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# SECOND-ORDER COVARIATION:

## AN ANALYSIS OF STUDENTS' REASONINGS AND TEACHER'S INTERVENTIONS WHEN MODELLING REAL PHENOMENA


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# ABSTRACT

This research project aims to investigate covariation understood not only as the ability to visualize two or more magnitudes while co-varying simultaneously (Thompson & Carlson, 2017), but in a broader epistemological sense, as the ability to grasp relationships of invariance between two quantities. The need to better characterize more complex forms of reasoning performed by students in mathematical modelling activities, led us to introduce *second-order covariation*, a form of covariation that consists in describing relations in which not only variables are involved but also parameters (Arzarello, 2019). These enable to represent families of relationships between variables that is classes of real phenomena characterized, from a mathematical standpoint, through parameters, which determine the specificities of the mathematical model.

The discussion of this theme arises not only from research needs in the field of Mathematics Education, i.e., the existence of a theoretical framework only partially useful to describe the covariational reasoning of students, but above all by its relevance in terms of teaching practices. There is a wide literature showing that in mathematical modelling situations the ability to reason covariationally is essential because it allows to visualize the invariant relationships that exist between quantities involved in dynamic situations (Thompson, 2011). The indications for teaching mathematics in high schools (MIUR, 2010) underline the relevance of introducing mathematical modelling as a representation of classes of real phenomena. However, despite the acknowledged relevance of covariation for the learning of numerous mathematical concepts, in the National Indications, as well as in most textbooks, references to this approach are generally absent. The teachers themselves have little specific knowledge of covariation and therefore struggle to introduce it into their teaching practices.

The data analyzed in this project come from three didactical experiments developed in some classes of a scientific high school and whose aim is the mathematical description of some real situations: specifically, the motion of a ball along an inclined plane and the relationship between temperature and humidity described in the so-called psychrometric diagram. Using appropriate technological tools, the students were guided in deriving a mathematical formula that described such phenomena and in recognizing the different role played by variables and parameters in the writing and reading of different registers of mathematical representations. Students' reasoning processes and the evolution of the different semiotic aspects (spoken, gestural,

representational) involved in the teaching-learning processes were analyzed; as well the support of technology and the role of the teacher in enhancing covariational reasoning through appropriate adaptive teaching strategies, were considered.

This study led us not only to the elaboration of a broader theoretical framework, which consistently includes second-order covariation, but also to hypothesize the existence of a third-order covariation. In addition, some research studies complementary to the main one described above, allowed us to explore the theme of assessment of covariation as a form of conceptual understanding and to elaborate a mathematical interpretation of the covariation construct using category theory (Mac Lane, 1978) and the cognitive mechanisms of conceptual blending (Fauconnier & Turner, 2002).

# BREVE DESCRIZIONE

Questo progetto di ricerca si propone di indagare la covariazione intesa non solo come capacità di visualizzare due o più grandezze mentre co-variano simultaneamente (Thompson & Carlson, 2017), ma in un più ampio senso epistemologico, come capacità di cogliere relazioni di invarianza tra due grandezze. L'esigenza di caratterizzare meglio forme di ragionamento più complesse messe in atto da studenti in attività di modellizzazione matematica, ci ha portato a introdurre la *covariazione al secondo ordine*, una forma di covariazione che consiste nel descrivere le relazioni in cui sono coinvolte non solo variabili ma anche parametri (Arzarello, 2019). Questi ultimi consentono di rappresentare famiglie di relazioni tra variabili cioè classi di fenomeni reali caratterizzati, da un punto di vista matematico, da parametri che determinano le specificità del modello matematico.

La trattazione di questo tema nasce non solo da esigenze a livello di ricerca nel settore della Didattica della Matematica, ovvero l'esistenza di un quadro teorico solo parzialmente utile a descrivere i ragionamenti covariazionali degli studenti, ma soprattutto da una sua rilevanza a livello di pratiche didattiche. Infatti, esiste un'ampia letteratura che mostra come in situazioni di modellizzazione matematica sia essenziale la capacità di ragionare in modo covariazionale poiché essa consente di visualizzare le relazioni invarianti che sussistono tra grandezze fisiche coinvolte in situazioni dinamiche (Thompson, 2011). Le indicazioni per l'insegnamento della matematica nei licei (MIUR, 2010) sottolineano l'importanza dell'introduzione alla modellizzazione matematica intesa come rappresentazione di classi di fenomeni reali eppure, nonostante la riconosciuta importanza della covariazione per l'apprendimento di numerosi concetti matematici, nelle Indicazioni Nazionali così come nella maggior parte dei libri di testo i riferimenti a questo approccio sono generalmente assenti. Gli insegnanti stessi hanno poche conoscenze in merito alla covariazione e quindi faticano a introdurla nelle loro pratiche didattiche.

I dati analizzati in questo progetto provengono da tre sperimentazioni didattiche condotte in alcune classi di un liceo scientifico e aventi come obiettivo la descrizione matematica di alcune situazioni reali quali, nello specifico, il moto di una pallina lungo un piano inclinato e la relazione tra temperatura e umidità descritta nel cosiddetto diagramma psicrometrico. Attraverso l'utilizzo di opportuni strumenti tecnologici, gli studenti sono stati guidati nel ricavare una formula matematica che descrivesse tali fenomeni e nel riconoscere il differente ruolo svolto da variabili e parametri nella scrittura e lettura di diversi registri di rappresentazione matematica. Sono stati analizzati i processi di ragionamento degli studenti, l'evoluzione dei diversi aspetti semiotici (parlato, gestualità, rappresentazioni) coinvolti nei processi di insegnamento-apprendimento, il supporto della tecnologia e il ruolo dell'insegnante nel favorire il ragionamento covariazionale adottando adeguate strategie didattiche adattive.

Questo studio ci ha portato non solo all'elaborazione di un più ampio quadro teorico che includesse in modo coerente la covariazione al secondo ordine, ma anche a ipotizzare l'esistenza di un terzo ordine di covariazione. Inoltre, alcuni studi di ricerca complementari a quello principale finora descritto, ci hanno permesso di esplorare il tema della valutazione della covariazione intesa come forma di apprendimento concettuale e ad elaborare un'interpretazione matematica del costrutto covariazione usando la teoria delle categorie (Mac Lane, 1978) e i meccanismi cognitivi del blending concettuale (Fauconnier & Turner, 2002).

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# 1 INTRODUCTION

International studies in Mathematics Education have deeply underlined and supported with evidence-based research the importance of covariational reasoning for a deep understanding of many mathematical concepts like proportion, rate of change, variable, periodic functions, exponential growth, and in particular functions of one and two variables (Thompson, 1994a; Thompson & Silverman, 2008; Thompson & Carlson, 2017), the conceptualization of dynamic situations (Carlson et al., 2002; Carlson, 1998), and a full comprehension of many physical magnitudes, for instance force, work, momentum and energy (Thompson et al., 2017). Covariational reasoning emerges when students are able to reason "about values of two or more quantities varying simultaneously" (Thompson & Carlson, 2017, p. 422), namely in case they are able to grasp "that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other" (Thompson & Carlson, 2017, p. 436). The issue of covariational reasoning has gained an increased educational interest in the last decades, and its theoretical framework is enlarging and detailing from many different points of view, but there are still many research gaps and perspectives that are worthy to be explored and we are going to describe and address all these issues in the following, underlining the scientific relevance of this research.

1. Covariation is essential for modelling activities

There are many reasons why students struggle with the concept of function, in mathematics as well as in all the STEM-areas. One main reason is that students are often introduced to the concept of function through a static definition, for instance the one of Bourbaki (1939) as a relationship between the elements of two sets, a definition adopted also in textbooks today widely used in school such as *Manuale.blu di matematica* (2020). Consequently, students do not understand the dynamic nature of functions conceptualized by covariation, that is how the independent and the dependent variables change together. Carlson's studies (1998) highlight the lack of a covariational approach may be one of the reasons why students are unable to interpret dynamic situations and to construct meaningful formulas suitable for representing one quantity as a function of another.

Covariation is crucial in activities of mathematical modelling because "the operations that compose covariational reasoning are the very operations that enable one to see invariant relationships among quantities in dynamic situations” (Thompson, 2011, p. 46), and so it reveals essential for entering into the so-called modelling cycle (Blum, 1996). Despite the recognized importance of covariation, most of mathematics curricula do not contain explicit references to covariation and therefore even teachers do not enhance it in their school practices.

2. The theoretical framework of covariation is worthy to be enlarged

The already existing theoretical framework about covariation deeply addresses how students reason when trying to co-vary two or more variables, and this kind of characterization relies on the cognitive features of students’ reasoning. In the last two decades new theoretical perspectives emerged and they mainly developed in two directions: the first one refers to the design principles of activities enhancing covariation and specifically the role of technology, e.g., DGS environment, in supporting covariational reasoning processes (e.g., Johnson et al., 2017; Johnson, 2013). This thread of research mainly provides a wider characterization of the covariation construct from a didactical point of view, investigating how supporting covariational reasoning at a task design level and teaching practices level. The second line of research, a more recent one, approaches covariation as a larger form of reasoning that considers mathematical objects jointly with their mathematical relations. This enlarged vision of covariation allows to coherently introduce a more complex form of reasoning, *second-order covariation* (Arzarello, 2019), the main object of investigation of this research project, which consists in the description of invariant relations in which are involved not only variables but also parameters. The latter allow to represent families of relations between variables, that is classes of real phenomena characterized, from the point of view of the mathematical representation, by peculiar parameters determining the specificities of the mathematical model. Hence, this broadening of the existing framework about covariation enables to include an improvement of the theoretical construct of covariation with the introduction of second-order covariation, and to widen the characterization of this construct not only on the cognitive side, but also from didactical and mathematical standpoints.

3. Teachers also struggle with covariation

Covariation is a complex type of reasoning, and many students meet difficulties in attaining and maintaining it (e.g., Adu-Gyamfi & Bossé, 2014; Carlson et al., 2002; Ellis et al., 2016). These same bodies of literature suggest that since covariational reasoning is uncommon among students, instruction should emphasize it in students' learning activities and should place an increased emphasis on teachers' instructions to reach individual students' cognitive needs for prompting covariational reasoning. The truth is that teachers also struggle with a covariational view of the concept of function and are usually not so efficient in teaching this concept, especially when a function within the same family of functions varies with respect to varying parameters.

As previously hinted, adaptive instruction may be a first answer to this issue: investigating how teachers interact with students to adapt their instruction to teach complex mathematical concepts and specifically the teacher's role in facilitating students' evolution towards the understanding of a specific mathematical content, covariation in our case, carries important theoretical, methodological, and pedagogical implications. But the key to a successful adaptive instruction is a deep knowledge and mastery of both first- and second-order covariation which would equip teachers with a flexible overview of handling adaptive teaching of covariation in school at a theoretical as well as at a practical level.

## 4. The assessment of covariation

Given the peculiarity and complexity of covariation as a theoretical and cognitive construct, and its rare presence in school practices, the assessment of this form of reasoning turns out to be challenging. Even studies on mathematical modelling do not propose a systematic exploration of the issue of its assessment, which should reflect not only the aims of applications and appropriate modelling (Blum, 2015), but also students' ability to reason in a covariational manner. In addition, covariational reasoning deals with a typical conceptual knowledge construct, which is more complex than a content area or procedure to be assessed because of the variety and complexity of students' forms of reasoning. Bisson et al. (2016) highlight the difficulties of using standard assessment practices when conceptual understanding is under the lens and even in Italy assessing conceptual understanding and the connections between different mathematical domains or between different subjects like math and physics is underrepresented in summative mathematics tests. While already existing method of assessment of covariation mainly focus on the features of the proposed tasks and its possible answers or on the taxonomy of covariational reasoning, we are not aware of studies that explore its assessment though the adoption of a holistic method that also supports teachers that are not so confident with all the facets of this form of reasoning. As it will be discussed later (Chapter 15), we identified in the comparative judgement a valuable tool, whose features could support the assessing of a complex construct as covariation, especially when it expresses forms of conceptual understanding.

The theoretical characterization of the construct of covariation, both from a cognitive, didactical, and mathematical standpoint, is the main object of investigation of this research and we are going to address the issues set out in points 1-2-3 of this Introduction in the main body of this work. The state of art about covariation with reference to existing literature will be exposed in detail in Chapter 2 along with some new perspectives about covariational reasoning arose in the last decades (Section 2.1) and some historical notes about the relevance of this form of reasoning in different fields of Mathematics (Section 2.2). In Chapter 3, the main conceptions and approaches to the concept of function in mathematics, including the covariational one, will be recalled together with the epistemological obstacles that students face when dealing with functions in modelling activities. Moreover, the support of technology is a valuable ally to introduce covariation in classroom activities and to instrument this kind of reasoning processes (Chapter 4). Chapter 5 contains a preliminary formulation of the research questions that led us in the design of this study both on an experimental, methodological, and epistemological level. The complexity of the topic of covariation and the numerous standpoints from which it is analyzed required various and suitable theoretical lenses to be networked: the four theoretical perspectives (semiotic bundle, commognition, conceptual blending and adaptive teaching) are presented in their main aspects in Chapter 6 jointly with their network. A clear definition of the theoretical framework enabled us to produce a definitive formulation of the research questions (Chapter 7). The methodology of our research is illustrated in Chapter 8 that is structured in the following four sections: the main features of a qualitative research based on teaching experiments (8.1), the design principles connoting our experimentations (8.2), the methodological tool of the Timeline (8.3) and finally some notes concerning the methods for data analysis (8.4). Moreover, given the relevance of the role of the teacher we collected some relevant information about her background and the school where the teaching experiments were conducted in Chapter 9. Chapters 10, 11, and 12 are devoted to the analysis of the three teaching experiments and the structure is recurrent: overview of the tasks and prospective analysis, data analysis of some selected episodes, discussion of the results according to four different layers of analysis, and some concluding remarks. Eventually Chapter 13 contains the answers to the research questions, some didactical implications and limitations of this study and some further purposes of research.

The last part of the thesis contains some extra chapters related to the topic of covariation which were not part of the original design of the research and respond to specific research purposes.

Indeed, the discussion of the results of the research revealed the need of a further investigation of the theoretical construct of covariation so to enlighten the characterization of second-order covariation and this interpretation emerged in particular in the last teaching experiment. These evidences led us to explore a mathematical interpretation adopting the universal language of category theory and the cognitive lens of conceptual blending. Preliminary results are exposed in Chapter 14.

The issue of the assessment of covariational reasoning in the form of conceptual understanding is specifically addressed in Chapter 15 adopting the technique of comparative judgement as method of assessment. Chapter 16 is a deeper exploration of the features of the notion of instrumented covariation and contains an analogy for variables and parameters based on C programming language.

