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Well-posedness and control in a hyperbolic–parabolic parasitoid–parasite system

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Abstract

We develop a time and space-dependent predator—prey model. The predators' equation is a nonlocal hyperbolic balance law, while the diffusion of prey obeys a parabolic equation, so that predators *"hunt"* for prey, while prey diffuse. A control term allows to describe the use of predators as parasitoids to limit the growth of prey–parasites. The general well-posedness and stability results here obtained ensure the existence of optimal pest control strategies, as discussed through some numerical integrations. The specific example we have in mind is that of *Trichopria drosophilæ* used to fight against the spreading of *Drosophila suzukii*.

KEYWORDS

nonlocal conservation laws, optimal control of conservation laws, predator–prey systems

1 | INTRODUCTION

We consider the following mixed system on $\mathbb{R}^n$

$$\begin{cases} \partial_t u + \nabla\cdot(u\, v(t,w)) = f(t,x,w)\, u + q(t,x) \\ \partial_t w - \mu\,\Delta w = g(t,x,u,w)\, w, \end{cases} \tag{1}$$

where $u = u(t,x)$ and $w = w(t,x)$ represent, respectively, the predator and the prey density at time $t \in \mathbb{R}_+$ and position $x \in \mathbb{R}^n$. We remark that in the vector field v the dependence on the prey density w is of a *functional* nature thus allowing, for instance, to describe predators that hunt for the prey they perceive within a given distance. The parameter μ, related to the prey diffusion speed, is assumed to be strictly positive.

Once the fundamental well-posedness and stability properties for (1) are obtained, we consider the problem to steer the solution to (1) to optimize a goal, typically represented by the minimization of a functional defined on the solutions to (1). In the driving example we have in mind, the term q in (1) represents the space and time-dependent deployment of parasitoids (predators) in the environment, aiming at limiting a given parasites (prey). In other words, (1) provides a possible structure for the search for an optimal strategy in biological pest control. Preliminary general numerical results are provided in Ref. 1.

A specific situation that fits the present framework is the current attempt to limit the spreading of *Drosophila suzukii* (a pest damaging fruits' cultivation) by means of ad hoc deployments of *Trichopria drosophilæ* (a parasitoid laying its eggs in the larvæ of the *Drosophila suzukii*), see Refs. 2–4. An obvious question risen by the adoption of these biological strategies is the search for the optimal time and space choices for the release of parasitoids in the environment. The present paper offers a framework to test and compare different strategies, see Section 3.

The range of applications of renewal equations, like the first in (1), and diffusion equations, like the second in (1), is extremely vast, in particular in the *"local"* version. To recall other fields where these equations play a role, we point to Ref. 5 as an optimal reference on renewal equations with applications to biology or also to Ref. 6, where age structured population models and epidemiological models are considered in detail.

From the analytic point of view, besides the introduction of the control, the mixed system (1) comprehends the one studied in Ref. 7 also by taking into account general source terms that may depend on the unknown variables, as well as on both t and x. Moreover, the flow $u\, v(t, w)$ in the first equation in (1) accounts for the velocity chosen by predators in response to the prey density distribution w. A key feature of the mixed system (1) is the nonlocality and nonlinearity of the function v with respect to the prey density. For instance, the choice

$$(v(t,w))(x) = \kappa(t,x)\,\frac{\nabla(w*\eta)(x)}{\sqrt{1+\|\nabla(w*\eta)(x)\|^2}}, \tag{2}$$

means that predators are directed toward regions where the concentration of prey is greater. Above, the positive function κ is the maximal speed of predators and may depend on time and space. For any fixed positive smooth mollifier η, the space-convolution product $(w(t)*\eta)(x)$ is an average of the prey density at time t around position x. The denominator in (2) acts as a smooth normalization factor.

The next section is devoted to the well-posedness and stability of the Cauchy Problem for (1). Then, we also deal with the optimal control of the solutions to (1) by means of the control q and aiming at the minimization of a given integral functional. A specific application of these theoretical results is in Section 3. All analytic proofs are deferred to Section 4.

2 | MAIN RESULTS

Below, we fix $T > t_o \geq 0$, possibly allowing the case $T = +\infty$, and correspondingly we set

$$I = [t_o, T] \text{ or } I = [t_o, +\infty[\quad \text{and} \quad J = \left\{(t_1, t_2) \in I^2 : t_1 < t_2\right\}. \tag{3}$$

The space dimension n is fixed throughout, as well as the parameter $\mu > 0$. For the heat kernel, we use the notation $H_\mu(t,x) = (4\pi\mu t)^{-n/2} \exp(-\|x\|^2/(4\mu t))$, where $t \in I$, $x \in \mathbb{R}^n$. As it is well known, $\|H_\mu(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} = 1$.

We recall below the definition of solution to (1), slightly extending that in Ref. 7, and adapting it to the present setting of time- and space-dependent coefficients.

Definition 1. A pair $(u,w) \in \mathbf{C}^0(I;\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^2))$ is a *solution* to problem (1) on I if

- setting $a(t,x) = g(t,x,u(t,x),w(t,x))$, w is a weak solution to $\partial_t w - \mu\Delta w = a\,w$;
- setting $b(t,x) = f(t,x,w(t,x))$ and $c(t,x) = (v(t,w(t)))(x)$, u is a weak solution to $\partial_t u + \nabla\cdot(u\,c) = b\,u + q$.

The extension of Definition 1 to Cauchy problems is immediate. For completeness, Definition 2 provides the definition of solution to the parabolic equation $\partial_t w - \mu\Delta w = a\,w$, while Definition 3 recalls the definition of solution to the balance law $\partial_t u + \nabla\cdot(u\,c) = b\,u + q$.

Introduce the spaces

$$\begin{array}{ll} \mathcal{U} = (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R}) & \mathcal{U}^+ = (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R}_+) \\ \mathcal{X} = \mathcal{U} \times \mathcal{U} & \mathcal{X}^+ = \mathcal{U}^+ \times \mathcal{U}^+ \end{array} \tag{4}$$

and the norm

$$\|(u,w)\|_{\mathcal{X}} = \|u\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} + \|w\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}. \tag{5}$$

We are now ready to state the key well-posedness and stability result of this paper.

Theorem 1. *Consider problem (1) under the following assumptions:*

($\boldsymbol{v}$) $v : I \times (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^n;\mathbb{R}) \to (\mathbf{C}^2 \cap \mathbf{W}^{1,\infty})(\mathbb{R}^n;\mathbb{R}^n)$ *admits two maps* $K_v \in \mathbf{L}^\infty_{\mathbf{loc}}(I;\mathbb{R}_+)$ *and* $C_v \in \mathbf{L}^\infty_{\mathbf{loc}}(I \times \mathbb{R}_+;\mathbb{R}_+)$ *weakly increasing in each argument and such that, for all* $t \in I$ *and* $w, w_1, w_2 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^n;\mathbb{R})$,

$$\|v(t,w)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le K_v(t)\,\|w\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})},$$

$$\|\nabla v(t,w)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})} \le K_v(t)\,\|w\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})},$$

$$\|v(t,w_1) - v(t,w_2)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le K_v(t)\,\|w_1 - w_2\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})},$$

$$\|\nabla(\nabla\cdot v(t,w))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} \le C_v(t,\|w\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})})\,\|w\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})},$$

$$\|\nabla\cdot(v(t,w_1) - v(t,w_2))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \le C_v(t,\|w_2\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})})\,\|w_1 - w_2\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}.$$

($\boldsymbol{f}$) $f : I \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ *admits a weakly increasing map* $K_f \in \mathbf{L}^\infty_{\mathbf{loc}}(I;\mathbb{R}_+)$ *such that, for a.e.* $t \in I$, *all* $w_1, w_2 \in \mathbb{R}_+$ *and all* $w \in \mathbf{BV}(\mathbb{R}^n;\mathbb{R})$,

$$\sup_{x\in\mathbb{R}^n} |f(t,x,w_1) - f(t,x,w_2)| \le K_f(t)\,|w_1 - w_2|\,,$$

$$\sup_{x\in\mathbb{R}^n} f(t,x,w_1) \le K_f(t)\,(1 + w_1),$$

$$\mathrm{TV}\, f(t,\cdot,w(\cdot)) \le K_f(t)(1+\|w\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}+\mathrm{TV}\,(w)).$$

(***g***) *$g : I\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ admits a weakly increasing map $K_g\in\mathbf{L}^\infty_{\mathbf{loc}}(I;\mathbb{R}_+)$ such that, for a.e. $t\in I$ and all $u_1,u_2,w_1,w_2\in\mathbb{R}_+$,*

$$\sup_{x\in\mathbb{R}^n}|g(t,x,u_1,w_1)-g(t,x,u_2,w_2)| \le K_g(t)\,(|u_1-u_2|+|w_1-w_2|),$$

$$\sup_{(x,u,w)\in\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{R}_+} g(t,x,u,w)\le K_g(t).$$

(***q***) *$q\in\mathbf{L}^\infty(I\times\mathbb{R}^n;\mathbb{R}_+)\cap\mathbf{L}^\infty(I;\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}_+))$ and $q(t)\in\mathbf{BV}(\mathbb{R}^n;\mathbb{R}_+)$, for a.e. $t\in I$.*

Then, for any initial datum $(u_o,w_o)\in\mathcal{X}^+$, problem (1) admits a unique solution

$$(u,w)\in\mathbf{C}^0\big(I,\mathbf{L}^1\big(\mathbb{R}^n;\mathbb{R}^2_+\big)\big)$$

in the sense of Definition 1 and, moreover,

(**T.1**) ***A priori estimates**: for all $t\in I$, we have*

$$\|w(t)\|_{\mathbf{L}^1(\mathbb{R}^n)} \le \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n)}\; e^{K_g(t)\,(t-t_o)},$$

$$\|w(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \le \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\; e^{K_g(t)\,(t-t_o)},$$

$$\begin{aligned}\|u(t)\|_{\mathbf{L}^1(\mathbb{R}^n)} \le{}& \Big(\|u_o\|_{\mathbf{L}^1(\mathbb{R}^n)}+\|q\|_{\mathbf{L}^1([t_o,t]\times\mathbb{R}^n)}\Big)\\ &\times\exp\Big[K_f(t)\,(t-t_o)\Big(1+\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)}e^{K_g(t)\,(t-t_o)}\Big)\Big],\end{aligned}$$

$$\begin{aligned}\|u(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \le{}& \Big(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)}+\|q\|_{\mathbf{L}^1([t_o,t];\mathbf{L}^\infty(\mathbb{R}^n))}\Big)\\ &\times\exp\Big[\big(K_f(t)+K_v(t)\big)(t-t_o)\Big(1+\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)}e^{K_g(t)\,(t-t_o)}\Big)\Big].\end{aligned}$$

(**T.2**) ***Lipschitz continuous dependence on the initial data**: for (u_o,w_o), $(\tilde u_o,\tilde w_o)\in\mathcal{X}^+$,*

$$\|(u(t),w(t))-(\tilde u(t),\tilde w(t))\|_{\mathcal{X}} \le C_o(t,r)\,\|(u_o,w_o)-(\tilde u_o,\tilde w_o)\|_{\mathcal{X}}, \tag{6}$$

where the locally bounded function C_o is defined in (61) and r is an upper bound for the $\mathbf{L}^1$ norm, the $\mathbf{L}^\infty$ norm, and the total variation of the initial data, see (43).

(**T.3**) ***Stability with respect to the control** $q,\tilde q$ satisfying (**q**), for all $t\in I$,*

$$\|(u(t),w(t))-(\tilde u(t),\tilde w(t))\|_{\mathcal{X}} \le C_q(t,r)\,\|q-\tilde q\|_{\mathbf{L}^1([t_o,t]\times\mathbb{R}^n)}, \tag{7}$$

where the locally bounded function C_q is defined in (64) and r is an upper bound for the $\mathbf{L}^1$ norm, the $\mathbf{L}^\infty$ norm, and the total variation of the initial data, see (43).

To prove Theorem 1, following the general lines of Ref. 7, we study separately, but symmetrically, the parabolic and the hyperbolic problems that constitute (1), namely,

$$\partial_t w - \mu\, \Delta w = a(t,x)\, w \quad \text{and} \quad \partial_t u + \nabla\cdot (c(t,x)\, u) = b(t,x)\, u + q(t,x),$$

with a, b and c as in Definition 1. All estimates use exclusively the $\mathbf{L}^1$ or $\mathbf{L}^\infty$ norms and the total variation in space.

The various hypotheses are rather technical and are motivated by our interest in providing sharp well-posedness and stability estimates, rather than aiming at the widest generality. In particular, we refrain from extending to the mere **BV** dependence on the space variable in the speed v, as it stems from assumption ($\boldsymbol{v}$), in particular from the fourth requirement therein.

The assumptions on the reaction functions f and g play a key role. Essentially, they require for both Lipschitz continuity in u and w, sublinearity for f, and boundedness for g. The latter condition on f is satisfied, for instance, if f has bounded variation in the space variable and is $\mathbf{C}^1$ in the latter one.

Remark 1. Note the different behaviors of f and g allowed by conditions ($\boldsymbol{f}$) and ($\boldsymbol{g}$), namely, $\sup_{x\in\mathbb{R}^n} f(t,x,w) \leq K_f(t)\,(1+w)$ and $\sup_{(x,u,w)\in\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{R}_+} g(t,x,u,w) \leq K_g(t)$. For instance, f may well increase in w, while g may decrease in both u and w. Thus, the classical Lotka–Volterra source terms $f(w) = \alpha\, w - \beta$ and $g(u) = \gamma - \delta\, u$ (for $\alpha, \beta, \gamma, \delta$ positive and constant) are compatible with ($\boldsymbol{f}$) and ($\boldsymbol{g}$), comprising the problem studied in Ref. 7 when $q \equiv 0$.

Theorem 1 allows to consider optimal control problems based on (1). To this aim, introduce a cost functional measuring the relevance of the presence of the pest, for instance quantifying its effect on cultivation. Inspired by Ref. [1, Section 4], we propose a cost of the general form

$$\mathcal{I} = \int_I \int_{\mathbb{R}^n} \Phi(t,x,u(t,x),w(t,x))\, d\,x\, d\,t. \tag{8}$$

It is clear that various assumptions on the function Φ ensure that the integral on the right-hand side of (8) is a continuous function of (u,w) in $\mathcal{X}$. Therefore, **(T.3)** in Theorem 1 ensures that $\mathcal{I}$ is a continuous function of the control q in $\mathbf{L}^1$.

