Mathematical Methods of Operations Research (2019) 90:1–22
https://doi.org/10.1007/s00186-018-0650-4

ORIGINAL ARTICLE

On f-domination: polyhedral and algorithmic results

Mauro Dell'Amico[1] · José Neto[2]

Received: 25 August 2017 / Accepted: 2 October 2018 / Published online: 20 October 2018

Abstract
Given an undirected simple graph G with node set V and edge set E, let f_v, for each node $v \in V$, denote a nonnegative integer value that is lower than or equal to the degree of v in G. An f-dominating set in G is a node subset D such that for each node $v \in V \backslash D$ at least f_v of its neighbors belong to D. In this paper, we study the polyhedral structure of the polytope defined as the convex hull of all the incidence vectors of f-dominating sets in G and give a complete description for the case of trees. We prove that the corresponding separation problem can be solved in polynomial time. In addition, we present a linear-time algorithm to solve the weighted version of the problem on trees: Given a cost $c_v \in \mathbb{R}$ for each node $v \in V$, find an f-dominating set D in G whose cost, given by $\sum_{v \in D} c_v$, is a minimum.

Keywords Domination · Polyhedral combinatorics · Tree · Linear-time algorithm

Mathematics Subject Classification 05C69 · 51M20 · 68Q25 · 68R10 · 90C10

1 Introduction

Let $G = (V, E)$ denote an undirected simple graph with node set $V = \{1, 2, \ldots, n\}$ and edge set E. An edge joining the nodes u and v is denoted by $\{u, v\}$. Given a node subset $S \subseteq V$, its *open neighborhood* is the set $N_G(S) = \{v \in V \backslash S \colon \exists u \in S \text{ such that } \{u, v\} \in E\}$, and its *closed neighborhood* is the set

This work was supported by EC-FP7 COST Action TD1207.

✉ José Neto
jose.neto@telecom-sudparis.eu

Mauro Dell'Amico
mauro.dellamico@unimore.it

1 Department of Sciences and Methods for Engineering, University of Modena and Reggio Emilia, Via Amendola 2, 42122 Reggio Emilia, Italy

2 Samovar, Telecom SudParis, CNRS, Université Paris-Saclay, 9 rue Charles Fourier, 91011 Evry, France

$N_G[S] = N_G(S) \cup S$. In case S consists of a single node $v \in V$ we may write $N_G(v)$ (resp. $N_G[v]$) instead of $N_G(\{v\})$ (resp. $N_G[\{v\}]$). Given some node $v \in V$, let d_v^G denote the degree of node v in G, i.e., $d_v^G = |N_G(v)|$. When G is clear from the context we shall more simply write d_v, and we may also omit G in the subscript of the notation used for neighborhoods as well. Let $c \in \mathbb{R}_+^n$ denote a vector of nonnegative node weights. Let $\mathcal{F}_G$ stand for the following set of vectors indexed on the nodes of G: $\{f \in (\mathbb{Z}_+)^n : 0 \leq f_v \leq d_v^G, v \in V\}$.

A node subset $D \subseteq V$ is called an *f-dominating set* if each node $v \in V \setminus D$ has at least f_v neighbors in D, i.e., $|N(v) \cap D| \geq f_v$. If, in addition, all the nodes in D (and thus all the nodes in V) also satisfy this inequality, then D is called a *total f-dominating set*. We will call f_v the *domination requirement* for node v.

The f-domination concept seems to appear first in the papers by Hedetniemi et al. (1985) and Stracke and Volkmann (1993). The concepts of f-domination and f-total domination generalize those of domination and total domination, respectively. The two latter correspond to the special case when $f_v = 1$, for all $v \in V$. (For further results related to the domination or total domination concepts and variants the reader may consult e.g. Haynes et al. (1998a), Haynes et al. (1998b), Henning and Yeo (2013), Bermudo et al. (2018) and Zhou (1996), and the references therein.) The authors previously investigated the polyhedral structures of polytopes related to the set of all total f-dominating sets in a graph (Dell'Amico et al. 2017). Here, we proceed to such investigations with respect to the f-dominating sets. Despite the apparent similarity of the two concepts, the polyhedral structures of the corresponding polytopes radically differ. And this also holds if we compare the polytopes corresponding, on the one hand, to the classical domination, and on the other, to the generalized f-domination concept we consider here.

In addition to a purely theoretical interest of these structures, the obtained results are relevant w.r.t. the problem which consists in determining a minimum weight f-dominating set in a graph G, where we define the weight of any node subset set $S \subseteq V$ as the quantity $\sum_{v \in S} c_v$. This problem, denoted by $[MW_f]$ is described hereafter together with an integer linear programming formulation.

$[MW_f]$ Minimum weight f-dominating set problem

Find a minimum weight f-dominating set of G, i.e., find a node subset $S \subseteq V$ such that S is an f-dominating set and the weight $\sum_{v \in S} c_v$ of S is minimum.

This problem may be formulated as the following integer program.

$$
(IP1) \begin{cases} \min \sum_{v \in V} c_v x_v \\ s.t. \\ \qquad f_v x_v + \sum_{u \in N(v)} x_u \geq f_v, \; v \in V, \\ \qquad x \in \{0, 1\}^n. \end{cases}
$$

With no loss of generality we may assume that all the weights (or costs) $(c_v)_{v \in V}$ are positive. If this is not the case, the nodes in V with a nonpositive weight can be fixed in any optimal solution. Let $\mathbf{1}$ stand for the n-dimensional all ones vector. The particular case when $f = c = \mathbf{1}$ corresponds to the classical minimum (cardinality) dominating set problem.

Given a node subset $S \subseteq V$, let $\chi^S \in \{0, 1\}^n$ denote its incidence vector: $\chi_v^S = 1$ if $v \in S$, and $\chi_v^S = 0$ otherwise. Let $\mathcal{D}_G^f$ denote the f-dominating set polytope, i.e., the convex hull in $\mathbb{R}^n$ of all the incidence vectors of the f-dominating sets in G. Then, problem $[MW_f]$ can be reformulated as the linear program: $\min\{c^t x : x \in \mathcal{D}_G^f\}$. Also, note that for any pair $(f, f') \in (\mathcal{F}_G)^2$ such that $f_v \leq f'_v$, for all $v \in V$, we have $\mathcal{D}_G^{f'} \subseteq \mathcal{D}_G^f$.

Motivation

Optimization problems involving dominating sets arise in many classical applications (see, e.g., Haynes et al. 1998a, b) that received renewed interest in the last twenty years. We refer, among others, to problems in telecommunication networks, such as clustering, backbone formation and intrusion detection in wireless ad-hoc and sensor networks (Wu and Li 1999; Chen and Liestman 2002; Subhadrabandhu et al. 2004), the gateway placement in wireless mesh networks (Aoun et al. 2006), and the deployment of wavelength division multiplexing in optical networks (Houmaidi et al. 2003). Data analysis problems such as information retrieval for multi document summarization (Shen and Li 2010) and query selection on web databases (Wu et al. 2006), are another domain of application where dominating sets are largely used.

More recently the minimum weighted dominating set problem, which is the natural extension of the minimum dominating set problem obtained by adding weights to the nodes, attracted several researchers and practitioners (see, e.g., Zou et al. 2011; Potluri and Singh 2013; Bouamama and Blum 2016).

The problem we consider in this paper corresponds to a further generalization. Given a wireless sensor network, represented by graph $G = (V, E)$, we wish to determine a minimum weight subset of nodes S to be upgraded as cluster heads, such that each node $v \in V \backslash S$ has at least f_v neighbors in S, where f_v is some given nonnegative integer. This consists in identifying a minimum weight fault-tolerant dominating set, i.e., a minimum weight node subset S such that if some set F of q nodes fail (for example, due to some environmental conditions), the set of nodes $v \in V \backslash F$ for which $f_v \geq q + 1$ or $v \in S \backslash F$ are still dominated by $S \backslash F$ (i.e., these nodes either belong to $S \backslash F$ or have at least one neighbor in $S \backslash F$). The case when all the nodes have unit weights and $f_v = k, v \in V$, for some given positive integer k is presented, e.g., in (Couture et al. 2006; Hwang and Chang 1991; Foerster 2013), see also the references therein.

Related work

The minimum cardinality dominating set problem can be solved in linear time if the graph G is series-parallel (Kikuno et al. 1983) or a cactus (Hedetniemi et al. 1986). It is known to be NP-hard in planar graphs with maximum node degree 3, regular planar graphs with nodes of degree 4 (Garey and Johnson 1979), chordal graphs (Booth 1980) and undirected path graphs (Booth and Johnson 1982). Also, for this problem, there exists a polynomial time $\left(H_{\Delta+1} - \frac{1}{2}\right)$-approximation algorithm, where Δ stands for the maximum degree in G, and, for some given positive integer q, $H_q = \sum_{i=1}^{q} \frac{1}{i}$ denotes the qth harmonic number (Chlebík and Chlebíková 2008; Duh and Fürer 1997). But it is NP-hard to approximate within a factor of $\ln(n) - c \ln\ln \Delta$ for general graphs with $\Delta \geq \Delta_0$, where $c > 0$ and $\Delta_0 \geq 3$ are absolute constants (Chlebík and Chlebíková 2008)). For the minimum cardinality f-dominating set in block graphs, Hwang and Chang 1991 presented a linear time algorithm. In this paper we extend their results to the weighted case for trees. For the case when $f_v = k$, $v \in V$, for some positive integer k, Foerster (2013) showed that the minimum cardinality f-dominating set problem is NP-hard to approximate with a factor better than $0.2267/k \ln(n/k)$, and he also provided a greedy-type algorithm with an approximation ratio of $\ln(\Delta + k) + 1 < \ln(\Delta) + 1.7$. For the case of unit disk graphs and $f_v = k$, $v \in V$, Couture et al. 2006 introduced an incremental algorithm having a constant deterministic performance ratio of six (see also the references therein).

