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On dispersion for Klein Gordon equation with periodic potential in 1D

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Abstract. By exploiting estimates on Bloch functions obtained in a previous paper, we prove decay estimates for Klein Gordon equations with a time independent potential periodic in space in 1D and with generic mass.

Key words: dispersive estimates, Klein Gordon equations, periodic Schroedinger operators.

1. Introduction

We consider Schrödinger operators of the form $H = H_0 + P(x)$ with, $H_0 = -d^2/dx^2$, $P(x)$ a smooth nonconstant real valued periodic function, $P(x+1) \equiv P(x)$, with spectrum $\Sigma(H) = \cup_{n\geq 0}\Sigma_n$, formed by bands $\Sigma_n = [A_n^+, A_{n+1}^-]$ with $A_n^+ < A_{n+1}^- \leq A_{n+1}^+$ for any $n \in \mathbb{N}\cup\{0\}$. We normalize H so that $A_0^+ = 0$. We then show:

Theorem 1.1 *Under the above hypotheses consider for* $\mu > 0$ *the solutions of the following Cauchy problem for the Klein Gordon equation*

$$u_{tt} + Hu + \mu u = 0, \quad u(0, x) \equiv 0, \quad u_t(0, x) = g(x). \tag{1.1}$$

Then there exists a bounded discrete set $\mathbb{D} \subset (0, +\infty)$ *such that for any* $\mu \in (0, +\infty)\backslash\mathbb{D}$ *there is a* $C_\mu > 0$ *such that the following dispersive estimate holds*:

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq C_\mu \langle t\rangle^{-1/3} \|g(\cdot)\|_{W^{1,1}(\mathbb{R})}. \tag{1.2}$$

Maybe $\mathbb{D}$ is empty. The exact condition defining $\mathbb{D}$ is given in Lemma 3.1 below. The proof is based on results in [C] where proofs are explicitly done only for the generic case when all the spectral gaps are nonempty. Since the generic case contains all the crucial difficulties, there is no problem at extending the results in [C] to the non generic case, and we will assume this as a fact (and if this is unconvincing the reader can assume that $\ell_n = n$

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below). To illustrate Theorem 1.1 consider $P(x) = 2\kappa^2 \operatorname{sn}^2(x, \kappa)$ for $\kappa \in (0, 1)$, with $\operatorname{sn}(x, \kappa)$ the Jacobian elliptic function. Then $\Sigma(H) = [\kappa^2, 1] \cup [1 + \kappa^2, +\infty)$ and by Theorem 1.1 for generic $\mu > -\kappa^2$ we get (1.2). Notice that for $\mu = 0$ this example resembles the flat Klein Gordon rather than the flat wave equation, because we have $A_0^+ = \kappa^2 > 0$. For $H = H_0$ the equation $u_{tt} + Hu - |u|^p = 0$ for any $p > 1$ is not globally well posed for small initial data in $C_0^\infty(\mathbb{R})$ while if $p \gg 1$ this is the case for H with $P(x) = 2\kappa^2 \operatorname{sn}^2(x, \kappa)$ or $P(x) = \sin^2(x)$. In the latter case all the gaps are non empty. The proof in this paper mixes results from [C] with a specific computation in Marshall *et al.* [MSW], specifically Lemma 5 therein.

2. Reformulation, spectrum, band and Bloch functions

We will prove:

Theorem 2.1 *Let H be as in* Theorem 1.1*, that is with a smooth periodic potential, and such that $A_0^+ = 0$. Then, there is a set $\mathbb{D}$ like in* Theorem 1.1 *such that for any $\mu \in (0, +\infty)\backslash\mathbb{D}$ there is a $C_\mu > 0$ such that the following dispersive estimate holds:*

$$\| \sin(t\sqrt{H+\mu})(H+\mu)^{-3/4} \colon L^1(\mathbb{R}) \to L^\infty(\mathbb{R})\| \le C_\mu \langle t\rangle^{-1/3}. \quad (2.1)$$

The u in (1.1) is, for $1/4 > \varepsilon > 0$, $G = (H+\mu)^{1/4}(H_0+1)^{-1/2+\varepsilon}$ and $h = (H_0+1)^{1/2-\varepsilon} g$,

$$u(t) = \frac{\sin(t\sqrt{H+\mu})}{(H+\mu)^{3/4}} Gh.$$

This implies

$$\|u(t)\|_\infty \le \left\| \frac{\sin(t\sqrt{H+\mu})}{(H+\mu)^{3/4}} \right\|_{L^1 \to L^\infty} \|G\|_{L^1 \to L^1} \|h\|_1.$$

We have $\|h\|_1 \le C\|g\|_{W^{1,1}}$, $\|G\|_{L^1\to L^1} \le C(\mu)$, with $C(0) = O(1)$, so (2.1) implies (1.2). (2.1) is a consequence of the following estimate:

Proposition 2.2 *There is a set $\mathbb{D}$ like in* Theorems 1.1–2 *such that for any $\mu \in (0, +\infty)\backslash\mathbb{D}$ there is a $C_\mu > 0$ such that the following estimate holds for any (t, x, y):*

$$\left| \left(\sin(t\sqrt{H+\mu})(H+\mu)^{-3/4} \right)(x, y) \right| \le C_\mu \langle t\rangle^{-1/3}. \quad (2.2)$$

We will prove Proposition 2.2 in the case when the spectrum $\Sigma(H)$ is formed by infinitely many bands, the finitely many bands case being easier. To prove (2.2) we express the integral kernel in (2.2) in terms of Bloch functions, see below. We express $\Sigma(H) = \cup_{n=0}^{\infty}\Sigma_n$, with $\Sigma_n = [A_n^+, A_{n+1}^-]$ with $A_n^+ < A_{n+1}^- \le A_{n+1}^+$ for any $n \in \mathbb{N} \cup \{0\}$, with $A_0^+ = 0$. Set for $n \ge 0$, $a_n^\pm = \sqrt{A_n^\pm}$ and $a_{-n}^\pm = -a_n^\pm$. For $n \ge 0$ set $\sigma_n = [a_n^+, a_{n+1}^-]$ and $\sigma_{-n} = -\sigma_n$. Set $\sigma = \cup_{n=-\infty}^{\infty}\sigma_n$, with each two intervals σ_n and σ_{n+1} separated by a non empty gap g_n. For $|g_n|$ the length of the gap g_n we have the following classical result, see [E] ch. 4:

Theorem 2.3 *Let $P(x)$ be smooth. Set $\sigma_n = [a_n^+, a_{n+1}^-]$ and $g_n =]a_n^-, a_n^+[$. Then $\exists$ a strictly increasing sequence $\{\ell_n \in \mathbb{Z}\}_{n\in\mathbb{Z}}$ and a fixed constant C such that*

$$|a_n^- - \ell_n\pi| + |a_n^+ - \ell_n\pi| \le C\langle \ell_n\rangle^{-1}.$$

$\forall\ N\ \exists$ a fixed constant C_N such that $|g_n| \le C_N\langle \ell_n\rangle^{-N}\ \forall\ n$.