In practice, the choice of a real strategy depends on a finite set of parameters, say $p \in \mathbb{R}^m$, defining, for instance, the (time/space) support of q, or the maximal value of q, or its (time/space) integral. We are thus lead to minimize a composition of maps of the type

$$\begin{array}{ccccccc} \mathbb{R}^m & \to & \mathbf{L}^\infty(I;\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})) & \to & \mathcal{X}^+ & \to & \mathbb{R}, \\ p & \to & q & \to & (u,w) & \to & \mathcal{I} \end{array}$$

to which, thanks to Theorem 1, Weierstraß Theorem can be applied, ensuring the existence of an optimal strategy p_*. The actual computation of p_* can be achieved through standard numerical procedures dedicated to the optimization of Lipschitz continuous functions. The next section is devoted to specific examples.

3 OPTIMIZED TIMING OF PARASITOIDS' RELEASES

We present below a sample of the possible behaviors of solutions to (1). Further examples can be found in Ref. 1.

Inspired by Refs. 3, 4, we address the problem of optimizing the timing and the location of parasitoids' (= predators') releases in the case of a parasite (= prey) whose reproduction is seasonal and geographically localized. To this aim, we consider the following instance of (1) in the case of $n = 2$ space dimensions

$$\begin{cases} \partial_t u + \nabla \cdot (u\, v(w)) = (\alpha\, w - \beta) u + q(t,x), \\ \partial_t w - \mu\, \Delta w = \left(\gamma\,(1 - \sin t)\, \chi_B(x)\left(1 - \dfrac{w}{C}\right) - \delta\, u\right) w. \end{cases} \tag{9}$$

Here, as usual, t is time and x is the space coordinate in $\mathbb{R}^2$. Moreover, $\alpha\, w$ is the predator natality due to predation, β is the predators' mortality, δ is the prey mortality due to predation, and C is the prey carrying capacity. The prey natality[1] $\gamma\,(1 - \sin t)\, \chi_B(x)$ is *seasonal*, that is, it is 2π-periodic in time, and *localized*, that is, it is supported in the ball B centered at the origin with radius 2. The speed v is chosen as in (2), with $\varkappa$ constant. The parasitoids predate hunting for parasites in the direction of the highest average prey density gradient within a radius ℓ, which hence measures the predator horizon. This parameter, determining the distance at which predators *feel* the prey, thus plays a key role. We refer to Ref. 8 for a detailed study on the dependence of the solution on this parameter in a similar framework.

We summarize here the choices of functions and parameters in (9)–(2), apart from q to be chosen below:

$$\begin{array}{lll} \alpha = 0.25 & \beta = 2.00 & \gamma = 9.00 \\ \delta = 0.50 & C = 10.0 & \ell = 0.80 \\ \varkappa = 2.00 & & \end{array} \qquad \eta(x) = \begin{cases} \dfrac{4}{\pi\,\ell^2}\left(1 - \dfrac{\|x\|^2}{\ell^2}\right)^3 & \|x\| \leq \ell\,, \\ 0 & \|x\| > \ell\,. \end{cases} \tag{10}$$

We now seek strategies $q = q(t,x)$ to release parasitoids so that the parasite population is kept small in the rectangle $R = [1,3] \times [-3,3]$, which we assume is the region where the presence of the parasites is most harmful. The regions B and R are chosen so that they are different but overlapping. Thus, for simplicity, we aim at the minimization of

$$\mathcal{I} = \int_{4\pi}^{12\pi} \int_R w(t,x)\, d\,x\; d\,t, \tag{11}$$

although within the present framework (8) more complex costs can be considered. Another natural choice, for instance, might be the minimization of the pest population w only in specific periods, for example, when fruits are ripening on the trees, as in the case of the *Drosophila suzukii*. As initial datum, we choose

$$u_o(x) \equiv 0, \qquad w_o(x) = 2\,\chi_B(x). \tag{12}$$

Clearly, Theorem 1 applies to (9)–(2)–(10)–(12) and the cost (11) fits into (8).

[1] χ_B is the characteristic function of the set B: $\chi_B(x) = 1 \iff x \in B$ and $\chi_B(x) = 0 \iff x \in \mathbb{R}^n \setminus B$.

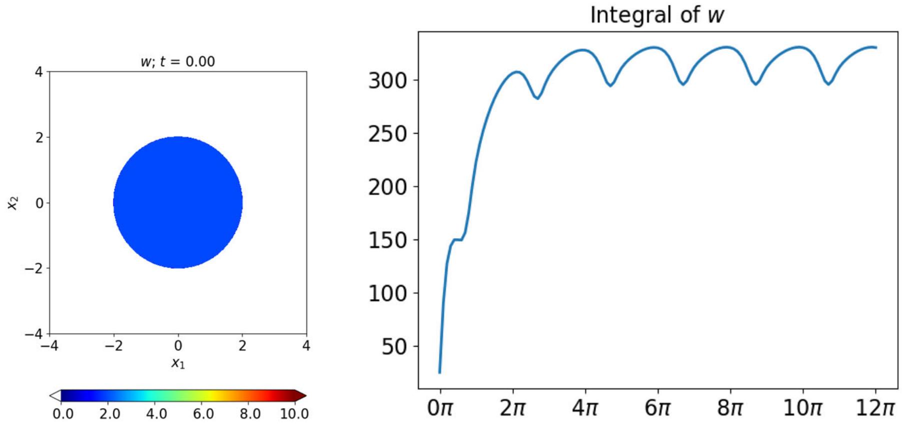

FIGURE 1 Left, the initial datum (12) for w in the x-plane and, right, the total amount of parasites $\int_{[-4,4]^2} w(t,x)\,dx$ on the whole physical domain as a function of time

In the examples below, we use the Lax–Friedrichs scheme (Ref. [9, Section 12.5]) to integrate the hyperbolic convective term and an explicit finite difference algorithm to deal with the parabolic equation. Furthermore, we exploit dimensional splitting Ref. [9, Section 19.5] and a further splitting to take care of the source terms (Ref. [9, Section 17.1]) which are computed through a second-order Runge–Kutta method (corresponding to $\alpha = 1/2$ in [Ref. 10, Section 12.5, p. 327]). Refer to Refs. 11–13 for alternative algorithms. The numerical domain is the rectangle $[-4-\ell, 4+\ell] \times [-4-\ell, 4+\ell]$ and we let the parameters α, β, and γ vanish outside the physical domain $[-4,4] \times [-4,4]$. The computations below were obtained with a uniform mesh consisting of $2^{10} \times 2^{10}$ points.

First, as a reference case, we integrate (9)–(2)–(10)–(12) with $q \equiv 0$. The results are displayed in Figure 1. Since parasitoids are absent, parasites evolve with a logistic growth with capacity C and a 2π-periodic natality. After two periods, the total number of parasites is approximately time periodic, with a high mean value.

We now assume that at time 4π measures need to be taken to reduce the presence of parasites. This is achieved through the release in the environment of the parasitoid u, which is described by the function q in (9). Different strategies correspond to different choices of q. The ones we consider below differ both in the space and time dependence: they may take place in the ball B where the parasites are born, or on the rectangle R where parasites are harmful. Moreover, they can take place uniformly in time (on $I_0 = [4\pi, 12\pi]$) or in the time intervals where parasites are more ($I_1 = \sin^{-1}([-1, -1/\sqrt{2}]) \cap I_0$), middle ($I_2 = \cos^{-1}([-1, -1/\sqrt{2}]) \cap I_0$), or less ($I_3 = \sin^{-1}([1/\sqrt{2}, 1]) \cap I_0$) prolific (see Figure 2). These strategies correspond to the following choices of q:

$$
\begin{aligned}
q_0^B(t) &= 3.166287\, \chi_{I_0}(t)\, \chi_B(x) & \qquad q_0^R(t) &= 3.315728\, \chi_{I_0}(t)\, \chi_R(x) \\
q_1^B(t) &= 12.66515\, \chi_{I_1}(t)\, \chi_B(x) & \qquad q_1^R(t) &= 13.26291\, \chi_{I_1}(t)\, \chi_R(x) \\
q_2^B(t) &= 12.66515\, \chi_{I_2}(t)\, \chi_B(x) & \qquad q_2^R(t) &= 13.26291\, \chi_{I_2}(t)\, \chi_R(x) \\
q_3^B(t) &= 12.66515\, \chi_{I_3}(t)\, \chi_B(x) & \qquad q_3^R(t) &= 13.26291\, \chi_{I_3}(t)\, \chi_R(x).
\end{aligned}
\tag{13}
$$

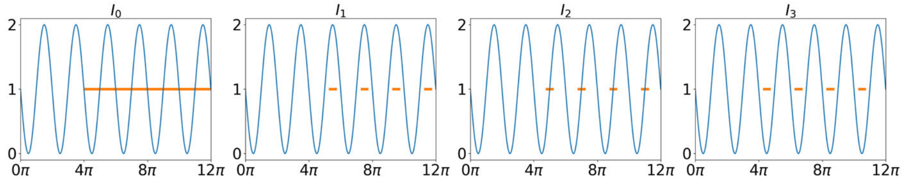

FIGURE 2 Characteristic functions of the time intervals, from left to right, I_0, I_1, I_2, and I_3 used in the definitions of the controls (13), plotted together with the map $t \to 1 - \sin t$ appearing in the natality of the parasite in (9)

The above values are chosen so that the amount of parasitoids inserted in the environment is constant, that is,

$$\int_0^{12\pi} \int_{\mathbb{R}^2} q_i^A(t,x)\, dx\, dt = 1000 \qquad \text{for } i = 0,1,2,3 \quad \text{and} \quad A = B, R.$$

The numerical integrations of (9)–(2)–(10)–(12) with the controls (13) yield the following values for the cost (11):

$\mathcal{I}$	0	1	2	3	
B	1179.05	1318.74	1332.75	1232.41	when $q \equiv 0$, $\mathcal{I} = 1866.98$
R	874.420	1068.13	1098.85	1080.19	

In the different cases of the controls in (13), the instantaneous costs $t \to \int_R w(t,x)\, dx$ are displayed in Figure 3. All solutions to (9)–(2)–(10)–(12) show a somewhat periodic behavior for $t > 4\pi$.

With respect to the cost (11), where the rectangle R obviously plays a key role, the most effective strategy consists in a constant release of parasitoids over the rectangle R, corresponding to the control q_0^R in (13). This solution is somewhat periodic and displays a maximum, respectively, a minimum, of the running cost at the time $t \approx 33.30$, respectively, $t \approx 30.79$: level plots of the corresponding solutions computed at these times are in Figure 4.

It is evident that the convective term in the first equation in (9) allows the parasitoids to move toward the region with the highest parasite concentration. On the other hand, the Laplace operator in the second equation makes the parasites diffuse everywhere.

We expect that a precise simulation of a real scenario requires a model more complex than (9)–(11), as well as the obvious tuning of the various parameters. For instance, also α, β, and δ are likely to be better substituted by *"seasonal"* (i.e., time periodic) functions. While such an experimental fitting is out of the scopes of the present work, we remark that the generality of the framework presented here, and in particular Theorem 1, allows to comprehend it.

Boundary conditions deserve a specific treatment on their own. At the modeling level, the immigration of parasites is neglected in the present work. At the analytic level, general well-posedness and stability results are currently apparently still missing, see Ref. 14 for recent preliminary results. The numerical algorithm to deal with boundary conditions would then be necessarily adapted.

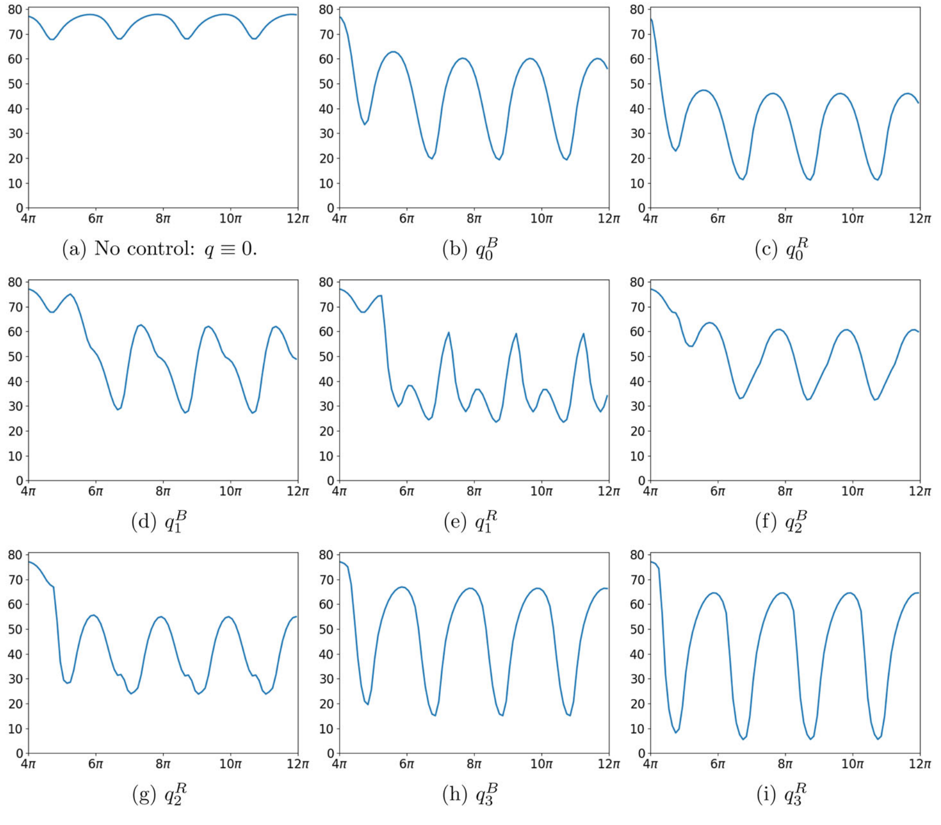

FIGURE 3 Graphs of the instantaneous cost $t \to \int_R w(t,x)\,dx$ corresponding to the controls (13) on the time interval $[4\pi, 12\pi]$. Figure 3(a) corresponds to the diffusion of parasites with no control. The most effective strategy, in the sense it minimizes (11), is in Figure 3(c)

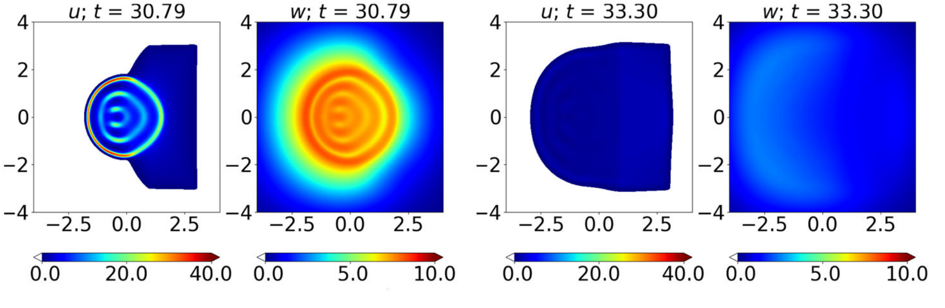

FIGURE 4 Contour plots of the solution to (9)–(10)–(12) corresponding to the best strategy q_0^R in (13). Left, at time $t = 30.79$ approximately corresponding to a maximum of the running cost (11) and, right, at time $t = 33.30$ approximately corresponding to a minimum

4 | ANALYTIC PROOFS

The following lemmas will be of use below. The proofs, where immediate, are omitted.