We now report some works dealing with the minimum weight dominating set problem, i.e., $[MW_{\mathbf{1}}]$ with linear programming based approaches. Given the afore mentioned complexity results, no complete description of the polytope $\mathcal{D}_G^{\mathbf{1}}$ is presently known for general graphs. For the case of strongly chordal graphs, the linear relaxation of $(IP1)$ provides such a description, i.e., we have $\mathcal{D}_G^{\mathbf{1}} = \{x \in [0, 1]^n : \sum_{u \in N[v]} x_u \geq 1, v \in V\}$, and this implies the polynomial time solvability of $[MW_{\mathbf{1}}]$ for this graph family (Farber 1984). A complete description of $\mathcal{D}_G^{\mathbf{1}}$ for the case of domishold graphs appears in (Mahjoub 1983). A graph is said to be *domishold* (Benzaken and Hammer 1978) if there exist real positive weights associated to the nodes so that a node subset is a dominating set if and only if the sum of the corresponding weights exceeds some threshold value. A complete formulation for cycles firstly appears in (Bouchakour et al. 2008). This work was extended in (Bianchi et al. 2010), leading to complete description for webs of the form $W^k_{s(2k+1)+r}$ with $s = 2, 3$ and $0 \leq r \leq s - 1$. Also recently, an exact extended formulation for cacti graphs was introduced in (Baïou and Barahona 2014), together with a polynomial-time algorithm to solve $[MW_{\mathbf{1}}]$ for cacti. Given two graphs $G_i = (V_i, E_i)$, $i = 1, 2$, such that the graph $(V_1 \cap V_2, E_1 \cap E_2)$ is a clique with cardinality $k \geq 1$, the graph $G = (V_1 \cup V_2, E_1 \cup E_2)$ is called the *k-sum* of G_1 and G_2. For the particular case when $k = 1$ and assuming complete formulations of the polytopes $\mathcal{D}_{G_i}^{\mathbf{1}}$, $i = 1, 2$, are known, a procedure for determining a complete formulation of $\mathcal{D}_G^{\mathbf{1}}$ is presented by Bouchakour and Mahjoub (1997). It follows from the latter that, in addition to the graph classes we mentioned just before (i.e., those for which a complete formulation is known), complete formulations are also known for the graphs which may be obtained as 1-sums of their members. We conclude our review on polyhedral works related to generalizations of the domination concept with a very special case: when $f_v = d_v^G$, for all $v \in V$, a node subset $S \subseteq V$ is an f-dominating set if and only if its complementary set $V \backslash S$ is a stable set. This implies that for this case it is equivalent to look for a maximum weight stable set and a minimum weight f-dominating set. From results on the stable set problem, it follows that $[MW_f]$ can be solved in polynomial time in perfect graphs for the particular case when $f_v = d_v$, for all $v \in V$, and that also a complete description of $\mathcal{D}_G^f$ is known for that case (Chvátal 1975; Padberg 1974).

To the best of the authors' knowledge no polyhedral results appear in the literature on the polyhedral structure of $\mathcal{D}_G^f$ for general $f \in \mathcal{F}_G$. The only works on the problem $[MW_f]$ for general $f \in \mathcal{F}_G$ we are aware of essentially focus on the case $c = \mathbf{1}$ and establish bounds on the optimal objective value of $[MW_f]$ for this particular case (Chen and Zhou 1998; Stracke and Volkmann 1993; Zhou 1996).

Our contribution

In this paper we introduce different results related to the polyhedral structure of the polytope $\mathcal{D}_G^f$. They namely include different families of facet-defining inequalities and a complete description of this polytope for trees. In general, this formulation may contain an exponential number of constraints (whereas it is linear in the number of nodes for $\mathcal{D}_G^{\mathbf{1}}$), but we show that the corresponding separation problem can be solved in polynomial time, thus implying the polynomial time solvability of $[MW_f]$ for trees, by the well-known results on the polynomial equivalence between optimization and separation established by Grötschel et al. (1981). In addition, we provide a linear time combinatorial algorithm for that case, thus extending some results in the literature on the classical domination concept.

Structure of the paper

In Sect. 2, we introduce different polyhedral results on $\mathcal{D}_G^f$ among which some families of facet-defining inequalities. In Sect. 3, we provide a complete description of the polytope $\mathcal{D}_G^f$ for the case of trees and show that the problem $[MW_f]$ can be solved in polynomial time in that case. Finally, in Sect. 4, we present a linear time algorithm for the case that G is a tree.

Additional notation

Before we close this section, let us mention some additional notation that will be used later. Given a graph $G = (V, E)$ and a node subset $S \subseteq V$, let $G[S]$ denote the subgraph of G that is induced by S, i.e., $G[S] = (S, E')$, where E' stands for the subset of edges in E having both endpoints in S. Given a positive integer n, I_n stands for the identity matrix with order n. We will also use $\mathbf{0}$ to denote a vector whose dimension will be clear from the context and having all its entries equal to zero.

2 Polyhedral results on the polytope $\mathcal{D}_G^f$

In this section, we assume that $G = (V, E)$ is a simple connected graph with $n \geq 2$. After some basic polyhedral properties are introduced, we present nontrivial facet-defining inequalities of $\mathcal{D}_G^f$.

2.1 Basic polyhedral properties of $\mathcal{D}_G^f$

In what follows, the inequalities of the form $x_v \geq 0$ and $x_v \leq 1$ with $v \in V$, will be called *trivial inequalities*.

Proposition 1 *Let $f \in \mathcal{F}_G$. Then the following holds.*

(i) *The polytope* $\mathcal{D}_G^f$ *has dimension n, i.e., it is full dimensional.*
(ii) *The inequality* $x_v \geq 0$ *is facet-defining for* $\mathcal{D}_G^f$ *iff* $d_w > f_w$*, for all* $w \in N[v]$*,* $v \in V$.
(iii) *The inequality* $x_v \leq 1$ *is facet-defining for* $\mathcal{D}_G^f$*, for all* $v \in V$.
(iv) *Every facet-defining inequality of* $\mathcal{D}_G^f$ *which is not trivial is of the form* $\sum_{v \in V} a_v x_v \geq b$*, with* $a_v \geq 0$*, for all* $v \in V$*. Moreover* $b > 0$*,* $|V_a = \{v \in V : a_v > 0\}| \geq 2$.

Proof The statements (i)–(iii) and the first part of statement (iv) follow from the work by Hammer et al. (1975). So we prove the last part of (iv) (which starts from "Moreover"). Also, for property (ii), we provide a simple and short alternate proof.

Proof of (ii). Let $F_u^\alpha = \mathcal{D}_G^f \cap \{x \in \mathbb{R}^n : x_u = \alpha\}$, for any $u \in V$ and $\alpha \in \{0, 1\}$.

[$\Rightarrow$] In case $d_v = f_v$, then necessarily $F_v^0 \subseteq \cap_{w \in N(v)} F_w^1$. Also, if there exists $w \in N(v)$ such that $d_w = f_w$, then $F_v^0 \subseteq F_w^1$. Thus, in both cases, the inequality $x_v \geq 0$ cannot define a facet.

[$\Leftarrow$] The incidence vectors of the n f-dominating sets: $V \backslash \{v\}$ and $V \backslash \{v, w\}$, for all $w \in V \backslash \{v\}$, are affinely independent and they all belong to F_v^0.

Proof of the last part of (iv): $b > 0$ and $|V_a| \geq 2$. The property $b > 0$ is implied by the fact that the inequality is not trivial (otherwise it would be dominated by the nonnegativity inequalities). If we had $V_a = \{v\}$, the inequality would correspond (up to multiplication by a positive scalar) to an inequality of the form $x_v \geq b$, with $b \geq 0$. Since $\mathcal{D}_G^f$ is full dimensional, there exists an f-dominating set D with $v \notin D$, thus implying $b = 0$, and the inequality would be trivial. $\square$

Let $a^t x \geq b$ denote a non trivial facet-defining inequality of $\mathcal{D}_G^f$, and let $G_a = G[V_a]$ be the subgraph of G that is induced by the node subset V_a. The following property was shown to hold for the dominating set polytope (Bouchakour and Mahjoub 1997), and its extension to f-dominating set polytopes is straightforward.

Proposition 2 (Bouchakour and Mahjoub 1997) *The graph* G_a *is connected.*

2.2 Non trivial facet-defining inequalities

We start with a definition that will be useful for presenting different families of inequalities that are valid for $\mathcal{D}_G^f$.

Definition 1 Given an undirected simple graph $G = (V, E)$ and $f \in \mathcal{F}_G$, an *f-clique* is a node subset $Q \subseteq V$, such that $|Q| \geq 2$ and satisfying the following two conditions:

(i) the nodes in Q are pairwise adjacent (i.e., Q is a clique in G), and
(ii) $|\{v \in Q : f_v = d_v\}| \geq |Q| - 1$.

The next three propositions deal with cases when some nodes of the graph satisfy $f_v = d_v$. Notice that, if for each edge $\{u, v\} \in E$ we have $f_u = d_u$ or (not exclusively) $f_v = d_v$, then the following holds: $D \subseteq V$ is an f-dominating set iff D is a vertex cover. So in that case, the vertex cover and f-dominating set polytopes coincide (and are both affinely equivalent to the stable set polytope). The next proposition directly follows from this correspondence, see, e.g., (Nemhauser and Trotter 1974).

