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ORIGINAL PAPER

Fractional absolute value equations

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Abstract

We introduce the fractional absolute value equation (FAVE), which is an absolute value equation containing fractional terms of the unknown variables. For simplicity, we first consider a simple 1-dimensional FAVE and then generalize the results to problems of arbitrary dimension *n*. In this context, we investigate the relationship between the FAVE and the weighted generalization of the horizontal linear complementarity problem. Moreover, we provide conditions that ensure the solvability of the FAVE by characterizing the number and the sign pattern of the solutions in several situations.

Keywords Absolute value equations · Complementarity problems · Uniqueness of solution · Solvability

1 Introduction

Aim of this paper is to introduce a fractional absolute value equation (FAVE). For simplicity, we first consider the 1-dimensional FAVE, which consists in finding the solution(s) to

$$ax - b|x| - \frac{c}{x} - \frac{d}{|x|} = q, \tag{1}$$

with $a, b, c, d \in \mathbb{R}$ assigned values, $q \in \mathbb{R}$ known term, and $x \in \mathbb{R}\setminus\{0\}$ unknown. We then generalize the results to the *n*-dimensional case

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$$A\boldsymbol{x} - B|\boldsymbol{x}| - C(\boldsymbol{e} \oslash \boldsymbol{x}) - D(\boldsymbol{e} \oslash |\boldsymbol{x}|) = \boldsymbol{q}, \tag{2}$$

where $A, B, C, D \in \mathbb{R}^{n \times n}$ are known matrices, $\boldsymbol{q} \in \mathbb{R}^n$ is a known vector, $\boldsymbol{e} = (1, 1, \dots 1)^T \in \mathbb{R}^n$ is a vector whose entries are all ones, $\boldsymbol{x} \in \mathbb{R}^n$ is an unknown vector of components $x_i \neq 0$, $i = 1, \dots, n$, and $\oslash$ denotes the Hadamard division. Hence, $\boldsymbol{e} \oslash \boldsymbol{x}$ represents the vector of components $\frac{1}{x_i}$, for $i = 1, \dots, n$.

While absolute value equations with fractional terms have not been studied in the literature, "standard" absolute value equations have excited much interest in recent years. In their most general form, the generalized absolute value equation (GAVE) is defined as [1]

$$A\boldsymbol{x} - B|\boldsymbol{x}| = \boldsymbol{q}, \tag{3}$$

where $A, B \in \mathbb{R}^{n \times n}$ are known matrices, $\boldsymbol{q} \in \mathbb{R}^n$ is a known vector, and $\boldsymbol{x} \in \mathbb{R}^n$ is unknown[1]. If $B = I$, the GAVE (3) reduces to an absolute value equation (AVE). The first works in this field are [1, 2], where the relationship between AVEs (GAVEs), linear complementarity problems (LCPs, [3, 4]), and bilinear programs was studied. Existence and non-existence conditions were provided as well. Many other works then followed, focusing on either proposing new existence conditions [5–9], devising new solution methods for AVEs and GAVEs [10–13], or formulating new generalizations of (3) [14–18]. The reader is referred to the recent review [19] for a complete overview of the main works on these aspects. Nonetheless, we highlight that none of the generalizations of the GAVE considered fractional terms: indeed, the generalizations of the GAVE in the literature focused on matrix equations [16] like the Sylvester-like AVE [14], stochastic AVEs [18], tensor equations [17], or variations of the GAVE [15].

One of the main motivations that led to the introduction of AVEs and GAVEs is their equivalence to (horizontal) linear complementarity problems [2, 20, 21]. In particular, it is known that the GAVE (3) is equivalent to the horizontal linear complementarity problem (HLCP) that consists in determining two unknown vectors $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^n$ such that

$$(A - B)\boldsymbol{\xi} - (A + B)\boldsymbol{\eta} = \boldsymbol{q} \text{ with } \boldsymbol{\xi} \geq \boldsymbol{0}, \boldsymbol{\eta} \geq \boldsymbol{0}, \boldsymbol{\xi}^T \boldsymbol{\eta} = 0. \tag{4}$$

This equivalence clearly includes the AVE when $B = I$ and can be reformulated in terms of a (non-horizontal) LCP. Furthermore, the equivalence was widely used for studying the solvability of AVEs/GAVEs and for proposing new solution methods (see, e.g., [2, 7]). Even the well-known modulus-based methods for LCPs and HLCPs [22, 23] can be seen, in practice, as based on solving a GAVE in place of the original complementarity problem.

Similarly, in this paper we show that the FAVE (2) has a strict relationship with a generalization of the HLCP, namely the *weighted* horizontal linear complementarity

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[1] In some works, including [1], the GAVE is defined with non-square matrices. In this case, the quantities in (3) are defined as $A, B \in \mathbb{R}^{m \times n}$, $\boldsymbol{q} \in \mathbb{R}^m$, and $\boldsymbol{x} \in \mathbb{R}^n$. However, consistently with much of the recent literature, by GAVE we here refer to the case with $m = n$.

problem (WHLCP, [24]). This problem consists in determining two unknown vectors $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^n$ such that

$$M_1\boldsymbol{\xi} - M_2\boldsymbol{\eta} = \boldsymbol{q} \text{ with } \boldsymbol{\xi} \geq \mathbf{0}, \boldsymbol{\eta} \geq \mathbf{0}, \boldsymbol{\xi} \circ \boldsymbol{\eta} = \boldsymbol{w}, \tag{5}$$

where $\boldsymbol{w} \geq \mathbf{0}$ is a non-negative weight vector, $\boldsymbol{q} \in \mathbb{R}^n$ is a known term, $M_1, M_2 \in \mathbb{R}^{n\times n}$ are two coefficient matrices, and $\circ$ denotes the Hadamard product. We denote the problem (5), which is a special instance of the weighted linear complementarity problem introduced in [25], as the WHLCP($M_1, M_2, \boldsymbol{q}$).

The relationship that we establish between FAVEs and WHLCPs contains and generalizes previous results in the literature. Indeed, both the FAVE and the WHLCP are larger settings than the GAVE and the HLCP, respectively. In fact, it is easy to verify that a FAVE (2) with $C = D = 0$ reduces to a GAVE, and a WHLCP with $\boldsymbol{w} = \mathbf{0}$ reduces to an HLCP. Thus, the known equivalence between the GAVE and the HLCP becomes a special instance of the analysis here performed with regard to the FAVE. In this perspective, it is interesting that a WHLCP with non-zero weight is no longer equivalent to a GAVE, but requires fractional terms to be present in the absolute value formalism to establish some kind of equivalence.

Furthermore, our analysis can foster the development of new solution methods for WHLCPs, similarly to the above-mentioned methods [22, 23] and as suggested by recent advances on modulus-based methods for WHLPs [26]. Finally, a last aim of the paper is to explore the solvability of the introduced FAVE. In this context, we introduce several conditions that ensure that the FAVE has solutions and particularly focus on their sign pattern.

Throughout our analysis, we are going to use the following two definitions, which we here particularize to the case of two matrices *A*, *B*. Indeed, this is the case that will occur in this paper, while the general definitions can be found in [27], where they were introduced.

Definition 1 [27] Let $A, B \in \mathbb{R}^{n\times n}$ and let $\mathcal{M} = \{A, B\}$. A matrix $C \in \mathbb{R}^{n\times n}$ is a column representative matrix of $\mathcal{M}$ if

$$\boldsymbol{c}_j \in \{\boldsymbol{a}_j, \boldsymbol{b}_j\}$$

for $j = 1, \ldots, n$, where $\boldsymbol{c}_j$ denotes the *j*-th column of *C* and $\boldsymbol{a}_j$, $\boldsymbol{b}_j$ denote the *j*-th column of *A* and *B*, respectively.

Definition 2 [27] The set $\mathcal{M} = \{A, B\}$ has the column $\mathcal{W}$-property if the determinants of the column representative matrices of $\mathcal{M}$ are all positive or all negative. If, instead, the determinants are all non-negative with at least one positive determinant (or all non-positive with at least one negative determinant), we say that $\mathcal{M}$ has the column $\mathcal{W}_0$-property.

In addition, we here introduce the following concept of *simultaneously column-representative* matrices.