Lemma 1 [15, Formula (1.8) and Remark 1.16]. *Let $\psi \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R})$. Then, there exists a sequence $\psi_h \in \mathbf{C}^\infty(\mathbb{R}^n;\mathbb{R})$ such that for $h \in \mathbb{N} \setminus \{0\}$*

$$\psi_h \underset{h\to+\infty}{\to} \psi \text{ in } \mathbf{L}^1(\mathbb{R}^n;\mathbb{R}), \quad \|\psi_h\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \le \|\psi\|_{\mathbf{L}^\infty(\mathbb{R}^n)}, \quad \mathrm{TV}\,(\psi_h) \underset{h\to+\infty}{\to} \mathrm{TV}\,(\psi). \tag{14}$$

Lemma 2. *Let $\psi \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R})$. Then, there exists a sequence $\psi_h \in \mathbf{C}^\infty(\mathbb{R}^n;\mathbb{R})$ such that for $h \in \mathbb{N} \setminus \{0\}$, $\psi_h \to \psi$ in $\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})$, so that also $\psi_h \to \psi$ in $\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}^n;\mathbb{R})$, and*

$$\|\psi_h\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \le \|\psi\|_{\mathbf{L}^\infty(\mathbb{R}^n)}, \quad \mathrm{TV}\,(\psi_h) \le \mathrm{TV}\,(\psi).$$

Proof. Let ρ be a mollifier: $\rho \in \mathbf{C}_c^\infty(\mathbb{R}^n,\mathbb{R})$, $\rho \ge 0$, $\operatorname{spt}\rho \subseteq \{x \in \mathbb{R}^2 : \|x\| \le 1\}$ and $\int_{\mathbb{R}^n} \rho = 1$. Define $\rho_h(x) = h^n\,\rho(h\,x)$ for $h \in \mathbb{N} \setminus \{0\}$ and set $\psi_h = \rho_h * \psi$. The $\mathbf{L}^1_{\mathbf{loc}}$ convergence follows from $\|\psi_h - \psi\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \to 0$, ensured by Ref. [16, Theorem 8.14]. The $\mathbf{L}^\infty$ estimate is a consequence of Ref. [16, Proposition 8.7]. Finally, Ref. [16, Proposition 8.68] implies the latter bound. ■

Lemma 3. *Let $\psi \in \mathbf{L}^\infty(I \times \mathbb{R}^n;\mathbb{R})$ be such that for all $t \in I$, $\psi(t) \in \mathbf{BV}(\mathbb{R}^n;\mathbb{R})$. Then, there exists a sequence $\psi_h \in \mathbf{L}^\infty(I \times \mathbb{R}^n;\mathbb{R})$ such that for all $h \in \mathbb{N} \setminus \{0\}$ and for a.e. $t \in I$, $\psi_h(t) \in (\mathbf{C}^\infty \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R})$, $\psi_h(t) \to \psi(t)$ in $\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}^n;\mathbb{R})$ and*

$$\|\psi_h(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \le \|\psi(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}, \quad \mathrm{TV}(\psi_h(t)) \le \mathrm{TV}(\psi(t)).$$

4.1 | About the parabolic equation $\partial_t w - \mu\,\Delta w = a(t,x)\,w$

We here focus on the parabolic problem:

$$\begin{cases} \partial_t w - \mu\,\Delta w = a(t,x)\,w \\ w(t_o,x) = w_o(x) \end{cases} \qquad (t,x) \in I \times \mathbb{R}^n. \tag{15}$$

Similarly to Ref. 7, solutions to (15) are sought as $\mathbf{L}^1$ functions defined on $\mathbb{R}^n$ and all estimates refer to the $\mathbf{L}^1$ or $\mathbf{L}^\infty$ norms, see (5), which is somewhat unusual in relation to (15).

Definition 2. Let $a \in \mathbf{L}^\infty(I \times \mathbb{R}^n;\mathbb{R})$ and $w_o \in \mathbf{L}^1(\mathbb{R}^n;\mathbb{R})$. A *solution* to problem (15) is a function $w \in \mathbf{C}^0(I;\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))$ such that

$$w(t,x) = (H_\mu(t) * w_o)(x) + \int_{t_0}^t (H_\mu(t-\tau) * (a(\tau)\,w(\tau)))(x)\,d\tau. \tag{16}$$

The above definition is classical, see, for instance, Ref. [17, Section 48.3]. The results below are instrumental in the sequel. We only quote them, providing full reference to the proofs.

Lemma 4 [7, Lemma 2.4]. *Let $a \in \mathbf{L}^\infty(I \times \mathbb{R}^n;\mathbb{R})$. Assume that $w_o \in \mathbf{L}^1(\mathbb{R}^n;\mathbb{R})$ and $w \in \mathbf{C}^0(I;\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))$. Then, the following statements are equivalent:*

1. *The function w solves (15) in the sense of Definition 2.*

2. *The function w is a weak solution to (15), that is, for all test functions $\varphi \in \mathbf{C}_c^2(I \times \mathbb{R}^n; \mathbb{R})$*

$$\int_{t_o}^{T} \int_{\mathbb{R}^n} (w\, \partial_t \varphi + \mu\, w\, \Delta\varphi + a\, w\, \varphi)\; d\,x\; d\,t = 0 \tag{17}$$

and $w(t_o, x) = w_o(x)$.

Proposition 1 [7, Proposition 2.5]. *Fix $a \in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R})$. Then, (15) generates the process*

$$\begin{array}{ccccccc} \mathcal{P} & : & J & \times & \mathbf{L}^1(\mathbb{R}^n;\mathbb{R}) & \to & \mathbf{L}^1(\mathbb{R}^n;\mathbb{R}) \\ & & (t_o, t) & , & w_o & \to & w \end{array}$$

with w defined as in (16), with the following properties, for a suitable $\mathcal{O} \in \mathbf{L}^\infty_{\mathbf{loc}}(I; \mathbb{R})$ that depends only on norms of the map a on $I \times \mathbb{R}^n$.

(P1) ***$\mathcal{P}$ is a Process**: $\mathcal{P}_{t,t} = \mathrm{Id}$ for all $t \in I$ and $\mathcal{P}_{t_2,t_3} \circ \mathcal{P}_{t_1,t_2} = \mathcal{P}_{t_1,t_3}$ for all $t_1, t_2, t_3 \in I$, with $t_1 \leq t_2 \leq t_3$.*

(P2) ***Regularity in time**: For all $w_o \in \mathbf{L}^1(\mathbb{R}^n; \mathbb{R})$, the map $t \to \mathcal{P}_{t_o,t} w_o$ is in $\mathbf{C}^0(I; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))$, and, moreover, for every $\vartheta \in]0, 1[$ and for all $\tau,\ t_1,\ t_2 \in I$, with $t_2 \geq t_1 \geq \tau > 0$,*

$$\|\mathcal{P}_{t_o,t_2} w_o - \mathcal{P}_{t_o,t_1} w_o\|_{\mathbf{L}^1(\mathbb{R}^n)} \leq \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n)} \left[\frac{n}{\tau - t_o} + \mathcal{O}(t_2)\right] |t_2 - t_1|^{\vartheta},$$

(P3) ***Regularity in space**: For all $t > t_o$, $w(t) \in \mathbf{C}^\infty(\mathbb{R}^n; \mathbb{R})$.*

(P4) ***Regularity in (t, x)**: If $w_o \in (\mathbf{L}^1 \cap \mathbf{C}^1)(\mathbb{R}^n; \mathbb{R})$, then $(t, x) \to (\mathcal{P}_{t_o,t} w_o)(x) \in \mathbf{C}^1(I \times \mathbb{R}^n; \mathbb{R})$.*

(P5) ***$\mathbf{L}^1$ continuous dependence on $\mathbf{w_o}$**: For all $t \in I$, the map $\mathcal{P}_{t_o,t} : \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}) \to \mathbf{L}^1(\mathbb{R}^n; \mathbb{R})$ is linear and continuous, with $\|\mathcal{P}_{t_o,t} w_o\|_{\mathbf{L}^1(\mathbb{R}^n)} \leq \mathcal{O}(t)\, \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n)}$.*

(P6) ***Stability with respect to a**: Let $a_1, a_2 \in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R})$ with $a_1 - a_2 \in \mathbf{L}^1(I \times \mathbb{R}^n; \mathbb{R})$ and call $\mathcal{P}^1, \mathcal{P}^2$ the corresponding processes. Then, for all $t \in I$ and for all $w_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^n; \mathbb{R})$,*

$$\|\mathcal{P}^1_{t_o,t} w_o - \mathcal{P}^2_{t_o,t} w_o\|_{\mathbf{L}^1(\mathbb{R}^n)} \leq \mathcal{O}(t)\, \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\, \|a_1 - a_2\|_{\mathbf{L}^1([t_o,t] \times \mathbb{R}^n)}.$$

(P7) ***$\mathbf{L}^\infty$ estimate**: For all $w_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^n; \mathbb{R})$, for all $t \in I$, $\|\mathcal{P}_{t_o,t} w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \leq \mathcal{O}(t)\, \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)}$.*

(P8) ***$\mathbf{W}^{1,1}$ estimate**: For all $w_o \in \mathbf{L}^1(\mathbb{R}^n; \mathbb{R})$, for all $t \in I$ with $t > t_o$,*

$$\|\nabla(\mathcal{P}_{t_o,t} w_o)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} \leq \frac{J_n}{\sqrt{\mu\,(t - t_o)}}\, \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n)} \times\left(1 + 2\,(t - t_o)\, \|a\|_{\mathbf{L}^\infty([t_o,t] \times \mathbb{R}^n)} e^{\int_{t_o}^{t} \|a(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)} d\tau}\right),$$

where $J_n = \frac{\Gamma((n+1)/2)}{\Gamma(n/2)}$ and Γ is the Gamma function.

In the statement above, we used the term *process* to emphasize the present nonautonomous setting: whenever a is not time-dependent, the process $\mathcal{P}$ is indeed a *semigroup*. The latter estimate in (**P8**) and (**P3**) provide a **BV** bound on the solution $\mathcal{P}_{t_o,t} w_o$ for $t > t_o$.

In the sequel, we need the following strengthened version of (**P8**).

Proposition 2. *Let $a \in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R})$ and assume $w_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R})$. Call w the solution to (15). Then, for all $t \in I$, $w(t) \in \mathbf{BV}(\mathbb{R}^n; \mathbb{R})$ and the following estimate holds:*

$$\mathrm{TV}\,(w(t)) \leq \mathrm{TV}\,(w_o) + \frac{2\,J_n}{\sqrt{\mu}}\;\mathcal{O}(t)\;\|a\|_{\mathbf{L}^\infty([t_o,t]\times\mathbb{R}^n)}\;\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n)}, \tag{18}$$

where $J_n = \frac{\Gamma((n+1)/2)}{\Gamma(n/2)}$ and Γ is the Gamma function.

Proof. Approximate w_o by means of a sequence w_o^h as defined in Lemma 1. Define w_h through (16) by

$$w_h(t,x) = \big(H_\mu(t) * w_o^h\big)(x) + \int_{t_0}^t (H_\mu(t-\tau) * (a(\tau)\,w_h(\tau)))(x)\;d\tau. \tag{19}$$

Let w be defined by (16) and compute

$$\|w_h(t) - w(t)\|_{\mathbf{L}^1(\mathbb{R}^n)} \leq \left\|w_o^h - w_o\right\|_{\mathbf{L}^1(\mathbb{R}^n)} + \int_{t_o}^t \|a(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\|w_h(\tau) - w(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)}\,d\tau.$$

An application of Gronwall Lemma (Ref. [18, Chapter I, 1.III]) yields

$$\|w_h(t) - w(t)\|_{\mathbf{L}^1(\mathbb{R}^n)} \leq \left\|w_o^h - w_o\right\|_{\mathbf{L}^1(\mathbb{R}^n)} \exp\left(\int_{t_o}^t \|a(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\,d\tau\right).$$

Thus, as h goes to $+\infty$, $w_h(t)$ converges to $w(t)$ in $\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})$ for a.e. $t \in I$.

It follows immediately from (19) and from the regularity of the heat kernel H_μ that $w_h(t) \in \mathbf{C}^\infty(\mathbb{R}^n;\mathbb{R})$ for a.e. $t \in I$. Moreover,

$$\nabla w_h(t,x) = (H_\mu(t) * \nabla w_o^h)(x) + \int_{t_o}^t \nabla H_\mu(t-\tau) * (a(\tau)\,w_h(\tau))(x)\,d\tau,$$

so that, using the properties of the heat kernel H_μ and (**P5**) in Proposition 1, we obtain

$$\begin{aligned}
\|\nabla w_h(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} \leq{}& \left\|\nabla w_o^h\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} \\
&+ \int_{t_o}^t \|\nabla H_\mu(t-\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}\|a(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\|w_h(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)}\,d\tau \\
\leq{}& \left\|\nabla w_o^h\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} + \mathcal{O}(t)\,\|a\|_{\mathbf{L}^\infty([t_o,t]\times\mathbb{R}^n)}\,\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n)} \int_{t_o}^t \frac{J_n}{\sqrt{\mu(t-\tau)}}\,d\tau
\end{aligned}$$

$$\leq \left\| \nabla w_o^h \right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} + \mathcal{O}(t) \, \|a\|_{\mathbf{L}^\infty([t_o,t]\times\mathbb{R}^n)} \, \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n)} \frac{2J_n}{\sqrt{\mu}} \sqrt{t-t_o} \, .$$

Let now $h \to +\infty$: Lemma 1 and the lower semicontinuity of the total variation imply that:

$$\begin{aligned} \mathrm{TV}\,(w(t)) &\leq \lim_{h\to+\infty} \mathrm{TV}\,(w_h(t)) = \lim_{h\to+\infty} \|\nabla w_h(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} \\ &\leq \mathrm{TV}\,(w_o) + \mathcal{O}(t)\, \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n)} \, \|a\|_{\mathbf{L}^\infty([t_o,t]\times\mathbb{R}^n)} \frac{2J_n}{\sqrt{\mu}} \sqrt{t-t_o}, \end{aligned}$$

completing the proof. ∎

In the case of positive initial data, we need the following improvements of the estimates in Propositions 1 and 2.