Proposition 3 *Let the graph $\mathcal{C} = (V(\mathcal{C}), E(\mathcal{C}))$ be an odd cycle such that each edge $\{u, w\} \in E(\mathcal{C})$ is an f-clique. Then, the inequality*

$$\sum_{v \in V(\mathcal{C})} x_v \geq \left\lceil \frac{|V(\mathcal{C})|}{2} \right\rceil \tag{1}$$

is facet-defining for $\mathcal{D}^f_{\mathcal{C}}$ iff $\mathcal{C}$ has no chord. (A chord is an edge joining two nonconsecutive nodes of the cycle).

The next two propositions illustrate the fact that, even though the property that "$f_u = d_u$ or $f_v = d_v$ for each edge $\{u, v\} \in E$" may hold only on some parts of the graphs, some well-known families of inequalities valid for the vertex cover polytope, may, under some conditions, define facets of f-dominating set polytopes. We now formulate a simple sufficient condition for an inequality of type (1) corresponding to a node-induced subgraph of an arbitrary graph to define a facet of $\mathcal{D}^f_G$.

Proposition 4 *Let $f \in \mathcal{F}_G$, and let $\mathcal{C} = (V(\mathcal{C}), E(\mathcal{C}))$ denote a node induced odd cycle in $G = (V, E)$ with no chord and such that each edge $\{u, w\} \in E(\mathcal{C})$ corresponds to an f-clique. If $|N(w) \cap V(\mathcal{C})| \leq 2$, for all $w \in V \backslash V(\mathcal{C})$, then the inequality (1) is facet-defining for $\mathcal{D}^f_G$.*

Proof Let F denote the face of $\mathcal{D}^f_G$ defined by (1). Let $a^t x \geq b$ denote a facet-defining inequality of $\mathcal{D}^f_G$ such that the facet $\overline{F}$ it defines contains F: $F \subseteq \overline{F}$. W.l.o.g, assume the nodes of the cycle $\mathcal{C}$ are $1, 2, \ldots, 2q+1$, in this order. For any positive integer i, let e_i denote the k-th unit vector in $\mathbb{R}^n$, with $k = i$ if $i \leq 2q+1$ and $k = 1 + (i \mod (2q+2))$ otherwise. Also, let $h \in \{0, 1\}^n$, such that $h_i = 1$ iff $i \notin V(\mathcal{C})$.

For $i = 1, 2, \ldots, 2q+1$, we define the vectors

$$y^i = h + e_i + e_{i+1} + e_{i+3} + \ldots + e_{i+2q-1}.$$

By assumption, it follows that all the points $(y^i)_{i=1}^{2q+1}$ correspond to incidence vectors of f-dominating sets and belong to F. For a fixed arbitrary $i \in \{1, 2, \ldots, 2q-1\}$, we have $y^{i+2} = y^i + e_{i+2} - e_{i+1}$. Also, $y^1 = y^{2q} + e_1 - e_{2q+1}$, and $y^2 = y^{2q+1} + e_2 - e_1$. We deduce $a_1 = a_{2q+1} = a_2$, and $a_{i+1} = a_{i+2}$, for all $i \in \{1, 2, \ldots, 2q-1\}$.

Consider now some node $k \in V \backslash V(\mathcal{C})$. And let $S \subset V(\mathcal{C})$ denote a subset of exactly $\left\lceil \frac{|V(\mathcal{C})|}{2} \right\rceil$ nodes of the cycle $\mathcal{C}$ such that $N(k) \cap V(\mathcal{C}) \subseteq S$, and the node subset $U = (V \backslash V(\mathcal{C})) \cup S$ is an f-dominating set. The existence of S follows from the assumption that $|N(k) \cap V(\mathcal{S})| \leq 2$. Since U and $U \backslash \{k\}$ correspond to f-dominating sets in G whose incidence vectors satisfy (1) with equality, we deduce $a_k = 0$.

It follows that the inequality $a^t x \geq b$ must correspond, up to multiplication by a positive scalar, to inequality (1). $\square$

Proposition 5 *Let $Q \subseteq V$ denote an f-clique. Then, the following inequality is valid for $\mathcal{D}_G^f$,*

$$\sum_{v \in Q} x_v \geq |Q| - 1, \tag{2}$$

and it is facet-defining if and only if Q is maximal (w.r.t. inclusion).

Proof The validity of (2) easily follows from the domination requirements. We now establish the necessary and sufficient conditions for it to be facet-defining. [$\Rightarrow$] Assume that the inequality (2) is facet-defining. If Q is not maximal, there exists some f-clique Q' such that $Q \subset Q'$. But then the inequality (2) is the sum of the f-clique inequality $\sum_{v \in Q'} x_v \geq |Q'| - 1$ and the trivial inequalities (multiplied by -1) $-x_v \geq -1$, for all $v \in Q' \backslash Q$, a contradiction. [$\Leftarrow$] Assume Q is a maximal f-clique. Let F denote the face of $\mathcal{D}_G^f$ which is defined by (2). Let $a^t x \geq b$ with $(a, b) \in (\mathbb{R}^n \backslash \{\mathbf{0}\} \times \mathbb{R})$, denote a facet-defining inequality of $\mathcal{D}_G^f$ such that the facet $\overline{F}$ it defines satisfies: $F \subseteq \overline{F}$.

Let $w \in V \backslash Q$. Assume firstly that there exists some node v in Q that is not a neighbor of w. Then, since $V \backslash \{v\}$ and $V \backslash \{v, w\}$ are f-dominating sets in G whose incidence vectors belong to F, we deduce $a_w = 0$.

Consider now the case of a node $w \in V \backslash Q$ such that $Q \subseteq N(w)$. Since Q is a *maximal* f-clique, there must exist some node $z \in Q$ satisfying $f_z < d_z$ and we must have $f_w < d_w$. Then, since $V \backslash \{z\}$ and $V \backslash \{z, w\}$ are f-dominating sets in G whose incidence vectors belong to F, we deduce $a_w = 0$.

For any pair $(v, w) \in Q^2$, with $v \neq w$, since the sets $V \backslash \{v\}$ and $V \backslash \{w\}$ are f-dominating in G, we deduce $a_v = a_w$.

It follows that the inequality $a^t x \geq b$ corresponds, up to multiplication by a positive scalar, to (2). □

We now introduce another family of valid inequalities for $\mathcal{D}_G^f$ that we will call *partial neighborhood inequalities*, a denomination that is suggested by their support which is a subset of a closed neighborhood of a node having a positive domination requirement. These inequalities may be seen as a generalization of the classical neighborhood inequalities (obtained by setting $q = f_v = 1$ in (3) given hereafter) used in linear formulations of the dominating set problem. Recall that these inequalities, together with the trivial inequalities, are sufficient to completely describe the dominating set polytope ($f_v = 1$, for all $v \in V$) in the case of strongly chordal graphs (Farber 1984). Their relevance will be further stressed later, when considering complete formulations of the f-dominating set polytope for trees.

Proposition 6 *Let $G = (V, E)$ denote an undirected simple graph, let $f \in \mathcal{F}_G$ and $u \in V$ such that $1 \leq f_u < d_u$. Then, the following* partial neighborhood inequality *is valid for $\mathcal{D}_G^f$:*

$$q x_u + \sum_{v \in N(u) \backslash Z_q} x_v \geq q, \tag{3}$$

with $q \in \{1, 2, \ldots, f_u\}$, $Z_q \subseteq N(u)$ such that $|Z_q| = f_u - q$.

Proof Any f-dominating set D not containing node u must contain at least f_u of its neighbors: $|D \cap N(u)| \geq f_u$. Let U denote a subset of at most $f_u - 1$ neighbors of u: $U \subset N(u)$ and $0 \leq |U| \leq f_u - 1$. Then, we must have $|D \cap (N(u) \backslash U)| \geq f_u - |U|$. With $q = f_u - |U|$, $Z_q = U$, we can check that the incidence vector of D satisfies (3). Also note that (3) is trivially satisfied by the incidence vector of any f-dominating set containing u. $\square$

We now formulate a simple sufficient condition for an inequality of type (3) to be facet-defining for $\mathcal{D}_G^f$.

Proposition 7 *Let $G = (V, E)$ denote an undirected simple graph and $f \in \mathcal{F}_G$. Let $u \in V$ such that $1 \leq f_u < d_u$, and assume that $|N(v) \backslash N[u]| \geq f_v$, for all $v \in V \backslash \{u\}$. Then the inequality (3) is facet-defining for $\mathcal{D}_G^f$, for all $q \in \{1, 2, \ldots, f_u\}$ and $Z_q \subseteq N(u)$ such that $|Z_q| = f_u - q$.*

Proof Assume all the conditions mentioned are satisfied, fix $q \in \{1, 2, \ldots, f_u\}$ and $Z_q \subseteq N(u)$ such that $|Z_q| = f_u - q$. Let F denote the face of $\mathcal{D}_G^f$ induced by (3), and assume that F is contained in a facet $\overline{F}$ of $\mathcal{D}_G^f$ that is defined by the inequality $a^t x \geq b$ with $(a, b) \in (\mathbb{R}^n \backslash \{\mathbf{0}\} \times \mathbb{R})$.

Let $w \in (V \backslash N[u]) \cup Z_q$. Since the node sets $(V \backslash N(u)) \cup Z_q$ and $((V \backslash N(u)) \cup Z_q) \backslash \{w\}$ are f-dominating sets (using the assumption that $|N(v) \backslash N[u]| \geq f_v$, for all $v \in V \backslash \{u\}$), both satisfying (3) with equality we deduce $a_w = 0$.

