Introduction to the papers of TWG04: geometry teaching and learning

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Keywords: Geometry education, tools, teacher training, geometry task.

Introduction

The TWG04 is concerned with research on geometry teaching and learning from kindergarten to university, including teacher education. In particular, it aims to contribute to the congress and research by discussing new and upcoming matters regarding the teaching and learning of geometry and, at the same time, continuing the discussions on the issues raised in the last CERME.

At virtual pre-CERME12 in 2021, two topics were discussed (*The specific aspects of mathematical activity in geometry* and *Teaching and learning geometry*), around questions such as “Which competencies (visual, reasoning, operational, and figural) should students/pupils have acquired/developed at the end of a specific school level?”, “Which is the impact of online teaching and learning geometry?”, and “What geometry our students should know when they move from primary to secondary to tertiary education?”. Crucial points highlighted by several contributions are the relevance of language from a very early age, the use of material and digital tools, and the focus on visualization. At CERME12, in TWG04, we intended to continue discussing the two issues proposed for pre-CERME12 and relaunch a third issue, *Teacher education in geometry*.

Around 25 researchers from South Africa, the Middle East, East/Central/West Europe, the USA, and South America participated in TWG04 during the online sessions. The TWG04 work was stimulated by and organized around nine research papers and three posters. This group’s exciting and relevant feature was its polyphony: a blend of experienced and young researchers discussing geometry teaching and learning. In the concluding survey, the participants indicated that participation in the TWG04 was an enriching experience.

Organization of the TWG04 at the congress

The papers and posters were divided into three topics, concerning their primary focus on *Teacher education in geometry*, *Use of tools in geometry teaching and learning*, and *Students*.

The fundamental elements of all the papers and posters were presented by the authors and discussed in groups. In particular, in each session, two papers/posters were introduced by a participant, and then a collective discussion followed. In turn, one participant was in charge of writing down the questions and comments arising during the discussion on an open document, as well as each participant could contribute by writing questions/comments on the same open document. At the end of the work on the set of papers/posters concerning a topic, the participants were divided into three subgroups to discuss related questions, which were formulated by the team of group co-leaders to guide them towards a synthesis of the previous collective discussion. Finally, the last session was dedicated to sharing the synthesis of all the discussions and small group works to prepare the final report of this working group.

Focus on teacher education in geometry

This topic includes contributions that have teacher training and reflections on teachers' beliefs on geometry as crucial elements.

In her paper, Kuzle investigates the value of geometry in mathematics instructions through a questionnaire proposed to a sample of 120 German in-service teachers. She analyses two items of that questionnaire: the first item aims to collect teachers' opinions on the role of geometry instruction in elementary school, and the second item asks to mark the most significant reasons for neglecting geometry education. A result of this research is that geometry is still insufficiently taught compared to other areas of mathematics, in particular arithmetic, even if positive changes are observable. In general, the predominance of arithmetic seems related to time-practical reasons rather than a disregard for geometry. Nevertheless, there seems to be a difference between prospective and in-service teachers, related to the insecurity that prospective teachers have with respect to geometry content knowledge. Two reasons could explain this difference: a lack of mathematics knowledge from school as learners and insufficient preparation at the university level as teachers. From this research emerges the question of knowing if there are some geometry topics teachers feel more insecure about teaching them. Further research could provide elements to plan teacher education from different perspectives.

Finally, the discussion on the paper confirmed the relevance of the investigated question "What role does geometry instruction have nowadays in school mathematics?" for all the teachers, and the relevance of taking into account how textbooks convey an image of geometry with respect to the other parts of mathematics curriculum (i.e., starting from the organization of chapters in the textbook).

Hausler and Kuzle's poster contributes to the discussion on teachers' images of geometry. By "Imagine you are an artist. A good friend asks you what geometry is. Draw a picture in which you explain to him or her what geometry is for you. Be creative in your ideas", prospective primary teachers attending a geometry course at the university were invited to perform two drawings, one at the beginning of the course and the other at the end. The two authors analyse the drawings to detect changes due to the attendance of the course taught by one of the authors.

Although research is carried out on teachers' beliefs (e.g., see other CERMEs groups), these two contributions emphasize that in teacher education, it is vital to consider not only primary teachers' beliefs about mathematics but also about particular fields of mathematics and its teaching, as in this case geometry. From this point of view, it might become fundamental to investigate which activities/tasks affect prospective teachers' attitudes towards geometry and its teaching and define criteria for identifying their impact on teachers.

