Nonlinear Differential Equations and Applications NoDEA

ELLIPTIC EQUATIONS IN DIMENSION 2 WITH DOUBLE EXPONENTIAL NONLINEARITIES

--Manuscript Draft--

Manuscript Number:	NDEA-D-16-00265
Full Title:	ELLIPTIC EQUATIONS IN DIMENSION 2 WITH DOUBLE EXPONENTIAL NONLINEARITIES
Article Type:	Original research
Keywords:	Keywords: Trudinger-Moser type inequalities; Critical exponents
Corresponding Author:	Marta Calanchi, Ph.D Universita degli Studi di Milano Dipartimento di Matematica Federigo Enriques Milano, Italy ITALY
Corresponding Author Secondary Information:	
Corresponding Author's Institution:	Universita degli Studi di Milano Dipartimento di Matematica Federigo Enriques
Corresponding Author's Secondary Institution:	
First Author:	Marta Calanchi, Ph.D
First Author Secondary Information:	
Order of Authors:	Marta Calanchi, Ph.D
	Bernhard Ruf, PhD
	Federica Sani, PhD
Order of Authors Secondary Information:	
Funding Information:	
Abstract:	A boundary value problem on the unit disk in R^2 is considered, involving an elliptic operator with a singular weight of logarithmic type and non- linearties which are subcritical or criticlal with respect to the associated gradient norm. The existence of non-trivial solutions is proved, relying on variational methods. In the critical case, the associated energy fucntional is non-compact. A suitable asymptotic condition allows to avoid the non-compactness levels of the functional.

ELLIPTIC EQUATIONS IN DIMENSION 2 WITH DOUBLE EXPONENTIAL NONLINEARITIES

MARTA CALANCHI, BERNHARD RUF AND FEDERICA SANI

ABSTRACT. A boundary value problem on the unit disk in $\mathbb{R}^2$ is considered, involving an elliptic operator with a singular weight of logarithmic type and nonlinearties which are subcritical or criticlal with respect to the associated gradient norm. The existence of non-trivial solutions is proved, relying on variational methods. In the critical case, the associated energy fucntional is non-compact. A suitable asymptotic condition allows to avoid the non-compactness levels of the functional.

1. INTRODUCTION

In this article we study the solvability of problems of the form

$$\begin{cases} L \ := \ -\mathrm{div}(w(x)\nabla u) = f(x,u) \qquad \text{in } \ B \\ \qquad\qquad\qquad u > 0 \quad \text{in } \ B \ , \quad u \ = \ 0 \quad \text{on } \ \partial B \end{cases} \tag{1.1}$$

where B is the unitary disk in $\mathbb{R}^2$, and the function $f(|x|, s)$ has a *maximal* growth in s with respect to the weighted gradient norm, where the radial positive weight $w(x)$ is of logarithmic type and will be specified below.

In some recent papers, the influence of weights on limiting inequalities of Trudinger-Moser type (for short, TM-inequalities) has been explored in some detail. In [3], [4], [6], [5], the interest is devoted in particular to the impact of weights in the Sobolev norm.

More precisely, let $\Omega \subset \mathbb{R}^N$ be a bounded domain. If $w \in L^1(\Omega)$ is a non-negative function, we introduce the weighted Sobolev space

$$H_0^1(\Omega, w) = cl\Big\{u \in C_0^\infty(\Omega) \ ; \ \int_\Omega |\nabla u|^2 w(x)dx < \infty\Big\} \tag{1.2}$$

A general embedding theory for such weighted Sobolev spaces has been developed in Kufner [16].

It turns out that for weighted Sobolev spaces of form (1.2) logarithmic weights have a particular significance, since they concern *limiting situations* of such embeddings. However, to obtain interesting results, one needs to restrict attention to radial functions. So let us consider the subspace of radial functions, i.e.

$$H_{0,rad}^1(B, w) = cl\Big\{u \in C_{0,rad}^\infty(B) \ ; \ \|u\|_w^2 := \int_\Omega |\nabla u|^2 w(x)dx < \infty\Big\}$$

with the specific weight

$$w(x) = \log \frac{e}{|x|} \tag{1.3}$$

On the space of radial functions $H^1_{0,rad}(B, w)$, the weight has a relevant impact on the corresponding embedding inequalities. In fact, the well-known Trudinger-Moser growth e^{u^2} is drastically increased: in [4] the following *double exponential* inequality was proved.

Theorem A (Calanchi-Ruf [4]). *Let w be given by* (1.3). *Then*

$$\int_B e^{e^{u^2}} dx \ < \ +\infty \quad , \quad \forall \ u \in H^1_{0,rad}(B, w) \tag{1.4}$$

and

$$\sup_{\substack{u \in H^1_{0,rad}(B,w) \\ \|u\|_w \leq 1}} \int_B e^{\beta e^{2\pi u^2}} dx < +\infty \iff \beta \leq 2 \tag{1.5}$$

We remark that if we consider the supremum in (1.5) on the whole Sobolev space $H^1_0(B)$, then the weight w has no effect and we remain with the standard Trudinger-Moser growth (see [19], [20], [22]). Indeed, as a consequence of the standard TM-inequality and [4, Proposition 8], we have:

$$\sup_{\substack{u \in H^1_0(B,w) \\ \|u\|_w \leq 1}} \int_B e^{\alpha u^2} dx < \infty \iff \alpha \leq 4\pi$$

Returning to equation (1.1), and in view of inequality (1.5), we say that f has *subcritical growth* at $+\infty$ if for all $\alpha > 0$

$$\lim_{s \to +\infty} \frac{|f(x,s)|}{\exp\{e^{\alpha s^2}\}} = 0 \tag{1.6}$$

and f has *critical growth* at $+\infty$ if there exists $\alpha_0 > 0$ such that

$$\lim_{s \to +\infty} \frac{|f(x,s)|}{\exp\{2e^{\alpha s^2}\}} = 0, \quad \forall \alpha > \alpha_0\,; \quad \lim_{s \to +\infty} \frac{|f(x,s)|}{\exp\{2e^{\alpha s^2}\}} = +\infty, \quad \forall \alpha < \alpha_0 \tag{1.7}$$

Remark 1. Note that, if there exist $\alpha_0 > 0$ and $\beta_0 > 0$ such that

$$\lim_{|t| \to +\infty} \frac{|f(t)|}{\exp\{\ \beta_0 e^{\alpha t^2}\ \}} = \begin{cases} +\infty & \text{for } 0 < \alpha < \alpha_0 \\ \\ 0 & \text{for } \alpha > \alpha_0 \end{cases}$$

then

$$\text{for all} \quad \beta > 0 \ : \qquad \lim_{|t|\to+\infty} \frac{|f(t)|}{\exp\{\ \beta e^{\alpha t^2}\ \}} = \begin{cases} +\infty & \text{for } 0 < \alpha < \alpha_0 \\ \\ 0 & \text{for } \alpha > \alpha_0 \end{cases}$$

Therefore the choice of $\beta > 0$ in the previous formulas has no influence. For convenience, we choose $\beta = 2$.

