.

HIGHER DIFFERENTIABILITY FOR SOLUTIONS TO A CLASS OF OBSTACLE PROBLEMS

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ABSTRACT. We establish the higher differentiability of integer and fractional order of the solutions to a class of obstacle problems assuming that the gradient of the obstacle possesses an extra (integer or fractional) differentiability property. We deal with the case in which the solutions to the obstacle problems satisfy a variational inequality of the form

$$\int_\Omega \langle \mathcal{A}(x, Du), D(\varphi - u)\rangle\, dx \geq 0 \qquad\qquad \forall \varphi \in \mathcal{K}_\psi(\Omega)$$

where $\mathcal{A}$ is a p-harmonic type operator, $\psi \in W^{1,p}(\Omega)$ is a fixed function called obstacle and $\mathcal{K}_\psi = \{w \in W^{1,p}(\Omega) : w \geq \psi \text{ a.e. in } \Omega\}$ is the class of the admissible functions. We prove that an extra differentiability assumption on the gradient of the obstacle transfers to Du with no losses in the natural exponent of integrability, provided the partial map $x \mapsto \mathcal{A}(x, \xi)$ possesses a suitable differentiability property measured or in the scale of the Sobolev space $W^{1,n}$ or in that of the critical Besov-Lipschitz spaces $B^{\alpha}_{\frac{n}{\alpha},q}$, for a suitable $1 \leq q \leq +\infty$.

1. INTRODUCTION

The aim of this paper is the study of the higher differentiability properties of the gradient of the solutions $u \in W^{1,p}(\Omega)$ to variational obstacle problems of the form

$$\min\left\{\int_\Omega F(x, Dw) : w \in \mathcal{K}_\psi(\Omega)\right\}. \tag{1.1}$$

The function $\psi : \Omega \to [-\infty, +\infty)$, called *obstacle*, belongs to the Sobolev class $W^{1,p}(\Omega)$ and the class $\mathcal{K}_\psi(\Omega)$ is defined as follows

$$\mathcal{K}_\psi(\Omega) := \left\{w \in W^{1,p}(\Omega) : w \geq \psi \text{ a.e. in } \Omega\right\}. \tag{1.2}$$

To avoid trivialities, we always assume that the set $\mathcal{K}_\psi$ is not empty.
It is worth observing that $u \in W^{1,p}(\Omega)$ is a *solution to the obstacle problem in* $\mathcal{K}_\psi(\Omega)$ if and only if $u \in \mathcal{K}_\psi$ solves the following variational inequality

$$\int_\Omega \langle \mathcal{A}(x, Du), D(\varphi - u)\rangle\, dx \geq 0, \tag{1.3}$$

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for all $\varphi \in \mathcal{K}_\psi(\Omega)$, where we set

$$\mathcal{A}(x,\xi) = D_\xi F(x,\xi).$$

We assume that there exist positive constants ν, L, ℓ and an exponent $p \geq 2$ such that the following p-ellipticity and p-growth conditions are satisfied:

$$\langle \mathcal{A}(x,\xi) - \mathcal{A}(x,\eta), \xi - \lambda \rangle \geq \nu |\xi - \eta|^2 (1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \qquad (\mathcal{A}1)$$

$$|\mathcal{A}(x,\xi) - \mathcal{A}(x,\eta)| \leq L\, |\xi - \eta| (1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \qquad (\mathcal{A}2)$$

$$|\mathcal{A}(x,\xi)| \leq \ell\, (1 + |\xi|^2)^{\frac{p-1}{2}} \qquad (\mathcal{A}3),$$

for a.e. $x \in \Omega$ and every $\xi, \eta \in \mathbb{R}^n$.

The study of the regularity theory for obstacle problems is a classical topic in Partial Differential Equations and Calculus of Variations. The obstacle problem appeared in the mathematical literature in the work by G. Stampacchia [42] in the special case $\psi = \chi_E$ and related to the capacity of a subset $E \subset\subset \Omega$; in an earlier independent work, G. Fichera [21] solved the first unilateral problem, the so-called *Signorini problem* in elastostatics.
It is well known that the solution to the obstacle problem cannot be of class $\mathcal{C}^2$ independently of how regular the obstacle is; this led to the origin of the concept of weak solution and to the theory of *variational inequalities*, after the fundamental work of J.L. Lions and G. Stampacchia [33] (for more details we refer to classical monographs [1], [12], [22], [29], [41]); these problems can generally be solved by applying methods of functional analysis, so the question to give conditions to establish that weak solutions are, in many cases, classical ones is of fundamental importance (see [10]).
It is usually observed that the regularity of solutions to the obstacle problems is influenced by the one of the obstacle; for example, for linear obstacle problems, obstacle and solutions have the same regularity ([7], [11], [29]). This does not apply in the nonlinear setting, hence along the years, there have been intense research activity for the regularity of the obstacle problem in this direction.
A first important result by J.H. Michael and W.P. Ziemer [34] establishes Hölder continuity of solutions to the obstacle problem when the obstacle itself is Hölder continuous. H. Choe [13] proved that if the gradient of obstacle is Hölder continuous, the same happens for the gradient of solutions. Other results that deserved to be quoted are [14], [32], [23], in the case of a single obstacle problem, and [4] in the case of double obstacle problems. Since then, many regularity results have been obtained in different situations: for instance we quote [16] in the setting of Morrey and Campanato spaces, [17], [19], [8], [35] in the setting of nonstandard growth conditions (see also [18], [9], [36] for Calderón-Zygmund case). Moreover we refer to [3], [37] for gradient continuity for nonlinear obstacle problems, [20] for global results up to the boundary, [5] for the parabolic case, [6] for the porous medium problem.
As far as we know, such analysis has not been carried out in case the gradient of the obstacle ψ possesses some extra differentiability properties.
Here, assuming that the gradient of the obstacle belongs to a suitable Sobolev class, of integer or fractional order, we are interested in finding conditions on the partial map $x \mapsto \mathcal{A}(x,\xi)$ in order to obtain that the extra differentiability property of the obstacle transfers to the gradient of the solution with no loss in the order of differentiability.
Our analysis comes from the fact that the regularity of the solutions to the obstacle problem (1.1) is strictly connected to the analysis of the regularity of the solutions to partial differential equations of the form

$$\text{div}\mathcal{A}(x,Du) = \text{div}\mathcal{A}(x,D\psi)$$

.
It is well known that no extra differentiability properties for the solutions of partial differential

equations of the type

$$\mathrm{div}\mathcal{A}(x, Du) = \mathrm{div}\,\mathcal{G}, \tag{1.4}$$

both in the setting of integer and fractional Sobolev spaces, can be expected even if $\mathcal{G}$ is smooth, unless some assumption is given on the x-dependence of $\mathcal{A}$.

On the other hand, recent results concerning the higher differentiability of solutions to (1.4) show that the weak differentiability of integer or fractional order of the map $\mathcal{A}$, as function of the x-variable, is a sufficient condition (see [38], [39], [40], [24], [25] for the case of Sobolev space with integer order and [2], [15] for the fractional one).

Indeed, in [38], [39], [24], the higher differentiability of the solutions to the equation in (1.4) is obtained assuming a $W^{1,n}$ type regularity on the partial map $x \mapsto \mathcal{A}(x,\xi)$ that is expressed through a pointwise condition on $\mathcal{A}(\cdot,\xi)$ that relies on the characterization of the Sobolev spaces due to P. Hajłasz ([27]).

More precisely, for Carathéodory functions $\mathcal{A}$ satisfying assumptions ($\mathcal{A}1$)-($\mathcal{A}3$), it is assumed that there exists a non negative function $\kappa \in L^n_{\mathrm{loc}}(\Omega)$ such that the following inequality

$$|\mathcal{A}(x,\xi) - \mathcal{A}(y,\xi)| \le (\kappa(x) + \kappa(y))|x-y|(1+|\xi|^2)^{\frac{p-1}{2}} \tag{$\mathcal{A}4$}$$

holds true for a.e. $x, y \in \Omega$ and for every $\xi \in \mathbb{R}^n$.

It turns out that this condition is sufficient also in our context of obstacle problems to prove that the differentiability of the gradient of the obstacle transfers to the gradient of the solution with no loss in the order of differentiability. More precisely, our first result is the following:

Theorem 1.1. *Let $\mathcal{A}(x,\xi)$ satisfy ($\mathcal{A}1$)–($\mathcal{A}4$) for an exponent $2 \le p < n$ and let $u \in \mathcal{K}_\psi(\Omega)$ be the solution to the obstacle problem* (1.3)*. Then we have*

$$D\psi \in W^{1,p}_{\mathrm{loc}}(\Omega) \Rightarrow (1+|Du|^2)^{\frac{p-2}{4}} Du \in W^{1,2}_{\mathrm{loc}}(\Omega). \tag{1.5}$$

The existing literature on the regularity of solutions to partial differential equations of the form (1.4) shows that the same phenomenon occurs also in case of a fractional Sobolev assumption on the data.

In fact, it has been proven that, if $\mathcal{G}$ belongs to a suitable Besov-Lipschitz space, the gradient of the solution to the equation (1.4) gains an extra fractional differentiability that can be expressed through its belonging to a suitable Besov space, provided the partial map $x \mapsto \mathcal{A}(x,\xi)$ belongs to a Besov space too. Also such Besov regularity of the map $\mathcal{A}(\cdot,\xi)$ can be expressed through the pointwise characterization of these spaces due to P. Koskela, D. Yan, Y. Zhou ([30]).

More precisely, given $0 < \alpha < 1$ and $1 \le q \le \infty$ we say that ($\mathcal{A}5$) *is satisfied* if there exists a sequence of measurable non-negative functions $g_k \in L^{\frac{n}{\alpha}}(\Omega)$ such that

$$\sum_k \|g_k\|^q_{L^{\frac{n}{\alpha}}(\Omega)} < \infty,$$

and at the same time

$$|\mathcal{A}(x,\xi) - \mathcal{A}(y,\xi)| \le \,(g_k(x) + g_k(y))\,|x-y|^\alpha\,(1+|\xi|^2)^{\frac{p-1}{2}} \tag{$\mathcal{A}5$}$$

for each $\xi \in \mathbb{R}^n$ and almost every $x, y \in \Omega$ such that $2^{-k}\mathrm{diam}(\Omega) \le |x-y| < 2^{-k+1}\mathrm{diam}(\Omega)$. We will shortly write then that $(g_k)_k \in \ell^q(L^{\frac{n}{\alpha}}(\Omega))$.

If $\mathcal{A}(x,\xi) = \gamma(x)|\xi|^{p-2}\xi$ and $\Omega = \mathbb{R}^n$ then ($\mathcal{A}5$) says that the function $\gamma(x)$ belongs to $B^\alpha_{\frac{n}{\alpha},q}$, see [30, Theorem 1.2] and Section 2 for more details.

Under assumption ($\mathcal{A}5$), we are able to prove that the extra fractional differentiability of the obstacle transfers to the solutions to the obstacle problem (1.3), both measured in the Besov scale. Indeed, we have the following:

Theorem 1.2. *Assume that $\mathcal{A}(x,\xi)$ satisfies $(\mathcal{A}1)$–$(\mathcal{A}3)$ and $(\mathcal{A}5)$ for an exponent $2 \le p < \frac{n}{\alpha}$. Let $u \in \mathcal{K}_\psi(\Omega)$ be the solution to the obstacle problem* (1.3). *Then the following implication*

$$D\psi \in B^{\alpha}_{p,q} \Rightarrow (1+|Du|^2)^{\frac{p-2}{4}} Du \in B^{\alpha}_{2,q}, \tag{1.6}$$

holds locally, provided $q \le p^*_\alpha = \frac{np}{n-\alpha p}$.