We now show $a_w = a_{w'}$, for all $(w, w') \in (N(u) \backslash Z_q)^2$. Let $A \subseteq N(u) \backslash Z_q$ such that $|A| = q$. Let $w \in A$, $w' \in N(v) \backslash (A \cup Z_q)$. Then, the incidence vectors of the node subsets $(V \backslash N[u]) \cup A \cup Z_q$ and $(V \backslash N[u]) \cup (A \backslash \{w\} \cup \{w'\}) \cup Z_q$ both correspond to f-dominating sets (using the assumption that $|N(v) \backslash N[u]| \geq f_v$, for all $v \in V \backslash \{u\}$, and because they both contain f_u neighbors of node u), and they satisfy (3) with equality. We deduce: $a_w = a_{w'}$.

Considering the incidence vectors of the node subsets $V \backslash N(u)$ and $(V \backslash N[u]) \cup A \cup Z_q$ (with A as defined before), since they correspond to f-dominating sets and their incidence vectors satisfy (3) with equality, we deduce $a_u = q a_w$, with $w \in A$.

It follows that the inequality $a^t x \geq b$ must correspond, up to multiplication by a positive scalar, to inequality (3). $\square$

3 Complete descriptions of $\mathcal{D}_G^f$ for trees

After we report some preliminary results in Sect. 3.1, we present a complete formulation of the f-dominating set polytope for stars in Sect. 3.2, and for trees in Sect. 3.3.

3.1 On leaves with domination requirement one

We start with an auxiliary result.

Lemma 1 *Let $G = (V, E)$ be an undirected simple graph with at least two nodes. Let $f \in \mathcal{F}_G$ and $\{u, w\} \in E$ be such that w is a leaf node and $f_w = d_w = 1$. If the* *constraint* $a^t x \geq b$ *is a facet-defining inequality for* $\mathcal{D}_G^f$ *that is not trivial and different from* $x_u + x_w \geq 1$*, then,* $a_w = 0$*.*

Proof Let $a^t x \geq b$ denote a facet-defining inequality satisfying the assumptions given in the statement of the lemma. Let $D \subseteq V$ denote an f-dominating set whose incidence vector satisfies $a^t x = b$, and such that u and w belong to D. Such an f-dominating set exists due to the fact that the inequality $a^t x \geq b$ is assumed to be facet-defining and different from $x_u + x_w \geq 1$. Since node w is a leaf and its only neighbor is $u \in D$, it follows that $D \backslash \{w\}$ is an f-dominating set. We deduce $a_w = 0$. (This is due to the fact that, by Proposition 1-(iv), we have $a_w \geq 0$, and if we had $a_w > 0$, then the incidence vector of $D \backslash \{w\}$ would violate $a^t x \geq b$, a contradiction.) □

Lemma 1 may be interesting when looking for a complete formulation of $\mathcal{D}_G^f$ when G contains a node w satisfying $f_w = d_w = 1$. This is illustrated by the next proposition whose proof relies on it.

Proposition 8 *Let* $G = (V, E)$ *be an undirected simple graph, and let* $f \in \mathcal{F}_G$*. Assume that there exists a node* $w \in V$ *such that* $f_w = d_w = 1$*, and let* $u \in V$ *be the unique neighbor of* w*. Let* $G' = G[V \backslash \{w\}] = (V', E')$ *and* $f' \in \mathcal{F}_{G'}$ *such that* $f'_v = f_v$*, for all* $v \in V' \backslash \{u\}$ *and* $f'_u = \max\{0, f_u - 1\}$*. Then, a complete formulation of* $\mathcal{D}_G^f$ *is obtained by adding to that of* $\mathcal{D}_{G'}^{f'}$ *the variable* x_w *and the inequalities:* $0 \leq x_w \leq 1$ *and* $x_u + x_w \geq 1$*.*

Proof Let $a^t x \geq b$ be a facet-defining inequality for $\mathcal{D}_{G'}^{f'}$ that is different from $x_u \geq 0$. Let $\widehat{a} \in \mathbb{R}^n$ be defined as follows: $\widehat{a}_v = a_v$, for all $v \in V'$, and $\widehat{a}_w = 0$. First, notice that the inequality $\widehat{a}^t x \geq b$ is valid for $\mathcal{D}_G^f$. (This follows from the fact that if D is an f-dominating set in G, then $D \backslash \{w\}$ is an f'-dominating set in G'.) Let F denote the face of $\mathcal{D}_G^f$ defined by $\widehat{a}^t x \geq b$. (Note that even though the inequality $x_u \geq 0$ may be facet-defining for $\mathcal{D}_{G'}^{f'}$, it cannot be facet-defining for $\mathcal{D}_G^f$ since the face it defines is contained in the one defined by $x_u + x_w \geq 1$.) We show that $\widehat{a}^t x \geq b$ is facet-defining for $\mathcal{D}_G^f$.

Let $D'_1, D'_2, \ldots, D'_{n-1}$ denote $n-1$ f'-dominating sets in G', whose incidence vectors in $\mathbb{R}^{V'}$ are affinely independent and satisfy the equation $a^t x = b$. Since $a^t x \geq b$ is assumed to be different from $x_u \geq 0$, there exists some index $k \in \{1, 2, \ldots, n-1\}$ such that $u \in D'_k$. Then, define $D_j = D'_j \cup \{w\}$, for all $j \in \{1, 2, \ldots, n-1\}$, and $D_n = D'_k$. Since the incidence vectors of the f-dominating sets $(D_i)_{i=1}^n$ are affinely independent and all lie in F, the inequality $\widehat{a}^t x \geq b$ defines a facet of $\mathcal{D}_G^f$.

Now let $g^t x \geq h$ be a facet-defining inequality of $\mathcal{D}_G^f$ with $g_w = 0$, $g \in \mathbb{R}^n \backslash \{0\}$. Let $\tilde{g}$ denote the restriction of the vector g to its entries indexed by V'. First note that the inequality $\tilde{g}^t x \geq h$ is valid for $\mathcal{D}_{G'}^{f'}$. (This follows from the fact that if D' is an f'-dominating set in G', then $D' \cup \{w\}$ is an f-dominating set in G.) Let F' denote the face of $\mathcal{D}_{G'}^{f'}$ defined by $\tilde{g}^t x \geq h$. We show F' is a facet of $\mathcal{D}_{G'}^{f'}$.

Assume, for a contradiction, that this is not the case. Then, there exists $(\overline{a}, \overline{b}) \in (\mathbb{R}^{n-1} \backslash \{0\} \times \mathbb{R})$ such that the inequality $\overline{a}^t x \geq \overline{b}$ defines a facet $\overline{F}$ of $\mathcal{D}_{G'}^{f'}$ satisfying $F' \subset \overline{F}$, and the vectors $\overline{a}$ and $\tilde{g}$ are linearly independent. From the first part of the proof, the inequality $\breve{a}^{\,t}x \geq b$ is facet-defining for $\mathcal{D}_G^f$, with $\breve{a} \in \mathbb{R}^n \backslash \{0\}$ defined by $\breve{a}_v = \overline{a}_v$, for all $v \in V'$, and $\breve{a}_w = 0$. This implies $\breve{a} = \lambda g$ for some $\lambda \in \mathbb{R}$ with $\lambda > 0$, and thus $\tilde{g} = \lambda \overline{a}$, a contradiction.

Then, by Lemma 1, no other facet-defining inequality with $a_w \neq 0$ exists, which concludes the proof. □

Considering the last proposition, in the rest of this section we will consider graphs with all degree-one nodes having zero domination requirement, unless otherwise stated.

3.2 Complete description of $\mathcal{D}_G^f$ when G is a star

Let $G = (V, E)$ be a star with $n \geq 3$ and having for center the node 1. Let $\overline{V} = V \backslash \{1\}$ and assume that $0 \leq f_1 < n-1$, $f_v = 1$, $v \in \overline{V}$. From Proposition 8, it follows that a complete description of $\mathcal{D}_G^f$ is given by the trivial inequalities, together with the constraints $x_1 + x_v \geq 1$, $v \in \overline{V}$. Note that this also holds for $\mathcal{D}_G^{f'}$ with f' defined as follows: $f_1' = n-1$ and $f_v' \in \{0, 1\}, v \in \overline{V}$, since $\mathcal{D}_G^f = \mathcal{D}_G^{f'}$. For the case when $f_v = 0$, $v \in V$, a description is given by the trivial inequalities. Together with Proposition 8, the next result leads to a complete description for the remaining cases, i.e., $0 \leq f_1 < n-1$, $f_v \in \{0, 1\}, v \in \overline{V}$, $f \neq \mathbf{0}$, and there exists one leaf node w with $f_w = 0$.

Proposition 9 *Let the graph $G = (V, E)$ be a star, with $n \geq 3$, and let 1 denote the center. Let $f \in \mathcal{F}_G$ with $1 \leq f_1 < n-1$ and $f_v = 0$, $v \in \overline{V}$. A complete description of $\mathcal{D}_G^f$ is then given by the trivial inequalities, together with the constraints (3) (taking $u = 1$ in the expression).*

Proof We prove that all the facet-defining inequalities of $\mathcal{D}_G^f$ correspond to trivial or type (3) inequalities.

