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Research Article

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Ambrosetti–Prodi Periodic Problem Under Local Coercivity Conditions

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Abstract: In this paper we focus on the periodic boundary value problem associated with the Liénard differential equation $x'' + f(x)x' + g(t, x) = s$, where s is a real parameter, f and g are continuous functions and g is T-periodic in the variable t. The classical framework of Fabry, Mawhin and Nkashama, related to the Ambrosetti–Prodi periodic problem, is modified to include conditions without uniformity, in order to achieve the same multiplicity result under local coercivity conditions on g. Analogous results are also obtained for Neumann boundary conditions.

Keywords: Liénard Equation, Periodic Solutions, Multiplicity Result, Coincidence Degree

MSC 2010: 34B15, 34C25

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1 Introduction

The aim of this paper is to study the periodic boundary value problem associated with the Liénard differential equation

$$x'' + f(x)x' + g(t, x) = s, \tag{1.1}$$

where s is a real parameter, f and g are continuous functions and g is T-periodic in the variable t.

A first motivation for our research is the study of the so-called Ambrosetti–Prodi problem for ODEs with periodic coefficients. Indeed, according to the list of open problems proposed in [2], Professor Prodi himself suggested such kind of investigations. In the seminal work [3], Ambrosetti and Prodi developed a method for studying nonlinear operator equations in Banach spaces in the presence of singularities. An application of their approach to the elliptic problem

$$\Delta u + g(u) = h, \quad x \in \Omega, \qquad u|_{\partial\Omega} = 0, \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with boundary of class $C^{2,\alpha}$ and $h \in C^{0,\alpha}(\overline{\Omega})$, with $0 < \alpha < 1$, led to the following result.

Theorem 1.1. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a function of class C^2 such that $g''(\xi) > 0$ for each $\xi \in \mathbb{R}$ and*

$$-\infty < \lim_{\xi\to-\infty} g'(\xi) < \lambda_1 < \lim_{\xi\to+\infty} g'(\xi) < \lambda_2, \tag{1.3}$$

with λ_1 and λ_2 being the first two eigenvalues of the associated linear problem. Then there exists a C^1 manifold $\mathcal{M}$ of codimension one in $C^{0,\alpha}(\overline{\Omega})$ which splits this space in two disjoint open sets $\mathcal{A}_0$ and $\mathcal{A}_2$ such that problem (1.2) has no solutions if $h \in \mathcal{A}_0$, exactly one solution if $h \in \mathcal{M}$, and exactly two solutions if $h \in \mathcal{A}_2$.

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Actually, with respect to [3], the assumption $0 < \lim_{\xi\to-\infty} g'(\xi)$ was also required in condition (1.3), but such a restriction was then removed in [16]. In the subsequent work [5], Berger and Podolak provided a reformulation of the conclusion in Theorem 1.1 by splitting h as

$$h = su_1 + p,$$

where u_1 is the normalized positive eigenfunction associated with the first eigenvalue λ_1 and p is the orthogonal component of h. In this case, they proved the existence of a unique real number $s_0(p)$ such that if $s < s_0$, then $h \in \mathcal{A}_0$, if $s = s_0$, then $h \in \mathcal{M}$, while if $s > s_0$, then $h \in \mathcal{A}_2$. By further results, due to Kazdan and Warner [14], Dancer [6], and Amann and Hess [1], it is known that the convexity assumption on g can be removed at the cost of the knowledge of the precise number of the solutions. In other words, one has at least one solution for $s = s_0$ and at least two solutions for $s > s_0$. These results are also valid in the case of more general second-order elliptic operators.

Looking at the periodic problem, for a fixed period $T > 0$ and the linear differential operator $-x''$ (or $-x'' - cx'$), it follows that $\lambda_1 = 0$. Moreover, in the splitting proposed by Berger and Podolak, we have that $u_1 \equiv 1$ and p is a T-periodic forcing term with mean value zero in a period. Notice that, in this situation, condition (1.3) implies $\lim_{|\xi|\to+\infty} g(\xi) = +\infty$. In this context, a main contribution comes from the work of Ortega [29]. Indeed, a version of Theorem 1.1 is obtained for the problem

$$x'' + cx' + g(x) = h(t), \quad x(0) = x(T), \quad x'(0) = x'(T), \tag{1.4}$$

with $c > 0$, under the convexity assumption on g and by assuming the condition

$$-\infty \le \lim_{\xi\to-\infty} g'(\xi) < 0 < \lim_{\xi\to+\infty} g'(\xi) \le \left(\frac{2\pi}{T}\right)^2 + \frac{c^2}{4},$$

which replaces (1.3) in the periodic setting. In [29] sharp results about the stability of T-periodic solutions were obtained as well. Moreover, still in the framework of $g'' > 0$, recent results about symbolic dynamics associated with the solutions of the equation in (1.4) have been obtained in [31]. Still in this context, we also recall the contribution of Vidossich [32], who achieved an Ambrosetti–Prodi type result for the first-order linear differential operator $x' + q(t)x$, with q being a continuous and T-periodic coefficient.

On the other hand, avoiding the convexity assumption and dealing with the more general equation (1.1), a relevant contribution is contained in the work of Fabry, Mawhin and Nkashama [9], who considered a general second-order parameter dependent differential equation of the form

$$x'' + F(t, x, x') = s,$$

with F satisfying a Bernstein–Nagumo condition. The study of this equation includes (1.1) for $F(t, x, y) = f(x)y + g(t, x)$ and the following result is proved in [9, Corollary 1].

Theorem 1.2. *Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions, and g is T-periodic in t and satisfies the hypothesis*

$$\lim_{|x|\to+\infty} g(t, x) = +\infty \quad \text{uniformly in } t. \tag{1.5}$$

Then, there exists a number s_0 such that
(1) *for $s < s_0$, equation* (1.1) *has no T-periodic solutions,*
(2) *for $s = s_0$, equation* (1.1) *has at least one T-periodic solution,*
(3) *for $s > s_0$, equation* (1.1) *has at least two T-periodic solutions.*

The results achieved in [9] have stimulated further investigations. For the sake of comparison, we recall that theorems analogous to Theorem 1.2 have been obtained in [10], involving nonlinearities $g(t, x)$ with singularities, and in [25] for a more general equation of the form

$$(|x'|^{p-1}x')' + f(x)x' + g(t, x) = s.$$

We also highlight that similar results have been carried out for first-order differential equations (see [8, 23, 28] and the references therein).