Problems of the above type in the non weighted case have been widely studied by several authors, see. e.g. [1],[2], [7], [8], [9], [12], [13], [14], [18], [21]. To our knowledge, the case with weighted Sobolev norm has not been considered, except for some particular applications of the above inequalities which can be found in [4], [6].

More precisely, in this paper we consider nonlinearities which have subcritical and critical growth in the sense of (1.6) and (1.7). We make the following assumptions throughout this paper:

(E1) $f : B \times \mathbb{R} \to \mathbb{R}$ is continuous, radial in x, and $f(x,t) = 0$ for $t \leq 0$

(E2) There exist $t_0 > 0$ and $M > 0$ such that

$$0 < F(x,t) = \int_0^t f(x,s)\ ds \leq M|f(x,t)|\ , \quad \text{for all} \quad t \geq t_0,\ \forall x \in B$$

(E3) $0 < F(x,t) \leq \dfrac{1}{2} f(x,t)t\ , \qquad \forall\, t > 0, \quad \forall x \in B$

(E4) $\displaystyle\limsup_{t\to 0} \frac{2F(x,t)}{t^2} < \lambda_1$, uniformly in $x \in B$, where $\lambda_1 > 0$ is the first eigenvalue associated to $(L, H^1_{0,rad}(B,w))$

(E5) $\displaystyle\lim_{t\to+\infty} \frac{f(x,t)t}{\exp\{2e^{\alpha_0 t^2}\}} \geq \gamma_0$ (uniformly in x) , with $\displaystyle\gamma_0 > \frac{1}{\alpha_0 e^2}$

Remark 2. Apparently, Remark 1 seems to downplay the importance of the critical exponent $\beta_0 = 2$. However, its crucial role is restored by assumption $(E5)$ which is the counterpart, in our weighted framework, of the growth condition [9, (H7)].

We prove the following results

Theorem 1.1. *Assume* $(E1)$, $(E2)$, $(E3)$, $(E4)$ *and that* f *has a subcritical growth at* $+\infty$. *Then problem (1.1) has a nontrivial radial solution.*

Theorem 1.2. *Assume* $(E1)$, $(E2)$, $(E3)$, $(E4)$, $(E5)$ *and that* f *has a critical growth at* $+\infty$. *Then problem (1.1) has a nontrivial radial solution.*

Example 1. Let $f = F'$, where $F(t) = k\left(e^{2(e^{t^2}-1)} - 1\right)$ for $t \geq 0$, and $k < \dfrac{\lambda_1}{4}$. Then f satisfies the hypotheses of Theorem 1.2, with $\alpha_0 = 1$.

The proof in both cases uses a variational approach and follows the schemes of [9]. In order to apply the classical Mountain Pass theorem of Ambrosetti-Rabinowitz, we first need to prove some geometric estimates. Theorem 1.1 then follows in a standard way, since we can prove compactness due to the subcritical growth. The proof of Theorem 1.2 is more delicate, since compactness is lost due to the critical growth of the nonlinearity. First, we show that the non-compactness levels of the functional are "quantized", i.e. they occur at discrete levels, and we determine the first such level. Then we prove, using a logarithmic concentrating sequence (*Moser sequence*) that the mountain-pass level of the functional avoids this non-compactness level, which requires new and delicate estimates. In this step the crucial assumption $(E5)$ is used.

We recall that in [10] and [11] (see also [15]) existence of solutions for elliptic equations with TM type nonlinearities were proved in the (slightly) supercritical regime, using the Lyapunov-Schmidt reduction method. We expect that with such methods solutions may be found also for the double exponential equation (1.1) in *non-symmetric domains* $\Omega \subset \mathbb{R}^2$, near the non-compactness level.

2. Preliminaries on the variational formulation

Let us consider the space $H := H^1_{0,rad}(B, w)$ endowed with the norm

$$\|u\| = \left(\int_B |\nabla u|^2 \log \frac{e}{|x|}\, dx\right)^{1/2}, \quad \forall u \in H,$$

and note that $\|\cdot\|$ comes from the inner product

$$\langle u, v\rangle = \int_B \nabla u \cdot \nabla v \, \log \frac{e}{|x|}\, dx\,, \quad \forall u, v \in H\,.$$

Let $\Phi : \; H \to \mathbb{R}$ be the functional defined by

$$\Phi(u) := \frac{1}{2}\|u\|^2 - \int_B F(x,u)\, dx\,, \quad \forall u \in H\,. \tag{2.1}$$

This functional is of class C^1, since the hypothesis on the growth of f (subcritical or critical growth) ensures the existence of positive constants c and C such that

$$|f(x,t)| \leq C \exp\{e^{c\,t^2}\}, \quad \forall x \in B,\ \forall t \in \mathbb{R} \tag{2.2}$$

First, we prove that the functional Φ has a mountain-pass geometry.

Proposition 2.1. *(Local minimum in $u = 0$). Assume $(E1)$, $(E2)$, $(E3)$ and $(E4)$. Then there exist $a > 0$ and $\rho > 0$ such that*

$$\Phi(u) \geq a \quad \forall u: \ \|u\| = \rho.$$

Proof. From $(E4)$, we have the existence of $\varepsilon_0 \in (0,1)$ and $\delta_0 > 0$ such that

$$F(x,t) \leq \frac{1}{2}\lambda_1(1-\varepsilon_0)t^2 \quad \text{for } |t| \leq \delta_0.$$

From (2.2), we obtain that for $q > 2$ there exists a constant $C_1 > 0$ such that

$$F(x,t) \leq C_1|t|^q e^{e^{\,ct^2}}, \quad \forall\ |t| \geq \delta_0,$$

so that

$$F(x,t) \leq \frac{1}{2}\lambda_1(1-\varepsilon_0)t^2 + C_1|t|^q e^{e^{\,ct^2}}, \quad \forall\ t \in \mathbb{R}\,.$$

Therefore, since $\lambda_1\|u\|_2^2 \leq \|u\|^2$,

$$\begin{aligned}\Phi(u) &\geq \frac{1}{2}\varepsilon_0\|u\|^2 - C_1\int_B e^{e^{\,cu^2}}|u|^q\,dx \\ &\geq \frac{1}{2}\varepsilon_0\|u\|^2 - C_1\Big(\int_B e^{2e^{\,cu^2}}\,dx\Big)^{1/2}\Big(\int_B |u|^{2q}\,dx\Big)^{1/2}\end{aligned}$$

Now, one can choose $\sigma > 0$ such that $c\,\sigma^2 \leq 2\pi$ so that

$$\int_B e^{2e^{\,cu^2}}\,dx = \int_B e^{2e^{\,c\|u\|^2\left(\frac{u}{\|u\|}\right)^2}}\,dx \leq C_2\,, \quad \forall u \in H \text{ with } \|u\| = \sigma,$$

in virtue of the weighted TM-inequality (1.5). Moreover, since $\|u\|_{2q} \leq C_3\|u\|$ (which follows easily by the radial Lemma in [4]), one has

$$\Phi(u) \geq \frac{1}{2}\varepsilon_0\|u\|^2 - C_4\|u\|^q\,, \quad \forall u \in H \text{ with } \|u\| = \sigma,$$

provided $\sigma > 0$ satisfies $c\,\sigma^2 \leq 2\pi$.