# 2 THE STATE OF ART ABOUT COVARIATION

Covariational reasoning started being considered and studied as a theoretical construct only in the late 1980s and early 1990s. Covariational reasoning entails thinking about how two quantities' values change together. However, there is no single understanding of what ways of thinking constitute covariational reasoning and in the following we are going to recall the main conceptions that contributed to a full conceptualization and definition of this construct.

Confrey (1991) and Confrey and Smith (1994) described a preliminary notion of covariation, where students coordinate a completed change in the values of $x$ with a completed change in the values of $y$. Hence, they characterized covariation in terms of coordinating two variables' values as they change.

Thompson and Thompson (1992) and Thompson (1994a; 1993), in the theory of quantitative reasoning, described a notion of covariation where students simultaneously track two quantities' varying values: a quantity is defined as someone's conceptualization of an object such that it has an attribute that could be measured; a person reasons covariationally when she envisions two quantities' values varying and then envisions them varying simultaneously.

Saldanha and Thompson (1998) further elaborated Thompson's notion of covariation. They explained that their notion of covariation is "of someone holding in mind a sustained image of two quantities values (magnitudes) simultaneously" (p. 299). The individual mentally forms a *multiplicative object*, a new conceptual object formed merging the attributes of the two initial quantities. According to the authors, this notion was derived from Piaget's notion of *and* as a multiplicative operator (1950), an operation that Piaget described as underlying operative classification and seriation in children's thinking. The authors clarify that their idea of multiplicative object differs from the one contained in Sfard's theory of reification (1991) because it should not be intended as a mathematical concept that a person can operate upon mentally, but as a specific cognitive act: hence, the focus is not on the resulting object, but on the cognitive process itself. To provide an example of multiplicative object, the conceptualization of torque requires to conceive the "amount of twist" thinking simultaneously to a force and the distance from a fulcrum to the force's point of application (Thompson & Saldanha, 2003), or again ordered pairs represented by points in the Cartesian plane are multiplicative objects when understood as values of two quantities that vary simultaneously.

Thompson et al. (2016) deeply investigated the relevance of creating a multiplicative object from two magnitudes to mastering a covariational meaning for graphs. They suggested that students'

difficulties with graphs could be partially attributed to not having conceived points on a graph as multiplicative objects that condense two measures simultaneously.
Carlson and colleagues (2002) described a developmental notion of covariation, where students begin by coordinating directional changes in the values of two quantities and eventually coordinate continuous change in one quantity with the instantaneous rate of change of another quantity. Moreover, Castillo-Garsow (2012) identified three distinctions of students' thinking about how a quantity's value varies: discrete, chunky continuous, and smooth continuous.

The current view of covariational reasoning as a theoretical construct, which has been widely presented in *The Compendium for Research in Mathematics Education* (Thompson & Carlson, 2017), retains emphasis on three main elements: (i) quantitative reasoning and multiplicative objects, (ii) coordination of changes in quantities' values, and (iii) the ways in which an individual conceives quantities as varying. It consists of a hierarchy of six different framework levels that we are going to present referring to "the bottle problem" as done in Thompson and Carlson (2017). In the bottle task, given a bottle of a certain shape, students are asked to sketch a graph of the height of the water in the bottle as a function of the amount of the poured water.

- A student at the *no coordination level* (L0) would not coordinate the height of the water with the amount of water contained in the bottle.
- At the *pre-coordination of values level* (L1), a student would observe that when some water is added to the bottle, the height of the water increases.
- At the *gross coordination of values* level (L2), the student would describe this covariation, saying that "the height increases as the volume increases".
- A student at the *coordination of values level* (L3) would coordinate the values of the water's height in the bottle with a certain increment of the quantity of added water.
- A student at the *chunky continuous covariation level* (L4) would envision the height of the water changing simultaneously with the amount of water, but these changes would refer to intervals of a fixed size, with the student unable to perceive the variables "height" and "volume" as passing through the intermediate values of the interval.
- In the end, a student at the *smooth continuous covariation level* (L5) would conceive height and volume as varying simultaneously through intervals in a smooth and continuous way.

These framework levels can be interpreted as descriptors of a class of behaviors or as the characteristics of a person's capacity to reason covariationally. A person showing a certain level of covariational reasoning means that she is able to reason reliably at lower levels but cannot reason reliably at higher levels. Co-variation necessarily involves two or more magnitudes varying simultaneously; when conceptualizing how a single quantity's values vary, the authors refers to it as *variational reasoning,* a framework consisting itself of six different cognitive levels:

- Variable understood only as *symbol*;
- *No variation* (L0) of the variable is perceived, it has a fixed value;
- *Discrete variation* (L1), when the variable is conceived as assuming specific values;
- *Gross variation* (L2), if the values of a variable are conceived as increasing or decreasing;
- *Chunky continuous variation* (L3), when the variable's values are intended as changing by intervals of a fixed size;
- *Smooth continuous variation* (L4), when the variable's values are intended as changing smoothly and continuously within those intervals.

Some other concluding remarks concerning the cognitive construct of covariation follow.
Saldanha and Thompson (1998) speak of global image of the simultaneous states of two co-varying quantities and graphs are intended as a modality to represent this image: in short, covariation is not just finalized to the reading and drawing of mathematical diagrams.
Thompson and colleagues (2017) have never investigated the problem of covariational reasoning among quantities whose values are related by a formula, but recent studies have been elaborated in this area involving students of algebra and analysis courses (Frank, 2016).
Finally, even if the existing literature until now is limited to a small number of subjects, its findings suggest that reasoning covariationally is uncommon among students and teachers, at least in the U.S where most research studies about covariation are carried out. Moreover, studies that investigate covariational reasoning either internationally or with a large, geographically diverse sample are not known (Thompson et al., 2017).

## 2.1 NEW PERSPECTIVES ABOUT COVARIATION

In the last years, new perspectives about covariation emerged as line of research in Mathematics Education and in this paragraph we are going to present three main contributions that have some contact points with our study.

1. Metavariation, Hoffkamp (2009; 2011)

In the German educational context the term *functional thinking* was introduced for the first time with the so-called *Meraner reform* of 1905 initiated by Felix Klein. In its initial and broad sense,

the term functional thinking addressed thinking in variations and functional dependencies and emphasized the aspects of change. According to Vollrath's definition (1989), functional thinking can be intended as the typical way to think when working with functions and in German mathematics curriculum the idea of functional dependency is one of the five central competencies, which form the mathematics education. Recently Hoffkamp (2009; 2011) analyzed how the use of Interactive Geometry Software (IGS) allows the visualization of the dynamic aspect of functional depencies simultaneously in different representations and offers the opportunity to experiment with them. Specifically, she underlined how some activities designed with IGS allow two levels of variation. The first level is that of covariation, when one can visualize the dynamic aspects varying within the given situation. To understand a dynamic situation one needs to construct an executable mental model to achieve mental simulation. The second level, called *metavariation*, arises when one changes the situation itself and watches the effects on the graph. Metavariation allows the user to investigate covariation in several scenarios. It is a variation within the function that maps the situation to the graph of the underlying functional dependency and changes the functional dependency itself. This leads to a more global view of the dependency. Therefore metavariation refers to the object view of the function, forces the detachment from concrete values, and leads to a qualitative view of the functional dependency and its local and global characteristics. Metavariation is intended as a step towards the perception of a function as an object.
Moreover, in the case studies analyzed by Hoffkamp in her research, the students were always asked to verbalize their observations when working with the applets in the IGS environment. Language acquires a role of mediator between the representations and the mental images of the students, so it is of special importance in the conceptualization process (Janvier, 1978).

2. Instrumented covariation and second-order covariation, Arzarello (2017; 2019)

In 2017 Arzarello remarked how covariation is an omnipresent idea within modern mathematical thinking. Therefore he suggested considering covariation not just as limited to the introduction of the concept of function but as a form of thinking with a larger epistemological and cognitive value, which considers mathematical objects together with their mutual relations. Moreover, he observed that covariation can be approached with a certain success since the first years of primary school thanks to the support of technological tools and analyzed some case studies supporting this claim (Arzarello, 2017; 2019).

Conceiving suitable didactical situations where students are introduced to covariational reasoning thanks to the mediation of appropriated artefacts constitutes a clear example of what he calls *instrumented covariation*, a didactical counterpart of the covariation construct that we are going to deepen in Chapter 4.

Moreover, analyzing the data from a secondary school teaching experiment about the modelling of a physical situation, i.e., a ball rolling along an inclined plane, Arzarello (2019) noticed that the law of the motion of the ball, a formula linking two variables and a parameter, was interpreted by students according two levels: the first one is that of the covariation between the two variables involved, time and distance, the second level is that of the covariation between the distance-time graph and the parameter depending on the angle of inclination of the plane. This second level is called *second-order covariation* and its identification gave rise to new research questions concerning its rigorous definition, its relationship with the already existing theoretical framework about covariation and its didactical implications. The denomination second-order covariation well fits with the terminology "second order functions" used by Bloedy-Vinner (2001) to address those functions whose argument is a parameter and whose corresponding outputs are equations or functions.

### 3. Covariation through a commognitive lens, Lisarelli (2019)

The Ph.D. research project conducted by Lisarelli (2019) aimed at investigating and describing students' learning of functions, when introduced to this notion through a particular dynamic approach, which stresses its covariational aspects: specifically, the dynamic aspects of the concept of variable and the dependency among variables.

The theoretical lens of commognition (Sfard, 2008) was adopted in this study to investigate covariation and it was a novelty in the research paramount. Commognition was used to:

- analyze the students' emerging discourses on covariation: categorizing the several instances of covariation in students' discourse, Lisarelli identified three different levels of covariation mirroring the expert discourse on covariation: (i) covariation of time and distance (the two variables involved in the tasks); (ii) covariation of the dependent variable with respect to the independent one; (iii) covariation of ratios that is "a description of the speed of a variable with respect to the speed of the other one" (Lisarelli, 2019, p. 204). In particular, the author states that "students expressed covariation by using verbs that refer to movement and changing over time, and with the help of dynamic visual mediators such as gestures, dragging and dragsturing[1] actions" (Lisarelli, 2019, p. 205);

- collect the expressions used by students to read *f(a)=b* and analyze how they use the word "function" in their discourses. The recognized expressions explicitly mentioned the dependence relation, contained some references to motion and the most reified realizations of correspondence were even without words.

As a didactical implication, the study suggested "a specific design of activities that can be employed in order to exploit the use of dynamic realizations of functions within a DIE [Dynamic Interactive Environment] to support the emergence of students' discourse on functions in terms of covariation of two quantities" (Lisarelli, 2019, p. 225). The analysis of the discourse of the involved students revealed numerous "references to the dynamic and temporal aspects of functions and graphs of functions and also their frequent use of non-formal mathematical words" (Lisarelli, 2019, p. 226). As stated by the author, an aspect that is absent in her study is a focus on the role of the teacher. In our research, given the relevance of the teacher in enhancing the transition to higher order covariational reasoning and the importance of teacher-led classroom discussion, we are going to deal with the analysis of emerging adaptive teaching strategies.

## 2.2 SOME HISTORICAL NOTES

Covariation became an explicit form of reasoning in mathematics around 1000 C.E., but for several centuries it had been considered only as a way of thinking and not as a mathematical concept. Attention to covariation from a theoretical standpoint was initiated by Klein's early twentieth century research[2] (2016) and then was explored in more recent papers (Janvier, 1978; Swan, 1985; Saldanha & Thompson, 1998) that deeply contributed to its theoretical definition toward the end of 20th century (Thompson & Carlson, 2017).

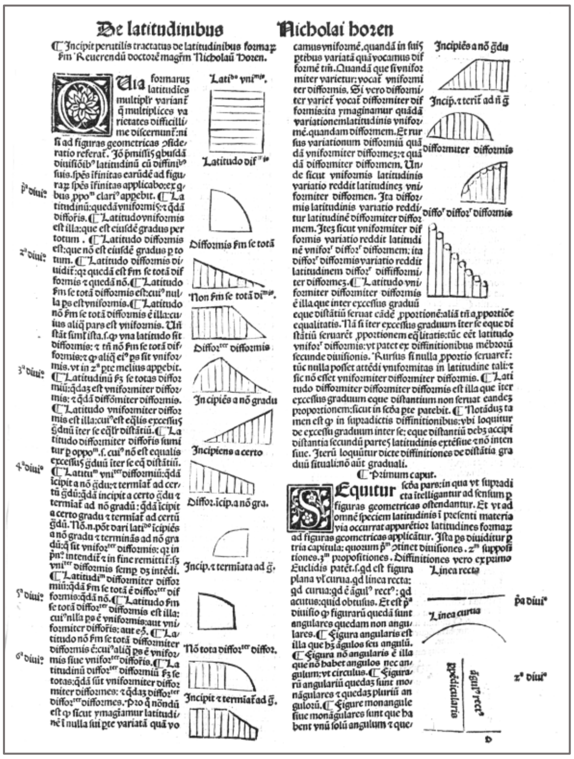
The notion of function mainly developed as a search for relationships between concrete, dynamic and continuous variables and to express the idea of change. Ancient scholars lacked a mathematical description of movement because they saw distance and time as measurable quantities, but not speed which was conceived only in its qualitative nature (Arzarello, 2008). Indeed, according to Aristotelian philosophy, qualities (*qualia*) and quantities (*quanta*) represented two distinct categories: qualities referred to the kind of subject or event considered;

[1] Dragsturing = action subsuming both dragging and gesturing characteristics.
[2] Its original work dates to 1908 but the first complete translation into English from the original German Edition of the three volumes dates to 2016.

quantities to something displaying the possibility of being measured and so being attributed numerical values. Ideas changed from the Middle Ages onwards, and it was in the XIV century that new revolutionary ideas developed at the Merton College at Oxford (1280-1340), and in Paris with Nicole Oresme (1325-1380). The Medieval philosophers realized that qualities also have an intensity, qualia began to be considered as quanta, and started that process of *quantification of qualities*, which developed through different approaches (Sylla, 1971).

The term Merton College or Merton School generally denotes four major Mertonians: John Dumbleton, Richard Swineshead, William Heytesbury, and Thomas Bradwardine. The Mertonian approach to the quantification of qualities that has been defined by historians as arithmetical and algebraic. Strongly influenced by the studies in the field of optics, Mertonians initiated an addition theory regarding only intensities of qualities, and often entirely neglecting extension. Thanks to the so-called Merton *mean degree theorem* or *mean speed theorem*, stating that “a body moving with a uniformly accelerated motion covers the same distance in a given time as a body moving for the same duration with a uniform speed equal to its mean (or average) speed”, they proved that “uniformly difform, i.e. linearly increasing or decreasing, qualities correspond to their mean degrees” (Sylla, 1971, p. 9). Nonetheless, Mertonians did not try to support their claim that their measures were additive with empirical experiments and nevertheless were aware that the additivity of their measures was an important issue. By the way, this approach lately contributed to a development of a quantification of qualities “based on intensity as an extensive and additive quality” (Sylla, 1971, p. 15).