The questions concerning the relation between teachers' content knowledge, their difficulty in teaching geometry, and didactical tasks are also investigated in other papers from different perspectives.

Ratnayake et al.’s paper focuses on secondary school prospective teachers and their difficulty in enacting geometry in the classroom. In the trend of research focusing on teachers’ mathematical knowledge, their research project aims to offer a set of criteria for developing aspects of examples in tasks to analyse teachers’ knowledge in geometry. In particular, with this work, the analysis of figures and their attributes in solving tasks is discussed.

Several questions emerged, such as which tasks can be considered emblematic for prospective teacher education in geometry. The papers presented by Bernabeu et al., Giménez et al., and Brunheira et al. contributed to the discussion on primary school teacher education. In particular, Bernabeu et al. and Giménez et al. propose classification tasks, the former on quadrilaterals and the latter on 3D figures. Brunheira et al. focus on the reasoning process in the context of a task on 3D figures. In all these papers, the features of the tasks allow highlighting the knowledge that students possess and the knowledge that teachers need to develop for geometric thinking.

Bernabeu et al. analyse quadrilateral classifications to bring out the definitions possessed by prospective primary teachers and to identify prospective teacher profiles. Although their research focuses on the mathematical knowledge of prospective teachers, the task analysed here can be interpreted between specialized content knowledge and pedagogical knowledge (Ball, Thames & Phelps, 2008). The researchers do not collect quadrilateral definitions and classifications with direct questions but as responses to a professional task given to prospective teachers. Specifically, they analyse teachers’ knowledge through their answer to the request to anticipate children’s answers for a classification task. The analysis shows that only around 30% of the teachers use hierarchical definitions involving the transitivity of inclusion relations (although with some non-economic definitions). The discussion of this paper also led to a comparison of quadrilateral classifications and definitions in the primary school of participants’ countries, showing how classifications can be relevant – referring to Usinkin and Griffin’s work (2008) – for comparing and examining curricula and textbooks (for instance, see the definition of rhomboid/parallelogram or trapezoid/trapezium).

Giménez et al. analyse the answers to the task of constructing polyhedra using material half-cubes and classifying them; they aim to identify which types of knowledge prospective teachers possess. As in Bernabeu et al.’s paper, they are interested in mathematical content knowledge; nevertheless, the authors also consider the meta-didactical knowledge on the value of the classification process in learning geometry. Indeed, the questions they ask students include important mathematical knowledge and invite discussion of the relevance of the processes activated in these tasks (for instance, “Why do you consider that when we study various types of figures, a fundamental process to work with is classification?”). The analysis highlights that even at the undergraduate level, the perceptual appearance dominates over certain conceptual aspects: most of the classification criteria are based on a description of the object with a low level of structuring. Despite having manipulative elements which could support them, prospective teachers are not capable of expressing mathematically well-formulated explanations. For instance, they use terms from plane geometry in their discourse on spatial geometry, and their arguments are very similar to those that pupils use. Thus, the question of what role the construction of tangible models plays in developing prospective teachers’ geometrical thinking and knowledge for teaching geometry remains relevant.

In Brunheira et al.’s paper, 3D geometry is the context in which the researchers study prospective (primary and secondary) teachers’ knowledge of reasoning process within the framework of specialized mathematics knowledge for teaching. The task for prospective teachers proposed in the paper consists of two parts: in the first part, the prospective teachers have to identify the reasoning processes involved in the task which will be proposed to students; in the second part, prospective teachers are asked to discuss on students’ reasoning that emerging in an excerpt of a dialogue between students and teacher. The analysis shows that this kind of task and context promotes the knowledge of reasoning processes.

The discussion which developed in the working group proposed additional issues, such as:

- It is significant to bring together the mathematical properties and visualization processes involved in solving a geometric problem.
- There is a need for more collaboration between researchers and teachers in the research field of geometry education.
- The role embodied context plays in geometry teaching and learning, particularly the use of tools, types of tasks, and activities.
- Research could focus on the learning trajectory between primary and secondary education and its relation to the education of prospective teachers.