Let $a^t x \geq b$ denote a facet-defining inequality of $\mathcal{D}_G^f$ that is not trivial. (So, by Proposition 1-(iv), we have: $|\{v \in V : a_v > 0\}| \geq 2$, $a \geq 0$, $b > 0$.) Let $P_a = \{v : a_v > 0 \text{ and } v \in \overline{V}\}$ and $Z_a = \left\{v : a_v = 0 \text{ and } v \in \overline{V}\right\}$. Note that, since $\{1\}$ is an f-dominating set, necessarily $a_1 > 0$, and any f-dominating set Q containing 1 and satisfying $a^t x = b$ must satisfy $Q \cap P_a = \emptyset$. Let D denote an f-dominating set such that $1 \notin D$ and whose incidence vector satisfies $a^t x = b$. Note that, since the constraint is assumed to be valid and nontrivial, $|P_a| \geq 1$. Also, necessarily, $Z_a \subset D$. (Otherwise, let $u \in D$ such that $a_u > 0$ and $w \in Z_a \backslash D$. The set $(D \backslash \{u\}) \cup \{w\}$ would be f-dominating but its incidence vector would violate the inequality $a^t x \geq b$, a contradiction). This also implies $|D \cap P_a| = f_1 - |Z_a|$.

Now, let u denote a node in D that is associated with the largest coefficient value of the inequality: $u \in \operatorname{argmax}\{a_v : v \in D\}$. Since for any node $w \in V \backslash D$, the node set $(D \backslash \{u\}) \cup \{w\}$ is an f-dominating set, we deduce $a_w \geq a_u > 0$, $w \in V \backslash D$.

Now, let u' stand for a node that is different from 1, is not in D, and is associated with the largest coefficient value in the inequality: $u' \in \operatorname{argmax}\{a_v : v \in \overline{V} \backslash D\}$. In particular, we have $a_{u'} \geq a_u$.

Let D' denote another f-dominating set satisfying $a^t x = b$, and such that $u' \in D'$. (Such an f-dominating set does exist, because otherwise, the face defined by $a^t x \geq b$ would be contained in the hyperplane $x_{u'} = 0$, a contradiction.) Considering the f-dominating set $(D' \backslash \{u'\}) \cup \{\overline{u}\}$ for any $\overline{u} \in D \backslash D'$, we deduce $a_{u'} \leq a_{\overline{u}} \leq a_u$, and thus $a_u = a_{u'} = a_w$, $w \in \overline{V} \backslash D$.

Let $u'' \in \text{argmin}\{a_v : v \in D \backslash Z_a\}$. Let D'' be an f-dominating set not containing u'' nor $\{1\}$, and such that its incidence vector satisfies $a^t x = b$. (D'' exists, because otherwise all the incidence vectors of the f-dominating sets satisfying $a^t x = b$ would belong to the hyperplane with equation $x_1 + x_{u''} = 1$. But the latter does not correspond to a supporting hyperplane of $\mathcal{D}_G^f$, since there exist f-dominating sets satisfying $x_1 + x_{u''} = 0$ and others satisfying $x_1 + x_{u''} = 2$). Since $|D'' \cap P_a| = |D \cap P_a| = f_1 - |Z_a|$ (see before), D'' contains some node $\overline{w} \in \overline{V} \backslash D$. Then, $(D'' \backslash \{\overline{w}\}) \cup \{u''\}$ is an f-dominating set, and we deduce $a_u \leq a_{\overline{w}} \leq a_{u''}$, thus implying $a_w = \alpha$, $w \in \overline{V} \backslash Z_a$, for some constant $\alpha > 0$.

Finally, considering an f-dominating set saturating the constraint and containing 1, we deduce $a_1 = (f_1 - |Z_a|)\alpha$. Thus, the inequality $a^t x \geq b$ corresponds (up to multiplication by a positive scalar), to an inequality of type (3). □

The former results are summarized in the next Proposition.

Proposition 10 *If the graph G is a star and $f \in \mathcal{F}_G$, then a complete description of $\mathcal{D}_G^f$ is given by the trivial inequalities ($0 \leq x_v \leq 1$, for all $v \in V$), by the f-clique inequalities (2) and the partial neighborhood inequalities (3).*

Remark 1 By Proposition 7, for the case of a star $G = (V, E)$ having node 1 as its center, and $f \in \mathcal{F}_G$ with $1 \leq f_1 < d_1$, the polytope $\mathcal{D}_G^f$ is full dimensional and the number of distinct facets of $\mathcal{D}_G^f$ defined by inequalities of type (3) may be exponential in n.

Next, we shall see how the results from this section may be extended to get a complete formulation for trees.

3.3 Complete description of $\mathcal{D}_G^f$ when G is a tree

Theorem 1 *If the graph $G = (V, E)$ is a tree and $f \in \mathcal{F}_G$, then a complete formulation of $\mathcal{D}_G^f$ is given by the following set of inequalities.*

$$\begin{cases} 0 \leq x_v \leq 1, & v \in V, \\ x_u + x_v \geq 1, & f\text{-clique } \{u, v\}, \\ q x_u + \sum_{v \in N(u) \backslash Z_q} x_v \geq q, & u \in V \text{ such that } 1 \leq f_u < d_u, q \in \{1, 2, \ldots, f_u\}, \\ & Z_q \subseteq N(u) \text{ with } |Z_q| = f_u - q. \end{cases}$$

Proof We proceed by contradiction for the proof which relies on the results established in the former sections (Propositions 8, 10). Let $\mathcal{P}_G^f$ denote the polytope which is defined by the set of inequalities given in the statement of the theorem. Naturally, we have

$\mathcal{D}_G^f \subseteq \mathcal{P}_G^f$. Let G denote a graph corresponding to a tree with a minimum number of nodes such that the result does not hold, i.e., such that the polytope $\mathcal{P}_G^f$ has a fractional extreme point x^*. Note that from Proposition 10, the graph G cannot be a star. This implies $n \geq 4$. Also, given the formulation of $\mathcal{P}_G^f$, the choice of G and Proposition 8, the domination requirement of each leaf in G must be equal to zero.

Let $\mathcal{E}$ denote a nonsingular subsystem of n equations defining x^*. Each equation of the subsystem corresponds to an inequality arising in the description of $\mathcal{P}_G^f$. In general, this subsystem is not uniquely defined. For our purposes, we will consider such a subsystem of n equations defining x^* *and having a maximum number of equations corresponding to trivial inequalities*. So, for each integral component x_v^* (if there exists one), the equation $x_v = 0$ or $x_v = 1$ appears in $\mathcal{E}$. Note that, necessarily, at least one equation corresponding to an inequality of type (3) must be contained in $\mathcal{E}$ (since otherwise, the matrix whose entries correspond to the left-hand side of the subsystem $\mathcal{E}$ is totally unimodular, and we get a contradiction with the fact that x^* is not an integer vector). And at least two variables in the support of this inequality must have positive fractional values (by the definition of the subsystem $\mathcal{E}$).

We then make use of the following auxiliary claims whose proofs are postponed, for clarity. The basic idea behind those claims is to identify some properties of the fractional solution x^* and of the graph G that would correspond to a minimal (w.r.t. the number of nodes) counterexample to the statement of the theorem. Their proofs heavily rely on this minimality assumption.

The first claim provides the information that the subsystem $\mathcal{E}$ cannot contain an equation corresponding to an f-clique inequality involving a leaf node. $\square$

Claim 1 *For each leaf node $w \in V$, we have: $x_u^* + x_w^* \neq 1$, where u denotes the unique neighbor of w.*

The second claim establishes that each entry of x^* corresponding to a non-leaf node has value strictly less than 1.

Claim 2 *For each node $w \in V$ that is not a leaf, we have: $x_w^* < 1$.*

The third claim states a property that is satisfied by any inequality of the type (3) *which belongs to the subsystem $\mathcal{E}$.*

Claim 3 *Assume the system $\mathcal{E}$ contains one equation corresponding to some inequality of type* (3)*:*

$$\overline{q} x_u + \sum_{w \in W = N(u) \setminus Z} x_w \geq \overline{q},$$

for some node $u \in V$, $\overline{q} \in \{1, 2, \ldots, f_u\}$, $Z \subseteq N(u)$ and $|Z| = f_u - \overline{q}$. Then $W \supseteq \{t \in N(u) \colon x_t^* = 0\}$.

The Claims 4 and 5 establish properties related to the leaves. The fourth claim states that all the entries of x^* corresponding to leaves must be fractional. This is used to establish Claim 5 stating that the neighbor of any leaf node must have a domination requirement strictly less than its degree.

Claim 4 *For each leaf node $w \in V$, we have: $0 < x_w^* < 1$.*

Claim 5 *G contains no leaf that is a neighbor of a node u satisfying $f_u = d_u$.*
The next two claims (whose proof relies on the former) establish the existence of a node in G that has degree two and is adjacent to exactly one leaf. This point will then be used to get a contradiction and prove the theorem.

Claim 6 *Each node in G is adjacent to at most one leaf.*

Claim 7 *The graph G has a node with degree 2 that is adjacent to one leaf.*

Let u be a node in G satisfying Claim 7 and let w denote its leaf neighbor. Then necessarily $f_u = 1$ (due to the definition of the system $\mathcal{P}_G^f$, the fact that $d_u = 2$ and Claim 5). Recall that x_w^* is fractional (by Claim 4), and by Claim 1, the subsystem $\mathcal{E}$ must contain an equation which corresponds to a partial neighborhood inequality with center u, namely $x_u + x_z + x_w = 1$, where z is the neighbor of u that is different from w. Also note that the variable x_w only occurs in this equation of $\mathcal{E}$, and at least one of the two quantities among x_u^* and x_z^* must be fractional.

Let $\mathcal{E}'$ denote the subsystem obtained from $\mathcal{E}$ by removing this equation and the column corresponding to x_w. Then $\mathcal{E}'$ is nonsingular and all the equations that are present in $\mathcal{E}'$ correspond to inequalities that are present in the description of $\mathcal{P}_{G'}^{f'}$, with $G' = G[V \backslash \{w\}]$, $f'_v = f_v$, for all $v \in V \backslash \{w, u\}$, $f'_u = 0$.

