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Strict Nash equilibria in non-atomic games with strict single crossing in players (or types) and actions

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Abstract In this paper, we study games where the space of players (or types, if the game is one of incomplete information) is atomless and payoff functions satisfy the property of strict single crossing in players (types) and actions. Under an additional assumption of quasipermodularity in actions of payoff functions and mild assumptions on the player (type) space—partially ordered and with sets of uncomparable players (types) having negligible size—and on the action space—lattice, second countable and satisfying a separation property with respect to the ordering of actions—we prove that every Nash equilibrium is essentially strict. Furthermore, we show how our result can be applied to incomplete information games, obtaining the existence of an evolutionary stable strategy, and to population games with heterogeneous players.

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14 1 Introduction

15 Strict Nash equilibrium is a solution concept that possesses desirable features.¹ In
 16 this paper, we identify a class of games where every pure-strategy Nash equilibrium
 17 is essentially strict. Only pure-strategy Nash equilibria are considered in the paper.
 18 Equilibria in mixed strategies might also be considered, but then a proper definition of
 19 mixed strategies should be carefully provided, tackling the difficulty of modeling the
 20 independence of a continuum of players. We refer the interested reader to [Khan et al.](#)
 21 [\(2015\)](#) for a possible solution. More precisely, we consider games with an atomless
 22 space of players (or types, if the game is of incomplete information), and action sets
 23 that are second countable and satisfy a mild separation property.^{2,3} In addition, we
 24 restrict attention to games where the payoff functions satisfy the strict single crossing
 25 property ([Milgrom and Shannon 1994](#)) in players (types) and actions. We are aware that
 26 this is a severe restriction. However, from the one hand, we relax such an assumption
 27 to some extent by considering, first, partial orders on the action sets together with
 28 quasisupermodular utility functions and, second, partial orders on the player (type)
 29 sets together with a comparability property that limits the numerosity of uncomparable
 30 players (types). On the other hand, we think that the strict single crossing property
 31 is less demanding when we come to applied models, where instead the possibility to
 32 work with action spaces such as the real line (or its intervals) is usually appreciated.

33 Our main contribution is the identification of conditions that guarantee that every
 34 Nash equilibrium is essentially strict ([Theorem 1](#)). However, the same conditions do
 35 not guarantee that a Nash equilibrium actually exists. To obtain existence of essen-
 36 tially strict Nash equilibria, one can apply our result together with one of the many
 37 equilibrium existence theorems that the literature provides. Actually, we follow this
 38 line in [Sect. 4](#), where in [Sects. 4.1](#) and [4.2](#) we provide applications of our main result to
 39 incomplete information games and large games. In particular, we show the existence
 40 of an evolutionarily stable strategy in a general class of incomplete information games,
 41 and strict Nash equilibrium in a class of population games with heterogeneous players.

42 The paper is organized as follows. In [Sect. 2](#), we introduce the assumptions. In
 43 [Sect. 3](#), we state our main result. We conclude with [Sect. 4](#), where we provide a
 44 discussion, first showing how to combine our main result with existence theorems and
 45 then commenting on the assumptions and the findings. The “Appendix” collects one
 46 technical lemma ([Lemma 1](#)), its proof, and the proof of [Proposition 2](#).

47 2 Assumptions

48 Let us consider a non-atomic game $\Gamma = \langle I, \{(T_i, \bar{T}_i, \tau_i)\}_{i \in I}, \{A_i\}_{i \in I}, \{u_i\}_{i \in I} \rangle$, where:

¹ When working with a finite set of actions, strict Nash equilibria have been proven to be evolutionary stable (see, e.g., [Crawford 1990](#)) and asymptotically stable (see, e.g., [Ritzberger and Weibull 1995](#)).

² Second countability implies a cardinality less than or equal to the cardinality of the continuum.

³ The separation property that we assume ensures that every two actions that can be strictly ordered can also be separated by a third action not greater than the largest of the two.

- 49 • I is a finite set of player groups or institutions;⁴
- 50 • for all $i \in I$, $(T_i, \mathcal{T}_i, \tau_i)$ is an atomless probability space with T_i set of players for
- 51 group/institution i , \mathcal{T}_i σ -algebra and τ_i probability measure;⁵
- 52 • for all $i \in I$, A_i is the set of actions for players in group i ;
- 53 • for all $i \in I$, $u_i : T_i \times F \rightarrow \mathbb{R}$ is the utility function for all players of group i ,
- 54 where $F = \prod_{j \in I} \prod_{t \in T_j} A_j$.

55 We call $f \in F$ a profile of actions, since it maps, for all $i \in I$, every player $t \in T_i$

56 into an action $f_t \in A_i$.⁶

57 We denote with f_{-t} the restriction of f to $F_{-t} = \prod_{j \in N} \prod_{t' \in T_j, t' \neq t} A_j$, and we

58 call it a profile of actions by players other than t .⁷ We write $u_i(t, a, f_{-t})$ to indicate

59 the utility accruing to player $t \in T_i$ if she chooses action $a \in A_i$ and faces a profile of

60 actions f_{-t} .

61 We now introduce assumptions on $\{(T_i, \mathcal{T}_i, \tau_i)\}_{i \in I}$ (collected in AT), on $\{A_i\}_{i \in I}$

62 (collected in AA), and on $\{u_i\}_{i \in I}$ (collected in AU).

63 **Assumption (AT).** For all $i \in I$:

64 AT1 (T_i, \leq_i^T) is a partial order;

65 AT2 for every $T' \subseteq T_i$ such that there do not exist $t, t' \in T'$ with either $t \leq_i^T t'$ or

66 $t' \leq_i^T t$, we have that (1) $T' \in \mathcal{T}_i$, and (2) $\tau_i(T') = 0$.

