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UNIMORE
UNIVERSITÀ DEGLI STUDI DI MODENA E REGGIO EMILIA

KIRBY DIAGRAMS, TRISECTIONS AND GEMS

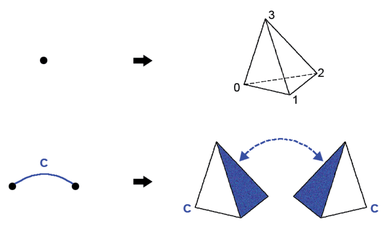
MARIA RITA CASALI - PAOLA CRISTOFORI

1. GEM = GRAPH ENCODING MANIFOLD

A ***(n + 1)-colored graph*** is (Γ, γ), with:

- $\Gamma = (V(\Gamma), E(\Gamma))$ regular graph of degree $n+1$
- $\gamma : E(\Gamma) \to \Delta_n = \{0, \dots, n\}$ such that $\gamma(e) \neq \gamma(f)$ for adjacent edges $e, f \in E(\Gamma)$ (*edge-coloration*)

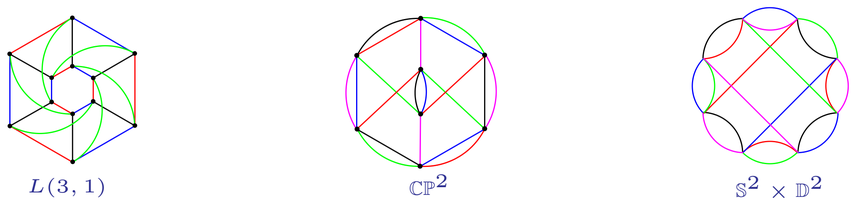
A ***colored pseudocomplex*** $K(\Gamma)$ is associated to (Γ, γ) :



from (Γ, γ) to $K(\Gamma)$, if $n = 3$

Theorem: *Any closed orientable (resp. non-orientable) PL manifold M^n admits a bipartite (resp. non-bipartite) colored graph (Γ, γ) such that $M^n \cong |K(\Gamma)|$, i.e. a* ***GEM*** *of M^n.*

Via *singular n-manifolds*, gems of manifolds with boundary can be obtained, too.

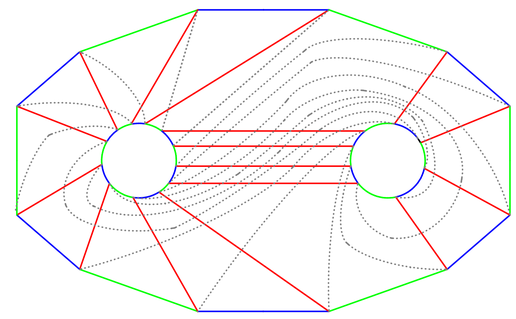


$L(3,1)$ $\quad \mathbb{CP}^2 \quad \mathbb{S}^2 \times \mathbb{D}^2$

2. REGULAR GENUS

For each $(n+1)$-colored graph (Γ, γ) and for every cyclic permutation ε of Δ_n, there exists a ***regular embedding*** of Γ onto a suitable surface F_ε.

Example: Regular embedding corresponding to $\varepsilon =$ (green, red, blue, grey)



$\rho_\varepsilon(\Gamma) = 2$, with Γ representing the Poincaré homology sphere

- ***regular genus of Γ with respect to ε***: $\rho_\varepsilon(\Gamma) = genus(F_\varepsilon)$ (or its half, if F_ε is non-orientable, i.e. Γ non-bipartite).
- ***regular genus of a PL n-manifold M^n***:

$$\mathcal{G}(M^n) = \min \{\rho_\varepsilon(\Gamma) \ / \ \Gamma \text{ gem of } M^n,\ \varepsilon \text{ cyclic permutation of } \Delta_n\}$$