We review band and Bloch functions. $\forall\ w \in \mathbb{C}_+$ (the open upper half plane) $\exists$ a unique $k \in \mathbb{C}_+$ such that there are two solutions of $(H - w^2)u = 0$ of the form $\tilde{\phi}_\pm(x, w) = e^{\pm ikx}\xi_\pm(x, w)$ with $\xi_\pm(x + 1, w) \equiv \xi_\pm(x, w)$ and with $\tilde{\phi}_\pm(0, w) = 1$. The correspondence between w and the "quasi-momentum" k is a conformal map between $\mathbb{C}_+$ and a "comb" $K = \mathbb{C}_+ - \cup_{n\neq 0}[\ell_n\pi, \ell_n\pi + ih_n]$ with ℓ_n satisfying the conclusions of Theorem 2.3, with $|g_n| \le 2h_n \le C|g_n|$ for a fixed C. For generic potentials, $\ell_n \equiv n$. The map $k(w)$ extends into a continuous map in $\overline{\mathbb{C}_+}$ with $k(\sigma_n) = [\ell_n\pi, \ell_{n+1}\pi]$, with $k(w)$ a one to one and onto map between σ_n and $[\ell_n\pi, \ell_{n+1}\pi]$, and with $k(g_n) =]\ell_n\pi, \ell_n\pi + ih_n]$. $k(w)$ extends into a conformal map from $\mathbb{C} - \cup_{n\neq 0}\overline{g_n}$ into $\mathcal{K} = \mathbb{C} - \cup_n\gamma_n$ with $\gamma_n = [\ell_n\pi - ih_n, \ell_n\pi + ih_n]$. Next set $N^2(w) = \int_0^1 \tilde{\phi}_+(x, w)\tilde{\phi}_-(x, w)dx$. We have $N^2(w) = \int_0^1 |\tilde{\phi}_\pm(x, w)|^2dx > 0$ for $w \in \sigma$, $N^2(w) \neq 0$ for any $w \in \mathbb{C}\backslash\cup_{n\neq 0}\overline{g_n}$. We set $m_+^0(x, w)m_-^0(y, w) = \xi_+(x, w)\xi_-(y, w)N^{-2}(w)$ and $m_\pm^0(x, w) = \xi_\pm^0(x, w)/N(w)$ with $N(w) > 0$ for $w \in \sigma$. We express $w = w(k)$ for $k \in \mathcal{K}$ and with an abuse of notation we write $\phi_\pm(x, k)$ for $\phi_\pm(x, w(k))$ and $m_\pm^0(x, k)$ for $m_\pm^0(x, w(k))$. We call $\phi_\pm(x, k) = e^{\pm ikx}\phi_\pm(x, k)$ Bloch functions. In [C] we had to work with w complex, but here we focus only on $w \in \sigma$. The band function is $E(k) = w^2(k)$. Now we have the following well known fact:

Theorem 2.4 *Set* $\hat{f}(k) = \int_{\mathbb{R}} \phi_+(y, k) f(y) dy$ *for any* $k \in \mathbb{R} \backslash \pi\mathbb{Z}$. *Then*:

$$\int_{\mathbb{R}} |f(y)|^2 dy = \int_{\mathbb{R}} |\hat{f}(k)|^2 dk, \quad f(x) = \int_{\mathbb{R}} \phi_-(x, k) \hat{f}(k) dk,$$
$$\widehat{Hf}(k) = E(k)\hat{f}(k).$$

In particular we have

$$\frac{\sin(t\sqrt{H+\mu})}{(H+\mu)^{3/4}}(x, y)$$
$$= \int_{\mathbb{R}} e^{-i(x-y)k} \frac{\sin(t\sqrt{E(k)+\mu})}{(E(k)+\mu)^{3/4}} m_-^0(x, k) m_+^0(y, k) dk. \tag{2.3}$$

We will show that the generalized integral (2.3) is a function which satisfies (2.2).

3. Estimates on band and Bloch functions

We set $\dot{f} = df/dk$, $f' = df/dw$ and $\eta(k) = \sqrt{E(k)+\mu}$. We compute

$$\dot{\eta} = \frac{\dot{E}}{2(E+\mu)^{1/2}}, \quad \ddot{\eta} = \frac{\ddot{E}}{2(E+\mu)^{1/2}} - \frac{\dot{E}^2}{4(E+\mu)^{3/2}};$$
$$\dddot{\eta} = \frac{\dddot{E}}{2(E+\mu)^{1/2}} - \frac{3\dot{E}\ddot{E}}{4(E+\mu)^{3/2}} + \frac{3\dot{E}^3}{8(E+\mu)^{5/2}} \tag{3.1}$$
$$= \frac{\dddot{E}}{2(E+\mu)^{1/2}} - \frac{3}{2}\dot{E}\ddot{\eta}.$$

Lemma 3.1 $\exists\ \mathbb{D} \subset (0, +\infty)$, *bounded and discrete, such that* $\forall\ \mu \in (0, +\infty) \backslash \mathbb{D}$ *the system* $\ddot{\eta}(k) = \dddot{\eta}(k) = 0$, *or equivalently* (3.2) *below, has no solutions in* $\mathbb{R}$:

$$\ddot{E} = \frac{\dot{E}^2}{2(E+\mu)}, \quad \dddot{E} = 0. \tag{3.2}$$

For the case $A_0^+ = 0$ and $\mu = 0$ see Korotyaev [K1].

Proof of Lemma 3.1. We start by focusing on *low energies*. $|E| \leq E_0$ implies $|k| \leq k_0$ for a fixed $k_0 = k_0(E_0)$. By [K2] we have the following two facts:

Lemma 3.2 (a) *On each band, $\dot{E} = 0$ holds exactly at the extremes of the band.*
(b) *On each band, there is exactly one solution of $\ddot{E} = 0$, contained in the interior of each band.*

Here recall we are assuming the bands to be bounded. By (a), for $|k| \le k_0$ (3.2) cannot hold near the extremes of the bands. So there is a fixed $c > 0$, such that, if k is a root of (3.2), then k is in the set, which we denote by J, formed by the k whose distance from the nearest edge is at least c. $\dddot{E} = 0$ has finitely many solutions in J. Indeed, $\dddot{E} \not\equiv 0$, is holomorphic in $\mathcal{K}$ and $\overline{J} \subset \mathcal{K}$. So except for at most finitely μ's with $\mu > 0$, (3.2) has no solutions for $|k| \le k_0$. $\square$

Next we consider Lemma 3.1 in the *high energy* case. Recall $E = w^2$ and assume that (3.2) is satisfied at some value w_0. Since E is even we can assume $w_0 > 0$, in particular $w_0 \in [a_n^+, a_{n+1}^-]$. We have $a_n^+ = w((\ell_n \pi)^+)$, with $\ell_n \in \mathbb{N}$. We have the following facts:

Lemma 3.3 (1) *There is a fixed $C > 0$ such that for $a_{n+1}^- - C\ell_{n+1}^{1/3}|g_{n+1}|^{2/3} < w \le a_{n+1}^-$ we have $\ddot{E} < 0$.*
(2) *For any given $C_1 \gg 1$ there are n_0 and $c_0 > 0$ such that for $n \ge n_0$ and for*

$$a_{n+1}^- - C_1\ell_{n+1}^{1/3}|g_{n+1}|^{2/3} \le w \le a_{n+1}^- - C\ell_{n+1}^{1/3}|g_{n+1}|^{2/3}$$

we have $|\dddot{E}| \ge c_0 \ell_{n+1}^{-1}|g_{n+1}|^{-2/3} \gg 1$.
(3) *If $a_{n+1}^- < \infty$ then there exists exactly one point w_1, $w_1 \in (a_n^+, a_{n+1}^-)$, with $k''(w_1) = 0$. For $w \in [a_n^+, w_1)$ we have $k''(w) < 0$. Furthermore, there are positive constants C_0, C_1, C_2, α such that for any $n \ge n_0$, for any $w > a_{n+1}^- - \alpha$ we have*

$$\frac{C_1}{\langle w\rangle^3} - \frac{\varphi(w)}{4} \ge -k''(w) \ge \frac{C_2}{\langle w\rangle^3} - \frac{C_0\varphi(w)}{4},$$
$$\varphi(w) := \frac{(a_{n+1}^+ - a_{n+1}^-)^2}{|w - a_{n+1}^-|^{3/2}|w - a_{n+1}^+|^{3/2}};$$