67 Assumption AT2 provides a bound on the cardinality of sets of uncomparable play-

68 ers, basically requiring for each T_i that any subset of players such that every pair is

69 uncomparable has negligible size. Indeed, the possibility that some players are not

70 comparable is left open by AT1, since the order may not be total. We observe that AT2

71 is trivially satisfied when (T_i, \leq_i^T) is a linear order. More interestingly, AT2 allows us

72 to consider other cases that might be of interest in applications. For instance, think of

73 T_i as made of a finite or countably infinite number of populations, where comparability

74 is within each population, but not across populations. This is not allowed if T_i is a

75 linear order, while it is compatible with our assumption. Moreover, AT2 is satisfied

76 if, for every $i \in I$, T_i is made by a subset of a multidimensional Euclidean space, as

77 shown in the Proof of Proposition 2.

78 **Assumption (AA).** For all $i \in I$:

79 AA1 (A_i, \leq_i^A) is a lattice, i.e., for every two actions $a, a' \in A_i$, there exists the least

80 upper bound $a \vee a'$, and the greatest lower bound $a \wedge a'$;

⁴ Here we follow the labeling proposed by Khan and Sun (2002), which allows to encompass both games with many players and games with incomplete information.

⁵ For games with incomplete information, the set I of groups/institutions has to be interpreted as the set of players, while the set of players T_i has to be interpreted as the set of types for player $i \in I$.

⁶ We note that, under this definition of F as uncountable cross product of action sets, measurability issues can emerge. These issues cannot be settled without imposing further structure, that is however unnecessary for our main result. Therefore, we choose to take care of measurability only in the applications of Sect. 4.

⁷ In case of incomplete information games (where i is interpreted as a player and T_i as her set of types), player i has already known her type t when computing expected utility. So, it is redundant to consider the actions that would be taken by types in $T_i \setminus \{t\}$, and hence, we have to require that $u_i(t, f)$ is constant over the actions chosen by types $t' \in T_i \setminus \{t\}$.

- 81 AA2 (A_i, \mathcal{S}_i) is a topological space;
 82 AA3 (A_i, \mathcal{S}_i) is second countable, i.e., there exists a countable base for topology \mathcal{S}_i ;
 83 AA4 (A_i, \mathcal{S}_i) is such that for every two actions $a, a' \in A_i$, with $a <_i^A a'$, there exists
 84 an open set $S \in \mathcal{S}_i$ such that $a' \in S$ and $a'' \notin S$ for every $a'' \leq_i^A a$.

85 Beyond imposing a lattice structure (AA1) and a topological structure (AA2) on the
 86 action space, AA contains two further topological properties: AA3, which is a standard
 87 assumption that imposes a bound on the topological size of the space, and AA4, which
 88 is about order separation with respect to the lattice structure and turns out to be a
 89 strengthening of the axiom of separation T0.⁸

90 **Assumption (AU).** For all $f \in F, i \in I, t, t' \in T_i$, and $a, a' \in A_i$:

- 91 AU1 u_i is quasisupermodular in actions, i.e., $u_i(t, a, f_{-i}) \geq u_i(t, a \wedge a', f_{-i})$ implies
 92 $u_i(t, a \vee a', f_{-i}) \geq u_i(t, a', f_{-i})$, and $u_i(t, a, f_{-i}) > u_i(t, a \wedge a', f_{-i})$ implies
 93 $u_i(t, a \vee a', f_{-i}) > u_i(t, a', f_{-i})$;
 94 AU2 u_i satisfies strict single crossing in players and actions, i.e., for all $t <_i^T t'$ and
 95 $a <_i^A a'$, we have that $u_i(t, a', f_{-i}) \geq u_i(t, a, f_{-i})$ implies $u_i(t', a', f_{-i}) >$
 96 $u_i(t', a, f_{-i})$.

97 Assumption AU1 is always satisfied when A_i is a total order, while it implies a sort of
 98 complementarity in own actions when A_i is a partial order, as for instance when $A_i =$
 99 $[0, 1]^k$ for some $k > 1$. Assumption AU2, instead, introduces a sort of complementarity
 100 between actions and players.⁹

101 Finally, we present some further definitions. A profile of actions $f \in F$ is said to
 102 be (essentially) a *Nash equilibrium in pure strategies*, or simply a *Nash equilibrium*,
 103 if, for all $i \in I$, for τ_i -almost all $t \in T_i$, we have that $u_i(t, f_i, f_{-i}) \geq u_i(t, a, f_{-i})$ for
 104 all $a \in A_i$. A Nash equilibrium f is said to be *essentially strict* if, for all $i \in I$, for
 105 τ_i -almost all $t \in T_i$, we have that $u_i(t, f_i, f_{-i}) > u_i(t, a, f_{-i})$ for $a \neq f_i$ such that
 106 $a_i \in A_i$, while it is said to be *monotone* if, for all $i \in I$, for all $t, t' \in T_i$, we have that
 107 $t' >_i^T t$ implies $f_{i'} \geq_i^A f_i$.

108 3 Main result

109 We are ready to state our main result.

110 **Theorem 1** *Let Γ be a game that satisfies AT, AA, and AU. Then, every Nash equi-*
 111 *librium of Γ is essentially strict and monotone.*

⁸ T0 requires that any two distinct points in a set are topologically distinguishable, i.e., the sets of neighborhoods of the two points differ one from the other.

⁹ We note that AU2 is slightly different from the standard definition of strict single crossing property since the profile of opponents' actions, which is a third argument of function u in addition to t and f_i , is not exactly the same in f_{-i} and $f_{-i'}$: Indeed, the behavior of players different from t and t' is the same, while the behavior of t is considered in f_{-i} but not in $f_{-i'}$, and the behavior of t' is considered in $f_{-i'}$ but not in f_{-i} . This difference disappears if, for instance, we assume individual negligibility (see discussion at the end of Sect. 3) or if we constrain players to care only about actions of groups/institutions different from theirs (as it happens, e.g., in games with incomplete information).