Definition 3 Let $A, B, C, D \in \mathbb{R}^{n\times n}$, and let $\mathcal{M}_1 = \{A, B\}$ and $\mathcal{M}_2 = \{C, D\}$. Let M_1 be a column representative matrix of $\mathcal{M}_1$ and let M_2 be a column representative matrix of $\mathcal{M}_2$. Let $\boldsymbol{m}_{1_i}$ and $\boldsymbol{m}_{2_i}$ denote the *i*-th column of M_1 and M_2, respectively, $i = 1, \ldots, n$. Finally, let $\mathcal{I}_A$ denote the set of indices where $\boldsymbol{m}_{1_i} = \boldsymbol{a}_i$, $\mathcal{I}_B$ the set of indices where $\boldsymbol{m}_{1_i} = \boldsymbol{b}_i$, $\mathcal{I}_C$ the set of indices where $\boldsymbol{m}_{2_i} = \boldsymbol{c}_i$, and $\mathcal{I}_D$ the set of indices where $\boldsymbol{m}_{2_i} = \boldsymbol{d}_i$, $i = 1, \ldots, n$. We say that M_1 and M_2 are simultaneously column representative of $\mathcal{M}_1$ and $\mathcal{M}_2$ if $\mathcal{I}_A = \mathcal{I}_C$ and $\mathcal{I}_B = \mathcal{I}_D$, which is if M_1 and M_2 are such that

$$\boldsymbol{m}_{1_i} = \boldsymbol{a}_i \text{ and } \boldsymbol{m}_{2_i} = \boldsymbol{c}_i \quad \text{or} \quad \boldsymbol{m}_{1_i} = \boldsymbol{b}_i \text{ and } \boldsymbol{m}_{2_i} = \boldsymbol{d}_i \tag{6}$$

for $i = 1, \ldots, n$.
Clearly, Definition 3 can readily be generalized to an arbitrary number of sets containing the same number of matrices. Furthermore, replacing the conditions over the columns with conditions over the rows, we get corresponding definitions for *simultaneously row representative* matrices. Both these cases are not of interest for the following analysis, but are naturally included in Definition 3.

Finally, the next proposition briefly discusses the number of simultaneously column representative matrices. The proof is evident, and hence not explicitly reported.

Proposition 1 *Let $\mathcal{M}_1, \mathcal{M}_2$ be two sets defined as in Definition 3. Then, the total number of simultaneously column representative matrices is 2^n.*

2 Relationship between the FAVE and the WHLCP

2.1 1-dimensional problems

We start by investigating the relationship between the FAVE (1) and the WHLCP focusing, at first, on a 1-dimensional case. Furthermore, we assume that fractional terms are actually present, i.e. that *c* and *d* are not both zero in (1). This implies that *x* is defined in $\mathbb{R}\backslash\{0\}$. A direct generalization of the next theorem to the GAVE (corresponding to $c = d = 0$) will be considered later.

Theorem 2 *Let $w > 0$ be a positive arbitrary value and let $\kappa = w$. Assume that c and d are not both zero in (1). If $\hat{x}$ is a positive solution to the 1-dimensional FAVE (1), then*

$$\hat{\xi} = \max\{0, \hat{x}\} + \max\left\{0, -\frac{w}{\hat{x}}\right\} \quad \textit{and} \quad \hat{\eta} = \max\{0, -\hat{x}\} + \max\left\{0, \frac{w}{\hat{x}}\right\} \tag{7}$$

solve the WHLCP

$$(a-b)\xi - \frac{1}{\kappa}(c+d)\eta = q \text{ with } \xi \geq 0, \eta \geq 0, \xi\eta = w. \tag{8}$$

If $\hat{x}$ is a negative solution to the FAVE (1), then $\hat{\xi}$ and $\hat{\eta}$ as in (7) solve the WHLCP

$$\frac{1}{\kappa}(c-d)\xi-(a+b)\eta=q \; with \; \xi\geq 0,\eta\geq 0,\xi\eta=w. \tag{9}$$

Conversely, if $\hat{\xi},\hat{\eta}$ solve (8), then $\hat{x}=\hat{\xi}$ is a positive solution to the FAVE (1); if $\hat{\xi},\hat{\eta}$ solve (9), then $\hat{x}=-\hat{\eta}$ is a negative solution to the FAVE (1).

Proof The assumption that *c* and *d* are not both zero in (1) implies that the FAVE (1) contains fractional terms in *x* and is defined only for $x\in\mathbb{R}\backslash\{0\}$. Hence, any solution $\hat{x}$ to the FAVE is either positive or negative. Recalling $w>0$, it follows that $\hat{\xi},\hat{\eta}$ as in (7) are positive for any possible solution $\hat{x}$.

Hence, let us separately explore the positive and the negative sets of solutions to (1). If $\hat{x}>0$, then the relations in (7) reduce to

$$\hat{\xi}=\hat{x};\quad \hat{\eta}=\frac{w}{\hat{x}};\quad \hat{\xi}\hat{\eta}=w \tag{10}$$

with $\hat{x}>0$ and $w>0$. Therefore, if we replace $\xi=x$ and $\eta=\frac{w}{x}$ into (1), we find that $\hat{\xi},\hat{\eta}$ as in (10) solve the WHLCP

$$a\xi-b\xi-\frac{1}{w}c\eta-\frac{1}{w}d\eta=q \text{ with } \xi\geq 0,\eta\geq 0,\xi\eta=w.$$

In the previous equation, we required $\xi\geq 0,\eta\geq 0$ for consistency with the general formulation of the WHLCP, but evidently any solution $\hat{\xi},\hat{\eta}$ must be strictly positive when $w>0$. Collecting terms and by $\kappa=w$ for $w\neq 0$, we finally obtain (8).

The inverse implication follows directly by considering $\hat{\xi},\hat{\eta}$ that solve (8) and by proceeding backwards. It is then easy to verify that $\hat{x}=\hat{\xi}$ is positive and satisfies the FAVE.

If, instead, $\hat{x}<0$ and $w>0$, then (7) reduces to

$$\hat{\xi}=-\frac{w}{\hat{x}};\quad \hat{\eta}=-\hat{x};\quad \hat{\xi}\hat{\eta}=w. \tag{11}$$

Hence, proceeding exactly as in the positive case, finding the negative solutions to the FAVE (1) can be written as finding the solutions $\hat{\xi},\hat{\eta}$ of the problem

$$\frac{1}{\kappa}(c-d)\xi-(a+b)\eta=q \text{ with } \xi\geq 0,\eta\geq 0,\xi\eta=w.$$

Again the inverse implication can be verified by proceeding backwards and setting $\hat{x}=-\hat{\eta}$, in accordance with (7). □

While the above theorem applies to the 1-dimensional FAVE with $c\neq 0$ and/or $d\neq 0$, the FAVE can be seen as a generalization of the GAVE if $c=d=0$ is allowed. Repeating the analysis of Theorem 2 allowing for $c=d=0$ leads to the following corollary, which can be applied to GAVEs as well. We remark that *x* can be zero in the special case $c = d = 0$, as all fractional terms disappear and the FAVE reduces to a GAVE.

Corollary 3 *Let $w = |c| + |d|$ and let $\kappa = w$ if $w \neq 0$ and κ equal to an arbitrary positive value if $w = 0$. If $\hat{x}$ is a non-negative solution to the 1-dimensional FAVE (1), then*

$$\hat{\xi} = \begin{cases} \max\{0, \hat{x}\} + \max\left\{0, -\frac{w}{\hat{x}}\right\} & \mathit{if}\ w \neq 0 \\ \max\{0, \hat{x}\} & \mathit{if}\ w = 0 \end{cases} \quad \mathit{and} \quad \hat{\eta} = \begin{cases} \max\{0, -\hat{x}\} + \max\left\{0, \frac{w}{\hat{x}}\right\} & \mathit{if}\ w \neq 0 \\ \max\{0, -\hat{x}\} & \mathit{if}\ w = 0 \end{cases} \tag{12}$$

solve the WHLCP (8). If $\hat{x}$ is a non-positive solution to the FAVE (1), then $\hat{\xi}$ and $\hat{\eta}$ as in (12) solve the WHLCP (9).

The inverse implications hold as well: if $\hat{\xi}, \hat{\eta}$ solve (8), then $\hat{x} = \hat{\xi}$ is a non-negative solution to the FAVE (1); if $\hat{\xi}, \hat{\eta}$ solve (9), then $\hat{x} = -\hat{\eta}$ is a non-positive solution to the FAVE (1).