Corollary 1. *Let $a \in \mathbf{L}^\infty(I\times\mathbb{R}^n;\mathbb{R})$, $w_o \in \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})$ with $w_o \geq 0$. Then,*

(**P9**) ***Positivity***: *$\mathcal{P}_{t_o,t}\, w_o \geq 0$ for all $t \in I$.*

(**P10**) ***A priori estimates***: *Assume that $w_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^n;\mathbb{R})$ and set, for all $t \in I$, $A(t) = \operatorname{ess\,sup}_{\xi\in\mathbb{R}^n} a(t,\xi)$. Then,*

$$\begin{aligned} \|\mathcal{P}_{t_o,t}\, w_o\|_{\mathbf{L}^1(\mathbb{R}^n)} &\leq \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n)} \exp \int_{t_o}^t A(\tau)\, d\tau \, , \\ \|\mathcal{P}_{t_o,t}\, w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} &\leq \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \exp \int_{t_o}^t A(\tau)\, d\tau . \end{aligned} \tag{20}$$

(**P11**) ***Stability with respect to a***: *Let $a_1, a_2 \in \mathbf{L}^\infty(I\times\mathbb{R}^n;\mathbb{R})$ with $a_1 - a_2 \in \mathbf{L}^1(I\times\mathbb{R}^n;\mathbb{R})$ and call $\mathcal{P}^1, \mathcal{P}^2$ the corresponding processes. Then, for all $t \in I$ and for all $w_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^n;\mathbb{R})$,*

$$\begin{aligned} &\|\mathcal{P}^1_{t_o,t} w_o - \mathcal{P}^2_{t_o,t} w_o\|_{\mathbf{L}^1(\mathbb{R}^n)} \\ \leq\ & \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \, e^{\int_{t_o}^t \left[\|a_1(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \|a_2(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\right] d\tau} \, \|a_1 - a_2\|_{\mathbf{L}^1([t_o,t]\times\mathbb{R}^n)} . \end{aligned} \tag{21}$$

(**P12**) **BV *estimate***: *If $w_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R})$, define $A(t) = \sup_{x\in\mathbb{R}^n} a(t,x)$, then*

$$\mathrm{TV}\left(\mathcal{P}_{t_o,t}\, w_o\right) \leq \mathrm{TV}\,(w_o) + \frac{2J_n}{\sqrt{\mu}} \sqrt{t-t_o}\, \|a\|_{\mathbf{L}^\infty([t_o,t]\times\mathbb{R}^n)} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n)} \, e^{\int_{t_o}^t A(\tau)d\tau}, \tag{22}$$

where $J_n = \frac{\Gamma((n+1)/2)}{\Gamma(n/2)}$ and Γ is the Gamma function.

Above, the apparent mismatch between the ess sup in (**P10**) with no modulus and of the norm in (**P11**) is due to the positivity of the solution, ensured by $w_o \geq 0$ in (**P10**).

Proof. The positivity (**P9**) follows from Ref. [7, Point 6 in Proposition 2.5] based on Ref. [19, Chapter 2, Section 4, Theorem 9].

Starting now from (16), we have

$$w(t,x) = (H_\mu(t) * w_o)(x) + \int_{t_o}^{t} \int_{\mathbb{R}^n} H_\mu(t-\tau, x-\xi)\, a(\tau,\xi)\, w(\tau,\xi)\, d\xi\, d\tau$$
$$\leq (H_\mu(t) * w_o)(x) + \int_{t_o}^{t} A(\tau) \int_{\mathbb{R}^n} H_\mu(t-\tau, x-\xi)\, w(\tau,\xi)\, d\xi\, d\tau.$$

In both cases of the $\mathbf{L}^1$ and $\mathbf{L}^\infty$ estimate, an application of Gronwall Lemma Ref. [18, Chapter I, 1.III] completes the proof of (**P10**).

Concerning the stability with respect to a, denote $w_i(t) = \mathcal{P}^i_{t_o,t} w_o$, for $i = 1, 2$ and $t \in I$, and using (16), compute

$$w_1(t,x) - w_2(t,x) = \int_{t_o}^{t} \int_{\mathbb{R}^n} H_\mu(t-\tau, x-\xi)(a_1(\tau,\xi)\, w_1(\tau,\xi) - a_2(\tau,\xi)\, w_2(\tau,\xi))\, d\xi\; d\tau$$
$$= \int_{t_o}^{t} \int_{\mathbb{R}^n} H_\mu(t-\tau, x-\xi)(a_1(\tau,\xi) - a_2(\tau,\xi)) w_1(\tau,\xi)\, d\xi\; d\tau$$
$$+ \int_{t_o}^{t} \int_{\mathbb{R}^n} H_\mu(t-\tau, x-\xi)\, a_2(\tau,\xi)(w_1(\tau,\xi) - w_2(\tau,\xi))\, d\xi\; d\tau,$$

so that

$$\|w_1(t) - w_2(t)\|_{\mathbf{L}^1(\mathbb{R}^n)} \leq \int_{t_o}^{t} \|a_1(\tau) - a_2(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)} \|w_1(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\, d\tau$$
$$+ \int_{t_o}^{t} \|a_2(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\, \|w_1(\tau) - w_2(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)}\, d\tau.$$

By Gronwall Lemma,

$$\|w_1(t) - w_2(t)\|_{\mathbf{L}^1(\mathbb{R}^n)}$$
$$\leq \int_{t_o}^{t} \|a_1(\tau) - a_2(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)} \|w_1(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\, d\tau \exp\left(\int_{t_o}^{t} \|a_2(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\, d\tau \right)$$
$$\leq \int_{t_o}^{t} \|a_1(\tau) - a_2(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)} \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \exp\left(\int_{t_o}^{\tau} \sup_{\xi \in \mathbb{R}^n} a_1(s,\xi)\, ds \right) d\tau$$
$$\times \exp\left(\int_{t_o}^{t} \|a_2(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\, d\tau \right)$$
$$\leq \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \exp\left(\int_{t_o}^{t} \left(\|a_1(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \|a_2(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \right) d\tau \right) \|a_1 - a_2\|_{\mathbf{L}^1([t_o,t] \times \mathbb{R}^n)}$$

completing the proof of (**P11**).

Finally, (**P12**) follows from Proposition 2, from (**P9**) and from the $\mathbf{L}^\infty$ bound (20). ■

4.2 | About the balance law $\partial_t u + \nabla\cdot (c(t,x)\, u) = b(t,x)\, u + q(t,x)$

We focus on the following Cauchy problem for a linear balance law

$$\begin{cases} \partial_t u + \nabla\cdot (c(t,x)\, u) = b(t,x)\, u + q(t,x) \\ u(t_o, x) = u_o(x). \end{cases} \tag{23}$$

Recall the following conditions on the functions defining problem (23):

- (**b**) $b \in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R})$.
- (**b**+) $b \in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R})$ and $b(t) \in \mathbf{BV}(\mathbb{R}^n; \mathbb{R})$ for $t \in I$.
- (**c1**) The map c satisfies $c \in (\mathbf{C}^0 \cap \mathbf{L}^\infty)(I \times \mathbb{R}^n; \mathbb{R}^n)$, $c(t) \in \mathbf{C}^1(\mathbb{R}^n; \mathbb{R}^n)$ for all $t \in I$ and $\nabla c \in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R}^{n\times n})$.
- (**c2**) The map c satisfies $c \in (\mathbf{C}^0 \cap \mathbf{L}^\infty)(I \times \mathbb{R}^n; \mathbb{R}^n)$; $c(t) \in \mathbf{C}^2(\mathbb{R}^n; \mathbb{R}^n)$ for all $t \in I$, $\nabla c \in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R}^{n\times n})$ and $\nabla\, \nabla\cdot c \in \mathbf{L}^1(I \times \mathbb{R}^n; \mathbb{R}^n)$.
- (**q−**) $q \in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R}) \cap \mathbf{L}^\infty(I; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))$.

The definition below is classical, see, for instance, Ref. [20, Chap. 3].

Definition 3. Let (**b**), (**c1**), and (**q−**) hold and choose $u_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^n; \mathbb{R})$. A *solution* to (23) is a function $u \in \mathbf{C}^0(I; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))$ such that

$$\begin{aligned} u(t,x) = {} & u_o(X(t_o;t,x)) \exp\left(\int_{t_o}^t \big(b(\tau, X(\tau;t,x)) - \nabla\cdot c(\tau, X(\tau;t,x))\big)\, d\tau\right) \\ & + \int_{t_o}^t q(s, X(s;t,x)) \exp\left(\int_s^t \big(b(\tau, X(\tau;t,x)) - \nabla\cdot c(\tau, X(\tau;t,x))\big)\, d\tau\right) ds, \end{aligned} \tag{24}$$

where

$$t \mapsto X(t; t_o, x_o) \quad \text{solves the Cauchy Problem} \quad \begin{cases} \dot X = c(t, X) \\ X(t_o) = x_o. \end{cases} \tag{25}$$

The next lemma clarifies the relations among different definitions of solutions.

Lemma 5 [7, Lemma 2.7] and [21, Lemma 5.1]. *Let* (**b**), (**c1**), (**q−**) *hold, Fix* $u_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^n; \mathbb{R})$ *and* $u \in \mathbf{C}^0(I; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))$. *Then, the following three statements are equivalent:*

1. *u is a Kružkov solution to (23), that is,* $u(t_o) = u_o$ *and for all* $k \in \mathbb{R}$ *and* $\varphi \in \mathbf{C}^1_c(\mathring{I} \times \mathbb{R}^n; \mathbb{R}_+)$,

$$\int_I \int_{\mathbb{R}^n} \left[(u-k)(\partial_t \varphi + c\cdot \nabla\varphi) + (b\, u + q - k\, \nabla\cdot c)\, \varphi\right] \operatorname{sgn}(u-k)\, dx\, dt \geq 0. \tag{26}$$

2. *u is a weak solution to (23), that is, $u(t_o) = u_o$ and for all $\varphi \in \mathbf{C}_c^1(\mathring{I} \times \mathbb{R}^n; \mathbb{R})$,*

$$\int_I \int_{\mathbb{R}^n} (u\, \partial_t \varphi + u\, c \cdot \nabla \varphi + (b\, u + q)\varphi)\, d\,x\, d\,t = 0\,. \tag{27}$$

3. *u solves (23) in the sense of Definition 3.*

The proof amounts to mix the techniques used in Ref. [7, Lemma 2.7] and Ref. [21, Lemma 5.1].

We recall a different approach to the study of linear balance laws of type (23), which is adopted in [22, Lemma 3.4]. That lemma guarantees the existence of a weak solution, in the sense of (27) in Lemma 5, and provides an explicit formula for the solution in terms of characteristics, corresponding exactly to (24). The regularity requirements in Ref. [22], on the functions defining problem (23) are the following: for $T \in \mathbb{R}$, $T > 0$,

$$u_o \in \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}), \quad b \in \mathbf{L}^1((0,T); \mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})), \quad q \in \mathbf{L}^1((0,T); \mathbf{L}^1(\mathbb{R}^n; \mathbb{R})),$$

and $c \in \mathbf{C}^0((0,T); \mathbf{C}^1(\mathbb{R}^n; \mathbb{R}^n))$ is globally Lipschitz continuous in space. Notice that, for $T \in \mathbb{R}$, $T > 0$, our assumptions ($\boldsymbol{b}$), (**c1**), and ($\boldsymbol{q}-$) are stronger than those required in Ref. [22, Lemma 3.4], allowing to apply that result in the present setting.

The next proposition is not only an extension of [7, Proposition 2.8] to the present setting, but it also improves it sharply. However, various similar statements are found in the literature: in the case $q = 0$, for instance, for the existence part refer to the results in Ref. [23], Other sources, detailed in the proof below, are Refs. 14, 21, 22.

Proposition 3. *Under the assumptions ($\boldsymbol{b}$), (**c1**), and ($\boldsymbol{q}-$), the Cauchy Problem (23) generates the map*

$$\begin{array}{ccccccc} \mathcal{H} & : & J & \times & \mathcal{U} & \to & \mathcal{U}, \\ & & (t_o, t) & , & u_o & \to & u, \end{array}$$

where u is defined by (24), with the following properties:

- **(H1)** ***$\mathcal{H}$ is a process**: $\mathcal{H}_{t,t} = \mathrm{Id}$ for all $t \in I$ and $\mathcal{H}_{t_2,t_3} \circ \mathcal{H}_{t_1,t_2} = \mathcal{H}_{t_1,t_3}$ for all $t_1, t_2, t_3 \in I$, with $t_1 \leq t_2 \leq t_3$.*
- **(H2)** ***Positivity**: If $q \geq 0$ and $u_o \in \mathcal{U}^+$, then $\mathcal{H}_{t_o,t}\, u_o \in \mathcal{U}^+$ for all $t \in I$.*
- **(H3)** **$\mathbf{L}^1$ *continuous dependence on $\mathbf{u_o}$***: *For all $t \in I$, the map $\mathcal{H}_{t_o,t} : \mathcal{U} \to \mathcal{U}$ is linear, continuous and*

$$\|\mathcal{H}_{t_o,t} u_o\|_{\mathbf{L}^1(\mathbb{R}^n)} \leq \left(\|u_o\|_{\mathbf{L}^1(\mathbb{R}^n)} + \|q\|_{\mathbf{L}^1([t_o,t]\times\mathbb{R}^n)}\right) \exp \int_{t_o}^t \|b(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\, d\,\tau\,.$$

Moreover, if $u_o \geq 0$ and $q \geq 0$, then

$$\|\mathcal{H}_{t_o,t} u_o\|_{\mathbf{L}^1(\mathbb{R}^n)} \leq \left(\|u_o\|_{\mathbf{L}^1(\mathbb{R}^n)} + \|q\|_{\mathbf{L}^1([t_o,t]\times\mathbb{R}^n)}\right) \exp \int_{t_o}^t \left(\sup_{x\in\mathbb{R}^n} b(\tau, x)\right) d\,\tau\,.$$

(H4) $\mathbf{L}^\infty$ ***estimate***: *For all $u_o \in \mathcal{U}$, for all $t \in I$,*

$$\|\mathcal{H}_{t_o,t} u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \leq \left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \|q\|_{\mathbf{L}^1([t_o,t];\mathbf{L}^\infty(\mathbb{R}^n))}\right) \times \exp \int_{t_o}^t \left(\|b(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \|\nabla\cdot c(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\right) d\tau.$$