112 *Proof* We first show that every Nash equilibrium is essentially strict. Let $R_{i,t}(f)$
 113 denote the set of best replies to f for player $t \in T_i$, namely $R_{i,t}(f) = \{a \in A_i : u_i(t, a, f_{-t}) \geq u(t, a', f_{-t})$ for all $a' \in A_i\}$. By Lemma 1 (see “Appendix I”), we
 114 know that, for all $i \in I$, the set $\{t \in T_i : \|R_{i,t}(f)\| > 1\}$ is a countable union of sets
 115 having measure zero. Since the countable union of zero-measure sets has measure zero,
 116 we can conclude that $\tau_i(\{t \in T_i : \|R_{i,t}(f)\| > 1\}) = 0$ for all $i \in I$. This, together
 117 with the observation that when f is a Nash equilibrium we have $\|R_{i,t}(f)\| > 0$ for
 118 τ_i -almost all $t \in T_i$ and for all $i \in I$, implies that $u_i(t, f_t, f_{-t}) > u_i(t, a, f_{-t})$ for
 119 τ_i -almost all $t \in T_i$ and for all $i \in I$.

121 We now show that every Nash equilibrium is monotone. Suppose that $t' >_i^T t$,
 122 $a \in R_{i,t}(f)$, $a' \in R_{i,t'}(f)$ and, ad absurdum, $a \not\leq_i^A a'$. Since $a \in R_t(f)$, we have
 123 that $u(t, a, f_{-t}) \geq u(t, a \wedge a', f_{-t})$, but then $u(t, a \vee a', f_{-t}) \geq u(t, a', f_{-t})$ by
 124 quasisupermodularity in actions, and $u(t', a \vee a', f_{-t}) > u(t', a', f_{-t})$ by strict single
 125 crossing property in players and actions and $a \vee a' \neq a'$, which in turn comes from
 126 $a \not\leq_i^A a'$. We simply observe that $u(t', a \vee a', f_{-t}) > u(t', a', f_{-t})$ is in contradiction
 127 with $a' \in R_{i,t'}(f)$. \square

128 The fact that f is essentially strict follows from $(T_i, \mathcal{T}_i, \tau_i)$ being atomless for all $i \in I$
 129 and from the set of weakly best responders being countable. Then, a straightforward
 130 application of the property of strict single crossing in players and actions allows
 131 establishing the monotonicity between players and actions in Nash equilibria—this
 132 result following basically from Theorem 4' of Milgrom and Shannon (1994).

133 Let us conclude with a remark on players' negligibility. In Theorem 1, utility
 134 depends on the actions of each single player $t \in T_i$, $i \in I$. We did this in order
 135 to state our findings in a setting which allows for a general form of utility functions.
 136 However, we note that when we have an atomless space of players, it may be reason-
 137 able to impose that any single player $j \neq t$ is negligible in terms of t 's utility.
 138 This assumption is particularly reasonable if one also assumes continuity of the utility
 139 function (see the discussion in Khan and Sun 2002, Section 2). To introduce negligi-
 140 bility in our framework, it suffices to impose that, for all $i \in I$, the utility function u_i
 141 is such that whenever $f, f' \in F$ agree on a set of measure one according to τ_i , we
 142 have that $u_i(t, f) = u_i(t, f')$ for every $t \in T_i$, such that $f_t = f'_t$. We observe that
 143 such kind of players' negligibility is implied in the applications of Sect. 4.

144 4 Discussion

145 The celebrated result in Harsanyi (1973) says that independently perturbing the payoffs
 146 of a finite normal form game produces an incomplete information game with a contin-
 147 uum of types where all equilibria are essentially pure and essentially strict¹⁰ and that
 for any regular equilibrium of the original game and any sequence of perturbed games

¹⁰ Note that strict Nash equilibria are called strong Nash equilibria in Harsanyi (1973).

148 converging to the original one, there is a sequence of essentially pure and essentially
 149 strict equilibria converging to the regular equilibrium.^{11, 12}

150 For Theorem 1 to have some bite, it needs to be coupled with a result guaranteeing
 151 the existence of a pure-strategy Nash equilibrium. The literatures on incomplete infor-
 152 mation games and large games have provided several of such existence results. We
 153 first discuss some known existence results in non-atomic games. Then, in Sects. 4.1
 154 and 4.2, we illustrate how our contribution can be used to shed light on the strictness
 155 of Nash equilibria in applications to incomplete information games and large games,
 156 respectively. Unless otherwise specified, any topological space in this section is under-
 157 stood to be equipped with its Borel σ -algebra, and the measurability is defined based
 158 on it. Finally, in Sect. 4.3, we comment on the assumptions used in the paper, arguing
 159 in favor of their tightness.

160 The use of single crossing properties is not new in the literature on games with
 161 many player types. [Athey \(2001\)](#) analyzes games of incomplete information where
 162 each agent has private information about her own type, and the types are drawn from
 163 an atomless joint probability distribution. The main result establishes the existence of
 164 pure Nash equilibria under an assumption called single crossing condition for games
 165 of incomplete information, which is a weak version of the single crossing property in
 166 [Milgrom and Shannon \(1994\)](#).¹³ In Sect. 4.3, we argue that such a property is not a
 167 sensible generalization for our purposes.

168 In a finite-player incomplete information game with diffused information, if in
 169 addition players' information is independent (instead of assuming an order structure),
 170 the existence of a pure Nash equilibrium can be established similarly to the one in a
 171 large game (with a non-atomic space of players). It is now well recognized (see [Khan
 172 et al. 2006](#)) that the purification principle due to [Dvoretzky et al. \(1951\)](#) guarantees
 173 the existence of pure Nash equilibria in non-atomic games¹⁴ when the action space is
 174 finite as, for example, in large games like [Schmeidler \(1973\)](#), or in games with diffused
 175 information as in [Radner and Rosenthal \(1982\)](#) and [Milgrom and Weber \(1985\)](#) (see
 176 [Khan and Sun 2002](#), for a survey on games with many players).^{15, 16} Existence of pure
 177 Nash equilibria does not extend, however, to general games. For action spaces that

¹¹ See also [Dubey et al. \(1980\)](#) for a related use of strict equilibria in large games.

¹² The work of [Harsanyi \(1973\)](#) has been extended by a series of contributions providing more general conditions for the existence of pure equilibria, but disregarding the issue of approachability and the existence of strict equilibria (see [Morris 2008](#) and references there in).