Proof When fractional terms are present in the FAVE (1), the definition of w ensures $w > 0$ and we fall directly within Theorem 2. In the special case $c = d = 0$, we instead have $w = 0$. In this situation, the FAVE reduces to the 1-dimensional GAVE

$$ax - b|x| = q, \tag{13}$$

and (12) becomes

$$\hat{\xi} = \max\{0, \hat{x}\} \quad \text{and} \quad \hat{\eta} = \max\{0, -\hat{x}\}. \tag{14}$$

Proceeding exactly as in the cases with nonzero w, finding the positive and the negative solutions to the GAVE (13) is equivalent to solving the HLCPs

$$\begin{aligned} (a - b)\xi = q \text{ with } \xi \geq 0, \eta \geq 0, \xi\eta = 0 \\ -(a + b)\eta = q \text{ with } \xi \geq 0, \eta \geq 0, \xi\eta = 0, \end{aligned} \tag{15}$$

respectively. These problems are evidently the same as (8) and (9) with $c = d = 0$, where the definition of κ ensures that we are not dividing by zero. The only difference with the proof of Theorem 2 is that we need to consider the case $\hat{x} = 0$, which is now also possible. This is easily analyzed by considering that $\hat{x} = 0$ is only possible if $c = d = 0$ and $q = 0$, and implies $\hat{\xi} = \hat{\eta} = 0$ by (12). Hence, a solution $\hat{x} = 0$ to (1) will necessarily satisfy both problems in (15) with $\hat{\xi} = \hat{\eta} = 0$. The inverse implication naturally follows: the first problem of (15) was obtained with $c = d = 0$ and is solved by $\hat{\xi} = 0$ if $q = 0$. This leads to $\hat{x} = \hat{\xi} = 0$ according to the theorem. Analogous considerations can be made with regard to the second problem of (15), which is solved by $\hat{\eta} = 0$ if $q = 0$, and leads to $\hat{x} = -\hat{\eta} = 0$. ☐

Remark 1 It is known that a GAVE is equivalent to a (non-weighted) HLCP. In particular, by [7, Proposition 1], the GAVE (1) is equivalent to the HLCP

$$(a-b)\xi-(a+b)\eta=q \text{ with } \xi\geq 0,\eta\geq 0,\xi\eta=0. \tag{16}$$

According to that equivalence result, if $\hat{x}$ is a solution to the GAVE, then $\xi=\hat{\xi},\eta=\hat{\eta}$ as in (14) solve (16).

Corollary 3 directly contains this equivalence. Indeed, the FAVE (1) becomes a GAVE when $c=d=0$. In this case, by the definition of w, we have $\xi\eta=0$, which, in turn, ensures that we necessarily have either $\hat{\xi}=0$ or $\hat{\eta}=0$ at the solution. Thus, solving the HLCP (16) is the same as solving the two HLCPs (15) of Corollary 3 with $\hat{\eta}=0$ for the first problem and with $\hat{\xi}=0$ for the second. The problems simply separate the positive and the negative solution sets of x.

2.2 Generalization to n-dimensional FAVEs

The results of the previous subsection can readily be generalized to FAVEs of arbitrary dimension. In this regard, it is useful to write the FAVE (2) by explicitly expressing the matrix–vector products as linear combinations of the columns of the matrices A, B, C, D. I.e., denoting by $\boldsymbol{a}_i$ the generic ith column of A and using an analogous notation for the columns of B, C, D, we can write the FAVE (2) in the form

$$\sum_{i=1}^{n}\boldsymbol{a}_i x_i-\sum_{i=1}^{n}\boldsymbol{b}_i|x_i|-\sum_{i=1}^{n}\boldsymbol{c}_i\frac{1}{x_i}-\sum_{i=1}^{n}\boldsymbol{d}_i\frac{1}{|x_i|}=\boldsymbol{q}, \tag{17}$$

Notice that the FAVE is defined for $x_i\neq 0$ when $\boldsymbol{c}_i$ and $\boldsymbol{d}_i$ are not both the zero vector, $i=1,\ldots,n$. Indeed, (17) highlights that fractional terms in x_i are present in this case. We first focus on this situation, and later generalize it to a FAVE where $\boldsymbol{c}_i=\boldsymbol{d}_i=\boldsymbol{0}$ for some i.

Theorem 4 *Let $\boldsymbol{w}\in\mathbb{R}^n>\boldsymbol{0}$ and let $K\in\mathbb{R}^{n\times n}$ be the positive diagonal matrix of entries $\kappa_{ii}=\frac{1}{w_i}$. Assume that $\mathbf{c}_i$ and $\mathbf{d}_i$ are not both the zero vector, for $i=1,\ldots,n$. Let $\hat{\boldsymbol{x}}\in\mathbb{R}^n$ be a solution to the FAVE (2) and define $\mathcal{I}_+$ and $\mathcal{I}_-$ as the sets of indices*

$$\mathcal{I}^+=\{i\mid \hat{x}_i>0\};\quad \mathcal{I}^-=\{i\mid \hat{x}_i<0\} \tag{18}$$

for $i=1,\ldots,n$. Then, the vectors

$$\hat{\boldsymbol{\xi}}=\max\{\boldsymbol{0},\hat{\boldsymbol{x}}\}+\max\{\boldsymbol{0},-\boldsymbol{w}\oslash\hat{\boldsymbol{x}}\};\quad \hat{\boldsymbol{\eta}}=\max\{\boldsymbol{0},-\hat{\boldsymbol{x}}\}+\max\{\boldsymbol{0},\boldsymbol{w}\oslash\hat{\boldsymbol{x}}\} \tag{19}$$

solve the WHLCP

$$M_1\boldsymbol{\xi}-M_2\boldsymbol{\eta}=\boldsymbol{q} \textit{ with } \boldsymbol{\xi}\geq\boldsymbol{0},\boldsymbol{\eta}\geq\boldsymbol{0},\boldsymbol{\xi}\circ\boldsymbol{\eta}=\boldsymbol{w}, \tag{20}$$

where $\{M_1,M_2\}$ are simultaneously column representative to the sets $\{A-B,(C-D)K\}$ and $\{(C+D)K,A+B\}$ with M_1 matrix of columns

$$\boldsymbol{m}_{1_i}=\boldsymbol{a}_i-\boldsymbol{b}_i \textit{ for } i\in\mathcal{I}^+;\quad \boldsymbol{m}_{1_i}=\kappa_{ii}(\boldsymbol{c}_i-\boldsymbol{d}_i) \textit{ for } i\in\mathcal{I}^-,\quad i=1,\ldots,n. \tag{21}$$

Conversely, if $\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}$ *solve the WHLCP (20) with* $\mathcal{I}^+$ *and* $\mathcal{I}^-$ *given sets of indices that define the matrices of the problem as in (21), then* $\hat{\boldsymbol{x}}$ *of components* $\hat{x}_i = \hat{\xi}_i$ *for any* $i \in \mathcal{I}^+$ *and* $\hat{x}_i = -\hat{\eta}_i$ *for any* $i \in \mathcal{I}^-$ *is a solution to the FAVE (2).*

Proof If $\boldsymbol{c}_i$ and $\boldsymbol{d}_i$ are not both zero for $i = 1, \dots, n$, then any solution to the FAVE (2) cannot have zero components. Therefore, given a solution $\hat{\boldsymbol{x}}$, any of its components is going to be either in $\mathcal{I}^+$ or in $\mathcal{I}^-$ as in (18). Furthermore, $\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}$ as in (19) are positive and satisfy $\hat{\boldsymbol{\xi}} \circ \hat{\boldsymbol{\eta}} = \boldsymbol{w}$. Similarly to the proof to Theorem 2, let us explore separately the positive and the negative sets of solutions to (1).

Assume that the FAVE (2) has a positive solution $\hat{\boldsymbol{x}} > \boldsymbol{0}$. Then, (19) reduces to

$$\hat{\boldsymbol{\xi}} = \hat{\boldsymbol{x}}; \quad \hat{\boldsymbol{\eta}} = \boldsymbol{w} \oslash \hat{\boldsymbol{x}}; \quad \hat{\boldsymbol{\xi}} \circ \hat{\boldsymbol{\eta}} = \boldsymbol{w}. \tag{22}$$

Using these expressions to replace $\hat{\boldsymbol{x}}$ in the FAVE (2), we easily find that $\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}$ solve the WHLCP

$$A\boldsymbol{\xi} - B\boldsymbol{\xi} - C\boldsymbol{\eta} \oslash \boldsymbol{w} - D\boldsymbol{\eta} \oslash \boldsymbol{w} = \boldsymbol{q} \text{ with } \boldsymbol{\xi} \geq \boldsymbol{0}, \boldsymbol{\eta} \geq \boldsymbol{0}, \boldsymbol{\xi} \circ \boldsymbol{\eta} = \boldsymbol{w},$$

which, by the definition of K, is

$$(A - B)\boldsymbol{\xi} - (C + D)K\boldsymbol{\eta} = \boldsymbol{q} \text{ with } \boldsymbol{\xi} \geq \boldsymbol{0}, \boldsymbol{\eta} \geq \boldsymbol{0}, \boldsymbol{\xi} \circ \boldsymbol{\eta} = \boldsymbol{w}.$$

As we did in the 1-dimensional case, in the previous equations we required $\boldsymbol{\xi} \geq \boldsymbol{0}, \boldsymbol{\eta} \geq \boldsymbol{0}$ for consistency with the general formulation of the WHLCP, but evidently any solution $\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}$ to these problems must be strictly positive when $\boldsymbol{w} > \boldsymbol{0}$.