(4) *For $a_n^- + |g_n|^{1/4} \le w \le a_{n+1}^- - |g_{n+1}|^{3/5}$ we have $|\dot{w} - 1 + Q_0/k^2| \le C|k|^{-3}$ for a fixed C, with $Q_0 := (1/2)\int_0^1 P(x)dx$. Furthermore, $\dot{E} = 2k + O(k^{-2})$ and $E = k^2 + 2Q_0 + O(k^{-2})$ with in either case $|O(k^{-2})| \le Ck^{-2}$ for a fixed C.*

(1) is a consequence of Lemma 4.2 [C], (2) is Lemma 4.3 [C]. In (3) the information on the sign of $k''(w)$ is in [K1] and the inequality is in Lemma 7.3 [C]. In (4) the inequalities for $\dot w$ and $\dot E$ are proved in Lemma 7.1 [C], the inequality for E is proved in Lemma 5.4 [C].
We return to Lemma 3.1. $\ddot E > 0$ at w_0 by (3.2). By (1) Lemma 3.3, $w_0 < a_{n+1}^- - C\ell_{n+1}^{1/3}|g_{n+1}|^{2/3}$. By $\dddot E = 0$ at w_0 and by (2) Lemma 3.3 then $w_0 \le a_{n+1}^- - C_1\ell_{n+1}^{1/3}|g_{n+1}|^{2/3}$ for some $C_1 \gg 1$. By (3) Lemma 3.3 in this region $k'' < 0$. Hence we have the inequality $\ddot E = 2(\dot w)^2 - 2w(\dot w)^3 k'' \ge 2(\dot w)^2$. Suppose $a_n^+ + C_1\ell_n^3|g_n| \le w_0$. By (4) Lemma 3.3 we have $\dot w = 1 - Q_0/k^2 + O(k^{-3})$. So

$$\ddot E \ge 2\Big(1 - 2\frac{Q_0}{k^2}\Big) + O(k^{-3}). \tag{1}$$

On the other hand for $w_0 = w(k)$ by (4) Lemma 3.3 we have

$$\begin{aligned}\ddot E = \frac{\dot E^2}{2(E+\mu)} &= 2\frac{(k+O(k^{-2}))^2}{k^2+2Q_0+\mu+O(k^{-2})}\\ &= 2 - \frac{4Q_0+\mu}{k^2} + O(k^{-3})\end{aligned}$$

The last formula is incompatible with (1) for $\mu \ge \mu_0 > 0$ with μ_0 fixed and for $|k| \gg 1/\mu$. Hence at large energies and for $a_n^+ + C_1\ell_n^3|g_n| \le w$, for some fixed $C_1 > 0$, there are no solutions of (3.2). Let $a_n^+ \le w \le a_n^+ + C_1\ell_n^3|g_n|$. The following lemma, see Lemma 4.3 [C], shows $w_0 \notin [a_n^+ + c|g_n|,\, a_n^+ + |g_n|^{3/5}]$ for $c \gg 1$ fixed:

Lemma 3.4 *For $a_n^+ + c|g_n| \le w \le a_n^+ + |g_n|^{3/5}$ for $c \gg 1$ a fixed large constant, $|\dddot E|$ is very large.*

Finally $w_0 \notin [a_n^+,\, a_n^+ + c|g_n|]$ because by the following lemma, see Lemmas 7.1 and 7.4 [C], and by Lemma 2.3 for $|u - a_n^+| \lesssim |g_n|$ then $\ddot E = \dot E^2/\{2(E+\mu)\}$ cannot hold:

Lemma 3.5 *For $|u - a_n^+| \le c|g_n|$ for $c > 0$ fixed there are n_0, $C_1 > 0$ and $C_2 > 0$ such that for any $n \ge n_0$ we have $|\ddot E| \ge C_1\ell_n|g_n|^{-1}$ and $|\dot E| \le C_2\ell_n\sqrt{u - a_n^+/|g_n|}$.*

Finally for later use we state the following, see Lemma 7.1 [C]:

Lemma 3.6 *$\exists\, C_1 > C_2 > 0$ such that $\forall m$ and $\forall w \in \sigma_m = [a_m^+,\, a_{m+1}^-]$ we have for*

$$\begin{aligned} A(w) &= \frac{|g_m|^2}{(w-a_m^+)^{1/2}(w-a_m^+ + |g_m|)^{3/2}} \\ &\quad + \frac{|g_{m+1}|^2}{(a_{m+1}^- - w)^{1/2}(a_{m+1}^- - w + |g_{m+1}|)^{3/2}} \\ 1 + C_2&\Big(A(w) + \frac{1}{\langle w\rangle^2}\Big) \ge k'(w) \ge 1 + C_1 A(w). \end{aligned}$$

Correspondingly for $k \in [\ell_m\pi,\, \ell_{m+1}\pi]$ and for $\dot{w} = dw/dk$ we have

$$\frac{1}{1 + C_2(A(w) + 1/\langle w\rangle^2)} \le \dot{w} \le \frac{1}{1 + C_1 A(w)}.$$

4. Decomposition of (2.3) and estimates on the single parts

We decompose $\big(\sin(t\sqrt{H+\mu})/(H+\mu)^{3/2}\big)(x\, y) = \sum_n K^n(t,\, x,\, y)$ with

$$K^n(t,\, x,\, y) := \int_{[\ell_n\pi, \ell_{n+1}\pi]} e^{-i(x-y)k} \frac{\sin(t\eta(k))}{\eta^{3/2}(k)} m_-^0(x,\, k) m_+^0(y,\, k) dk. \quad (4.1)$$

A basic ingredient in the proof is the stationary phase theorem, see p. 334 [S]:

Lemma 4.1 *Suppose $\phi(x)$ is real valued and smooth in $[a,b]$ with $|\phi^{(m)}(x)| \ge c_m > 0$ in $]a,\, b[$ for $m \ge 1$. For $m = 1$ assume furthermore that $\phi'(x)$ is monotonic in $]a,\, b[$. Then we have for $C_m = 5 \cdot 2^{m-1} - 2$:*

$$\Big| \int_a^b e^{i\mu\phi(x)} \psi(x) dx \Big| \le C_m (c_m \mu)^{-1/m} \Big[\min\{|\psi(a)|,\, |\psi(b)|\} + \int_a^b |\psi'(x)| dx \Big].$$

The following two lemmas are special cases of Lemmas 4.4 & 4.5 in [C]:

Lemma 4.2 *There are fixed constants $C > 0$, $C_3 > 0$, $\Gamma > 0$ and $c > 0$ such that for all x, all n we have:*

(1) *$\forall w \in [a_n^+ + C_3 \ell_n^5 |g_n|,\, a_{n+1}^- - C_3 \ell_{n+1}^5 |g_{n+1}|]$ we have $|m_+^0(x,\, k) m_-^0(y,\, k) - 1| \le C/\langle k\rangle$;*

(2) $\exists$ *fixed* $C > 0$ *such that forall* $k \in \mathbb{R}$ *we have* $|m_+^0(x, k) m_-^0(y, k)| \le C$.

