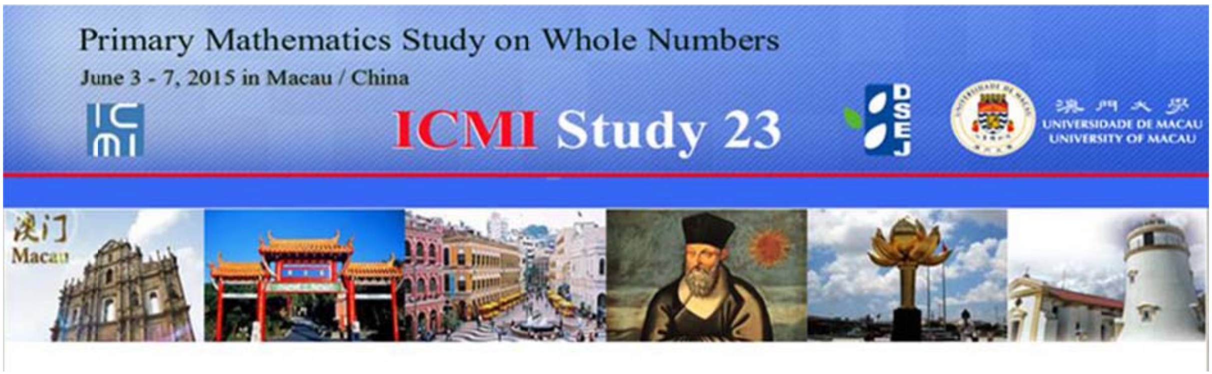
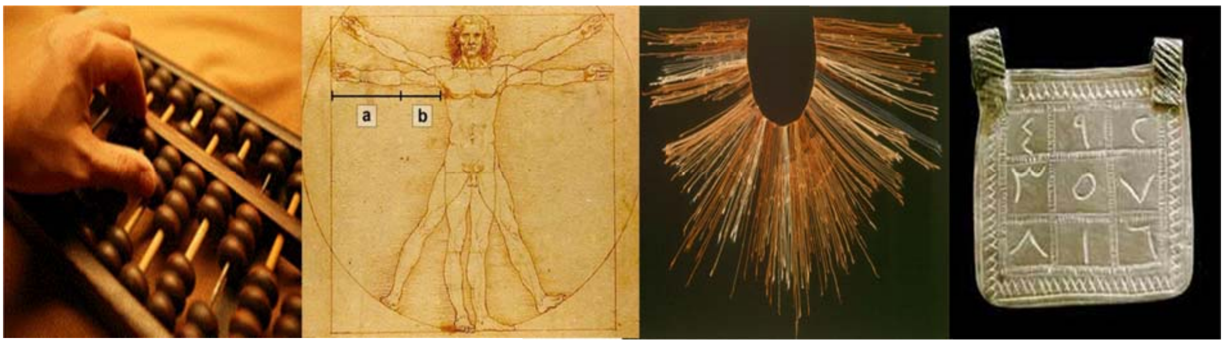


CONFERENCE PROCEEDINGS OF ICMI STUDY 23 : PRIMARY MATHEMATICS STUDY ON WHOLE NUMBERS



Editors: Xuhua Sun , Berinderjeet Kaur , Jarmila Novotná

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NUMBER SYSTEM AND COMPUTATION WITH A DUO OF ARTEFACTS: THE PASCALINE AND THE E-PASCALINE

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Abstract

We are using a duo of artefacts, constituted by a mechanical arithmetic machine and its digital counterpart, to enable six-year old French students to learn about numbers. The experiment shows the separate conceptualisation processes involving numbers as sign of a quantity and number sequences on one side and recursive addition, computation and its effect on the decimal code for numbers on the other side. The duo of artefacts enabled the design of situations that required these processes to be connected. We observed how students and teachers used the duo and discuss the results concerning the conceptualisation of number.

Key words: duo of artefacts, e-pascaline, number system, pascaline

Teaching number decimal system and computation

One of the aims of the first year of compulsory education for 6-year old French students is to learn the decimal system of writing numbers and to use it to perform computation. In their review of studies about whole numbers, Nunes and Bryant point out the key question: 'how do children come to understand that any number in the counting sequence is equal to the preceding number plus 1?' (2007, p. 4). We reformulate this in a more general way by asking how children connect what they know about numbers, number sequences and the manipulation of quantities with what they know about computation, performing addition or subtraction within the number system. The question also concerns the representation of number, taking into account that the number sequence is often learnt as an oral sequence, while the number system is a symbolic written system.

