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The Pairing-Hamiltonian property in graph prisms

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ABSTRACT

Let G be a graph of even order, and consider K_G as the complete graph on the same vertex set as G. A perfect matching of K_G is called a pairing of G. If for every pairing M of G it is possible to find a perfect matching N of G such that $M \cup N$ is a Hamiltonian cycle of K_G, then G is said to have the Pairing-Hamiltonian property, or PH-property, for short. In 2007, Fink (2007) [4] proved that for every $d \geq 2$, the d-dimensional hypercube $\mathcal{Q}_d$ has the PH-property, thus proving a conjecture posed by Kreweras in 1996. In this paper we extend Fink's result by proving that given a graph G having the PH-property, the prism graph $\mathcal{P}(G) = G\square K_2$ of G has the PH-property as well. Moreover, if G is a connected graph, we show that there exists a positive integer k_0 such that the k^{th}-prism of a graph $\mathcal{P}^k(G)$ has the PH-property for all $k \geq k_0$.

1. Introduction

The problem of extending perfect matchings of a graph to a Hamiltonian cycle has been first considered by Las Vergnas [9] and Häggkvist [7] in the 1970s. They both proved Ore-type conditions which ensure that every perfect matching of a graph having some initial conditions can be extended to a Hamiltonian cycle.

Some years later, Kreweras [8] conjectured that any perfect matching of the hypercube $\mathcal{Q}_d$, $d \geq 2$, can be extended to a Hamiltonian cycle. This conjecture was proved in 2007 by Fink [4]. Actually, he proved a stronger version of the problem. Given a graph G, let K_G denote the complete graph on the same vertex set $V(G)$ of G. Fink shows that every perfect matching of $K_{\mathcal{Q}_d}$, and not only the perfect matchings of $\mathcal{Q}_d$, can be extended to a Hamiltonian cycle of $K_{\mathcal{Q}_d}$, by using only edges of $\mathcal{Q}_d$. More generally, for a graph G of even order, a perfect matching of K_G is said to be a *pairing* of G. Given a pairing M of G, we say that M can be *extended* to a Hamiltonian cycle H of K_G if we can find a perfect matching N of G such that $M \cup N = E(H)$, where $E(H)$ is the set of edges of H.

A graph G is said to have the *Pairing-Hamiltonian property* (or, the PH-property for short), if every pairing M of G can be extended to a Hamiltonian cycle as described above. For simplicity, we shall also say that a graph G is PH if it has the PH-property. This notation was introduced in [2], where amongst other results, a classification of cubic graphs that admit

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the PH-property was given: these are the complete graph K_4, the complete bipartite graph $K_{3,3}$, and the cube $\mathcal{Q}_3$. We remark that this was the first non-trivial classification of graphs (having regular degree) admitting the PH-property, as, the only 2-regular graph admitting the PH-property is the cycle on 4 vertices, which happens to be $\mathcal{Q}_2$. We also remark that there is an infinite number of 4-regular graphs having the PH-property (see [2,6]). Following such a terminology we can state Fink's result from [4] as follows.

Theorem 1.1 *(Fink, [4] 2007). The hypercube $\mathcal{Q}_d$ has the PH-property, for every $d \geq 2$.*

Recall that the *Cartesian product* $G \square H$ of two graphs G and H is a graph whose vertex set is $V(G) \times V(H)$, and two vertices (u_1, v_1) and (u_2, v_2) are adjacent precisely if $u_1 = u_2$ and $v_1 v_2 \in E(H)$, or $u_1 u_2 \in E(G)$ and $v_1 = v_2$.

Given a graph G, the *prism operator* $\mathcal{P}(G) = G \square K_2$ is the Cartesian product of G with K_2. Note that it consists of two copies G_1 and G_2 of G with the same vertex labelling as in G, and an edge between the vertices having the same label. Throughout the paper, we refer to these subgraphs, G_1 and G_2, as the main copies of G in $\mathcal{P}(G)$. The result of a single application of the operator is usually called the *prism graph* $\mathcal{P}(G)$ of G (see [3]), and repeated applications shall be denoted by powers, with $\mathcal{P}^k(G)$ being the prism graph of $\mathcal{P}^{k-1}(G)$. If needed we shall assume that $\mathcal{P}^0(G) = G$.

It is worth noting that for $d \geq 2$, $\mathcal{Q}_d = \mathcal{P}^{d-2}(\mathcal{Q}_2)$. Hence, Theorem 1.1 is equivalent to saying that for each $k \geq 0$, $\mathcal{P}^k(\mathcal{Q}_2)$ admits the PH-property. One might wonder whether it is possible to replace $\mathcal{Q}_2$ with some other initial graph. The main contribution of this paper is Theorem 2.1, which generalises Theorem 1.1. We obtain a much larger class of graphs with the PH-property by proving that for every graph G having the PH-property, the graph $\mathcal{P}^k(G)$ has the PH-property for each $k \geq 0$. Hence, Kreweras' Conjecture, and therefore Theorem 1.1, turn out to be special consequences of Theorem 2.1 obtained starting from $G = \mathcal{Q}_2$, which is trivially PH.