Finally, choose $\rho > 0$ as the point where $g(\sigma) = \frac{1}{2}\varepsilon_0\sigma^2 - C_4\sigma^q$ assumes its maximum on $[0, \sqrt{2\pi/c}\,]$ and $a = \Phi(\rho)$. □

To complete the geometric requirements of the Mountain Pass Theorem we need the following Proposition:

Proposition 2.2. *Suppose $(E1)$ and $(E2)$ hold. Then there exists $\bar{u}$, with $\|\bar{u}\| > \rho$, such that $\Phi(\bar{u}) < 0$.*

Proof. Let $u_0 \in H \cap L^\infty(B)$ such that $\|u_0\|_\infty = 1$. Let us define

$$\psi(t) := \Phi(tu_0) = \frac{t^2}{2}\|u_0\|^2 - \int_B F(x,tu_0)\,dx.$$

From $(E1)$ and $(E2)$, by integration, there exists a constant C such that

$$F(x,t) \geq Ce^{\frac{1}{M}|t|}, \quad \forall |t| \geq t_0.$$

In particular, for $p > 2$, there exists C such that

$$F(x,t) \geq C|t|^p - C, \quad \forall t \in \mathbb{R},\ x \in B.$$

Therefore

$$\psi(t) \leq \frac{t^2}{2}\|u_0\| - |t|^p\|u_0\|_p - C_3 \to -\infty, \text{ for } t \to +\infty,$$

and the result easily follows. □

3. The compactness levels.

In this section we prove a compactness result (Proposition 3.2). We first need a Lions' type result [17] concerning an improved TM-inequality for weakly convergent sequences, adapted to the double exponential case.

Lemma 3.1. *Let $\{u_k\}_k \in H$ be such that $\|u_k\| = 1$. If $u_k \rightharpoonup u$ in H and $u \neq 0$ then*

$$\sup_k \int_B e^{2e^{2\pi p\, u_k^2}}\, dx < +\infty$$

for any $1 < p < P$ where

$$P := \begin{cases} \dfrac{1}{1 - \|u\|^2} & \text{if } \|u\| < 1 \\ +\infty & \text{if } \|u\| = 1 \end{cases}$$

Proof. First, we recall the following elementary inequality

$$e^{2e^{a+b}} \leq \frac{1}{q}\, e^{2e^{qa}} + \frac{1}{q'}\, e^{2e^{q'b}} \quad \forall a,\ b \in \mathbb{R},\ q > 1 \text{ and } \frac{1}{q} + \frac{1}{q'} = 1 \tag{3.1}$$

which is a direct consequence of Young's inequality. In fact,

$$e^{2e^{a+b}} = e^{2e^a e^b} \leq e^{2\left(\frac{1}{q}\, e^{qa} + \frac{1}{q'}\, e^{q'b}\right)} = e^{\frac{2}{q}\, e^{qa}} e^{\frac{2}{q'}\, e^{q'b}} \leq \frac{1}{q}\, e^{2e^{\,qa}} + \frac{1}{q'}\, e^{2e^{\,q'b}}$$

Also, we may estimate

$$u_k^2 = (\, u_k - u + u\,)^2 \leq (1+\varepsilon)(u_k - u)^2 + \left(1 + \frac{1}{\varepsilon}\right)u^2 \quad \forall \varepsilon > 0 \tag{3.2}$$

Therefore, for any $p > 1$, we have

$$\int_B e^{2e^{2\pi p u_k^2}}\, dx \leq \frac{1}{q}\int_B e^{2e^{2\pi pq(1+\varepsilon)(u_k-u)^2}}\, dx + \frac{1}{q'}\int_B e^{2e^{2\pi pq'\left(1+\frac{1}{\varepsilon}\right)u^2}}\, dx$$

where we used (3.2) and (3.1). Since by (1.4)

$$\int_B e^{2e^{2\pi pq'\left(1+\frac{1}{\varepsilon}\right)u^2}}\, dx < +\infty\,,$$

the proof is complete if we show that when $1 < p < P$ then

$$\sup_k \int_B e^{2e^{2\pi pq(1+\varepsilon)(u_k-u)^2}}\, dx < +\infty \tag{3.3}$$

for some $\varepsilon > 0$ and $q > 1$.

We assume $\|u\| < 1$, the proof in the case $\|u\| = 1$ is similar. When $\|u\| < 1$, for

$$p < P = \frac{1}{1 - \|u\|^2}$$

there exists $\delta > 0$ such that

$$p(1 - \|u\|^2)(1 + \delta) < 1 \tag{3.4}$$

We have

$$\lim_{k\to+\infty} \|u_k - u\|^2 = 1 + \|u\|^2 - 2 \lim_{k\to+\infty} \langle u_k, u\rangle = 1 - \|u\|^2$$

and for any $\varepsilon > 0$ there exists $\overline{k} = \overline{k}(\varepsilon) \geq 1$ such that

$$\|u_k - u\|^2 \leq (1 + \varepsilon)(1 - \|u\|^2) \quad \forall k \geq \overline{k}$$

Then, for $q = 1 + \varepsilon$ with $\varepsilon := \sqrt[3]{1 + \delta} - 1 > 0$, and for any $k \geq \overline{k}$

$$pq(1 + \varepsilon)\|u_k - u\|^2 \leq p(1 + \varepsilon)^3(1 - \|u\|^2) = p(1 + \delta)(1 - \|u\|^2) < 1$$

Thus

$$\begin{aligned}\int_B e^{2e^{2\pi pq(1+\varepsilon)(u_k-u)^2}}\,dx &= \int_B e^{2e^{2\pi pq(1+\varepsilon)\|u_k-u\|^2\left(\frac{u_k-u}{\|u_k-u\|}\right)^2}}\,dx \\ &\leq \sup_{v\in H,\,\|v\|\leq 1} \int_B e^{2e^{2\pi v^2}}\,dx < +\infty \quad \forall k \geq \overline{k}\end{aligned}$$

and this yields (3.3) in the case $\|u\| < 1$. □

Next, we identify the first non-compactness level of the functional Φ.