Lemma 4.3 *There are fixed constants* $C > 0$ *and* $C_4 > 0$, *with* $C_4 < C_2$, *such that for all* x, *all* n *we have*:
(1) *for all* $a_n^+ + |g_n|^{1/4} \le k \le (a_n^+ + a_{n+1}^-)/2$, *then* $|\partial_k(m_-^0(x, k) m_+^0(y, k))| \le C/(k|k - \pi\ell_n|)$;
(2) *for all* $(a_n^+ + a_{n+1}^-)/2 \le k \le a_{n+1}^- - |g_{n+1}|^{3/5}$, *then* $|\partial_k(m_-^0(x, k) m_+^0(y, k))| \le C/(k|k - \pi\ell_{n+1}|)$;
(3) *for* w *in the remaining part of* $[a_n^+, a_{n+1}^-]$ *we have for* $m = n$ *(resp.* $m = n+1$*) near* a_n^+ *(resp.* a_{n+1}^-*)* $|\partial_k(m_-^0(x, k) m_+^0(y, k))| \le C/(|k - \pi\ell_m| + |g_m|)$.

By Lemmas 4.1–3 and Lemma 3.1 we conclude:

Lemma 4.4 $\exists\, \mathbb{D} \subset (0, +\infty)$, *bounded discrete, such that* $\forall\, \mu \in (0, +\infty)\backslash\mathbb{D}$ *and and for any* n_0 *bands then there exists a* $C = C(\mu, n_0) > 0$ *such that for all* x, y *and for all* $t \ge 0$ *we have* $\left|\sum_{|n| \le n_0} K_n(t, x, y)\right| \le C\langle t\rangle^{-\frac{1}{3}}$.

As a consequence of Lemma 4.4, in order to prove Proposition 2.2 it is enough to look at K^n in (4.1) with large n. It is not restrictive to sum over $n \gg 1$. We split further in (4.1). We introduce a smooth, even, compactly supported cutoff $\chi_0(t) \in [0, 1]$ with $\chi \equiv 0$ near 1 and $\chi_0 = 1$ near 0. Set $\chi_1 = 1 - \chi_0$. For $c \gg 1$ fixed we split each K^n in (4.1) as $K^n = \sum_1^5 K_j^n$ partitioning the identity in $\sigma_n = [a_n^+, a_{n+1}^-]$,

$$
\begin{aligned}
1_{\sigma_n}(w) = \chi_0\Big(\frac{w - a_n^+}{c|g_n|}\Big) + \chi_1\Big(\frac{w - a_n^+}{c|g_n|}\Big)\chi_0\Big(\frac{w - a_n^+}{|g_n|^{1/4}}\Big) \\
+ \chi_1\Big(\frac{w - a_n^+}{|g_n|^{1/4}}\Big)\chi_1\Big(\frac{a_{n+1}^- - w}{|g_{n+1}|^{3/5}}\Big) + \chi_0\Big(\frac{a_{n+1}^- - w}{|g_{n+1}|^{3/5}}\Big)\chi_1\Big(\frac{a_{n+1}^- - w}{c|g_{n+1}|}\Big) \\
+ \chi_0\Big(\frac{a_{n+1}^- - w}{c|g_{n+1}|}\Big).
\end{aligned}
$$

By $c \gg 1$ we have $\dot{w} \approx 1$ for $w \in [a_n^+ + c|g_n|, a_{n+1}^- - c|g_{n+1}|]$, Lemma 3.6.

Lemma 4.5 $\exists$ *a fixed* $C > 0$ *s.t.* $|K_1^n| \le Ct^{-1/2}|g_n|^{1/2}$ *and* $|K_5^n| \le Ct^{-1/2}|g_{n+1}|^{1/2}$.

Lemma 4.6 *There are an* $\epsilon > 0$ *and* C_ϵ *such that* $|K_2^n| \le C_\epsilon t^{-1/3}|g_n|^\epsilon$.

Lemma 4.7 *There are an* $\epsilon > 0$ *and* C_ϵ *such that* $|K_4^n| \le C_\epsilon t^{-1/3}|g_{n+1}|^\epsilon$.

Lemmas 4.5–7 imply $\sum_n \sum_{j\neq 3, j=1}^{5} |K_j^n(t, x, y)| \leq C\max\{t^{-1/3}, t^{-1/2}\}$. Turning to K_3^n, set $K_3 = \sum_n K_3^n$. The following lemma completes the proof of Proposition 2.2:

Lemma 4.8 *There is a fixed C such that $|K_3(t, x, y)| \leq C\langle t\rangle^{-1/3}$.*

We prove Lemmas 4.5–7 in § 5 and Lemma 4.8 in § 6.

5. Proof of Lemmas 4.5–7

For all the $j \neq 3$ and for $\psi(k)$ the corresponding cutoff, we consider

$$H_j^n(t, x, y) = \int_{[\ell_n\pi, \ell_{n+1}\pi]} e^{-i(x-y)k \pm it\eta(k)} m_-^0(x, k) \times m_+^0(y, k)\eta^{-3/2}(k)\psi(k)dk.$$

Lemma 4.5 is an immediate consequence of:

Lemma 5.1 *$\exists$ a fixed C such that $|H_1^n| \leq Ct^{-1/2}|g_n|^{1/2}$ and $|H_5^n| \leq Ct^{-1/2}|g_{n+1}|^{1/2}$.*

Proof. We will prove the $j = 1$ case. Recall from formula (3.1)

$$\ddot{\eta} = 2^{-1}\ddot{E}(E+\mu)^{-1/2} - 4^{-1}\dot{E}^2(E+\mu)^{-3/2}.$$

For $0 \leq w - a_n^+ \lesssim |g_n|$ by Lemma 3.6 we have $0 \leq \dot{w} \lesssim (w-a_n^+)^{1/2}|g_n|^{-1/2} \lesssim 1$ and so in particular $\dot{E}^2 \lesssim \ell_n^2$. By Lemma 3.5 we have $|\ddot{E}| \geq c\ell_n|g_n|^{-1}$ for some fixed $c > 0$. Hence $|\ddot{\eta}| \gtrsim |g_n|^{-1}$. Then, by Lemmas 4.1–3 we obtain

$$|H_1^n(t, x, y)| \leq \frac{C\sqrt{|g_n|}}{\sqrt{t}} \int_{a_n^+}^{a_n^+ + c|g_n|} \frac{(dk/dw)dw}{|k-\pi\ell_n| + |g_n|}.$$

We have $dk/dw \approx \sqrt{|g_n|}(w-a_n^+)^{-1/2}$ and so $|k-\pi\ell_n| \approx \sqrt{|g_n|}\sqrt{w-a_n^+}$. Hence

$$|H_1^n(t, x, y)| \leq \frac{C_1}{\sqrt{|g_n|t}} \int_{a_n^+}^{a_n^+ + c|g_n|} \frac{\sqrt{|g_n|}}{\sqrt{w-a_n^+}} dw \leq \frac{C_2\sqrt{|g_n|}}{\sqrt{t}}.$$

The argument for H_5^n is the same, by $\dot{E}^2 \lesssim \ell_{n+1}^2$ and $|\ddot{E}| \geq c\ell_{n+1}|g_{n+1}|^{-1}$. □

Lemma 4.6 is an immediate consequence of the following lemma:

Lemma 5.2 *There is $C > 0$ such that*
$|H_2^n| \le C \min\{\langle \ell_n\rangle^{3/2} t^{-1/2} \log(1/|g_n|),\ |g_n|^{1/4}\}$.