An immediate consequence of Theorem 1.2 is the following.

Corollary 1.1. *Let $f, \phi \colon \mathbb{R} \to \mathbb{R}$ be continuous functions and suppose that*

$$\lim_{|x|\to+\infty} \phi(x) = +\infty. \tag{1.6}$$

Let $a, p \colon \mathbb{R} \to \mathbb{R}$ be continuous and T-periodic functions, with $\min_{t\in[0,T]} a(t) > 0$. Then, for the equation

$$x'' + f(x)x' + a(t)\phi(x) = s + p(t), \tag{1.7}$$

there exists a number s_0 such that
(1) *for $s < s_0$, equation (1.7) has no T-periodic solutions,*
(2) *for $s = s_0$, equation (1.7) has at least one T-periodic solution,*
(3) *for $s > s_0$, equation (1.7) has at least two T-periodic solutions.*

At this point, from Theorem 1.2 and Corollary 1.1, the natural question arises whether the uniform condition in (1.5) can be weakened. In particular, with respect to Corollary 1.1, one could wonder if the result still holds when $a(t) \geq 0$ and it vanishes somewhere. The present paper is devoted to give an answer to these questions. Indeed, we shall prove that the uniform conditions (1.5) and (1.6) can be improved to treat cases in which the coercivity condition $g(t, x) \to \infty$ holds only locally, namely, on a sub-interval of $[0, T]$.

First positive answers to these questions, although not complete, were obtained in [30] for the Neumann problem. Let us observe that solving the Neumann problem, on an interval of length $T/2$, can provide solutions also for the T-periodic problem associated with ODEs presenting suitable symmetries in the variable t. The results in [30] are based on the shooting method, which requires the uniqueness of the solutions for the initial value problems and their continuability. In the present paper we adopt a completely different approach based on topological degree theory in function spaces. Actually, we borrow some arguments already developed in [9], where the method of upper and lower solutions is exploited along with coincidence degree theory. Some steps of our proof closely follows those performed in the above quoted paper and so, at the beginning of Section 2, we recall some basic notions about coincidence degree. We stress the fact that the uniform condition (1.5) permits to find lower solutions in a direct way. Without uniformity, in the limits at infinity, we cannot proceed in the same manner, but, nevertheless, we can provide some lower bounds for the solutions by using a Villari type condition (see Definition 2.1). This sentence is technically expressed in Theorem 2.2, which will be the main tool for achieving our results about existence and multiplicity of solutions. This way, we are able to obtain a version of Theorem 1.2 without the uniformity condition (see Theorem 3.2). As a consequence, we will present an improvement of Corollary 1.1 (see Corollary 4.1), where the same conclusion holds by assuming, for the term $a(t)$, that

$$a(t) \geq 0, \quad \text{with } a \not\equiv 0.$$

We finally observe that all our results can be stated in the Carathéodory setting (see Sections 2 and 3) and extended to the Neumann problem (see Section 4).

2 Preliminaries

In this section we recall the fundamental properties of Mawhin's coincidence degree, which is needed to treat the periodic boundary value problem associated with the second-order differential equation

$$x'' + f(x)x' + \psi(t, x) = 0. \tag{2.1}$$

Such properties follow from general results, which can be found in the classical monographs [11, 20, 24] and adapted to the present situation following [18]. Moreover, we establish a priori bounds for periodic solutions of (2.1) that are useful for deriving results both of existence and multiplicity for such kind of equation.

2.1 Mawhin's Coincidence Degree

Mawhin's coincidence degree is a powerful tool to deal with coincidence equations of the form

$$Lx = Nx, \quad x \in \operatorname{dom} L \cap \Omega, \tag{2.2}$$

where $L\colon X \supseteq \operatorname{dom} L \to Z$ is a linear non-invertible operator and $N\colon X \to Z$ is a nonlinear operator. In our setting, X and Z are real normed spaces and Ω is an open bounded set in X. More specifically, we assume that L is a linear Fredholm mapping of index zero. We also consider two linear and continuous projections $P\colon X \to \ker L$ and $Q\colon Z \to \operatorname{Im} L$, as well as, the (continuous) right inverse of L, denoted by $K_P\colon \operatorname{Im} L \to \operatorname{dom} L \cap X_0$, where $X_0 := \ker P \equiv X/\ker L$ is a complementary subspace of $\ker L$ in X. Notice that equation (2.2) is equivalent to the fixed point problem

$$x = \Phi(x) := Px + JQNx + K_P(I - Q)Nx, \quad x \in \Omega, \tag{2.3}$$

where $J\colon \operatorname{coker} L = \operatorname{Im} Q \equiv Z/\operatorname{Im} L \to \ker L$ is a linear isomorphism. We further suppose that N is a continuous operator which maps bounded sets to bounded sets and such that for any bounded set B in X, the set $K_P(I - Q)N(B)$ is relatively compact (namely, N is L-completely continuous [24]). These assumptions imply that the operator Φ, defined in (2.3), is completely continuous.

If we suppose that

$$Lx \neq Nx \quad \text{for all } x \in \operatorname{dom} L \cap \partial\Omega,$$

then also $I - \Phi$ never vanishes on $\partial\Omega$ and, therefore, we can define the *coincidence degree*

$$D_L(L - N, \Omega) := \deg(I - \Phi, \Omega, 0),$$

where "deg" denotes the Leray–Schauder degree. To avoid ambiguity of sign, sometimes the convention is to consider only $|D_L(L - N, \Omega)|$. Otherwise, we can fix an orientation on $\ker L$ and $\operatorname{coker} L$, so that we choose J in the class of orientation preserving isomorphisms (see [24]). In any case, for our application, the choice of P, Q and J is obvious and no ambiguity will arise.