In France, before they are six, students begin to learn the numbers up to 30 at "école maternelle". For these children, number is used to indicate a specific characteristic of a collection, which is the quantity of its elements, or a position in a list (Margolinas and Wosniack, 2014). Comparing numbers involves going back to the collections and operating on their objects. The number name is a label that signals the number of elements. These names are ranked and can be recited in a given order. Children count by using the oral list of number names, which turns to be an action on the objects of the collection. They may even perform some kind of addition, which is, in fact, the union of two collections of objects. Once the two collections are unified, children can determine the number of its objects. Even if they use digits to write a representation of numbers, they use these as an icon and do not manipulate the decimal number system.

When children start "elementary school", during the first year of compulsory education, they have to learn the decimal number system used to represent numbers up to 100. The aim is that they understand how this system is linked to the collections of objects they have used to count, in a more precise manner than just as different names for different kinds of collection, and that one can operate on numbers by operating on their digits. They have to build the relationship between successive numbers in the number sequence; for instance the fact that the number next in the list is the previous one plus one unit.

We are investigating the use of a duo of artefacts, with a physical machine (the pascaline) and its digital counterpart (the e-pascaline), in the learning of numbers and computation. Following Italian research about mathematical machines (Maschietto and Bartolini Bussi, 2009), we assume that the physical machine enables the action-perception loop linking eyes and hands which is important for mathematical conceptualisation (Edwards, Radford and Arzarello, 2009; Kalenine, Pinet and Gentaz, 2011). However, using only a limited range of physical material may also lock students into procedures that require the presence of the physical artefacts, even when the didactical sequence and the teacher try to take this possibility into account and to facilitate the transfer and generalisation of procedures. Therefore, we have extended the physical artefact by a digital version of the machine that enables students to use their procedures in another context (Maschietto and Soury-Lavergne, 2013).

The aim of this paper is to describe how the duo of artefacts offers students a way to learn about the number system by solving problems that require flexibility in moving between number writing and computation. We have designed the e-pascaline in order to enlarge the mathematical experience of the students and to make complementary activities with the two kinds of artefacts. This paper discusses a French teaching experiment carried out in ordinary classrooms with voluntary teachers (Soury-Lavergne, 2014).

Materials and methods

A duo of artefacts: the pascaline and the e-pascaline

The pascaline is an arithmetic machine composed of gears analogous to the famous machine, called Pascaline, invented by the French mathematician Blaise Pascal in 1642. It is a crucial tool in the history of European mathematics because it represents the first example of addition performed independent of the human intellect. When the pascaline is introduced in the classroom, the reference to Pascal and his motivation for the construction of the machine plays an important role in the vision of mathematics as a cultural product. It provides a symbolic representation of the whole numbers from 0 to 999 and enables arithmetic operations to be performed. Each of the five wheels has ten teeth. The digits from 0 to 9 are written on the lower yellow wheels, which display units, tens and hundreds from the right to the left (Fig. 1 shows the pascaline displaying the number 122). When the units wheel (respectively the tens wheel) turns a complete rotation clockwise, the right upper wheel (respectively the left upper wheel) makes the tens wheel (respectively the hundreds wheel) go one step forward. The jerky motion of the wheel supports the recursive approach to number as it rotates one tooth at a time, adding 1 according to clockwise rotation. Anticlockwise rotation, corresponding to subtract 1, allows the operations of addition and subtraction to be linked as inverses of each other.

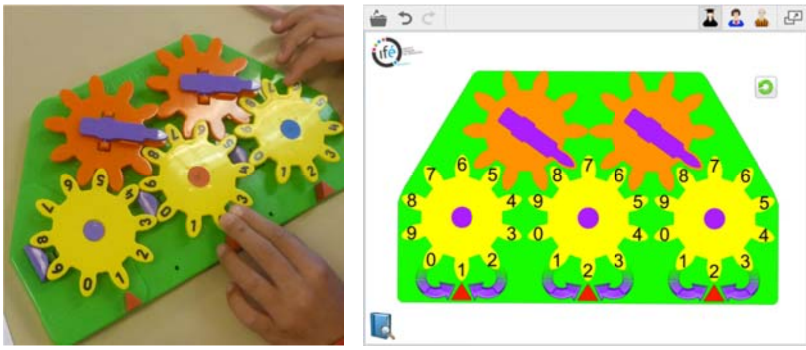


Fig. 1: The pascaline (left) and the e-pascaline (right) are displaying the number 122