Our proofs are achieved by means of the difference quotient method, that is quite natural when trying to establish higher differentiability results. Here the difficulties come from the set of admissible test functions that have to take into account the presence of the obstacle. In order to overcome this problem, we need to consider difference quotient involving both the solution and the obstacle, so that the test function satisfies the constraint of belonging to the admissible class $\mathcal{K}_\psi$. Moreover, in the case of Besov coefficients, the proof relies on the fact that the Besov spaces $B^{\alpha}_{\frac{n}{\alpha},q}$, for every $1 \le q \le +\infty$, continuously embed into the VMO space of Sarason (see [28] and Section 2 below). Obstacle problems with VMO coefficients are known to have a nice L^p theory (see [9]) and we take advantage from this result through the embedding Theorems in Besov spaces ([28] and Section 2.) In case the order of fractional differentiability of the coefficients differs from that of the obstacle, we have that the gradient of the solution still inherits an extra fractional differentiability which is the minimum between the one of the coefficients and the one of the obstacle, all measured in the scale of Besov spaces. Indeed, combining the inclusion between Besov spaces (see Lemma 2.3 in Section 2) with Theorem 1.2, allows us to establish the following:

Corollary 1.3. *Assume that $\mathcal{A}(x,\xi)$ satisfies $(\mathcal{A}1)$–$(\mathcal{A}3)$ and $(\mathcal{A}5)$. Let $u \in \mathcal{K}_\psi(\Omega)$ be the solution to the obstacle problem* (1.3). *Then the following implication*

$$D\psi \in B^{\beta}_{p,q} \Rightarrow (1+|Du|^2)^{\frac{p-2}{4}} Du \in B^{\min\{\alpha,\beta\}}_{2,q}, \tag{1.7}$$

holds locally, provided $q \le p^*_\beta = \frac{np}{n-\beta p}$.

It is also worth mentioning that, in the scale of Besov spaces, a regularity of the type $B^{\alpha}_{p,\infty}$ is the weakest one to assume both on the coefficients and on the gradient of the obstacle (see Lemma 2.3 in Section 2 below). In this case, we still have that the differentiability assumption on the gradient of the obstacle transfers to the solutions, but with a small loss. The main difference is that an analogous of the Sobolev imbedding Theorem holds but in a weaker form. Actually, we have the following:

Theorem 1.4. *Assume that $\mathcal{A}(x,\xi)$ satisfies $(\mathcal{A}1)$–$(\mathcal{A}3)$ for an exponent $2 \le p < \frac{n}{\alpha}$ and let $u \in \mathcal{K}_\psi(\Omega)$ be the solution to the obstacle problem* (1.3). *If there exist $0 < \alpha < 1$ and $g \in L^{\frac{n}{\alpha}}_{\mathrm{loc}}(\Omega)$ such that*

$$|\mathcal{A}(x,\xi) - \mathcal{A}(y,\xi)| \le (g(x)+g(y))\, |x-y|^\alpha \,(1+|\xi|^2)^{\frac{p-1}{2}}, \tag{$\mathcal{A}6$}$$

for a.e. $x, y \in \Omega$ and for every $\xi \in \mathbb{R}^n$, then the following implication

$$D\psi \in B^{\beta}_{p,\infty} \Rightarrow (1+|Du|^2)^{\frac{p-2}{4}} Du \in B^{\alpha}_{2,\infty}, \tag{1.8}$$

holds locally, provided $0 < \alpha < \beta < 1$.

Previous regularity results concerning local minimizers of integral functionals of the Calculus of Variations, under the assumption $(\mathcal{A}6)$, have been obtained by J. Kristensen and G. Mingione in [31].
The paper is organized as follows: Section 2 contains some notations and preliminary results, Section 3 is devoted to the proof of Theorem 1.1 while Section 4 is concerned with the proof of Theorem 1.2 and Corollary 1.3 and Section 5 with that of Theorem 1.4.

2. Notations and Preliminary Results

In this paper we shall denote by C or c a general positive constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies will be suitably emphasized using parentheses or subscripts. In what follows, $B(x,r) = B_r(x) = \{y \in \mathbb{R}^n : \ |y-x| < r\}$ will denote the ball centered at x of radius r. We shall omit the dependence on the center and on the radius when no confusion arises. For a function $u \in L^1(B)$, the symbol

$$\fint_B u(x)\,dx = \frac{1}{|B|}\int_B u(x)\,dx$$

will denote the integral mean of the function u over the set B.
It is convenient, to introduce an auxiliary function

$$V_p(\xi) := \Big(1+|\xi|^2\Big)^{\frac{p-2}{4}}\xi, \tag{2.1}$$

defined for all $\xi \in \mathbb{R}^n$. For the auxiliary function V_p, we recall the following estimate (see the proof of [26, Lemma 8.3]):

Lemma 2.1. *Let $1 < p < \infty$. There exists a constant $c = c(n,p) > 0$ such that*

$$c^{-1}\Big(1+|\xi|^2+|\eta|^2\Big)^{\frac{p-2}{2}} \le \frac{|V_p(\xi)-V_p(\eta)|^2}{|\xi-\eta|^2} \le c\Big(1+|\xi|^2+|\eta|^2\Big)^{\frac{p-2}{2}}$$

for any ξ, $\eta \in \mathbb{R}^n$.

2.1. Besov-Lipschitz spaces.

Given $h \in \mathbb{R}^n$ and $v : \mathbb{R}^n \to \mathbb{R}$, let us introduce the notation $\tau_h v(x) = v(x+h) - v(x)$. As in [28, Section 2.5.12], given $0 < \alpha < 1$ and $1 \le p,q < \infty$, we say that v belongs to the Besov space $B^\alpha_{p,q}(\mathbb{R}^n)$ if $v \in L^p(\mathbb{R}^n)$ and

$$\|v\|_{B^\alpha_{p,q}(\mathbb{R}^n)} = \|v\|_{L^p(\mathbb{R}^n)} + [v]_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)} < \infty,$$

where

$$[v]_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|v(x+h)-v(x)|^p}{|h|^{\alpha p}}dx\right)^{\frac{q}{p}}\frac{dh}{|h|^n}\right)^{\frac{1}{q}} < \infty. \tag{2.2}$$

Equivalently, we could simply say that $v \in L^p(\mathbb{R}^n)$ and $\frac{\tau_h v}{|h|^\alpha} \in L^q\left(\frac{dh}{|h|^n}; L^p(\mathbb{R}^n)\right)$. As usually, if one simply integrates for $h \in B(0,\delta)$ for a fixed $\delta > 0$ then an equivalent norm is obtained, because

$$\left(\int_{\{|h|\ge\delta\}}\left(\int_{\mathbb{R}^n}\frac{|v(x+h)-v(x)|^p}{|h|^{\alpha p}}dx\right)^{\frac{q}{p}}\frac{dh}{|h|^n}\right)^{\frac{1}{q}} \le c(n,\alpha,p,q,\delta)\,\|v\|_{L^p(\mathbb{R}^n)}.$$

Similarly, we say that $v \in B^\alpha_{p,\infty}(\mathbb{R}^n)$ if $v \in L^p(\mathbb{R}^n)$ and

$$[v]_{\dot{B}^\alpha_{p,\infty}(\mathbb{R}^n)} = \sup_{h\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|v(x+h)-v(x)|^p}{|h|^{\alpha p}}dx\right)^{\frac{1}{p}} < \infty. \tag{2.3}$$

Again, one can simply take supremum over $|h| \le \delta$ and obtain an equivalent norm. By construction, $B^\alpha_{p,q}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$. One also has the following version of Sobolev embeddings (a proof can be found at [28, Prop. 7.12]).

Lemma 2.2. *Suppose that $0 < \alpha < 1$.*

(a) *If $1 < p < \frac{n}{\alpha}$ and $1 \le q \le p^*_\alpha := \frac{np}{n-\alpha p}$, then there is a continuous embedding $B^\alpha_{p,q}(\mathbb{R}^n) \subset L^{p^*_\alpha}(\mathbb{R}^n)$.*

(b) *If $p = \frac{n}{\alpha}$ and $1 \le q \le \infty$, then there is a continuous embedding $B^\alpha_{p,q}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$, where BMO denotes the space of bounded mean oscillations* [26, Chapter 2].

For further needs, we recall the following inclusions ([28, Proposition 7.10 and Formula (7.35)]).

Lemma 2.3. *Suppose that $0 < \beta < \alpha < 1$.*

(a) *If $1 < p \le +\infty$ and $1 \le q \le r \le +\infty$ then $B^\alpha_{p,q}(\mathbb{R}^n) \subset B^\alpha_{p,r}(\mathbb{R}^n)$.*
(b) *If $1 < p \le +\infty$ and $1 \le q, r \le +\infty$ then $B^\alpha_{p,q}(\mathbb{R}^n) \subset B^\beta_{p,r}(\mathbb{R}^n)$.*
(c) *If $1 \le q \le \infty$, then $B^\alpha_{\frac{n}{\alpha},q}(\mathbb{R}^n) \subset B^\beta_{\frac{n}{\beta},q}(\mathbb{R}^n)$.*

Given a domain $\Omega \subset \mathbb{R}^n$, we say that v belongs to the local Besov space $B^\alpha_{p,q,\mathrm{loc}}$ if $\varphi\, v$ belongs to the global Besov space $B^\alpha_{p,q}(\mathbb{R}^n)$ whenever φ belongs to the class $\mathcal{C}^\infty_c(\Omega)$ of smooth functions with compact support contained in Ω. It is worth noticing that one can prove suitable versions of Lemma 2.2 and of Lemma 2.3, by using local Besov spaces.
The following Lemma is an easy exercise and its proof can be found in [2].

Lemma 2.4. *A function $v \in L^p_{\mathrm{loc}}(\Omega)$ belongs to the local Besov space $B^\alpha_{p,q,\mathrm{loc}}$ if and only if*

$$\left\| \frac{\tau_h v}{|h|^\alpha} \right\|_{L^q\left(\frac{dh}{|h|^n}; L^p(B)\right)} < \infty$$

for any ball $B \subset 2B \subset \Omega$ with radius r_B. Here the measure $\frac{dh}{|h|^n}$ is restricted to the ball $B(0, r_B)$ on the h-space.

It is known that Besov-Lipschitz spaces of fractional order $\alpha \in (0,1)$ can be characterized in pointwise terms. Given a measurable function $v : \mathbb{R}^n \to \mathbb{R}$, a *fractional α-Hajłasz gradient for v* is a sequence $(g_k)_k$ of measurable, non-negative functions $g_k : \mathbb{R}^n \to \mathbb{R}$, together with a null set $N \subset \mathbb{R}^n$, such that the inequality

$$|v(x) - v(y)| \le \, (g_k(x) + g_k(y))\, |x-y|^\alpha$$

holds whenever $k \in \mathbb{Z}$ and $x, y \in \mathbb{R}^n \setminus N$ are such that $2^{-k} \le |x-y| < 2^{-k+1}$. We say that $(g_k) \in \ell^q(\mathbb{Z}; L^p(\mathbb{R}^n))$ if

$$\|(g_k)_k\|_{\ell^q(L^p)} = \left(\sum_{k\in\mathbb{Z}} \|g_k\|^q_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{q}} < \infty.$$

The following result was proved in [30].

