^1([0,T],H_n): f(t)\in G_n(t,q(t)) \text{ for a.e. } t\in[0,T]\}\neq\emptyset;$$

moreover, it is well known that, for each $q\in Q_n,\lambda\in[0,1]$ and $f\in\mathcal{R}_{nq}$, the linear initial value problem

$$\begin{cases} x'(t)=\lambda[f(t)-\phi(q(t))(v_{\overline{K}}(t)\chi_{\theta_m}(t)+\frac{1}{m})], \text{ for a.e. } t\in[0,T],\\ x(0)=\lambda P_nM_nq, \end{cases} \tag{7}$$

has a unique solution denoted by $\mathcal{H}_{nm}(f,\lambda)$.

Let us introduce now the multimap $\mathcal{T}_{nm} : Q_n\times[0,1]\multimap C([0,T],H_n)$, defined as $\mathcal{T}_{nm}(q,\lambda)=\{\mathcal{H}_{nm}(f,\lambda): f\in\mathcal{R}_{nq}\}$. According to (F1) it has convex values. We prove now that $\mathcal{T}_{nm}$ has a closed graph in $Q_n\times[0,1]\times C([0,T],H_n)$. Assume that $\lambda_k\to\lambda\in[0,1], q_k\to q$ in Q_n and $x_k\to x$ in $C([0,T],\overline{K}\cap H_n)$ with $x_k\in\mathcal{T}_{nm}(q_k,\lambda_k)$ for all k and let $f_k\in\mathcal{R}_{nq_k}$ be such that

$$x'_k(t)=\lambda_k\left[f_k(t)-\phi(q_k(t))\left(v_{\overline{K}}(t)\chi_{\theta_m}(t)+1/m\right)\right].$$

According to the convergence of $\{q_k\}$, condition (F4) and by the Dunford-Pettis Theorem there exists $f\in L^1([0,T],H_n)$ and a suitable subsequence of $\{f_k\}$ denoted as the sequence, such that $f_k\rightharpoonup f$. Moreover, the continuity of ϕ implies that

$$x'_k\rightharpoonup l: t\to\lambda\left[f(t)-\phi(q(t))\left(v_{\overline{K}}(t)\chi_{\theta_m}(t)+1/m\right)\right]$$

in $L^1([0,T],H_n)$. From (M) we get that $x_k(0)=\lambda_kP_nMq_k\to\lambda P_nMq$ and by the finite dimension of H_n we then obtain that

$$x_k(t)\to y(t):=\lambda P_nM(q)+\int_0^t l(s)\,ds$$

for all $t \in [0,T]$. According to the uniqueness of the limit we get $x = y$. Finally, the upper semicontinuity of G_n in $[0,T] \setminus \theta_m \times (\overline{K} \cap H_n)$ for every m implies that $f \in \mathcal{R}_{nq}$, i.e. the closure of the graph of $\mathcal{T}_{nm}$ for every $m \in \mathbb{N}$.

Given now $x \in \mathcal{T}_{nm}(Q_n \times [0,1])$, we have that $x(t) = \lambda P_n M q + \lambda \int_0^t [f(s) - \phi(q(s))(v_{\overline{K}}(s)\chi_{\theta_m}(s) + \frac{1}{m})]dt$, for some $q \in Q_n, \lambda \in [0,1]$ and $f \in \mathcal{R}_{nq}$. The boundedness and equicontinuity of $\mathcal{T}_{nm}(Q_n \times [0,1])$ follow from (F4), the boundedness of Q_n and the continuity of ϕ, thus the Ascoli-Arzelá theorem implies the compactness of $\mathcal{T}_{nm}$.

Since trivially $\mathcal{T}_{nm}(Q_n \times \{0\}) = \{0\} \subset \operatorname{int} Q_n$, to apply the continuation principle it remains to prove that $\mathcal{T}_{nm}(\cdot, \lambda)$ is fixed point free on ∂Q_n for every $\lambda \in (0,1)$. We reason by a contradiction and assume the existence of $\lambda \in (0,1)$, $q \in \partial Q_n$ and $t_0 \in [0,T]$ such that $q \in \mathcal{T}_{nm}(q, \lambda)$ and $q(t_0) \in \partial K$. If we assume that $t_0 = 0$, by (M) we have the contradictory conclusion $R_0 = \|q(0)\|_H = \lambda \|P_n M q\|_H < \|M\| \|q\|_C \leq R_0$. Hence $t_0 > 0$, thus there is $h > 0$ such that $q(t) \in \{x \in H_n : R_0 - \frac{\varepsilon}{2} \leq \|x\|_H \leq R_0\}$ for all $t \in [t_0 - h, t_0]$. According to (F4), we then obtain