¹³ [Reny and Zamir \(2004\)](#) prove the existence of pure-strategy Nash equilibria under a slightly weaker condition. [McAdams \(2003\)](#) further extends the analysis to multidimensional type spaces and action spaces, while [Reny \(2011\)](#) extends it to more general partially ordered type spaces and action spaces.

¹⁴ Interest in games with many players has recently spanned across different settings (see, e.g., [Alós-Ferrer and Ritzberger 2013](#), for extensive form games and [Balbus et al. 2013](#), for games with differential information), and different notions of equilibrium (see, e.g., [Correa and Torres-Martínez 2014](#), can exist when the make for essential equilibria).

¹⁵ [Mas-Colell \(1984\)](#) deals with the issue of [Schmeidler \(1973\)](#) using a different approach based on distributions rather than measurable functions. See [Khan et al. \(2013b\)](#) for a recent discussion of related issues.

¹⁶ Approximated versions of the result in [Schmeidler \(1973\)](#) have been given for a large but finite number of players ([Rashid 1983](#); [Carmona 2004, 2008](#)).

178 are countable and compact, conditions for the existence of pure Nash equilibrium are
 179 given in [Khan and Sun \(1995\)](#) and then generalized in [Yu and Zhang \(2007\)](#). When
 180 the action space is an uncountable compact metric space, saturated probability spaces
 181 can be used to guarantee the existence of a pure-strategy Nash equilibrium, as shown
 182 in [Keisler and Sun \(2009\)](#) and [Khan et al. \(2013a\)](#).¹⁷

183 4.1 An application to incomplete information games

184 We now show how [Theorem 1](#) can be used to shed light on the strictness of a Nash
 185 equilibrium in a Bayesian setting. We use the setup given by [McAdams \(2003\)](#),¹⁸
 186 which is a generalization of the one in [Athey \(2001\)](#). More precisely, we consider the
 187 incomplete information game $\Gamma^I = \langle I, ([0, 1]^h, \phi), A, \{u_i\}_{i \in I} \rangle$, where:

- 188 • I is the set of players with cardinality $|I| = n \in \mathbb{N}$;
- 189 • for all $i \in I$, $([0, 1]^h, \phi)$ describes the h -dimensional common type space, with
 190 $\phi : \mathbb{R}^{nh} \rightarrow \mathbb{R}_{++}$ the positive and bounded joint density on type profiles;
- 191 • for all $i \in I$, $A \subset \mathbb{R}^k$ is the set of actions for types of player i ,¹⁹ with A being
 192 either a finite sublattice with respect to the product order or $[0, 1]^k$;
- 193 • for all $i \in I$, $u_i^I(t_i, a_i, \alpha_{-i}) = \int_{[0, 1]^{h(n-1)}} U_i(a_i, \alpha_{-i}(\mathbf{t}_{-i})) \phi(\mathbf{t}_{-i} | t_i) d\mathbf{t}_{-i}$ is the util-
 194 ity function for all types of i , where $\alpha_{-i}(\mathbf{t}_{-i})$ is the vector of others' actions as a
 195 function of their type, \mathbf{t}_{-i} is the vector of others' types, $\phi(\mathbf{t}_{-i} | t_i)$ is the conditional
 196 density of \mathbf{t}_{-i} given t_i , and U_i is bounded, Lebesgue measurable and, if $A = [0, 1]^k$,
 197 also continuous in $\mathbf{a} \in A^n$.

198 In Γ^I a strategy for player i can be described by function $\alpha_i : [0, 1]^h \rightarrow A$. So, we can
 199 say that a strategy profile $(\alpha_1, \dots, \alpha_n)$ is a *Nash equilibrium* of game Γ^I if it induces
 200 a profile of actions such that for all $i \in I$, for all $t \in [0, 1]^h$, $u_i(t, \alpha_i(t), \alpha_{-i}) \geq$
 201 $u_i(t, a, \alpha_{-i})$ for all $a \in A$.

202 By construction, Γ^I satisfies AA and AT. So, if Γ^I also satisfies AU, then by virtue
 203 of our [Theorem 1](#) every Nash equilibrium of Γ^I is essentially strict, and monotone in
 204 types and actions. Moreover, existence of a Nash equilibrium follows from [Theorem 1](#)
 205 in [McAdams \(2003\)](#) that can be applied since AU2 implies the single crossing
 206 condition—which is required by the [Theorem](#).

207 Perhaps more interestingly, we can use the setup of incomplete information games
 208 to show what [Theorem 1](#) can say from the perspective of evolutionary game theory.²⁰
 209 Indeed, although the notion of evolutionarily stable strategy remains a prominent
 210 solution concept in evolutionary game theory, its use has some shortcomings when

¹⁷ See [Carmona and Podczeck \(2009\)](#) for a discussion on the relationship between alternative formalizations of non-atomic games and existence results, with a focus on large games. See also [Fu and Yu \(2015\)](#) for a discussion of the connection between the class of large games and the class of finite-player Bayesian games.

¹⁸ [McAdams \(2006\)](#) applies and extends this setup to prove existence of pure Nash equilibria in multiunit auctions.

¹⁹ As noted by [McAdams \(2003\)](#), the assumptions of a common support for types and a common set for actions are just for notational simplicity and can be safely removed.

²⁰ Evolution in the context of Bayesian games is analyzed in [Ely and Sandholm \(2005\)](#) and [Sandholm \(2007\)](#).

211 continuous strategy spaces are employed.²¹ If an order structure is imposed on types,
 212 our Theorem 1 can allow to tackle the issue. This follows a seminal idea in Riley (1979),
 213 where incomplete information and a form of the strict single crossing property are used
 214 to show existence of an evolutionarily stable strategy in the “war of attrition”.

215 For this purpose, we restrict attention to a game Γ^I that is symmetric, i.e., we focus
 216 on game $\Gamma^{IS} = \langle I, ([0, 1]^h, \phi), A, u \rangle$. We also provide some further useful notation
 217 and definitions.