Proposition 3.2. *The functional Φ satisfies the Palais-Smale condition at level c, i.e. the $(PS)_c$-condition, for any $c < \frac{\pi}{\alpha_0}$.*

Proof. Let $\{u_n\}_n \subset H$ be a $(PS)_c$ – sequence, i.e.

$$\Phi(u_n) = \frac{1}{2}\|u_n\|^2 - \int_B F(x, u_n)\,dx \to c, \;\; n \to +\infty \tag{3.5}$$

and

$$|\Phi'(u_n) \cdot v| = \left|\langle u_n, v\rangle - \int_B f(x, u_n)v\,dx\,\right| \leq \varepsilon_n\|v\|, \;\; \forall v \in H \tag{3.6}$$

where $\varepsilon_n \to 0$, as $\; n \to +\infty$.

From $(E2)$ it follows that for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$F(x, t) \leq \varepsilon t f(x, t), \quad \forall\; |t| \geq t_\varepsilon, \text{ uniformly in } x \in B. \tag{3.7}$$

Then, for any $\varepsilon > 0$,

$$\begin{aligned}\frac{1}{2}\|u_n\|^2 &\le C + \int_B F(x,u_n)\ dx \\ &\le C + \int_{\{|u_n|\le t_\varepsilon\}} F(x,u_n)\ dx + \varepsilon \int_B\ f(x,u_n)u_n\ dx \\ &\le C_0 + \varepsilon\,\varepsilon_n\|u_n\| + \varepsilon\|u_n\|^2\end{aligned}$$

where we also used (3.5) in the first inequality and (3.6) in the last one. Therefore

$$\Big(\frac{1}{2} - \varepsilon\Big)\|u_n\|^2 \le C_0 + \varepsilon\,\varepsilon_n\|u_n\|\ ,$$

so that $\{u_n\}$ is bounded in H, and hence $u_n \to u$ weakly in H and strongly in $L^q(B)$ for any $q \ge 1$.

Moreover, from (3.5), (3.6) and (3.8)

$$(a)\quad \Big|\int_B f(x,u_n)u_n\ dx\Big| \le \varepsilon_n\|u_n\| + \|u_n\|^2 \le C$$

$$(b)\quad \Big|\int_B F(x,u_n)\ dx\Big| \le C_0 + \varepsilon\Big|\int_B f(x,u_n)u_n\ dx\Big| \le C$$

and applying Lemma 2.1 of [9]

$$f(x,u_n) \to f(x,u), \quad \text{in} \quad L^1(B) \quad \text{as} \quad n \to +\infty, \tag{3.8}$$

Since

$$0 < F(x,t) \le M|f(x,t)|\ , \quad \forall |t| \ge t_0, \text{ uniformly in } x,$$

we may apply the generalized Lebesgue dominated convergence Theorem to conclude that

$$F(x,u_n) \to F(x,u) \quad \text{in} \quad L^1(B) \quad \text{as} \quad n \to +\infty. \tag{3.9}$$

From (3.5) we have

$$\lim_{n\to\infty} \frac{1}{2}\|u_n\|^2 = c + \int_B F(x,u)\ dx$$

and by (E3) and (3.6)

$$\lim_{n\to\infty} 2\int_B F(x,u_n)\ dx \le \lim_{n\to\infty}\int_B f(x,u_n)u_n\ dx = 2\Big(c + \int_B F(x,u)\ dx\Big) \tag{3.10}$$

so that $c \ge 0$. Moreover, from (3.6) and (3.8),

$$\int_B \nabla u \cdot \nabla v\ \log\frac{e}{|x|}\ dx = \int_B f(x,u)v\ dx \quad \forall v \in C_0^\infty(B) \tag{3.11}$$

and u is a weak (and strong, via standard regularity results) solution of the problem. In particular, testing the equation with $v = u$,

$$\int |\nabla u|^2 \log \frac{e}{|x|}\, dx = \int f(x,u)u\, dx \geq 2 \int F(x,u), \text{ which implies } \Phi(u) \geq 0.$$

We will distinguish three cases according to

(I) $c = 0$

(II) $c > 0,\ u = 0$

(III) $c > 0,\ u > 0$

Case (I): $c = 0$. We prove that $u_n \to u$ strongly in H. Indeed,

$$\begin{aligned} 0 &\leq \Phi(u) = \frac{1}{2}\|u\|^2 - \int_B F(x,u)\, dx \\ &\leq \liminf_{n\to\infty} \frac{1}{2}\|u_n\|^2 - \int_B F(x,u)\, dx = \liminf_{n\to\infty} \Phi(u_n) = 0 \end{aligned}$$

which implies that $\frac{1}{2}\|u\|^2 = \int_B F(x,u)\, dx$ and $\|u_n\| \to \|u\|$.

Case (II): $c > 0$, $u = 0$. In this case, (3.5) and (3.6) read as

$$\lim_{n\to\infty} \|u_n\|^2 = 2c \quad \text{and} \quad \lim_{n\to\infty} \int_B f(x,u_n)u_n\, dx = 2c$$

We show that this case cannot happen.

We begin by assuming that there exists $q > 1$ such that

$$\int_B |f(x,u_n)|^q\, dx \leq C. \tag{3.12}$$

Then using (3.6) with $v = u_n$

$$\left| \|u_n\|^2 - \int f(x,u_n)u_n dx \right| \leq C\varepsilon_n$$

which implies

$$\|u_n\|^2 \leq C\varepsilon_n + \left(\int_B |f(x,u_n)|^q dx \right)^{1/q} \left(\int |u_n|^{q'} \right)^{1/q'} \to 0, \quad \text{as } n \to +\infty,$$

where q and q' are conjugate exponents with $q > 1$. The strong convergence $u_n \to 0$ in H leads to $c = 0$, which contradicts the assumption $c > 0$.

So it remains to prove (3.12). Since f has a subcritical or critical growth, for every $\varepsilon > 0$ and $q > 1$ there exist $t_\varepsilon > 0$ and $C_{\varepsilon,q} > 0$ such that

$$|f(x,t)|^q \leq C_{\varepsilon,q} \exp\left\{2e^{\alpha_0(1+\varepsilon)t^2}\right\} \quad , \ \forall\ |t| \geq t_\varepsilon, \ \text{ uniformly in } x\,. \tag{3.13}$$

Therefore

$$\begin{aligned}\int_B |f(x,u_n)|^q\ dx \ &= \int_{\{|u_n|\leq t_\varepsilon\}} |f(x,u_n)|^q\ dx + \int_{\{|u_n|>t_\varepsilon\}} |f(x,u_n)|^q dx \\ &\leq \pi \max_{B\times[-t_\varepsilon,t_\varepsilon]} |f(x,s)|^q + C_{\varepsilon,q}\int_B \exp\left\{2e^{\alpha_0(1+\varepsilon)u_n^2}\right\} dx\end{aligned}$$

We have to estimate this second integral.