Other results on this topic, dealing with the Cartesian product of graphs, were also obtained in [2] and [6]. In particular, we state the following theorem which shall be needed in Section 3.

Theorem 1.2 *(Alahmadi et al., 2015 [2]). Let P_q be a path of length q. The graph $P_q \square \mathcal{Q}_d$ admits the PH-property, for $d \geq 5$.*

The above theorem is stated as Theorem 5 in [2], where some other results apart from the statement above are proved. We use this result to obtain a similar one for every connected graph G (see Theorem 3.5). More precisely, we prove that for every arbitrary connected graph G, the graph $\mathcal{P}^k(G)$ has the PH-property for a sufficiently large k, depending on the minimum number of leaves over all spanning trees of G. We refer the reader to [1] and [10] for other papers dealing with the Pairing-Hamiltonian property and related concepts under some graph operations.

2. Generalising Fink's result

As stated in the introduction, this section shall be devoted to generalising Theorem 1.1.

Theorem 2.1. *Let G be a graph having the PH-property. Then, for each $k \geq 0$, $\mathcal{P}^k(G)$ admits the PH-property.*

Proof. Consider $\mathcal{P}(G)$ and let G_1 and G_2 be the two main copies of the graph G in $\mathcal{P}(G)$. Then, a pairing P of $\mathcal{P}(G)$ can be partitioned into three subsets $P_1 \cup P_2 \cup X$ where:

$$P_i = \{xy \in P \mid \{x, y\} \subset V(G_i), \text{ for each } i \in \{1, 2\}\}; \text{ and}$$
$$X = \{xy \in P \mid x \in V(G_1),\ y \in V(G_2)\}.$$

Note that $|X| \equiv 0 \pmod 2$ since each G_i admits the PH-property and so are both of even order. We shall distinguish between two cases: whether X is empty or not.

Case 1. $|X| = 0$.

In this case, $P = P_1 \cup P_2$. Since G_1 has the PH-property, there exists a perfect matching M of G_1 such that $P_1 \cup M$ is a Hamiltonian cycle of K_{G_1}. We shall denote the corresponding vertex of $x \in V(G_1)$ by $x' \in V(G_2)$. Let M' be the perfect matching of G_2 such that $x'y' \in M'$ if and only if $xy \in M$. In other words, M' is the copy of M in G_2. We observe that $P_2 \cup M'$ consists of the union of cycles of even length, say $C_1, \ldots, C_t$. Note that cycles of length 2 shall be allowed in the sequel as they arise when $P_2 \cap M' \neq \emptyset$. For each $i \in \{1, \ldots, t\}$, we choose an edge $e_i' = x_i' y_i' \in M' \cap C_i$ and we denote the corresponding edge in M by $e_i = x_i y_i$. Consequently, the set

$$N = (M \setminus \{e_1, \ldots, e_t\}) \cup (M' \setminus \{e_1', \ldots, e_t'\}) \cup \{x_i x_i', y_i y_i' \mid i \in \{1, \ldots, t\}\}$$

is a perfect matching of $\mathcal{P}(G)$ such that $P \cup N$ is a Hamiltonian cycle of $K_{\mathcal{P}(G)}$. We note that the vertex x_i' in G_2 corresponds to the vertex x_i in G_1, see Fig. 1.

Case 2. $|X| = 2r > 0$.

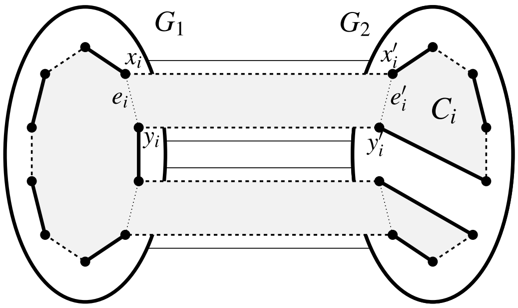

Fig. 1. An extension of the pairing P, depicted in bold, when $|X| = 0$. The dashed edges represent those in N, whereas the dotted edges are the edges e_i and e'_i.