$$\begin{aligned}
&0 \leq \|q(t_0)\|_H^2 - \|q(t_0 - h)\|_H^2 = 2 \int_{t_0-h}^{t_0} < q(s), q'(s) > ds = \\
&2\lambda \int_{[t_0-h,t_0]\cap\theta_m} [< q(s), f(s) > -(v_{\overline{K}}(s) + 1/m)\|q(s)\|_H] ds + \\
&2\lambda \int_{[t_0-h,t_0]\setminus\theta_m} [< q(s), f(s) > -1/m \|q(s)\|_H] ds \leq \\
&2\lambda \int_{[t_0-h,t_0]\cap\theta_m} [\|q(s)\|_H v_{\overline{K}}(s) - (v_{\overline{K}}(s) + 1/m)\|q(s)\|_H] ds + \\
&2\lambda \int_{[t_0-h,t_0]\setminus\theta_m} [< q(s), f(s) > -1/m \|q(s)\|_H] ds < \\
&2\lambda \int_{[t_0-h,t_0]\setminus\theta_m} [< q(s), f(s) > -1/m \|q(s)\|_H] ds.
\end{aligned} \tag{8}$$

Now, if $t_0 \in \theta_m$, we can choose h sufficiently small such that $[t_0 - h, t_0] \subset \theta_m$, because θ_m is open and we get $\int_{[t_0-h,t_0]\setminus\theta_m} [< q(s), f(s) > -\frac{1}{m}\|q(s)\|_H] ds = 0$, so (8) gives a contradiction. Otherwise, let us consider the map $\mathcal{J} : [0,T] \setminus \theta_m \times (\overline{K} \cap H_n) \multimap \mathbb{R}$ defined as

$$(t, x) \multimap \{< x, w > -1/m \|x\|_H \, : \, w \in G_n(t, x)\}.$$

Since $\|q(t_0)\|_H = R_0$, assumption (3) implies that

$$< q(t_0), w > \leq 0, \text{ for every } w \in G_n(t_0, q(t_0)) \subset P_n F(t_0, q(t_0)).$$

Therefore, $j \leq -\frac{1}{m} R_0 < 0$ for every $j \in \mathcal{J}(t_0, q(t_0))$ and since the multimap G_n is u.s.c. in $[0,T] \setminus \theta_m \times (\overline{K} \cap H_n)$ and q is continuous, we can choose h sufficiently small such that $< q(s), f(s) > -\frac{1}{m}\|q(s)\|_H < 0$ on all $[t_0 - h, t_0] \setminus \theta_m$ and (8) gives again a contradiction.

Therefore it follows the existence of $q_{nm} \in Q_n$ with $q_{nm} \in \mathcal{T}_{nm}(q_{nm}, 1)$, i.e. of a solution of problem (6).

Step 4. *Solvability of problems* (4). For every $n \in \mathbb{N}$, we got a sequence $\{q_{nm}\} \subset AC([0,T], \overline{K} \cap H_n)$ such that

$$q'_{nm}(t) = f_m(t) - \phi(q_{nm}(t))\,(v_{\overline{K}}(t)\chi_{\theta_m}(t) + 1/m) \text{ for a.a. } t \in [0,T] \tag{9}$$

with $f_m \in \mathcal{R}_{nq_{nm}}$. Hence, $\{q_{nm}\}_m$ is bounded. Moreover, according to (F4), $\{f_m\}_m$ is integrably bounded, thus the boundedness of ϕ implies the integrable boundedness of $\{q'_{nm}\}$. The Ascoli–Arzelà Theorem then implies the existence of $q_n \in AC([0,T], \overline{K} \cap H_n)$ such that $\{q_{nm}\}$ has a subsequence, again denoted as the sequence, with $q_{nm} \to q_n$ uniformly in $[0,T]$, and $q'_{nm} \rightharpoonup q'_n$ in $L^1[0,T]$. Notice, moreover, that since ϕ is bounded and $\lim_{m\to\infty} \chi_{\theta_m}(t) = 0$

for every $t \notin \cap_{m=1}^{\infty} \theta_m$, and $\tau\left(\cap_{m=1}^{\infty} \theta_m\right) = 0$,

$$\phi(q_{nm}(t))\left(v_{\overline{K}}(t)\chi_{\theta_m}(t) + 1/m\right) \to 0, \text{ for a.a. } t \in [0,T].$$