Since $2c < \frac{2\pi}{\alpha_0}$, there exists $\delta \in (0,1/2)$ such that $2c = (1-2\delta)\frac{2\pi}{\alpha_0}$. Therefore, since $\|u_n\|^2 \to 2c$, there exists n_δ such that $\|u_n\|^2 \leq (1-\delta)\frac{2\pi}{\alpha_0}$, $n \geq n_\delta$, and $\alpha_0(1+\varepsilon)\|u_n\|^2 \leq (1+\varepsilon)(1-\delta)2\pi$. Now, choosing $\varepsilon > 0$ sufficiently small, we see that $\alpha_0(1+\varepsilon)\|u_n\|^2 \leq 2\pi$. The integral which we have to estimate can be rewritten as

$$\int_B \exp\left\{2e^{\alpha_0(1+\varepsilon)\|u_n\|^2\left(\frac{u_n}{\|u_n\|}\right)^2}\right\}\ dx \leq \int_B \exp\left\{2e^{2\pi\left(\frac{u_n}{\|u_n\|}\right)^2}\right\}\ dx$$

which is uniformly bounded in view of the weighted TM-inequality (1.5).

Case (III): $c > 0$ and $u \neq 0$. We will prove that $\Phi(u) = c$ and this yields

$$\lim_{n\to+\infty} \|u_n\|^2 = 2\Big(c + \int F(x,u)\ dx\Big) = 2\Big(\Phi(u) + \int F(x,u)\,dx\Big) = \|u\|^2$$

To prove that $\Phi(u) = c$, note that

$$\Phi(u) \leq \frac{1}{2}\liminf_{n\to\infty} \|u_n\|^2 - \int_B F(x,u)\,dx = c$$

We argue by contradiction assuming that $\Phi(u) < c$, or equivalently

$$\|u\|^2 < 2\Big(c + \int_B F(x,u)\,dx\Big) \tag{3.14}$$

Let $v_n = \frac{u_n}{\|u_n\|}$ and

$$v = \frac{u}{\sqrt{2\big(c + \int_B F(x,u)dx\big)}}$$

Since $v_n \rightharpoonup v$ weakly in H, $v \neq 0$, $\|v_n\| = 1$ and $\|v\| < 1$, we may apply the Lions-type Lemma (Lemma 3.1) to obtain

$$\sup_n \int_B e^{2e^{p2\pi v_n^2}} dx < +\infty, \quad \text{for } \ 1 < p < \frac{1}{1-\|v\|^2}$$

Next, we estimate the L^q–norm of $\{f(x,u_n)\}$ with $q>1$. We have, using (3.13) as above,

$$\begin{aligned}\int_B |f(x,u_n)|^q &= \int_{\{|u_n|\le t_\varepsilon\}} |f(x,u_n)|^q + \int_{\{|u_n|>t_\varepsilon\}} |f(x,u_n)|^q dx \\ &\le \tilde{C}_\varepsilon + C_{\varepsilon,q}\int_B \exp\left\{2e^{\alpha_0(1+\varepsilon)\|u_n\|^2 v_n^2}\right\}\, dx < C\end{aligned}$$

provided that $\alpha_0(1+\varepsilon)\|u_n\|^2 < 2\pi p$, for some $1<p<\frac{1}{1-\|v\|^2}$. Indeed, note that by the definition of v

$$\frac{1}{1-\|v\|^2} = \frac{2(c+\int_B F(x,u)dx)}{2c+2\int_B F(x,u)dx - \|u\|^2} = \frac{c+\int_B F(x,u)dx}{c-\Phi(u)}$$

Since

$$\lim_{n\to+\infty}\|u_n\|^2 = 2\Big(c+\int_B F(x,u)dx\Big)$$

we have

$$\frac{\alpha_0}{2\pi}(1+\varepsilon)\|u_n\|^2 \le \frac{\alpha_0}{2\pi}(1+2\varepsilon)\, 2\left(c+\int_B F(x,u)dx\right)$$

and, to obtain the desired estimate, it is enough to show that we can choose $\varepsilon>0$ sufficiently small such that

$$\frac{\alpha_0}{2\pi}(1+2\varepsilon)2 < \frac{1}{c-\Phi(u)}\ ,$$

that is

$$(1+2\varepsilon)(c-\Phi(u)) < \frac{\pi}{\alpha_0} \tag{3.15}$$

Since $\Phi(u)\ge 0$, and $c<\frac{\pi}{\alpha_0}$, it is indeed possible to choose $\varepsilon>0$ such that (3.15) is valid.

From the boundedness of $\{f(x,u_n)\}$ in $L^q(B)$ for $q>1$, we will deduce that $u_n\to u$ strongly in H. In fact

$$\|u_n-u\|^2 = \langle u_n, u_n-u\rangle - \langle u, u_n-u\rangle = \langle u_n, u_n-u\rangle + o(1)$$

Moreover, from (3.6)

$$\begin{aligned}|\langle u_n, u_n-u\rangle| &\le \varepsilon_n\|u_n-u\| + \Big|\int_B f(x,u_n)(u_n-u)\ dx\Big| \\ &\le C\varepsilon_n + \Big(\int_B |f(x,u_n)|^q dx\Big)^{1/q}\|u_n-u\|_{q'} \to 0,\ \text{as } n\to+\infty\end{aligned}$$

Therefore

$$2\Big(c+\int_B F(x,u)\ dx\Big) = \lim_{n\to+\infty}\|u_n\|^2 = \|u\|^2$$

which contradicts (3.14). $\square$

4. Proof of Theorems 1.1 and 1.2.

The proof in the subcritical case follows easily from an application of the Mountain Pass Theorem, since in this case the functional satisfies the $(PS)_c$-condition for all $c \in \mathbb{R}$.