On the other hand, Oresme initiated an approach to the quantification of qualities that can be labelled as geometrical: he distinguished between qualitative intensity and extension and proposed geometrical methods of representing the configurations of qualities. In his work, *Tractatus de configurationibus qualitatum et motuum* (1350), he used the term *configuratio* with two different meanings: the first meaning refers to the use of geometrical representations to graph intensities in qualities and velocities in motion. “Thus the base line of such figures is the subject when discussing linear qualities or the time when discussing velocities, and the perpendiculars raised on the base line represent the intensities of the quality from point to point in the subject, or they represent the velocity from instant to instant in the motion” (Clagett, 1970, p. 226) (**Figure 1**). The resulting figure represents the whole distribution of intensities in the quality i.e., the *quantity of the quality*, or in case of motion the total velocity or the total space traversed in the given time. To provide an example, a quality of uniform intensity is thus represented by a rectangle that is its configuration, while a right triangle represents a quality of uniformly nonuniform intensity starting from zero intensity (**Figure 1**). Concerning the law of falling bodies, Oresme suggested that the speed of the fall of bodies is directly proportional to the time of fall, rather than the distance of fall. He did not apply the Merton rule of the measure of uniform acceleration by its mean speed, as done by Galileo lately, but he knew this rule and provided the first geometric proof of it in one of his works (Clagett, 1970).


De latitudinibus Nicholai horen

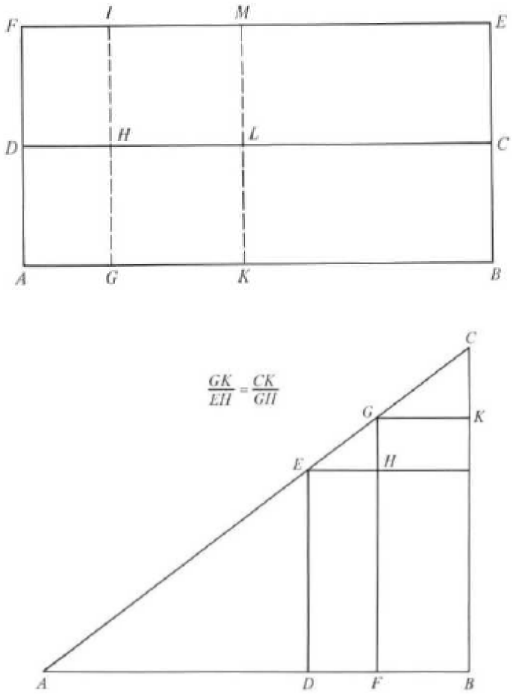



**Figure 1** – On the left, a page of *Tractatus de latitudinibus formarum* (1505), a reduced version of the original book by Nicole Oresme. On the right, a rectangle and a right triangle representing respectively the *configuratios* of a quality of uniform intensity and a quality of uniformly nonuniform intensity

Covariational reasoning appeared dramatically in mathematics with the birth and development of modern algebra through the works of Viète, Descartes, and others. The methods of analysis and synthesis in algebra, borrowed from the geometry of the Greeks, introduced a revolutionary way of approaching the problems of mathematics, which Lagrange in the early 19th century could summarize as follows: "Algebra taken in the most extensive sense, is the art of determining unknowns by functions of known quantities, or which are regarded as known" (Lagrange, 1806, p. vii).

Not only, the method of covariational reasoning with physical quantities enabled to express mathematically the physical laws and so made possible the birth of modern science with the *sensate esperienze e necessarie dimostrazioni* [sensible experiences and mathematical demonstrations] of Galileo. With his master work *Discorsi e dimostrazioni matematiche intorno a due nuove scienze attenenti alla meccanica e ai movimenti locali* [Discourses and Mathematical Demonstrations Relating to Two New Sciences] (1638), Galileo decreed the end of the medieval theory of mechanics and of the entire Aristotelian cosmology founded on it. He has the merit of having shown that natural phenomena are not always trivial, but they can actually be really complex. After having formulated some hypotheses on the motion of falling bodies and having foreseen the trend of a ball rolling along an inclined plane, he performed some experiments to confirm his assumptions. After hundreds of replications of his sophisticated experiment, Galileo could state that "gli spazii passati esser tra loro come i quadrati de i tempi, e questo in tutte le inclinazioni del piano, cioè del canale nel quale si faceva scender la palla; dove osservammo ancora i tempi delle scese per diverse inclinazioni mantener esquisitamente tra loro quella proporzione[3]" (Galileo, 1638).

The conception of quantities' values varying continuously deeply contributed to the arise of calculus as a body of thought (Kaput, 1994) and so continuous covariation can be seen as central to the development of the mathematical notion of function (Thompson & Carlson, 2017). Just to quote a relevant author, Newton in his *The method of fluxions and infinite series* (1736) explicitly spoke of quantities flowing from one value to another and so assuming specific values.

According to Boyer (1946), the final stage in the development of the concept of function started with the definition of function introduced by Dirichlet (1838) basing on a precise law of correspondence between variables: "If a variable $y$ is so related to a variable $x$ that whenever a numerical value is assigned to $x$, there is a rule according to which a unique value of $y$ is determined, then $y$ is said to be a function of the independent variable $x$". Nowadays in schools, the mathematical definition of function is that of Dirichlet translated into Cartesian products and ordered pairs according to Bourbaki's definition (1939): "Let $E$ and $F$ be two sets, which may or may not be distinct. A relation between a variable element $x$ of $E$ and a variable element $y$ of $F$ is called a functional relation in $y$ if, for all $x \in E$, there exists a unique $y \in F$ which is in the given relation with $x$. We give the name of function to the operation which in this way associates with every element $x \in E$ the element $y \in F$ which is in the given relation with $x$, and the function is said to be determined by the given functional relation. Two equivalent functional relations

[3] "Distances traversed are proportional to the square of fall times, and this in all inclinations of the plane, that is of the channel in which the ball rolled down; where we observed again the times of the descents for various inclinations to maintain between them exactly that proportion".

determine the same function”. Both these definitions do not leave space to the idea covariation between variables’ values: they are static approaches that freeze the dynamic nature of functions in the static language of set theory.

# 3 CONCEPTIONS OF VARIABLES AND FUNCTIONS

In this chapter we are going to explore in synthesis the main conceptions of variables and functions which are both relevant for a clear theoretical framing and definition of first- and second- order covariation. Moreover, some hints about the teaching and learning of these concepts in school practice are outlined. Finally, the last section addresses the relevance of covariation in the mathematics curricula.

## 3.1 VARIABLES AND PARAMETERS IN THEORY AND IN TEACHING PRACTICE

The concept of variable assumes different roles in algebra (Bernardi, 1994; Küchemann, 1981); in Arcavi et al. (2016), in agreement with a larger literature, five facets of meaning are identified:

- A variable can be intended as a *placeholder* of a numerical value, a blank space to be filled with a specific value within the formulation of an algebraic expression describing a certain situation.
- A second facet is that of *unknown*, hence a number to be found like in the case of equations.
- A third meaning is that of *varying quantity*: in this case the literal symbol does not represent a single value but rather a domain of possible values and hence incorporates and idea of motion and dynamicity.
- A variable is a *generalized number* when it is used to describe general properties like physical laws or the formulas for the area and volume of geometrical figures.
- Eventually, a *parameter* can be intended as a higher order variable in the sense that its value determines a situation as a whole.

The meaning of these terms is therefore related to a specific context, but in the teaching practice it is not always defined or sometimes it is not because a rigorous definition would require a certain level of knowledge in the field of mathematical logic.

Moreover, there exists tacit conventions on the choice of the letters to be used in the different contexts: $x$, $y$, $z$ for unknows or undetermined variables; $a$, $b$, $c$ for the coefficient of a curve in its canonical form; $h$, $k$ for the parameters within families of lines or conics (Chiarugi et al., 1995). This fact involves the risk that such choices induce to understand the literal symbols as rigid designators (Arzarello, Bazzini & Chiappini, 1994, p. 37) and so that students facing an equation in which the unknown is denoted with the letter $a$ instead of $x$, may find themselves in difficulty.

Among the various educational problems which students struggle with there are certainly the mastery of the manipulation of the symbols of algebraic language and particularly the distinction between variables and parameters (Reggiani, 2002; Furinghetti & Paola, 1994). These difficulties are related not only to the dependence on the role assumed by the letters according to the context, but also on the logical complexity required to explain the difference between their roles (Bloedy-Vinner, 2001).

One of the first topic in which students face variables and parameters already in the first two years of high secondary school is the discussion of literal equations. An analysis of the textbooks currently in use revealed that the presence of a definition of the term parameter is of recent introduction. For instance, in *Algebra.blu* (2016), today of wide use in the school practice in its various reissues, the term parameter is used for the first time when introducing the literal problems and it is defined as "a term that represents a known value and that is not an unknown" (p. 502). And again, in the paragraph devoted to literal equations "the letters" present "in addition to the unknown" (p. 506), are called parameters. This attempt of definition recalls those conventional choices described above and to the possible difficulties linked to them.

## 3.2 Conceptions and representations of functions

In the literature in Mathematics Education three main views of student conceptions of functions have been distinguished (Vollrath, 1989; Slavit, 1997) that collect various approaches. Now we are going to present them briefly including also the covariational approach in this classification:

1. Action/operational views of function: this is the view initially acquired by students (Sfard, 1989); it consists in computing the numeric values for a given input adopting an algorithm or a rule of association. In this view, functions are intended as non-permanent objects and the focus is on the computational aspects.
2. Object-Oriented views of function: this view can be interpreted as an evolution of students' action-oriented view into an object-oriented notion, i.e., a more permanent one. Three main approaches fall into this category:

- *Correspondence* or *mapping approach*, the one introduced by Confrey and Smith (1994) where quantity 1 is assigned to quantity 2, or equivalently asking for the value of $f$ at $x_1$ and $x_2$. Functional relationships describe connections between two quantities' values, without making the quantities explicit.

- *Covariation approach*, deeply explained in Chapter 12, that consists in the understanding of the way in which quantity 1 co-varies with quantity 2, or equivalently asking for how *f* changes with *x.* Thompson and Carlson (2017) introduced a definition of function that is based on this approach and according to them it was something missing in the literature. The meaning they propose is the following:

  > a function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly the value of the other. (Thompson & Carlson, 2017, p. 436)

  The authors remark that a function is a conception, hence their definition does not use the terms independent and dependent variable because they entirely depend on the person's conception of the situation.
- *Holistic approach*: the function is conceived as an abstract object resulting from a reification of mathematical operations and is described by holistic features (Sfard, 1991).

3. Property-oriented view of function: this view of functions "deals with the gradual awareness of specific functional growth properties of a local and global nature, followed by the ability to recognize and analyze functions by identifying the presence or absence of this growth properties" (Slavit, 1997, p. 266). Functional properties can be classified into *global* such as symmetry and periodicity, or *local* like intercepts or points of inflections. Some properties such as continuity, transcend this possible classification. A student reifies the notion of function as mathematical object possessing or not possessing certain properties. This property-oriented view differs from the covariational approach because less emphasis is put on the way in which variables change and more emphasis is on the properties resulting from these changes.

In addition to this *vertical* development of the concept of function, in which there is an evolution from process aspects to the function concept intended as an object, there is a *horizontal* development of the same concept that is obtained relating different representations. These two dimensions are called *depth* and *breadth* respectively of the function concept, where increasing depth means higher levels of cognitive abstraction (DeMarois & Tall, 1996). The breadth dimension consists of various representations such as the *numeric one* using tables, the *geometric* using graphs, and the *symbolic* using equations: these three facets have been widely discussed in the literature (e.g., Thompson, 1994b). Other facets are the written and verbal descriptions of function, the function notation, the colloquial facet using the notion of function machine and the kinesthetic aspect might be represented by asking students to act out their understanding about function. The use of multiple representations strongly supports the learning of mathematical concept, but they are non-trivial for students to relate and identify connections (Ainsworth, 2006). More details concerning this issue will be provided in Section 8.2.1.

To conclude, some didactical remarks follow. In line with the definition of function established last century (see Bourbaki's definition reported in Section 2.2), in school practices and school textbooks, the notion of function is mainly introduced as a generalization of the concept of relationship. Here, an emblematic example from *Manuale.blu di matematica* (2020): "Una relazione dall'insieme *A* all'insieme *B* è una funzione se a *ogni* elemento di *A* associa *uno ed un solo* elemento di *B*"[4] (Vol. 1, p. 237). This approach is extremely static and does not leave space to an underlining idea of motion or dynamicity. The same holds for the possible representations of relationships that can be consequently applied to the specific case of functions. In the same book (p. 229), four different types of representation are reported:

- *Enumeration*, that is writing the set of all ordinate pairs of the elements that are in relationship e.g., R= {(2;1), (4;2), (6;3)}.
- *Sagittal representation* or *arrow diagram* using Euler-Venn's set and making visible the correspondence between elements of the sets using arrows.
- *Double entry tables* in which the elements of the first set are disposed in vertical, the elements of the second horizontally and the pairs in relations are marked by a cross. In the specific case of functions, can be translated into the two columns table used to represent a function by points.
- The *cartesian graph* in which pairs in relation are detected by points.

As already highlighted in Chapter 2, paired couples are an example of multiplicative objects but only when understood as values of two quantities that vary simultaneously, and this approach is not typically enhanced in school practices but most frequently reduced to the correct representation of some numerical values in a law of correspondence.

[4] A relationship from set *A* to set *B* is a function if to *each* element of *A* associates *one and only one* element of *B.*

## 3.3 RELEVANCE OF COVARIATIONAL REASONING IN THE MATHEMATICS CURRICULA

Despite the recognized importance of covariation, school practices and mathematics curricula rarely focus on covariational reasoning, even in the U.S. where covariation is an extremely relevant research topic (Thompson et al., 2017). Exceptions can be found in a few Eastern countries like South Korea, Russia, and Japan (e.g., see Japanese Ministry of Education, 2008). Even the curricula for Italian secondary schools, both for second cycle MIUR (2010a, 2010b, 2010c) and for first cycle MIUR (2012), do not contain explicit references to covariation. Concerning the curricula for scientific-oriented secondary school, an implicit reference can be found in the following statement, which could be interpreted according to a covariational reasoning perspective: "Un tema importante di studio sarà il concetto di velocità di variazione di un processo rappresentato mediante una funzione"[5] (MIUR, 2010a, p. 340). From the early years of high school, functions are studied, especially as representations of real phenomena, and the Italian curricula stress the relevance of mathematical modelling activities in teaching practices. The words *model* and *modelling* appear nearly 12 times, under the headings *General Guidelines*, *Change and relationships* and *Uncertainty and data*. In particular, it states that modeling consists in the "possibilità di rappresentare la stessa classe di fenomeni mediante differenti approcci"[6] (MIUR, 2010a, p. 337) and that the study of the language of functions is the "primo passo all'introduzione del concetto di modello matematico"[7] (MIUR, 2010a, p. 339). This lack of explicit references is also reflected in the fact that, apart from a minority of textbooks which in recent years have introduced covariation as a thinking tool (Paola & Impedovo, 2014), most Italian textbooks currently in use do not foster this approach when dealing with the modelling of classes of phenomena or conceptualization of dynamic problems. For instance, when teachers introduce the concept of function, they mainly adopt a static definition as the ones recalled in the previous section. Literature suggests that the lack of a covariational approach may be one reason why students are unable to interpret dynamic situations and to construct meaningful formulas suitable for representing one quantity as a function of another (Carlson, 1998).