Moreover, if $u_o \geq 0$ and $q \geq 0$, then

$$\|\mathcal{H}_{t_o,t} u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \leq \left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \|q\|_{\mathbf{L}^1([t_o,t];\mathbf{L}^\infty(\mathbb{R}^n))}\right) \times \exp \int_{t_o}^t \left(\left(\sup_{x\in\mathbb{R}^n} b(\tau,x)\right) + \|\nabla\cdot c(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\right) d\tau.$$

(H5) ***Stability with respect to $\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{q}$***: *If $b, \tilde b$ satisfy $(\boldsymbol{b}+)$ with $b - \tilde b \in \mathbf{L}^1(I\times\mathbb{R}^n;\mathbb{R})$; $c, \tilde c$ satisfy $(\boldsymbol{c2})$ with $\nabla\cdot(c-\tilde c) \in \mathbf{L}^1(I\times\mathbb{R}^n;\mathbb{R})$, and $q, \tilde q$ satisfy $(\boldsymbol{q})$. Call $\mathcal{H}, \tilde{\mathcal{H}}$ the corresponding processes. Then, for all $t \in I$ and for all $u_o \in \mathcal{U}$,*

$$\begin{aligned}
&\|\mathcal{H}_{t_o,t} u_o - \tilde{\mathcal{H}}_{t_o,t} u_o\|_{\mathbf{L}^1(\mathbb{R}^n)} \\
\leq\ & \mathcal{O}_1(t)\,\|c-\tilde c\|_{\mathbf{L}^1([t_o,t];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n))} \Bigg[\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \mathrm{TV}\,(u_o) \\
&+ \int_{t_o}^t \left(\max\left\{\|q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}, \|\tilde q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\right\} + \max\left\{\mathrm{TV}\,(q(\tau)),\ \mathrm{TV}\,(\tilde q(\tau))\right\}\right) d\tau\Bigg] \\
&+ \mathcal{O}_2(t)\,\|q-\tilde q\|_{\mathbf{L}^1([t_o,t]\times\mathbb{R}^n)} \\
&+ \mathcal{O}_2(t)\left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \int_{t_o}^t \max\left\{\|q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}, \|\tilde q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\right\} d\tau\right) \\
&\times\left(\|b-\tilde b\|_{\mathbf{L}^1([t_o,t]\times\mathbb{R}^n)} + \|\nabla\cdot(c-\tilde c)\|_{\mathbf{L}^1([t_o,t]\times\mathbb{R}^n)}\right),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{O}_1(t) =\ & \exp\int_{t_o}^t \max\left\{\|b(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}, \|\tilde b(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\right\} d\tau \\
&\times \exp\int_{t_o}^t \max\left\{\|\nabla c(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}, \|\nabla \tilde c(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\right\} d\tau \\
&\times\left[1 + \int_{t_o}^t \max\left\{\begin{array}{l}\mathrm{TV}\,(b(s)) + \|\nabla\,\nabla\cdot c(s)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}, \\ \mathrm{TV}\,(\tilde b(s)) + \|\nabla\,\nabla\cdot \tilde c(s)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}\end{array}\right\} ds\right], \\
\mathcal{O}_2(t) =\ & \exp\int_{t_o}^t \max\left\{\|b(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}, \|\tilde b(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\right\} d\tau.
\end{aligned}$$

(H6) ***Total variation bound***: *Let* (***b***+), (***c*2**), *and* (***q***) *hold. If* $u_o \in \mathcal{U}$, *then, for all* $t \in I$,

$$\mathrm{TV}\,(\mathcal{H}_{t_o,t} u_o) \leq \mathcal{O}(t)\left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \mathrm{TV}\,(u_o) + \int_{t_o}^t \left(\|q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \mathrm{TV}\,(q(\tau)) \right) d\,\tau \right),$$

where

$$\mathcal{O}(t) = \exp\left(\int_{t_o}^t \left(\|b(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \|\nabla c(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})} \right) d\,\tau \right)$$
$$\times \left(1 + \int_{t_o}^t \left(\mathrm{TV}\,(b(\tau)) + \|\nabla\,\nabla\cdot c(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} \right) d\,\tau \right).$$

(H7) ***Regularity in time***: *Let* (***b***+), (***c*2**), *and* (***q***) *hold. For all* $u_o \in \mathcal{U}$, *the map* $t \to \mathcal{H}_{t_o,t} u_o$ *is in* $\mathbf{C}^{0,1}(I;\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))$, *moreover for all* $t_1, t_2 \in I$, *with* $\mathcal{O}(t)$ *as above,*

$$\|\mathcal{H}_{t_o,t_2} u_o - \mathcal{H}_{t_o,t_1} u_o\|_{\mathbf{L}^1(\mathbb{R}^n)} \leq \mathcal{O}(\max\{t_1,t_2\}) \left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \mathrm{TV}\,(u_o) \right.$$
$$\left. + \int_{t_o}^{\max\{t_1,t_2\}} \left(\|q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \mathrm{TV}\,(q(\tau)) \right) d\,\tau \right) |t_2 - t_1|.$$

(H8) ***Finite propagation speed***: *If, for all* $t \in I$, *the map* $x \to q(t,x)$ *is compactly supported and* $u_o \in \mathcal{U}$ *has compact support, then, for* $t \in I$ *also,* $\operatorname{spt} \mathcal{H}_{t_o,t} u_o$ *is compact.*

Proof. Statement (**H1**) directly follows from Definition 3, Lemma 5, and Ref. [22, Lemma 3.4] thanks to (***b***), (***c*1**), and (***q***−). Using (24), points (**H2**), (**H4**), and (**H8**) are ensured.

To get the $\mathbf{L}^1$ bound (**H3**), exploit the change of variable $y = X(s;t,x)$, see also Ref. [21, Section 5.1]. Denoting the Jacobian of this change of variable by $J(t,y) = \det(\nabla_x X(t;s,y))$, J solves

$$\frac{d\,J(t,y)}{d\,t} = \nabla\cdot c(t,X(t;s,y))\,J(t,y) \qquad \text{with } J(s,y) = 1.$$

Thus, $J(t,y) = \exp(\int_s^t \nabla\cdot c(\tau, X(\tau;s,y))\,d\,\tau)$, so that $J(t,y) > 0$ for $t \in I$ and (**H3**) follows.

To prove the remaining points, we exploit the techniques used in the proof of Ref. [14, lemmas 4.4 and 4.6] for an initial boundary value problem for a conservation law, thus without source term. To this aim, we approximate b, respectively, q, by a sequence b_h, respectively, q_h, as in Lemma 3. Regularize also the initial datum u_o and call $u_o^h \in \mathbf{C}^\infty(\mathbb{R}^n;\mathbb{R})$ the sequence defined by Lemma 1. Using (24), define the corresponding sequence u_h of solutions to

$$\begin{cases} \partial_t u_h + \nabla\cdot(c(t,x)\,u_h) = b_h(t,x)\,u_h + q_h(t,x) \\ u_h(t_o,x) = u_o^h(x)\,, \end{cases}$$

so that

$$u_h(t,x) = u_o^h(X(t_o;t,x)) \exp\left(\int_{t_o}^t (b_h(\tau,X(\tau;t,x)) - \nabla\cdot c(\tau,X(\tau;t,x)))\,d\,\tau \right) \tag{28}$$

$$+\int_{t_o}^{t} q_h(s, X(s;t,x)) \exp\left(\int_s^t (b_h(\tau, X(\tau;t,x)) - \nabla\cdot c(\tau, X(\tau;t,x)))\, d\tau\right) d s,$$

where X is defined in (25). Observe that for a.e. $t \in I$, the map $x \to u_h(t,x)$ is of class $\mathbf{C}^1$, due to Lemma 3, applied to both b and q, and to (***c*2**).

Pass now to (**H6**). Differentiate the solution to (25) with respect to the initial point, that is, for $\tau \in [t_o, t]$,

$$\nabla_x X(\tau;t,x) = \mathrm{Id} + \int_t^{\tau} \nabla_x c(s, X(s;t,x))\, \nabla_x X(s;t,x)\, d s,$$

$$\|\nabla_x X(\tau;t,x)\| \leq 1 + \int_{\tau}^{t} \|\nabla_x c(s, X(s;t,x))\|\, \|\nabla_x X(s;t,x)\|\, d s,$$

so that, by Gronwall Lemma,

$$\|\nabla_x X(\tau;t,x)\| \leq \exp\left(\int_{\tau}^{t} \|\nabla_x c(s)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\right) d s. \tag{29}$$

By (28) and the properties of u_o^h, the gradient $\nabla u_h(t)$ is well defined and continuous:

$$\begin{aligned}\nabla u_h(t,x) = {} & \exp\left(\int_{t_o}^{t} (b_h - \nabla\cdot c)(\tau, X(\tau;t,x))\, d\tau\right)\Bigg(\nabla u_o^h(X(t_o;t,x))\, \nabla_x X(t_o;t,x) \\ & + u_o^h(X(t_o;t,x)) \int_{t_o}^{t} \nabla(b_h - \nabla\cdot c)(\tau, X(\tau;t,x))\, \nabla_x X(\tau;t,x)\, d\tau\Bigg) \\ & + \int_{t_o}^{t} \exp\left(\int_s^t (b_h - \nabla\cdot c)(\tau, X(\tau;t,x))\, d\tau\right)\Bigg(\nabla q_h(s, X(s;t,x))\, \nabla_x X(s;t,x) \\ & + q_h(s, X(s;t,x)) \int_s^t \nabla(b_h - \nabla\cdot c)(\tau, X(\tau;t,x))\, \nabla_x X(\tau;t,x)\, d\tau\Bigg) d s.\end{aligned}$$

Therefore, for every $t \in I$, we use the change of variable described at the beginning of the proof together with (29) to get

$$\begin{aligned}& \|\nabla u_h(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} \\ \leq {} & \exp\left(\int_{t_o}^{t} \|b_h(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\, d\tau\right) \exp\left(\int_{t_o}^{t} \|\nabla c(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\, d\tau\right) \\ & \times\Bigg[\|\nabla u_o^h\|_{\mathbf{L}^1(\mathbb{R}^n)} + \int_{t_o}^{t} \|\nabla q_h(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}\, d\tau\end{aligned} \tag{30}$$

$$+\left(\|u_o^h\|_{\mathbf{L}^\infty(\mathbb{R}^n)}+\int_{t_o}^t \|q_h(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\,d\tau\right)\int_{t_o}^t \|\nabla(b_h-\nabla\cdot c)(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}\,d\tau\Bigg].$$

Let u be defined as in (24): Lemmas 3 and 1 imply that $u_h \to u$ in $\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})$. By the lower semicontinuity of the total variation, by (30) and (14), for $t \in I$ we obtain

$$\begin{aligned}
\mathrm{TV}\,(u(t)) &\le \lim_h \mathrm{TV}\,(u_h(t)) = \lim_h \|\nabla u_h(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} \\
&\le \exp\left(\int_{t_o}^t \left(\|b(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}+\|\nabla c(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\right)d\tau\right)\left[\mathrm{TV}\,(u_o)+\int_{t_o}^t \mathrm{TV}\,(q(\tau))\,d\tau\right.\\
&\quad \left.+\left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)}+\int_{t_o}^t \|q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\,d\tau\right)\int_{t_o}^t \left(\mathrm{TV}\,(b(\tau))+\|\nabla\,\nabla\cdot c(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}\right)d\tau\right],
\end{aligned} \tag{31}$$

concluding the proof of (**H6**).

The proof of (**H7**), is entirely analogous, leading to

$$\|u(t_2)-u(t_1)\|_{\mathbf{L}^1(\mathbb{R}^n)} \le \mathrm{TV}\,(u(\max\{t_1,t_2\}))|t_2-t_1|.$$

To prove (**H5**), we follow the idea of the proof of Ref. [14, Lemma 4.6], adapting it to the present setting. With obvious notation, we denote by b_h and $\tilde b_h$ sequences of functions converging to b and $\tilde b$, with the properties in Lemma 3. Similarly, we denote by q_h and $\tilde q_h$ sequences of functions converging to q and $\tilde q$, with the properties in Lemma 3. Consider also the regularization of the initial datum $u_o^h \in \mathbf{C}^\infty(\mathbb{R}^n;\mathbb{R})$ provided by Lemma 1. For $\vartheta \in [0,1]$, set

$$\begin{aligned}
&b_h^\vartheta(t,x) = \vartheta\, b_h(t,x) + (1-\vartheta)\,\tilde b_h(t,x), \quad c^\vartheta(t,x) = \vartheta\, c(t,x) + (1-\vartheta)\,\tilde c(t,x),\\
&q_h^\vartheta(t,x) = \vartheta\, q_h(t,x) + (1-\vartheta)\,\tilde q_h(t,x).
\end{aligned}$$

Let u_h^ϑ be the solution to

$$\begin{cases}\partial_t u_h^\vartheta + \nabla\cdot\left(c^\vartheta(t,x)\,u_h^\vartheta\right) = b_h^\vartheta(t,x)\,u_h^\vartheta + q_h^\vartheta(t,x),\\ u_h^\vartheta(t_o,x) = u_o^h(x),\end{cases} \quad \text{where } \begin{cases}\dot X^\vartheta = c^\vartheta(t,X^\vartheta),\\ X^\vartheta(t_o)=x_o,\end{cases}$$

that is,

$$\begin{aligned}
u_h^\vartheta(t,x) \;=\;& u_o^h(X^\vartheta(t_o;t,x))\,\exp\left(\int_{t_o}^t \left(b_h^\vartheta-\nabla\cdot c^\vartheta\right)\left(\tau,X^\vartheta(\tau;t,x)\right)d\tau\right)\\
&+\int_{t_o}^t q_h^\vartheta\left(s,X^\vartheta(s;t,x)\right)\exp\left(\int_s^t \left(b_h^\vartheta-\nabla\cdot c^\vartheta\right)\left(\tau,X^\vartheta(\tau;t,x)\right)d\tau\right)ds.
\end{aligned} \tag{32}$$

Compute the derivative of X^ϑ with respect to ϑ, recalling that $X^\vartheta(t;t,x)=x$ for all ϑ:

$$\begin{cases}\partial_t\partial_\vartheta X^\vartheta(\tau;t,x) = c(\tau,X^\vartheta(\tau;t,x)) - \tilde c(\tau,X^\vartheta(\tau;t,x)) + \nabla c^\vartheta(\tau,X^\vartheta(\tau;t,x))\,\partial_\vartheta X^\vartheta(\tau;t,x),\\ \partial_\vartheta X^\vartheta(t;t,x)=0.\end{cases}$$