The restriction $\overline{x}^*$ of x^* to its components corresponding to nodes in $V \backslash \{w\}$ belongs to $\mathcal{P}_{G'}^{f'}$ and also satisfies $\mathcal{E}'$. It follows that $\overline{x}^*$ corresponds to a fractional extreme point of $\mathcal{P}_{G'}^{f'}$, contradicting the hypothesis that G has the minimum number of nodes. □

We now give the details of the proofs of the seven claims used in the proof of Theorem 1.

Proof of Claim 1 Assume, for a contradiction that there exists an edge $\{u, w\} \in E$ such that w is a leaf and $x_u^* + x_w^* = 1$. Let $\overline{x}^*$ denote the restriction of x^* to its entries indexed on $V \backslash \{w\}$. Let $G' = G[V \backslash \{w\}]$, and define $f' \in \mathcal{F}_{G'}$ as follows: $f'_v = f_v$, for all $v \in V \backslash \{u, w\}$ and $f'_u = f_u - 1$. Then, one can easily check that $\overline{x}^* \in \mathcal{P}_{G'}^{f'}$. And from our minimality assumption on G, $\overline{x}^*$ can be expressed as a convex combination of incidence vectors of f'-dominating sets in G':

$$\overline{x}^* = \sum_{i=1}^{q} \lambda_i \chi^{S_i},$$

where q denotes a positive integer, the sets $(S_i)_{i=1}^q$ are f'-dominating sets in G' and $\lambda \in \mathbb{R}_+^q$ satisfies $\sum_{i=1}^q \lambda_i = 1$. Let us now consider the sets $(\hat{S}_i)_{i=1}^q$ defined as follows: $\hat{S}_i = S_i \cup \{w\}$ if $u \notin S_i$ and $\hat{S}_i = S_i$ otherwise, for each $i \in \{1, 2, \ldots, q\}$. It can be checked that the sets $(\hat{S}_i)_{i=1}^q$ correspond to f-dominating sets in G. But this also implies that $x^* = \sum_{i=1}^q \lambda_i \chi^{\hat{S}_i}$, contradicting the fact that x^* is a fractional extreme point of $\mathcal{P}_G^f$. □

Proof of Claim 2 Assume, for a contradiction, that there exists a non-leaf node $w \in V$ such that $x^*_w = 1$. From our assumption on the system $\mathcal{E}$ (it contains a maximum number of equations corresponding to trivial inequalities), it contains $x_w = 1$ and it cannot contain an equation corresponding to a partial neighborhood inequality with center w.

Let C denote the node set of a connected component of $G' = G[V \backslash \{w\}]$ such that C contains a node corresponding to a fractional component of x^*. Recall that each equation present in the system $\mathcal{E}$ only involves variables corresponding to nodes which belong to a single component of G', and possibly x_w. Then, let $\overline{C} = C \cup \{w\}$, $G'' = G[\overline{C}]$, and define $f' \in \mathbb{R}^{\overline{C}}$ as follows: $f'_v = f_v$, $v \in C$ and $f'_w = 0$. Let $x^{1,*}$ (resp. $x^{2,*}$) denote the restriction of x^* to its entries with index in $\overline{C}$ (resp. $V \backslash \overline{C}$). Note that $x^{1,*} \in \mathcal{P}^{f'}_{G''}$ (using the definition of $\mathcal{P}^{f'}_{G''}$ and the fact that $x^{1,*}_w = 1$). And from the minimality assumption on G, $x^{1,*} \in \mathcal{D}^{f'}_{G''}$. If $x^{1,*}$ is not an extreme point of $\mathcal{P}^{f'}_{G''}$, then there exist extreme points of $\mathcal{P}^{f'}_{G''}$: $y^1, y^2, \ldots, y^p$ such that $x^{1,*} = \sum_{i=1}^{p} \lambda_i y^i$, with $\lambda \in \mathbb{R}^p_+$, $\sum_{i=1}^{p} \lambda_i = 1$, p a positive integer. This would imply $x^* = \sum_{i=1}^{p} \lambda_i (y^i, x^{2,*})$. Note that for each $i \in \{1, 2, \ldots, p\}$ the point $(y^i, x^{2,*})$ belongs to $\mathcal{P}^f_G$ (using the property that $x^*_w = 1$). This contradicts the fact that x^* is an extreme point of $\mathcal{P}^f_G$. So, necessarily, $x^{1,*}$ is an extreme point of $\mathcal{P}^{f'}_{G''}$ with a fractional entry (from our choice of C), a contradiction with our minimality assumption on G. $\square$

Proof of Claim 3 Assume, for a contradiction, that there exists some node $z \in N(u)$ such that $x^*_z = 0$ and $z \notin W$. Note that in this case, necessarily, $\overline{q} < f_u$. Let $W' = W \cup \{z\}$. Then, since the inequality

$$(\overline{q} + 1)x_u + \sum_{w \in W'} x_w \geq \overline{q} + 1$$

must also be satisfied by x^*, this implies $x^*_u = 1$ and $x_w = 0$, for all $w \in W$. But this contradicts our definition of $\mathcal{E}$ (which is assumed to contain a maximum number of trivial inequalities). $\square$

Proof of Claim 4 Let $w \in V$ denote a leaf in G, and let u denote its unique neighbor. Let $G' = (V', E') = G[V \backslash \{w\}]$ denote the subgraph which is induced by $V \backslash \{w\}$. Let $\overline{x}^*$ denote the restriction of x^* to its components indexed by V'.

Assume firstly that $x^*_w = 1$. Then one can easily check that $\overline{x}^*$ is a fractional extreme point of $\mathcal{P}^{f'}_{G'}$ with $f' \in \mathcal{F}_{G'}$ such that $f'_v = f_v$, for all $v \in V' \backslash \{u\}$, and $f'_u = \max\{0, f_u - 1\}$. This contradicts the minimality of G.

Then assume that $x^*_w = 0$. Let $f'' \in \mathcal{F}_{G'}$ be defined as follows: $f''_v = f_v$, for all $v \in V'$. For the case when $f_u < d_u - 1$, one can easily check that $\overline{x}^* \in \mathcal{P}^{f''}_{G'}$ and that it must correspond to an extreme point (since otherwise we could express x^* as a convex combination of the incidence vectors of f-dominating sets in G, a contradiction). Then, for the case $f_u = d_u - 1$, note that the formulation of $\mathcal{P}^{f''}_{G'}$ does not contain any inequality of type (3) with center u. The latter are replaced in the formulation of $\mathcal{P}^{f''}_{G'}$ by f''-clique inequalities involving the variable x_u. Note that the system defining

$\mathcal{P}_G^f$ contains the inequalities $x_u + x_z + x_w \geq 1$, for all $z \in N(u)\backslash\{w\}$. And since we have $x_w^* = 0$, $\overline{x}^*$ also satisfies the f''-clique inequalities present in the description of $\mathcal{P}_{G'}^{f''}$ and involving x_u. It then follows that $\overline{x}^* \in \mathcal{P}_{G'}^{f''}$. And it must correspond to a fractional extreme point of $\mathcal{P}_{G'}^{f''}$ (since otherwise x^* could also be expressed as a convex combination of different feasible solutions in $\mathcal{P}_G^f$), thus contradicting the minimality of G. □

Proof of Claim 5 Assume, for a contradiction, that there exists in G a leaf w that is a neighbor of some node u satisfying $f_u = d_u$. In this situation, there are only 3 inequalities in the description of $\mathcal{P}_G^f$ involving x_w: $x_w \geq 0$, $x_w \leq 1$ and $x_u + x_w \geq 1$. By Claim 1, we know that $\mathcal{E}$ does not contain any equation corresponding to an f-clique inequality involving a leaf, which implies $x_w^* \in \{0, 1\}$, a contradiction to Claim 4. □

Proof of Claim 6 Assume for a contradiction that there exists some node $u \in V$ that is adjacent to (at least) two leaves v_1 and v_2. By Claim 5, we have $f_u < d_u$. Non-singularity of the system $\mathcal{E}$ together with the fact that $x_{v_i}^* \in]0, 1[$ for $i = 1, 2$ (due to Claim 4) imply the existence of at least one equation corresponding to a partial neighborhood inequality with center u and such that its support contains exactly one variable among x_{v_1}, x_{v_2} (since otherwise, x^* can be expressed as a convex combination of two other feasible solutions of $\mathcal{P}_G^f$, thus contradicting the fact that x^* is an extreme point), w.l.o.g. assume that it contains x_{v_1} and not x_{v_2}. Then, there must exist another partial neighborhood inequality with center u and whose support contains x_{v_2} (at least). W.l.o.g. let I_1 denote an equation of $\mathcal{E}$ corresponding to a partial neighborhood inequality whose support contains x_{v_1} and *not* x_{v_2}. Then, let I_2 denote an equation of the system $\mathcal{E}$ whose support contains x_{v_2}. So the inequality I_i ($i \in \{1, 2\}$) has the following form :

$$q_i x_u + \sum_{w \in W_i = N(u)\backslash Z_i} x_w \geq q_i,$$

with $q_i \in \{1, 2, \ldots, f_u\}$, $Z_i \subseteq N(u)$ with $|Z_i| = f_v - q_i$, $v_1 \in W_1$, $v_2 \notin W_1$, $v_2 \in W_2$.

Consider the inequality obtained using $W_1' = W_1 \cup \{v_2\}$. Remark that this inequality belongs to the description of $\mathcal{P}_G^f$. And since $x^* \in \mathcal{P}_G^f$, we have $(q_1 + 1)x_u^* + \sum_{W_1 \cup \{v_2\}} x_w^* \geq q_1 + 1$, thus implying $x_u^* + x_{v_2}^* \geq 1$.