218 The following expression denotes ex-ante utility for a player choosing strategy α
 219 when all other players choose strategy α' :

$$220 \quad V(\alpha, \alpha') = \int_{[0,1]^h} \left(\int_{[0,1]^{h(n-1)}} U(\alpha(t), \alpha'_{-i}(\mathbf{t}_{-i})) \phi(\mathbf{t}_{-i}|t) d\mathbf{t}_{-i} \right) \phi_i(t) dt,$$

221 where $\phi_i(t)$ is the marginal density function of types for player i .

222 Given two strategies α, α' , we define $D(\alpha, \alpha')$ as the set of types that pick different
 223 actions in α and α' , i.e., $D(\alpha, \alpha') = \{t \in [0, 1]^h : \alpha(t) \neq \alpha'(t)\}$.

224 The following definition adapts the standard definition of evolutionarily stable strategy
 225 to our setup. A strategy α is an *evolutionarily stable strategy* (henceforth, ESS) if
 226 and only if there exists $\epsilon > 0$ such that, for all α' such that $\int_{D(\alpha, \alpha')} \phi_i > 0$:

$$227 \quad (1 - \epsilon)V(\alpha, \alpha) + \epsilon V(\alpha, \alpha') > (1 - \epsilon)V(\alpha', \alpha) + \epsilon V(\alpha', \alpha').$$

228 Basically, the above definition requires that a strategy performs strictly better than any
 229 invading strategy that differs non-negligibly from the incumbent strategy.

230 While an evolutionarily stable strategy may not exist in general, we are able to
 231 prove the following result (see “Appendix 2” for the proof).

232 **Proposition 2** *Suppose Γ^{IS} satisfies AU. Then, (1) every pure-strategy Nash equi-*
 233 *librium is an evolutionarily stable strategy, and (2) an evolutionarily stable strategy*
 234 *exists.*

235 We observe that our Proposition 2 is not implied by the Harsanyi’s purification theorem,
 236 which applies only to games with a finite number of strategies for each player, while
 237 we allow for continuous strategies as well.

238 4.2 An application to large games

239 A pure Nash equilibrium is not necessarily a strict Nash equilibrium, so our Theorem 1
 240 can be usefully employed to establish Nash strictness in games where this is a desirable
 241 property (e.g., in games where the local stability of a Nash equilibrium is a crucial
 242 property). Below, we provide an example of such applicability.

243 Consider the following game, which is an instance of the class of games considered
 244 in Khan et al. (2013a) (see discussion at p. 1130), and that represents a slight gen-
 245 eralization of a static population game (see Sandholm 2010, for a formal definition

²¹ Alternative notions of evolutionary stability have been proposed in the literature (Vickers and Canning 1987; Bomze and Pötscher 1989; Oechssler and Riedel 2001, 2002).

of population games). There is a large population of heterogeneous players whose characteristics consist of both an individual payoff structure and an ordered numerical trait, with a player's payoff depending on own action and societal summary of actions traits. In particular, a player's payoff depends on her own action and type as well as the sum of the traits of the players choosing each action.²² Formally, consider the game $\Gamma^P = \langle ([\underline{t}, \bar{t}], \phi), (B, \beta), A, u^P \rangle$ where:

- there is a unit-mass population of players distributed over $[\underline{t}, \bar{t}]$ according to the positive and bounded probability density ϕ ;
- $B = \{b_1, \dots, b_n\}$ is a finite and totally ordered set of traits, with $\beta : [\underline{t}, \bar{t}] \rightarrow B$ a measurable function that assigns each player to a trait;
- $A = \{1, \dots, m\}$ is a finite and totally ordered set of actions, common to all players;
- $u^P(t, a, \alpha) = U(t, a, (\sigma_{11}, \dots, \sigma_{mn}))$ is agents' utility function, which we assume to be measurable in t and continuous in $(\sigma_{11}, \dots, \sigma_{mn})$, and where $\alpha : [\underline{t}, \bar{t}] \rightarrow A$ is a measurable function representing the actions chosen by every player in the population, and $\sigma_{jk} = \int_{(\alpha, \beta)^{-1}(j, b_k)} \phi t$ measures the amount of players with trait b_k who play action $j \in A$.

We observe that if, in addition to AA and AT which are satisfied by construction, Γ^P also satisfies AU, then Theorem 1 implies that every Nash equilibrium of Γ^P is essentially strict, and monotone in players and actions.²³ So, we know that all Nash equilibria of Γ^P are locally stable with respect to dynamics typically applied in population games (see e.g., Sandholm 2015).

We think that considering the heterogeneity of characteristics in a population is a natural addition to population games. Also, assuming the strict single crossing property in players and actions appears to us, at least in some cases, a reasonable hypothesis. Think of this variant of a congestion game, where the trait is the length of the car possessed, and the congestion along a route depends on the overall length of cars in that route. If a longer route is preferred by the owner of some car, then it means that the shorter route has heavier traffic. Hence, it is reasonable to assume that the owners of longer cars prefer a fortiori the shorter route, since a larger car typically performs relatively worse under heavy traffic.

4.3 Discussion of assumptions

Negligibility of sets of uncomparable players (AT2) This assumption cannot be dispensed with, in the sense that a positive measure of uncomparable players would allow the existence of Nash equilibria that are not essentially strict. Indeed, if there exists a non-negligible set of players such that every pair cannot be ordered, then the strict single crossing property cannot be employed to rule out that all such players are weakly best responders in equilibrium, and therefore, Nash equilibria need not be essentially strict. The following example illustrates why. Let $\|I\| = 1$, and let the set of actions

²² This last assumption can be easily generalized to any form of trait aggregation, in the same way as it is typically done for aggregative games (see, e.g., Acemoglu and Jensen 2013).

²³ We also note that the existence of a Nash equilibrium is not an issue in this game, e.g., one can invoke Theorem 1, point (i), in Khan et al. (2013a).