Addition is performed by two different procedures which both start from displaying the first term on the pascaline. The iteration procedure consists in repeating the operation of pushing the units wheel, one tooth at a time clockwise, until the number of clicks correspond to the second term in the sum. For example, adding 26 by iteration takes place by the user clicking 26 times on the units wheel). Then, the pascaline use is based on a 'counting on' process, which is relevant for linking the knowledge of part-whole with the counting sequence (Nunes and Bryant, 2007). The decomposition procedure in contrast consists in pushing each of the three wheels by a number of clicks equals to the corresponding digit of the second term. For instance adding 26 by decomposition occurs when the user clicks 6 times on the units wheel and 2 times on the tens wheel. The iteration procedure is based on the quantity represented by the number while the decomposition procedure is based on the decimal coding of the number and the signification of the digits. The evolution of students' procedures hence indicates an evolution of the mathematical signification associated with the digit code of a number.

We have designed the e-pascaline, a digital version of the pascaline (Fig. 1, on the right), to build a complementary duo of artefacts, in which each component adds value to the other (Maschietto and Soury-Lavergne, 2013). The e-pascaline is not a simulation of the pascaline, as close as possible to the model, but is rather a separate artefact which is close enough to the physical one to enable students to transfer some schemes of use, but also different enough (in appearance or in behavior) to reduce components that have inadequate semiotic potential for mathematic learning. The e-pascaline is a component of a Cabri Elem e-book developed with the Cabri Elem technology (Mackrell, Maschietto, and Soury-Lavergne, 2013). Later in the paper, two e-books will be presented.

The didactical sequence

The sequence is composed of four teaching units. The first concerns the exploration and use of the pascaline alone, while in the other units students and teachers use both the pascaline and the e-pascaline. Each teaching unit contains tasks for students working in small group with the pascaline or the e-books, individual tasks, and collective discussions with the teacher. The activities of the units were planned within the Theory of Semiotic Mediation (Bartolini Bussi and Mariotti, 2008) the Instrumental Approach (Rabardel and Bourmaud, 2003) and the Theory of Didactical Situations (Brousseau, 1997). In particular, the activities in the e-books were constructed by considering the didactical variables in different situations and the different kinds of feedback that the software enabled to be introduced (Mackrell et al., 2013).

Analysis of specific tasks: addition and writing numbers under constraints

We analyse the e-books "adding with the e-pascaline" and "counting the e-pascaline clicks" which are linking computation and number writing.

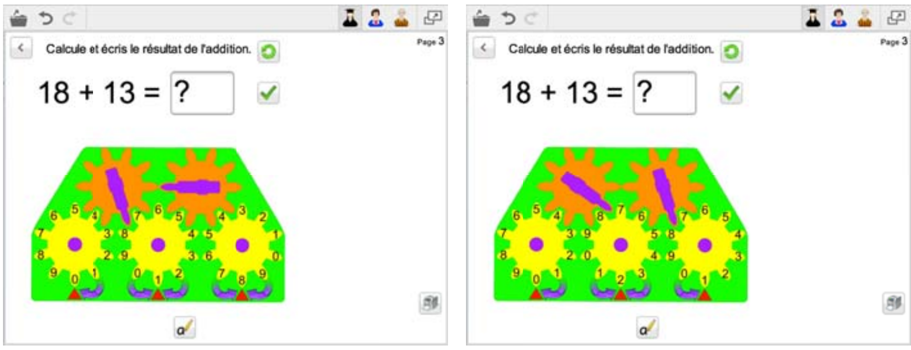


Fig. 2: On the left, the first term 18 is written. On the right, the second term is added, by using the units wheel. After 3 clicks, the adding units arrow disappears.