Theorem 2.5. *Let $0 < \alpha < 1$, $1 \le p < \infty$ and $1 \le q \le \infty$. Let $v \in L^p(\mathbb{R}^n)$. One has $v \in B^\alpha_{p,q}(\mathbb{R}^n)$ if and only if there exists a fractional α-Hajłasz gradient $(g_k)_k \in \ell^q(\mathbb{Z}; L^p(\mathbb{R}^n))$ for v. Moreover,*

$$\|v\|_{B^\alpha_{p,q}(\mathbb{R}^n)} \simeq \inf \|(g_k)_k\|_{\ell^q(L^p)},$$

where the infimum runs over all possible fractional α-Hajłasz gradients for v.

2.2. **Difference quotient.** We recall some properties of the finite difference operator that will be needed in the sequel. We start with the description of some elementary properties that can be found, for example, in [26].

Proposition 2.6. *Let F and G be two functions such that $F, G \in W^{1,p}(\Omega)$, with $p \ge 1$, and let us consider the set*

$$\Omega_{|h|} := \{x \in \Omega : dist(x, \partial\Omega) > |h|\}.$$

Then

(*d*1) $\tau_h F \in W^{1,p}(\Omega)$ *and*

$$D_i(\tau_h F) = \tau_h(D_i F).$$

(*d*2) *If at least one of the functions* F *or* G *has support contained in* $\Omega_{|h|}$, *then*

$$\int_\Omega F\,\tau_h G\,dx = -\int_\Omega G\,\tau_{-h} F\,dx.$$

(*d*3) *We have*

$$\tau_h(FG)(x) = F(x+h)\tau_h G(x) + G(x)\tau_h F(x).$$

The next result about finite difference operator is a kind of integral version of Lagrange Theorem.

Lemma 2.7. *If* $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $1 < p < +\infty$, *and* $F, DF \in L^p(B_R)$ *then*

$$\int_{B_\rho} |\tau_h F(x)|^p \; dx \le c(n,p)|h|^p \int_{B_R} |DF(x)|^p \; dx.$$

Moreover

$$\int_{B_\rho} |F(x+h)|^p \; dx \le \int_{B_R} |F(x)|^p \; dx.$$

Now, we recall the fundamental Sobolev embedding property.

Lemma 2.8. *Let* $F : \mathbb{R}^n \to \mathbb{R}^N$, $F \in L^p(B_R)$ *with* $1 < p < n$. *Suppose that there exist* $\rho \in (0,R)$ *and* $M > 0$ *such that*

$$\sum_{s=1}^n \int_{B_\rho} |\tau_{s,h} F(x)|^p \, dx \le M^p |h|^p,$$

for every h *with* $|h| < \frac{R-\rho}{2}$. *Then* $F \in W^{1,p}(B_\rho) \cap L^{\frac{np}{n-p}}(B_\rho)$. *Moreover*

$$||DF||_{L^p(B_\rho)} \le M$$

and

$$||F||_{L^{\frac{np}{n-p}}(B_\rho)} \le c\left(M + ||F||_{L^p(B_R)}\right),$$

with $c = c(n,N,p)$.

For the proof see, for example, [26, Lemma 8.2].
We conclude this subsection recalling a fractional version of previous Lemma (see [31]).

Lemma 2.9. *Let* $F \in L^2(B_R)$. *Suppose that there exist* $\rho \in (0,R)$, $0 < \alpha < 1$ *and* $M > 0$ *such that*

$$\sum_{s=1}^n \int_{B_\rho} |\tau_{s,h} F(x)|^2 \, dx \le M^2 |h|^{2\alpha},$$

for every h *such that* $|h| < \frac{R-\rho}{2}$. *Then* $F \in L^{\frac{2n}{n-2\beta}}(B_\rho)$ *for every* $\beta \in (0,\alpha)$ *and*

$$||F||_{L^{\frac{2n}{n-2\beta}}(B_\rho)} \le c\left(M + ||F||_{L^2(B_R)}\right),$$

with $c = c(n,N,R,\rho,\alpha,\beta)$.

2.3. **VMO coefficients.** We shall use the fact that if $\mathcal{A}$ satisfies $(\mathcal{A}1),(\mathcal{A}2),(\mathcal{A}3)$ and $(\mathcal{A}5)$ or $(\mathcal{A}6)$ then it is locally uniformly in VMO. More precisely, given a ball $B\subset\Omega$, let us introduce the operator

$$\mathcal{A}_B(\xi)=\fint_B \mathcal{A}(x,\xi)\,dx.$$

One can easily check that $\mathcal{A}_B(\xi)$ also satisfies assumptions $(\mathcal{A}1),(\mathcal{A}2)$ and $(\mathcal{A}3)$. Setting

$$V(x,B)=\sup_{\xi\neq 0}\frac{|\mathcal{A}(x,\xi)-\mathcal{A}_B(\xi)|}{(\mu^2+|\xi|^2)^{\frac{p-1}{2}}}, \tag{2.4}$$

we will say that $x\mapsto\mathcal{A}(x,\xi)$ is *locally uniformly in* VMO if for each compact set $K\subset\Omega$ we have that

$$\lim_{R\to 0}\sup_{r<R}\sup_{x_0\in K}\fint_{B_r(x_0)}V(x,B)\,dx=0. \tag{2.5}$$

Next Lemma will be a key tool in the proof of our results. Its proof can be found in [15, Lemma 12]

Lemma 2.10. *Let $\mathcal{A}$ be such that $(\mathcal{A}1),(\mathcal{A}2),(\mathcal{A}3)$ and $(\mathcal{A}5)$ or $(\mathcal{A}6)$ hold. Then $\mathcal{A}$ is locally uniformly in VMO, that is* (2.5) *holds.*

The following Theorem is a Calderón-Zygmund type estimate for solutions to the obstacle problem with VMO coefficients.

Theorem 2.11. *Let $2\le p\le n$, and $q>p$. Assume that $(\mathcal{A}1),(\mathcal{A}2),(\mathcal{A}3)$ hold, and that $x\mapsto \mathcal{A}(x,\xi)$ is locally uniformly in VMO. Let $u\in\mathcal{K}_\psi(\Omega)$ be the solution to the obstacle problem* (1.3)*. Then the following implication holds*

$$D\psi\in L^q_{loc}(\Omega)\quad\Rightarrow\quad Du\in L^q_{loc}(\Omega).$$

Moreover, there exists a constant $C=C(n,\nu,\ell,L,p,q)$ such that the following inequality

$$\fint_{B_R}|Du|^q\,dx\le C\left(1+\fint_{B_{2R}}|D\psi|^q\,dx\right)$$

holds for any ball B_R such that $B_{2R}\Subset\Omega$.

The proof of previous Theorem can be deduced by that of [8, Theorem 2.6] in the particular case $p(x)=p$, observing that VMO functions obviously satisfy the required smallness condition on the BMO-norm of the coefficients.

3. Proof of Theorem 1.1

Existence of solutions to the obstacle problem (1.1) can be easily proved through classical results regarding variational inequalities, so in this paper we will mainly concentrate on the regularity results. In particular, this Section is devoted to the proof of Theorem 1.1. The main point will be the choice of suitable test functions φ in (1.3) that involve the difference quotient of the solution but at the same time turns to be admissible for the obstacle class $\mathcal{K}_\psi(\Omega)$.

Proof. Let us consider $\varphi:=u+tv$ for a suitable $v\in W^{1,p}_0(\Omega)$ such that

$$u-\psi+t\,v\ge 0\qquad\text{for } t\in[0,1). \tag{3.1}$$

It is easy to see that such function φ belongs to the obstacle class $\mathcal{K}_\psi(\Omega)$, because $\varphi=u+tv\ge\psi$. Let us fix a ball B_R such that $B_{2R}\Subset\Omega$ and a cut off function $\eta\in C^\infty_0(B_R)$, $\eta\equiv 1$ on $B_{\frac{R}{2}}$ such that $|\nabla\eta| \leq \frac{c}{R}$. Due to the local nature of our results, there is no loss of generality in supposing $R \leq 1$, that we will do from now on. Then, for $|h| < \frac{R}{4}$, we consider

$$v_1(x) = \eta^2(x)[(u-\psi)(x+h) - (u-\psi)(x)]. \tag{3.2}$$

From the regularity of u and ψ, it is immediate to check that $v_1 \in W_0^{1,p}(\Omega)$. Moreover v_1 fulfills (3.1). Indeed, for a.e. $x \in \Omega$ and for any $t \in [0,1)$

$$\begin{aligned} & u(x) - \psi(x) + tv_1(x) \\ = \; & u(x) - \psi(x) + t\eta^2(x)[(u-\psi)(x+h) - (u-\psi)(x)] \\ = \; & t\,\eta^2(x)(u-\psi)(x+h) + (1-t\eta^2(x))(u-\psi)(x) \geq \; 0, \end{aligned}$$

because $u \in \mathcal{K}_\psi(\Omega)$.
By using in (1.3) as an admissible test function $\varphi = u + tv$, with v_1 chosen in (3.2) in place of v, we obtain

$$0 \leq \int_\Omega \langle \mathcal{A}(x, Du(x)), D[\eta^2(x)[(u-\psi)(x+h) - (u-\psi)(x)]] \rangle \, dx. \tag{3.3}$$

On the other hand, if we define

$$v_2(x+h) = \eta^2(x)[(u-\psi)(x) - (u-\psi)(x+h)] \tag{3.4}$$

then $v_2 \in W_0^{1,p}(\Omega)$ and (3.1) still is trivially satisfied (calculated in $x+h$ instead of in x), due to the fact that

$$\begin{aligned} & u(x+h) - \psi(x+h) + tv_2(x+h) \\ = \; & u(x+h) - \psi(x+h) + t\eta^2(x)[(u-\psi)(x) - (u-\psi)(x+h)] \\ = \; & t\,\eta^2(x)(u-\psi)(x) + (1-t\eta^2(x))(u-\psi)(x+h) \geq \; 0. \end{aligned}$$

By means of a simple change of variable, inequality (1.3) becomes

$$0 \leq \int_{\Omega-|h|} \langle \mathcal{A}(x+h, Du(x+h)), D(\varphi - u)(x+h) \rangle \, dx,$$

for a generic $\varphi \in \mathcal{K}_\psi(\Omega)$. Then, by choosing in the previous variational inequality $\varphi = u + tv_2$, where v_2 is defined in (3.4), we obtain

$$0 \leq \int_{\Omega-|h|} \langle \mathcal{A}(x+h, Du(x+h)), D[\eta^2(x)[(u-\psi)(x) - (u-\psi)(x+h)]] \rangle \, dx. \tag{3.5}$$

Taking into account that $\mathrm{supp}\eta \subset \Omega - |h|$, we can add (3.3) and (3.5), thus obtaining

$$\begin{aligned} 0 \;\; \leq & \;\; \int_\Omega \langle \mathcal{A}(x, Du(x)), D[\eta^2(x)[(u-\psi)(x+h) - (u-\psi)(x)]] \rangle \, dx \\ & + \int_\Omega \langle \mathcal{A}(x+h, Du(x+h)), D[\eta^2(x)[(u-\psi)(x) - (u-\psi)(x+h)]] \rangle \, dx \\ = & \;\; \int_\Omega \langle \mathcal{A}(x, Du(x)) - \mathcal{A}(x+h, Du(x+h)), D[\eta^2(x)[(u-\psi)(x+h) - (u-\psi)(x)]] \rangle \, dx, \end{aligned}$$

which implies

$$\begin{aligned} 0 \;\; \geq & \;\; \int_\Omega \langle \mathcal{A}(x+h, Du(x+h)) - \mathcal{A}(x, Du(x)), \eta^2(x) D[(u-\psi)(x+h) - (u-\psi)(x)] \rangle \, dx \\ & + \int_\Omega \langle \mathcal{A}(x+h, Du(x+h)) - \mathcal{A}(x, Du(x)), 2\,\eta(x)\, D\eta(x)\, [(u-\psi)(x+h) - (u-\psi)(x)] \rangle \, dx. \end{aligned}$$