The solution to the above problem satisfies

$$\begin{aligned}\partial_\vartheta X^\vartheta(\tau;t,x) &= \int_t^\tau \exp\left(\int_s^\tau \nabla c^\vartheta(\sigma,X^\vartheta(\sigma;t,x))\,d\sigma\right)(c-\tilde c)(s,X^\vartheta(s;t,x))\,ds\\ &= \int_\tau^t \exp\left(\int_\tau^s -\nabla c^\vartheta(\sigma,X^\vartheta(\sigma;t,x))\,d\sigma\right)(\tilde c-c)(s,X^\vartheta(s;t,x))\,ds. \end{aligned} \tag{33}$$

Derive (32) with respect to ϑ:

$$\begin{aligned}
&\partial_\vartheta u_h^\vartheta(t,x)\\
&= \exp\left(\int_{t_o}^t (b_h^\vartheta - \nabla\cdot c^\vartheta)(\tau,X^\vartheta(\tau;t,x))\,d\tau\right)\Big\{\nabla u_o^h(X^\vartheta(t_o;t,x))\,\partial_\vartheta X^\vartheta(t_o;t,x)\\
&\quad + u_o^h(X^\vartheta(t_o;t,x))\int_{t_o}^t \big(b_h - \tilde b_h - \nabla\cdot(c-\tilde c)\big)(\tau,X^\vartheta(\tau;t,x))\,d\tau\\
&\quad + u_o^h(X^\vartheta(t_o;t,x))\int_{t_o}^t \nabla(b_h^\vartheta - \nabla\cdot c^\vartheta)(\tau,X^\vartheta(\tau;t,x))\,\partial_\vartheta X^\vartheta(\tau;t,x)\,d\tau\Big\}\\
&\quad + \int_{t_o}^t \exp\left(\int_s^t (b_h^\vartheta - \nabla\cdot c^\vartheta)(\tau,X^\vartheta(\tau;t,x))\,d\tau\right)\\
&\quad \times\Big\{(q_h-\tilde q_h)(x,X^\vartheta(s;t,x)) + \nabla q_h^\vartheta(s,X^\vartheta(s;t,x))\,\partial_\vartheta X^\vartheta(s;t,x)\\
&\quad + q_h^\vartheta(s,X^\vartheta(s;t,x))\int_s^t \big(b_h - \tilde b_h - \nabla\cdot(c-\tilde c)\big)(\tau,X^\vartheta(\tau;t,x))\,d\tau\\
&\quad + q_h^\vartheta(s,X^\vartheta(s;t,x))\int_s^t \nabla(b_h^\vartheta - \nabla\cdot c^\vartheta)(\tau,X^\vartheta(\tau;t,x))\,\partial_\vartheta X^\vartheta(\tau;t,x)\,d\tau\Big\}\,ds\\
&\le \exp\left(\int_{t_o}^t (b_h^\vartheta - \nabla\cdot c^\vartheta)(\tau,X^\vartheta(\tau;t,x))\,d\tau\right)\Big\{\int_{t_o}^t (q_h-\tilde q_h)(x,X^\vartheta(s;t,x))\,ds\\
&\quad + \left(\nabla u_o^h(X^\vartheta(t_o;t,x)) + \int_{t_o}^t \nabla q_h^\vartheta(\tau,X^\vartheta(\tau;t,x))\,d\tau\right)\\
&\quad \times \int_{t_o}^t \exp\left(\int_{t_o}^s -\nabla c^\vartheta(\sigma,X^\vartheta(\sigma;t,x))\,d\sigma\right)(\tilde c - c)(s,X^\vartheta(s;t,x))\,ds\\
&\quad + \left(u_o^h(X^\vartheta(t_o;t,x)) + \int_{t_o}^t q_h^\vartheta(\tau,X^\vartheta(\tau;t,x))\,d\tau\right)\\
&\quad \times\Big[\int_{t_o}^t \big(b_h - \tilde b_h - \nabla\cdot(c-\tilde c)\big)(\tau,X^\vartheta(\tau;t,x))\,d\tau
\end{aligned}$$

$$+\int_{t_o}^{t} \nabla(b_h^\vartheta - \nabla\cdot c^\vartheta)(\tau, X^\vartheta(\tau;t,x))$$

$$\times\left[\int_{\tau}^{t} \exp\left(\int_{\tau}^{s} -\nabla c^\vartheta(\sigma, X^\vartheta(\sigma;t,x))\,d\sigma\right)(\tilde{c}-c)(s, X^\vartheta(s;t,x))\,ds\right]d\tau\Bigg]\Bigg\},$$

where we made use of (33). Call u_h and $\tilde{u}_h$ the functions defined by (32) for $\vartheta = 0$ and $\vartheta = 1$, that is, $u_h = u_h^{\vartheta=0}$ and $\tilde{u}_h = u_h^{\vartheta=1}$. Compute

$$\|u_h(t) - \tilde{u}_h(t)\|_{\mathbf{L}^1(\mathbb{R}^n)} \le \int_{\mathbb{R}^n}\left|\int_0^1 \partial_\vartheta u_h^\vartheta(t,x)\,d\vartheta\right|dx \le \int_0^1\int_{\mathbb{R}^n} |\partial_\vartheta u_h^\vartheta(t,x)|\,dx\,d\vartheta. \tag{34}$$

Exploiting the change of variable introduced at the beginning of the proof, compute

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\partial_\vartheta u_h^\vartheta(t,x)|\,dx\\
&\le \exp\left(\int_{t_o}^{t} \|b_h^\vartheta(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\,d\tau\right)\Bigg\{\int_{t_o}^{t} \|(q_h-\tilde{q}_h)(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)}\,d\tau\\
&\quad+\left(\int_{\mathbb{R}^n} |\nabla u_o^h(y)|\,dy + \int_{t_o}^{t}\int_{\mathbb{R}^n} |\nabla q_h^\vartheta(\tau,y)|\,dy\,d\tau\right)\\
&\quad\times\exp\left(\int_{t_o}^{t} \|\nabla c^\vartheta(\sigma)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\,d\sigma\right)\int_{t_o}^{t} \|(c-\tilde{c})(s)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)}\,ds\\
&\quad+\left(\|u_o^h\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \int_{t_o}^{t} \|q_h^\vartheta(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}(\tau)\,d\tau\right)\int_{t_o}^{t} \|(b_h-\tilde{b}_h-\nabla\cdot(c-\tilde{c}))(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)}\,d\tau\\
&\quad+\left(\|u_o^h\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \int_{t_o}^{t} \|q_h^\vartheta(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}(\tau)\,d\tau\right)\int_{t_o}^{t} \|\nabla(b_h^\vartheta-\nabla\cdot c^\vartheta)(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}\,ds\\
&\quad\times\exp\left(\int_{t_o}^{t} \|\nabla c^\vartheta(\sigma)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\,d\sigma\right)\int_{t_o}^{t} \|(c-\tilde{c})(s)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)}\,ds\Bigg\}.
\end{aligned}$$

Inserting the result above in (34), by the definitions of b_h^ϑ, q_h^ϑ and their properties as stated in Lemma 3, we have

$$\begin{aligned}
&\|u_h(t) - \tilde{u}_h(t)\|_{\mathbf{L}^1(\mathbb{R}^n)}\\
&\le \exp\left(\int_{t_o}^{t} \max\left\{\|b(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}, \|\tilde{b}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\right\}d\tau\right)\Bigg\{\int_{t_o}^{t} \|(q_h-\tilde{q}_h)(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)}\,d\tau\\
&\quad+\left(\|u_o^h\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \int_{t_o}^{t} \max\left\{\|q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}, \|\tilde{q}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\right\}d\tau\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \int_{t_o}^{t} \|(b_h - \tilde b_h - \nabla\cdot(c-\tilde c))(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)}\, d\tau \\
&+\exp\left(\int_{t_o}^{t} \max\left\{\|\nabla c(s)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}, \|\nabla \tilde c(s)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\right\} ds\right) \\
&\times \int_{t_o}^{t} \|(c-\tilde c)(s)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)}\, ds \\
&\times \Bigg[\int_{\mathbb{R}^n} |\nabla u_o^h(y)|\, dy + \int_{t_o}^{t} \max\left\{\|\nabla q_h(s)\|_{\mathbf{L}^1(\mathbb{R}^n)}, \|\nabla \tilde q_h(s)\|_{\mathbf{L}^1(\mathbb{R}^n)}\right\} ds \\
&+\left(\|u_o^h\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \int_{t_o}^{t} \max\left\{\|q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}, \|\tilde q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\right\} d\tau\right) \\
&\times \int_{t_o}^{t} \max\left\{\|\nabla(b_h - \nabla\cdot c)(s)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}, \|\nabla(\tilde b_h - \nabla\cdot \tilde c)(s)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}\right\} ds\Bigg]\Bigg\}.
\end{aligned}
$$

Let now h tend to $+\infty$. We have:

$$
\begin{array}{lll}
\|u_h(t) - \tilde u_h(t)\|_{\mathbf{L}^1(\mathbb{R}^n)} \to & \|u(t) - \tilde u(t)\|_{\mathbf{L}^1(\mathbb{R}^n)} & \\
\|u_o^h\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \leq & \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} & \text{by (14)} \\
\|(q_h - \tilde q_h)(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)} \to & \|(q-\tilde q)(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)} & \text{by Lemma 3} \\
\|(b_h - \tilde b_h - \nabla\cdot(c-\tilde c))(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)} \to & \|(b - \tilde b - \nabla\cdot(c-\tilde c))(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)} & \text{by Lemma 3} \\
\|\nabla u_o^h\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} \to & \mathrm{TV}\,(u_o) & \text{by (14)} \\
\|\nabla b_h(s)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} \leq & \mathrm{TV}\,(b(s)) & \text{by Lemma 3} \\
\|\nabla \tilde b_h(s)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} \leq & \mathrm{TV}\,\big(\tilde b(s)\big) & \text{by Lemma 3} \\
\|\nabla q_h(s)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} \leq & \mathrm{TV}\,(q(s)) & \text{by Lemma 3} \\
\|\nabla \tilde q_h(s)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} \leq & \mathrm{TV}\,(\tilde q(s)) & \text{by Lemma 3.}
\end{array}
$$

Therefore,

$$
\begin{aligned}
&\|u(t) - \tilde u(t)\|_{\mathbf{L}^1(\mathbb{R}^n)} \qquad (35)\\
\leq\ &\exp\left(\int_{t_o}^{t} \max\left\{\|b(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}, \|\tilde b(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\right\} d\tau\right)\Bigg\{\int_{t_o}^{t} \|(q-\tilde q)(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)}\, d\tau \\
&+\left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \int_{t_o}^{t} \max\left\{\|q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}, \|\tilde q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\right\} d\tau\right) \\
&\times \int_{t_o}^{t} \|(b - \tilde b - \nabla\cdot(c-\tilde c))(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)}\, d\tau \\
&+\exp\left(\int_{t_o}^{t} \max\left\{\|\nabla c(s)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}, \|\nabla \tilde c(s)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\right\} ds\right)
\end{aligned}
$$

$$\times \int_{t_o}^{t} \|(c-\tilde{c})(s)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)}\, d\,s$$
$$\times \left[\mathrm{TV}\,(u_o) + \int_{t_o}^{t} \max\left\{\mathrm{TV}\,(q(s)),\ \mathrm{TV}\,(\tilde{q}(s))\right\} d\,s\right.$$
$$+ \left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \int_{t_o}^{t} \max\left\{ \|q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}, \|\tilde{q}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \right\} d\,\tau \right)$$
$$\left.\left. \times \int_{t_o}^{t} \max\left\{ \mathrm{TV}\,(b(s)) + \|\nabla\,\nabla\cdot c(s)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}, \mathrm{TV}\,\left(\tilde{b}(s)\right) + \|\nabla\,\nabla\cdot \tilde{c}(s)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} \right\} d\,s \right]\right\}.$$

This completes the proof. ■

4.3 | Proof of the main result

Proof of Theorem 1. Choose an initial datum $(u_o, w_o) \in \mathcal{X}^+$. Define $u_0(t,x) = u_o(x)$ and $w_0(t,x) = w_o(x)$ for $(t,x) \in I \times \mathbb{R}^n$. Then, construct recursively for $i = 1, 2, \ldots$ the following sequences of functions:

$$\begin{array}{ll} a_i(t,x) = g(t,x,u_{i-1}(t,x), w_{i-1}(t,x)); & u_i \text{ solves } \begin{cases} \partial_t u_i + \nabla\cdot(c_i(t,x)u_i) = b_i(t,x)u_i + q(t,x), \\ u_i(t_o,x) = u_o(x); \end{cases} \\ b_i(t,x) = f(t,x,w_{i-1}(t,x)); & \\ c_i(t,x) = (v(t,w_{i-1}(t)))(x); & w_i \text{ solves } \begin{cases} \partial_t w_i - \mu\,\Delta w_i = a_i(t,x)w_i, \\ w_i(t_o,x) = w_o(x). \end{cases} \end{array} \tag{36}$$

The existence part of the proof amount to verify that (u_i, w_i) is a Cauchy sequence in a suitable complete metric space and that its limit solves (1). We divide the proof into several steps.

Step 0: For all $i \in \mathbb{N}$, (u_i, w_i) is well defined and

$$\begin{array}{lll} \text{for all } t \in I\ u_i(t) \in \mathcal{U}^+ & \text{and} & u_i \in \mathbf{C}^{0,1}(I;\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}_+)), \\ \text{for all } t \in I\ w_i(t) \in \mathcal{U}^+ & \text{and} & w_i \in \mathbf{C}^{0}(I;\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}_+)). \end{array} \tag{37}$$

Proof of Step. 0 For $i = 0$, the thesis holds true due to the choice of the initial data and the definition of u_0 and w_0. We proceed by induction.

Assume now that the claim holds for $i-1$, with $i \geq 1$. Then, $a_i \in \mathbf{L}^\infty(I \times \mathbb{R}^n;\mathbb{R})$ for all $t \in I$, by ($\boldsymbol{g}$) and by the inductive hypothesis. Propositions 1 and 2 and Corollary 1 hence ensure that w_i is well defined, with $w_i(t) \in \mathcal{U}^+$ for all $t \in I$. Similarly, b_i satisfies ($\boldsymbol{b}$+) by ($\boldsymbol{f}$) and c_i satisfies ($\boldsymbol{c2}$) by ($\boldsymbol{v}$). An application of Proposition 3 ensures the existence of u_i, with $u_i(t) \in \mathcal{U}^+$ for all $t \in I$. The time regularity of w_i follows from (**P2**) in Proposition 1 and, for u_i, from (**H7**) in Proposition 3.