Instead, if $\hat{\boldsymbol{x}} < \boldsymbol{0}$, then (19) reduces to

$$\hat{\boldsymbol{\xi}} = -\boldsymbol{w} \oslash \hat{\boldsymbol{x}}; \quad \hat{\boldsymbol{\eta}} = -\hat{\boldsymbol{x}}; \quad \hat{\boldsymbol{\xi}} \circ \hat{\boldsymbol{\eta}} = \boldsymbol{w}. \tag{23}$$

Proceeding as in the previous paragraph, the FAVE (2) can be written as finding $\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}$ that solve

$$(C - D)K\boldsymbol{\xi} - (A + B)\boldsymbol{\eta} = \boldsymbol{q} \text{ with } \boldsymbol{\xi} \geq \boldsymbol{0}, \boldsymbol{\eta} \geq \boldsymbol{0}, \boldsymbol{\xi} \circ \boldsymbol{\eta} = \boldsymbol{w}.$$

Obviously, we can also have solutions that are not completely positive nor completely negative. In this regard, define $\mathcal{I}^+$ and $\mathcal{I}^-$ as in (18). Then, (19) reduces to

$$\begin{aligned} &\hat{\xi}_i = \hat{x}_i; \quad \hat{\eta}_i = \frac{w_i}{\hat{x}_i}; \quad \hat{\xi}_i\hat{\eta}_i = w_i \quad \text{for } i \in \mathcal{I}^+ \\ &\hat{\xi}_i = -\frac{w_i}{\hat{x}_i}; \quad \hat{\eta}_i = -\hat{x}_i; \quad \hat{\xi}_i\hat{\eta}_i = w_i \quad \text{for } i \in \mathcal{I}^-. \end{aligned} \tag{24}$$

Consequently, expressing the FAVE in the form (17) and using (24) to replace suitable components of $\hat{\boldsymbol{x}}$ based on their sign, we get that $\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}$ as in (24) solve the WHLCP

$$\sum_{i\in\mathcal{I}^+} \boldsymbol{a}_i\xi_i - \sum_{i\in\mathcal{I}^-} \boldsymbol{a}_i\eta_i - \sum_{i\in\mathcal{I}^+} \boldsymbol{b}_i\xi_i - \sum_{i\in\mathcal{I}^-}\boldsymbol{b}_i\eta_i - \sum_{i\in\mathcal{I}^+}\boldsymbol{c}_i\frac{\eta_i}{w_i} + \sum_{i\in\mathcal{I}^-}\boldsymbol{c}_i\frac{\xi_i}{w_i} - \sum_{i\in\mathcal{I}^+}\boldsymbol{d}_i\frac{\eta_i}{w_i} - \sum_{i\in\mathcal{I}^-}\boldsymbol{d}_i\frac{\xi_i}{w_i} = \boldsymbol{q},$$

with $\boldsymbol{\xi} \geq \boldsymbol{0}, \boldsymbol{\eta} \geq \boldsymbol{0}, \boldsymbol{\xi} \circ \boldsymbol{\eta} = \boldsymbol{w}$. Collecting terms and by $\kappa_{ii} = \frac{1}{w_i}$, we further get

$$\sum_{i\in\mathcal{I}^+}(\boldsymbol{a}_i - \boldsymbol{b}_i)\xi_i - \sum_{i\in\mathcal{I}^-}(\boldsymbol{a}_i + \boldsymbol{b}_i)\eta_i - \sum_{i\in\mathcal{I}^+}(\boldsymbol{c}_i + \boldsymbol{d}_i)\kappa_{ii}\eta_i + \sum_{i\in\mathcal{I}^-}(\boldsymbol{c}_i - \boldsymbol{d}_i)\kappa_{ii}\xi_i = \boldsymbol{q}, \tag{25}$$

with $\boldsymbol{\xi} \geq \boldsymbol{0}, \boldsymbol{\eta} \geq \boldsymbol{0}, \boldsymbol{\xi} \circ \boldsymbol{\eta} = \boldsymbol{w}$. Passing to the matrix form and using the definition of *K*, the previous equation can be written compactly as (20) with M_1 column representative matrix of the set $\{A - B, (C - D)K\}$ of columns

$$\boldsymbol{m}_{1_i} = \boldsymbol{a}_i - \boldsymbol{b}_i \text{ if } x_i \in \mathcal{I}^+; \quad \boldsymbol{m}_{1_i} = \kappa_{ii}(\boldsymbol{c}_i - \boldsymbol{d}_i) \text{ if } i \in \mathcal{I}^- \tag{26}$$

and M_2 defined consequently as a simultaneously column representative matrix to M_1 in the set $\{(C + D)K, A + B\}$.

In all cases, the inverse implication follows by repeating the same passages backwards, similarly to the proof to Theorem 2 for the 1-dimensional case. □

Nonetheless, we may have $\boldsymbol{c}_i = \boldsymbol{d}_i = \boldsymbol{0}$ for some index *i*. If this happens for every $i = 1, \ldots, n$, the FAVE reduces to a GAVE. Otherwise, we still have a FAVE, but we do not have fractional terms at the generic *i*-th rows where $\boldsymbol{c}_i = \boldsymbol{d}_i = \boldsymbol{0}$. For these indices, $x_i = 0$ is possible. We treat both these situations in the following corollary, which includes the case of zero components in the solution to a FAVE by allowing $\boldsymbol{c}_i = \boldsymbol{d}_i = \boldsymbol{0}$. The hypotheses of the corollary will still prevent unfeasible solutions and allow to view the following results as generalizations of existing conditions for standard GAVEs and AVEs.

Corollary 5 *Let* $\boldsymbol{w} = (|C| + |D|)^T\boldsymbol{e}$ *and let* $K \in \mathbb{R}^{n\times n}$ *be the diagonal matrix of entries* $\kappa_{ii} = \frac{1}{w_i}$ *if* $w_i \neq 0$ *and* κ_{ii} *equal to an arbitrary positive value if* $w_i = 0$. *Let* $\hat{\boldsymbol{x}} \in \mathbb{R}^n$ *be a solution to the FAVE (2) and define* $\mathcal{I}_+$ *and* $\mathcal{I}_-$ *as the sets of indices*

$$\mathcal{I}^+ = \{i \mid \hat{x}_i \geq 0\}; \quad \mathcal{I}^- = \{i \mid \hat{x}_i < 0\} \tag{27}$$

for $i = 1, \ldots, n$. *Then, the non-negative vectors* $\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}} \in \mathbb{R}^n$ *of components*

$$\begin{aligned} \hat{\xi}_i &= \begin{cases} \max\{0, \hat{x}_i\} + \max\{0, -\frac{w_i}{\hat{x}_i}\} & if\ w_i \neq 0 \\ \max\{0, \hat{x}_i\} & if\ w_i = 0 \end{cases} \\ and\ \hat{\eta}_i &= \begin{cases} \max\{0, -\hat{x}_i\} + \max\{0, \frac{w_i}{\hat{x}_i}\} & if\ w_i \neq 0 \\ \max\{0, -\hat{x}_i\} & if\ w_i = 0, \end{cases} \end{aligned} \tag{28}$$

$i = 1, \ldots, n$, *solve the WHLCP (20), where* $\{M_1, M_2\}$ *are simultaneously column representative to the sets* $\{A - B, (C - D)K\}$ *and* $\{(C + D)K, A + B\}$ *with* M_1 *matrix of columns as in (21) with* $\mathcal{I}^+, \mathcal{I}^-$ *as in (27). Conversely, if* $\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}$ *solve the WHLCP (20) with* $\mathcal{I}^+$ *and* $\mathcal{I}^-$ *given sets of indices that define the matrices of the problem as in (21), then* $\hat{\boldsymbol{x}}$ *of components* $\hat{x}_i = \hat{\xi}_i$ *for any* $i \in \mathcal{I}^+$ *and* $\hat{x}_i = -\hat{\eta}_i$ *for any* $i \in \mathcal{I}^-$ *is a solution to the FAVE (2).*