Consequently, a standard limiting argument (see e.g. [13, page 88]) implies that q_n is a solution of (4).

Step 5. *Solvability of problem* (1). The conclusion follows as in [3, Theorem 7].

3. **Applications.** We apply the developed abstract theory to the Cauchy multi-point problem (10) associated to a nonlocal diffusion process. The multivalued nature of the nonlinear integro-differential equation in (10) depends on the integral kernel, which can be unknown and can only be chosen in a suitable family of functions.

Let $\Omega \subset \mathbb{R}^k$ $(k \geq 2)$ be an open bounded domain with Lipschitz boundary. Consider the multi-point problem

$$\begin{cases} u_t = u(t,\xi)\displaystyle\int_\Omega v(\xi,\eta)u(t,\eta)d\eta - bu(t,\xi) + f(t,u(t,\xi)) \\ v \in \mathcal{S} \\ u(0,\xi) = \sum_{i=1}^p \alpha_i u(t_i,\xi), \ \alpha_i \in \mathbb{R}, \ i = 1,2,...,p, \ 0 < t_1 < ... < t_p \leq 1, \end{cases} \tag{10}$$

for a.a. $t \in [0,1]$ and all $\xi \in \Omega$, where $b > 0$, $\sum_{i=1}^p |\alpha_i| \leq 1$, $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and

$$\mathcal{S} = \left\{ v \in W^{1,2}(\Omega \times \Omega, \mathbb{R}) \ : \begin{array}{c} \exists\, \beta > 0 \text{ such that} \\ |v(\xi,\eta)| + \|\nabla v(\xi,\eta)\|_{\mathbb{R}^{2k}} \leq \beta \text{ for a.e. } (\xi,\eta) \in \Omega \times \Omega \end{array} \right\},$$

where the symbol ∇ stands for the derivative with respect to $(\xi,\eta) \in \Omega \times \Omega$.

Assume that

(f) the partial derivative $\frac{\partial f}{\partial z}\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and there is a positive constant N, such that

$$\left|\frac{\partial f(t,z)}{\partial z}\right| \leq N \text{ for all } (t,z) \in [0,1] \times \mathbb{R}.$$

We assume that $b = N + \sqrt{6\delta|\Omega|\beta}$, where $\delta = \max\limits_{t\in[0,1]} |f(t,0)|$. The symbol D stands for the derivative (i.e. the gradient) with respect to the variables in the vector ξ and for $(t,z) \in [0,1] \times \mathbb{R}$, we denote $f_2'(t,z) = \frac{\partial f(t,z)}{\partial z}$.

By a solution to (10) we mean a continuous function $u\colon [0,1] \times \Omega \to \mathbb{R}$ whose partial derivative $\frac{\partial u(t,\xi)}{\partial t}$ exists, for a.a. $t \in [0,1]$, and it satisfies (10).

Let $H = W^{1,2}(\Omega,\mathbb{R})$ and $E = L^2(\Omega,\mathbb{R})$. It is clear that H is a separable Hilbert space which is compactly embedded in E. By means of a reformulation of this problem we will prove the existence of a continuous function $u(t,\xi)$ such that at every value t the function $u(t,\cdot)$ belongs to the Sobolev space $W^{1,2}(\Omega,\mathbb{R})$. To this aim, for each $t \in [0,1]$, set $x(t) = u(t,\cdot)$. Then we can substitute (10) with the following problem

$$\begin{cases} x'(t) \in F(t,x(t)), \text{ for a.e. } t \in [0,1], \\ x(0) = \sum_{i=1}^p \alpha_i x(t_i) \text{ with } \alpha_i \in \mathbb{R}, \ i = 1,2,...,p, \ 0 < t_1 < ... < t_p \leq 1, \end{cases} \tag{11}$$

where $F\colon [0,1] \times H \multimap H$, $F(t,w) = G(w) + \overline{f}(t,w)$ with

$$G\colon H \multimap H, G(w) = \left\{ g \in H \ : \ g(\xi) = w(\xi)\int_\Omega v(\xi,\eta)w(\eta)\,d\eta - bw(\xi)\ v \in \mathcal{S} \right\},$$

$$\overline{f}\colon [0,1] \times H \to H, \overline{f}(t,w)(\xi) = f(t,w(\xi)) \text{ for a.a. } \xi \in \Omega.$$