To conclude the proof of Theorem 1.2 we have to show that the mountain-pass level c satisfies $c < \frac{\pi}{\alpha_0}$, and for this it is sufficient to show that there exists $w \in H$, with $\|w\| = 1$, such that

$$\max\{\Phi(tw): \quad t \geq 0\} < \frac{\pi}{\alpha_0}$$

To this aim, we consider the functions $w_k = w_k(x)$ defined by means of the identity

$$\psi_k(t) =: \sqrt{2\pi}\, w_k(x), \quad \text{with} \quad |x| = e^{-t}$$

where $\{\psi_k\}_k$ is the Moser-type sequence introduced in [4]. More precisely,

$$\psi_k(t) = \begin{cases} \dfrac{\log(1+t)}{\log^{1/2}(1+k)} & , \quad 0 \leq t \leq k \\ \log^{1/2}(1+k) & , \quad t \geq k \end{cases} \tag{4.1}$$

Then

$$\text{(i)} \qquad \|w_k\|^2 = \int_0^{+\infty} |\psi_k(t)|^2 (1+t) dt = 1$$

$$\text{(ii)} \qquad \lim_{k\to+\infty} \int_B \exp\{\, 2e^{2\pi w_k^2}\,\}\, dx = \lim_{k\to+\infty} 2\pi \int_0^{+\infty} \exp\{\, 2e^{\psi_k^2} - 2t\,\}\, dt$$

that is $w_k(t)$ is a normalized sequence which tends pointwise and weakly to zero in H (and likewise for $\psi_k(s)$), and which blows up in 0 (at $+\infty$, respectively). In the following Lemma 4.1 we calculate the limit in (ii), which will be crucial in the proof of Theorem 1.2. It also shows that the integral in (ii) (and hence in (1.5)) is not weakly continuous in 0, since in $w = 0$ the value of the integral in (ii) is πe^2, while the following Lemma 4.1 shows that the limit in (ii) is $3\pi e^2$.

Lemma 4.1. *Let* ψ_k *as in* (4.1). *Then*

$$\lim_{k\to+\infty} \int_0^{+\infty} \exp\{\, 2e^{\psi_k^2} - 2t\,\}\, dt = \frac{3}{2}\, e^2 \tag{4.2}$$

Proof. Performing the changes of variable $s = 1 + t$ and $j = k + 1$, we may rewrite

$$\begin{aligned}\int_0^{+\infty} \exp\Big\{\, 2e^{\psi_k^2} - 2t \,\Big\}\, dt &= \frac{e^2}{2} + \int_0^k \exp\Big\{\, 2(1+t)^{\frac{\log(1+t)}{\log(1+k)}} - 2t \,\Big\}\, dt \\ &= \frac{e^2}{2} + \int_1^{k+1} \exp\Big\{\, 2s^{\frac{\log s}{\log(1+k)}} - 2(s-1) \,\Big\}\, ds \\ &= \frac{e^2}{2} + e^2 \int_1^{j} \exp\Big\{\, 2s^{\frac{\log s}{\log j}} - 2s \,\Big\}\, ds\end{aligned}$$

and, to complete the proof of (4.2), we need to show that

$$\lim_{j\to+\infty} \int_1^{j} \exp\Big\{\, 2s^{\frac{\log s}{\log j}} - 2s \,\Big\}\, ds = 1$$

For any fixed $j > 4$, let $\eta_j(s) := 2s^{\frac{\log s}{\log j}} - 2s$ with $s \geq 1$, and split the interval $[1, j]$ as

$$[1, j] = [1, \sqrt{j}] \cup [\sqrt{j}, j - \sqrt{j}] \cup [j - \sqrt{j}, j]$$

Since

$$\chi_{[1,\sqrt{j}]}(s)\, e^{\eta_j(s)} \leq e^{2\sqrt{s}-2s} \in L^1([1, +\infty))$$

and

$$\chi_{[1,\sqrt{j}]}(s)\, e^{\eta_j(s)} \to e^{2-2s} \quad \text{for a.e. } s \in [1, +\infty), \text{ as } j \to +\infty,$$

the Lebesgue dominated convergence theorem yields

$$\lim_{j\to+\infty} \int_1^{\sqrt{j}} \exp\Big\{\, 2s^{\frac{\log s}{\log j}} - 2s \,\Big\}\, ds = \lim_{j\to+\infty} \int_1^{+\infty} \chi_{[1,\sqrt{j}]}(s)\, e^{\eta_j(s)}\, ds = \frac{1}{2} \tag{4.3}$$

Next, we will study the limit of the integrals on $[\sqrt{j}, j - \sqrt{j}]$ and $[j - \sqrt{j}, j]$, respectively. To this aim, we begin by computing

$$\eta_j(\sqrt{j}) = -2\sqrt{j}\Big(\, 1 - \frac{1}{\sqrt[4]{j}} \,\Big)$$

and we observe that

$$\eta_j(\sqrt{j}) \leq -\sqrt{j} \quad , \quad \forall j \geq 2^4 \tag{4.4}$$

Also,

$$\begin{aligned}\eta_j(j - \sqrt{j}) &= 2\exp\Big\{\, \frac{1}{\log j}\Big[\, \log j + \log\Big(\, 1 - \frac{1}{\sqrt{j}} \,\Big) \,\Big]^2 \,\Big\} - 2(j - \sqrt{j}) \\ &= 2\exp\Big\{\, \log j - \frac{2}{\sqrt{j}} - \frac{1}{j} + o\Big(\frac{1}{j}\Big) \,\Big\} - 2(j - \sqrt{j}) \\ &= 2j\Big[\, \exp\Big\{\, -\frac{2}{\sqrt{j}} - \frac{1}{j} + o\Big(\frac{1}{j}\Big) \,\Big\} - 1 \,\Big] + 2\sqrt{j}\end{aligned}$$

and we conclude that for any $\varepsilon \in (0, 1)$ there exists $j_\varepsilon \geq 1$ such that

$$\eta_j(j - \sqrt{j}) \leq -2(1 - \varepsilon)\sqrt{j} \quad , \quad \forall j \geq j_\varepsilon \tag{4.5}$$

Let us fix $j \geq 1$ and assume j is sufficiently large. A qualitative study of η_j on $[1, +\infty)$ shows that there exists a unique $s_j \in (1, j)$ such that $\eta_j'(s_j) = 0$ and hence

$$\int_{\sqrt{j}}^{j-\sqrt{j}} e^{\eta_j(s)}\, ds \leq (j - 2\sqrt{j})\, e^{\max\{\, \eta_j(\sqrt{j}),\, \eta_j(j-\sqrt{j})\,\}}$$

Moreover, from (4.4) and (4.5) with $\varepsilon = \frac{1}{2}$, we deduce

$$\max\Big\{\ \eta_j(\sqrt{j}),\ \eta_j(j - \sqrt{j})\ \Big\} \leq -\sqrt{j}$$

provided j is sufficiently large. Therefore, there exists $\overline{j} \geq 1$ such that

$$\int_{\sqrt{j}}^{j-\sqrt{j}} e^{\eta_j(s)}\, ds \leq (j - 2\sqrt{j}) e^{-\sqrt{j}} \quad , \quad \forall j \geq \overline{j}$$

and hence

$$\lim_{j\to+\infty} \int_{\sqrt{j}}^{j-\sqrt{j}} \exp\Big\{\ 2s^{\frac{\log s}{\log j}} - 2s\ \Big\}\, ds = 0 \tag{4.6}$$