The e-book for addition takes into account the crucial passage from the iteration procedure to the decomposition procedure, corresponding to the transition from a procedure based on the quantity represented by the number (adding by counting one by one) to a procedure based on the decimal system. It is a delicate passage because most six-year old students apply the iteration procedure even with large numbers. The e-book consists of three pages with the same structure and components (e-pascaline, reload button, evaluation button, arrows for navigation…). The differences from one page to another concern the size of the proposed numbers for addition (up to 30 in pages 1 and 2, up to 69 in page 3) and the type of feedback given by the e-pascaline in response to students' procedures. We have implemented feedback to compel the evolution of students' procedures from iteration to decomposition. To be precise, the e-pascaline wheels turn in the direction of the arrow when the user clicks on one of the two arrows beside the red triangles (Fig. 1). When the addition arrow is not displayed, the user cannot add units. The possibility of hiding the arrows is used to force the students to choose a different wheel from the units one. In the first page all procedures are possible, to support appropriation and devolution of the task, while in the following pages, the unit wheel can only be used the number of times equal to the sum of the unit digits of the two terms. For example, to add 18 + 13 (Fig. 2), the user can click 8+3 times on the units wheel before the addition arrow disappears. The iteration procedure, which needs 13 clicks on the units wheel, is hence no longer possible. In such a way, the student has to look for another strategy to perform the addition.

The e-book “Counting the e-pascaline clicks” contains a task, which consists in minimising the number of clicks required to write a number on the e-pascaline. This task appears to only concern the writing of numbers, but in fact, it requires the exploration of different ways of reaching a number through combinations of additions and subtractions. Starting with the e-pascaline displaying 0, there are three possible procedures to display a given number on the e-pascaline. Let’s consider an example. The number 17 can be written by iteration (17 clicks) or by decomposition (8 clicks), but the minimum of clicks is obtained by a computation 20-3 (5 clicks). This third strategy requires knowledge about the decomposition of numbers and also for the students to change their point of view on the problem and to move from writing the number to computing it.

Experimental setting

The didactical sequence has been developed within a French project gathering teachers, researchers and teacher educators <http://ife.ens-lyon.fr/sciences21/ressources/sequences-et-outils/pascaline-CP>. It has been experimented during the last school year by a team of eight teachers who did not participate in the initial project. During the twelve week experiment, we were able to directly observe working sessions with two classes of six-year old students. We also collected information from the teachers through interviews and written reports throughout the experiment. The results refer either to direct observations of students’ behaviours (with teachers Stina and Nelly) or to the teachers’ reports and interviews (Stina and Cleo).

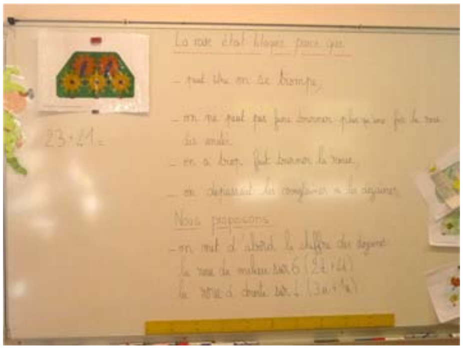
Results

In the teaching sequence, students first worked on addition with the pascaline alone and then used the e-book. In almost every teaching experiment with the pascaline, students’ initial strategies to add two terms below ten were: (i) the two terms were written on two separate wheels and the result was expected to appear on the third wheel; (ii) the addition was done by mental calculation and the result was written on the pascaline. With the first strategy, analogous to the use of a calculator, the user transferred the main part of the work to the pascaline. With the second one, the user performed the main part of the work. The iteration procedure appeared only after teacher interventions and discussion about the two previous strategies.

Adding numbers with the pascaline and the e-pascaline

In Cleo’s class, the iteration procedure appeared after she suggested using the units wheel alone. Then, with terms over ten, mistakes did not increase enough to make the students look for another strategy, even when the teacher suggested that they did. Only one of Cleo’s twenty-three students found the decomposition procedure. When Cleo first set the addition e-book to the students as an individual activity, they still had the possibility to use the pascaline to do addition. Many of them hence used the pascaline, on which the iteration procedure was possible. Then, Cleo compelled the students to perform addition using the e-book. The analysis showed: (i) the students' resistance to changing their strategy; (ii) that the cognitive processes of identifying the decomposition procedure and manipulating symbolic writing are complex. For some students, there was no evidence of the decomposition of the written number into units and tens for doing addition.

In Stina's class, pairs of students used the e-book "addition" on laptops. They used the iteration procedure and were completely unable to progress when the arrow disappeared. The session after, Stina transformed this incident into an opportunity to understand the functioning of the e-pascaline and to share ideas regarding solutions. She led a collective discussion and summarised it on the whiteboard (Fig. 3).