First we remark that G is well-defined. Indeed for $w \in H$ and $v \in \mathcal{S}$, by Fubini's Theorem we have that $\{v(\xi, \cdot)\} \in L^2(\Omega, \mathbb{R})$ for a.e. $\xi \in \Omega$; so g can be defined and for a.e. $\xi \in \Omega$ it holds

$$Dg(\xi) = Dw(\xi)\left(\int_\Omega v(\xi,\eta)w(\eta)\, d\eta\right) + w(\xi)\left(\int_\Omega Dv(\xi,\eta)w(\eta)\, d\eta\right) - bDw(\xi).$$

Trivially, by the definition of the set $\mathcal{S}$ it follows that $g \in H$.

From (f) it follows that

$$|f(t,z)| \le |f(t,0)| + \int_0^z |f_2'(t,\eta)|\, d\eta \le |f(t,0)| + N|z|, \tag{12}$$

for all $(t,z) \in [0,1] \times \Omega$. Moreover

$$Df(t,w(\xi)) = f_2'(t,w(\xi))Dw(\xi) \text{ for } \xi \in \Omega.$$

So $\overline{f}$ is well-defined as well. Trivially G has bounded and convex values. To prove that G has closed values we have to show that given $w \in H$ and $\{g_n\} \subset H$ such that $g_n \in G(w)$ for any $n \in \mathbb{N}$ and $g_n \overset{H}{\to} g$, it follows $g \in G(w)$. First notice that w.l.o.g. $\{g_n\}$ almost pointwise converges to g. By the definition of the multimap G there exists a sequence of functions $\{v_n\} \subset W^{1,2}(\Omega \times \Omega, \mathbb{R})$ such that g_n satisfies

$$g_n(\xi) = w(\xi)\int_\Omega v_n(\xi,\eta)w(\eta)\, d\eta - bw(\xi),\ \xi \in \Omega.$$

By the weak compactness of the set $\mathcal{S}$ there exists a subsequence, denoted as the sequence, such that $v_n \rightharpoonup v_0, v_0 \in \mathcal{S}$ in $W^{1,2}(\Omega \times \Omega, \mathbb{R})$. By the compact embedding of $W^{1,2}(\Omega \times \Omega, \mathbb{R})$ into $L^2(\Omega \times \Omega, \mathbb{R})$, the weak convergence of $\{v_n\}$ in $W^{1,2}(\Omega \times \Omega, \mathbb{R})$ implies its strong convergence in $L^2(\Omega \times \Omega, \mathbb{R})$ and hence the almost pointwise convergence of a suitable subsequence. Denote with $h : \Omega \to \mathbb{R}$ the function:

$$h(\xi) = w(\xi)\int_\Omega v_0(\xi,\eta)w(\eta)\, d\eta - bw(\xi),\ \xi \in \Omega.$$

From the dominated almost pointwise convergence of $\{v_n\}$ to v_0 it follows that $|g_n(\xi) - h(\xi)|$ goes to zero as n goes to ∞ for a.e. $\xi \in \Omega$. Therefore by the uniqueness of the limit we have that $g(\xi) = h(\xi)$ for a.e. $\xi \in \Omega$, hence $g \in G(w)$.

Now let $w_n \overset{E}{\to} w_0$. We have

$$\begin{aligned}\|\overline{f}(t,w_n) - \overline{f}(t,w_0)\|_E^2 &= \int_\Omega |\overline{f}(t,w_n(\xi)) - \overline{f}(t,w_0(\xi))|^2 d\xi \\ &= \int_\Omega \left|\int_{w_0(\xi)}^{w_n(\xi)} f_2'(t,\tau)d\tau\right|^2 d\xi \\ &\le N^2 \int_\Omega |w_n(\xi) - w_0(\xi)|^2 d\xi = N^2\|w_n - w_0\|_E^2.\end{aligned}$$

Hence $\overline{f}(t,w_n) \overset{E}{\to} \overline{f}(t,w_0)$ and then $\overline{f}(t,\cdot)$ is $E-E$ continuous.