We are now ready to state the celebrated Mawhin's continuation theorem as follows (see [17, 19]), where by "$\deg_B$" we denote the (finite dimensional) Brouwer degree.

Theorem 2.1. *Let L and N be as above and let $\Omega \subseteq X$ be an open and bounded set. Suppose that*

$$Lx \neq \lambda Nx \quad \textit{for all } x \in \operatorname{dom} L \cap \partial\Omega \textit{ and all } \lambda \in\,]0, 1],$$

and

$$QN(x) \neq 0 \quad \textit{for all } x \in \partial\Omega \cap \ker L.$$

Then

$$D_L(L - N, \Omega) = \deg_B(-JQN|_{\ker L}, \Omega \cap \ker L, 0).$$

As a consequence, if $\deg_B(-JQN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$, *then* (2.2) *has at least one solution.*

2.2 Application of the Coincidence Degree and A Priori Bounds for Equation (2.1)

We consider now equation (2.1) and assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function and $\psi\colon [0, T] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. As usual, by a T-periodic solution of (2.1) we mean a generalized solution $x\colon [0, T] \to \mathbb{R}$ of equation (2.1) which satisfies the boundary condition

$$x(0) = x(T), \quad x'(0) = x'(T).$$

Equivalently, one could extend the map $\psi(\,\cdot\,, x)$ on $\mathbb{R}$ by T-periodicity, and then consider T-periodic solutions $x\colon \mathbb{R} \to \mathbb{R}$, with x' being absolutely continuous (AC).

The standard setting to enter in the framework of the coincidence degree is the following. Let

$$X = C^1_T := \{x \in C^1([0,T]) : x(0) = x(T),\ x'(0) = x'(T)\},$$

endowed with the norm

$$\|x\|_X := \|x\|_\infty + \|x'\|_\infty$$

and $Z = L^1([0,T])$ with the norm $\|x\|_Z := \|x\|_1$. Let $L\colon X \supseteq \operatorname{dom} L \to Z$ be defined as $Lx := -x''$, with

$$\operatorname{dom} L = W^{2,1}_T := \{x \in X : x' \in \mathrm{AC}\}.$$

In accord to [18], a natural choice for the projections is given by

$$Qx := \frac{1}{T}\int_0^T x(t)\,dt \quad \text{for all } x \in Z, \qquad Px = Qx \quad \text{for all } x \in X.$$

This way, we have $\ker L = \operatorname{Im} P = \mathbb{R}$ and $\operatorname{coker} L = \operatorname{Im} Q = \mathbb{R}$. Moreover, we take J as the identity in $\mathbb{R}$. Notice that for each $w \in Z$, the vector $v = K_P(I-Q)w$ is the (unique) solution of the linear boundary value problem

$$\begin{cases} -v''(t) = w(t) - \dfrac{1}{T}\displaystyle\int_0^T w(t)\,dt, \\ v(0) = v(T), \quad v'(0) = v'(T), \quad \displaystyle\int_0^T v(t)\,dt = 0. \end{cases}$$

Lastly, as nonlinear operator N, we take the associated Nemytskii operator, namely,

$$(Nx)(t) := f(x(t))x'(t) + \psi(t, x(t)) \quad \text{for all } x \in X.$$

By a standard argument, it is possible to verify that the operator N is L-completely continuous and, moreover, that a function $\tilde{x}(\cdot)$ is a T-periodic solution of (2.1) if and only if $\tilde{x} \in \operatorname{dom} L$, with $L\tilde{x} = N\tilde{x}$. Analogously, solutions to the abstract equation $Lx = \lambda Nx$, with $0 < \lambda \le 1$, correspond to T-periodic solutions of

$$x'' + \lambda f(x)x' + \lambda\psi(t,x) = 0, \quad 0 < \lambda \le 1. \tag{2.4}$$

In the next two lemmas we provide some a priori bounds for the solutions of the parameter dependent equation (2.4) that will be useful for the application of Theorem 2.1 to equation (2.1).

Lemma 2.1. *Let $\psi\colon [0,T]\times\mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the following:*
(H0) *There exists $\gamma \in L^1([0,T],\mathbb{R}^+)$ such that $\psi(t,x) \ge -\gamma(t)$ for all $x \in \mathbb{R}$ and a.e. $t \in [0,T]$.*
Then there exists a constant $K_0 = K_0(\gamma)$ such that any T-periodic solution x of (2.4) satisfies $\max x - \min x \le K_0$.

Proof. Without loss of generality, we suppose that the map $\psi(\cdot, x)$ is extended by T-periodicity on the whole real line and the solutions satisfy $x(t+T) = x(t)$ for all $t \in \mathbb{R}$. Let t^* be such that $x(t^*) = x_{\max} := \max x$. We also define $u(t) := x_{\max} - x(t)$, which satisfies $u' = -x'$ and $u'' = -x''$. From (2.4) we have

$$-u''(t) = x''(t) = -\lambda f(x(t))x'(t) - \lambda\psi(t,x(t)) \le \lambda f(x(t))u'(t) + \gamma(t) \quad \text{for a.e. } t.$$

Multiplying the previous inequality by $u(t) \ge 0$ and integrating on $[t^*, t^*+T]$, we obtain, after an integration by parts,

$$\begin{aligned} \|u'\|^2_{L^2} &= \int_0^T u'(t)^2\,dt = \int_{t^*}^{t^*+T} u'(t)^2\,dt = -\int_{t^*}^{t^*+T} u''(t)u(t)\,dt \\ &\le \lambda\int_{t^*}^{t^*+T} f(x(t))u(t)u'(t)\,dt + \int_{t^*}^{t^*+T} \gamma(t)u(t)\,dt \le \|\gamma\|_{L^1}\|u\|_\infty. \end{aligned}$$