Proof. H_2^n is defined by an integral for $w \in [a_n^+ + c|g_n|,\ a_n^+ + |g_n|^{1/4}]$. We claim we have $|\ddot\eta| \gtrsim \langle k\rangle^{-3}$. Assume this inequality. By Lemma 3.6 we have $dk/dw \approx 1$, $w - a_n^+ \approx k - \pi\ell_n$. So by Lemmas 4.1–3, by $k - \pi\ell_n \gtrsim |g_n|$ and proceeding as in Lemma 5.1

$$|H_3^n| \le C_1 t^{-1/2} \langle \ell_n\rangle^{3/2} \int_{a_n^+ + c|g_n|}^{a_n^+ + |g_n|^{1/4}} \frac{(dk/dw)dw}{|k - \pi\ell_n| + |g_n|}$$
$$\le C_2 t^{-1/2} \langle \ell_n\rangle^{3/2} \int_{a_n^+ + c|g_n|}^{a_n^+ + |g_n|^{1/4}} \frac{dw}{w - a_n^+} \le C_3 t^{-1/2} \langle \ell_n\rangle^{3/2} \log \frac{1}{|g_n|}.$$

By Lemma 4.2 we have also $|K_3^n| \le C|g_n|^{1/4}$. To prove $|\ddot\eta| \gtrsim \langle k\rangle^{-3}$ we write $4\ddot\eta = (2(E+\mu)\ddot E - \dot E^2)(E+\mu)^{-3/2}$ with $E = k^2 + 2Q_0 + O(k^{-2})$, $\dot E^2 = 4k^2 + O(1/k)$:

$$\ddot\eta = \frac{(2\ddot E - 4)k^2 + 2\ddot E(2Q_0 + \mu) + O(1/k)}{4(E+\mu)^{3/2}}.$$

For $w \in [a_n^+ + c|g_n|,\ a_n^+ + |g_n|^{1/4}]$ we have $k'' < 0$ and so as in Lemma 3.3

$$\ddot E = 2(\dot w)^2 - 2w(\dot w)^3 k'' \ge 2(\dot w)^2 = 2(1 - 2Q_0 k^{-2} + O(k^{-3})).$$

So we get $\ddot\eta \ge (\mu + O(k^{-1}))(E+\mu)^{-3/2}$ and our claim is proved. □

Lemma 4.7 is an immediate consequence of the following lemma.

Lemma 5.3 *There is a C s.t. $|H_4^{n-1}| \le Ct^{-1/3}|g_{n-1}|^{1/30}$.*

Proof. H_4^{n-1} is defined by an integral for $w \in [a_n^- - |g_n|^{3/5},\ a_n^- - c|g_n|]$. By Lemma 4.3 [C] we have $|\dddot\eta| \gtrsim |g_n|^{-1/10}$. By Lemma 3.6 we have $dk/dw \approx 1$, $a_n^- - w \approx \pi\ell_n - k$, and so by Lemmas 4.1–3 and proceeding as in Lemma 5.1

$$|H_4^n| \le Ct^{-1/3}|g_n|^{1/30} \int_{a_n^- - |g_n|^{3/5}}^{a_n^- - \ell_n|g_n|} \frac{dw}{a_n^- - w}$$
$$\le C_1 t^{-1/3}|g_n|^{1/30} \log \frac{1}{|g_n|}.$$

□

6. Proof of Lemma 4.8

For $K_3 = \sum_n K_3^n$ we show $|K_3(t, x, y)| \le C\langle t\rangle^{-1/3}$ for C fixed by reducing to the flat case of Lemma 5 [MSW], whose proof permeates this section. Set $\chi_{\text{int}}(k) = \sum_n \chi_1(w - a_n^+/|g_n|^{1/4})\chi_1(a_{n+1}^- - w/|g_{n+1}|^{3/5})$ supported inside the union of sets $a_n^+ + |g_n|^{1/4} \le w \le a_{n+1}^- - |g_{n+1}|^{3/5}$. Then for $R = x - y$

$$K_3(t, x, y) = 2\int_0^\infty \cos(Rk)\sin(t\eta(k))\eta^{-3/2}(k) \\ \times m_-^0(x, k)m_+^0(y, k)\chi_{\text{int}}(k)dk.$$

We split the integral between $[0, t]$ and $[t, \infty)$. By Lemma 4.2 and by $\eta(k) \approx \langle k\rangle$ the $[t, \infty)$ integral has absolute value less than $C\langle t\rangle^{-1/2}$ for a fixed $C > 0$. Next write

$$\begin{aligned} h_\pm(k) &= t\eta(k) \pm Rk \\ I_\pm(t) &= \int_0^t e^{ih_\pm(k)}\eta^{-3/2}(k)m_-^0(x, k)m_+^0(y, k)\chi_{\text{int}}(k)dk \end{aligned} \tag{6.1}$$

It is not restrictive to assume $R = x - y > 0$. We start with $I_-(t)$.

Lemma 6.1 *There is a fixed C such that $|I_-(t)| \le C\langle t\rangle^{-1/3}$.*

Proof. The proof ends in Lemmas 6.16. We set $I_-(t) = I_1(t) + I_2(t)$ with

$$\begin{aligned} I_1(t) &= \int_0^t e^{ih(k)}\eta^{-3/2}(k)(m_-^0(x, k)m_+^0(y, k) - 1)\chi_{\text{int}}(k)dk \\ I_2(t) &= \int_0^t e^{ih(k)}\eta^{-3/2}(k)\chi_{\text{int}}(k)dk \end{aligned} \tag{6.2}$$

To prove Lemma 6.1 we use:

Lemma 6.2 *In* $\text{supp}(\chi_{\text{int}}) \cap [0, t]$ *we have for fixed constants*:
(1) $0 < 1 - \dot\eta(k) < Q_0k^{-2} + O(k^{-3})$;
(2) $\ddot\eta(k) \gtrsim \langle k\rangle^{-3}$ *and so* $|\ddot h(k)| = t|\ddot\eta(k)| \gtrsim t\mu\eta^{-3} \approx t\langle k\rangle^{-3}$.
(3) *There is a fixed $C > 0$ such that for any n sufficiently large we have*

$$|\dot E(k) - 2k| \le \frac{C}{\langle k\rangle^2} \quad \textit{for} \quad a_n^+ + \ell_n^3|g_n| \le w \le a_{n+1}^- - \ell_{n+1}^3|g_{n+1}|.$$

(4) *There are fixed constants $C > 0$, $C_1 > 0$ and $c > 0$ such that for any*

n and any $k \in [\ell_n \pi, \ell_{n+1}\pi]$, *that is for any* $w \in [a_n^+, a_{n+1}^-]$, *we have*:

$$a_n^+ + c|g_n| \le w \le a_{n+1}^- - C_1 \ell_{n+1}^{1/3} |g_{n+1}|^{2/3} \Rightarrow \ddot{E} \approx \frac{1}{2} + \frac{\ell_n |g_n|^2}{|w - a_n^+|^3}.$$