Proof When $w_i > 0$, the proof is exactly as in Theorem 4. If, instead, the generic *i*-th column of *C* and *D* is zero, we have $w_i = 0$. The analysis is, nonetheless, analogous. Indeed, if $\hat{x}_i > 0$, (28) reduces to $\hat{\xi}_i = x_i, \hat{\eta}_i = 0, \hat{\xi}_i\hat{\eta}_i = 0$. This means that the *i*-th term of the left-hand side of (17) reduces to

$$\boldsymbol{a}_i x_i - \boldsymbol{b}_i |x_i|,$$

and the $i-$th term of the left-hand side of (25) reduces to $(\boldsymbol{a}_i - \boldsymbol{b}_i)\xi_i$. Accordingly, the corollary would prescribe to consider the WHLCP (20) with

$$\boldsymbol{m}_{1_i} = (\boldsymbol{a}_i - \boldsymbol{b}_i); \quad \boldsymbol{m}_{2_i} = [\kappa_{ii}(\boldsymbol{c}_i + \boldsymbol{d}_i)] = \boldsymbol{0}.$$

Similar considerations apply to $\hat{x}_i < 0$: (28) reduces to $\hat{\xi}_i = 0, \hat{\eta}_i = -\hat{x}_i, \hat{\xi}_i\hat{\eta}_i = 0$. Finally, for $w_i = 0$ we need to also consider the special case $\hat{x}_i = 0$. Nonetheless, this leads to $\hat{\xi}_i = 0, \hat{\eta}_i = 0, \hat{\xi}_i\hat{\eta}_i = 0$. Hence, the *i*-th term of the WHLCP (20) vanishes just as the corresponding term of the left-hand side of (17). □

Remark 2 Notice that the choice $\boldsymbol{w} = (|C| + |D|)^T \boldsymbol{e}$ was made out of convenience. More in general, $\boldsymbol{w}$ can be defined as an arbitrary non-negative vector whose generic *i*-th component is zero only if $\boldsymbol{c}_i = \boldsymbol{d}_i = \boldsymbol{0}$.

Example 1 Consider the FAVE (2) of matrices

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{q} = \begin{pmatrix} -\frac{4}{3} \\ \frac{7}{3} \end{pmatrix}. \tag{29}$$

It can be verified that such FAVE admits the positive solution $\hat{\boldsymbol{x}}_1 = (1, 3)^T$. Next, set $\boldsymbol{w} = (|C| + |D|)^T \boldsymbol{e} = (2, 3)^T$ (or any arbitrary positive vector according to Theorem 4) and define *K* accordingly. Based on Theorem 4, and especially on (19), the vectors

$$\hat{\boldsymbol{\xi}}_1 = (1, 3)^T, \quad \hat{\boldsymbol{\eta}}_1 = (2, 1)^T$$

solve the WHLCP($M_1, M_2, \boldsymbol{q}$) with

$$M_1 = A - B = \begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}, \quad M_2 = (C + D)K = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{pmatrix}.$$

It can easily be verified that $\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\eta}}_1$ really solve this WHLCP. Similarly, it can be verified that the FAVE (2) with terms as in (29) also admits the following solutions (here rounded to the second decimal digit)

$$\hat{\boldsymbol{x}}_2 = (1.07, -0.30)^T, \quad \hat{\boldsymbol{x}}_3 = (-0.47, 2.62)^T, \quad \hat{\boldsymbol{x}}_4 = (-0.41, -0.33)^T.$$

It is easy to verify that the WHLCPs defined according to the rules of Theorem 4 are solved by the vectors

$$\hat{\boldsymbol{\xi}}_2 = (1.07, 9.93)^T,\ \hat{\boldsymbol{\eta}}_2 = (1.87, 0.30)^T$$
$$\hat{\boldsymbol{\xi}}_3 = (4.27, 2.62)^T,\ \hat{\boldsymbol{\eta}}_3 = (0.47, 1.15)^T$$
$$\hat{\boldsymbol{\xi}}_4 = (4.90, 9.14)^T,\ \hat{\boldsymbol{\eta}}_4 = (0.41, 0.33)^T,$$

which are immediately computed via (19) using the (non-rounded) values of $\hat{\boldsymbol{x}}_2, \hat{\boldsymbol{x}}_3, \hat{\boldsymbol{x}}_4$.

Remark 3 In the special case $C = D = 0$, the FAVE (2) reduces to the GAVE

$$A\boldsymbol{x} - B\boldsymbol{x} = \boldsymbol{q}. \tag{30}$$

It is known that this GAVE is equivalent to the HLCP

$$(A - B)\boldsymbol{\xi} - (A + B)\boldsymbol{\eta} = \boldsymbol{q} \text{ with } \boldsymbol{\xi} \geq \boldsymbol{0}, \boldsymbol{\eta} \geq \boldsymbol{0}, \boldsymbol{\xi}^T\boldsymbol{\eta} = 0, \tag{31}$$

with $\boldsymbol{\xi} = \max\{\boldsymbol{0}, \boldsymbol{x}\}, \boldsymbol{\eta} = \max\{\boldsymbol{0}, -\boldsymbol{x}\}$. For instance, see [7, Proposition 1]. This equivalence is contained in Corollary 5. Indeed, if $\boldsymbol{w} = \boldsymbol{0}$, Corollary 5 implies that all the solutions to the GAVE can be found by solving (20) for any couple $\{M_1, M_2\}$ simultaneously column representative to the sets $\{A - B, 0\}$ and $\{0, A + B\}$ for which it admits a solution. Notice that this contains the solutions to (31). Indeed, at any solution of (31), $\hat{\xi}_i$ or $\hat{\eta}_i$ needs to be zero in order to satisfy the complementarity condition, $i = 1, \ldots, n$. The i-th column either of $A - B$ or of $A + B$ in (31) will therefore multiply a zero term at the solution. It follows that any solution to (20) with M_1, M_2 simultaneously column representative to $\{A - B, 0\}$ and $\{0, A + B\}$, and with $\hat{\xi}_i = 0$ for $i \in \mathcal{I}^-$ and $\hat{\eta}_i = 0$ for $i \in \mathcal{I}^+$, must solve (31) as well. The inverse implication holds as well.

3 Existence of solutions to FAVEs

3.1 Existence of solutions to 1-dimensional FAVEs

Based on Theorem 2, the existence of solutions to (1) can be analyzed via the solvability of (8) and (9). Hence, we have the following result, which, for simplicity, is at first given with regard to the 1-dimensional case.

Theorem 6 *The FAVE (1) has exactly one positive solution for any q if and only if $a - b$ and $c + d$ are either both positive or both negative. Similarly, (1) has exactly one negative solution for any q if and only if $a + b$ and $c - d$ are either both positive or both negative.*

Proof By Theorem 2, the existence of positive solutions to the FAVE (1) is linked to the existence of solutions to the WHLCP (8). Furthermore, the solution to the WHLCP (8) is unique for any q if and only if the coefficients $a - b$ and $c + d$ are either both positive or both negative, as can be seen by direct analysis of (8). This is also a special 1-dimensional instance of the $\mathcal{W}$-property, which ensures the unique existence of the WHLCP solution for any q and for any $w \geq 0$ according to [24, Theorem 3]. Hence, the first statement of the theorem is readily proved.

As regards the second statement, the proof is analogous, considering that Theorem 2 links the existence of negative solutions to the FAVE (1) to the existence of solutions to the WHLCP (9). ☐

Outside the assumptions of Theorem 6, a FAVE may or may not have solutions depending on the known term q. Moreover, the solution may not be unique for some values of q. Nonetheless, notice that the two statements of the theorem are independent from one another: it may very well happen that the FAVE has a unique positive solution for any q (i.e., the first statement of Theorem 6 is satisfied), while negative solutions exist only for some values of q (i.e., the second statement is not satisfied). To illustrate this, we can consider some numerical experiments. In this regard, in Fig. 1 we represent the left-hand side of a selection of 1D FAVEs (blue curves). To aid the interpretation of the plots, we also represent the value of a special choice of the right-hand side (i.e., $q = 2$) by a red line. Evidently, the solutions to the FAVE (1) for the special choice $q = 2$ are the intersections between the red and the blue lines of Fig. 1.

As a first example, consider the FAVE (1) with $a = c = 1$, $b = -2$, $d = 2$. Evidently, $\{a - b, c + d\} = \{3, 3\}$ are both positive and $\{a + b, c - d\} = \{-1, -1\}$ are both negative. Hence, by Theorem 6, we expect the FAVE (1) to have exactly one positive and one negative solution for every choice of q. This is indeed what happens, as we can observe in Fig. 1a. Beyond the special choice $q = 2$, it is easy to verify that we always obtain one positive and one negative solution for any q (i.e., when the red line is moved upwards of downwards).