Notice that

$$\int_{t^*}^{t^*+T} f(x(t))u(t)u'(t)\,dt = \int_{t^*}^{t^*+T} f(x(t))x(t)x'(t)\,dt - x_{\max}\int_{t^*}^{t^*+T} f(x(t))x'(t)\,dt = 0.$$

Using the fact that $u(t^*) = 0$, and therefore $\|u\|_\infty \le c_1\|u'\|_{L^2}$ (for a suitable embedding constant c_1), we have $\|u'\|_{L^2} \le c_1\|\gamma\|_{L^1}$, and then

$$\|u\|_\infty \le K_0 := c_1^2\|\gamma\|_{L^1}.$$

This concludes the proof, since $\max x - \min x = \|u\|_\infty$. □

Lemma 2.2. *Let $\psi\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function. Let $[a,b]\subset\mathbb{R}$ and $\rho\in L^1([0,T],\mathbb{R}^+)$ be such that $|\psi(t,x)|\le\rho(t)$ for all $x\in[a,b]$ and a.e. $t\in[0,T]$. Then, there exists a constant $\kappa=\kappa(a,b,\rho)$ such that any T-periodic solution x of* (2.4)*, with $a\le x(t)\le b$ for all $t\in[0,T]$, satisfies $\|x'\|_\infty\le\kappa$.*

The proof follows straightway, since the term $\mathcal{F}(t,x,x') := f(x)x' + \psi(t,x)$ satisfies a Bernstein–Nagumo condition (see [7, 9, 21]).

For the main result of this section, it will be useful to introduce the following definition, which deals with Villari's conditions adapted here from [33]. For more information about these conditions, as well as generalizations in different contexts, we refer to [4, 15, 26].

Definition 2.1. *We say that $\psi(t,x)$ satisfies Villari's condition at $-\infty$ (respectively, at $+\infty$) if there exists a constant $d_0>0$ such that for $\delta=\pm1$, we have*

$$\delta\int_0^T \psi(t,x(t))\,dt > 0$$

for each $x\in C^1_T$, with $x(t)\le -d_0$ for every $t\in[0,T]$ (respectively, $x(t)\ge d_0$ for every $t\in[0,T]$).

In the sequel, it will be convenient to call a *strict upper solution* for equation (2.1) any function $\beta\in W^{2,1}_T$ satisfying

$$\beta''(t) + f(\beta(t))\beta'(t) + \psi(t,\beta(t)) < 0 \quad \text{for a.e. } t\in[0,T] \tag{2.5}$$

and, moreover, such that if x is any T-periodic solution of (2.1) with $x\le\beta$, then $x(t)<\beta(t)$ for all t. We note that this definition is a particular case of the one considered in [7]. We also observe that if ψ is a continuous function (T-periodic in t) and $\beta\in C^2_T$ satisfies (2.5) for all t, then β is strict. In other words, it is always true that $x<\beta$ whenever x is a T-periodic solution of (2.1) with $x\le\beta$ (see [7, Chapter 3, Proposition 1.2]).

Theorem 2.2. *Let $f\colon\mathbb{R}\to\mathbb{R}$ be a continuous function. Let $\psi\colon[0,T]\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function satisfying* (H0) *in Lemma 2.1 and Villari's condition at $-\infty$ with $\delta=1$. Suppose there exists $\beta\in W^{2,1}_T$ which is a strict upper solution for equation* (2.1)*. Then* (2.1) *has at least a T-periodic solution $\tilde{x}$ such that $\tilde{x}<\beta$. Moreover, there exist $R_0\ge d_0$ and $M_0>0$ such that for each $R>R_0$ and $M>M_0$, we have*

$$D_L(L-N,\Omega) = 1$$

for

$$\Omega = \Omega(R,\beta,M) := \{x\in C^1_T : -R < x(t) < \beta(t) \text{ for all } t\in[0,T],\ \|x'\|_\infty < M\}.$$

Proof. We follow a standard procedure based on a truncation argument, as usual in the theory of upper and lower solutions. Accordingly, we define the truncated function

$$\hat{\psi}(t,x) := \begin{cases} \psi(t,x) & \text{for } x\le\beta(t),\\ \psi(t,\beta(t)) & \text{for } x\ge\beta(t),\end{cases}$$

and consider the parameter dependent equation

$$x'' + \lambda f(x)x' + \lambda\hat{\psi}(t,x) = 0, \quad 0<\lambda\le 1. \tag{2.6}$$

First of all, as a consequence of (H0), we remark that $\hat{\psi}(t,x) \geq -\gamma(t)$ for all $x \in \mathbb{R}$ and a.e. $t \in [0,T]$, and so $\hat{\psi}$ satisfies (H0), too. Therefore, according to Lemma 2.1 (applied to $\hat{\psi}$ in place of ψ), any T-periodic solution x of (2.6) satisfies $\max x - \min x \leq K_0$ for some constant K_0 (depending on γ).

Next, we choose a constant $d_1 \geq d_0$ with $d_1 > \|\beta\|_\infty$ and we claim that $\max x > -d_1$. Indeed, if we suppose by contradiction that $x(t) \leq -d_1$ for all $t \in [0,T]$, then $x(t) < \beta(t)$ for all $t \in [0,T]$, and so $x(t)$ is a T-periodic solution of (2.4). Hence, an integration on $[0,T]$ of (2.4) (divided by $\lambda > 0$) yields $\int_0^T \psi(t,x(t))\,dt = 0$, which clearly contradicts Villari's condition at $-\infty$ as $-d_1 \leq -d_0$. Having proved that $x(t) > -d_1$ for some $t \in [0,T]$ and hence $\max x > -d_1$, we immediately obtain that

$$\min x > -R_0 \quad \text{for } R_0 := K_0 + d_1.$$

At this point, we claim that there exists $\bar{t} \in [0,T]$ such that $x(\bar{t}) < \beta(\bar{t})$. If, by contradiction, $x(t) \geq \beta(t)$ for all $t \in [0,T]$, then x turns out to be a T-periodic solution of

$$x'' + \lambda f(x)x' + \lambda\psi(t,\beta(t)) = 0, \quad 0 < \lambda \leq 1.$$