Proof. For (3) see Lemma 7.1 [C], for (4) see Lemmas 4.2 and 7.4 [C]. (2) is proved as in Lemma 5.2. By construction $\operatorname{supp}(\chi_{\text{int}}) \cap [0, \infty) \subset [\widetilde{k}, \infty)$ for some $\widetilde{k} \approx n_0 \gg 1$. The following for $k \gg 1$, which uses (3) here and (4) Lemma 3.3, proves (1):

$$\dot{\eta} = \frac{\dot{E}}{2(E+\mu)^{1/2}} = \frac{2k + O(k^{-2})}{2(k^2 + 2Q_0 + O(k^{-2}))^{1/2}} = 1 - Q_0 k^{-2} + O(k^{-3}).$$

□

The following lemma coincides with Lemma 4.7 [C], with the proof scattered in Lemmas 5.2, 7.1, 7.4 and 7.6 [C]:

Lemma 6.3 *In the support of* χ_{int} *we have* $w \approx k$, $\dot{w} = 1 + O(k^{-2})$, $\ddot{w} = O(k^{-3})$. *We can extend* w *from the support of* χ_{int} *to the whole of* $\mathbb{R}$ *so that the extension (which we denote again with* w*) satisfies the same relations and is an odd function.*

Thanks to Lemma 6.3 we obtain:

Lemma 6.4 *We can extend* $\eta(k) = \sqrt{w^2(k) + \mu}$ *to all* $\mathbb{R}$ *so that there are fixed positive* c_1, c_2 *so that* $\ddot{\eta}(k) \ge c_1 \langle k \rangle^{-3}$ *and* $|1 - \dot{\eta}(k)| \le c_2 \langle k \rangle^{-2}$, *and positive* c_3, c_4 *such that in* $\mathbb{R} \backslash [-1, 1]$, $c_3 \ge \dot{\eta}(k) \ge c_4$. *Furthermore, from* $\dot{\eta} = w\dot{w}/\sqrt{w^2 + \mu}$ *where* $\mu \ge \mu_0 > 0$, *from* Lemma 6.3, $n_0 = n_0(\mu_0)$ *can be chosen and the extension in* Lemma 6.3 *be done so that* $|\dot{\eta}| < 1$ *in* $\mathbb{R}$.

In the rest of the paper by $\eta(k)$ we will mean this extension and we will set $h(k) = t\eta(k) - Rk$. We have:

Lemma 6.5 *Consider the* $h(k)$ *just introduced.*

(1) *If* $t \le R$ *then* $|\dot{h}(k)| = R - t\dot{\eta}(k) \ge ct|k|^{-2}$ *for a fixed* $c > 0$.

(2) *If* $t > R$ *then* $\dot{h}(k)$ *has exactly one zero in* $[0, +\infty)$ *which we denote by* k_0.

(3) *In case* (2), *if* $k_0 > 2$, *for* $1 \le k < k_0/2$ *and for* $k > 2k_0$ *we have* $|\dot{h}(k)| \ge ct|k|^{-2}$. *If* $k_0 \le 2$ *for* $k > 2k_0$ *we have* $|\dot{h}(k)| \ge ct|k|^{-2}$.

Proof. $|\dot\eta| < 1$ implies (1). Consider $t > R$. By $\dot h = tw\dot w(w^2+\mu)^{-1/2} - R$ we have $\dot h(0) = -R$ and by Lemma 6.3 $\dot h \approx t - R > 0$ for $k \to \infty$. So there is a zero which by $\ddot h = t\ddot\eta \geq c_1 t\langle k\rangle^{-3} > 0$ is unique. We denote it by k_0. This gives us (2). We set $[a, b] = [1, t] \cap [k_0/2, 2k_0]$. For $k \in (1, a)$ by Lemma 6.5 we have $\dot h(k) < \dot h(2k) < 0$. So for some $\widetilde k \in [k, 2k]$

$$|\dot h(k)| > \dot h(2k) - \dot h(k) = \ddot h(\widetilde k)k > ctk^{-2}.$$

For $k > b \geq 2k_0$ we have $\dot h(k) > \dot h(k/2) > 0$ and for some $\widetilde k \in [k/2, k]$

$$|\dot h(k)| > \dot h(k) - \dot h(k/2) = \ddot h(\widetilde k)k/2 > ctk^{-2}.$$

□

Lemmas 4.1 and 6.2–4 imply:

Lemma 6.6 *Let $\dot H(k) = e^{ih(k)}$ with $H(0) = 0$. Then for a fixed $c > 0$ we have $|H(k)| \leq ct^{-1/2}\langle k\rangle^{3/2}$ for all $k \in [0, t]$.*

Next, we have the following analogue of Lemma 5 [MSW]:

Lemma 6.7 *For $|g(k)| = O(\langle k\rangle^{-5/2})$ we have*

$$\Big|\int_0^t H(k)g(k)dk\Big| \leq C\langle t\rangle^{-1/2}. \tag{1}$$

Proof. By Lemma 6.6, $|H(k)| \lesssim \langle t\rangle^{-1/2}$ for $|k| \leq 2$. If $R \geq t$ by Lemma 6.4 we have $|\dot h(k)| \geq ctk^{-2}$. Then by Lemma 4.1 for $k \geq 1$ we have $|H(k) - H(2)| \leq ct^{-1}k^2$. By $|H(2)| \leq C\langle t\rangle^{-1/2}$ and by $|g(k)| = O(\langle k\rangle^{-5/2})$ we obtain (1). If $R < t$ by Lemma 6.4 $\dot h(k)$ has one zero which we denote by k_0. If $k_0 \leq 2$ we can repeat the above argument. If $k_0 > 2$ set $[a, b] = [1, t] \cap [k_0/2, 2k_0]$ as in Lemma 6.5. For $k \in (1, a)$ and $p < k$ by Lemma 6.5

$$|\dot h(p)| > ctp^{-2} > Ctk^{-2}.$$

By Lemmas 4.1 and 6.5 $|H(k)| \leq ck^2t^{-1}$. Then we get

$$\Big|\int_1^a H(k)g(k)dk\Big| \leq Ct^{-1/2}.$$

For $k > p > b \geq 2k_0$ by Lemma 6.5

$$|\dot h(p)| > ctp^{-2} > Ctk^{-2}.$$

By Lemmas 4.1 and 6.6 $|H(k)-H(b)| \le ck^2t^{-1}$. Then we get

$$\Big|\int_b^t (H(k)-H(b))g(k)dk\Big| \le Ct^{-1/2}.$$

By Lemma 6.6

$$\Big|\int_b^t H(b)g(k)dk\Big| \le Ct^{-1/2}\langle b\rangle^{3/2}b^{-3/2}.$$

Finally

$$\begin{aligned}\Big|\int_a^b H(k)g(k)dk\Big| &\le Ct^{-1/2}\int_{k_0/2}^{2k_0}\frac{dk}{k}\\ &\le Ct^{-1/2}2\log 2.\end{aligned}$$

□

Lemma 6.8 *There is a fixed C such that for the $I_2(t)$ in* (6.2), $|I_2(t)| \le C\langle t\rangle^{-1/3}$.