[5] An important topic of study will be the concept of rate of change of a process represented through a function.
[6] The possibility to represent the same class of phenomena through different approaches.
[7] First step to the introduction to the concept of a mathematical model.

# 4 INSTRUMENTED COVARIATION

The term *instrumented covariation* has been introduced by Arzarello (2017) and it refers to the exploration of certain mathematical problems or situations with the use of appropriate artefacts supporting the learning of covariational aspects.

The adjective instrumentation is recovered from the instrumental approach by Vérillon and Rabardel (1995), in which it is underlined the distinction between artefact (material or abstract object produced by the human activity) and instrument (mixed entity characterized by an artefact component and cognitive component consisting of the utilization schemes).

Numerous research studies in the field of Mathematics Education (Trouche, 2005; Drijvers, 2019) are oriented to the exploration of the didactical modalities in which the modern digital technologies can foster the learning of mathematics: instrumented covariation may help teachers in designing suitable activities, which could initiate and support students to covariational reasoning, for instance within Dynamic Geometry Environments (DGE) or Computer Algebra System (CAS).

Some details concerning the covariational approach applied to the formulation of an open problem are provided in Arzarello (2017) and Arzarello (2019). In the Theoretical framework of Euclidean Geometry (TGE), a covariational approach should have the following requirements:

- Leading to a “different” geometrical form (e.g., that typical of open problems where the formulation of hypotheses is explicitly required);
- The implication of an epistemological change with respect to TGE;
- Some cognitive consequences;
- Some didactical consequences in the classroom environment.

“In particular, the environment of dynamic geometry is an artefact which amplifies the phenomena that depend on the formulation of the problem and allow their instrumentation” (Arzarello, 2017, p. 12). Moreover, as example of instrumented covariation is quoted the teaching experiment conducted in Bari (Italy) and deeply described in Faggiano, Montone, and Rossi (2017): a duo of artefact, in the sense by Maschietto and Soury-Lavergne (2013), is adopted to produce an instrumented understanding of covariation in a situation of geometric symmetry between two points. The analysis of the classroom discussion mainly revealed the internalization of the covariation of the symmetric figures and the synergic effect of the two artefacts. These findings reveal that covariation can be approached with a certain success ever since first years of primary school thanks to the mediation of suitable technological supports (Arzarello, 2017). These results are groundbreaking with respect to the traditional perspective emerging in the literature. For instance, in the study by Johnson et al. (2017) we can read that "[c]ovariational reasoning, entailing the individuals' conceptions of change and variation, is a critical form of mathematical reasoning for secondary students to use when studying the gatekeeping concepts of rate and function", or again "[a]lthough covariational reasoning can be challenging even for successful university students [...], secondary students can engage in covariational reasoning" (p. 852).

Concerning the design of activities aiming at promoting students' covariational reasoning, researchers have often designed tasks in dynamic computer environments displaying animations and graphs (Ellis et al., 2016; Johnson, 2013; Saldanha & Thompson, 1998), but literature reveals that students working on such tasks may not show the intended reasoning as in designers' intentions (Carlson et a., 2002). In particular, Moore et al. (2013) suggest providing students opportunities to interpret and represent quantities using different axes in Cartesian representation. Finally, "[i]f students only view features of dynamic computer environments and related tasks as physical features, rather than attributes possible to measure, it can inhibit their development of more complex forms of mathematical reasoning" (Johnson et al., 2017, p. 862).

Recently, even more sophisticated technologies than the ones previously quoted are employed and studied so to support the conceptualization of the dynamic aspects involved in tasks of mathematical modelling: one of these technologies is the Augmented Reality, an environment that can help to bring together both continuous dynamic features of a real phenomenon and its mathematical representations. A relevant study in which Augmented Reality is used to engage students in covariational reasoning is Swidan, Schacht et al. (2019).

Some examples related to the instrumentation of the concepts of variables and parameters with different approaches are proposed in Section 16.2.

## 4.1 A DIFFERENT APPROACH TO THE INSTRUMENTATION OF COVARIATION[8]

Despite the definition of instrumented covariation and all its features presented in broad terms in the previous paragraph and that we are going to deepen during the development of this

[8] We would like to thank Prof. Nathalie Sinclair for sharing with us her point of view about this issue and suggesting us interesting insights.

research study, a different approach to instrumentation could be grasped from the already existing literature. Past times mathematical activities related to covariational reasoning have always been performed just adopting the tools available at the time, meaning pencil and paper or computer. In both cases, students had the chance to interact only through one single input (the pen or the mouse pointer) which nowadays seems really limiting at least in a task involving covariational reasoning where, by definition, it is required to coordinate simultaneously the values of two different quantities. The remarkable technological developments of the last decades, such as touchscreen tablets, are bringing sensorimotor interaction back into mathematics learning activities and offer today some novelty approaches that long ago we could not have expected. Research in Mathematics Education offers some interesting examples of a concrete form of instrumentation of certain mathematical reasoning processes achieved not only using a technological support but also using hands and their movements so to directly interact with the technology and perceive on a bodily-sensorial level the mathematical process that is instrumented. If we wanted to distinguish this approach to the interpretation previously presented, we could speak of an *embodied instrumented covariation*. Some examples are presented in the following lines.

A first one is described by Abrahamson and Sánchez-García (2016) and consists in the use of the Mathematical Imagery Trainer for Proportion (MIT-P). The MIT-P set at a certain ratio, for instance at a 1:2 ratio, displays a green background, meaning a favorable sensory stimulus, when the student positions the right hand twice as high along the monitor as the left hand (see **Figure 2**) and a red stimulus when the hands position is incorrect**.**
A second example is provided by some software applications (apps) that exploit affordances of multi-touch devices for fostering the learning of certain mathematical concepts: some emblematic studies in this sense are that by Bairral and Arzarello (2015) exploring students' geometrical reasoning using a free online touch Device, the Geometric Constructer (GC) software, and that by Sinclair and Ferrara (2021) in which primary school students experience the concepts of numbers and quantities in a Digital Multitouch Environment, i.e., using a multi-touch iPad application called TouchCounts (see **Figure 3**).

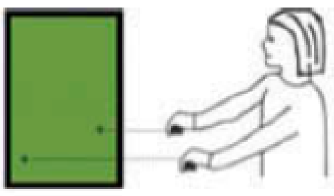


**Figure 2** - Schematic description of a possible student's interaction with MIT-P: hands are positioned correctly thus the screen is green. It becomes red when hands are positioned incorrectly

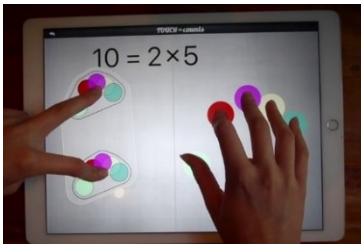



**Figure 3 –** A screen of the TouchCounts application

# 5 RESEARCH QUESTIONS: A PRELIMINARY FORMULATION

The main issue of this study is the investigation of students' understanding of the covariation among magnitudes involved in the modelling of real phenomena. In 2017 a first teaching experiment was conducted in an Italian 9th grade classroom. The main purpose of the experiment was about the so-called Galileo experiment (Galileo, 1638), a ball running along an inclined plane, in order to model the law of the motion with the support of technological tools. The analysis of the classroom discussion revealed some unexpected ways of covariational reasoning: students tried not only to covary the main variables, time ($t$) and distance ($s$), but also to understand how the distance-time graph changed according to the inclination of the plane. These two forms of covariations manifested in a double way of reading the formula of the law of the motion $s= k \cdot t^2$: the first order is the $s$-$t$ covariation, the second-order covariation between $(s,t)$ graph and the parameter $k$ depending on the angle of inclination of the plane. It seemed to us that this aspect does not completely fit with the framework of covariation developed by Thompson and Carlson (2017). The results of this work, partially illustrated in Arzarello (2019), specifically guided us to the formulation of a preliminary definition of this more complex form of covariational reasoning, second-order covariation, and that seems to consist in *grasping a further relationship in a family of invariant relations among two or more varying quantities, where this family is characterized by the presence of one or more parameters*. This preliminary definition will be validated and revised thanks to the results from three teaching experiments, one of which is the one quoted before: we have reanalyzed it in detail in view of our research questions and the adopted theoretical framework.

This second-order covariation is not the only contribution that seems to widen the already existing framework of covariation: in Section 2.1 we already introduced metavariation, a construct that seems to address covariation from the tasks design point of view but has some points in common with our vision of second-order covariation. Moreover, in Swidan, Sabena and Arzarello (2020), the authors speak of *covariation of a covariation* when students consider functions globally and focus on how the changes in one graph are linked to the changes in the graph of its slope, and conversely. Also, this way of reasoning covariationally reveals to be more cognitively demanding than the one presented in the six levels taxonomy and differs from our second-order covariation. Hence, this is a contribution worthy to be kept into account in the perspective of a theoretical enlargement that aims at coherently including more complex forms of covariational reasoning and opens up to the possibility of existence of other orders of covariation.

The scientific relevance of covariation as a research topic, the main research gaps we identified in the literature and the numerous standpoints from which covariation can be addressed have already been widely discussed in the Introduction. All these considerations and findings led us to a preliminary formulation of some research questions that can be condensed as follows:

1. How can the theoretical framework of covariational reasoning be enriched to better explicit the relationships among a plurality of variables involved in a physical-mathematical problem?
2. Is it possible to identify some gestural, linguistic, and semiotic markers that connote the way students can enter into each of the levels of first- and, possibly, second-order covariation?
3. Which could be the role of the teacher in enhancing covariational reasoning in her school practices?

These questions will be refined and reformulated in the Chapter 7 in light of the theoretical lenses that we are going to make explicit in the next chapter.

# 6 THEORETICAL FRAMEWORKS

In order to investigate covariational reasoning from various standpoints, we are going to adopt various theoretical lenses and as a consequence, networking of theories becomes essential to elaborate a coherent framework. In this chapter, after having recalled the main features of the networking of theories (6.1), we are going to describe in detail the multiple theoretical lenses adopted in this study: the *semiotic bundle* (6.2), the *commognition* (6.3), the *conceptual blending* (6.4) and the *adaptive teaching* (6.5). The networking of all these theoretical contributions will be performed using mainly the strategy of coordinating them to look at the complex phenomenon of covariation from different educational points of view. The elements of commonalities and dissimilarities between the different lenses will be pointed out jointly with their specific use so to answer to research goals.

## 6.1 NETWORKING OF THEORIES

The complexity of the teaching-learning processes developing within the mathematics classroom and the variety of research purposes guiding our study require suitable theoretical frameworks to give justice to the multi-faceted phenomenon under the lens of analysis and a suitable methodological tool should be drawn up to respond to research aims.

Literature in Mathematics Education offers different definitions of *theory* but in the following we are going to intend it according to Radford's approach (2008), i.e., a way to produce understanding and modes of actions based on: a system of principles P characterized by a strong relationship between many of its elements; a methodology M characterized by operability and coherence with respect to the principles, and some research questions Q. Hence theories can be identified as a triplet (P, M, Q): they bring the footprint of the initial research questions they tried to answer, but then they emerge as response to specific problems.

The diversity of theories characterizing the educational research is a source of richness but in order to allow a comparison between these theoretical frameworks a metalanguage is required, enabling to speak of commonalities and differences between the theoretical lenses. According to Radford (2008), this semiotic environment can be identified in *Lotman's semiosphere* (1990): it is a multi-cultural space necessary to the existence and functioning of languages; outside of the semiosphere there can be neither communication nor language. It is a space where all the existing theories are embedded and allows them to dialogue with each other. It is within this semiotic space that is possible realize a *networking of two or more theories* (Prediger et al., 2008): this term denotes a collection of “research practices that aim[s] at creating a dialogue and establishing relationships between parts of theoretical approaches while respecting the identity of the different approaches” (Bikner-Ahsbahs & Prediger, 2014, p. 118). There are many different ways and degrees to bring theoretical approaches into dialogue (Prediger et al., 2008) but we are going to recall only four main strategies:

- *Coordinating* and *combining* are the strategies “mostly used for a networked understanding of an empirical phenomenon or a piece of data” (Prediger et al., 2008, p. 172). Whereas combining theories can be intended as a juxtaposition of theories and does not require a full compatibility of them, coordinating theories instead refers to the use of analytical elements of different theories so to investigate specific research problem;
- *Comparing* and *contrasting* are the most widely used strategies and the difference between them is subtle. While comparing theories means highlighting in a neutral way both commonalities and dissimilarities between strategies, contrasting them consists in underlining mainly the differences between them.

## 6.2 The semiotic bundle

One suitable semiotic lens we are going to adopt in this study is that of the *semiotic bundle.* Introduced by Arzarello (2006), it is a good tool to grasp the interplay within the semiotic activities, productions, and interactions in the mathematics classroom. It arises as

> a system of signs […] that is produced by one or more interacting subjects and that evolves in time. Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a mathematical question. Possibly, the teacher too participates in this production, and so the semiotic bundle may include also the signs produced by the teacher. (Arzarello et al., 2009, p. 100)

In particular, the signs can be produced by artefacts that are used during the interactions, from old ruler and compass to more sophisticated technological devices, and these artefacts are included in the learning environment. Typically, the semiotic bundle embraces students’ and teacher’s perceptuo-motor activities and productions: from language (utterances, written texts, etc.) and extra-linguistic modes of expression (gestures and glances) to different types of inscriptions (drawings, sketches, graphs, etc.), that is all the semiotic resources produced or acted on to think and communicate in the classroom environment. The semiotic bundle, constituted by both a collection of semiotic components and their mutual relations, is a dynamic structure which changes in time because of the semiotic activities of the subjects, and allows to describe these multimodal semiotic activities in a holistic way. Hence “the semiotic bundle considers the semiotic resources in a unifying analysis tool” (Bikner-Ahsbahs et al., 2015, p. 168). Moreover, the semiotic bundle dynamics can be analyzed in two different and complementary ways: a *synchronic* analysis, which considers the relationships among different semiotic resources simultaneously in a specific moment in time; a *diachronic* analysis, focusing on the evolution of signs and of their relationships over time. “Together, synchronic and diachronic analysis allow us to foreground the roles that the different types of signs (gestures, speech, inscriptions) play in students’ cognitive processes” (Arzarello et al., 2009, p. 101).