Moreover the multimap G is $E-E$ u.s.c.. Indeed, if $w_n \overset{E}{\to} w_0$, w.l.o.g. $\{w_n\}$ almost pointwise converges to w_0 and the convergence is dominated in E. Let moreover $g_n \in G(w_n)$, implying the existence of $\{v_n\} \subset \mathcal{S}$ such that

$$g_n(\xi) = w_n(\xi)\int_\Omega v_n(\xi,\eta)w_n(\eta)\, d\eta - bw_n(\xi),\ \xi \in \Omega. \tag{13}$$

As above there is a subsequence, denoted as the sequence, and $v_0 \in \mathcal{S}$ such that $\{v_n\}$ almost pointwise converges to v_0 and the convergence is dominated in $L^2(\Omega \times \Omega, \mathbb{R})$. Put

$$g_0(\xi) = w_0(\xi)\int_\Omega v_0(\xi,\eta)w_0(\eta)\, d\eta - bw_0(\xi),\ \xi \in \Omega. \tag{14}$$

Notice that

$$
\begin{aligned}
|g_n(\xi) - g_0(\xi)| \leq & |w_n(\xi)| \int_\Omega |v_n(\xi,\eta) - v_0(\xi,\eta)||w_n(\eta)|\, d\eta + \\
& |w_n(\xi)| \int_\Omega |v_0(\xi,\eta)||w_n(\eta) - w_0(\eta)|\, d\eta + \\
& |w_n(\xi) - w_0(\xi)| \int_\Omega |v_0(\xi,\eta)||w_0(\eta)|\, d\eta + \\
& b|w_n(\xi) - w_0(\xi)|, \quad \xi \in \Omega,
\end{aligned}
$$

implying that $|g_n(\xi) - g_0(\xi)| \to 0$ for a.a. $\xi \in \Omega$ and the convergence is dominated in E. Hence $g_n \stackrel{E}{\to} g_0$ and then the multimap G is quasicompact. Following the same reasonings it follows that it is $E - E$ closed. By [10, Theorem 1.1.12] we have that G is an u.s.c. multimap.

So $w \longmapsto F(t,w)$ is u.s.c. from E into itself, for each $t \in [0,1]$ and condition (F3) is satisfied.

Now, let $t \in [0,1]$. To prove the $H - H$ continuity of the map $\overline{f}(t,\cdot)$ we assume by contradiction that there exists a sequence $\{\widetilde{w}_n\}$ such that $\widetilde{w}_n \stackrel{H}{\to} w_0$ and $\overline{\varepsilon} > 0$ such that $\|\overline{f}(t,\widetilde{w}_n) - \overline{f}(t,w_0)\|_H > \overline{\varepsilon}$ for any $n \in \mathbb{N}$. Thus we have

$$
\begin{aligned}
&\overline{\varepsilon}^2 < \|\overline{f}(t,\widetilde{w}_n) - \overline{f}(t,w_0)\|_H^2 = \\
&\qquad \int_\Omega \left(|f(t,\widetilde{w}_n(\xi)) - f(t,\widetilde{w}_0(\xi))|^2 + \|Df(t,\widetilde{w}_n(\xi)) - Df(t,w_0(\xi))\|_{\mathbb{R}^k}^2 \right) d\xi.
\end{aligned}
$$

For the continuity in E of the map $\overline{f}(t,\cdot)$, w.l.o.g. has to be

$$
\int_\Omega \|Df(t,\widetilde{w}_n) - Df(t,w_0)\|_{\mathbb{R}^k}^2\, d\xi > \overline{\varepsilon}^2 \ \ \forall n \in \mathbb{N}. \tag{15}
$$