Proof. Write $\chi_{\rm int}(k) = 1-\chi_{\rm ext}(k)$ and correspondingly $I_2(t) = I_{21}(t) - I_{22}(t)$ with $I_{21}(t)=\int_0^t e^{ih(k)}\eta^{-3/2}(k)dk$ and $I_{22}(t)=\int_0^t e^{ih(k)}\chi_{\rm ext}(k)\eta^{-3/2}(k)dk$. Then:

Lemma 6.9 *There is a fixed C such that $|I_{21}(t)| \le C\langle t\rangle^{-1/2}$.*

Lemma 6.10 *There is a fixed C such that $|I_{22}(t)| \le C\langle t\rangle^{-1/3}$.*

Proof of Lemma 6.9. We have

$$I_{21}(t) = \frac{3}{2}\int_0^t H(k)\eta^{-5/2}(k)\dot\eta(k)dk + H(t)\eta^{-3/2}(t).$$

$|H(t)| \le t$ and $\eta(t)\approx\langle t\rangle$ imply $|H(t)\eta^{-3/2}(t)| \lesssim \langle t\rangle^{-1/2}$. We have $\eta(k)\approx k$ and $|\dot\eta| \lesssim 1$. So $g(k) := \eta^{-5/2}(k)\dot\eta(k) = O(\langle k\rangle^{-5/2})$. Then Lemma 6.7 implies Lemma 6.9. □

Proof of Lemma 6.10. For $J_n = [\ell_n\pi - C|g_n|^{3/5},\ \ell_n\pi + C|g_n|^{1/4}]$ for some fixed $C \gtrsim 1$, set

$$I_{22}(t) = \int_0^t e^{ih(k)}\eta^{-3/2}(k)\chi_{\rm ext}(k)dk = -I_{221}(t) - I_{222}(t),$$

$$I_{221}(t) := \sum_{n=0}^{\infty} \int_{[0,t]\cap J_n} H_n(k)\chi_{\text{ext}}(k)\frac{d}{dk}\eta^{-3/2}(k)dk,$$

$$I_{222}(t) := \sum_{n=0}^{\infty} \int_{[0,t]\cap J_n} H_n(k)\eta^{-3/2}(k)\dot{\chi}_{\text{ext}}(k)dk,$$

with $\dot{H}_n(k) = e^{ih(k)}$, $H_n(\ell_n) = 0$. By $\ddot{h}(k) = t\ddot{\eta}(k)$, $c_1\langle k\rangle^{-3} \le \ddot{\eta}(k)$, $|\dot{\eta}| < 1$ and Lemma 6.6 which implies $|H_n(k)| \le Ct^{-1/2}\langle k\rangle^{3/2}$ for C fixed,

$$|H_n(k)\chi_{\text{ext}}(k)\eta^{-5/2}(k)\dot{\eta}(k)| \le Ct^{-1/2}\langle k\rangle^{3/2}\langle k\rangle^{-5/2}$$

so $|I_{221}(t)| \le Ct^{-1/2}\sum_{n=1}^{[t]}\langle n\rangle^{-1} \lesssim t^{-1/2}|\log t|$. By $|H_n(k)| \le C\min\{t^{-1/2}\langle k\rangle^{3/2}), |g_n|^{1/4}\}$

$$\int_{[0,t]\cap J_n} |H_n(k)\eta^{-3/2}(k)\dot{\chi}_{\text{ext}}(k)|dk \le \min\{t^{-1/2}, |g_n|^{1/4}\langle\ell_n\rangle^{-3/2}\}.$$

This by Theorem 2.3 implies $|I_{222}(t)| \le Ct^{-1/3}$. □

Lemma 6.11 *There is a fixed C such that $|I_1(t)| \le C\langle t\rangle^{-1/3}$.*

Proof. We fix some small $\varepsilon > 0$ and split $I_1(t) = I_{11}(t) + I_{12}(t)$ with

$$\widetilde{\chi}_{\text{int}}(k) := \sum_n \chi_1\big(\varepsilon^{-1}(k-\pi\ell_n)\big)\chi_1\big(\varepsilon^{-1}(\ell_{n+1}\pi - k)\big)$$

$$I_{11}(t) := \int_0^t e^{ih(k)}\eta^{-3/2}(k)\big(m_-^0(x,\, k)m_+^0(y,\, k) - 1\big)\widetilde{\chi}_{\text{int}}(k)dk$$

$$I_{12}(t) := \int_0^t e^{ih(k)}\eta^{-3/2}(k)\big(m_-^0(x,\, k)m_+^0(y,\, k) - 1\big) \times\chi_{\text{int}}(k)(1-\widetilde{\chi}_{\text{int}}(k))dk.$$

□

Lemma 6.12 *There is a fixed C such that $|I_{11}(t)| \le C\langle t\rangle^{-1/2}$.*

Proof. We have $m_-^0(x,k)m_+^0(y,k) - 1 = O(k^{-1})$ and $\partial_k(m_-^0(x,k)m_+^0(y,k)) = O(k^{-1})$ in the support of $\widetilde{\chi}_{\text{int}}(k)$. Then Lemma 6.7 implies Lemma 6.12. □

Lemma 6.13 *There is a fixed C such that $|I_{12}(t)| \le C\langle t\rangle^{-1/3}$.*

Proof. We consider $I_{12}(t) = \sum_n I_{12}^n(t)$, $I_{12}^n(t) := \int_{\ell_n\pi}^{\ell_{n+1}\pi} e^{ih(k)}f(k)dk$

$$f(k) := \Psi_n(k)\eta^{-3/2}(k)\big(m_-^0(x,\, k)m_+^0(y,\, k) - 1\big)\text{where}$$

$$\Psi_n(k) := \chi_1\Big(\frac{w-a_n^+}{|g_n|^{3/5}}\Big)\chi_1\Big(\frac{a_{n+1}^- - w}{|g_{n+1}|^{1/4}}\Big)\chi_0\Big(\frac{k-\pi\ell_n}{\varepsilon}\Big)\chi_0\Big(\frac{\ell_{n+1}\pi - k}{\varepsilon}\Big).$$

□

Observe that $\Psi_n(k) = \Psi_{n1}(k) + \Psi_{n2}(k)$ with $\Psi_{n1}(k)$ supported in $|g_n|^{1/4} \lesssim k - \pi\ell_n \lesssim \varepsilon$ and with $\Psi_{n2}(k)$ supported in $\varepsilon \gtrsim \pi\ell_{n+1} - k \gtrsim |g_{n+1}|^{3/5}$. Correspondingly write $f = f_1 + f_2$ and $I_{12}^n = I_{12}^{n1} + I_{12}^{n2}$. We have:

Lemma 6.14 *For a fixed* C *and for* $j = 1, 2$: $\sum_n |I_{12}^{nj}(t)| \le C\langle t\rangle^{-1/2}|\log t|$.

Proof. We focus on I_{12}^{n1}, the proof for I_{12}^{n2} being almost the same. We have

$$I_{12}^{n1}(t,\, x,\, y) = \int_{\ell_n\pi}^{(\ell_n+1/2)\pi} \dot H_n(k) f_1(k) dk \quad \text{with} \qquad H_n(k) = \int_{\ell_n\pi}^{k} e^{ih(k')}dk'.$$

For I_{12}^{n2} the proof is the same but with $H_n(k) = \int_{\ell_{n+1}\pi}^{k} e^{ith(k')}dk'$. We get

$$\begin{aligned} I_{12}^{n1}(t) &= -I_{121}^{n1}(t) - I_{122}^{n1}(t) \quad \text{with} \\ I_{121}^{n1}(t) &= \int_{\ell_n\pi}^{(\ell_n+1/2)\pi} H_n(k)\partial_k\big(\Psi_{n1}(k)\eta^{-3/2}(k)\big) \\ &\qquad\qquad \times\big(m_-^0(x,\, k)m_+^0(y,\, k) - 1\big)dk \\ I_{122}^{n1}(t) &= \int_{\ell_n\pi}^{(\ell_n+1/2)\pi} H_n(k)\Psi_{n1}(k)\eta^{-3/2}(k) \\ &\qquad\qquad \times\partial_k\big(m_-^0(x,\, k)m_+^0(y,\, k) - 1\big)dk. \end{aligned}$$

□

Lemma 6.15 *There is a fixed* $C > 0$ *such that* $\sum_{n=1}^{\infty} |I_{121}^{n1}(t)| \le C\langle t\rangle^{-1/3}$.