The wheel was blocked because:
- maybe we are wrong
- one cannot turn the wheel more than one unit
- we have turned the wheel too much
- we overcome the twenties or the tens

We propose:
- to first put the tens digit

Fig. 3: Students' explanations for the missing arrow and solutions to compute 23 + 41

She then asked her students to look for additive decompositions of numbers. They worked in small groups and had to write 23 and 41 in different ways, using the pascaline to check. They obtained the following decompositions for 23: 20 + 3; 13 + 10; 10 + 10 + 3; 10 + 5 + 5 + 3; 10 + 10 + 2 + 1; 5 + 5 + 5 + 5 + 2 + 1 and even 11 + 12 and 14 + 9.

When we observed her students, most of them were able to use the two procedures and to compare their efficiency.

Writing a number on the pascaline with a minimum of clicks

The e-book "Counting the e-pascaline clicks" represented a challenge for every student. At the beginning, Cleo's students considered the task easy to perform because of the way it was formulated; it was about writing numbers. The condition of minimising the number of clicks transformed the easy task into a real problem, however. The e-book functioned well in provoking the use of the decomposition procedure. Finding additive or subtractive decompositions of a number were difficult for her class, however. Hence the children adopted other strategies, such as controlling the position of the digits, and reading them near but not above the red triangles.

When Stina's students were first exploring the e-book, they did not succeed in finding the right solutions (indicated by obtaining "smileys" when evaluating their solution). At the next session, Stina raised the problem and asked them to answer the question *"Why haven't we succeeded in obtaining 'smileys'?"*. She also used an intermediate task, with the pascaline, to make pupils find decompositions of numbers. With this intermediate situation, she introduced the solution by asking students to complete a partial decomposition with a subtraction, such as 28 = 30 – … . After this episode, most of her pupils were able to solve the problem.

We directly observed the first use of the e-book in Nelly's class and we made two important observations. First, students didn't use the iteration procedure. They directly used decomposition, starting with the tens digit. This meant that they reproduced on the e-pascaline the spatial organisation of the digits in the written number. Moreover, two numbers, 9 and 19, provoked different procedures, although the successful procedure asked for computation. Students failed to write 9 with the minimum of clicks while they succeeded with 19, first writing 1 on the tens wheel and then turning the units wheel one click in the anticlockwise direction, after having observed that the tooth with the digit 9 was close to the red triangle. They finished by adjusting the tens wheel (one click) when they observed that it had returned to 0. Their procedure could be represented by the computation 19 = 10 – 1 + 10. The fact that they didn't do this for 9 illustrated that they are not in the process of computing but in the process of writing the number (and adjusting the wheels if needed). Their procedure is not equivalent to 19 = 20 – 1, which could have been transferred to 9.

Discussion and conclusion

We have elaborated the duo of artefacts and the e-books to build didactical situations that require the evolution from the iteration strategy to the decomposition one. With the "minimum of clicks" e-book, a third strategy requires computing. It is worth remarking that on one hand, the addition e-book explicitly required computation, but, as in any process of computing, it requires taking into account the way numbers are written with digits and not just adding units one by one (Nunes and Bryant, 2007). On the other hand, the "minimum of clicks" e-book explicitly required writing numbers, but a successful strategy required the students to compute. Both situations were rather difficult for students and successes resulted from teachers' interventions. They revealed that the concept of number, its properties and the signification of its digits code are not stable. They compel the students to make connections between the number designing a quantity and the number represented by its digits code. This relationship frames the fundamental conceptual understanding of whole numbers. Another example of this relationship is the connection between two successive numbers in the sequence, the operation +1 and its effect on the digits of the numbers codes.

In this learning process, both artefacts of the duo played a crucial role. The pascaline was used to produce a sequence of clicks that can be counted. Meanwhile it displays numbers code and ask for operating on these codes. The e-pascaline provided different constraints and feedback that led students to change their strategies and to deal with complementary conceptualisation of numbers.

Further observations are needed to deepen our understanding of the different aspects of numbers that are developed by students while using the duo of artefacts. We have planned to conduct further experiments in France as well as in Italy, to record students' strategies with the pascaline and the e-pascaline and to address some cultural issues.

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