We can write previous inequality as follows

$$
\begin{aligned}
0 \;\ge\;& \int_\Omega \langle \mathcal{A}(x+h,Du(x+h))-\mathcal{A}(x+h,Du(x)),\eta^2(Du(x+h)-Du(x))\rangle\,dx\\
&-\int_\Omega \langle \mathcal{A}(x+h,Du(x+h))-\mathcal{A}(x+h,Du(x)),\eta^2(D\psi(x+h)-D\psi(x))\rangle\,dx\\
&+\int_\Omega \langle \mathcal{A}(x+h,Du(x+h))-\mathcal{A}(x+h,Du(x)),2\eta\,D\eta\tau_h(u-\psi)\rangle\,dx\\
&+\int_\Omega \langle \mathcal{A}(x+h,Du(x))-\mathcal{A}(x,Du(x)),\eta^2(Du(x+h)-Du(x))\rangle\,dx\\
&-\int_\Omega \langle \mathcal{A}(x+h,Du(x))-\mathcal{A}(x,Du(x)),\eta^2(D\psi(x+h)-D\psi(x))\rangle\,dx\\
&+\int_\Omega \langle \mathcal{A}(x+h,Du(x))-\mathcal{A}(x,Du(x)),2\eta\,D\eta\tau_h(u-\psi)\rangle\,dx\\
=:\;& I+II+III+IV+V+VI, \qquad (3.6)
\end{aligned}
$$

that yields

$$I \le |II|+|III|+|IV|+|V|+|VI|. \qquad (3.7)$$

The ellipticity assumption expressed by $(\mathcal{A}1)$ implies

$$I \ge \nu \int_\Omega \eta^2|\tau_h Du|^2(1+|Du(x+h)|^2+|Du(x)|^2)^{\frac{p-2}{2}}\,dx. \qquad (3.8)$$

By virtue of assumption $(\mathcal{A}2)$ and Young's inequality, we get

$$
\begin{aligned}
|II| \;\le\;& L\int_\Omega \eta^2|\tau_h Du|(1+|Du(x+h)|^2+|Du(x)|^2)^{\frac{p-2}{2}}\,|\tau_h D\psi|dx\\
\le\;& \varepsilon\int_\Omega \eta^2|\tau_h Du|^2(1+|Du(x+h)|^2+|Du(x)|^2)^{\frac{p-2}{2}}\,dx\\
&+C_\varepsilon(L)\int_\Omega \eta^2|\tau_h D\psi|^2(1+|Du(x+h)|^2+|Du(x)|^2)^{\frac{p-2}{2}}\,dx\\
\le\;& \varepsilon\int_\Omega \eta^2|\tau_h Du|^2(1+|Du(x+h)|^2+|Du(x)|^2)^{\frac{p-2}{2}}\,dx\\
&+C_\varepsilon(L)\left(\int_{B_R}|\tau_h D\psi|^p\,dx\right)^{\frac{2}{p}}\left(\int_{B_{2R}}(1+|Du(x)|^p)\,dx\right)^{\frac{p-2}{p}}, \qquad (3.9)
\end{aligned}
$$

where we used Hölder's inequality, the properties of η and the second estimate of Lemma 2.7. Using the assumption $D\psi \in W^{1,p}$ and first estimate of Lemma 2.7 in the second integral of the right hand side of (3.9), we obtain

$$
\begin{aligned}
|II| \;\le\;& \varepsilon\int_\Omega \eta^2|\tau_h Du|^2(1+|Du(x+h)|^2+|Du(x)|^2)^{\frac{p-2}{2}}\,dx\\
&+C_\varepsilon(L,n,p)|h|^2\left(\int_{B_{2R}}|D^2\psi|^p\,dx\right)^{\frac{2}{p}}\left(\int_{B_{2R}}(1+|Du(x)|^p)\,dx\right)^{\frac{p-2}{p}}. \qquad (3.10)
\end{aligned}
$$

Arguing analogously, we get

$$|III| \;\le\; 2L\int_\Omega |\tau_h Du|\,|D\eta|\eta(1+|Du(x+h)|^2+|Du(x)|^2)^{\frac{p-2}{2}}\,|\tau_h(u-\psi)|\,dx$$

$$
\begin{aligned}
&\leq \varepsilon \int_\Omega \eta^2|\tau_h Du|^2 \left(1+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}} dx \\
&\quad + C_\varepsilon(L) \int_\Omega |D\eta|^2 (1+|Du(x+h)|^2+|Du(x)|^2)^{\frac{p-2}{2}} |\tau_h(u-\psi)|^2 \, dx \\
&\leq \varepsilon \int_\Omega \eta^2|\tau_h Du|^2 \left(1+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}} dx \\
&\quad + \frac{C_\varepsilon(L)}{R^2}\left(\int_{B_{2R}} (1+|Du(x)|)^p \, dx\right)^{\frac{p-2}{p}} \left(\int_{B_R} |\tau_h(u-\psi)|^p \, dx\right)^{\frac{2}{p}},
\end{aligned}
$$

since $|D\eta| \leq \frac{c}{R}$. Since $u-\psi \in W^{1,p}(\Omega)$, we may use the first estimate of Lemma 2.7 to control last integral in the right hand side of previous estimate, obtaining that

$$
\begin{aligned}
|III| &\leq \varepsilon \int_\Omega \eta^2|\tau_h Du|^2 \left(1+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}} dx \\
&\quad + |h|^2 \frac{C_\varepsilon(L,n,p)}{R^2}\left(\int_{B_{2R}} (1+|Du(x)|)^p \, dx\right)^{\frac{p-2}{p}} \left(\int_{B_{2R}} |D(u-\psi)(x)|^p \, dx\right)^{\frac{2}{p}} \\
&\leq \varepsilon \int_\Omega \eta^2|\tau_h Du|^2 \left(1+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}} dx \\
&\quad + |h|^2 \frac{C_\varepsilon(L,n,p)}{R^2}\int_{B_{2R}} (1+|Du(x)|)^p \, dx + |h|^2 \frac{C_\varepsilon(L,n,p)}{R^2}\int_{B_{2R}} |D\psi(x)|^p \, dx, \qquad (3.11)
\end{aligned}
$$

where we used also Young's inequality. In order to estimate the integral IV, we use assumption $(\mathcal{A}4)$, Young's and Hölder's inequalities as follows

$$
\begin{aligned}
|IV| &\leq |h| \int_\Omega \eta^2 (\kappa(x+h)+\kappa(x))(1+|Du(x)|^2)^{\frac{p-1}{2}} |\tau_h Du| dx \\
&\leq \varepsilon \int_\Omega \eta^2|\tau_h Du|^2 \left(1+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}} dx \\
&\quad + C_\varepsilon |h|^2 \int_{B_R} (\kappa(x+h)+\kappa(x))^2 (1+|Du(x)|^2)^{\frac{p}{2}} \, dx \\
&\leq \varepsilon \int_\Omega \eta^2|\tau_h Du|^2 \left(1+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}} dx \\
&\quad + C_\varepsilon |h|^2 \left(\int_{B_R} (\kappa(x+h)+\kappa(x))^n \, dx\right)^{\frac{2}{n}} \left(\int_{B_R} (1+|Du(x)|)^{\frac{np}{n-2}} \, dx\right)^{\frac{n-2}{n}}, \qquad (3.12)
\end{aligned}
$$

where we also used that $\text{supp}\eta \subset B_R$. Therefore, Theorem 2.11 and classical Sobolev embedding Theorem imply

$$
\fint_{B_R} |Du(x)|^{p^*} \, dx \leq C\left(1+\fint_{B_{2R}} |D\psi(x)|^{p^*} \, dx\right), \qquad (3.13)
$$

for a constant $C = C(n,\nu,L,\ell,p,R)$. Observing that

$$
\frac{np}{n-2} \leq \frac{np}{n-p} \iff p \geq 2,
$$

we have

$$
\left(\int_{B_R} (1+|Du(x)|)^{\frac{np}{n-2}} \, dx\right)^{\frac{n-2}{n}} \leq C\, R^{p-2} \left(\int_{B_{2R}} (1+|D\psi(x)|)^{p^*} \, dx\right)^{\frac{n-p}{n}}. \qquad (3.14)
$$

Inserting (3.14) in (3.12), we get

$$\begin{aligned}|IV| &\le \varepsilon\int_\Omega \eta^2|\tau_h Du|^2\left(1+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}dx\\&+C_\varepsilon|h|^2\left(\int_{B_R}(\kappa(x+h)+\kappa(x))^n\,dx\right)^{\frac{2}{n}}\\&+\left(\int_{B_{2R}}(1+|D\psi(x)|)^{p^*}dx\right)^{\frac{n-p}{n}},\end{aligned}\tag{3.15}$$

where now $C_\varepsilon=C(\varepsilon,n,\nu,L,\ell,p,R)$.
Assumption $(\mathcal{A}4)$ also entails

$$\begin{aligned}|V| &\le |h|\int_\Omega \eta^2(\kappa(x+h)+\kappa(x))(1+|Du|^2)^{\frac{p-1}{2}}|\tau_h D\psi|dx\\&\le |h|\left(\int_\Omega|\tau_h D\psi|^p\eta^2\,dx\right)^{\frac1p}\left(\int_\Omega\eta^2(\kappa(x+h)+\kappa(x))^{\frac{p}{p-1}}(1+|Du(x)|^2)^{\frac p2}dx\right)^{\frac{p-1}{p}}\\&\le |h|\left(\int_{B_R}|\tau_h D\psi|^p\,dx\right)^{\frac1p}\left(\int_{B_R}(\kappa(x+h)+\kappa(x))^n\,dx\right)^{\frac1n}\\&\quad\cdot\left(\int_{B_R}(1+|Du(x)|)^{\frac{np(p-1)}{n(p-1)-p}}dx\right)^{\frac{n(p-1)-p}{np}}\\&\le CR^{p-2}|h|\left(\int_{B_R}|\tau_h D\psi|^p\,dx\right)^{\frac1p}\left(\int_{B_R}(\kappa(x+h)+\kappa(x))^n\,dx\right)^{\frac1n}\\&\quad\cdot\left(\int_{B_{2R}}(1+|D\psi(x)|)^{p^*}dx\right)^{\frac{n-p}{np'}}\end{aligned}\tag{3.16}$$

where we used the properties of η, Hölder's inequality and, in the last line, we used estimate (3.13), since $D\psi\in L^{p^*}$ and since

$$\frac{np(p-1)}{n(p-1)-p}\le\frac{np}{n-p}\quad\Longleftrightarrow\quad p\ge 2.$$