Thus, an integration on $[0,T]$ of the previous equation (divided by $\lambda > 0$) yields $\int_0^T \psi(t,\beta(t))\,dt = 0$. However, since β is T-periodic and satisfies (2.5), an integration of (2.5) on $[0,T]$ gives $\int_0^T \psi(t,\beta(t))\,dt < 0$, which leads to a contradiction. Having proved that $x(t) < \|\beta\|_\infty$ for some $t \in [0,T]$ and hence $\min x < \|\beta\|_\infty$, we immediately obtain that

$$\max x < \|\beta\|_\infty + K_0.$$

From Lemma 2.2 (applied to $\hat{\psi}$ in place of ψ), we find a constant m_0, which depends on R_0, $\|\beta\|_\infty + K_0$ and an L^1-function bounding $|\hat{\psi}(t,x)|$ on $[0,T] \times [-R_0, \|\beta\|_\infty + K_0]$, such that $\|x'\|_\infty \leq m_0$.

Writing the equation

$$-x'' = f(x)x' + \hat{\psi}(t,x) \tag{2.7}$$

as a coincidence equation of the form $Lx = \hat{N}x$ in the space C^1_T, from the a priori bounds, we find that the coincidence degree $D_L(L - \hat{N}, \mathcal{O})$ is well defined for any open and bounded set $\mathcal{O} \subset C^1_T$ of the form

$$\mathcal{O} := \{x \in C^1_T : -R < x(t) < C \text{ for all } t \in [0,T],\ \|x'\|_\infty < m\},$$

where $R \geq R_0$, $C \geq \|\beta\|_\infty + K_0$ and $m > m_0$.

Finally, we consider the averaged scalar map

$$\hat{\psi}^\# : \mathbb{R} \to \mathbb{R}, \qquad \hat{\psi}^\#(\xi) := \frac{1}{T}\int_0^T \hat{\psi}(t,\xi)\,dt \quad \text{for all } \xi \in \mathbb{R},$$

and we observe that the following holds:

$$-JQ\hat{N}|_{\ker L} = -\hat{\psi}^\#.$$

Indeed, the kernel of the differential operator L is made by the constant functions which are identified with the real numbers. Moreover, we have

$$\hat{\psi}^\#(-R) > 0 > \hat{\psi}^\#(C).$$

In fact, the first inequality comes from Villari's condition and the choice $R \geq d_1$, while the second inequality follows from $\int_0^T \psi(t,\beta(t))\,dt < 0$ and the choice $C \geq \|\beta\|_\infty$. Thus, an application of Theorem 2.1 guarantees that $D_L(L - \hat{N}, \mathcal{O}) = 1$, and hence equation (2.7) has a T-periodic solution $\tilde{x}$, with $-R < \tilde{x}(t) < C$ for all $t \in [0,T]$.

To conclude with the proof, we have only to check that $\tilde{x} < \beta$. This is standard from the theory of strict upper solutions, and so we give a sketch just for completeness. We have already proved that any T-periodic solution of (2.6) is below β, at least for some t. Since $\tilde{x}$ is a solution of (2.6) for $\lambda = 1$, we have that there exists t_* such that $\tilde{x}(t_*) < \beta(t_*)$. Suppose by contradiction that there exists t^* such that $\tilde{x}(t^*) > \beta(t^*)$. By the T-periodicity of $v(t) := \tilde{x}(t) - \beta(t)$, there exists an interval $[t_1, t_2]$ such that $t_1 < t^* < t_2$, with $v(t) > 0$ for all $t \in]t_1, t_2[$ and, moreover, $v(t_1) = v(t_2) = 0$, with $v'(t_1) \geq 0 \geq v'(t_2)$. On the interval $[t_1, t_2]$, we have that $\tilde{x}''(t) + f(\tilde{x}(t))\tilde{x}'(t) + \psi(t, \beta(t)) = 0$. Therefore, recalling (2.5), we have

$$v''(t) + f(\tilde{x}(t))\tilde{x}'(t) - f(\beta(t))\beta'(t) > 0 \quad \text{for a.e. } t \in [t_1, t_2].$$

An integration on $[t_1, t_2]$ gives a contradiction, because

$$\int_{t_1}^{t_2} v''(t)\,dt = v'(t_2) - v'(t_1) \leq 0$$

and, for $F' = f$, we have

$$\int_{t_1}^{t_2} f(\tilde{x}(t))\tilde{x}'(t)\,dt = F(\tilde{x}(t_2)) - F(\tilde{x}(t_1)) = F(\beta(t_2)) - F(\beta(t_1)) = \int_{t_1}^{t_2} f(\beta(t))\beta'(t)\,dt.$$

We have thus proved that $\tilde{x}(t) \leq \beta(t)$ for all $t \in [0, T]$, and therefore $\tilde{x}$ is a T-periodic solution of (2.1) satisfying $\tilde{x} \leq \beta$. By the hypothesis about β being strict, we conclude that $\tilde{x}(t) < \beta(t)$ for all t.

Applying again Lemma 2.2, we find a positive constant M_0, depending on R_0, $\|\beta\|_\infty$ and an L^1-function bounding $|\psi(t, x)|$ on $[0, T] \times [-R_0, \|\beta\|_\infty]$ such that $\|x'\|_\infty \leq M_0$. Finally, the excision property of the coincidence degree (see [20]) gives the conclusion. □

3 Existence and Multiplicity Theorems

Here we discuss the number of T-periodic solutions for the parameter dependent equation (1.1). Throughout this section we suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions. Taking into account [7, Chapter 3, Proposition 1.5], we also assume that the following condition holds:

(A) For all $t_0 \in [0, T]$, $u_0 \in \mathbb{R}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|t - t_0| < \delta,\ |u - u_0| < \delta \Rightarrow |g(t, u) - g(t, u_0)| < \varepsilon.$$

Moreover, in the sequel, the following hypotheses will be considered as well:

(G0) There exists $\gamma_0 \in L^1([0, T], \mathbb{R}^+)$ such that $g(t, x) \geq -\gamma_0(t)$ for all $x \in \mathbb{R}$ and a.e. $t \in [0, T]$.