Finally, two suggestions emerged for further research contributions. The first suggestion is the following research questions: “Do teacher training programs in different countries have geometry/ teaching geometry courses (and to what extent)?" and "How can this choice influence prospective teachers’ efficiency/preparation for geometry instruction?”. The second suggestion concerns the development of research on secondary teacher education in geometry in connection with primary teacher education.

As pointed out in the discussion of some of the papers in this section, tasks on 3D geometry seem to be a good context for exploring geometrical thinking. The papers in the next section also focus on this content using DGS and material (tangible) models.

Focus on the use of tools in geometry teaching and learning

The group discussed this topic based on three contributions: one paper concerns using the 3D environment of GeoGebra DGS, and the other papers report two experiments with material (tangible) models of 3D figures made with different tools (3D pens and construction kits). All these papers refer to secondary school. They are based on different theoretical frameworks (i.e., the theory of semiotic mediation and embodied learning).

In Sua et al.’s paper, 3D DGE mediates the relationships between 2D space and 3D space. The task described in the paper involves geometrical constructions and their meaning in plane geometry and pays attention to the meaning of correct constructions (corresponding to several solutions) for constructions in 3D space. The dimension of space is the only variable of the task: the request for constructing triangles (isosceles and equilateral ones) is the same in the two spaces, 2D and 3D, starting from the same kind of initial object (the side of the triangles). While adapting the 2D procedure, the students consider the new dimension of the space passing from circles to spheres in their constructions, but they do not control the intersections between the drawn spheres as they do with the circles. On the other hand, the syntax of some commands in 3D DGE does not match 2D procedures; the transition between environments also requires a different and new conceptualization of 2D objects by students. This paper opens several research questions on: 3D visualization mediated by the DGE, cognitive and epistemological control in the transition from 2D to 3D spaces, awareness of this transition for students (it seems to observe a misattributing of 2D properties to 3D objects). Finally, in general, which is the challenge of this kind of task for secondary students?

The two other papers investigate 3D geometry using 3D pens and construction kits.

Palatnik and Abrahamson develop an enactivist argument for learning 3D geometry by constructing tangible models in their paper. They describe the gap between the geometry taught at school, mainly 2D geometry with 2D tools, and the 3D space where students live and experience 3D objects. From an embodied perspective in mathematics education, the authors investigate students' cognitive processes in 3D tasks with tools for constructing 3D models of solids (i.e., cube and tetrahedron) in small and medium sizes (hand-held and human-scale models) (Herbst, Fujita, Halverscheid & Weiss, 2017). These tasks foster the transition between 2D and 3D figures and models and, in general, between 2D and 3D space differently from that analysed in Sua et al.'s paper. Students' physical actions led to shifts in perceptual-motor attention, which involve a refinement of geometric reasoning.

Rosenski et al.'s research focuses on emotional aspects and the influence of affect in 3D problem solving using 3D pens. Their results show the relevance of those aspects in learning geometry with tools and the potential of the 3D pen for engaging students in spatial geometry tasks. Compared to the construction kits, the 3D pen is an emerging technology for mathematics education. Subsequent group work discussed the role of the novelty of material and digital tools in student engagement, particularly when the novelty of a tool is neutralized by regular use in the classroom.

The discussion in the group raised several questions, such as: "What is the added value of tangible and/or digital/virtual manipulatives in geometry instruction?", "Is it important to start a teaching experiment from material manipulatives with students?", "What are the affordances and constraints of using material manipulatives?". For students in different age groups, we asked: "What constitutes learning of geometry when they use physical models?", "When is it useful to propose material tools, and when is it necessary to let students work only with their mental images?", "What are principles of task design with manipulatives?", "What makes work with manipulatives (material and virtual) exciting for students (except the novelty of medium)?". Furthermore, "Why is 3D DSE especially important as a tool of semiotic mediation?". These questions remain open for research.

Our group agreed on the importance of using material tools to teach geometry in elementary school, high school, and even at university level. Nevertheless, the idea that manipulatives and digital tools are age-sensitive emerged: some tools (or some ways of using them) are adequate for younger or older students; so, while moving between different manipulatives or a manipulative and symbolic forms, the meanings that need to be constructed have to be in focus. Material and digital tools offer a combination of representations, which is very important in geometry teaching and learning (Herbst, Fujita, Halverscheid & Weiss, 2017). In some cases, as with 3D pens, boundaries between material and digital tools, novel and traditional mediums, 2D and 3D, are constantly crossed; thus, it becomes interesting to study them. Although the novelty of a tool is a factor of motivation and reinforcement in the learning process, it seems that students' motivation in activities with tools cannot be ascribed to their novelty alone; the creative and collaborative nature of activities could maintain student's interest because they made something together, they have to communicate, and so on.