284 A be equal to the real segment $[0, 1]$. Also, let the set T be such that no $t, t' \in T$
 285 are comparable, so that AT1 is trivially satisfied while AT2 fails. Finally, suppose that
 286 $u_i(t, f) = \tau(\{t' : f_{t'} = f_t\})$, meaning that t 's payoff only depends on the fraction
 287 of players coordinating on her action f_t . It is straightforward to see that any profile
 288 where a measure of $\tau(T)/k$ players coordinate on k distinct actions (with k a natural
 289 number) is a Nash equilibrium, since each t obtains a payoff of $\tau(T)/k$ which cannot
 290 be improved upon by deviating. However, for $k \geq 2$, all $t \in T$ are indifferent between
 291 any of the k actions played, and so the Nash equilibrium is not essentially strict.

292 *Separability versus second countability (AA3)* A space is called separable if it contains
 293 a countable dense subset. Separability is a topological property which is weaker than
 294 second countability but plays a similar role: It constrains the topological size of the
 295 space.

296 However, if we assume that the action sets are separable instead of second countable,
 297 then our results fail. The following example, which is a modification of a standard
 298 argument to illustrate that a separable space need not be second countable, shows
 299 that if we replace second countability with separability then there may exist Nash
 300 equilibria that are not essentially strict. We consider a unique group of players, and we
 301 let the set T be the real line, denoted with \mathbb{R} . We let the action set A be the Cartesian
 302 product $\mathbb{R} \times \{0, 1\}$. We give A the lexicographic order, i.e., $(r, i) < (s, j)$ if either
 303 $r < s$ or else $r = s$ and $i < j$. For every profile of actions f , t 's utility function is
 304 $u(t, f) = -(t - f_t)^2$, where $f_t = s$ if $f_i = (s, i)$. In the order topology, A is separable:
 305 The set of all points $(q, 0)$ with q rational is a countable dense set. However, f such
 306 that $f_t = (t, 0)$ for all $t \in T$ is a Nash equilibrium that is not essentially strict since
 307 every agent t is indifferent between $(t, 0)$ and $(t, 1)$.

308 *Axioms of separation (T0, T1) versus order separation (AA4)* Intuitively, our assumption
 309 on order separation ensures that different weakly best responders can be assigned
 310 to actions that are substantially different, in the sense that each action can be associated
 311 with a distinct base set. Then, second countability of the action set ensures that this
 312 function relating actions to base sets is enumerable.

313 One might hope to weaken our assumption to something that is more in line with
 314 standard separation axioms (like T0 or T1): For all $a > a'$, there exists an open set
 315 $S(a, a')$ such that $a \in S(a, a')$ and $a' \notin S(a, a')$. However, we stress that this attempt
 316 would contrast with our technique of proof. Indeed, following the Proof of Lemma 1
 317 (see the ‘‘Appendix’’), a t that is a weakly best responder might be associated with a
 318 set \widehat{S} obtained as $\bigcap_{t' \in R_{i,t}(f), t > t'} S(g_{i,1}(t), g_{i,1}(t'))$. But then an infinite intersection
 319 of open sets need not be open, and this does not allow us to conclude that a base set
 320 exists that is included in \widehat{S} and contains the action $g_{i,1}(t)$.

321 *Single crossing versus strict single crossing (AU2)* Games of incomplete information
 322 are a very important class of games where single crossing properties are usually
 323 assumed in order to prove existence of pure Nash equilibria. In these cases, we can
 324 apply our Theorem 1 to obtain the existence of an essentially strict Nash equilibrium
 325 (see Sect. 4.1). We stress that this result is based on a strict version of the single cross-

ing property, while existence results in games of incomplete information (Athey 2001; McAdams 2003) use weaker assumptions. In particular, they are weaker under two respects. First, they assume single crossing instead of strict single crossing. Second, they require that the property of single crossing holds on a smaller domain: for each player, whenever all other players adopt strategies such that higher types take higher actions. Therefore, one may wonder whether our results still hold if we consider each of the two weakenings of strict single crossing. With respect to the first weakening, the following straightforward counterexample shows that single crossing is not enough. Assume that every agent has a constant utility function, so that everyone is always indifferent between any of her actions. Single crossing property is satisfied, and whatever profile of actions is a weak Nash equilibrium. This trivial example also shows that we cannot recover our main result even if we replace the property of single crossing with the stronger one of increasing differences—i.e., for all $f \in F, i \in I, t' >_i^T t$ and $a' >_i^A a$, we have that $u(t, a', f_{-i}) - u(t, a, f_{-i}) \leq u(t', a', f_{-i'}) - u(t', a, f_{-i'})$.

With respect to the second weakening, we observe that restricting the domain to profiles that are monotone in types and actions for other players is a clever generalization of single crossing when the purpose is to prove the existence of pure Nash equilibria. However, a strict version of this weaker property of single crossing does not work when we want to show that every Nash equilibrium is essentially strict. The reason is that it would allow the existence of some weak Nash equilibrium with a profile of actions for which no property of strict single crossing must hold.

Strict increasing difference versus strict single crossing (AU2) One may wonder whether the result in Theorem 1 can be refined to prove strict monotonicity instead of monotonicity. It turns out that this is not the case, even if we adopt the stronger property of strict increasing differences in players (or types) and actions—i.e., for all $f \in F, i \in I, t' >_i^T t$ and $a' >_i^A a$, we have that $u(t, a', f_{-i}) - u(t, a, f_{-i}) < u(t', a', f_{-i'}) - u(t', a, f_{-i'})$ —instead of strict single crossing. The following example illustrates why. Let $\|I\| = 1$ and let both set T and set A be equal to the real segment $[0, 1]$. For every profile of actions f , the utility function of t is $u(t, f) = (1 + t)f_t$. It is clear that there exists a unique Nash equilibrium where everybody plays action 1. Hence, monotonicity holds, but strict monotonicity does not.

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Appendix 1: Lemma 1 and its proof

A key result for the Proof of Theorem 1 is that any set of weakly best responders is a countable union of sets having measure zero. Lemma 1 below provides such result.