(G1) There exists $g_0 \in \mathbb{R}$ such that $g(t, 0) \leq g_0$ for a.e. $t \in [0, T]$.

(G2) For each σ, there exists $d_\sigma > 0$ such that $\frac{1}{T}\int_0^T g(t, x(t))\,dt > \sigma$ for each $x \in C^1_T$, with $x(t) \leq -d_\sigma$ for all $t \in [0, T]$.

(G3) For each σ, there exists $d_\sigma > 0$ such that $\frac{1}{T}\int_0^T g(t, x(t))\,dt > \sigma$ for each $x \in C^1_T$, with $x(t) \geq d_\sigma$ for all $t \in [0, T]$.

The main result in this section reads as follows.

Theorem 3.1. *Assume* (A) *and* (G0)–(G2). *Then there exists $s_0 \in \mathbb{R}$ such that*

(1) *for $s < s_0$, equation* (1.1) *has no T-periodic solutions,*

(2) *for $s > s_0$, equation* (1.1) *has at least one T-periodic solution.*

Proof. The proof follows the scheme proposed in [9, Theorem 1], which is adapted from [14]. For any given parameter $s \in \mathbb{R}$, we set

$$\psi_s(t, x) := g(t, x) - s, \tag{3.1}$$

so that equation (1.1) is of the form (2.1).

Let us start by fixing a parameter $s_1 > g_0$. In this situation, the constant function $\beta(t) \equiv 0$ is a strict upper solution. Indeed, we have

$$\beta''(t) + f(\beta(t))\beta'(t) + g(t, \beta(t)) - s_1 = g(t, 0) - s_1 \leq -(s_1 - g_0) < 0,$$

and then property (A) guarantees our claim, according to [7, Section 3, Proposition 1.6] (applied to the function $-f(u)v - g(t, u) - s_1$). On the other hand, for $\sigma = s_1$, condition (G2) implies Villari's condition at $-\infty$ with $\delta = 1$. Hence, an application of Theorem 2.2 guarantees the existence of at least one T-periodic solution x of (1.1) with $x < 0$, for $s = s_1$.

As a second step, we claim that if for some $\tilde{s} < s_1$, the equation has a T-periodic solution (that we will denote by w), then equation (1.1) has a T-periodic solution for each $s \in [\tilde{s}, s_1]$. Clearly, it will be sufficient to prove this assertion for s, with $\tilde{s} < s < s_1$. Writing equation (1.1) as

$$x'' + f(x)x' + g(t, x) - \tilde{s} - (s - \tilde{s}) = 0,$$

we find that $\beta(t) \equiv w(t)$ is a strict upper solution of (1.1). Indeed, we have

$$\beta''(t) + f(\beta(t))\beta'(t) + g(t, \beta(t)) - s = w'' + f(w(t))w'(t) + g(t, w(t)) - s = -(s - \tilde{s}) < 0.$$

and then property (A) guarantees our claim, according to [7, Section 3, Proposition 1.6] (applied to the function $-f(u)v - g(t, u) - s$). On the other hand, for $\sigma = s$, condition (G2) implies Villari's condition at $-\infty$ with $\delta = 1$. Again, an application of Theorem 2.2 guarantees the existence of at least one T-periodic solution x of (1.1), with $x < w$, and the claim is proved.

If x is any T-periodic solution of (1.1), then, taking the average of the equation on $[0, T]$, we have $\frac{1}{T}\int_0^T g(t, x(t))\,dt = s$ and, using (G0), we obtain

$$s \geq \nu_0 := -\frac{1}{T}\int_0^T \gamma_0(t)\,dt. \tag{3.2}$$

Hence, if $s < \nu_0$, equation (1.1) has no T-periodic solution.

At this point, we have proved that the set of the parameters s for which equation (1.1) has T-periodic solutions is an interval which is bounded from below. Let

$$s_0 := \inf\{s \in \mathbb{R} : (1.1) \text{ has at least one } T\text{-periodic solution}\}.$$

By the previous discussion, we know that $\nu_0 \leq s_0 \leq g_0$ and the thesis follows. □

Remark 3.1. In Theorem 3.1 no information is given about the existence or nonexistence of solutions to (1.1) for $s = s_0$. Indeed, in this case without supplementary conditions, we are not able to determine whether the equation has T-periodic solutions. For instance, the T-periodic solutions of the equation

$$x'' + g(x) = s, \quad \text{with } g(x) = 2\alpha(\sqrt{1 + x^2} - x) \text{ for } 0 < \alpha < (\pi/T)^2,$$

are only the constant ones, namely, the real solutions of $g(x) = s$. In this case, $s_0 = 0$ and no solutions exist for $s = s_0$. Similar elementary examples of equations having T-periodic solutions for $s = s_0$, can be provided too.

Theorem 3.2. *Assume* (A) *and* (G0)–(G3). *Then, there exists* $s_0 \in \mathbb{R}$ *such that*
(1) *for* $s < s_0$, *equation* (1.1) *has no* T*-periodic solutions,*
(2) *for* $s = s_0$, *equation* (1.1) *has at least one* T*-periodic solution,*
(3) *for* $s > s_0$, *equation* (1.1) *has at least two* T*-periodic solutions.*

Proof. Without loss of generality, we can suppose that the map $\sigma \mapsto d_\sigma$ is defined on $[0, +\infty)$ and is monotone non-decreasing. The proof of our result follows that in [9, Theorem 2]. As before, using (3.1), we write equation (1.1) in the form of (2.1). Following the functional-analytic approach introduced in Section 2, we also denote by N_s the corresponding Nemytskii operator, namely,

$$(N_s x)(t) := f(x(t))x'(t) + \psi_s(t, x(t)) \quad \text{for all } x \in X.$$

Let us start by fixing a parameter $s_1 > \max\{0, g_0\}$. We claim that the following property is satisfied:
(P) there exists a positive constant $\Lambda = \Lambda(s_1)$ such that for each $s \leq s_1$, any solution of $Lx = \lambda N_s x$, with $0 < \lambda \leq 1$, satisfies $\|x\|_\infty < \Lambda$.