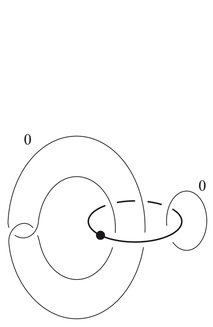
3. FROM KIRBY DIAGRAMS TO GEMS

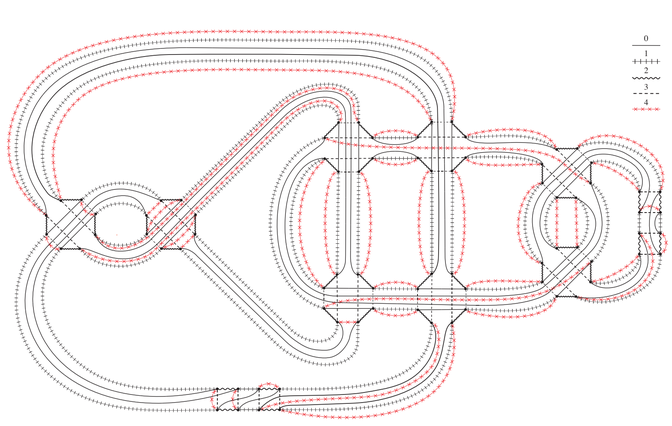
Kirby diagrams represent 4-manifolds: *dotted components* describe 1-handles, while *framed components* describe 2-handles (and, in case of closed 4-manifolds, 3- and 4-handles may be understood).

For any Kirby diagram (L, d), an algorithmic procedure allows to construct a gem of the associated 4-manifold $M(L, d)$ directly "drawn over" a planar diagram of L.

As a consequence, if (L, d) has l components, whose first $m \geq 1$ are dotted, s crossings and $\bar{s}$ undercrossings on the framed components:

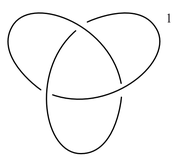
$$\mathcal{G}(M(L,d)) \leq\ s + \bar{s} + (l - m) + 1$$

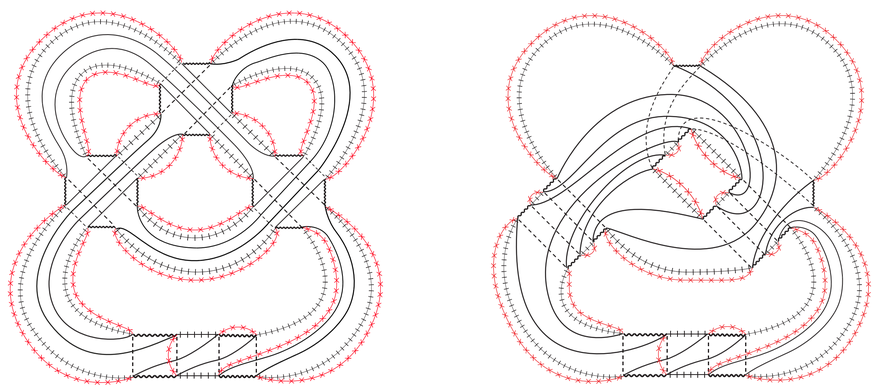




If (L, d) has no dotted components (i.e. it is a ***framed link***) and m_α is the number of α-colored regions in a chess-board coloration of L, by further simplifying the obtained gem:

$$\mathcal{G}(M(L,d)) \leq m_\alpha + l$$





Remark: The above procedure (also implemented in *Regina* software package) has been applied to get triangulations of *exotic pairs*, as well as triangulations of *corks*.

4. FROM GEMS TO TRISECTIONS

For any gem (Γ, γ) of a compact 4-manifold M with empty or connected boundary and for each cyclic permutation ε of Δ_4, it is possible to construct a decomposition (H_0, H_1, H_2) of M into three submanifolds with pairwise disjoint interiors such that:

- H_1, H_2 are 4-*dimensional handlebodies*, $H_0 \cong \partial M \times I$ (iff $\partial M \neq \emptyset$) or $H_0 \cong \mathbb{D}^4$
- $H_0 \cap H_1$ and $H_0 \cap H_2$ are 3-*dimensional handlebodies* of genus $\rho_\varepsilon(\Gamma_{\hat{4}})$, where $\Gamma_{\hat{4}}$ is the gem obtained from Γ by deleting its 4-colored edges
- $\Sigma = H_0 \cap H_1 \cap H_2$ is a *closed connected surface*