### 6.2.1 The importance of gestures

It is well-established in the literature that cognitive processes of students emerging during mathematical activities manifest not only through oral productions. Indeed, we can adopt multiple modes to communicate meanings to others (words, sounds, sketches, gestures) and these modes typically are active together in an integrated way: the term *multimodality* addresses exactly this variety of modalities of communication (Kress, 2004).

Gestures are relevant both from a communicational and a cognitive perspective (Goldin-Meadow, 2003; McNeill, 1992). Gestures belong to the wide world of nonverbal communication, and they are defined by McNeill (1992) as “movements of the hands and arms that we see when people talk” (p. 1). The author also identifies four dimensions of gestures: (1) *deictic*, used in concrete or abstract pointing; (2) *iconic* or *representational*, arm or hand movements with a perceptual relation with the concrete object that is represented; (3) *metaphoric*, similar to iconic gestures but referring to abstract objects and (4) *beats*, up and down flicks of the hand or tapping motions. These dimensions should be seen as overlapping, rather than discrete categories: “Most gestures are multifaceted: iconicity is combined with deixis, deixis is combined with metaphoricity, and so forth. Rather than categories we should think in terms of dimensions” (McNeill, 1992, p. 38). In addition, we consider another overlapping dimension, that of *writing* gestures (Shein, 2012; Alibali & Nathan, 2007), generating indelible marks on figural representations. Gestures may also assume the following functions (Roth, 2001): *narrative function* (especially for iconic and metaphoric gestures), when gestures connect the gestural and verbal narrative to the pictorial background; *interactive function* (concerning beats) for those gestures that serve to regulate the rhythm of the speech or *grounding function* (in particular deictic ones) when gestures allow to relate mental structures to external objects (Nathan, 2008;

Roth, 2001). Moreover, research has shown how gestures and speech work together to express meanings (Kendon, 2004) and the way in which they express largely non-overlapping semantic information have been referred to with different terminology as "complementary" (McNeill, 1992) or "mismatching" (Church & Goldin-Meadow, 1986). In this work, we are going to focus in particular on the detection of *non-redundant* gestures with respect to speech (Alibali et al., 2000), that is on gestures that communicate additional information with respect to those expressed orally.

Concerning significant glances, Kendon (1967) identified three main social functions of gazes during a conversation and in particular focusing on the gazes of speakers toward their listeners. Gaze patterns analyzed send signals to another person with a regulatory, monitoring, or expressive intent but they are mainly sent without awareness. In this study we are going to focus on the features and functions of gazes emerging during face-to-face interactions where both subjects involved can perceive and signal information. According to the cognitive model of Interpersonal Gaze Processing (IGP) developed by Cañigueral and Hamilton (2019) and focusing specifically on social interactions, a gaze may assume an *active sensing* function when the movement of eyes aims to gain useful information from the environment, or a *social signaling* function when expressing a clear communicative intent to others. In particular the IGP model distinguishes the behavior of gazes depending on the kind of social stimuli, i.e., when the social stimulus is an inanimate object (picture or video) or a real person, but we will not dwell on this distinction.

### 6.2.2 The semiotic game

The investigation through the semiotic bundle will also allow us to enter into what in the literature is called *semiotic game*[9] (Arzarello & Paola, 2007; Arzarello et al., 2009), a specific aspect of teacher's revoicing (O'Connor & Michaels, 1996). The semiotic games are typical communication strategies among subjects, who share the same semiotic resources in a specific situation (Arzarello & Paola, 2007). In a semiotic game the teacher exploits the potentialities expressed by semiotic resources adopted by the students to enhance the construction of mathematical knowledge and scientifically shared meanings. For instance, a semiotic game

[9] The expression semiotic game may recall the *language-game* introduced by Wittgenstein in his *Philosophical Investigations* (1953). It is a philosophical concept referring to simple examples of language use and the actions into which the language is woven. Wittgenstein argued that a word or even a sentence has meaning only as a result of the "rule" of the "game" being played.

happens when students communicate their thoughts through two different semiotic channels: speech and gestures. Typically, the speech is confused, but the gestures in their iconic-metaphorical dimension suggest to the teacher that students are near to the right understanding. Hence the teacher imitates the students' gestures but suitably changes their utterances dressing their gesture with appropriate linguistic expressions and explanations (Arzarello & Paola, 2007). Teacher's interventions are imitative-based: the teacher imitates the student's gestures and accompanies them with certain scientific meanings, in order that in the following, the student will be able to imitate the teacher's words.

Such semiotic games can develop if the students produce something meaningful with respect to the problem at hand: words, gestures, drawings, inscriptions, etc. It is apt for the teacher to seize these moments to enact the semiotic game. Even a vague gesture of the students can really indicate a certain comprehension level, even when students have not yet the words to express themselves at this level. In a Vygotskian frame, the semiotic game is useful for the student's cognitive development, if student-teacher interactions are developing in a suitable *zone of proximal development* (ZPD) for a certain concept (Vygotsky, 1978). As pointed out by Radford (2010, p. 3), "[t]he ZPD is not a kind of well-delimited and rigid region that belongs to one particular student but a social, complex system in motion with evolving tensions" between the teacher and the students. Hence this complex relationship must be built in the classroom through the semiotic interactions between the teacher, the students, and the instruments. It is within the semiotic bundle that semiotic games can arise (Arzarello et al., 2009). Semiotic games can happen and develop because of joint teacher's and students' semiotic productions and productive interactions. Both are the actors of a semiotic game, which is a typical interactional construct that could not exist without the 'tuned' contribution of all its actors, teacher and students.

### 6.2.3 The role of artefacts

The semiotic bundle will also allow to properly analyze the relationships with the artefacts used in our teaching experiments and contributing to the development of teaching-learning processes.

In particular, we will underline two different modalities of student-artefact interactions that can be observed: an *ascending* modality, when students use the tool with an explorative approach, looking for relationships and mathematical properties; a *descending* modality emerges when the students already have a conjecture in mind and use the tool to verify it, searching for data supporting their hypothesis. These two different cognitive approaches were identified by Saada-Robert (1989) regarding the control schemes activated by the subject when facing a given problematic situation. Moreover, the students' interactions with different artefacts possibly contribute to produce a synergy or a conflict between them. The notion of *synergy* between artefacts has been introduced and studied by Faggiano, Montone, and Mariotti (2018): there is synergy when the use of different artefacts "can foster the integration of different and complementary meanings providing a rich support to the development of the expected mathematical meaning", namely "the combined, intentional and controlled use of the […] artefacts may develop a synergy, so that each activity enhances the potential of the other[s]" (p. 1). We speak of *conflict* between artefacts, when the use of different artefacts to face the same situation fosters different, not converging, or even apparently contradictory meanings for the situation. The term conflict has here an intuitive meaning and has no epistemic connotation: in a sense, it is the opposite of synergy.

What we call instrumented covariation will be the result of an evolution, where all the components of the semiotic bundle model prove essential for students' knowledge formation.

## 6.3 THE THEORY OF COMMOGNITION

Given the importance of multimodal aspects for our analysis, including the communicational features of the teaching-learning process and the purpose to explore the evolution from first-order of covariation towards second-order of reasoning, we retain suitable that the lens of commognition, which perfectly fits with the Vygotskian basis of the semiotic bundle, could contribute to a deeper analysis of some discursive elements highlighting the progress of the mathematical modelling activity in relation to covariational reasoning.

According to the commognitive perspective, which refuses a dualist vision of learning in favor of a communicative approach, mathematics is conceived as a historically established discourse and learning mathematics means becoming a participant in this specific discursive activity (Sfard, 2008). One of the peculiar aspects of the mathematical discourse lies in its autopoietic nature: mathematics creates all the elements of its discourse and the process of construction of new mathematical objects is called *objectification* (Sfard, 2020). Four features are relevant to consider a discourse as "mathematical": *keywords*, *visual mediators* that are visible objects that are operated upon as a part of the process of communication such as symbolic artefacts, *narratives*, i.e., any sequence of utterances that is "a description of objects, of relations between objects, or of processes with or by objects" (Sfard, 2008, p.134), and finally repetitive patterns characteristic of the given discourse called *routines.*

Considering knowledge as a multimodal form of communication means, from the research perspective, studying the processes of development and evolution of mathematical discourses (Sfard, 2020). The learning is intended as a social and collective process more than an individual one and it happens through three different degrees of discursive involvement: *explorations*, tasks requiring a reformulation of a specific kind of mathematical narrative; *routines*, action patterns with mathematical objects reflecting human tendency to repetitions and *rituals*, recapitulation of actions of previous performers process taking place thanks to the interactions with others. Moreover, learning can happen on two distinct levels: on an *object-level*, it consists in the enlargement of already existing narratives about familiar mathematical objects; on a *meta-level,* learning means subsuming an old discourse into a new one changing the meta-rules of the discourse. If object-level learning can be led simply by the learner's interest in the outcome, the meta-level learning requires the intervention of a participant, recognized as an expert by all participants in the discourse, who has the fundamental role to foster the overcoming of the encounter with an incommensurable discourse, what is called *commognitive conflict*. It originates from apparently incompatible narratives: for instance, on a semantic level, that incommensurability can be generated by a same word intended and used in different ways. The commognitive conflict is resolved by a rational argumentation and the gradual acceptance of others' discourses.

Many studies adopting a commognitive approach are focused on the detection of the developmental levels, whether ontogenetic or historical, of mathematical discourses. Caspi and Sfard (2012) worked on the levels of elementary algebra, conceived as a meta-discourse of arithmetic; Kim et al. (2012) investigated the learning about infinity and limits comparing data by Korean and American students. These studies are conducted in the perspective that mathematical discourse annexes its own successive meta-discourses, and the process of growth results in a "hierarchy of increasingly complex, increasingly reified, and possibly mutually incommensurable discourses on a specific mathematical topic" (Kim et al., 2012, p. 89). In accordance with those studies, we will direct our efforts to the identification of a hierarchy of levels of the mathematical discourse concerning the conceptualization and modelling of real phenomena.

Finally, Sfard and Kieran (2001) developed a powerful tool that reveals helpful to evaluate the real interest of the interlocutors in creating a dialogue with their partners. It is the *interactivity* *flowchart*: it represents as arrows the interactions between the different subjects involved in a communication. The analysis of the arrows, which can represent reactive or proactive interactions, allows to understand if the subjects are really aiming at creating a communicative channel with their partners or their communicative efforts are more directed to a private channel, that is a communication with themselves.

It is important to underline that the two used theoretical lenses presented until now, semiotic bundle and commognition, are coherent each other, since both are based on similar founding principles, and consider communication as "the glue that holds human collectives together" (Sfard, 2008, p. 81). The coordination of these two frameworks is effective because on the one hand the semiotic bundle is a coherent extension of the commognitive approach that allows to highlight some components, which are only hinted in Sfard's approach. In a sense, the semiotic bundle model is a concrete instantiation of what Sfard calls *signifiers' realizations* in mathematical discourse. Among what she calls visual modalities of these signifiers' realizations, in contraposition to spoken and written words, we also find the gestural component, as well as what we called inscriptions. But let us sketch a subtle difference between the two models. One of the basic assumptions for the semiotic bundle model is the unity between its three main components (speech, gestures, inscriptions); in this sense it broadens McNeill's claim (1992) that "gesture and the spoken utterance [are] different sides of a single underlying mental process" (p. 1), namely that "gesture and language are one system" (p. 2). In fact, "the unitary nature of processes within the semiotic bundle shows that under mental processes, there is a richer and more complex system" (Arzarello et al., 2009, p. 108).

In the commognitive approach, even if the different modalities of signifiers' realizations are acknowledged, a form of differentiation remains between symbolic, basically verbal and sequential means of representation, and the visual ones, e.g., gestures:

> The symbolic means, on the other hand, are basically verbal and thus sequential and as such exert greater demands on one's memory. And yet, what is lost in simplicity is gained in generalizability and applicability. The process-object duality of symbolic mediators is a basis for compression and the subsequent extension of mathematical discourse, and it renders this discourse independent of external, situation-specific visual means. All this ensures a very wide applicability of the discourse. (Sfard, 2008, p. 162)

On the other hand, the added value of commognition to the semiotic bundle, at least according to our point of view, can be identified in the relevance attributed to interactions between subjects involved that further remarks the social aspects of the process of learning and makes them visible thanks to the interaction flowchart. Moreover, the commognitive framework puts the spotlights on the mathematical discourse and suggests which elements should be pinpointed for both a punctual and holistic analysis.

## 6.4 Conceptual Blending

During the last of the three teaching experimentations that we are going to present in this research, the need for a new theoretical lens emerged. This necessity was motivated by the inability to fully describe from a cognitive standpoint students' forms of reasoning, especially when dealing with more than one representation of the same phenomenon. These findings led us to consider introducing conceptual integration. Called also *conceptual blending*, it is a mental operation which is essential for the construction of meaning. It is "an invisible and unconscious activity involved in every aspect of human life" (Fauconnier & Turner, 2002, p. 18). The functioning of the conceptual integration can be described through a network of relationships (see **Figure 4**), meaning a schematic diagram representing four mental spaces and their related interconnections. The mental spaces, represented by circles, are small conceptual packets that we construct while thinking and talking so to understand and act in relation to what is said; elements in the mental spaces are represented by points or icons, and their connections are represented by lines. "In a neural interpretation of these cognitive processes, mental spaces are sets of activated neuronal assemblies, and the lines between elements correspond to coactivation-bindings of a certain kind" (Fauconnier & Turner, 2002, p. 40). The four mental spaces characterizing the conceptual integration network are:

- *Input spaces*, at least two of them, containing the input elements of the two spaces which are going to be blended;
- *Cross-space mapping,* connecting counterparts in the input mental spaces;
- *Generic space,* mapping onto each of the inputs and containing what the inputs have in common. "A given element in the generic space maps onto paired counterparts in the two inputs spaces" (Fauconnier & Turner, 2002, p. 47);
- *Blended space*, also called the *blend*, containing the projection of the structure from the two input mental spaces. A phenomenon of selective projection verifies because not all the elements contained in the inputs are projected onto the blend. The blended space is related to the generic space since it contains the generic structure of the generic space, but also a more complex and specific one.

The *emergent structure* of the blend is not copied from the inputs but is obtained through three different mental operations:

- *Composition* of projections from the inputs so to create relations that do not exist in the separate inputs;
- *Completion* that brings additional structure to the blend. It is the most basic kind of recruitment of background knowledge and structure that are brought into the blend unconsciously. It is based on frames and scenarios recruited independently;
- *Elaboration* of the blend is obtained by treating it as a simulation and running it imaginatively according to the principles established for the blend.