In order to prove property (P), we observe that, by (G0) and $s \le s_1$, it follows that $\psi_s(t,x) \ge -\gamma_0(t) - s_1$ for a.e. $t \in [0,T]$. In this manner, condition (H0) of Lemma 2.1 holds for $\gamma := \gamma_0(t) - |s_1|$, and there exists a positive constant $K = K(s_1)$ such that, any possible T-periodic solution of

$$x'' + \lambda f(x)x' + \lambda\psi_s(t,x) = 0, \quad 0 < \lambda \le 1, \tag{3.3}$$

satisfies

$$\max x - \min x \le K.$$

Next, we observe that any possible T-periodic solution of (3.3) satisfies

$$\max x > -d_{s_1}.$$

Indeed, if $x(t) \le -d_{s_1}$ for all t, then, taking the average of the equation on $[0,T]$ (and dividing by $\lambda > 0$), we obtain

$$0 = \frac{1}{T}\int_0^T \psi_s(t,x(t))\,dt = \frac{1}{T}\int_0^T g(t,x(t))\,dt - s \ge \frac{1}{T}\int_0^T g(t,x(t))\,dt - s_1 > 0,$$

as a consequence of (G2), and so a contradiction is achieved. Similarly, from (G3), it follows that

$$\min x < d_{s_1}.$$

By the above inequalities we conclude that

$$\|x\|_\infty < \Lambda(s_1) := K(s_1) + d_{s_1},$$

proving property (P).

As a next step, we observe that there is no T-periodic solution for equation (3.3) for $s < \nu_0$, where ν_0 is the constant introduced in (3.2) in Theorem 3.1.

Let us fix now a constant $s_2 < \nu_0$. Let also ρ_g be a non-negative L^1-Carathéodory function bounding $|g(t,x)|$ for $|x| \le \Lambda(s_1)$, so that

$$|\psi_s(t,x)| \le \rho_g(t) + \max\{s_1, |s_2|\} \quad \text{for a.e. } t \in [0,T], \text{ all } s \in [s_2,s_1] \text{ and all } x \in [-\Lambda(s_1), \Lambda(s_1)].$$

An application of Lemma 2.2, along with property (P), leads to the existence of a constant $\eta(s_1,s_2) > 0$ such that for each $s \in [s_2,s_1]$, any solution of $Lx = \lambda N_s x$, with $0 < \lambda \le 1$, satisfies $\|x'\|_\infty < \eta(s_1,s_2)$.

Following [9], we define the set

$$\Omega_1 = \Omega_1(R_1,R_2) := \{x \in C^1_T : \|x\|_\infty < R_1,\ \|x'\|_\infty < R_2\},$$

which is open and bounded in C^1_T. Putting $\lambda = 1$ and varying $s \in [s_2,s_1]$ as a homotopic parameter, we obtain that

$$D_L(L - N_{s_1}, \Omega_1) = D_L(L - N_{s_2}, \Omega_1) = 0 \quad \text{for all } R_1 \ge \Lambda(s_1) \text{ and all } R_2 \ge \eta(s_1,s_2).$$

From Theorem 3.1 we already know that for $s = s_1$, there is at least one solution and, if there is a solution for some $\tilde{s} < s_1$, then also for every $s \in [\tilde{s}, s_1]$ a solution exists. We claim now that a second solution exists for $s \in \,]\tilde{s}, s_1]$.

Let w be a T-periodic solution of (1.1) for some $s = \tilde{s} < s_1$. Let us also fix $\tilde{s} < s \le s_1$. Writing equation (1.1) as

$$x'' + f(x)x' + g(t,x) - \tilde{s} - (s - \tilde{s}) = 0,$$

we have that $\beta(t) \equiv w(t)$ is a strict upper solution of (1.1) (as proved in Theorem 3.1). On the other hand, for $\sigma = s$, condition (G2) implies Villari's condition at $-\infty$ with $\delta = 1$. Given any constant $R_1 \ge \Lambda(s_1) + 1$ and by fixing a constant $R_2 \ge \eta(s_1,s_2)$, we have that

$$\Omega := \Omega(R_1, w, R_2) \subseteq \Omega_1 := \Omega_1(R_1,R_2),$$

with

$$D_L(L - N_s, \Omega) = 1, \quad D_L(L - N_s, \Omega_1) = 0.$$

Then, the additivity property of the coincidence degree theory implies that, besides a solution $w_s^{(1)} \in \Omega$, there exists also a second solution $w_s^{(2)} \in \Omega_1 \setminus \overline{\Omega}$.

As in the proof of Theorem 3.1, let us define again

$$s_0 := \inf\{s \in \mathbb{R} : (1.1) \text{ has at least one } T\text{-periodic solution}\}.$$

By the above discussion, it follows that $v_0 \le s_0 \le g_0$ and, moreover, for every $s < s_0$, there is no T-periodic solution of (1.1), and for every $s > s_0$, there are at least two T-periodic solutions of (1.1). We conclude the proof (following again the argument in [9]) by showing that for $s = s_0$, there is at least one T-periodic solution.