Consider now the inequality I_2. By Claim 1, necessarily $q_2 \geq 2$. Also, x^* satisfies

$$(q_2 - 1)x_u + \sum_{w \in W_2 \backslash \{v_2\}} x_w \geq q_2 - 1.$$

which implies (with the inequality we established above) $x_{v_2}^* + x_u^* = 1$, a contradiction with Claim 1. □

Proof of Claim 7 At the beginning of the proof, we mentioned that G is a tree that is not a star and with at least four nodes. Let G' denote the graph obtained from G by removing all the leaves (and their incident edges). Necessarily, G' is a tree with at least two nodes (using the assumption that G is not a star). Let v denote a leaf in G'. Given the definition of G' and Claim 6, v satisfies the claim. □

At this point, Theorem 1 is proved, and we now discuss some related properties.

The LP formulation of the problem $[MW_f]$ stemming from Theorem 1 may contain an exponential number of constraints (see Remark 1 at the end of Sect. 3.2). Despite this fact, we next show that $[MW_f]$ can be solved in polynomial time if the graph G is a tree. To do so, we can resort to the fundamental result that is the polynomial equivalence between optimization and separation in linear programming (Grötschel et al. 1981).

The separation problem with respect to the f-dominating set polytope $\mathcal{D}_G^f$ is denoted by $[SEP\_DS_f]$ in what follows. It consists, for some given point $\widehat{x} \in \mathbb{R}^n$, in determining whether it belongs to $\mathcal{D}_G^f$, and, if not, in giving an inequality which is valid for $\mathcal{D}_G^f$ but is violated by $\widehat{x}$.

Proposition 11 *Problem $[SEP\_DS_f]$ can be solved in polynomial time if the graph G is a tree.*

Proof Let $\widehat{x} \in \mathbb{R}^n$ be some given point. In order to determine whether $\widehat{x}$ belongs to $\mathcal{D}_G^f$ we check if $\widehat{x}$ satisfies all the inequalities mentioned in Theorem 1.

One can check in linear time (in $\mathcal{O}(n)$) if the trivial inequalities are satisfied. Since the graph is a tree, the number of f-clique inequalities is at most the number of edges, which is linear in the number of nodes. And since each f-clique is composed of two nodes, checking their violation can also be done in linear time. In order to determine whether $\widehat{x}$ satisfies all the partial neighborhood inequalities, we may proceed as follows, for each node $v \in V$ satisfying $1 \leq f_v < d_v$. First, we order the neighbors of v by increasing value $\widehat{x}_u, u \in N(v)$. (This can be done in time $\mathcal{O}(d_v \log(d_v))$.) Assume that the neighbors of v are $u_1, u_2, \ldots, u_{d_v}$, after reordering. For $k = f_v - 1, \ldots, 1, 0$, we evaluate the quantity $(f_v - k)\widehat{x}_v + \sum_{i=1}^{d_v - k} \widehat{x}_{u_i}$. If this quantity is strictly lower than $(f_v - k)$, a violated inequality has been found. If, instead, no violated inequality exists after all the nodes have been processed, then we can conclude $\widehat{x} \in \mathcal{D}_G^f$. The whole complexity of such a procedure for separation is then $\mathcal{O}(n \log n)$. □

From the equivalence between optimization and separation (Grötschel et al. 1981), Theorem 1 and Proposition 11, it follows that $[MW_f]$ can be solved in polynomial time if the graph G is a tree. In the next section, we present a combinatorial algorithm, showing that this can be done in linear time.

4 A linear-time combinatorial algorithm to solve $[MW_f]$ for trees

We consider the problem $[MW_f]$ for the particular case when the graph $G = (V, E)$ is a rooted tree, extending results by Natarajan and White (1978) on the classical dominating set problem $[MW_\mathbf{1}]$, and Hwang and Chang (1991) on the minimum cardinality f-dominating set problem. Let $r \in V$ denote the root node, and, for each node $v \in V$, let $\sigma(v)$ denote the unique ancestor of node v, in the path from r

to v. For a given $v \in V$, let $T_v = (V_v, E_v)$ denote the subtree of G with root v, i.e., the collection of all the paths from v to the leaves, that do not contain the root node r. Finally, let $H_v = \{u \in V : \sigma(u) = v\}$ be the set of immediate successors (or children) of node v.

To compute an optimum f-dominating set we use a recursive dynamic programming method which associates three labels to each node $v \in V$:

- $C_1(v)$: minimum weight of an f-dominating set in T_v,
- $C_2(v)$: minimum weight of an f-dominating set in T_v containing node v,
- $C_3(v)$: minimum weight of an f^v-dominating set in T_v,

where $f_u^v = f_u$, for all $u \in V \backslash \{v\}$, and $f_v^v = \max\{f_v - 1, 0\}$. The next lemma shows that we can compute these labels using a bottom-up recursion, from the leaves to the root r. Label $C_1(r)$ gives the optimal solution value for the entire graph G.

Lemma 2 *Given a tree $G = (V, E)$, rooted at node r, $c \in \mathbb{R}^n_+$, and a subtree T_v rooted at node v, one can compute the labels $C_1(v)$, $C_2(v)$ and $C_3(v)$ as follows. If node v is a leaf, we have*

$$C_1(v) = \begin{cases} c_v, & \text{if } f_v = 1, \\ 0, & \text{otherwise}, \end{cases}$$

while $C_2(v) = c_v$ and $C_3(v) = 0$. Otherwise,

$$C_1(v) = \min\left\{C_2(v), \min_{\substack{I \subseteq H_v \\ |I| = f_v}} \left\{\sum_{u \in I} C_2(u) + \sum_{u \in H_v \backslash I} C_1(u)\right\}\right\}, \tag{4}$$

$$C_2(v) = c_v + \sum_{u \in H_v} C_3(u), \tag{5}$$

$$C_3(v) = \min\left\{C_2(v), \min_{\substack{I \subseteq H_v \\ |I| = \max\{f_v - 1, 0\}}} \left\{\sum_{u \in I} C_2(u) + \sum_{u \in H_v \backslash I} C_1(u)\right\}\right\}. \tag{6}$$

where the minimization subproblem over I in the expressions of $C_1(v)$ and $C_3(v)$ takes value $+\infty$ in case $f_v > |H_v|$ or $f_v - 1 > |H_v|$, respectively. Also the sum over an empty set of indices is assumed to take value 0.

Proof Consider a node $v \in V$. If v is a leaf, the correctness of the labels is straightforward, so let us suppose $d_v > 1$. Let D denote a minimum cost f-dominating set in T_v. Label $C_1(v)$ gives the value of an optimal f-dominating set for T_v. If $v \in D$, then $C_1(v) = C_2(v)$, by definition, otherwise f_v nodes from H_v must belong to D. The inner minimum in (4), for each possible choice of f_v successors of v, considers the optimal costs of the corresponding subtrees, plus the optimal cost for the subtrees rooted at the remaining successors. The outer minimum chooses the best option between the two above cases: $v \in D$ and $v \notin D$.

$C_2(v)$ gives the optimal solution value for T_v if $v \in D$. Therefore, it accounts for c_v, plus the sum of the optimal cost of each subtree rooted at a node $u \in H_v$, provided that

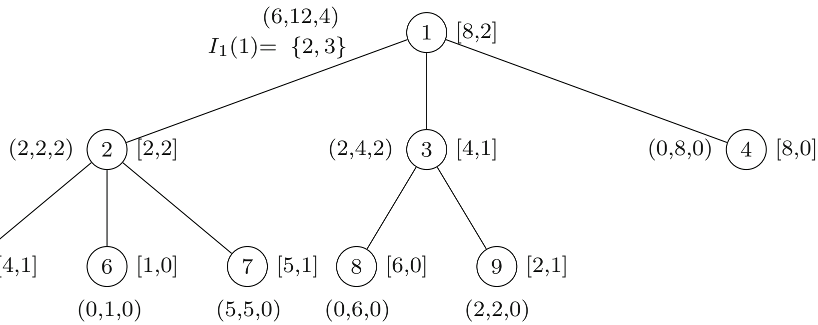

Fig. 1 Labels computed by the linear time algorithm

u is already dominated by $v \in D$. Therefore $C_3(u)$ is the required value for subtree T_u.

$C_3(v)$ is the optimal solution value for T_v if the domination requirement of v is set to $\max\{f_v - 1, 0\}$. The formula is similar to the one used to compute the label $C_1(v)$. The outer minimum chooses between the case $v \in D$ (where $C_3(v) = C_2(v)$ gives the optimal value), and the case $v \notin D$, where $\max\{f_v - 1, 0\}$ successors of v must be inserted in D. The inner minimum in (6) operates as the equivalent minimum in (4), but considering subsets of H_v with cardinality $\max\{f_v - 1, 0\}$. □

From Lemma 2, we deduce a dynamic programming algorithm to compute both the optimal objective value of $[MW_f]$ and an optimal solution (by storing the sets generating the minima in the definition of the labels). This is illustrated by the next example.