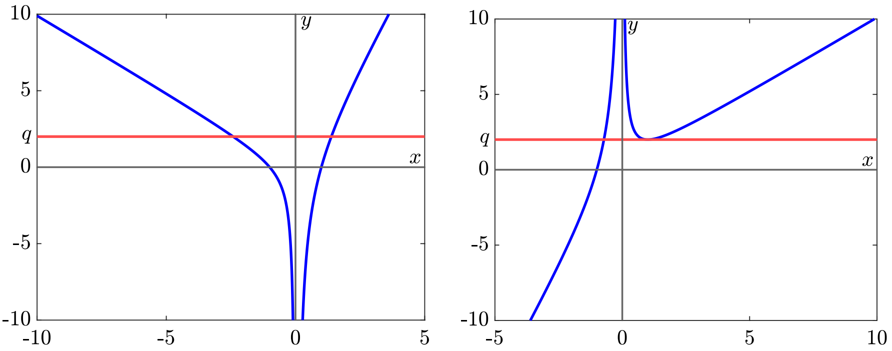
Instead, if we take $a = 2, b = c = 1, d = -2$, the values $\{a - b, c + d\} = \{1, -1\}$ do not have the same sign. Hence, there is not a unique positive solution for any q. At the same time, $\{a + b, c - d\} = \{3, 3\}$ are both positive, which ensures that there exists exactly one negative solution for any q. Indeed, this is what we observe in Fig. 1b, where the negative solution is unique for any q, while the positive solution does not exist for $q < 2$, is unique for $q = 2$, and is not unique for $q > 2$. The opposite situation is represented if Fig. 1c, where we set $a = d = 1$, $b = -5$, and $c = 2$. In this case, the positive solution is unique for any q, while the negative solution exists only for $q \geq 4$.

Finally, Fig. 1d represents the case with $a = 1.5$, $b = d = -2$, $c = 1$. As none of the uniqueness conditions of Theorem 6 are satisfied, one or more solutions may or may not exist depending on the value of q.

3.2 Generalization to n-dimensional FAVEs

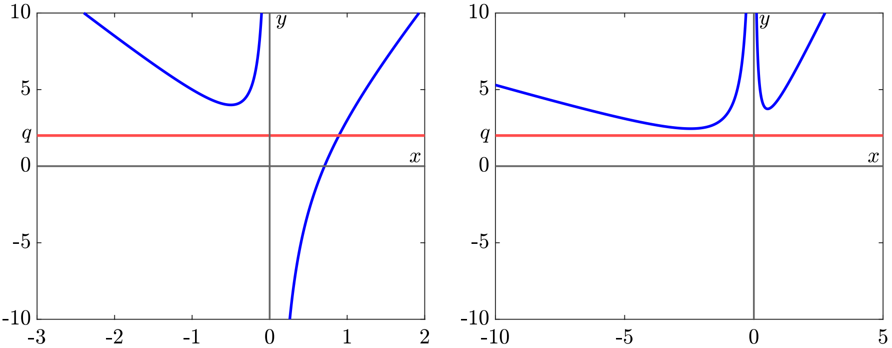
As regards the generalization to multiple dimensions, we first notice the following.

Proposition 7 *All the solutions to the FAVE (2) can be traced back to the solution of, at most, 2^n WHLCPs.*



(a) $a = c = 1$, $b = -2$, $d = 2$. Exactly one positive and one negative solution exist for any q.

(b) $a = 2$, $b = c = 1$, $d = -2$. Only the negative solution exists and is unique for any q.



(c) $a = d = 1$, $b = -5$, and $c = 2$. Only the positive solution exists and is unique for any q.

(d) $a = 1.5$, $b = d = -2$, $c = 1$. Positive and negative solutions may or may not exist, depending on q.

Fig. 1 Plots of a selection of FAVEs. The blue lines represent the left-hand side of the FAVE. The solutions to the FAVE (if any) are given by the intersections between the blue line and the line $y = q$. As an example, the red line represents the special choice $q = 2$

Proof The proof follows directly from Theorem 4 with Proposition 1. $\square$

We can then analyze the existence and the number of solutions to the FAVE (2). In particular, we have the following theorem, which expresses a condition for the existence of solutions with a specific sign pattern.

Theorem 8 *Let* $\boldsymbol{w} \in \mathbb{R}^n$ *and* $K \in \mathbb{R}^{n \times n}$ *be as in Theorem 4. Let* M_1, M_2 *be simultaneously column representative to the sets* $\{A - B, (C - D)K\}$ *and* $\{(C + D)K, A + B\}$*, respectively. Furthermore, let* $\mathcal{I}$ *denote the set of indices* $i = 1, \ldots, n$ *and*

$$\mathcal{I}_1 = \{i \in \mathcal{I} \mid \boldsymbol{m}_{1_i} = \boldsymbol{a}_i - \boldsymbol{b}_i\}; \qquad \mathcal{I}_2 = \{i \in \mathcal{I} \mid \boldsymbol{m}_{1_i} = \kappa_{ii}(\boldsymbol{c}_i - \boldsymbol{d}_i)\}. \quad (32)$$

Then, if $\{M_1, M_2\}$ *has the column* $\mathcal{W}$*-property, the FAVE (2) has a unique solution* $\hat{\boldsymbol{x}}$ *of components* $\hat{x}_i \in \mathbb{R} \backslash \{0\}$*,* $i = 1, \ldots, n$*, with*

$$\hat{x}_i > 0 \; for \; i \in \mathcal{I}_1; \quad \hat{x}_i < 0 \; for \; i \in \mathcal{I}_2 \tag{33}$$

for any $\boldsymbol{q}$. The uniqueness of the solution equivalently holds if either of the following conditions is satisfied in place of the column $\mathcal{W}$-property:

1. *for arbitrary non-negative diagonal matrices $D_1, D_2 \in \mathbb{R}^{n\times n}$ with $diag(D_1 + D_2) > 0$,*

$$\det(M_1 D_1 + M_2 D_2) \neq 0;$$

2. *M_1 is invertible and $\{I, M_1^{-1} M_2\}$ has the column $\mathcal{W}$-property.*

Proof Let us focus on the first claim. The fact that $\{M_1, M_2\}$ has the column $\mathcal{W}$ -property implies that $(|C| + |D|)^T \boldsymbol{e} > \boldsymbol{0}$. Otherwise, a column of either M_1 or M_2 would be zero, implying that $\{M_1, M_2\}$ could satisfy, at most, the column $\mathcal{W}_0$-property. Two consequences immediately follow:

- the FAVE contains fractional terms in all the components of $\boldsymbol{x}$, and is therefore defined for $x_i \in \mathbb{R}\backslash\{0\}$, $i = 1, \ldots, n$;
- the WHLCP (20) can only have strictly positive solutions, in order to satisfy $\hat{\boldsymbol{\xi}} \circ \hat{\boldsymbol{\eta}} = \boldsymbol{w} > \boldsymbol{0}$.

Furthermore, if $\{M_1, M_2\}$ has the column $\mathcal{W}$-property, then the WHLCP (20) has a unique solution for that specific choice of M_1, M_2 by [24, Theorem 3]. To such unique, strictly positive solution $\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}$ there will correspond a unique solution $\hat{\boldsymbol{x}}$ to the FAVE according to Theorem 4. By how this solution is defined in the statement of Theorem 4, it cannot have any zero component when $\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}$ are strictly positive. Furthermore, by comparing (18)–(21) with the definition of $\mathcal{I}_1, \mathcal{I}_2$, it is immediate to realize that $\hat{\boldsymbol{x}}$ has components whose signs must satisfy (33).

As regards the assumptions equivalent to the column $\mathcal{W}$-property, they come directly from [27, Theorem 2]. □

While the previous theorem (and subsequent results) employs an arbitrary $\boldsymbol{w} > \boldsymbol{0}$ for generality, note that $K = I$ if we choose $\boldsymbol{w} = \boldsymbol{e}$. Thus, in practice, the requirements of Theorem 8 involve only the matrices of the FAVE. This can also be viewed through the fact that the set of *P*-matrices is closed under positive diagonal multiplication.

Example 2 Consider the FAVE of Example 1. It is easy to compute

$$A - B = \begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}, \quad A + B = \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}, \quad (C - D)K = \begin{pmatrix} \frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{2} & \frac{2}{3} \end{pmatrix}, \quad (C + D)K = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{pmatrix}.$$

By the definition of strictly and irreducibly diagonally dominant column representative sets (see [28]), it is easy to verify that all possible combinations of simultaneously column representative matrices to the sets $\{A - B, (C - D)K\}$ and $\{(C + D)K, A + B\}$ satisfy the column $\mathcal{W}$-property. Hence, the FAVE has 4 solutions with unique sign pattern for any choice of $\boldsymbol{q}$. This is consistent with the results presented in Example 1.