The logic of the Proof of Lemma 1 goes as follows. The joint use of quasisupermodularity in actions (AU1) and strict single crossing in players and actions (AU2) is similar to that in Theorem 4 of Milgrom and Shannon (1994), and it allows to arrange multiple best replies of different players in a linear order. The crucial economic assumption is the strict single crossing property in players and actions, which

implies that the sets of weakly best replies of any two distinct players intersect at most at an extreme point and hence are—roughly speaking—rather separated one from the other. The technical assumptions on countability (AA3) and separation (AA4) complete the job, allowing at most a countable number of such sets (see Sect. 4.3 for a discussion on the importance of the countability and separation properties). Therefore, there can exist only a countable number of comparable players that are weakly best responders; for any such player, there can be many (even uncountable) players that are all uncomparable and weakly best responders, but for the comparability assumption (AT2) their measure is null. This leads to conclude that the set of weakly best responders is formed by countably many sets having measure zero, and hence, its measure is zero as well.

Preliminarily, we define $R_{i,t}(f)$ as the set of best replies to f for $t \in T_i$, namely $R_{i,t}(f) = \{a \in A_i : u_i(t, a, f_{-t}) \geq u(t, a', f_{-t}) \text{ for all } a' \in A_i\}$.

Lemma 1 *Let Γ be a game that satisfies AT, AA, and AU. Then, for every $i \in I$, $\{t \in T_i : \|R_{i,t}(f)\| > 1\}$ is a countable union of sets having measure zero.*

Proof This is the outline of the proof. For a generic $i \in I$, first we define a function g_i that maps every $t \in \{t \in T_i : \|R_{i,t}(f)\| > 1\}$ into a pair (a, a') of her best replies, then we define a function h_i , and we use it to assign (a, a') to a base set. We show that function h_i is injective and that function g_i is such that any set of players assigned to the same pair of actions has measure zero. Finally, we invoke the fact that there exists only a countable number of base sets to obtain the desired result.

For each $i \in I$, we consider the partial orders assumed in AA1 (lattice structure) and AT1 (partial ordering) and we take a function $g_i : \{t \in T_i : \|R_{i,t}(f)\| > 1\} \rightarrow A_i^2$ such that $g_i(t) = (g_{i,0}(t), g_{i,1}(t))$ with $g_{i,0}(t), g_{i,1}(t) \in R_{i,t}(f)$, $g_{i,0}(t) \leq_i^A g_{i,1}(t)$, and $g_{i,1}(t) \leq_i^A g_{i,0}(t')$ for $t' >_i^T t$. The following two arguments show that such a function exists for each $i \in I$. First, $a \in R_{i,t}(f)$ and $a' \in R_{i,t}(f)$ imply $a \vee a' \in R_{i,t}(f)$, so that we can set $g_{i,0}(t) = a$ and $g_{i,1}(t) = a \vee a'$, with $a \vee a'$ existing thanks to AA1 (lattice structure). In fact, $u_i(t, a, f_{-t}) \geq u_i(t, a \wedge a', f_{-t})$ since $a \in R_{i,t}(f)$, and hence, $u_i(t, a \vee a', f_{-t}) \geq u_i(t, a', f_{-t})$ by AU1 (quasisupermodularity in actions), which in turn implies that $u_i(t, a \vee a', f_{-t}) = u_i(t, a, f_{-t}) = u_i(t, a', f_{-t})$ since $a \in R_{i,t}(f)$ and $a' \in R_{i,t}(f)$. Second, $a \in R_{i,t}(f)$ and $a' \in R_{i,t'}(f)$ for $t' >_i^T t$ imply $a \leq_i^A a'$. This is true since $u_i(t, a, f_{-t}) \geq u_i(t, a \wedge a', f_{-t})$ due to $a \in R_{i,t}(f)$, and hence, $u_i(t, a \vee a', f_{-t}) \geq u_i(t, a', f_{-t})$ by AU1 (quasisupermodularity in actions), and therefore, $u_i(t', a \vee a', f_{-t'}) > u_i(t', a', f_{-t'})$ by AU2 (strict single crossing in players and actions), with $a \wedge a'$ existing thanks to AA1 (lattice structure).

For all $i \in I$, by AA2 (topology structure), A_i has a topology and by AA3 (second countability) we can take a countable base \mathcal{B}_i for such a topology. For each $i \in I$, we take a function $h_i : g_i(\{t \in T_i : \|R_{i,t}(f)\| > 1\}) \rightarrow \mathcal{B}_i$ such that $a_1 \in h_i(a_0, a_1)$ and $a \notin h_i(a_0, a_1)$ for all $a \leq_i^A a_0$. To see that such a function h_i exists, note that by AA4 (order separation) for each $(a_0, a_1) \in g_i(\{t \in T_i : \|R_{i,t}(f)\| > 1\})$ there exists some open set $S_{a_1} \subset A_i$ such that $a_1 \in S$ and $a \notin S$ for all $a \leq_i^A a_0$; since \mathcal{B}_i is a base, there must exist some $B_{a_1} \in \mathcal{B}_i$ such that $a_1 \in B_{a_1}$ and $B_{a_1} \subseteq S_{a_1}$. We set $h_i(a_0, a_1) = B_{a_1}$.

410 We check that, for all $i \in I$, g_i is such that, for all $(a, a') \in A_i^2$, $g_i^{-1}(a, a')$ has
 411 measure zero. For all $t, t' \in \{t \in T_i : \|R_{i,t}(f)\| > 1\}$, $t <_i^T t'$, we have that
 412 $g_{i,0}(t) < g_{i,1}(t) \leq g_{i,0}(t') < g_{i,1}(t')$ from the definition of function g_i . Therefore,
 413 $t, t' \in g_i^{-1}(a, a')$ implies $t \not<_i^T t'$ and $t' \not<_i^T t$, and AT2 (negligibility of sets of
 414 uncomparable players) guarantees that $\tau_i(g_i^{-1}(a, a')) = 0$.