By the convergence of $\{\widetilde{w}_n\}$ to w_0 in H there exists a subsequence $\{\widetilde{w}_{n_k}\}$ such that $\widetilde{w}_{n_k}(\xi) \to w_0(\xi)$ and $D\widetilde{w}_{n_k}(\xi) \to Dw_0(\xi)$ for a.e. $\xi \in \Omega$ and the convergence is dominated. We have the following estimation

$$
\begin{aligned}
\int_\Omega \|Df(t,\widetilde{w}_n) - Df(t,w_0)\|_{\mathbb{R}^k}^2\, d\xi &= \int_\Omega \|f_2'(t,\widetilde{w}_{n_k}(\xi))D\widetilde{w}_{n_k}(\xi) - f_2'(t,w_0(\xi))Dw_0(\xi)\|_{\mathbb{R}^k}^2\, d\xi \\
&\leq 2\int_\Omega |f_2'(t,\widetilde{w}_{n_k}(\xi))|^2 \|D\widetilde{w}_{n_k}(\xi) - Dw_0(\xi)\|_{\mathbb{R}^k}^2\, d\xi \\
&\quad + 2\int_\Omega |f_2'(t,\widetilde{w}_{n_k}(\xi)) - f_2'(t,w_0(\xi))|^2 \|Dw_0(\xi)\|_{\mathbb{R}^k}^2\, d\xi.
\end{aligned}
$$

By the continuity of the map f_2' it follows $f_2'(t,\widetilde{w}_{n_k}(\xi)) \to f_2'(t,w_0(\xi))$ for a.e. $\xi \in \Omega$. Moreover by hypothesis (f) we have

$$
|(f_2'(t,\widetilde{w}_{n_k}(\xi)) - f_2'(t,w_0(\xi))|^2 \|Dw_0(\xi)\|_{\mathbb{R}^k}^2 \leq 4N^2 \|Dw_0(\xi)\|_{\mathbb{R}^k}^2
$$

and

$$
|f_2'(t,\widetilde{w}_{n_k}(\xi))|^2 \|D\widetilde{w}_{n_k}(\xi) - Dw_0(\xi)\|_{\mathbb{R}^k}^2 \leq N^2 \|D\widetilde{w}_{n_k}(\xi) - Dw_0(\xi)\|_{\mathbb{R}^k}^2.
$$

Thus by the convergence of $\{\widetilde{w}_n\}$ to w_0 in H and by the Lebesgue's Convergence Theorem

$$
\int_\Omega \|Df(t,\widetilde{w}_n) - Df(t,w_0)\|_{\mathbb{R}^k}^2\, d\xi \text{ goes to zero as } n \to \infty,
$$

obtaining a contradiction with (15). Hence for any sequence $\{w_n\}$ such that $w_n \stackrel{H}{\to} w_0$ it follows $\overline{f}(t,w_n) \stackrel{H}{\to} \overline{f}(t,w_0)$.

The multimap G is $H - H^\omega$ closed. Indeed, assume that there exists $w_n \stackrel{H}{\to} w_0$, $g_n \in G(w_n)$ such that $g_n \stackrel{H}{\rightharpoonup} g$, we shall prove that $g \in G(w_0)$.

By the definition of G, the function g_n satisfies (13) with $v_n \in \mathcal{S}$ for every $n \in \mathbb{N}$. Reasoning as above we can show the existence of $v_0 \in \mathcal{S}$ and a subsequence, denoted as the sequence, such that v_n almost pointwise converges to v_0 in $\Omega \times \Omega$ and the convergence is dominated in $L^2(\Omega \times \Omega, \mathbb{R})$. Moreover $g_n \stackrel{E}{\to} g_0$ with g_0 defined as in (14). The weak convergence of $\{g_n\}$ to g in H also implies that $g_n \stackrel{E}{\to} g$. By the uniqueness of the limit we have that $g \equiv g_0$, i.e. $g \in G(w)$. Hence $w \longmapsto F(t,w)$ is closed from H into H^ω, for each $t \in [0,1]$ and condition (F2) is satisfied.