Notice that the inverse implication of Theorem 8 (namely, that the uniqueness of the solution of the FAVE implies the column $\mathcal{W}$-property) does not hold in the general FAVE case. Indeed, suppose that $\hat{\boldsymbol{x}}$ as in (33) is unique for any $\boldsymbol{q}$. Then, following the proof of Theorem 8, we can prove that $\hat{\boldsymbol{\xi}}$ and $\hat{\boldsymbol{\eta}}$, as given in (19), are unique for any $\boldsymbol{q}$ with a specific weight $\boldsymbol{w} > \mathbf{0}$. However, the column $\mathcal{W}$-property holds if $\hat{\boldsymbol{\xi}}$ and $\hat{\boldsymbol{\eta}}$ are unique for any $\boldsymbol{q}$ and for any $\boldsymbol{w} \geq \mathbf{0}$ (or for the special case $\boldsymbol{w} = \mathbf{0}$) [24, Theorem 3]. Thus, the uniqueness of the solution for a specific $\boldsymbol{w} > \mathbf{0}$ is, in general, insufficient, even when the solution is unique for any $\boldsymbol{q}$. In this regard, consider the following counterexample.

Example 3 Consider the FAVE of matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}. \tag{34}$$

The positive solutions of this FAVE are the values $x_1 > 0, x_2 > 0$ which solve the nonlinear system

$$\begin{cases} x_1 + x_2 - \frac{4}{x_1} = q_1 \\ x_1 + x_2 - \frac{3}{x_2} = q_2. \end{cases} \tag{35}$$

Denoting by s the sum $s := x_1 + x_2$, the system reduces to

$$x_1 = \frac{4}{s - q_1}; \quad x_2 = \frac{3}{s - q_2} \tag{36}$$

where s is the solution of

$$s = x_1 + x_2 = \frac{4}{s - q_1} + \frac{3}{s - q_2},$$

with $s > \max\{q_1, q_2\}$ to ensure the positivity of x_1, x_2. Therefore, s can be determined by solving

$$f(s) := \frac{4}{s - q_1} + \frac{3}{s - q_2} - s = 0. \tag{37}$$

with the restriction $s > \max\{q_1, q_2\}$. The function $f(s)$ is defined for all $s > \max\{q_1, q_2\}$, with $f(s) \to \infty$ as $s \to \max\{q_1, q_2\}^+$ and $f(s) \to -\infty$ as $s \to \infty$. Moreover, $f(s)$ is continuous for $s > \max\{q_1, q_2\}$, and its derivative

$$f'(s) = -\frac{4}{(s - q_1)^2} - \frac{3}{(s - q_2)^2} - 1$$

is negative, indicating that $f(s)$ is strictly decreasing for $s > \max\{q_1, q_2\}$. These properties hold for any $q_1, q_2 \in \mathbb{R}$. Thus, there exists exactly one $s > \max\{q_1, q_2\}$

satisfying (37), ensuring that the FAVE defined by (34) has a unique positive solution $\hat{\boldsymbol{x}}$ for any $\boldsymbol{q}$.

By Theorem 4, any such $\hat{\boldsymbol{x}} > \mathbf{0}$ forms, via Eq. (19), two vectors $\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}$ which solve the WHLCP

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad \boldsymbol{\xi} > \mathbf{0},\ \boldsymbol{\eta} > \mathbf{0}\ \boldsymbol{\xi} \circ \boldsymbol{\eta} = \boldsymbol{e}, \qquad (38)$$

where we have set $\boldsymbol{w} = \boldsymbol{e}$ and $K = I$ for simplicity. The matrices of this problem are M_1, M_2, which have been computed as $M_1 = A - B$ and $M_2 = C + D$, in accordance with the previous theorems. The relationship FAVE–WHLCP guarantees that (38) has a unique solution for any $\boldsymbol{q}$ as well. However, the matrices M_1, M_2 as in (38) do not satisfy the column $\mathcal{W}$-property, as the determinant of M_1 is zero. Instead, they satisfy only the weaker column $\mathcal{W}_0$-property.

Thus, the column $\mathcal{W}$-property is not necessary for the unique solvability of the FAVE with a specific sign pattern, nor for the uniqueness of the solution of a WHLCP with a specific weight $\boldsymbol{w} > \mathbf{0}$ for any $\boldsymbol{q}$.

On the other hand, it is well known that the column $\mathcal{W}$-property is necessary and sufficient for the uniqueness of the solution of an HLCP for any $\boldsymbol{q}$ [27, Theorem 2]. This is relevant to particularize Theorem 8 (together with Corollary 5) to the existence of GAVE solutions with a given sign pattern. Indeed, the case of a GAVE implies $C = D = 0$. In this case, M_1 and M_2 are simultaneously column representative to the sets $\{A - B, 0\}$ and $\{0, A + B\}$. Because of the zero columns of M_1, M_2, they cannot satisfy the column $\mathcal{W}$-property. Hence, there does not exist any $\{M_1, M_2\}$ simultaneously column representative to $\{A - B, 0\}$ and $\{0, A + B\}$ for which the HLCP($M_1, M_2, \boldsymbol{q}$) has a unique solution for any $\boldsymbol{q}$. Even though this was expected (as multiple solutions of the HLCP($M_1, M_2, \boldsymbol{q}$) may map to the same GAVE solution), it suggests a more general non-existence result. Indeed, assume that $\hat{\boldsymbol{x}}$ is the unique solution that was obtained with a specific known term $\hat{\boldsymbol{q}}$ and let $D_s = \mathrm{diag}(\mathrm{sgn}(\hat{\boldsymbol{x}}))$. If the solution with the same sign pattern as $\hat{\boldsymbol{x}}$ is unique for any $\boldsymbol{q}$, it means that there exists exactly one solution of the form $\boldsymbol{x} = (A - BD_s)^{-1}\boldsymbol{q}$ for any $\boldsymbol{q}$ with the unchanging sign pattern prescribed by D_s. However, such relation is linear in $\boldsymbol{q}$, implying that, at some point, changing $\boldsymbol{q}$ changes the sign of the components of $\boldsymbol{x}$. This directly leads to the following corollary:

Corollary 9 *A GAVE cannot have a unique solution which preserves the same sign pattern for all* $\boldsymbol{q}$.

Notice that this does not mean that the solution to a GAVE cannot be unique for all $\boldsymbol{q}$ (which would contradict known results), but it means that a GAVE whose solution is unique must change the sign pattern for some value of $\boldsymbol{q}$. In this context, we remark that many results in the literature focused on the existence of solutions (and especially on unique solvability), while results on the sign pattern and the number of solutions are scarce, with the notable exceptions of [2, 29].

Several other existence results follow. For instance, the following corollary provides a sufficient condition that ensures a minimum number of solutions to the FAVE for any $\boldsymbol{q}$.

Corollary 10 *Let $\boldsymbol{w} \in \mathbb{R}^n$ and $K \in \mathbb{R}^{n\times n}$ be as in Theorem 4. Let $\{M_1, M_2\}$ satisfy the column $\mathcal{W}$-property for m matrices M_1, M_2 simultaneously column representative to the sets $\{A - B, (C - D)K\}$ and $\{(C + D)K, A + B\}$. Then, for any $\boldsymbol{q}$, the FAVE (2) has at least m solutions characterized by a different sign pattern.*

Proof It follows directly from Theorem 8 with *m* couples of simultaneously column representative matrices that satisfy the column $\mathcal{W}$-property. ☐

The previous corollary only gives a sufficient condition of existence for any $\boldsymbol{q}$. Indeed, when $\{M_1, M_2\}$ do not satisfy the column $\mathcal{W}$-property, one or more solutions may or may not exist depending on $\boldsymbol{q}$. This is evident in the 1-dimensional examples analyzed in Sect. 3.1. Corollary 10 just provides a lower bound to the minimum number of solutions that exist for any $\boldsymbol{q}$ based on the number of simultaneously column representative matrices that satisfy the column $\mathcal{W}$-property.

In the special case when all possible choices of M_1 and M_2 satisfy the column $\mathcal{W}$ -property, we instead obtain the following condition.