Let $s_2 < s_0 < s_1$ be fixed and let θ_n be a decreasing sequence of parameters with $\theta_n \to s_0$ and $\theta_n \in]s_0, s_1]$ for all n. By the previous estimates, we know that for each n, there exists at least one (actually two) T-periodic solution w_n of equation

$$x'' + f(x)x' + g(t, x) = \theta_n,$$

with

$$\|w_n\|_\infty \le \Lambda(s_1), \quad \|w_n'\|_\infty \le \eta(s_1, s_2).$$

An application of the Ascoli–Arzelà theorem, passing to the limit as $n \to \infty$, provides the existence of at least one T-periodic solution of (1.1) for $s = s_0$. This completes the proof. □

Remark 3.2. We observe that if $g\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and T-periodic in the variable t, then hypothesis (G1) and condition (A) are always satisfied (see [7]); moreover, the solutions of (1.1) are of class C^2. This is the situation considered in [9]. With respect to the quoted paper, we do not require $g(t, x) \to +\infty$ for $|x| \to +\infty$, uniformly in t. In our case, the uniform condition is replaced by assuming the existence of a lower bound as in (G0) and the Villari type conditions (G2) and (G3).

4 Final Remarks

This final section is concerned to describe some possible consequences of the results developed in the present paper. In the first part, we gave an improvement of Corollary 1.1 as a direct application of Theorem 3.2 to equation (1.7). In the second part, we considered the analogous Ambrosetti–Prodi problem under Neumann boundary conditions and showed how to achieve the same kind of existence and multiplicity results by means of simple modifications of our approach.

4.1 Application to a Weighted Liénard Equation

The study of the periodically forced generalize Liénard equation

$$x'' + f(x)x' + g(x) = e(t)$$

is a classical research topic which has been widely investigated (see, for instance, [12] and the references therein). Equation (1.7) could be seen both as a generalized Liénard equation with a weighted restoring term, and also as an example of a forced Liénard–Mathieu equation. The interest in the study of these equations has been growing up in recent years (see [13]), and it can be traced back to Minorsky's work [27]. In this case, a multiplicity result reads as follow.

Corollary 4.1. *Let $f, \phi\colon \mathbb{R} \to \mathbb{R}$ be continuous functions and suppose that* (1.6) *holds. Let $a, p \in L^\infty([0, T])$, with $a(t) \ge 0$ for a.e. $t \in [0, T]$ and $\int_0^T a(t)\,dt > 0$. Then, there exists $s_0 \in \mathbb{R}$ such that*
(1) *for $s < s_0$, equation* (1.7) *has no T-periodic solutions,*
(2) *for $s = s_0$, equation* (1.7) *has at least one T-periodic solution,*
(3) *for $s > s_0$, equation* (1.7) *has at least two T-periodic solutions.*

Proof. We apply Theorem 3.2 for

$$g(t, x) := a(t)\phi(x) - p(t).$$

Let us set $\phi_0 := \min_{x \in \mathbb{R}} \phi(x)$. For any $d > \max\{\phi_0, 0\}$, we introduce the following constants:

$$\zeta^-(d) := \min\{\phi(x) : x \le -d\}, \quad \zeta^+(d) := \min\{\phi(x) : x \ge d\}.$$

From (1.6), it follows that $\zeta^{\pm}(d) \to +\infty$ for $d \to +\infty$. For convenience, we denote by $\bar{\ell} := T^{-1}\int_0^T \ell(t)\,dt$ the mean value of a T-periodic function $\ell(t)$ in a period. Let $x \in C^1_T$ be such that $|x(t)| \geq d$ for all $t \in [0, T]$. Clearly, $x(t) \leq -d$ or $x(t) \geq d$ for all t. In the former case, we have that

$$\frac{1}{T}\int_0^T g(t, x(t))\,dt = \frac{1}{T}\int_0^T a(t)\phi(x(t))\,dt - \bar{p} \geq \bar{a}\,\zeta^-(d) - \bar{p}.$$

Analogously, we have also

$$\frac{1}{T}\int_0^T g(t, x(t))\,dt \geq \bar{a}\,\zeta^+(d) - \bar{p}$$

in the other case. This way, both the Villari type conditions (G2) and (G3) are satisfied. Condition (G0) is satisfied by choosing as $\gamma_0(t)$ the positive part of $p(t) - a(t)\phi_0$. On the other hand, (G1) holds for any constant $g_0 \geq \|a\|_\infty \phi(0) + \|p\|_\infty$. Lastly, we observe that condition (A) holds for this special choice of $g(t, x)$ (see [7]). □

4.2 Remarks on the Neumann Boundary Value Problem

In [22], Mawhin presented a comprehensive overview of the Ambrosetti–Prodi type problems for first and second-order differential equations with different boundary conditions. In particular, a version of Theorem 1.2 was obtained for the Neumann boundary value problem

$$x'' + g(t, x) = s, \quad x'(0) = x'(T) = 0. \tag{4.1}$$

In this context, for our functional-analytic setting, we introduce the space

$$X = C^1_\# := \{x \in C^1([0, T]) : x'(0) = x'(T) = 0\}.$$

Thanks to minimal changes to our results in Section 2 and Section 3, one can prove the following.

Theorem 4.1. *Let $g\colon [0, T] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying* (A), (G0) *and* (G1). *Assume also the Villari type conditions* (G2) *and* (G3) *with reference to $x \in C^1_\#$. Then there exists $s_0 \in \mathbb{R}$ such that*
(1) *for $s < s_0$, problem* (4.1) *has no solutions,*
(2) *for $s = s_0$, problem* (4.1) *has at least one solution,*
(3) *for $s > s_0$, problem* (4.1) *has at least two solutions.*

From Theorem 4.1, the next consequence can be immediately deduced for the problem

$$x'' + a(t)\phi(x) = s + p(t), \quad x'(0) = x'(T) = 0. \tag{4.2}$$

Corollary 4.2. *Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying* (1.6). *Let $a, p \in L^\infty([0, T])$, with $a(t) \geq 0$ for a.e. $t \in [0, T]$ and $\int_0^T a(t)\,dt > 0$. Then there exists $s_0 \in \mathbb{R}$ such that*
(1) *for $s < s_0$, problem* (4.2) *has no solutions,*
(2) *for $s = s_0$, problem* (4.2) *has at least one solution,*
(3) *for $s > s_0$, problem* (4.2) *has at least two solutions.*

Note that Corollary 4.2 improves some results in [30], which were obtained by means of a different approach.

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