Step 1: For all $i \in \mathbb{N}$, for all $t \geq t_o$

$$\|w_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n)} \leq \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n)}\, e^{K_g(t)(t-t_o)}, \quad \|w_i(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \leq \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\, e^{K_g(t)(t-t_o)}, \tag{38}$$

$$\begin{aligned}\|u_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n)} \leq & \Big(\|u_o\|_{\mathbf{L}^1(\mathbb{R}^n)} + \|q\|_{\mathbf{L}^1([t_o,t]\times\mathbb{R}^n)} \Big) \\ & \times \exp\left[K_f(t)\,(t-t_o) \Big(1 + \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \mathrm{e}^{K_g(t)\,(t-t_o)}\Big)\right],\end{aligned} \tag{39}$$

$$\begin{aligned}\|u_i(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \leq & \Big(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \|q\|_{\mathbf{L}^1([t_o,t];\mathbf{L}^\infty(\mathbb{R}^n))} \Big) \\ & \times \exp\left[\big(K_f(t) + K_v(t)\big)(t-t_o) \Big(1 + \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} e^{K_g(t)\,(t-t_o)}\Big)\right].\end{aligned} \tag{40}$$

(The $\mathbf{L}^1$ and $\mathbf{L}^\infty$ estimates on w are *independent* of u. This fact plays a key role throughout, in particular in **Step 6** below.)

Proof of Step 1. By ($\boldsymbol{g}$) and (36), with obvious notation, for all $\tau \in [t_o, t]$,

$$A_i(\tau) := \sup_{\xi\in\mathbb{R}^n} a_i(\tau,\xi) = \sup_{\xi\in\mathbb{R}^n} g(\tau,\xi,u_{i-1}(\tau,\xi), w_{i-1}(\tau,\xi)) \leq K_g(\tau) \leq K_g(t)\,.$$

Hence, (38) follows by (**P10**) in Corollary 1.

Proceeding now similarly, using (36), ($\boldsymbol{f}$), and (38), compute for $\tau \in [t_o, t]$,

$$\begin{aligned}\sup_{x\in\mathbb{R}^n} b_i(\tau,x) = \sup_{x\in\mathbb{R}^n} f(\tau,x,w_{i-1}(\tau,x)) &\leq \sup_{x\in\mathbb{R}^n} K_f(\tau)(1 + w_{i-1}(\tau,x)) \\ &\leq K_f(\tau)\Big(1 + \|w_i(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\Big) \leq K_f(t)\Big(1 + \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\,\mathrm{e}^{K_g(t)\,(t-t_o)}\Big).\end{aligned}$$

Estimate (39) now follows from (**H3**) in Proposition 3 and (36). Moreover, by ($\boldsymbol{v}$),

$$\|\nabla\cdot c(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \leq K_v(\tau)\,\|w_{i-1}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \leq K_v(t)\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\,e^{K_g(t)\,(t-t_o)}.$$

Using now (**H4**) in Proposition 3 and (36), the bound (40) follows.

Step 2: There exists $\mathcal{G} \in \mathbf{C}^0(I;\mathbb{R}_+)$ such that for all $t \in I$ and $i \in \mathbb{N}$, $\mathrm{TV}\,(w_i(t)) \leq \mathcal{G}(t)$.

Proof of Step 2. By the definition of a_i given in (36), by ($\boldsymbol{g}$) and by (**P12**) in Corollary 1 we obtain $\mathrm{TV}\,(w_i(t)) \leq \mathcal{G}(t)$ where

$$\mathcal{G}(t) = \mathrm{TV}\,(w_o) + \frac{2J_n\sqrt{t-t_o}}{\sqrt{\mu}}\,K_g(t)\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\,\mathrm{e}^{K_g(t)\,(t-t_o)}.$$

Step 3: There exists $\mathcal{F} \in \mathbf{C}^0(I;\mathbb{R}_+)$ such that, for all $t \in I$ and all $i \in \mathbb{N}$, $\mathrm{TV}\,(u_i(t)) \leq \mathcal{F}(t)$.

Proof of Step 3. Exploiting the definitions of b_i and c_i given in (36), by ($\boldsymbol{v}$), for $\tau \in [t_0, t]$,

$$\|\nabla c_i(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})} \leq K_v(\tau)\|w_{i-1}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \leq K_v(\tau)\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\,\mathrm{e}^{K_g(\tau)\,(\tau-t_o)},$$

$$\|\nabla\,\nabla\cdot c_i(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} \leq C_v\Big(\tau, \|w_{i-1}(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)}\Big)\|w_{i-1}(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)}$$

$$\leq C_v\left(\tau, \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n)}\, e^{K_g(\tau)\,(\tau-t_o)}\right)\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n)} e^{K_g(\tau)\,(\tau-t_o)}.$$

and by (***f***), (38), and **Step 2**,

$$\begin{aligned}\mathrm{TV}\,(b_i(\tau)) &= \mathrm{TV}\,(f(\tau,\cdot,w_{i-1}(\tau,\cdot)))\\ &\leq K_f(\tau)\Big(1+\|w_{i-1}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}+\mathrm{TV}\,(w_{i-1}(\tau))\Big)\\ &\leq K_f(\tau)\left(1+\mathrm{TV}\,(w_o)+\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\left(1+\frac{2J_n\sqrt{\tau-t_o}}{\sqrt{\mu}}\,K_g(\tau)\right)e^{K_g(\tau)\,(\tau-t_o)}\right).\end{aligned}$$

Insert the latter estimates above in (**H6**) of Proposition 3 to get $\mathrm{TV}\,(u_i(t)) \leq \mathcal{F}(t)$, where

$$\begin{aligned}\mathcal{F}(t) = &\left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)}+\mathrm{TV}\,(u_o)+\int_{t_o}^t\Big(\|q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}+\mathrm{TV}\,(q(\tau))\Big)\,d\tau\right)\\ &\times\exp\left(\int_{t_o}^t\Big(K_f(\tau)+\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\big(K_f(\tau)+K_v(\tau)\big)e^{K_g(\tau)(\tau-t_o)}\Big)\,d\tau\right)\\ &\times\Bigg(1+\int_{t_o}^t C_v\Big(\tau,\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n)}e^{K_g(\tau)(\tau-t_o)}\Big)\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n)}\,e^{K_g(\tau)(\tau-t_o)}\,d\tau\\ &+K_f(t)(t-t_o)\left(1+\mathrm{TV}\,(w_o)+\frac{4J_n}{3\sqrt{\mu}}\sqrt{t-t_o}\,K_g(t)\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\,e^{K_g(t)(t-t_o)}\right)\Bigg)\end{aligned}$$

concluding the proof of **Step 3**.

Observe for later use that, due to (***f***), (***g***), and (***v***), on a bounded time interval $[t_o, T]$

$$a_{i+1}-a_i,\; b_{i+1}-b_i,\; \nabla\cdot(c_{i+1}-c_i)\in\mathbf{L}^1([t_o,T]\times\mathbb{R}^n;\mathbb{R})\,. \tag{41}$$

Step 4: Referring to (4), (5), **Step 2**, and **Step 3**, consider the complete metric space

$$\begin{aligned}&\mathcal{Y}_T=\big\{(u,w)\in\mathbf{C}^0([t_o,T];\mathcal{X}^+)\,:\,\mathrm{TV}\,(u(t))\leq\mathcal{F}(t)\text{ and }\mathrm{TV}\,(w(t))\leq\mathcal{G}(t)\text{ for all }t\in[t_o,T]\big\},\\ &d((u_1,w_1),(u_2,w_2))=\sup_{t\in[t_o,T]}\|(u_1(t)-u_2(t),\,w_1(t)-w_2(t))\|_{\mathcal{X}}.\end{aligned}\tag{42}$$

Moreover, for $r > 0$ introduce the following subset of $\mathcal{X}^+$:

$$\mathcal{X}_r^+=\left\{(u,w)\in\mathcal{X}^+\,:\,\begin{array}{lll}\|u\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\leq r, & \mathrm{TV}\,(u)\leq r, & \\ \|w\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\leq r, & \|w\|_{\mathbf{L}^1(\mathbb{R}^n)}\leq r, & \mathrm{TV}\,(w)\leq r\end{array}\right\}. \tag{43}$$

Then, given $(u_o, w_o) \in \mathcal{X}_r^+$, there exists a continuous function $\mathcal{K}_r : [t_o, T] \to \mathbb{R}_+$, for a suitable $T \in I$ with $T > t_o$, such that for all $i \in \mathbb{N}$

$$d((u_{i+1},w_{i+1}),(u_i,w_i))\leq\mathcal{K}_r(T)\,(T-t_o)\,d((u_i,w_i),(u_{i-1},w_{i-1})). \tag{44}$$

Proof of Step 4. In the following, we make use of the bounds (38)–(40). Start from (**P11**) in Corollary 1: for all $t \in [t_o, T]$, using (36) and ($\boldsymbol{g}$), we obtain

$$\begin{aligned} & \|w_{i+1}(t) - w_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n)} \\ \le\ & K_g(t)\,(t-t_o)\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} e^{2\,(t-t_o)K_g(t)} \sup_{\tau\in[t_o,t]} \|(u_i(\tau)-u_{i-1}(\tau),\, w_i(\tau)-w_{i-1}(\tau))\|_{\mathcal{X}} \\ \le\ & \mathcal{K}_r^w(T)\,(T-t_o)\; d((u_i,w_i),(u_{i-1},w_{i-1})), \end{aligned} \tag{45}$$

with

$$\mathcal{K}_r^w(T) = r\,K_g(T)\,e^{2\,(T-t_o)K_g(T)}. \tag{46}$$

Now consider (**H5**) in Proposition 3: by ($\boldsymbol{v}$) and ($\boldsymbol{f}$), setting

$$\begin{aligned} \tilde{\mathcal{O}}_1(t) = {} & \exp\left(K_f(t)(t-t_o) + \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\,(t-t_o)\big(K_f(t)+K_v(t)\big)e^{K_g(t)\,(t-t_o)}\right) \\ & \times\Bigg[1 + (t-t_o)\,C_v(t, \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n)} e^{K_g(t)(t-t_o)})\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n)} e^{K_g(t)(t-t_o)} \\ & + K_f(t)(t-t_o)\left(1 + \mathrm{TV}\,(w_o) + \frac{4\,J_n}{3\,\sqrt{\mu}}\,\sqrt{t-t_o}\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\,K_g(t)\,e^{K_g(t)(t-t_o)}\right)\Bigg], \end{aligned} \tag{47}$$

$$\tilde{\mathcal{O}}_2(t) = \exp\left(K_f(t)\,(t-t_o)\Big(1 + \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} e^{K_g(t)(t-t_o)}\Big)\right), \tag{48}$$

we get

$$\begin{aligned} & \|u_{i+1}(t) - u_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n)} \\ \le & \Bigg[\tilde{\mathcal{O}}_1(t)(t-t_o)\left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \mathrm{TV}\,(u_o) + \int_{t_o}^t \Big(\|q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \mathrm{TV}\,(q(\tau))\,d\tau\Big)\right)K_v(t) \\ & + \tilde{\mathcal{O}}_2(t)(t-t_o)\left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \int_{t_o}^t \|q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\,d\tau\right)\left(K_f(t) + C_v\Big(t, \|w_o\|_{\mathbf{L}^\infty} e^{K_g(t)(t-t_o)}\Big)\right)\Bigg] \\ & \times \sup_{\tau\in[t_o,t]} \|w_i(\tau) - w_{i-1}(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)} \\ \le & \mathcal{K}_r^u(T)\,(T-t_o) \sup_{\tau\in[t_o,T]} \|w_i(\tau)-w_{i-1}(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)}, \end{aligned} \tag{49}$$

with

$$\mathcal{K}_r^u(T) = \left(r + \int_{t_o}^T \|q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\,d\tau\right)\exp\left(K_f(T)(T-t_o)\Big(1 + r\,e^{K_g(T)(T-t_o)}\Big)\right)$$

$$
\begin{aligned}
&\times\left[K_f(T)+C_v\left(T, r\,e^{K_g(T)(T-t_o)}\right)\right]\\
&+\left(2r+\int_{t_o}^{T}\left(\|q(\tau)_{\mathbf{L}^\infty(\mathbb{R}^n)}\|+\mathrm{TV}\,(q(\tau))\right)d\tau\right)\\
&\times\exp\left(K_f(T)(T-t_o)\left(1+r\,e^{K_g(T)(T-t_o)}\right)\right) \qquad (50)\\
&\times K_v(T)\exp\left(K_v(T)\,(T-t_o)\,r\,e^{K_g(T)(T-t_o)}\right)\\
&\times\left(1+r\,(T-t_o)\,C_v(T, r\,e^{K_g(T)(T-t_o)})e^{K_g(T)(T-t_o)}\right.\\
&\left.+(T-t_o)\,K_f(T)\left(1+r+\frac{4J_n}{3\sqrt{\mu}}K_g(T)\sqrt{T-t_o}\,r\,e^{K_g(T)(T-t_o)}\right)\right).
\end{aligned}
$$

Thus, collecting together (45) and (49),

$$
\begin{aligned}
d((u_{i+1}, w_{i+1}), (u_i, w_i)) &= \sup_{t\in[0,T]}\left(\|u_{i+1}(t)-u_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n)}+\|w_{i+1}(t)-w_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n)}\right)\\
&\le (\mathcal{K}_r^u(T)+\mathcal{K}_r^w(T))\,(T-t_o)\,d((u_i, w_i), (u_{i-1}, w_{i-1})).
\end{aligned}
$$

This proves (44), with $\mathcal{K}_r(T)=\mathcal{K}_r^u(T)+\mathcal{K}_r^w(T)$ and **Step 4** is completed.

Step 5: For any $r>0$, there exists a $T_r>0$ such that for all $(u_o, w_o)\in\mathcal{X}_r^+$, the sequence (u_i, w_i) converges in $\mathcal{Y}_{T_r}$ to a (u_*, w_*) solving (1) in the sense of Definition 1.

Proof of Step 5. Choose $T_r>t_o$ such that $\mathcal{K}_r(T_r)\,(T_r-t_o)<1$. Thanks to (44), the sequence (u_i, w_i) defined through (36) is a Cauchy sequence and converges in the complete metric space $(\mathcal{Y}_{T_r}, d)$ defined in (42). Call (u_*, w_*) the limit. Clearly, $u_*\in\mathbf{C}^0([t_o, T_r]; \mathcal{U}^+)$ and $w_*\in\mathbf{C}^0([t_o, T_r]; \mathcal{U}^+)$. It remains to prove that (u_*, w_*) is a solution to (1) in the sense of Definition 1. By Lemmas 4 and 5, it is sufficient to prove that u_* is a weak solution to (23) and w_* is a weak solution to (15) with

$$
a(t,x)=g(t,x,u_*(t,x),w_*(t,x)),\quad b(t,x)=f(t,x,w_*(t,x)),\quad c(t,x)=(v(t,w_*(t)))(x).
$$

The initial condition is satisfied: $(u_*, w_*)(0)=(u_o, w_o)$. Using the weak formulations (27) and (17), applying the Dominated Convergence Theorem, thanks to ($\boldsymbol{f}$) and ($\boldsymbol{g}$), we obtain that (u_*, w_*) solves (1) on $[t_o, T_r]$, with initial datum (u_o, w_o), in the sense of Definition 1.