Theorem 11 *Let $\boldsymbol{w} \in \mathbb{R}^n$ and $K \in \mathbb{R}^{n\times n}$ be as in Theorem 4. If $\{M_1, M_2\}$ satisfies the column $\mathcal{W}$-property for any M_1, M_2 simultaneously column representative to the sets $\{A - B, (C - D)K\}$ and $\{(C + D)K, A + B\}$, then the FAVE (2) has exactly 2^n solutions for any $\boldsymbol{q}$. Furthermore, each solution has no zero components and a distinct sign pattern.*

Proof It comes from Theorem 8 together with Proposition 1, which states that there are 2^n simultaneously column representative matrices to two sets of two matrices. To each choice of $\{M_1, M_2\}$ there will exist a unique solution to the equivalent WHLCP and, analogously, a solution $\hat{\boldsymbol{x}}$ with a unique sign pattern (indeed, notice that there are 2^n possible sign patterns in $\hat{\boldsymbol{x}} \in \mathbb{R}^n$ with $\hat{x}_i \neq 0$, $i = 1, \ldots, n$). Thus, the theorem is immediately proved. ☐

For instance, Example 1–2 can again be considered. As mentioned above, all the simultaneously column representative matrices satisfy the column $\mathcal{W}$-property. Therefore, there exist $2^n = 2^2$ solutions to the FAVE for any $\boldsymbol{q}$. There are no zero components and the sign pattern is different in each solution, as can be seen in Example 1 for a special choice of $\boldsymbol{q}$.

In the case of a GAVE, we have already noticed that we would need to consider (non-weighted) HLCPs of matrices $\{M_1, M_2\}$ simultaneously column representative to $\{A - B, 0\}$ and $\{0, A + B\}$. Moreover, Corollary 9 remarks that a GAVE cannot have a unique solution which preserves the same sign pattern for any $\boldsymbol{q}$. It directly follows that there cannot be GAVEs which have exactly 2^n solutions for any $\boldsymbol{q}$, each with a different sign pattern.

Theorem 11 directly leads to the following corollary, which gives a sufficient condition that ensures that 2^n solutions exist for any $\boldsymbol{q}$.

Corollary 12 *Let* $\boldsymbol{w} \in \mathbb{R}^n$ *and* $K \in \mathbb{R}^{n\times n}$ *be as in Theorem 4. If the set* $\mathcal{M} = \{A - B, A + B, (C - D)K, (C + D)K\}$ *satisfies the column* $\mathcal{W}$*-property, then the FAVE (2) has exactly* 2^n *solutions for any* $\mathbf{q}$*. Furthermore, there is no solution with zero components, and there is exactly one solution for each possible pattern of signs of the components.*

Proof If $\mathcal{M}$ satisfies the column $\mathcal{W}$-property, then every possible couple of simultaneously column representative matrices M_1, M_2 to the sets $\{A - B, (C - D)K\}$ and $\{(C + D)K, A + B\}$ satisfies the column $\mathcal{W}$-property. Hence, the proof follows by Theorem 11. ☐

Other sufficient conditions can then be obtained for special forms of the matrices of the problem. For instance, we prove the following corollary.

Corollary 13 *Let* $A - B, A + B, C - D$*, and* $C + D$ *be either*

- strictly diagonally dominant by columns
- or constitute an irreducibly diagonally dominant column representative set [28].

Furthermore, let $sgn(a_{ii} - b_{ii}) = sgn(a_{ii} + b_{ii}) = sgn(c_{ii} - d_{ii}) = sgn(c_{ii} + d_{ii})$ for $i = 1, \ldots, n$. Then, the FAVE (2) has exactly 2^n solutions for any $\boldsymbol{q}$. Furthermore, there is no solution with zero components, and there is exactly one solution for each possible combination of signs of the components.

Proof Under the assumptions of the theorem, $\mathcal{M} = \{A - B, A + B, (C - D)K, (C + D)K\}$ satisfies the column $\mathcal{W}$-property, for any K as defined in Theorem 4. See [28, Lemma 1–Corollary 1]. This fact together with Theorem 11 directly proves the corollary. ☐

Notice that this last corollary allows to easily assess the number and sign pattern of the solutions to the FAVE of Example 1 via the assumption of irreducibly diagonally dominant column representative set.

Finally, we briefly discuss the following special case, where all the solutions to the FAVE map to the unique solution of a WHLCP.

Theorem 14 *Let* $K \in \mathbb{R}^{n\times n}$ *be a positive diagonal matrix such that* $A - B = (C - D)K$ *and* $A + B = (C + D)K$*. Furthermore, let* $\{A - B, A + B\}$ *satisfy the column* $\mathcal{W}$*-property and let* $\boldsymbol{w} \in \mathbb{R}^n$ *be the vector of components* $w_i = \frac{1}{\kappa_{ii}}$*,* $i = 1, \ldots, n$*. Then, the FAVE (2) has* 2^n *solutions for any* $\mathbf{q}$*, each with a different sign pattern. Each solution can be obtained as the* 2^n *distinct vectors of components*

$$\hat{x}_i = \hat{\xi}_i \quad or \quad \hat{x}_i = -\hat{\eta}_i, \quad for\ i = 1, \ldots, n \tag{39}$$

where $(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}})$ denote the solution of the WHLCP

$$(A - B)\boldsymbol{\xi} - (A + B)\boldsymbol{\eta} = \boldsymbol{q} \textit{ with } \boldsymbol{\xi} \geq \mathbf{0}, \boldsymbol{\eta} \geq \mathbf{0}, \boldsymbol{\xi} \circ \boldsymbol{\eta} = \boldsymbol{w} \tag{40}$$

Such solution is unique and strictly positive for any $\boldsymbol{q}$.

Analogously, given any solution $\hat{\boldsymbol{x}}$ of the FAVE, the unique solution to (40) can be found via (19).

Proof Under the assumptions of the theorem, the simultaneously column representative matrices to the sets $\{A - B, (C - D)K\} = \{A - B, A - B\}$ and $\{(C + D)K, A + B\} = \{A + B, A + B\}$. Therefore, it directly follows $M_1 = A - B$ and $M_2 = A + B$ irrespective of the sign pattern of $\hat{\boldsymbol{x}}$. The assumption that $\{A - B, A + B\}$ satisfies the column $\mathcal{W}$-property is the same as requiring that any M_1, M_2 simultaneously column representative to the sets $\{A - B, (C - D)K\}$ and $\{(C + D)K, A + B\}$ have the column $\mathcal{W}$-property. Then, the FAVE (2) has 2^n solutions for any $\boldsymbol{q}$ by Theorem 11 and the WHLCP (40) has a unique solution for any $\boldsymbol{q}$ by [24].

All the 2^n solutions to the FAVE must therefore map to this unique solution of the WHLCP in accordance with Theorem 4. Equation (39) follows directly from this. Indeed, we will always have $M_1 = A - B$ and $M_2 = A + B$ in Theorem 4, irrespective of the sign pattern of the solution. □

Example 4 Consider the FAVE (2) with

$$A = \begin{pmatrix} 3 & -1 \\ 0 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 6 & -3 \\ 0 & 12 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 3 \\ -2 & 6 \end{pmatrix} \quad \boldsymbol{q} = \begin{pmatrix} -2 \\ -13 \end{pmatrix} \tag{41}$$

Evidently, the vectors $\{\boldsymbol{a}_i, \boldsymbol{c}_i\}$ are linearly dependent and so are $\{\boldsymbol{b}_i, \boldsymbol{d}_i\}$, for $i = 1, \ldots, n$. Therefore, the respective sums are also going to be linearly dependent, and we can easily find

$$A - B = \begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix} = (C - D)K \quad A + B = \begin{pmatrix} 4 & 0 \\ -1 & 6 \end{pmatrix} = (C + D)K$$

when we select

$$K = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}.$$

It is easy to verify that $\{A - B, A + B\}$ the column $\mathcal{W}$-property (as all column representative matrices have positive diagonal elements and are either strictly diagonally dominant or irreducibly diagonally dominant). Moreover, it can be found that the unique solution of the WHLCP (40) with $\boldsymbol{w} = (2, 3)^T$ (obtained directly from definition by the performed choice of *K*) is

$$\hat{\boldsymbol{\xi}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \hat{\boldsymbol{\eta}} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Thus, based on Theorem 14, we find that the FAVE has the following 4 solutions with unique sign pattern

$$\hat{\boldsymbol{x}}_1 = \begin{pmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \hat{\boldsymbol{x}}_2 = \begin{pmatrix} \hat{\xi}_1 \\ -\hat{\eta}_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \quad \hat{\boldsymbol{x}}_3 = \begin{pmatrix} -\hat{\eta}_1 \\ \hat{\xi}_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \hat{\boldsymbol{x}}_4 = \begin{pmatrix} -\hat{\eta}_1 \\ -\hat{\eta}_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

It can easily be verified by substitution that these vectors actually solve the FAVE (2) with matrices as in (41).

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