By virtue of the assumption $D\psi\in W^{1,p}_{\rm loc}(\Omega)$, we can use the first inequality of Lemma 2.7 to estimate the first integral in the right hand side of (3.16) and the second inequality of Lemma 2.7 to estimate the second one, thus getting

$$|V|\le C|h|^2\left(\int_{B_{2R}}|D^2\psi|^p\,dx\right)^{\frac1p}\left(\int_{B_{2R}}\kappa^n(x)\,dx\right)^{\frac1n}\left(\int_{B_{2R}}(1+|D\psi(x)|)^{p^*}dx\right)^{\frac{n-p}{np'}},\tag{3.17}$$

with $C=C(n,\nu,L,\ell,p,R)$.
Finally, arguing as we did for the estimate of V we get

$$\begin{aligned}|VI| &\le 2|h|\int_\Omega\eta|D\eta|(\kappa(x+h)+\kappa(x))(1+|Du|^2)^{\frac{p-1}{2}}|\tau_h(u-\psi)|dx\\&\le 2|h|\left(\int_{B_R}|\tau_h(u-\psi)|^p\,dx\right)^{\frac1p}\left(\int_{B_R}|D\eta|^{\frac{p}{p-1}}(\kappa(x+h)+\kappa(x))^{\frac{p}{p-1}}(1+|Du(x)|^2)^{\frac p2}dx\right)^{\frac{p-1}{p}}\\&\le \frac{C\,|h|}{R}\left(\int_{B_R}|\tau_h(u-\psi)|^p\,dx\right)^{\frac1p}\end{aligned}$$

$$\cdot\left(\int_{B_R}(\kappa(x+h)+\kappa(x))^n\,dx\right)^{\frac{1}{n}}\left(\int_{B_R}(1+|Du(x)|)^{\frac{np(p-1)}{n(p-1)-p}}\,dx\right)^{\frac{n(p-1)-p}{np}},$$

where we used the fact that $|D\eta|\le\frac{C}{R}$. Using Lemma 2.7 and (3.13) in the right hand side of previous estimate, we get

$$\begin{aligned}|VI| \;\le\; & C|h|^2\left(\int_{B_{2R}}|D(u-\psi)(x)|^p\,dx\right)^{\frac{1}{p}}\\ & \cdot\left(\int_{B_{2R}}\kappa^n(x)\,dx\right)^{\frac{1}{n}}\left(\int_{B_{2R}}(1+|D\psi(x)|)^{p^*}\,dx\right)^{\frac{n-p}{np'}},\end{aligned}\tag{3.18}$$

with $C=C(n,\nu,L,\ell,p,R)$.
Inserting estimates (3.8), (3.10), (3.11), (3.15), (3.17) and (3.18) in (3.7), we infer the existence of constants $C_\varepsilon\equiv C_\varepsilon(\varepsilon,\nu,L,\ell,n,p,R)$ and $C\equiv C(\nu,L,\ell,n,p,R)$ such that

$$\begin{aligned}&\nu\int_\Omega\eta^2|\tau_hDu|^2(1+|Du(x+h)|^2+|Du(x)|^2|)^{\frac{p-2}{2}}\,dx\\
&\le 3\varepsilon\int_\Omega\eta^2|\tau_hDu|^2(1+|Du(x+h)|^2+|Du(x)|^2|)^{\frac{p-2}{2}}\,dx\\
&+C_\varepsilon\,|h|^2\left(\int_{B_{2R}}|D^2\psi|^p\,dx\right)^{\frac{2}{p}}\left(\int_{B_{2R}}(1+|Du(x)|)^p\,dx\right)^{\frac{p-2}{p}}\\
&+C_\varepsilon|h|^2\int_{B_{2R}}|D\psi(x)|^p\,dx+\frac{C_\varepsilon(L,n,p)}{R^2}|h|^2\int_{B_{2R}}(1+|Du(x)|)^p\,dx\\
&+C_\varepsilon|h|^2\left(\int_{B_{2R}}\kappa^n(x)\,dx\right)^{\frac{2}{n}}\left(\int_{B_{2R}}(1+|D\psi(x)|)^{p^*}\,dx\right)^{\frac{n-p}{n}}\\
&+C|h|^2\left(\int_{B_{2R}}|D^2\psi|^p\,dx\right)^{\frac{1}{p}}\left(\int_{B_{2R}}\kappa^n(x)\,dx\right)^{\frac{1}{n}}\left(\int_{B_{2R}}(1+|D\psi(x)|)^{p^*}\,dx\right)^{\frac{n-p}{np'}}\\
&+C|h|^2\left(\int_{B_R}|Du(x)|^p+|D\psi(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_{B_{2R}}\kappa^n(x)\,dx\right)^{\frac{1}{n}}\left(\int_{B_{2R}}(1+|D\psi(x)|)^{p^*}\,dx\right)^{\frac{n-p}{np'}},\end{aligned}$$

where we also used Lemma 2.7. Choosing $\varepsilon=\frac{\nu}{6}$ we get

$$\begin{aligned}&\nu\int_\Omega\eta^2|\tau_hDu|^2(1+|Du(x+h)|^2+|Du(x)|^2|)^{\frac{p-2}{2}}\,dx\\
\le\;& C\,|h|^2\left(\int_{B_R}\left(1+|Du|^p+|D\psi|^p\right)dx+\int_{B_{2R}}|D^2\psi|^p\,dx+\int_{B_{2R}}|D\psi|^{p^*}\,dx+\int_{B_{2R}}\kappa^n(x)\,dx\right),\end{aligned}$$

also by virtue of Young's inequality. Using Lemma 2.1 in the left hand side of previous estimate and recalling that $\eta\equiv1$ on $B_{\frac{R}{2}}$, we get

$$\begin{aligned}&\nu\int_{B_{\frac{R}{2}}}|\tau_hV_p(Du)|^2\,dx\\
\le\;& C\,|h|^2\left(\int_{B_R}\left(1+|Du|^p+|D\psi|^p\right)dx+\int_{B_{2R}}|D^2\psi|^p\,dx\right.\\
&\left.+\int_{B_{2R}}|D\psi|^{p^*}\,dx+\int_{B_{2R}}\kappa^n(x)\,dx\right)=:H|h|^2.\end{aligned}\tag{3.19}$$

Lemma 2.8 implies that

$$\int_{B_{\frac{R}{2}}} |D\big(V_p(Du)\big)|^2 \leq CH$$

and the conclusion follows recalling the definition of $V_p(\xi)$ in (2.1).

□

4. Proof of Theorem 1.2 and Corollary 1.3

This section is devoted to the proof of Theorem 1.2 and Corollary 1.3. The proof of Theorem 1.2 goes along the lines of the one presented in the previous Section until the estimate of the first three terms, I, II, III in (3.6). Differences come when starting estimate the last three integrals, in which the different assumption on the partial map $x \mapsto \mathcal{A}(x, \cdot)$ and on the obstacle come into the play.

Proof. Our starting point is the following estimate

$$\begin{aligned} & \nu \int_{\Omega} \eta^2 |\tau_h Du|^2 (1 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}}\, dx \\ \leq \; & 2\varepsilon \int_{\Omega} \eta^2 |\tau_h Du|^2 \, (1 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}}\, dx \\ & + C_\varepsilon(L, n, p) \left(\int_{B_R} |\tau_h D\psi|^p \, dx \right)^{\frac{2}{p}} \left(\int_{B_{2R}} (1 + |Du(x)|^p) \, dx \right)^{\frac{p-2}{p}} \\ & + |h|^2 \frac{C_\varepsilon(L, n, p)}{R^2} \int_{B_{2R}} (1 + |Du(x)| + |D\psi(x)|)^p \, dx \\ & + IV + V + VI, \end{aligned} \tag{4.1}$$

that is obtained inserting (3.8), (3.9) and (3.11) in (3.7) of previous Section and where IV, V and VI are those defined in (3.6) in the previous Section. In order to estimate the integral IV, we use this time assumption ($\mathcal{A}5$), Young's and Hölder's inequalities as follows

$$\begin{aligned} |IV| \; & \leq \; |h|^\alpha \int_{\Omega} \eta^2 (g_k(x+h) + g_k(x))(1 + |Du(x)|^2)^{\frac{p-1}{2}} |\tau_h Du| dx \\ & \leq \; \varepsilon \int_{\Omega} \eta^2 |\tau_h Du|^2 \, \left(1 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p-2}{2}} \, dx \\ & \quad + C_\varepsilon |h|^{2\alpha} \int_{B_R} (g_k(x+h) + g_k(x))^2 (1 + |Du(x)|^2)^{\frac{p}{2}} \, dx \\ & \leq \; \varepsilon \int_{\Omega} \eta^2 |\tau_h Du|^2 \, \left(1 + |Du(x)|^2 + |Du(x+h)|^2\right)^{\frac{p-2}{2}} \, dx \\ & \quad + C_\varepsilon |h|^{2\alpha} \left(\int_{B_R} (g_k(x+h) + g_k(x))^{\frac{n}{\alpha}} \, dx \right)^{\frac{2\alpha}{n}} \left(\int_{B_R} (1 + |Du(x)|)^{\frac{np}{n-2\alpha}} \, dx \right)^{\frac{n-2\alpha}{n}}, \end{aligned} \tag{4.2}$$

where we also used that supp$\eta \subset B_R$ and where $2^{-k}\frac{R}{4} \leq |h| \leq 2^{-k+1}\frac{R}{4}$, for $k \in \mathbb{N}$. Note that the assumption $D\psi \in B^{\alpha}_{p,q,\mathrm{loc}}(\Omega)$ with $q \leq p^*_\alpha$ yields that $D\psi \in L^{p^*_\alpha}_{\mathrm{loc}}(\Omega)$ locally, by virtue of Lemma 2.2. Therefore, Theorem 2.11 implies

$$⨍_{B_R} |Du(x)|^{p^*_\alpha} \, dx \leq C \left(1 + ⨍_{B_{2R}} |D\psi(x)|^{p^*_\alpha} \, dx \right) \tag{4.3}$$

with a constant $C = C(n, \nu, L, \ell, p, \alpha, R)$. Observing that

$$\frac{np}{n-2\alpha} \leq \frac{np}{n-\alpha p} \iff p \geq 2,$$

by Hölder's inequality and (4.3), we have

$$\left(\int_{B_R} (1+|Du(x)|)^{\frac{np}{n-2\alpha}}\, dx\right)^{\frac{n-2\alpha}{n}} \leq C\, R^{\alpha(p-2)} \left(\int_{B_{2R}} (1+|D\psi(x)|)^{p^*_\alpha}\, dx\right)^{\frac{n-\alpha p}{n}}. \tag{4.4}$$

Inserting (4.4) in (4.2), we get

$$\begin{aligned} |IV| \;&\leq\; \varepsilon \int_\Omega \eta^2 |\tau_h Du|^2 \left(1+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}} dx \\ &\quad + C_\varepsilon\, |h|^{2\alpha} \left(\int_{B_R} (g_k(x+h)+g_k(x))^{\frac{n}{\alpha}}\, dx\right)^{\frac{2\alpha}{n}} \\ &\quad \cdot \left(\int_{B_{2R}} (1+|D\psi(x)|)^{p^*_\alpha}\, dx\right)^{\frac{n-\alpha p}{n}}, \end{aligned} \tag{4.5}$$

with a constant $C_\varepsilon = C_\varepsilon(\varepsilon, n, \nu, L, \ell, p, \alpha, R)$.
For h such that $2^{-k}\frac{R}{4} \leq |h| \leq 2^{-k+1}\frac{R}{4}$, $k \in \mathbb{N}$, assumption ($\mathcal{A}5$) also yields that