The discussion proposed additional issues, such as:

- Taking into account instrumental genesis for students and teachers and the importance of students' experience with tools.
- Identifying the tool's novelty effect on students' engagement and motivation.
- Considering the change of didactical contract when a new tool is introduced in the class for planning activities and identifying students' engagement.
- Emphasizing connections between geometry education and its applications in the real world (e.g., architecture, mechanical engineering, and art).

Focus on students

The discussion on the third topic was based on three papers concerning students in grades 5, 6, and 9. These papers are developed within different theoretical frameworks (e.g., embodied cognition, realistic geometry, van Hiele Levels, PISA framework for competences) and focus respectively on abstraction in geometry (Boonstra), problem solving requested by online DGS tasks in the time of COVID (Edamus et al.), and assessment of geometrical competencies (Bočková et al.).

The contributions of Boonstra and Edamus et al. discuss the same mathematical content, the geometric reflection, and pay attention to static and dynamic aspects concerning the reflection in two different environments. Grounded in the realistic mathematics education approach, Boonstra focuses on the processes activated by 5-6-grade pupils facing two problems on reflection through the use of a mirror ("Move the chess towers in front and behind the see-through mirror and let them meet in the mirror"; "Place a see-through mirror in the correct position on the line") to investigate how embodied activities support abstraction. This paper suggests discussing which characteristics and properties the tasks should have to support abstraction and which is the role of the tool (e.g., tasks with a mirror and tasks without a mirror) within the embodied research design.

Edamus et al. analyse the resolution of a task on the composition of two reflections with parallel axes to study students' understanding of reflection from a functional point of view. In particular, after constructing the symmetrical figures, two students have to collaborate online to solve the task of moving axes and describing/explaining what happens to the symmetrical figures. In the discussion on students' processes, the questions on why students must move between static and dynamic aspects for understanding reflection and how this helps them in their learning process are asked. The setting of the experiment is interesting for fostering communication between the students, the emergence and development of mathematical language: in pair work, one student gets access to the online DGS file and shares his screen; only this student can manipulate the DGS figure while the other one is forced to verbalize his ideas and give instructions.

The third contribution brought in the discussion on the assessment of geometrical competencies. The heart of the matter is which strategies and methods researchers and teachers have at their disposal to assess students' competencies in geometry. The assessment involves choosing what is to be evaluated, a framework for geometry learning and competencies, and a format in which questions are asked. The crucial question arises as to a suitable framework for constructing a good test to measure geometric thinking.

In the discussion, additional issues emerged, such as:

- Different types of tasks and content (i.e., the inclusion of non-Euclidean geometry; topology; task with manipulatives) can surface different aspects of students' geometric thinking, reasoning, behaviour, argumentation, …
- Engaging students in collective visualization is challenging in the absence of physical manipulatives.

Finally, other questions emerged for further research contributions, for instance, concerning social aspects of students learning of geometry (i.e., gender differences in participation)

Future Directions for TWG04

The discussions in this TWG were vibrant and, as a good discussion, posed more questions than answers. In this paper, we leave some other suggestions for the next TWG on geometry teaching and learning.

In the frame of the geometry education research community considering the possible international study on the geometry content knowledge necessary for (primary/secondary) teachers:

- What research questions should be considered?
- Which differences and similarities of national curricula should be taken into account?
- Which theoretical perspectives on geometry teaching and learning should be taken into account?
- What tools does the current research provide for teacher education in geometry?

In the paper and poster presentations, we observed three essential processes involved in students' work when they engage with a geometric problem/task: visualization, argumentation, and the transitions between different representations (e.g., 3D geometrical objects and their 2D representation). As researchers and/or teachers, how can we become aware of/observe when our students engage in those different processes? Finally, how can we foster/promote the activation of those processes?

We hope that this TWG on geometry teaching and learning continues to advance the research on geometry education, paying attention to the changes in scholarly contexts and the new educational needs that arise.

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