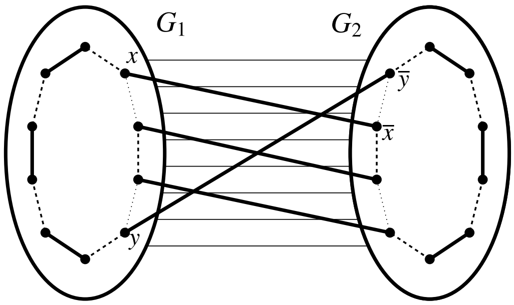

Fig. 2. An extension of the pairing P, depicted in bold, when $|X| = 2r > 0$. The dashed edges represent those in M and M', whereas the dotted edges are those in L and R.

In this case we consider an analogous argument to the one used by Fink to prove Theorem 1.1. Since $|X| \neq 0$, P_1 is a matching of K_{G_1} which is not perfect, as there are $2r$ unmatched vertices. Let L be an arbitrary set of r edges of K_{G_1} such that $P_1 \cup L$ is a pairing of G_1. Since G_1 has the PH-property, there exists a perfect matching M, of G_1, such that $P_1 \cup L \cup M$ is a Hamiltonian cycle of K_{G_1}. Next we define the following set

$$R = \left\{ \overline{x}\,\overline{y} \in E(K_{G_2}) \;\middle|\; \begin{array}{l} \exists\, x, y \in V(G_1) \text{ with } \{x\overline{x}, y\overline{y}\} \subseteq X \text{ and} \\ \exists \text{ an } (x, y)\text{-path contained in } P_1 \cup M \end{array} \right\},$$

such that $P_2 \cup R$ is a pairing of G_2. Note that $x\,\overline{x}$ and $y\,\overline{y}$ are edges in K_G since $|X| \neq 0$, and their endvertices might not be corresponding vertices in G_1 and G_2, as they were in the former case. Since G_2 has the PH-property there exists a perfect matching M' of G_2, such that $P_2 \cup R \cup M'$ is a Hamiltonian cycle of G_2. It follows that $P_1 \cup P_2 \cup X \cup M \cup M'$ is a Hamiltonian cycle of $K_{\mathcal{P}(G)}$ in which $M \cup M'$ is a perfect matching of $\mathcal{P}(G)$, see Fig. 2.

This proves that $\mathcal{P}(G)$ has the PH-property and thus, by iterating the prism operator, the result follows. $\square$

3. Convergence of general graph prisms to the PH-property

In this section we show that given any connected graph G, there exists a sufficiently large integer k such that $\mathcal{P}^k(G)$ has the PH-property. In other words, after iterating the prism operator a sufficient number of times, the resulting graph will have the PH-property. We remark that if a graph contains a spanning subgraph admitting the PH-property, then the graph itself admits the PH-property. Hence, by Theorem 1.2, the next corollary follows.

Corollary 3.1. *Let G be a traceable graph. For $k \geq 5$, the graph $\mathcal{P}^k(G)$ has the PH-property.*

Recall that a *traceable* graph is a graph admitting a Hamiltonian path. Next, we show that starting from an arbitrary connected graph G, we can always obtain a traceable graph by iterating the prism operator a suitable number of times. To this purpose, we need the following definition and lemma.

Definition 3.2. Let G be a connected graph. The *minimum leaf number* of G, denoted by $\mathrm{ml}(G)$, is the minimum number of leaves over all spanning trees of G.

Clearly, for any connected graph G, $\mathrm{ml}(G) \geq 2$, and $\mathrm{ml}(G) = 2$ if and only if G is traceable.

Lemma 3.3. *Let G be a connected graph with* $\mathrm{ml}(G) > 2$. *Then,* $\mathrm{ml}(G) > \mathrm{ml}(\mathcal{P}(G))$.

Proof. Suppose that $\mathrm{ml}(G) = t > 2$ and let G_1 and G_2 be the two copies of G in $\mathcal{P}(G)$. Let R_1, R_2 be two copies of a spanning tree of G with t leaves in G_1 and G_2, respectively. Let $S = \{e_0, e_1, \dots, e_{t-1}\}$ be the set consisting of the t edges which connect a leaf of R_1 to the corresponding leaf of R_2. Consequently, we have that $T_0 = (R_1 \cup R_2) + e_0$ is a spanning tree of $\mathcal{P}(G)$ with $2t-2$ leaves. Moreover, $T_0 + e_1$ has exactly one cycle, say C_1. Since $\mathrm{ml}(G) > 2$, C_1 is a proper subgraph of $T_0 + e_1$ and there exists a vertex v of C_1 such that $deg_{T_0+e_1}(v) > 2$. We note that the removal of an edge of C_1, say f_1, which is incident to v gives rise to a spanning tree $T_1 = T_0 + e_1 - f_1$ of $\mathcal{P}(G)$ with at most $2t-3$ leaves. Then, for every $j \in \{2, \dots, t-1\}$, starting from $j = 2$ and continuing consecutively up to $t-1$, we choose an edge f_j from $E(T_{j-1} + e_j)$ lying on the unique cycle in $T_{j-1} + e_j$ and incident to a vertex of degree at least 3 in $T_{j-1} + e_j$. We then let T_j be equal to $T_{j-1} + e_j - f_j$, which by a similar argument to the above is a spanning tree of $\mathcal{P}(G)$ with at most $2t-2-j$ leaves. Therefore, T_{t-1} has at most $t-1$ leaves and $\mathrm{ml}(\mathcal{P}(G)) \le t-1 < \mathrm{ml}(G)$. $\square$

From the above statements, it is easy to obtain the following result.