$$\begin{aligned} |V| \;&\leq\; |h|^\alpha \int_\Omega \eta^2 (g_k(x+h)+g_k(x))(1+|Du|^2)^{\frac{p-1}{2}} |\tau_h D\psi| dx \\ &\leq\; |h|^\alpha \left(\int_\Omega \eta^2 |\tau_h D\psi|^p\, dx\right)^{\frac{1}{p}} \left(\int_\Omega \eta^2 (g_k(x+h)+g_k(x))^{\frac{p}{p-1}} (1+|Du(x)|^2)^{\frac{p}{2}}\, dx\right)^{\frac{p-1}{p}} \\ &\leq\; |h|^\alpha \left(\int_{B_R} |\tau_h D\psi|^p\, dx\right)^{\frac{1}{p}} \left(\int_{B_R} (g_k(x+h)+g_k(x))^{\frac{n}{\alpha}}\, dx\right)^{\frac{\alpha}{n}} \\ &\qquad \cdot \left(\int_{B_R} (1+|Du(x)|)^{\frac{np(p-1)}{n(p-1)-p\alpha}}\, dx\right)^{\frac{n(p-1)-p\alpha}{np}} \\ &\leq\; C \left(\int_{B_R} |\tau_h D\psi|^p\, dx\right)^{\frac{1}{p}} \left(\int_{B_R} (g_k(x+h)+g_k(x))^{\frac{n}{\alpha}}\, dx\right)^{\frac{\alpha}{n}} \\ &\qquad \cdot \left(\int_{B_{2R}} (1+|D\psi(x)|)^{p^*_\alpha}\, dx\right)^{\frac{n-p\alpha}{np'}}, \end{aligned} \tag{4.6}$$

with a constant $C = C(n, \nu, L, \ell, p, \alpha, R)$; here we used the properties of η, Hölder's inequality and, in the last line, we used estimate (4.3), since $D\psi \in L^{p^*_\alpha}$ and since

$$\frac{np(p-1)}{n(p-1)-\alpha p} \leq \frac{np}{n-\alpha p} \iff p \geq 2.$$

Finally, arguing as we did for the estimate of V, we get

$$\begin{aligned} |VI| \;&\leq\; 2|h|^\alpha \int_\Omega \eta |D\eta| (g_k(x+h)+g_k(x))(1+|Du|^2)^{\frac{p-1}{2}} |\tau_h(u-\psi)| dx \\ &\leq\; 2|h|^\alpha \left(\int_{B_R} |\tau_h(u-\psi)|^p\, dx\right)^{\frac{1}{p}} \left(\int_{B_R} |D\eta|^{\frac{p}{p-1}} (g_k(x+h)+g_k(x))^{\frac{p}{p-1}} (1+|Du(x)|^2)^{\frac{p}{2}}\, dx\right)^{\frac{p-1}{p}} \\ &\leq\; \frac{C\,|h|^\alpha}{R} \left(\int_{B_R} |\tau_h(u-\psi)|^p\, dx\right)^{\frac{1}{p}} \end{aligned}$$

$$\cdot\left(\int_{B_R}(g_k(x+h)+g_k(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha}{n}}\left(\int_{B_R}(1+|Du(x)|)^{\frac{np(p-1)}{n(p-1)-\alpha p}}\,dx\right)^{\frac{n(p-1)-\alpha p}{np}},$$

since $|D\eta|\le\frac{C}{R}$. Using the first estimate of Lemma 2.7 and (4.3) in the right hand side of previous estimate, we get

$$\begin{aligned}|VI| \;\le\; & C|h|^{1+\alpha}\left(\int_{B_{2R}}|D(u-\psi)(x)|^p\,dx\right)^{\frac{1}{p}}\\ &\cdot\left(\int_{B_R}(g_k(x+h)+g_k(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha}{n}}\left(\int_{B_{2R}}(1+|D\psi(x)|)^{p^*_\alpha}\,dx\right)^{\frac{n-\alpha p}{np'}}, \quad (4.7)\end{aligned}$$

with a constant $C=C(n,\nu,L,\ell,p,\alpha,R)$.
Inserting estimates (4.5), (4.6) and (4.7) in (4.1), we infer the existence of constants $C_\varepsilon=C_\varepsilon(\varepsilon,n,\nu,L,\ell,p,\alpha,R)$ and $C=C(n,\nu,L,\ell,p,\alpha,R)$ such that

$$\begin{aligned}&\nu\int_\Omega\eta^2|\tau_hDu|^2(1+|Du(x+h)|^2+|Du(x)|^2|)^{\frac{p-2}{2}}\,dx\\ &\le 3\varepsilon\int_\Omega\eta^2|\tau_hDu|^2(1+|Du(x+h)|^2+|Du(x)|^2|)^{\frac{p-2}{2}}\,dx\\ &+C_\varepsilon\left(\int_{B_R}|\tau_hD\psi|^p\,dx\right)^{\frac{2}{p}}\left(\int_{B_{2R}}(1+|Du(x)|)^p\,dx\right)^{\frac{p-2}{p}}\\ &+C_\varepsilon|h|^2\int_{B_{2R}}(1+|Du(x)|+|D\psi(x)|)^p\,dx\\ &+C_\varepsilon|h|^{2\alpha}\left(\int_{B_R}(g_k(x+h)+g_k(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{2\alpha}{n}}\left(\int_{B_{2R}}(1+|D\psi(x)|)^{p^*_\alpha}\,dx\right)^{\frac{n-p\alpha}{n}}\\ &+C|h|^{\alpha}\left(\int_{B_R}|\tau_hD\psi|^p\,dx\right)^{\frac{1}{p}}\left(\int_{B_R}(g_k(x+h)+g_k(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha}{n}}\\ &\qquad\cdot\left(\int_{B_{2R}}(1+|D\psi(x)|)^{p^*_\alpha}\,dx\right)^{\frac{n-p\alpha}{np'}}\\ &+C|h|^{1+\alpha}\left(\int_{B_{2R}}|Du(x)|^p+|D\psi(x)|^p\,dx\right)^{\frac{1}{p}}\\ &\qquad\cdot\left(\int_{B_R}(g_k(x+h)+g_k(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha}{n}}\left(\int_{B_{2R}}(1+|D\psi(x)|)^{p^*_\alpha}\,dx\right)^{\frac{n-\alpha p}{np'}}.\end{aligned}$$

Choosing $\varepsilon=\frac{\nu}{6}$ yields

$$\begin{aligned}&\nu\int_\Omega\eta^2|\tau_hDu|^2(1+|Du(x+h)|^2+|Du(x)|^2|)^{\frac{p-2}{2}}\,dx\\ \le\;& C\left(\int_{B_R}|\tau_hD\psi|^p\,dx\right)^{\frac{2}{p}}\left(\int_{B_{2R}}(1+|Du(x)|)^p\,dx\right)^{\frac{p-2}{p}}\\ &+C|h|^2\int_{B_{2R}}(1+|Du(x)|+|D\psi(x)|)^p\,dx\\ &+C|h|^{2\alpha}\left(\int_{B_R}(g_k(x+h)+g_k(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{2\alpha}{n}}\left(\int_{B_{2R}}(1+|D\psi(x)|)^{p^*_\alpha}\,dx\right)^{\frac{n-p\alpha}{n}}\end{aligned}$$

$$
\begin{aligned}
&+C|h|^{\alpha}\left(\int_{B_R}|\tau_h D\psi|^p\,dx\right)^{\frac{1}{p}}\left(\int_{B_R}(g_k(x+h)+g_k(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha}{n}}\\
&\qquad\cdot\left(\int_{B_{2R}}(1+|D\psi(x)|)^{p^*_\alpha}\,dx\right)^{\frac{n-p\alpha}{np'}}\\
&+C|h|^{1+\alpha}\left(\int_{B_{2R}}|Du(x)|^p+|D\psi(x)|^p\,dx\right)^{\frac{1}{p}}\\
&\qquad\quad\cdot\left(\int_{B_R}(g_k(x+h)+g_k(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha}{n}}\left(\int_{B_{2R}}(1+|D\psi(x)|)^{p^*_\alpha})\,dx\right)^{\frac{n-\alpha p}{np'}}.
\end{aligned}
$$

Using Lemma 2.1 in the left hand side of previous estimate, recalling that $\eta\equiv 1$ on $B_{\frac{R}{2}}$ and dividing both sides by $|h|^{2\alpha}$, we get

$$
\begin{aligned}
&\nu\int_{B_{\frac{R}{2}}}\frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha}}\,dx\\
\le\ & C\left(\int_{B_R}\frac{|\tau_h D\psi|^p}{|h|^{\alpha p}}\,dx\right)^{\frac{2}{p}}\left(\int_{B_{2R}}(1+|Du(x)|)^p\,dx\right)^{\frac{p-2}{p}}\\
&+C|h|^{2-2\alpha}\int_{B_{2R}}(1+|Du(x)|+|D\psi(x)|)^p\,dx\\
&+C\left(\int_{B_R}(g_k(x+h)+g_k(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{2\alpha}{n}}\left(\int_{B_{2R}}(1+|D\psi(x)|)^{p^*_\alpha}\,dx\right)^{\frac{n-p\alpha}{n}}\\
&+C\left(\int_{B_R}\frac{|\tau_h D\psi(x)|^p}{|h|^{\alpha p}}\,dx\right)^{\frac{1}{p}}\left(\int_{B_R}(g_k(x+h)+g_k(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha}{n}}\\
&\qquad\cdot\left(\int_{B_{2R}}(1+|D\psi(x)|)^{p^*_\alpha}\,dx\right)^{\frac{n-p\alpha}{np'}}\\
&+C|h|^{1-\alpha}\left(\int_{B_R}|Du(x)|^p+|D\psi(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_{B_R}(g_k(x+h)+g_k(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha}{n}}\\
&\qquad\cdot\left(\int_{B_{2R}}(1+|D\psi(x)|)^{p^*_\alpha})\,dx\right)^{\frac{n-\alpha p}{np'}}.
\end{aligned}\tag{4.8}
$$

In order to conclude, we need now to take the L^q norm with the measure $\frac{dh}{|h|^n}$ restricted to the ball $B(0,\frac{R}{4})$ on the h-space of the L^2 norm of the difference quotient of order α of the function $V_p(Du)$. We obtain the following estimate

$$
\begin{aligned}
&\int_{B_{\frac{R}{4}}(0)}\left(\int_{B_{\frac{R}{2}}}\frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha}}\,dx\right)^{\frac{q}{2}}\frac{dh}{|h|^n}\\
\le\ & C\left(\int_{B_{2R}}(1+|Du(x)|)^p\,dx\right)^{\frac{(p-2)q}{2p}}\int_{B_{\frac{R}{4}}(0)}\left(\left(\int_{B_R}\frac{|\tau_h D\psi|^p}{|h|^{\alpha p}}\,dx\right)^{\frac{2}{p}}\right)^{\frac{q}{2}}\frac{dh}{|h|^n}\\
&+C\left(\int_{B_{2R}}(1+|Du(x)|+|D\psi(x)|)^p\,dx\right)^{\frac{q}{2}}\int_{B_{\frac{R}{4}}(0)}\left(|h|^{1-\alpha}\right)^q\frac{dh}{|h|^n}
\end{aligned}
$$

$$
\begin{aligned}
&+C\left(\int_{B_{2R}}(1+|D\psi(x)|)^{p^*_\alpha}\,dx\right)^{\frac{(n-p\alpha)q}{2n}}\int_{B_{\frac{R}{4}}(0)}\left(\int_{B_R}(g_k(x+h)+g_k(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha q}{n}}\frac{dh}{|h|^n}\\
&+C\int_{B_{\frac{R}{4}}(0)}\left(\int_{B_R}\frac{|\tau_h D\psi(x)|^p}{|h|^{\alpha p}}\,dx\right)^{\frac{q}{2p}}\left(\int_{B_R}(g_k(x+h)+g_k(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha q}{2n}}\frac{dh}{|h|^n}\\
&\qquad\cdot\left(\int_{B_{2R}}(1+|D\psi(x)|)^{p^*_\alpha}\,dx\right)^{\frac{(n-p\alpha)q}{2np'}}\\
&+C\left(\int_{B_R}|Du(x)|^p+|D\psi(x)|^p\,dx\right)^{\frac{q}{2p}}\cdot\left(\int_{B_{2R}}(1+|D\psi(x)|)^{p^*_\alpha})\,dx\right)^{\frac{(n-\alpha p)q}{2np'}}\\
&\qquad\cdot\int_{B_{\frac{R}{4}}(0)}|h|^{\frac{q(1-\alpha)}{2}}\left(\int_{B_R}(g_k(x+h)+g_k(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha q}{2n}}\frac{dh}{|h|^n}.
\end{aligned}
\tag{4.9}
$$