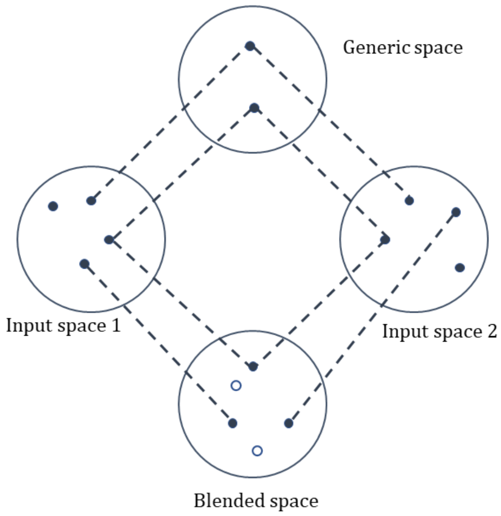



**Figure 4 –** Diagram showing the functioning of conceptual network integration

The conceptual integration network presented until now is a minimal diagram: blending networks actually "can have several input spaces and even multiple blended spaces" (Fauconnier & Turner, 2002, p. 47). According to this theoretical lens, blending knowledge in different mental spaces is the way in which students make sense of new information. We would like to quote a few interesting studies valuing this framework: for instance, Apkarian and colleagues (2019) used conceptual blending to reveal the processes and structure of students' reasoning when dealing with the Sierpinski triangle. During their analysis, they leveraged on the three constituent elements of blending, and this enabled them to grasp and emphasize students' unusual ideas when reasoning about complex mathematical concepts. A less recent contribution applied to physics is the one by Hu and Rebello (2013) who made a significant study analyzing in detail a physics problem on resistance: adopting the tool of conceptual blending, they investigated how students bring together specific information to set up mathematical integrals in physics and so how they blend knowledge from the domains of calculus and physics. This kind of analysis led the authors to better understand students' difficulties on this topic.

The presence of blends is typically revealed by the linguistic forms employed and in this sense, conceptual blending seems to be a suitable lens to shed light on the inputs provided by the various representations adopted and to grasp how they are blended in students' reasoning. This framework, with respect to the semiotic bundle and commognition, works as a magnifying lens on the verbal component that enables to see the input spaces of knowledge and the new emerging blends revealing some forms of covariational reasoning. Indeed, its main contribution from an analytical point of view mainly resides on the instrumentation level: it will help us to evaluate the influence and specifically the benefits or disadvantages coming from the employment of several representations. This kind of analysis will emerge predominantly in all those aspects of classroom discussions related to the mathematical discourse.

## 6.5 Adaptive Teaching

In order to reach individual students' cognitive necessities for prompting covariational reasoning, teachers should resort to adaptive instruction that is specifically geared to meet the needs of the individually different students (Gallagher et al., 2020). Adaptive instruction is an ongoing process in which teachers continuously respond to interactions observed in the classroom, rather than following a predetermined lesson plan with standardized materials. Helping students meet their individual learning needs requires adaptive teachers to be proficient in a range of practices (e.g., asking useful questions, requesting clarifications, facilitating class discussions). The adaptive instruction featured in this study is oriented to a specific, and particularly challenging content, which adds an additional layer to the complexity of the instruction process.

The idea of adaptive instruction is long standing in the literature. Dewey, in his 1902 essay, *Child and Curriculum*, expressed his concerns about the current emphasis on a single kind of curriculum development that produced a uniform, inflexible sequence of instruction that ignored or minimized the child's individual peculiarities, whims, and experiences. Following Dewey, Wang and Lindvall (1984) defined adaptive instruction as "an educational approach that incorporates alternative procedures and strategies for instruction and resource utilization and has the built-in flexibility to permit students to take various routes to, and amounts of time for, learning" (p. 161). Corno and Snow (1986) defined adaptive instruction as instructional approaches and techniques that are geared to meet the needs of individually different students. In a similar way, Park and Lee (2003) described adaptive instruction as "educational interventions aimed at effectively accommodating individual differences in students while helping each student develop the knowledge and skills required to learn a task" (p. 651). Although several definitions of adaptive instruction exist in the literature, the common thread of these theoretical considerations is that the instruction inside the classroom should be flexible enough to meet the student's specific learning needs.

Researchers of adaptive instruction mainly distinguish between two types of adaptiveness: macro-adaptations and micro-adaptations (Corno, 2008; Randi, & Corno, 2005). *Macro-adaptations* refer to the teacher's efforts outside the educational setting to redesign instructional and curricular plans in light of new information about students learning. *Micro-adaptations* occur when teachers flexibly respond to students' demands in the moment of teaching. More recently, other scholars (e.g., Maskiewicz & Winters, 2012, Vaughn & Parsons, 2013) have proposed that adaptive teaching is also demonstrated through responsive guidance, where a teacher "works first to engage students in the pursuit of [their authentic questions], and then to support them in their pursuit in ways that afford progress toward canonical practices and ideas" (Hammer et al., 2012, p. 55). In this form, teachers seek out the queries of their students, and then adaptively constructs learning activities that address these student interests and curiosities, eventually tying them back to broader scientific concepts.

For teachers to develop adaptive instruction, literature suggests that they need a strong pedagogical and content knowledge, a vision of ideal teaching, and a deep understanding of and familiarity with their students (Fairbanks et al., 2010). This means that teachers need to constantly learn about who their students are moment-to-moment, what their students can and want to do with guidance from their teacher, and how and what their students think about the content. Thus, adaptive instruction requires teachers to learn continually about students and to develop ways and strategies to teach them. Not only, adaptive instruction is crucial in the teaching process to foster students' learning especially when inquiry-based learning is implemented in the classroom, where learning is both an individual and a social process, including work by individual students, work by small groups of students, and teacher-led whole classroom discussions.

Gallagher and colleagues (2020), in a review of the literature, found that mathematics educations researchers have focused mainly on these aspects related to adaptive teaching: (a) how curricula can serve as stimuli aiding adaptive teaching practices (e.g., Choppin, 2011; van Es, 2012)); (b) the primary factors to which adaptive teachers respond - namely stimuli from students, the learning trajectory, or their own actions (e.g., Scherrer & Stein, 2013; Wager, 2014; Weiland et al., 2014); and (c) additional teacher responses, including orchestrating classroom discourse, modifying curricular materials, or selecting teaching aids (Huang & Li, 2012; van Es & Conroy, 2009). Adaptive instruction on the micro level of teacher-student interaction has rarely been investigated with respect to students' content-specific learning pathways or with respect to the utterances, gestures, and observations of the students (Jacobs & Empson, 2016). Less attention has been paid to the adaptive teaching oriented to a specific mathematical content. Even scanter attention has been paid to adaptive instruction of a specific content in a teaching situation based on digital technology, and to the teacher's role in facilitating students' evolution towards the understanding of a specific mathematical content.

In this research, we try to address this gap in knowledge by examining a teacher adaptive instruction while using several artefacts and focusing on the development of a specific content, meaning covariational reasoning, adding an additional layer to the complexity of the instruction process. We do believe that understanding how teachers interact with students to adapt their instruction to teach complex mathematical concepts carries important theoretical, methodological, and pedagogical implications.

In Section 9.2 we are going to show the reasons why the teacher involved in this research represents a good example to speak of adaptive teaching and lately, over the three teaching experiments, some adaptive teaching strategies suitable to foster covariational reasoning within a digitally rich environment will be described and discussed.

# 7 RESEARCH QUESTIONS: DEFINITIVE FORMULATION

In the Introduction (Chapter 1) we highlighted the recognized relevance of covariational reasoning for a deep understanding of many mathematical concepts and specifically its importance in modelling tasks; we underlined difficulties met by both students and teachers in secondary school when dealing with the teaching and learning of the complex concept of function, in particular when tasks enhance dynamic aspects requiring a covariational approach that goes beyond the static definition of function. Moreover, a strong revision of the already existing literature brought to light some contributions remarking that the framework of covariation finalized by Thompson and Carlson seems not enough to fully address the variety of reasoning's displayed by students when working with functions. Starting from these observations and findings, we elaborated three teaching experiments involving modelling activities concerning classes of real phenomena to introduce students to families of functions characterized by the presence of parameters. This approach was supported by the use of suitable technological tools which can enhance covariational reasoning processes and help in appreciating the dynamic nature of functions.

The theoretical framework elaborated and presented in the previous chapter consists of four different theoretical lenses: the semiotic bundle, the theory of commognition, conceptual blending, and adaptive teaching. We suitably networked those theories coordinating them and this clarified framework allows us to introduce a second and definitive formulation of the research questions of this study. We aim at deepening students' understanding of covariation not only between variables, but also between variables and parameters, in the context of mathematical modelling of classes of real phenomena through the use of technology enhancing a dynamic approach; we are going to focus especially on students' discursive and linguistic productions and on the teaching strategies that can foster covariational reasoning.

In the following we give a specific formulation of the four research questions guiding our study, now addressing them through the theoretical lenses we have already introduced.

## 7.1 RESEARCH QUESTION 1

How can the Thompson and Carlson's theoretical framework about covariation be enlarged so to encompass second-order covariation in a unique and coherent construct?

This first question reflects the main purpose of our research: it has a theoretical value that will be addressed throughout the entire study and will find an answer only at the end, considering from a global standpoint all the findings of this research. As outlined before, we aim at enlarging the already existing framework about covariation including not only the still under construction construct of second-order covariation, but also all the other hints coming from a revision of the literature and that do not fit completely the Thompson and Carlson hierarchy. We would like to extend a theorization based predominantly on a cognitive characterization, also to other standpoints, specifically the mathematical and didactical ones.

## 7.2 Research question 2

Is it possible to identify some levels connoting second-order covariation?

As done in Thompson and Carlson (2017), we aim at identifying some levels connoting second-order covariation: since we will mainly analyze classroom discussions and brief working group activities, and not individual interviews, we hypothesize that we will not be able to elaborate a rigorous taxonomy of cognitive levels but we aim at least at identifying some mathematical connotations, obviously determined by the mathematical context of our teaching experiments, in which second-order covariation manifests.

## 7.3 Research question 3

Which linguistic markers connote specifically each of the levels of students' covariational reasoning?

From a linguistic standpoint, we aim at conducting a strict analysis of the recurrent syntactical structures or lexical markers that can be associated to the different levels of covariational reasoning. Despite the limitations of a merely qualitative analysis restricted to a limited sample of students in a well-defined contest, this investigation could provide interesting insights on the way in which students express when reasoning covariationally.

## 7.4 Research question 4

Which levels characterizing the discourse about modelling of real phenomena can be distinguished and how do they relate to covariation?

We are interested in investigating how the students' emerging discourse about modelling of real phenomena manifests: we aim at classifying the different levels that can be recognized according to a commognitive perspective and grasp how the use of various technological tools influences and manifests in these various levels of the mathematical discourse. Moreover, we would like to investigate how these levels relate to the evolution of students' covariational reasoning.

## 7.5 Research question 5

Which adaptive teaching strategies does the teacher use to responsively guide the students to engage in covariational reasoning within classroom activities?

Finally, with this last question, we would like to clarify which is the role of the teacher when dealing with classroom activities that involve covariational reasoning: adopting the definition by Maskiewicz and Winters (2012) who consider that adaptive teaching is demonstrated through responsive guidance, we aim at identifying some relevant adaptive teaching strategies that our expert teacher uses in her school practices and from which students benefit because these strategies help them in better explicating covariational reasoning. The identification of these strategies could be helpful also for other teachers because their adoption in their school practices could support students' covariational processes.

All these questions will be addressed in Chapters 10, 11 and 12 and will have a well formulated answer in Chapter 13. Moreover, Chapter 14 will contain some additional and preliminary results helpful to enrich the answer to research question 1.

# 8 METHODOLOGY

## 8.1 TEACHING EXPERIMENTS: A QUALITATIVE RESEARCH

The research presented in this dissertation has all the features of a qualitative study in the sense that its main goal is the discover and analysis of the meanings and interpretations elaborated by students (Gall et al., 1996) when dealing with tasks requiring covariational reasoning and specifically related to mathematical modelling situations. At the beginning of each *teaching experiment (T.E.)*, we made a hypothetical planning of our experimentations and we implemented it in the classroom setting over a period of few weeks. During the implementation, we made careful observations about the efficacy of the original plan, and we used them to refine our planning along the way according to students' reaction to the proposed activities. Our observations and the analysis of the collected data enabled us to gain insights on the students' mathematical conceptualization of the administered tasks and to reflect on how teacher's choices influenced students' understanding. Despite the qualitative nature of our research, it displays many of the characteristics of a scientific research (Groth, 2010) and in particular:

- The adoption of a conceptual framework originated by the networking of several theoretical lenses deeply presented in Chapter 6 and specifically: the semiotic bundle, commognition, conceptual blending, and adaptive instruction;
- The direct and empirical observation of some specific and relevant research questions: their formulation and motivation have been illustrated in Chapter 7;
- A rich description of the setting in which the study was carried out (see Section 9.3, and Chapters 10, 11, and 12);
- A variety of qualitative data collected through classroom observations (video and audio recordings) and some interviews as described in the *Data collection* paragraphs and Section 8.4.

Moreover, since we, as researchers, are not only investigating teachers' attitude, but also theorizing about others' (those of students and in some sense also of the teacher) cognitions related to covariation, we are contributing to the phenomenon we are analyzing: *reflexivity*, according to Steier's definition (1995), means first of all to be aware that our involvement as researchers helps to create the behavior we wish to study. While we struggle with sense-making of others' mathematical understanding, "[t]he images, goals, and intensions guiding [our] actions appear explicitly in [our] image of another's understanding, and [we] attempt to take those aspects of others' experiences into account as [we] try to understand their realities" (Thompson, 1995, p. 124). Reflexive research, as is our research, also implies that we can never capture our current understanding; at best we can capture where we have been, so "we always reflect retrospectively on our contributions to the phenomena of interest" (Thompson, 1995, p. 125). This explains why we felt the need to reformulate in a finer way our research questions during the study and why the preliminary definition of second-order covariation was partially revised and enriched during each of the teaching experiments but only at the end of the whole study we will be able to provide a coherent characterization. Furthermore, as already mentioned, it is only during the analysis of data from the third T.E. that we felt the need to introduce an additional theoretical lens.

In addition to this point, we have to clearly state that the introduction of our enlarged framework about covariational reasoning presents many constraints related to the design and context of our research:

- The context in which we investigate covariational reasoning is that of mathematical modelling intended as mathematical representation of classes of real phenomena;
- We investigate the problem of covariational reasoning among quantities whose values are related by a formula which is something that Thompson and colleagues actually never dealt with, but recent studies have been elaborated in this area involving students of algebra and analysis courses (Frank, 2016);
- The specific kind of instrumentation we propose in our teaching experiments deeply influences the path leading students to a full grasping of second-order covariation.