Proof. Set

$$\begin{aligned} \partial_k\big(\Psi_{n1}(k)\eta^{-3/2}(k)\big) &= -\frac{3}{2}\Psi_{n1}(k)\eta^{-5/2}(k)\dot\eta(k) + \dot\Psi_{n1}(k)\eta^{-3/2}(k) \\ &=: a(k) + b(k). \end{aligned}$$

We have $a(k) = O(k^{-5/2})$ and

$$|b(k)| \le C(\varepsilon^{-1}k^{-3/2}\chi_{I_1}(k) + |g_n|^{-1/4}k^{-3/2}\chi_{I_2}(k))$$

with length of $I_1 \approx \varepsilon$ and length of $I_2 \approx |g_n|^{1/4}$. Recall by Lemma 4.2,

$$\big(m_-^0(x,\, k)m_+^0(y,\, k) - 1\big) = O(k^{-1}).$$

For $t \le R$ then $\dot h \ge ct\langle k\rangle^{-2}$. From the estimates on $a(k)$ and $b(k)$ we get $|I_{121}^{n1}(t)| \le Ct^{-1}\ell_n^{-1/2}$. Summing up over $\ell_n \lesssim t$ we get much less than $t^{-1/3}$. For $t > R$ and for the critical point $k_0 > 2$ (otherwise proceed as above) distinguish between two cases

Case 1: $\pi\ell_n$ outside $[k_0/2-\pi,\, 2k_0+\pi]$. Then $|\dot h| \ge ct\langle k\rangle^{-2}$ by Lemma 6.5 and $|I_{121}^{n1}(t)| \le Ct^{-1}\ell_n^{-1/2}$. Summing up over $\ell_n \le t$ we get $O(t^{-1/2})$.

Case 2: $\pi\ell_n$ inside $[k_0/2-\pi,\, 2k_0+\pi]$. Then by Lemma 6.4 we have $|H_n(k)| \le Ct^{-1/2}\langle k\rangle^{3/2}$ and $|I_{121}^{n1}(t)| \le Ct^{-1/2}\ell_n^{-1}$. Summing up over $\ell_n \approx k_0$ we get $O(t^{-1/2})$. □

Lemma 6.16 *There is a fixed* $C > 0$ *such that* $\sum_{n=1}^{\infty}|I_{122}^{n1}(t)| \le C\langle t\rangle^{-1/3}$.

(1) Suppose $t > R$. Then $|\dot h(k)| \ge ct\langle k\rangle^{-2}$. For a fixed C by Lemma 4.1

$$\Big|\int_{\ell_n\pi}^{k} e^{ih(k')}dk'\Big| \le \min\{C\langle k\rangle^2 t^{-1},\, |k-\pi\ell_n|\}. \tag{6.3}$$

Next we split $I_{122}^{n}(t) = \int_{\ell_n\pi}^{\ell_n\pi+t^{-1}}\cdots + \int_{\ell_n\pi+t^{-1}}^{(\ell_n+1/2)\pi}\cdots$. By Lemma 4.3

$$\begin{aligned}&\Big|\int_{\ell_n\pi}^{\ell_n\pi+t^{-1}} H_n(k)\Psi_{n1}(k)\eta^{-3/2}(k)\partial_k\big(m_-^0(x,\, k)m_+^0(y,\, k)\big)\Big|\\ &\le C\int_{\ell_n\pi}^{\ell_n\pi+t^{-1}} |k-\pi\ell_n||k-\pi\ell_n|^{-1}\langle\ell_n\rangle^{-5/2}dk = Ct^{-1}\langle\ell_n\rangle^{-5/2}\end{aligned} \tag{6.4}$$

and

$$\Big|\int_{\ell_n\pi+t^{-1}}^{(\ell_n+1/2)\pi}\cdots\Big| \le \int_{\ell_n\pi+t^{-1}}^{(\ell_n+1/2)\pi} Ct^{-1}\langle k\rangle^2|k-\pi\ell_n|^{-1}\langle\ell_n\rangle^{-5/2} \tag{6.5}$$

and so $|I_{122}^{n}(t)| \le C\langle\ell_n\rangle^{-1/2}t^{-1}\log t$. Then $\sum|I_{122}^{n}(t)| \le Ct^{-1/2}\log t$.
(2) Suppose $t < R$. Then there is a unique $k_0 > 0$ with $\dot h(k_0) = 0$. If $k_0 \le 2$ we have $\dot h(k) \ge ct\langle k\rangle^{-2}$ in the support of the integrands and we can apply the argument in (1). If $k_0 > 2$ set $[a,\, b] = [1,\, t]\cap[k_0/2,\, 2k_0]$. Then consider $\ell_n \le a - \pi/2$. Then for a fixed C we get (6.3) and by proceeding as in the case $t > R$ we can split again and obtain estimates (6.4–5). Same is true for $\ell_n \ge b$. Summing up over all these $\ell_n \lesssim t$ we get $\sum|I_{122}^{n}(t)| \le$

$Ct^{-1/2}\log t$. For $a-\pi/2<\ell_n<b$ for a fixed C

$$\Big|\int_{\ell_n\pi}^{k} e^{ith(k')}dk'\Big| \le \min\{Ct^{-1/2}k^{3/2},\, |k-\pi\ell_n|\}.$$

Next we split

$$I_{122}^n(t)=\int_{\ell_n\pi}^{\ell_n\pi+t^{-1/2}\ell_n^{3/2}}\cdots+\int_{\ell_n\pi+t^{-1/2}\ell_n^{3/2}}^{(\ell_n+1/2)\pi}\cdots$$

But now

$$\Big|\int_{\ell_n\pi}^{\ell_n\pi+t^{-1/2}\ell_n^{3/2}}\cdots\Big| \le C\int_{\ell_n\pi}^{\ell_n\pi+t^{-1/2}\ell_n^{3/2}} |k-\pi\ell_n||k-\pi\ell_n|^{-1}\langle\ell_n\rangle^{-5/2} = Ct^{-1/2}\ell_n^{-1}$$

and

$$\Big|\int_{\ell_n\pi+t^{-1/2}\ell_n^{3/2}}^{(\ell_n+1/2)\pi}\cdots\Big| \le \int_{\ell_n\pi+t^{-1/2}\ell_n^{3/2}}^{(\ell_n+1/2)\pi} t^{-1/2}|k-\pi\ell_n|^{-1}\langle\ell_n\rangle^{-1}$$

and so $|I_{122}^n(t)|\le C\langle\ell_n\rangle^{-1}t^{-1/2}\log t$. Then $\sum|I_{122}^n(t)|\le Ct^{-1/2}\log t$.

To complete the proof of Lemma 4.8 we have to prove the following lemma whose proof is analogous to the proof for $I_-(t)$ in the easier case $t<R$ and which we skip:

Lemma 6.17 *There is a fixed C such that $|I_+(t)|\le C\langle t\rangle^{-1/3}$.*

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