Setting

$$
\tilde H=\int_{B_{2R}}(1+|Du(x)|^p+|D\psi(x)|^p+|D\psi(x)|^{p^*_\alpha})\,dx,
$$

inequality (4.9) can be simplified as follows

$$
\begin{aligned}
&\int_{B_{\frac{R}{4}}(0)}\left(\int_{B_{\frac{R}{2}}}\frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha}}\,dx\right)^{\frac{q}{2}}\frac{dh}{|h|^n}\\
\le\ & C\int_{B_{\frac{R}{4}}(0)}\left(\int_{B_R}\frac{|\tau_h D\psi|^p}{|h|^{\alpha p}}\,dx\right)^{\frac{q}{p}}\frac{dh}{|h|^n}+C\int_{B_{\frac{R}{4}}(0)}\left(|h|^{1-\alpha}\right)^q\frac{dh}{|h|^n}\\
&+C\int_{B_{\frac{R}{4}}(0)}\left(\int_{B_R}(g_k(x+h)+g_k(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha q}{n}}\frac{dh}{|h|^n}\\
&+C\int_{B_{\frac{R}{4}}(0)}\left(\int_{B_R}\frac{|\tau_h D\psi(x)|^p}{|h|^{\alpha p}}\,dx\right)^{\frac{q}{2p}}\left(\int_{B_R}(g_k(x+h)+g_k(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha q}{2n}}\frac{dh}{|h|^n}\\
&+C\int_{B_{\frac{R}{4}}(0)}|h|^{\frac{q(1-\alpha)}{2}}\left(\int_{B_R}(g_k(x+h)+g_k(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha q}{2n}}\frac{dh}{|h|^n},
\end{aligned}
\tag{4.10}
$$

where now the constant $C=C(\nu,\ell,L,n,p,q,\alpha,R,H)$. Furthermore, the use of Young's inequality with exponent 2 in the third and fourth lines of estimate (4.10) yields

$$
\begin{aligned}
&\int_{B_{\frac{R}{4}}(0)}\left(\int_{B_{\frac{R}{2}}}\frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha}}\,dx\right)^{\frac{q}{2}}\frac{dh}{|h|^n}\\
\le\ & C\int_{B_{\frac{R}{4}}(0)}\left(\int_{B_R}\frac{|\tau_h D\psi|^p}{|h|^{\alpha p}}\,dx\right)^{\frac{q}{p}}\frac{dh}{|h|^n}+C\int_{B_{\frac{R}{4}}(0)}\left(|h|^{1-\alpha}\right)^q\frac{dh}{|h|^n}\\
&+C\int_{B_{\frac{R}{4}}(0)}\left(\int_{B_R}(g_k(x+h)+g_k(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha q}{n}}\frac{dh}{|h|^n}\\
=:\ & I_1+I_2+I_3.
\end{aligned}
\tag{4.11}
$$

Note that I_1 is controlled by the norm in the Besov space $B^{\alpha}_{p,q}$ on B_R of the gradient of the obstacle. More precisely

$$I_1 = C||D\psi||^q_{B^{\alpha}_{p,q}(B_R)}$$

which is finite by the assumptions. The integral I_2 can be easily calculated in polar coordinates as follows

$$I_2 = C\int_0^{R/4} \rho^{q-\alpha q-1}\, d\rho = C(n,\alpha,q,R),$$

since $\alpha \in (0,1)$. We now write the integral I_3 in polar coordinates, so $h \in B(0,\frac{R}{4})$ if and only if $h = r\xi$ for some $0 \le r < \frac{R}{4}$ and some ξ in the unit sphere S^{n-1} on $\mathbb{R}^n$. We denote by $d\sigma(\xi)$ the surface measure on S^{n-1}. We bound the term I_3 by

$$\begin{aligned}
&\int_0^{\frac{R}{4}}\int_{S^{n-1}}\left(\int_{B_R}(g_k(x+r\xi)+g_k(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha q}{n}} d\sigma(\xi)\,\frac{dr}{r}\\
&\qquad=\sum_{k=0}^{\infty}\int_{r_{k+1}}^{r_k}\int_{S^{n-1}}\left(\int_{B_R}(g_k(x+r\xi)+g_k(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha q}{n}} d\sigma(\xi)\,\frac{dr}{r}\\
&\qquad=\sum_{k=0}^{\infty}\int_{r_{k+1}}^{r_k}\int_{S^{n-1}}\|(\tau_{r\xi}g_k+g_k)\|^q_{L^{\frac{n}{\alpha}}(B_R)}\,d\sigma(\xi)\,\frac{dr}{r},
\end{aligned}$$

where we set $r_k = \frac{R}{4}\frac{1}{2^k}$. We note that for each $\xi \in S^{n-1}$ and $r_{k+1} \le r \le r_k$

$$\|(\tau_{r\xi}g_k+g_k)\|_{L^{\frac{n}{\alpha}}(B_R)} \le \|g_k\|_{L^{\frac{n}{\alpha}}(B_R-r_k\xi)} + \|g_k\|_{L^{\frac{n}{\alpha}}(B_R)} \le 2\|g_k\|_{L^{\frac{n}{\alpha}}\left(B_{R+\frac{R}{4}}\right)},$$

Hence

$$I_3 \le C(n,\alpha,q)\,\|\{g_k\}_k\|^q_{\ell^q\left(L^{\frac{n}{\alpha}}\left(B_{2R}\right)\right)}$$

Inserting the estimate of I_i, $i = 1,2,3$ in (4.11) and taking the power $\frac{1}{q}$ to both sides, we get

$$\begin{aligned}
\left\|\frac{\Delta_h(V_p(Du))}{|h|^{\alpha}}\right\|_{L^q\left(\frac{dh}{|h|^n};L^2\left(B_{\frac{R}{2}}\right)\right)} &\le C\left(1+||D\psi||_{B^{\alpha}_{p,q}(B_R)}\right)\\
&\quad + C(n,\alpha,q)\,\|\{g_k\}_k\|_{\ell^q\left(L^{\frac{n}{\alpha}}\left(B_{2R}\right)\right)}.
\end{aligned}$$

Lemma 2.4 now yields that $V_p(Du) \in B^{\alpha}_{2,q}$ locally, i.e. estimate (1.8).

□

We are in position to give the

Proof of Corollary 1.3. We start observing that assumption ($\mathcal{A}5$) is equivalent to $x \mapsto \mathcal{A}(x,\xi) \in B^{\alpha}_{\frac{n}{\alpha},q}$, by virtue of Theorem 2.5. Therefore, in case $\alpha > \beta$, it suffices to use the inclusion of Lemma 2.3

$$B^{\alpha}_{\frac{n}{\alpha},q} \subset B^{\beta}_{\frac{n}{\beta},q}$$

to deduce that the partial map $x \mapsto \mathcal{A}(x,\xi) \in B^{\beta}_{\frac{n}{\beta},q}$. At this point, applying Theorem 1.2 with β in place of α we deduce that

$$V_p(Du) \in B^{\beta}_{2,q},$$

and so

$$V_p(Du) \in B^{\min\{\alpha,\beta\}}_{2,q}.$$

In case $\beta > \alpha$, we just use Theorem 1.2 since

$$D\psi \in B^{\beta}_{p,q} \subset B^{\alpha}_{p,q},$$

again by Lemma 2.3. □

5. Proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4 that is achieved using the arguments of previous sections, with the modifications that take into account the different assumptions on the gradient of the obstacle and on the map $x \mapsto \mathcal{A}(x,\xi)$.

Proof. The proof goes exactly as that of Theorem 1.2 until we arrive at estimate (4.1) of previous Section. We need to treat differently only the integrals IV, V and VI in which the new assumption ($\mathcal{A}6$) comes into the play. Indeed, assumption ($\mathcal{A}6$) together with Young's and Hölder's inequalities yields that

$$\begin{aligned}
|IV| &\le |h|^{\alpha}\int_{\Omega}\eta^2(g(x+h)+g(x))(1+|Du(x)|^2)^{\frac{p-1}{2}}|\tau_h Du|dx\\
&\le \varepsilon\int_{\Omega}\eta^2|\tau_h Du|^2\left(1+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}dx\\
&\quad +C_\varepsilon|h|^{2\alpha}\int_{B_R}(g(x+h)+g(x))^2(1+|Du(x)|^2)^{\frac{p}{2}}dx\\
&\le \varepsilon\int_{\Omega}\eta^2|\tau_h Du|^2\left(1+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}dx\\
&\quad +C_\varepsilon|h|^{2\alpha}\left(\int_{B_R}(g(x+h)+g(x))^{\frac{n}{\alpha}}dx\right)^{\frac{2\alpha}{n}}\left(\int_{B_R}(1+|Du(x)|)^{\frac{np}{n-2\alpha}}dx\right)^{\frac{n-2\alpha}{n}}, \qquad (5.1)
\end{aligned}$$

where we also used that supp$\eta \subset B_R$. Observe that the assumption $D\psi \in B^{\beta}_{p,\infty,\mathrm{loc}}(\Omega)$ implies, by means of the definition (2.3) that there exists a constant $\Psi = \Psi\big(||D\psi||_{B^{\beta}_{p,\infty}}(B_{2R})\big)$ such that

$$\int_{B_R}\frac{|D\psi(x+h)-D\psi(x)|^p}{|h|^{p\beta}} \le \Psi, \qquad (5.2)$$

for every $h \in \mathbb{R}^n$ such that $|h| < \frac{R}{4}$. Since $p \ge 2$, the use of Hölder's inequality implies

$$\begin{aligned}
\int_{B_R}|\tau_h V_p(D\psi)|^2\,dx &\le C\int_{B_R}|\tau_h D\psi|^2\left(1+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p-2}{2}}dx\\
&\le C\left(\int_{B_R}|\tau_h D\psi|^p\,dx\right)^{\frac{2}{p}}\left(\int_{B_R}\left(1+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{p-2}{p}},
\end{aligned}$$

where the first inequality is given by Lemma 2.1. Therefore, dividing previous inequality by $|h|^{2\beta}$ and using (5.2), we get

$$\begin{aligned}
\int_{B_R}\frac{|\tau_h V_p(D\psi)|^2}{|h|^{2\beta}}dx &\le C\left(\int_{B_R}\frac{|\tau_h D\psi|^p}{|h|^{p\beta}}dx\right)^{\frac{2}{p}}\left(\int_{B_R}\left(1+|D\psi(x)|^2+|D\psi(x+h)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{p-2}{p}}\\
&\le C\Psi^{\frac{2}{p}}\left(\int_{B_{2R}}\left(1+|D\psi(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{p-2}{p}}\\
&\le C\Psi^{\frac{2}{p}}(1+||D\psi||_p)^{\frac{p-2}{p}} =: \widetilde{\Psi},
\end{aligned}$$

where $\widetilde{\Psi}=\widetilde{\Psi}\big(||D\psi||_{B^{\beta}_{p,\infty}},||D\psi||_p\big)$ and where we used also the second inequality of Lemma 2.7. Lemma 2.9 implies that

$$|V_p(D\psi)|\in L^{\frac{2n}{n-2\vartheta}}_{\mathrm{loc}}(\Omega),$$

for every $\vartheta<\beta$ and so, in particular that

$$|V_p(D\psi)|\in L^{\frac{2n}{n-2\alpha}}_{\mathrm{loc}}(\Omega).$$

Since, by definition (2.1),

$$|D\psi|^{\frac{p}{2}}\le |V_p(D\psi)|,$$

we deduce that

$$|D\psi|\in L^{\frac{np}{n-2\alpha}}_{\mathrm{loc}}(\Omega). \tag{5.3}$$

Therefore, by Theorem 2.11, we get

$$⨍_{B_R}|Du(x)|^{\frac{np}{n-2\alpha}}\,dx\le C\left(1+⨍_{B_{2R}}|D\psi(x)|^{\frac{np}{n-\alpha p}}\,dx\right), \tag{5.4}$$

where $C=C(n,\nu,L,\ell,p,\alpha,\beta,R)$. Inserting (5.4) in (5.1), we get

$$\begin{aligned}|IV| &\le \varepsilon\int_\Omega \eta^2|\tau_h Du|^2\left(1+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}\,dx\\ &\quad+C_\varepsilon|h|^{2\alpha}\left(\int_{B_R}(g(x+h)+g(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{2\alpha}{n}}\left(\int_{B_{2R}}(1+|D\psi(x)|)^{\frac{np}{n-2\alpha}}\,dx\right)^{\frac{n-2\alpha}{n}} \end{aligned}\tag{5.5}$$