To verify condition $(F1)$ it is sufficient to prove that $\overline{f}(\cdot, w)$ is measurable, for every $w \in H$. In fact the multimap G does not depend on $t \in [0,1]$ and so it is trivially measurable with respect to t. We will prove that $\overline{f}(\cdot, w)$ is continuous. In fact, let $t_0 \in [0,1]$ and $\{t_n\} \subset [0,1]$ such that $t_n \to t_0$. According to (f) we obtain that $f(t_n, w(\xi)) \to f(t_0, w(\xi))$ and $Df(t_n, w(\xi)) = f_2'(t_n, w(\xi))Dw(\xi) \to f_2'(t_0, w(\xi))Dw(\xi) = Df(t_0, w(\xi))$ for all $\xi \in \Omega$. As a consequence of (12), the previous convergences are also dominated in E, implying that $\overline{f}(t_n, w) \stackrel{H}{\to} \overline{f}(t_0, w)$. Therefore, $\overline{f}(\cdot, w)$ is continuous, and hence, it is measurable.

Now let $\Theta \subset H$ be bounded, $w \in \Theta$ and $t \in [0,1]$. If $z \in F(t,w)$, hence $z = g + \overline{f}(t,w)$ with $g \in G(w)$. Therefore there exists $v \in \mathcal{S}$ such that

$$\begin{aligned}
\|z\|_H^2 = & \int_\Omega \left| w(\xi) \left(\int_\Omega v(\xi,\eta) w(\eta)\, d\eta \right) - bw(\xi) + f(t, w(\xi)) \right|^2 d\xi \\
& + \int_\Omega \Bigg\| w(\xi) \left(\int_\Omega Dv(\xi,\eta) w(\eta)\, d\eta \right) + Dw(\xi) \left(\int_\Omega v(\xi,\eta) w(\eta)\, d\eta \right) \\
& - bDw(\xi) + f_2'(t, w(\xi)) Dw(\xi) \Bigg\|_{\mathbb{R}^k}^2 d\xi \\
& \le 7\beta^2 |\Omega| \|w\|_H^4 + 4b^2 \|w\|_H^2 + 6|f(t,0)|^2 |\Omega| + 6N^2 \|w\|_H^2.
\end{aligned}$$

So condition (F4) is satisfied.

Now, let $w \in H$ and $g \in G(w)$; by virtue of (f) and (12) the following estimation is true

$$\begin{aligned}
\langle w, g + \overline{f}(t,w) \rangle \le & -b\|w\|_H^2 + \int_\Omega |w(\xi)| \left(|f(t,0)| + N|w(\xi)| \right) d\xi \\
& + N \int_\Omega \|Dw(\xi)\|_{\mathbb{R}^k}^2 d\xi + \beta \left(\int_\Omega |w(\xi)|^2 d\xi \right) \left(\int_\Omega |w(\eta)|\, d\eta \right) \\
& + \beta \left(\int_\Omega \|Dw(\xi)\|_{\mathbb{R}^k} |w(\xi)|\, d\xi \right) \left(\int_\Omega |w(\eta)|\, d\eta \right) \\
& + \beta \left(\int_\Omega \|Dw(\xi)\|_{\mathbb{R}^k}^2 d\xi \right) \left(\int_\Omega |w(\eta)|\, d\eta \right) \\
& \le (-b + N) \|w\|_H^2 + \delta |\Omega|^{1/2} \|w\|_H + \beta |\Omega|^{1/2} \|w\|_H^2 \|w\|_E \\
& + \frac{1}{2} \beta |\Omega|^{1/2} \|w\|_E \|w\|_H^2 \\
& \le \frac{3}{2} \beta |\Omega|^{1/2} \|w\|_H^3 + (-b+N) \|w\|_H^2 + \delta |\Omega|^{1/2} \|w\|_H = 0,
\end{aligned}$$

provided $\|w\|_H = \dfrac{b-N}{3\beta|\Omega|^{1/2}}$.

Applying Theorem 1.1 we obtain the existence of a solution to (11), and therefore, of (10).

Notice that the last term of the previous inequality is grater than zero for any $w \in H$ with $\|w\|_H \neq \dfrac{b-N}{3\beta|\Omega|^{1/2}}$. Thus, from [3, Theorem 7] it is not possible to deduce the existence of a solution of problem (10) from the above estimation.

Remark 1. We remark that different nonlocal conditions can be replaced in (10) such as the mean value condition, i.e.,

$$u(0,\xi) = \int_0^1 u(s,\xi)\,ds,\ \xi \in \Omega.$$

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E-mail address: `irene.benedetti@dmi.unipg.it`
E-mail address: `luisa.malaguti@unimore.it`
E-mail address: `valentina.taddei@unimore.it`