Next, note that for fixed $j \geq 1$ sufficiently large, η_j is convex on $[j - \sqrt{j}, +\infty)$. Using this and the fact that $\eta_j(j) = 0$, we may estimate

$$\eta_j(s) \leq \frac{j - s}{\sqrt{j}}\, \eta_j(j - \sqrt{j})\,, \quad s \in [j - \sqrt{j}, j]$$

In view of (4.5), if $\varepsilon \in (0, 1)$ and $j \geq j_\varepsilon$, we have

$$\eta_j(s) \leq 2(1 - \varepsilon)(s - j)\,, \quad s \in [j - \sqrt{j}, j] \tag{4.7}$$

On the other hand, the convexity of η_j on $[j - \sqrt{j}, +\infty)$ and the fact that $\eta_j'(j) = 2$ yields

$$\eta_j(s) \geq \eta_j(j) + \eta_j'(j)(s - j) = 2(s - j)\,, \quad s \in [j - \sqrt{j}, j] \tag{4.8}$$

Hence, combining (4.7) and (4.8), we get

$$\frac{1}{2} \leq \lim_{j\to+\infty} \int_{j-\sqrt{j}}^{j} e^{\eta_j(s)}\, ds \leq \frac{1}{2(1 - \varepsilon)}$$

and, since $\varepsilon \in (0, 1)$ is arbitrarily fixed, we conclude

$$\lim_{j\to+\infty} \int_{j-\sqrt{j}}^{j} \exp\Big\{\ 2s^{\frac{\log s}{\log j}} - 2s\ \Big\}\, ds = \frac{1}{2} \tag{4.9}$$

Joining (4.3), (4.6) and (4.9) we conclude. $\square$

Lemma 4.1 will play a crucial role to derive the following estimate

$$\max\{\Phi(tw_k) : \ \ t \geq 0\} < \frac{\pi}{\alpha_0} \quad \text{ for some } k \geq 1 \tag{4.10}$$

To prove (4.10), we argue by contradiction assuming

$$\max\{\Phi(tw_k):\ \ t\geq 0\}\geq \frac{\pi}{\alpha_0}\quad,\quad \forall k\geq 1$$

Then, for any $k\geq 1$, there exists $t_k>0$ satisfying

$$\frac{\pi}{\alpha_0}\leq \max_{t\geq 0}\Phi(tw_k)=\Phi(t_kw_k)=\frac{1}{2}t_k^2-\int_B F(x,t_kw_k)\,dx$$

and

$$0=\frac{d}{dt}\Phi(tw_k)\Big|_{t=t_k}=t_k-\int_B f(x,t_kw_k)w_k\,dx$$

Hence, we have

$$t_k^2\geq\frac{2\pi}{\alpha_0}\quad\text{ and }\quad t_k^2=\int_B f(x,t_kw_k)t_kw_k\,dx \tag{4.11}$$

The growth condition $(E5)$, i.e.

$$\lim_{t\to+\infty}\frac{f(x,t)t}{\exp\{2e^{\alpha_0t^2}\}}\geq\gamma_0,\ \ \text{uniformly in }\ x,\quad\text{with}\quad \gamma_0>\frac{1}{\alpha_0e^2}$$

will be be crucial to reach a contradiction. Note that from $(E5)$, for any fixed $\varepsilon>0$, we deduce the existence of $\bar t=\bar t(\varepsilon)>0$ such that

$$f(x,t)t\geq(\gamma_0-\varepsilon)\exp\Big\{\ 2e^{\alpha_0t^2}\ \Big\}\quad\forall|t|\geq\bar t,\ \ \text{uniformly in } x \tag{4.12}$$

First, $(E5)$ will enable us to show that the sequence $\{t_k\}_k$ is bounded. To this aim, using (4.11) and the definition of w_k, we estimate

$$t_k^2=\int_B f(x,t_kw_k)t_kw_k\,dx\geq 2\pi\int_k^{+\infty}f\Big(e^{-s},t_k\frac{\psi_k}{\sqrt{2\pi}}\Big)\,t_k\frac{\psi_k}{\sqrt{2\pi}}\,e^{-2s}\,ds$$

As a consequence of the definition of ψ_k and (4.11), we have

$$t_k\frac{\psi_k}{\sqrt{2\pi}}=t_k\left(\frac{\log(1+k)}{2\pi}\right)^{\frac12}\geq\left(\frac{\log(1+k)}{\alpha_0}\right)^{\frac12}$$

Therefore, for any $k\geq\overline{k}$ with $\overline{k}=\overline{k}(\varepsilon)\geq 1$ sufficiently large, we may apply (4.12) to deduce that

$$\begin{aligned}
t_k^2&\geq 2\pi\int_k^{+\infty}f\Big(e^{-s},t_k\frac{\psi_k}{\sqrt{2\pi}}\Big)\,t_k\frac{\psi_k}{\sqrt{2\pi}}\,e^{-2s}\,ds\\
&\geq 2\pi(\gamma_0-\varepsilon)\int_k^{+\infty}\exp\Big\{\ 2e^{\alpha_0\left(t_k\frac{\psi_k}{\sqrt{2\pi}}\right)^2}-2s\ \Big\}\,ds\\
&=\pi(\gamma_0-\varepsilon)\exp\Big\{\ 2e^{\frac{\alpha_0}{2\pi}t_k^2\log(1+k)}-2k\ \Big\}
\end{aligned}\tag{4.13}$$

In particular,

$$1\geq\pi(\gamma_0-\varepsilon)\exp\Big\{\ 2e^{\frac{\alpha_0}{2\pi}t_k^2\log(1+k)}-2k-2\log t_k\ \Big\}\quad,\quad\forall k\geq\overline{k}$$

and thus $\{t_k\}_k$ must be bounded.