Example 1 Figure 1 shows a small example with a tree of nine nodes and root 1. Near each node v we report: (a) in square brackets, the weight c_v and the domination requirement f_v, respectively; (b) in parenthesis, the three labels $C_1(v)$, $C_2(v)$ and $C_3(v)$. When the value of C_1 or C_3 is determined by the inner minimum and is used in the second phase (described later) to determine an optimal solution, we report also the set which produces the minimum. In a first phase, the algorithm computes the labels recursively, starting from the leaves and moving toward the root, using formulas (4)–(6). The value of the minimum f-dominating set is given by $C_1(1) = 6$. Then, in a second phase, we can identify a set D of nodes corresponding to an optimal solution, by starting from root 1, with $D = \emptyset$. The fact that there is no ancestor for the root, and that $C_1(1) < C_2(1)$ indicate that node 1 is not in the solution. Since $f_1 = 2$ we know that two of its successors have been used when computing the inner minimum in (4). Set $I_1(1)$, stored in the first phase, contains the two nodes giving the minimum, namely nodes 2 and 3. We set $D = \{2, 3\}$ and we consider each of these nodes in turn. The fact that they have been fixed in the solution implies that labels $C_2(2)$ and $C_2(3)$ have been chosen by the recursion for these nodes, and using (5) we know that labels $C_3(j)$ have been chosen for $j = 5, \ldots, 9$. All these labels have value zero, so none of these nodes is in the solution. It remains to consider node 4. Its ancestor, the root, is not in D and $C_1(4) < C_2(4)$, so the node is not in solution.

The next result directly follows and extends the one presented by Farber 1984 showing $[MW_{\mathbf{1}}]$ can be solved in linear time for trees.

Theorem 2 *If $G = (V, E)$ is a tree rooted at r, then problem $[MW_f]$ can be solved in linear time.*

Proof Lemma 2 shows that label $C_1(r)$, computed using (4)–(6), gives the optimal solution value. The second phase of the dynamic programming algorithm retrieves the nodes of the set D which produce the value $C_1(r)$, i.e., an optimal set. In order to complete the proof, it is sufficient to show that the labels can be computed in time $\mathcal{O}(d_v)$, for each node $v \in V$. This is trivial for the case when v is a leaf or $f_v = 0$. So, let v denote a node that is not a leaf and such that $f_v > 0$. Observe that the argument of the inner minimization in (4) and (6) can be rewritten as

$$\sum_{u \in I} (C_2(u) - C_1(u)) + \sum_{u \in H_v} C_1(u).$$

Let S_1 (resp. S_2) denote the node subset of H_v corresponding to the f_v (resp. $f_v - 1$) smallest values in the set $\mathcal{C} = \{C_2(u) - C_1(u) : u \in H_v\}$. It can be easily checked that S_1 (resp. S_2) corresponds to an optimal solution for the minimization problem over I in the expression of $C_1(v)$ (resp. $C_3(v)$).

Using the results by Blum et al. 1973, the f_v-th smallest number $\widehat{w}$ in $\mathcal{C}$ can be found in time $\mathcal{O}(|H_v|)$. Then, iterating (in an arbitrary order) over H_v, putting into S_1 all the nodes corresponding to a value smaller than $\widehat{w}$, and then filling up S_1 with nodes associated with the quantity $\widehat{w}$ to have $|S_1| = f_v$, we deduce that the inner minimization problem in (4) can be found in time $\mathcal{O}(d_v)$. The argument to prove that $C_3(v)$ can be determined within the same time complexity is analogous. □

5 Conclusions

In this paper, we presented descriptions of the f-dominating set polytope when the graph is a star or a tree. They namely lead to the polynomial time solvability of the problem $[MW_f]$ for these graphs. In addition, we presented a linear time algorithm for this problem on trees. Further research work will be directed towards extensions of these results for other graph families. The development of efficient methods and further investigations on the polyhedral structures of polytopes related to generalizations of other variants of domination are under work.

References

Aoun B, Boutaba R, Iraqi Y, Kenward G (2006) Gateway placement optimization in wireless mesh networks with QoS constraints. IEEE J Sel Areas Commun 24(11):2127–2136

Baïou M, Barahona F (2014) The dominating set polytope via facility location. In: ISCO 2014, LNCS 8596, pp 38–49

Benzaken C, Hammer PL (1978) Linear separation of domination sets in graphs. Ann Discrete Math 3:1–10

Bermudo S, Hernandez-Gomez JC, Sigaretta JM (2018) On the total k-domination in graphs. Discuss Math Graph Theory 38:301–317

Bianchi S, Nasini G, Tolomei P (2010) The set covering problem on circulant matrices: polynomial instances and the relation with the dominating set problem on webs. Electronic Notes Discrete Math 36:1185–1192

Blum M, Floyd RW, Pratt V, Rivest RL, Tarjan RE (1973) Time bounds for selection. J Comput Syst Sci 7(4):448–461

Booth KS (1980) Dominating sets in chordal graphs. Technical Report CS-80-34. Univ. Waterloo, Waterloo, Ontario, Canada

Booth KS, Johnson JH (1982) Dominating sets in chordal graphs. SIAM J Comput 11:191–199

Bouamama S, Blum C (2016) A hybrid algorithmic model for the minimum weight dominating set problem. Simul Model Pract Theory 64:57–78

Bouchakour M, Mahjoub AR (1997) One-node cutsets and the dominating set polytope. Discrete Math 165(166):101–123

Bouchakour M, Contenza TM, Lee CW, Mahjoub AR (2008) On the dominating set polytope. Eur J Comb 29:652–661

Chen YP, Liestman AL (2002) Approximating minimum size weakly-connected dominating sets for clustering mobile ad hoc networks. In: Proceedings of the 3rd ACM international symposium on mobile ad hoc networking & computing (MobiHoc '02), ACM, New York, pp 165–172

Chen B, Zhou S (1998) Upper bounds for f-domination number of graphs. Discrete Math 185:239–243

Chlebík M, Chlebíková J (2008) Approximation hardness of dominating set problems in bounded degree graphs. Inf Comput 206:1264–1275

Couture M, Barbeau M, Bose P, Kranakis E (2006) Incremental construction of k-dominating sets in wireless sensor networks. In: Proceedings of the 10th international conference on principles of distributed systems, pp 202–214

Chvátal V (1975) On certain polytopes associated with graphs. J Comb Theory (B) 18:138–154

Dell'Amico M, Neto J (2017) On total f-domination: polyhedral and algorithmic results. Technical report. University of Modena and Reggio Emilia, Italy

Duh R, Fürer M (1997) Approximation of k-set cover by semi-local optimization. In: Proceedings of the 29th ACM symposium on theory of computing, STOC, pp 256–264

Farber M (1984) Domination, independent domination, and duality in strongly chordal graphs. Discrete Appl Math 7:115–130

Foerster KT (2013) Approximating fault-tolerant domination in general graphs. In: Proceedings of the tenth workshop on analytic algorithmics and combinatorics (ANALCO), pp 25–32

Garey MR, Johnson DS (1979) Computers and intractability: a guide to the theory of NP-completeness. Freeman, San Francisco

Grötschel M, Lovàsz L, Schrijver A (1981) The ellipsoid method and its consequences in combinatorial optimization. Combinatorica 1(2):169–197

Hammer PL, Johnson EL, Peled UN (1975) Facet of regular 0–1 polytopes. Math Program 8:179–206

Haynes TW, Hedetniemi ST, Slater JB (1998a) Fundamentals of domination in graphs. Marcel Dekker, New York City

Haynes TW, Hedetniemi ST, Slater JB (1998b) Domination in graphs: advanced topics. Marcel Dekker, New York City

Hedetniemi S, Hedetniemi S, Laskar R (1985) Domination in trees: models and algorithms. Graph theory with applications to algorithms and computer science. Wiley, New York, pp 423–442

Hedetniemi ST, Laskar R, Pfaff J (1986) A linear algorithm for finding a minimum dominating set in a cactus. Discrete Appl Math 13:287–292

Henning M, Yeo A (2013) Total domination in graphs. Springer monographs in mathematics. Springer, Berlin

Houmaidi ME, Bassiouni MA (2003), K-weighted minimum dominating sets for sparse wavelength converters placement under non-uniform traffic. In: Proceedings of MASCOTS'03, pp 56–61

Hwang SF, Chang GJ (1991) The k-neighbor domination problem. Eur J Oper Res 52:373–377

Kikuno T, Yoshida N, Kakuda Y (1983) A linear algorithm for the domination number of a series-parallel graph. Discrete Appl Math 5:299–311

Mahjoub AR (1983) Le polytope des absorbants dans une classe de graphe à seuil. Ann Discrete Math 17:443–452

Natarajan KS, White LJ (1978) Optimum domination in weighted trees. Inf Process Lett 7(6):261–265

Nemhauser GL, Trotter LE (1974) Properties of vertex packing and independence system polyhedra. Math Program 6:48–61
Padberg MW (1974) Perfect zero-one matrices. Math Program 6:180–196
Potluri A, Singh A (2013) Hybrid metaheuristic algorithms for minimum weight dominating set. Appl Soft Comput 13:76–88
Shen C, Li T (2010) Multi-document summarization via the minimum dominating set. In: Proceedings of the 23rd international conference on computational linguistics, pp 984–992
Stracke C, Volkmann L (1993) A new domination conception. J Graph Theory 17:315–323
Subhadrabandhu D, Sarkar S, Anjum F (2004) Efficacy of misuse detection in adhoc networks. In: Proceedings of the first annual IEEE communications society conference on sensor and ad hoc communications and networks, pp 97–107
Wu J, Li H (1999) On calculating connected dominating set for efficient routing in ad-hoc wireless networks. In: Proceedings of DIALM '99, ACM, New York, pp 7–14
Wu P, Wen JR, Liu H, Ma WY (2006) Query selection techniques for efficient crawling of structured web sources. Proceedings of ICDE'06, pp 47
Zhou SM (1996) On f-domination number of a graph. Czechoslov Math J 46(3):489–499
Zou F, Wang Y, Xu X, Li X, Du H, Wan P, Wu W (2011) New approximations for minimum-weighted dominating sets and minimum-weighted connected dominating sets on unit disk graphs. Theor Comp Sci 412:198–208