Proposition 3.4. *Let G be a connected graph. Then, $\mathcal{P}^k(G)$ is traceable for all $k \ge$* $\mathrm{ml}(G) - 2$.

Proof. If we start from G and apply the prism operator $\mathrm{ml}(G) - 2$ times, by Lemma 3.3, the graph $\mathcal{P}^{\mathrm{ml}(G)-2}(G)$ has $\mathrm{ml}(\mathcal{P}^{\mathrm{ml}(G)-2}(G)) = 2$. Consequently, it admits a Hamiltonian path. $\square$

Combining Theorem 1.2 and Proposition 3.4 we obtain the following.

Theorem 3.5. *Let G be a connected graph with $m =$* $\mathrm{ml}(G)$, *then $\mathcal{P}^{m+3}(G)$ has the PH-property.*

Proof. If G is traceable, then $m = 2$, and so, from Theorem 1.2 we have that $\mathcal{P}^5(G)$ has the PH-property. On the other hand, if G is not traceable, then $m > 2$. By Theorem 3.4, the graph $\mathcal{P}^{m-2}(G)$ is traceable. Hence, by Theorem 1.2, $\mathcal{P}^5(\mathcal{P}^{m-2}(G)) = \mathcal{P}^{m+3}(G)$ admits the PH-property. $\square$

Let us note that the condition given in Proposition 3.4, and subsequently in Theorem 3.5, is only a sufficient condition, though we do not believe it is necessary for sufficiently large values of $\mathrm{ml}(G)$. Establishing an optimal lower bound for k in Proposition 3.4 in terms of $\mathrm{ml}(G)$ would be an interesting problem to consider.

4. Final remarks

Several open problems were posed in [2]. In particular, proving that the graph $P_q \square \mathcal{Q}_d$ has the PH-property for $d = 3, 4$ and an arbitrary q is still open. It is dutiful to note that we are aware that in case of a positive answer, Theorem 3.5 should be refined accordingly.

A much more ambitious problem is to wonder whether it is enough for two graphs G and H to have the PH-property, for $G \square H$ to have the PH-property as well. This latter question seems very difficult to prove. Here, we have shown, in Theorem 2.1, that it holds when H is the hypercube, which is an iteration of the prism operator. In Theorem 3.5, we see that even if G does not have the PH-property, but is connected, a *large enough* number of iterations of the prism operator make it *converge* to a graph with the PH-property. As a matter of fact, we can define the parameter $\mathfrak{p}(G)$ as the smallest non-negative integer $\mathfrak{p} = \mathfrak{p}(G)$ such that $\mathcal{P}^{\mathfrak{p}}(G)$ admits the PH-property. It trivially follows that $\mathfrak{p}(G) = 0$ if and only if G is PH. Henceforth, the parameter $\mathfrak{p}(G)$ can be considered as a measure of how far a graph G is from having the PH-property, with respect to the prism operator. Determining the behaviour of $\mathfrak{p}(G)$ for some special classes of graphs could be of interest in the study of the PH-property.

We could also wonder if there are other graphs that speed up the convergence to the PH-property under the Cartesian product, or on the other hand if there are other products under which the convergence to the PH-property is guaranteed. It seems so if we consider the strong product of graphs. The *strong product* $G \boxtimes H$ is a graph whose vertex set is the Cartesian product $V(G) \times V(H)$ of $V(G)$ and $V(H)$, and two vertices (u_1, v_1), (u_2, v_2) are adjacent if and only if they are adjacent in $G \square H$ or if $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$. It is trivial that $G \square H$ is a subgraph of $G \boxtimes H$; hence, if $G \square H$ has the PH-property, then $G \boxtimes H$ will inherit the same property as well.

A result from [5] on accordion graphs easily implies that in the case of Hamiltonian graphs, only one occurrence of the strong product with K_2 is enough to obtain a graph with the PH-property.

Theorem 4.1. *Let G be a Hamiltonian graph, then $G \boxtimes K_2$ has the PH-property.*

This suggests that the strong product may have a convergence to the PH-property in few steps also for general graphs.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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