Again by virtue of assumption ($\mathcal{A}6$), we get

$$\begin{aligned}|V| &\le |h|^\alpha\int_\Omega \eta^2(g(x+h)+g(x))(1+|Du|^2)^{\frac{p-1}{2}}|\tau_h D\psi|dx\\ &\le |h|^\alpha\left(\int_\Omega\eta^2|\tau_hD\psi|^p\,dx\right)^{\frac1p}\left(\int_\Omega\eta^2(g(x+h)+g(x))^{\frac{p}{p-1}}(1+|Du(x)|^2)^{\frac p2}\,dx\right)^{\frac{p-1}{p}}\\ &\le |h|^\alpha\left(\int_{B_R}|\tau_hD\psi|^p\,dx\right)^{\frac1p}\left(\int_{B_R}(g(x+h)+g(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha}{n}}\\ &\qquad\cdot\left(\int_{B_R}(1+|Du(x)|)^{\frac{np(p-1)}{n(p-1)-\alpha p}}\,dx\right)^{\frac{n(p-1)-\alpha p}{np}}\\ &\le C|h|^\alpha\left(\int_{B_R}|\tau_hD\psi|^p\,dx\right)^{\frac1p}\left(\int_{B_R}(g(x+h)+g(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha}{n}}\\ &\qquad\cdot\left(\int_{B_{2R}}(1+|D\psi(x)|)^{\frac{np}{n-2\alpha}}\,dx\right)^{\frac{n-2\alpha}{np'}} \end{aligned}\tag{5.6}$$

where we used the properties of η, Hölder's inequality and, in the last line, we used estimate (5.4), since (5.3) holds, and since

$$\frac{np(p-1)}{n(p-1)-\alpha p}\le\frac{np}{n-2\alpha}\quad\Longleftrightarrow\quad p\ge 2.$$

By virtue of (5.2), we may estimate the first integral in the right hand side of (5.6) as follows

$$|V|\ \le\ C|h|^{\alpha+\beta}\left(\int_{B_{2R}}g^{\frac{n}{\alpha}}(x)\,dx\right)^{\frac{\alpha}{n}}\left(\int_{B_{2R}}(1+|D\psi(x)|)^{\frac{np}{n-2\alpha}}\,dx\right)^{\frac{n-2\alpha}{np'}}, \tag{5.7}$$

with a constant C depending also on Ψ, and where we also used the second inequality of Lemma 2.7.
Finally, arguing as we did for the estimate of V we get

$$\begin{aligned}
|VI| &\le 2|h|^{\alpha}\int_{\Omega}\eta|D\eta|(g(x+h)+g(x))(1+|Du|^2)^{\frac{p-1}{2}}|\tau_h(u-\psi)|dx\\
&\le 2|h|^{\alpha}\left(\int_{B_R}|\tau_h(u-\psi)|^p\,dx\right)^{\frac{1}{p}}\left(\int_{B_R}|D\eta|^{\frac{p}{p-1}}(g(x+h)+g(x))^{\frac{p}{p-1}}(1+|Du(x)|^2)^{\frac{p}{2}}\,dx\right)^{\frac{p-1}{p}}\\
&\le \frac{C\,|h|^{\alpha}}{R}\left(\int_{B_R}|\tau_h(u-\psi)|^p\,dx\right)^{\frac{1}{p}}\\
&\qquad\cdot\left(\int_{B_R}(g(x+h)+g(x))^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha}{n}}\left(\int_{B_R}(1+|Du(x)|)^{\frac{np(p-1)}{n(p-1)-\alpha p}}\,dx\right)^{\frac{n(p-1)-\alpha p}{np}},
\end{aligned}$$

where we used that $|D\eta|\le\frac{C}{R}$. Using Lemma 2.7 and (5.4) in the right hand side of previous estimate, we get

$$\begin{aligned}
|VI| &\le C|h|^{\alpha+1}\left(\int_{B_{2R}}|D(u-\psi)(x)|^p\,dx\right)^{\frac{1}{p}}\\
&\qquad\cdot\left(\int_{B_{2R}}g^{\frac{n}{\alpha}}(x)\,dx\right)^{\frac{\alpha}{n}}\left(\int_{B_{2R}}(1+|D\psi(x)|)^{\frac{np}{n-2\alpha}}\,dx\right)^{\frac{n-2\alpha}{np'}}.
\end{aligned}\tag{5.8}$$

Inserting estimates (5.5), (5.7) and (5.8) in (4.1) we infer the existence of constants $C_\varepsilon\equiv C(\varepsilon,\nu,L,\ell,n,p,q,R,||D\psi||_{B^{\beta}_{p,\infty}})$ and $C\equiv C(\nu,L,\ell,n,p,q,R,||D\psi||_{B^{\beta}_{p,\infty}})$ and

$$\begin{aligned}
&\nu\int_{\Omega}\eta^2|\tau_hDu|^2(1+|Du(x+h)|^2+|Du(x)|^2|)^{\frac{p-2}{2}}\,dx\\
&\le 3\varepsilon\int_{\Omega}\eta^2|\tau_hDu|^2(1+|Du(x+h)|^2+|Du(x)|^2|)^{\frac{p-2}{2}}\,dx\\
&+C_\varepsilon\left(\int_{B_{2R}}|\tau_hD\psi|^p\,dx\right)^{\frac{2}{p}}\left(\int_{B_{2R}}(1+|Du(x)|)^p\,dx\right)^{\frac{p-2}{p}}\\
&+C_\varepsilon|h|^2\int_{B_{2R}}(1+|Du(x)|+|D\psi(x)|)^p\,dx\\
&+C_\varepsilon\,|h|^{2\alpha}\left(\int_{B_{2R}}g(x)^{\frac{n}{\alpha}}\,dx\right)^{\frac{2\alpha}{n}}\left(\int_{B_{2R}}(1+|D\psi(x)|)^{\frac{np}{n-2\alpha}}\,dx\right)^{\frac{n-2\alpha}{n}}\\
&+C|h|^{\alpha+\beta}\left(\int_{B_{2R}}g^{\frac{n}{\alpha}}(x)\,dx\right)^{\frac{\alpha}{n}}\left(\int_{B_{2R}}(1+|D\psi(x)|)^{\frac{np}{n-2\alpha}}\,dx\right)^{\frac{n-2\alpha}{np'}}\\
&+C|h|^{1+\alpha}\left(\int_{B_{2R}}|Du(x)|^p+|D\psi(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_{B_{2R}}g^{\frac{n}{\alpha}}(x)\,dx\right)^{\frac{\alpha}{n}}\left(\int_{B_{2R}}(1+|D\psi(x)|)^{\frac{np}{n-2\alpha}}\,dx\right)^{\frac{n-2\alpha}{np'}},
\end{aligned}$$

where we also used Lemma 2.7. Using inequality (5.2) and the fact that $\alpha<\beta<1$ from previous estimate we deduce

$$\begin{aligned}
&\nu\int_{\Omega}\eta^2|\tau_hDu|^2(1+|Du(x+h)|^2+|Du(x)|^2|)^{\frac{p-2}{2}}\,dx\\
&\le 3\varepsilon\int_{\Omega}\eta^2|\tau_hDu|^2(1+|Du(x+h)|^2+|Du(x)|^2|)^{\frac{p-2}{2}}\,dx
\end{aligned}$$

$$+C_\varepsilon\,|h|^{2\alpha}\left(\int_{B_{2R}}(1+|Du(x)|)^p\,dx\right)^{\frac{p-2}{p}}$$
$$+C_\varepsilon|h|^{2\alpha}\int_{B_{2R}}(1+|Du(x)|+|D\psi(x)|)^p\,dx$$
$$+C_\varepsilon\,|h|^{2\alpha}\;\left(\int_{B_{2R}}g(x)^{\frac{n}{\alpha}}\,dx\right)^{\frac{2\alpha}{n}}\left(\int_{B_{2R}}(1+|D\psi(x)|)^{\frac{np}{n-2\alpha}}\,dx\right)^{\frac{n-2\alpha}{n}}$$
$$+C|h|^{2\alpha}\left(\int_{B_{2R}}g^{\frac{n}{\alpha}}(x)\,dx\right)^{\frac{\alpha}{n}}\left(\int_{B_{2R}}(1+|D\psi(x)|)^{\frac{np}{n-2\alpha}}\,dx\right)^{\frac{n-2\alpha}{np'}}$$
$$+C\frac{|h|^{2\alpha}}{R}\left(\int_{B_{2R}}|Du(x)|^p+|D\psi(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_{B_{2R}}g^{\frac{n}{\alpha}}(x)\,dx\right)^{\frac{\alpha}{n}}\;\left(\int_{B_{2R}}(1+|D\psi(x)|)^{\frac{np}{n-2\alpha}}\,dx\right)^{\frac{n-2\alpha}{np'}},$$

Choosing $\varepsilon=\frac{\nu}{6}$ yields

$$\nu\int_\Omega\eta^2|\tau_hDu|^2(1+|Du(x+h)|^2+|Du(x)|^2|)^{\frac{p-2}{2}}\,dx$$
$$\le\quad C|h|^{2\alpha}\left(\int_{B_{2R}}\left(1+|Du|^p+|D\psi|^p\right)dx+\int_{B_{2R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}\,dx+\int_{B_{2R}}g^{\frac{n}{\alpha}}(x)\,dx\right),$$

where we used Young's inequality in the right hand side. Using Lemma 2.1 in the left hand side of previous estimate, recalling that $\eta\equiv1$ on $B_{\frac{R}{2}}$ and dividing both sides by $|h|^{2\alpha}$, we get

$$\nu\int_{B_{\frac{R}{2}}}\frac{|\tau_hV_p(Du)|^2}{|h|^{2\alpha}}\,dx$$
$$\le\quad C\left(\int_{B_R}\left(1+|Du|^p+|D\psi|^p\right)dx+\int_{B_{2R}}|D\psi(x)|^{\frac{np}{n-2\alpha}}\,dx+\int_{B_{2R}}g^{\frac{n}{\alpha}}(x)\,dx\right).\qquad(5.9)$$

The conclusion follows, recalling the definition of the auxiliary function $V_p(\xi)$ and of Besov space $B^\alpha_{2,\infty}$.

□

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