Step 6: The solution constructed above can be uniquely extended to all I.

Proof of Step 6. The uniform continuity in time of (u_*, w_*) on $[t_o, T_r]$ ensures that $(u_*(T_r), w_*(T_r))=\lim_{t\to T_r-}(u_*(t), w_*(t))$ is in $\mathcal{X}^+$. The above results can be iteratively applied, proving that (u_*, w_*) can be uniquely extended to a maximal time interval $[t_o, T_*[$.

The $\mathbf{L}^1$ and $\mathbf{L}^\infty$ bounds in (38), together with the **BV** bound in **Step 2**, ensure that the limit $\lim_{t\to T_*-} w_*(t)$ exists and is in $\mathcal{U}^+$, so that we can define $w_*(T_*)=\lim_{t\to T_*-} w_*(t)$. Similarly, Proposition 3, allows to uniquely extend u_* in T_*, setting $u_*(T_*) = \lim_{t\to T_*-} u_*(t)$ with $u_*(T_*) \in \mathcal{U}^+$. A further application of the steps above then allows to further prolong (u_*, w_*) beyond time T_*, unless $T_* = \sup I$, completing the proof of this step.

Step 7: Let $r > 0$. Given (u_o, w_o), $(\tilde u_o, \tilde w_o) \in \mathcal{X}_r^+$, call (u, w) and $(\tilde u, \tilde w)$ the corresponding solutions to (1). Then, for all $t \in I$, (6) holds, with C_o defined in (61).

Proof of Step 7. Define for $(t, x) \in I \times \mathbb{R}^n$ the following functions:

$$\begin{aligned} a(t,x) &= g(t,x,u(t,x),w(t,x)), & \tilde a(t,x) &= g(t,x,\tilde u(t,x),\tilde w(t,x)),\\ b(t,x) &= f(t,x,w(t,x)), & \tilde b(t,x) &= f(t,x,\tilde w(t,x)),\\ c(t,x) &= (v(t,w(t)))(x), & \tilde c(t,x) &= (v(t,\tilde w(t)))(x). \end{aligned} \tag{51}$$

Let $\hat w$ be the solution to (15) with a in the source term and initial datum $\tilde w_o$, and let $\hat u$ be the solution to (23) with coefficients b, c, and initial datum $\tilde u_o$. More precisely,

$$\begin{cases} \partial_t \hat w - \mu\,\Delta \hat w = a(t,x)\,\hat w\\ \hat w(t_o,x) = \tilde w_o(x) \end{cases} \quad\text{and}\quad \begin{cases} \partial_t \hat u + \nabla\cdot(c(t,x)\,\hat u) = b(t,x)\,\hat u + q(t,x)\\ \hat u(t_o,x) = \tilde u_o(x). \end{cases} \tag{52}$$

By (5), we need to compute

$$\begin{aligned} \|(u(t),w(t)) - (\tilde u(t),\tilde w(t))\|_{\mathcal{X}} &= \|u(t)-\tilde u(t)\|_{\mathbf{L}^1(\mathbb{R}^n)} + \|w(t)-\tilde w(t)\|_{\mathbf{L}^1(\mathbb{R}^n)}\\ &\le \|u(t)-\hat u(t)\|_{\mathbf{L}^1(\mathbb{R}^n)} + \|\hat u(t)-\tilde u(t)\|_{\mathbf{L}^1(\mathbb{R}^n)} \end{aligned} \tag{53}$$

$$+ \|w(t)-\hat w(t)\|_{\mathbf{L}^1(\mathbb{R}^n)} + \|\hat w(t)-\tilde w(t)\|_{\mathbf{L}^1(\mathbb{R}^n)}. \tag{54}$$

Compute each term in (53) separately. The first one is the $\mathbf{L}^1$-distance between solutions to balance laws of the type (23) with different initial data. Exploiting (24) for the solution to these balance laws and the bounds obtained in the proof of **Step 1**, we get

$$\|u(t)-\hat u(t)\|_{\mathbf{L}^1(\mathbb{R}^n)} \le \|u_o - \tilde u_o\|_{\mathbf{L}^1(\mathbb{R}^n)} \exp\left[K_f(t)\,(t-t_o)\Big(1 + \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\,e^{K_g(t)(t-t_o)}\Big)\right]. \tag{55}$$

The second term in (53) is the $\mathbf{L}^1$-distance between solutions to balance laws of the type (23) with different coefficients b, c and same initial datum. Exploiting the computations in the proof of **Step 4**, as well as (**H5**) in Proposition 3, we get

$$\begin{aligned} &\|\hat u(t)-\tilde u(t)\|_{\mathbf{L}^1(\mathbb{R}^n)}\\ &\le \Bigg\{ \hat{\mathcal{O}}_1(t,r)\Bigg(\|\tilde u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \mathrm{TV}(\tilde u_o) + \int_{t_o}^t \Big(\|q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \mathrm{TV}(q(\tau))\Big)\,d\tau\Bigg)K_v(t)\\ &\quad + \hat{\mathcal{O}}_2(t,r)\Bigg(\|\tilde u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \int_{t_o}^t \|q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\,d\tau\Bigg)\Big(K_f(t) + C_v\Big(t, \|\tilde w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} e^{K_g(t)(t-t_o)}\Big)\Big)\Bigg\} \end{aligned}$$

$$\times \int_{t_o}^{t} \|w(\tau) - \tilde{w}(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)}\, d\tau, \tag{56}$$

with

$$\hat{\mathcal{O}}_1(t,r) = \exp\left(K_f(t)\,(t-t_o) + r\,(t-t_o)\,\mathrm{e}^{K_g(t)(t-t_o)}\big(K_f(t)+K_v(t)\big)\right)$$
$$\times\Bigg[1 + (t-t_o)\,C_v(t, r\,e^{K_g(t)(t-t_o)})\,r\,\mathrm{e}^{K_g(t)(t-t_o)}$$
$$+ K_f(t)\,(t-t_o)\left(1 + r + \frac{4\,J_n}{3\,\sqrt{\mu}}\,r\,K_g(t)\,\sqrt{t-t_o}\,\mathrm{e}^{K_g(t)(t-t_o)}\right)\Bigg], \tag{57}$$

$$\hat{\mathcal{O}}_2(t,r) = \exp\left(K_f(t)\,(t-t_o)\left(1 + r\,\mathrm{e}^{K_g(t)(t-t_o)}\right)\right). \tag{58}$$

The first term in (54) is the $\mathbf{L}^1$-distance between solutions to equations of the type (15) with different initial data. Since $\mathcal{P}$ as defined in Proposition 1 is linear, by **Step 1** we obtain

$$\|w(t) - \hat{w}(t)\|_{\mathbf{L}^1(\mathbb{R}^n)} \le \|w_o - \tilde{w}_o\|_{\mathbf{L}^1(\mathbb{R}^n)}\, \exp\left(K_g(t)(t-t_o)\right). \tag{59}$$

The second term in (54) is the $\mathbf{L}^1$-distance between solutions to the parabolic equation (15) with different coefficients in the source term and the same initial datum. Exploiting the computations in the proof of **Step 4**, as well (**P11**) in Corollary 1, we get

$$\|\hat{w}(t) - \tilde{w}(t)\|_{\mathbf{L}^1(\mathbb{R}^n)} \le \|\tilde{w}_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} K_g(t)\,\mathrm{e}^{2\,K_g(t)(t-t_o)} \int_{t_o}^{t} \|(u(\tau) - \tilde{u}(\tau),\, w(\tau) - \tilde{w}(\tau))\|_{\mathcal{X}}\, d\tau. \tag{60}$$

Hence, (55), (56), (59), and (60) yield

$$\|(u(t), w(t)) - (\tilde{u}(t), \tilde{w}(t))\|_{\mathcal{X}} \le \mathcal{K}_1(t,r)\, \|(u_o, w_o) - (\tilde{u}_o, \tilde{w}_o)\|_{\mathcal{X}}$$
$$+\mathcal{K}_2(t,r) \int_{t_o}^{t} \left(\|(u(\tau) - \tilde{u}(\tau),\, w(\tau) - \tilde{w}(\tau))\|_{\mathcal{X}}\right) d\tau,$$

where we set

$$\mathcal{K}_1(t,r) = \exp\left(\max\left\{K_f(t)\,(t-t_o)\left(1 + r\,\mathrm{e}^{K_g(t)(t-t_o)}\right),\ \mathrm{e}^{K_g(t)(t-t_o)}\right\}\right),$$

$$\mathcal{K}_2(t,r) = \hat{\mathcal{O}}_1(t,r)\left(2\,r + \int_{t_o}^{t} \left(\|q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \mathrm{TV}\,(q(\tau))\right) d\tau\right) K_v(t)$$
$$+\hat{\mathcal{O}}_2(t,r)\left(r + \int_{t_o}^{t} \|q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\, d\tau\right)\left(K_f(t) + C_v\left(t, r\,\mathrm{e}^{K_g(t)(t-t_o)}\right)\right)$$
$$+r\,K_g(t)\,\mathrm{e}^{2\,K_g(t)(t-t_o)}.$$

An application of Gronwall Lemma yields:

$$\|(u(t),w(t))-(\tilde u(t),\tilde w(t))\|_{\mathcal{X}} \le \|(u_o,w_o)-(\tilde u_o,\tilde w_o)\|_{\mathcal{X}} \int_{t_o}^{t} \mathcal{K}_1(s,r)\,\exp\left(\int_s^t \mathcal{K}_2(\tau,r)\,d\tau\right)d\,s,$$

proving **Step 7** with

$$C_o(t,r)=\int_{t_o}^{t} \mathcal{K}_1(s,r)\,\exp\left(\int_s^t \mathcal{K}_2(\tau,r)\,d\tau\right)d\,s. \tag{61}$$

Step 8: Given q, $\tilde q$ satisfying ($\boldsymbol{q}$), call (u,w) and $(\tilde u,\tilde w)$ the solutions to (1) with the same initial datum $(u_o,w_o)\in\mathcal{X}_r^+$. Then, for all $t\in I$, (7) holds with C_q defined in (64).

Proof of Step 8. Define for $(t,x)\in I\times\mathbb{R}^n$ the functions $a,\tilde a,\ b,\ \tilde b,\ c,\ \tilde c$ as in (51).

The $\mathbf{L}^1$-distance between $w(t)$ and $\tilde w(t)$ can be computed as in (60), leading to

$$\|w(t)-\tilde w(t)\|_{\mathbf{L}^1(\mathbb{R}^n)} \le \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)} K_g(t)\,e^{2K_g(t)(t-t_o)}\int_{t_o}^{t}\|(u-\tilde u,\ w-\tilde w)(\tau)\|_{\mathcal{X}}\,d\tau. \tag{62}$$

To compute the $\mathbf{L}^1$-distance between $u(t)$ and $\tilde u(t)$, we exploit (**H5**) in Proposition 3 and the computations in the proofs of **Step 4** and **Step 7**, to get

$$\begin{aligned}
&\|u(t)-\tilde u(t)\|_{\mathbf{L}^1(\mathbb{R}^n)}\\
\le\ & \hat{\mathcal{O}}_1(t,r)\Bigg[\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)}+\mathrm{TV}\,(u_o)\\
&+\int_{t_o}^{t}\Big(\max\Big\{\|q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)},\ \|\tilde q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\Big\}+\max\{\mathrm{TV}\,(q(\tau)),\ \mathrm{TV}\,(\tilde q(\tau))\}\Big)\,d\tau\Bigg]\\
&\times K_v(t)\int_{t_o}^{t}\|w(\tau)-\tilde w(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)}\,d\tau\\
&+\hat{\mathcal{O}}_2(t,r)\left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n)}+\int_{t_o}^{t}\max\Big\{\|q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)},\ \|\tilde q(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}\Big\}\,d\tau\right)\\
&\times\Big(K_f(t)+C_v\Big(t,\|w_o\|_{\mathbf{L}^\infty}e^{K_g(t)(t-t_o)}\Big)\Big)\\
&\times\int_{t_o}^{t}\|w_i(\tau)-w_{i-1}(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n)}\,d\tau+\hat{\mathcal{O}}_2(t,r)\,\|q-\tilde q\|_{\mathbf{L}^1([t_o,t]\times\mathbb{R}^n)},
\end{aligned} \tag{63}$$

where $\hat{\mathcal{O}}_1(t,r)$ and $\hat{\mathcal{O}}_2(t,r)$ are as in (57)–(58). Collecting together (62) and (63) and an application of Gronwall Lemma completes the proof of **Step 8** with

$$C_q(t,r) = \hat{\mathcal{O}}_2(t,r) \int_{t_o}^{t} \exp \int_{s}^{t} \Bigg\{ r\, K_g(\tau)\, e^{2 K_g(\tau)(\tau - t_o)} + K_v(\tau)\, \hat{\mathcal{O}}_1(\tau,r) \Bigg[2r \tag{64}$$

$$+ \int_{t_o}^{\tau} \left[\max\left\{ \|q(\sigma)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}, \|\tilde{q}(\sigma)\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \right\} + \max\left\{ \mathrm{TV}\,(q(\sigma)), \mathrm{TV}\,(\tilde{q}(\sigma)) \right\} \right] d\sigma \Bigg]$$

$$+ \hat{\mathcal{O}}_2(\tau,r) \left[r + \int_{t_o}^{\tau} \max\left\{ \|q(\sigma)\|_{\mathbf{L}^\infty(\mathbb{R}^n)}, \|\tilde{q}(\sigma)\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \right\} d\sigma \right]$$

$$\times \left[K_f(\tau) + C_v\left(\tau, r e^{K_g(\tau)(\tau - t_o)}\right) \right] \Bigg\} d\tau\, ds.$$

■

5 | CONCLUSIONS

We introduced a predator–prey model amenable to describe the use of biological strategies in fighting pests' growth. Parasitoids–predators hunt for parasites–prey moving toward regions with high pests' density. The introduction of parasitoids in the environment is the space- and time-dependent control parameter. Numerical integrations show the wide differences in pests' control outcome, due to different control strategies.

From the analytical point of view, we deal with a mixed system of hyperbolic–parabolic nonlocal partial differential equations, whose well-posedness and stability properties are ensured. Thus, sufficient conditions for the existence of time- and space-dependent optimal controls are at hand, while necessary conditions are still currently unknown.

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DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

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