Theorem: *After $0 \leq k \leq \rho_\varepsilon(\Gamma) - \rho_\varepsilon(\Gamma_{\hat{4}})$ stabilizations along 4-colored edges, also $H_1 \cap H_2$ becomes a 3-dimensional handlebody.*
In particular, if M is closed:

- *a trisection of M of genus $\rho_\varepsilon(\Gamma_{\hat{4}}) + k$ is obtained;*
- *$k = 0$ (for a suitable gem) iff M admits a handle-decomposition with no 1- or 3-handles.*

5. ESTIMATION OF THE TRISECTION GENUS

Trisection genus of a closed 4-manifold M:

$$g_T(M) \ = \ \text{minimum genus among all trisections of } M$$

Theorem: *Let M be a compact 4-manifold whose boundary is either empty or a connected sum of sphere bundles over $\mathbb{S}^1$ and let $\bar{M}$ be its uniquely associated closed 4-manifold. Then:*

$$g_T(\bar{M}) \leq \mathcal{G}(M)$$

In particular, if M is closed:

$$g_T(M) \leq \mathcal{G}(M)$$

Theorem: *Let $\bar{M}$ be a closed orientable 4-manifold and (L, d) a (connected) Kirby diagram of $\bar{M}$ having s crossings and whose dotted components - if any - are in good position[a]. Then:*

$$g_T(\bar{M}) \leq s + 1$$

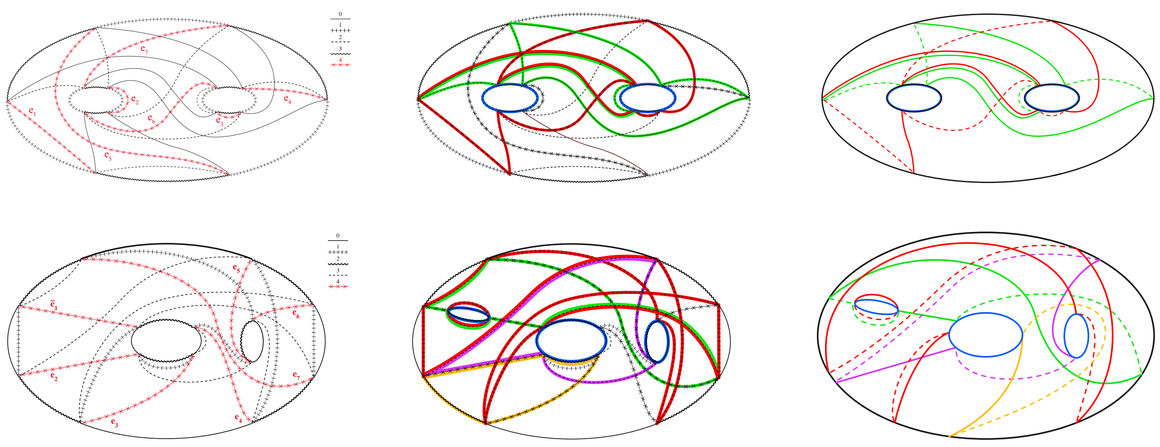
Furthermore, if (L, d) has no dotted components and m_α is the number of α-colored regions in a chess-board coloration of L :

$$g_T(\bar{M}) \leq m_\alpha$$

[a] They are unknotted, unlinked and on each of them overcrossings and undercrossings never alternate

6. FROM GEMS TO TRISECTION DIAGRAMS

From each gem (Γ, γ) of a closed 4-manifold M, a ***trisection diagram*** of M may be directly obtained: it consists of suitable subsets of *bicolored cycles* of Γ, plus - in case stabilizations are required - *additional pairs of length two cycles on some 4-colored edges*.



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