415 We check that, for all $i \in I$, h_i is injective. For all $(a_0, a_1), (a'_0, a'_1) \in g_i(\{t \in T_i : \|R_{i,t}(f)\| > 1\})$, $(a_0, a_1) \neq (a'_0, a'_1)$, we know that either $a_0 < a_1 \leq a'_0 < a'_1$ or
 416 $a'_0 < a'_1 \leq a_0 < a_1$. Suppose, without loss of generality, that $a_0 < a_1 \leq a'_0 < a'_1$.
 417 Then, by the definition of function h_i , we know that $a_1 \in h_i(a_0, a_1)$, $a'_1 \in h_i(a'_0, a'_1)$,
 418 and $a_1 \notin h_i(a'_0, a'_1)$ since $a_1 \leq a'_0$. Hence, $h_i(a_0, a_1) \neq h_i(a'_0, a'_1)$.

419 Therefore, $g \circ h$ maps $\{t \in T_i : \|R_{i,t}(f)\| > 1\}$ into \mathcal{B}_i in such a way that
 420 for every $B \in \mathcal{B}_i$ such that there exists $t \in T_i$ with $h(g(t)) = B$, we have that
 421 $\tau_i(\{t \in T_i : h(g(t)) = B\}) = 0$. Since \mathcal{B}_i is countable, we can conclude that $\{t \in T_i : \|R_{i,t}(f)\| > 1\}$ is the countable union of sets having measure zero. \square

424 **Appendix 2: Proof of Proposition 2**

425 We start by checking that Theorem 1 can be applied to Γ^{IS} . Clearly, Γ^{IS} is a special
 426 case of Γ^I . First, we note that Γ^I is a specific instance of Γ . To see this, set i 's type
 427 space $T_i = [0, 1]^h$, with associated probability space $(T_i, \mathcal{T}_i, \tau_i)$ where \mathcal{T}_i is the sigma
 428 algebra of all Lebesgue measurable subsets of T_i and measure τ_i is the one induced by
 429 ϕ_i , implying that τ_i is atomless since ϕ_i is bounded. Furthermore, set i 's action space
 430 $A_i = A$. Finally, note that utility u_i^I is a special case of u_i where the utility of type t
 431 does not depend on the actions chosen by other types of the same player role.

432 We next check that all hypotheses of Theorem 1 are satisfied.

433 AU is satisfied by assumption.

434 We check AT. Since $[0, 1]^h$ is a partial order, AT1 is satisfied. Take a set $\widehat{T} \subseteq$
 435 $[0, 1]^h$ which is made of types that are all uncomparable. For any $(t_1, t_2, \dots, t_{h-1}) \in$
 436 $[0, 1]^{h-1}$, there exists at most one $t_h \in [0, 1]$ such that $(t_1, t_2, \dots, t_{h-1}, t_h) \in \widehat{T}$;
 437 otherwise, we would have two elements belonging to \widehat{T} that are comparable. This
 438 shows that \widehat{T} is contained in the graph of a function from $[0, 1]^{h-1}$ to $[0, 1]$, which
 439 constitutes an hypersurface in $[0, 1]^h$. We know that an hypersurface has Lebesgue
 440 measure equal to zero and hence \widehat{T} as well. Therefore, the measure of \widehat{T} according to
 441 the marginal density function ϕ_i is null, since the integration of ϕ_i over a zero-measure
 442 set is zero. So, AT2 is satisfied.

443 We check AA. If A is a finite lattice, then AA1–AA4 hold trivially. If $A = [0, 1]^k$,
 444 then AA1 and AA2 are satisfied by considering, respectively, the standard order and
 445 the Euclidean topology on $[0, 1]^k$. It is well known that the Euclidean space (and any
 446 of its subsets) is second countable (it is enough to consider as base the set of all open
 447 balls with rational radii and whose centers have rational coordinates). So AA3 is also
 448 satisfied. Finally, consider $a, a' \in [0, 1]^k$ such that $a'_i \geq a_i$, $a' \neq a$. Then take an
 449 open ball centered at a' with radius lower than the Euclidean distance between a' and
 450 a ; clearly, a' belongs to the ball, while every $a'' \in [0, 1]^k$ such that $a''_i \leq a_i$ does not
 451 belong to the ball. This shows that AA4 is satisfied.

452 So, we can apply Theorem 1 to conclude that every pure-strategy Nash equilibrium
453 must be essentially strict and monotone in types and actions.

454 Consider now a symmetric pure-strategy Nash equilibrium where every player
455 chooses strategy α . Consider also any strategy α' , with $\alpha' \neq \alpha$. We have already shown,
456 by exploiting Theorem 1, that α is essentially strict, and so $u^I(t, \alpha(t), \alpha_{-i}(\mathbf{t}_{-i})) >$
457 $u^I(t, \alpha'(t), \alpha_{-i}(\mathbf{t}_{-i}))$ for almost all $t \in [0, 1]^h$. Therefore,

$$458 \int_{[0,1]^h} (u(t, \alpha(t), \alpha'_{-i}(\mathbf{t}_{-i}))) \phi_i(t) dt > \int_{[0,1]^h} (u(t, \alpha'(t), \alpha_{-i}(\mathbf{t}_{-i}))) \phi_i(t) dt, \quad (1)$$

459 which means that $V(\alpha, \alpha) > V(\alpha', \alpha)$. Hence, for ϵ small enough, we can conclude
460 that $(1-\epsilon)V(\alpha, \alpha) + \epsilon V(\alpha, \alpha') > (1-\epsilon)V(\alpha', \alpha) + \epsilon V(\alpha', \alpha')$. We have so established
461 that α is an ESS.

462 Finally, to show that an ESS exists, we can rely on Theorem 1 in [McAdams \(2003\)](#)
463 that can be applied since AU2 implies the single crossing condition—which is required
464 by the Theorem. Such theorem, if applied to symmetric games, establishes the existence
465 of a symmetric pure-strategy Nash equilibrium.²⁴ By the previous argument, we
466 conclude that the strategy played in such equilibrium must be an ESS.

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²⁴ Even if we have not found a precise reference, it follows almost directly from the Proof of Theorem 1 in [McAdams \(2003\)](#) that, if we restrict attention to symmetric profiles in a symmetric game, then we are still able to show existence of an isotone pure-strategy equilibrium, which is hence symmetric.

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