## 8.2 Design principles

The three teaching experiments we designed as research group have as main topic the modelling of real phenomena describable through a mathematical formula and interpretable through various representations. The phenomena we chose to investigate locate in a perspective of multidisciplinariety, but the focus of the proposed tasks is always that of mathematical interpretation. Hence, the most suitable perspective to read these activities is that of mathematical modelling in the sense intended by Blum and Niss (1991). The design of the activities proposed in our teaching experiments makes explicit use of various artefacts and technological supports which allow an instrumentation of covariational processes and that constitute *multiple external representations* (*MERs*) of the same phenomenon of interest. The teacher has always been involved in designing all the tasks and in structuring the three experiments in their entirety: her wise experience in teaching has been a precious element in determining the time required for each of the activities proposed and in remodulating on the spot the requests of the students according to the trend of achievements and difficulties.

### 8.2.1 Modelling

Mathematical modelling has not a unique definition in Mathematics Education literature. According to the PISA mathematics framework, the mathematical modelling competence is intended as a lens onto the real world (Niss & Højgaard, 2019): mathematical models represent an ideal conceptualization of a real-life or scientific phenomenon; they are formulated in mathematical language and use a wide variety of mathematical tools and results. Moreover, summarizing Blum and Ferri (2009), mathematical modelling is a tool that: (a) helps students to better understand the world; (b) supports mathematics learning, understanding and motivation; (c) contributes to developing various mathematical competencies; and (d) contributes to an adequate picture of mathematics. From the didactical standpoint, many scholars and researchers have underlined the usefulness of modeling tasks for developing mathematical competencies in students (Zbiek & Conner, 2006; Watson & Ohtani, 2015; Arcavi & Friedlander, 2018). Specifically, modeling physical phenomena can produce a useful joint pedagogical fulfillment of the inquiry-based epistemologies of science and mathematics: an interesting concrete didactical example of this approach is given by the *Fibonacci Project* (Harlen, 2012). Commenting this experience, Artigue and Blomhøj (2013) focus some features of this kind of approach to mathematics teaching-learning. Among them: the epistemological relevance of the questions from a mathematical perspective; the modelling dimension of the inquiry process; the experimental dimension of mathematics; the autonomy and responsibility given to students, from the formulation of questions to the production and validation of answers; the guiding role of the teacher and teacher–students dialogic interactions; the collaborative dimension of the inquiry process. Starting from this background, which clearly reveals the validity of introducing modelling in mathematics classroom, we share the vision of mathematical modelling emerging from Italian Indicazioni Nazionali (2010a), meaning as representation of the same class of real phenomena.

To describe the steps of mathematical modelling process, we need to refer to the modelling cycle (**Figure 5**). Inspired by Blum (1996), it consists of two parallel chambers, *reality* and *mathematics*, which comprise four other building blocks: *real situation*, which is the starting stage of the process, *real world model*, *mathematical model*, and *results* which are the numerical solution to a given problem. Other modelling cycles can be found in the literature (Sokolowski, 2015), but they all begin with a real situation and conclude with an attained unique solution. The process of modelling is generally characterized by removing noise, meaning removing all those disturbing elements that could make challenging pass from the real situation to a mathematical model: a first cleaning phase happens when passing from the real situation to the scientific simulation or real world model; the second round happens during the transition toward the mathematical model which necessarily requires some simplifications so that numerical mathematical results are obtainable and computable. Therefore, a mathematical model may present a conceptualization that is understood to be an approximation or an intentional simplification of the object phenomenon. The passages previously described are underlying the arrows of the modelling cycle.

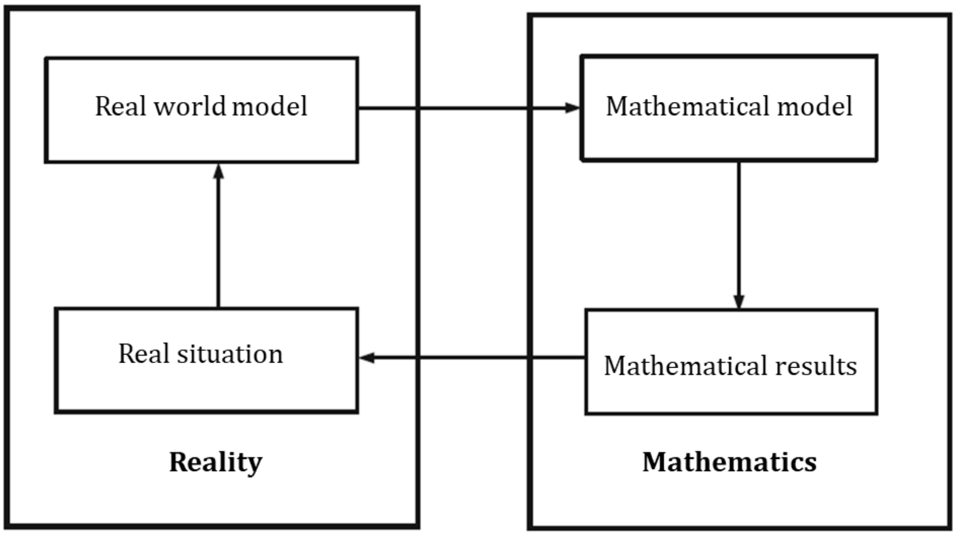



**Figure 5** – Mathematical modelling cycle inspired by Blum (1996)

From the description of the tasks proposed in our teaching experiments, it will clearly emerge that we are not so rigorous in the application of the steps of the modelling cycle: in 2020 teaching experiment, students start with an experiment conducted in classroom under the guidance of their teacher and the results will reveal that it has a strong influence on students' reasoning; in 2017 and 2019 teaching experiments instead do not start from a real situation but rather from a simulation or a reproduction of the same through a video or a GeoGebra applet. In these cases, the reproduction of the experiment in the physics laboratory was the last step of their modelling activities which allowed to verify the validity of their mathematical model.

As we mentioned before, we locate these activities in a perspective of multidisciplinarity, but the disciplines other than mathematics, i.e., physics or science, just constitute the background where the modelling situations under investigation arose: the specific method of investigation of those subjects does not interfere in the experimentation. Students are guided to look at the contest and inputs always with the eyes of the mathematician. If we wanted to condense the aim of our teaching experiments in a few words, those of Dirac (1939) reveal highly appropriate:

> The mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by Nature, but as time goes on it becomes increasingly evident that the rules which the mathematician finds interesting are the same as those which Nature has chosen. (P. A. M. Dirac, From Lecture delivered on presentation of the James Scott prize, (6 Feb 1939), 'The Relation Between Mathematics and Physics', printed in Proceedings of the Royal Society of Edinburgh (1938-1939), 59, Part 2, 124)

### 8.2.2 Multiple External Representations (MERs)

The acronym MERs stands for Multiple External Representations, meaning different modalities of representations so to enhance the learning of the characterizing aspects of the analyzed phenomenon (Ainsworth, 1999). Multi-representational systems employ at least two representations, but commonly many more are available and typical multi-media system can display pictures, text, animations, sound, equations, and graphs. Combinations of representations can play at least three different functions in supporting learning (Ainsworth, 2006):

- *Complementary function*: when MERs complement each other, they do so because they differ either in the processes each supports or in the information each contains. By combining representations that differ in these ways, it is hoped that learners will benefit from the advantages of each of the individual representations;
- *Constrain interpretation*: certain combinations of representations can help learning when one representation constrains the interpretation of a second representation;
- *Construct deeper understanding*: multiple representations support the construction of deeper understanding when learners integrate information from MERs to achieve insights that would be difficult to achieve with only a single representation.

Literature shows that the use of multiple representations strongly supports the learning of mathematical concepts but, although multiple representations are beneficial to the learners, they are non-trivial for students to relate and identify connections (Ainsworth, 2008). In our T.E.s our teacher will play a crucial role in mediating between the involved representations. The theoretical lens of conceptual blending will enable us to analyze how MERs affect cognitively students' reasoning: the analysis of students' emerging mathematical discourses during classroom discussions or working group activities will help us to focus on the level of instrumentation and to observe how the blending of the inputs from the several MERs adopted will reveal and shape students' covariational reasoning.

The representations adopted in designing the proposed tasks have mainly a complementary/construct deeper understanding role, but all the details will be provided in the following sections while describing the specific tasks for each case study.

### 8.2.3 Overview of the whole experimentation and general sequence of tasks

The data presented and discussed in this study come from three teaching experiments. Now we are going to provide briefly some details concerning logistic information, the topic and goals of the tasks proposed:

1) *Teaching experiment 2017* (Chapter 10): it was conducted in a 9$^{th}$ grade classroom, and it was about leading students to construct the mathematical meaning of the quadratic function starting from the simulation of a real phenomenon that is a ball rolling along an inclined plane, the so-called Galileo experiment. Dr Osama Swidan, from the Ben-Gurion University of the Negev, while he was visiting fellow in Turin, deeply contributed to the experimentation, both with the design of the GeoGebra applet and with the logistic organization of video recordings. In this T.E., covariational reasoning processes were instrumented thanks to the use of two main artefacts: a video reproducing the experiment and a GeoGebra applet simulating it (**Figure 6**). The two artefacts provided complementary information that will be presented in detail in Section 10.1.

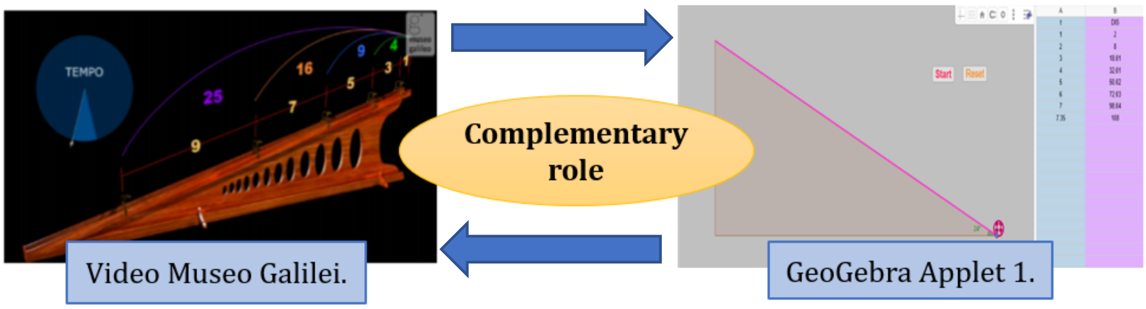



**Figure 6** – The two main artefacts used in the 2017 T.E. instrumentation process

2) *Teaching experiment 2019* (Chapter 11): it took place in a 10th grade classroom, and it was a replication of the 2017 teaching experiment with some other additional tasks. The aim of the activities was to obtain the law of the motion of the ball running along an inclined plane and in particular: (a) obtain the formula describing the motion of the ball $s = k \cdot t^2$; (b) explore the relationship between the angle of inclination of the plane and the *(s, t)* graph.
The instrumentation of the modelling cycle in this case was made possible through four main artefacts: a video and a GeoGebra applet, as done in 2017 T.E., a second applet containing additional information (both numerical and analytical), and finally the reproduction of the experiment in the physics laboratory so to validate the mathematical model (**Figure 7**). The experiment constituted the point of arrival in this cycle and so, it was not so influent in students' reasoning, but the comparison with reality enabled students to verify the validity of the mathematical results obtained from the video or the applets.

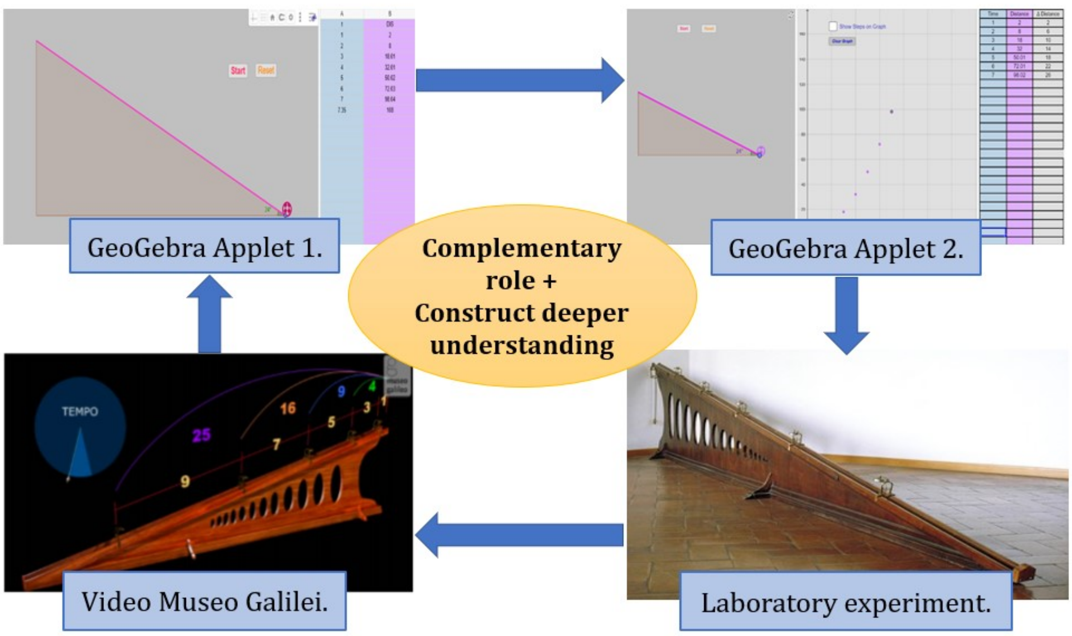



**Figure 7** – Instrumentation of 2019 T.E. modelling process

Globally, we can claim that the MERs adopted have a complementary role when providing additional information, also on a different semiotic register, and help to construct a deeper understanding when enabling to get in touch with different aspects of the phenomenon under investigation.

3) *Teaching experiment 2020* (Chapter 12): it was conducted in a 11th grade classroom, the same of the previous teaching experiment, and had three main goals: (a) investigating the relationship between humidity and temperature; (b) being able to read and interpret the *psychrometric chart*[10] in order to explain real phenomena concerning temperature and humidity; (c) distinguishing the role of variables and parameters in reading charts. Even in this case, the instrumentation of the modelling process and covariational reasoning is supported by four main artefacts: a classroom experiment, a real psychrometric chart and two GeoGebra applets (**Figure 8**). In this case the classroom experiment constituted the starting point of the whole modelling activity and, as the data analysis will reveal, it was a solid reference point throughout the whole T.E., both in reading the psychrometric chart and the graphs showed in the GeoGebra environment: it constituted the element which enabled students to interpret the mathematical representations from a physical point of view. Even in this T.E., the MERs adopted have mainly a complementary or construct deeper understanding function.