Note also that, if

$$\lim_{k\to+\infty} t_k^2 > \frac{2\pi}{\alpha_0} \tag{4.14}$$

then (4.13) would yield a contradiction with the boundedness of $\{t_k\}_k$. Hence, (4.14) cannot hold and, this together with (4.11) gives

$$\lim_{k\to+\infty} t_k^2 = \frac{2\pi}{\alpha_0}$$

We will use this information to reach a contradiciton with condition (E5). For this reason, we introduce the sets

$$A_k := \left\{\, x \in B \mid t_k w_k(x) \geq \bar{t} \,\right\} \quad \text{and} \quad C_k := B \setminus A_k$$

where $\bar{t} = \bar{t}(\varepsilon) > 0$ is given by (4.12). By construction,

$$\begin{aligned}
t_k^2 &= \int_B f(x, t_k w_k) t_k w_k \, dx \\
&\geq (\gamma_0 - \varepsilon) \int_{A_k} \exp\Big\{\, 2e^{\alpha_0 t_k^2 w_k^2} \,\Big\}\, dx + \int_{C_k} f(x, t_k w_k) t_k w_k \, dx \\
&= (\gamma_0 - \varepsilon) \int_B \exp\Big\{\, 2e^{\alpha_0 t_k^2 w_k^2} \,\Big\}\, dx - (\gamma_0 - \varepsilon) \int_{C_k} \exp\Big\{\, 2e^{\alpha_0 t_k^2 w_k^2} \,\Big\}\, dx \\
&\quad + \int_{C_k} f(x, t_k w_k) t_k w_k \, dx
\end{aligned}$$

Since $w_k \to 0$ a.e. in B and $\chi_{C_k} \to 1$ a.e. in B, applying Lebesgue's dominated convergence theorem, we get

$$\frac{2\pi}{\alpha_0} = \lim_{k\to+\infty} t_k^2 \geq (\gamma_0 - \varepsilon) \lim_{k\to+\infty} \int_B \exp\Big\{\, 2e^{\alpha_0 t_k^2 w_k^2} \,\Big\}\, dx - (\gamma_0 - \varepsilon)\pi\, e^2$$

In view of (4.11) and Lemma 4.1, we have

$$\lim_{k\to+\infty} \int_B \exp\Big\{\, 2e^{\alpha_0 t_k^2 w_k^2} \,\Big\}\, dx \geq \lim_{k\to+\infty} \int_B \exp\Big\{\, 2e^{2\pi w_k^2} \,\Big\}\, dx = 3\pi e^2$$

and hence we conclude that

$$\frac{2\pi}{\alpha_0} \geq (\gamma_0 - \varepsilon) 2\pi e^2$$

But $\varepsilon > 0$ is arbitrarily fixed and we can pass to the limit as $\varepsilon \to 0$, reaching a contradiction with $(E5)$.

This concludes the proof of Theorem 1.2.

References

[1] Adimurthi, *Existence results for the semilinear Dirichlet problem with critical growth for the n-Laplacian*, Houston J. Math. **7**, (1991), 285–298.

[2] C. Alves, L. de Freitas, *Multiplicity of nonradial solutions for a class of quasilinear equations on annulus with exponential critical growth*, Topol. Methods Nonlinear Anal. **39** (2012), no. 2, 243–262.

[3] M. Calanchi, *"Some weighted inequalities of Trudinger-Moser Type"* in Analysis and Topology in Nonlinear Differential Equations, Progress in Nonlinear Differential Equations and Appl., Birkhauser, vol 85, (2014) 163–174.

[4] M. Calanchi, B. Ruf, *On Trudinger-Moser type inequalities with logarithmic weights*. J. Differential Equations 258 (2015), 1967–1989.

[5] M. Calanchi, B. Ruf *Trudinger-Moser Type Inequalities with Logarithmic Weights in dimension N*, Nonlinear Analysis, Series A: Theory, Methods and Applications,121 (2015) 403–411.

[6] M. Calanchi, B. Ruf *Weighted Trudinger-Moser inequalities and Applications*, Bulletin of the South Ural State University. Ser. Mathematical Modelling, programming and Computer Software, (2015), vol. 8, no. 3, 42–55

[7] M. Calanchi, E. Terraneo, *Non-radial Maximizers For Functionals With Exponential Nonlinearity in $\mathbb{R}^2$* Advanced Nonlinear Studies **5** (2005); 337–350

[8] D.G. de Figueiredo, J.M. do Ó, and B. Ruf, *On an inequality by N. Trudinger and J. Moser and related elliptic equations*, Comm. Pure Appl. Math. **55** (2) (2002); 135–152.

[9] D.G. de Figueiredo, O.H. Miyagaki, and B. Ruf, *Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range*, Calc. Var. Partial Differential Equations **3** (2) (1995); 139–153.

[10] M. del Pino, M. Musso, B. Ruf, *New solutions for TrudingerMoser critical equations in $\mathbb{R}^2$*, Journal of Functional Analysis **258** (2010); 421-457.

[11] M. del Pino, M. Musso, B. Ruf, *Beyond the TrudingerMoser supremum*, Calc. Var. Partial Differential Equations **44** (2012); 543-576.

[12] J.M. do Ó, M.A. Souto, *On a class of nonlinear Schrdinger equations in $\mathbb{R}^2$ involving critical growth*, J. Differential Equations **174** (2001), no. 2, 289-311.

[13] N. Lam, G. Lu, *Existence and multiplicity of solutions to equations of N -Laplacian type with critical exponential growth in $\mathbb{R}^N$*, J. Funct. Anal. **262**, (2012), no. 3, 1132-1165.

[14] N. Lam, G. Lu, *Elliptic equations and systems with subcritical and critical exponential growth without the Ambrosetti-Rabinowitz condition* J. Geom. Anal. **24** (2014), 118–143.

[15] T. Lamm, F. Robert, M. Struwe, *The heat flow with a critical exponential nonlinearity*, J. Funct. An.**257** (2009), 2951-2998

[16] A. Kufner, Weighted Sobolev spaces, John Wiley & Sons Ltd, 1985

[17] P.-L. Lions, *The Concentration-compactness principle in the Calculus of Variations, Part 1*, Revista Iberoamericana **11** (1985), 185-201

[18] N. Masmoudi, F. Sani, *Trudinger-Moser inequalities with the exact growth condition in $\mathbb{R}^N$ and applications*. Comm. Partial Differential Equations **40** (2015), no. 8, 1408–1440.

[19] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1970/71); 1077–1092.

[20] S. I. Pohozaev, *The Sobolev embedding in the case $pl = n$*, Proc. of the Technical Scientific Conference on Advances of Scientific Research 1964?-1965, Mathematics Section, (Moskov. Energet. Inst., Moscow), (1965), 158?-170.

[21] Y. Yang, *Existence of positive solutions to quasi-linear elliptic equations with exponential growth in the whole Euclidean space*, J. Funct. Anal. **262** (2012), no. 4, 1679-1704.

[22] N. S. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech., **17** (1967); 473–483.

Dipartimento di Matematica, Università degli Studi di Milano, via C. Saldini 50, 20133 Milano, Italy

E-mail address: marta.calanchi@unimi.it, bernhard.ruf@unimi.it, federica.sani@unimi.it