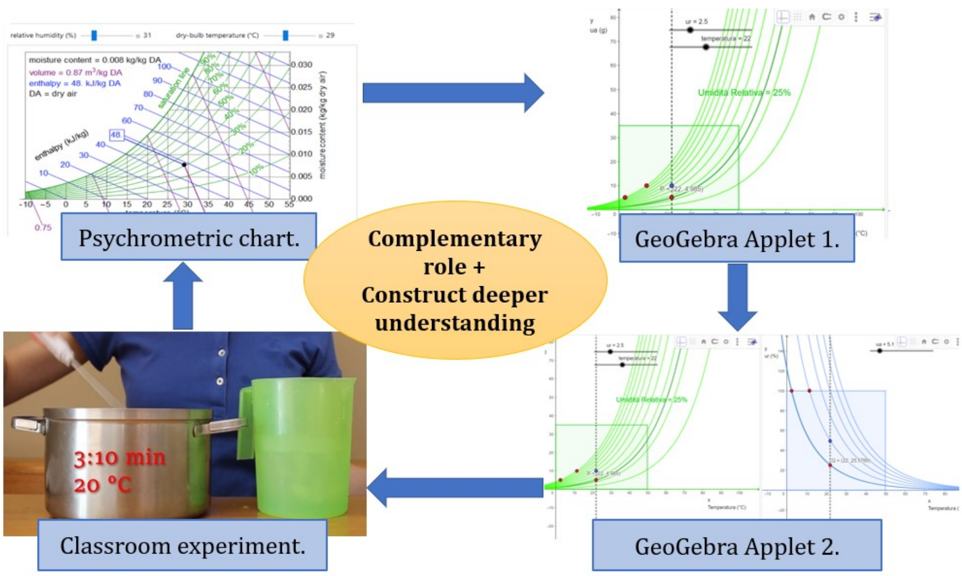



**Figure 8** – Instrumentation of 2020 T.E. modelling process

[10] A *psychrometric chart*, or *Carrier diagram*, is a graph of the thermodynamic parameters of moist air at a constant pressure, often equated to an elevation relative to sea level. Although the principles of psychrometry apply to any physical system consisting of gas-vapor mixtures, the most common system of interest is the mixture of water vapor and air, because of its application in heating, ventilation, air-conditioning, and meteorology.

All the three projects have seen the involvement of the same classroom teacher, Silvia, both during the design phase and during the classroom implementation. Some details concerning her background formation and her teaching method will be presented in Chapter 9.
From a methodological point of view, the teaching experiments consisted in an alternating of working group activities to teacher-led discussions. During the working group phase, students mainly focused on a single representation (video, GeoGebra applet, chart…) and, thanks to some instruction provided by a worksheet, students were guided through an exploratory phase consisting of the formulation of some hypotheses about the phenomenon under investigation. During the explorational phases the role of the teacher was primarily that of supervising students' work and helping the groups in case of problems with the technology involved. Her role became determinant during classroom discussions, which always followed the working group activities and during which the teacher typically started the debate from the written answers of the different groups. Since the teacher had the chance to read the works before starting the discussion, the order in which answers were read and commented on was not casual but somehow planned in order to encourage the discussion, starting from the answers less rich of information to the ones more deepened and somehow near to the correct answer. We expect the teacher to start from reading the conjectures elaborated by the different groups and inviting the members of the group in explaining them. During the following phases, thanks to the complementary or additional information provided by the newly introduced artefact, students were asked to validate or falsify their previous assumptions and reformulate them in a more correct and complete form.

## 8.3 THE TIMELINE: A TOOL TO DESCRIBE WHAT HAPPENS WITHIN THE MATHEMATICS CLASSROOM

The theoretical lenses presented in Section 6.2 allow to properly explore the didactical phenomena that happen in the classroom but given their numerosity and complexity, a specific tool which can condense all the sought information and help the researchers in a proficient analysis is required. Starting from the theoretical perspective of the semiotic bundle, a suitable tool of analysis, called Timeline, has already been presented in Arzarello et al. (2010), in the Italian book *Matematica: Non è solo questione di testa* (2011), and in Sabena et al. (2012): it arises from the need to describe in detail the didactic situations and the dense intertwining of relationships between the variables present in the classroom. It is supposed to be used as a microanalysis tool which provides a global view on the different semiotic registers of the semiotic bundle: speech, body, and inscriptions. In line with the characterization of the semiotic bundle construct, two kinds of analysis may be carried out with the Timeline: a *diachronic* one, focusing on the evolution of the various components over time; and a *synchronic* one, allowing to grasp the relations of the components in a specific moment in time. The Timeline, a table with many rows and columns, is both a powerful and complex tool of analysis and deserves some words to be spent in a detailed description of its various rows dedicated to the analysis of the specific aspects of the semiotic and discursive students' and teachers' productions, including their interactions with tools. Given the density of details, the Timeline is suitable for an accurate analysis of short episodes. However, one of the advantages of this tool is that it can be easily adapted and integrated to respond to different research purposes, for example also for a macroanalysis of longer episodes (Javorski & Potari, 2009).

In this study, in order to satisfy our purposes of research, we propose a new version of the Timeline aiming at integrating the semiotic bundle with the other theoretical lenses we introduced. The improved version of the Timeline we are going to present in this section allows to represent the dynamic flow of the various episodes in a condensed form and to underline the specific main components according to the different theoretical standpoints in each of them. In these rich cognitive processes, all the different components of the semiotic bundle, enriched by a more subtle analysis made possible by the networking with other theoretical frameworks, are active in a unitary way and in the end the second-order covariation is conquered by many students within this unitary process itself.

This enhanced Timeline allows to point out with great detail:

- how the complex interactions between the teacher and the students, suitably represented in the interaction flowchart, intertwine and enhance the exploration and learning of mathematical concepts;
- how artefact interactions can instrument the steps through which the students approach second-order covariation within the communicational environment of the classroom discussion and their evolution in time. Sometimes the students use instruments to get answers to their inquiry processes, sometimes they base on instruments' semiotic productions to settle contrasting conjectures in discussions with their mates or with the teacher. Other times it is the teacher herself that suitably echoes and develops such productions at the blackboard to trigger, support, or provoke students' conjectures;

- the deep structure of what happens in the classroom and shows like in a movie students' growing understanding of the second-order covariation thanks to the adaptive instruction method used by the teacher;
- the evolution of classroom discourse about modelling of real phenomena, the way in which the various external representations of the situations proposed emerge in students' discourse revealing a blending of their knowledge of it and which linguistic features connote their claims involving covariation;
- a tentative classification of the orders and levels of covariational reasoning.

All the details concerning the information collected in the Timeline in order to respond to our research goals are presented in the following: the interface of the Timeline is also shown in

Figure 9 with a brief description of the function of each of its rows.

(i) *Interaction flowchart* - The first row of the Timeline is the flowchart of interactions between the teacher and the students during classroom discussions or among students during working-group sessions. The interactivity flowchart, already introduced in 6.3, arises within a commognitive perspective and was firstly elaborated by Sfard and Kieran (2001) with the aim to collect all the oral interactions and reveal the real interest of the interlocutors in communicating with their partners. The interactions, represented by arrows, can be of three kinds: reactive (diagonally back arrow), proactive (diagonally forward arrow) and both reactive and proactive (both diagonally back and forward arrow). Lately, Liljedahl and Andrà (2014) have further developed the interactivity flowchart including gazes produced in social interactions, specifically their direction and intensity (denoted by the different thickness of the arrows). Our innovative contribution consists of including non-verbal interactions that is the main gestures produced by the subjects involved in the discussion. In particular, the following symbology, reported also in **Table 1**, is adopted: blue arrows for gestures, dotted blue arrows for gazes and dashed-dotted blue arrows to refer to writing gestures; black arrows for oral interactions, dashed arrows to denote questions, and double arrows to underline the semiotic games.

(ii) *Utterances* – This section collects the oral utterances, and it is divided into two rows devoted to the teacher and the students. Revoiced sentences are reported in italics. In addition, we indicate with T→[11] when the teacher asks the students something and use the notation T⇓ when the teacher repeats (revoices) a sentence pronounced by a student to underline the importance or the correctness, possibly even using gestures.

(iii) *Linguistic analysis* – This row re-proposes students' utterances revealing covariational reasoning. The translation into English is preceded by the original statements in Italian. The main goal of this Timeline row is to highlight some linguistic markers, meaning recurrent linguistic structures and features, connoting the various forms of covariational reasoning. Moreover, when needed, a finer grain attention was paid to the identification of terms afferent to different mental input spaces. Linguistic markers, jointly with the other components of the semiotic bundle, e.g., gestures, inscriptions, and interactions with artefacts, can be indicators of subsequent moments in the development of the second-order covariation as will be explained below in the microanalysis and, with greater detail, in the analysis of some episodes.

(iv) *Discourse levels* - Basing on the commognition framework, a row of the Timeline is reserved to the analysis of the discourse on mathematical modelling and the identification of some levels connoting its evolution. Some benchmark studies are those developed by Caspi and Sfard (2012) about the levels of algebraic discourses and those by Kim et al. (2012) on the levels of discourse on infinity. While these studies adopt a historical or ontogenetic approach, we are instead going to focus on the narratives connoting the mathematical discourse.

(v) *Gestures* - Gestures are analyzed according to the four dimensions (deictic, iconic, metaphoric, and beats) identified by McNeill (1992) and adapted to the analysis of mathematics teaching-learning processes as in Arzarello et al. (2015). In addition, we consider another overlapping dimension, that of writing gestures (Shein, 2012; Alibali & Nathan, 2007). The main function of gestures, narrative, interactive or grounding, is reported in red boxes. In particular, non-redundant gestures with respect to speech (Alibali et al., 2000) are marked with a red triangle in the Timeline. Concerning significant gazes, their features and function, sensing or signaling, according to the Interpersonal Gaze Processing Model (Cañigueral & Hamilton, 2019), are reported in blue boxes.

(vi) *Inscriptions* - The row dedicated to inscriptions shows the writings produced by the teacher or students on the blackboard, the interactive whiteboard or on their worksheets.

[11] In this dissertation we are going to use this arrow, instead of a dashed one as done in the Timelines, to denote teacher's questioning.

(vii) *Artefacts interactions* - In this row we record how the students and the teacher refer and interact with the different artefacts, denoted with a capital letter, along the time. Basing on the Saada-Robert microgenesis model of the representation of a problem (1989), further elaborated by Arzarello and colleagues (2002), we distinguish two main modalities which feature the interactions with artefacts, already presented in 6.2.3: ascending control (X→) or descending control (→X).

(viii) *Covariation (order and level)* – Eventually, this row is devoted to a tentative analysis of the orders and levels of covariational reasoning. Specifically, we consider an extension of the already existing taxonomy of Thompson and Carlson (2017) consisting of: first-order covariation (COV 1) and its six levels hierarchy (L0: No coordination, L1: Pre-coordination of values, L2: Gross coordination of values, L3: Coordination of values, L4: Chunky continuous covariation, L5: Smooth continuous covariation) and the new construct of second-order covariation (COV 2). Those rare examples of reasoning revealing variational reasoning will be denoted with letter V. Its sublevels are recalled in Chapter 2.

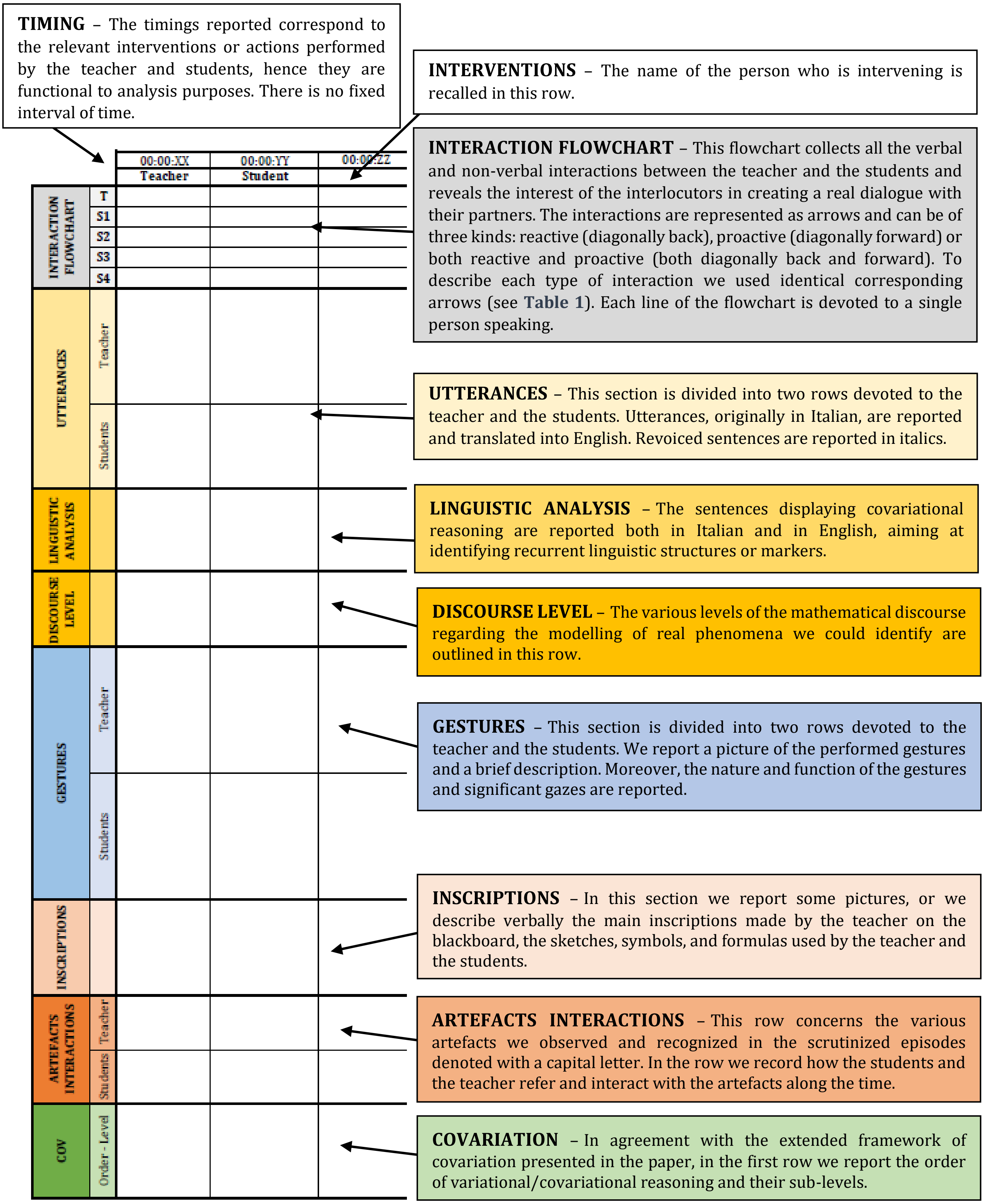



**Figure 9** - Timeline interface

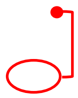
| Section of the timeline | Symbol | Meaning | Comments |
|---|---|---|---|
| INTERACTION FLOWCHART | → | Sentences | Short arrows denote utterances addressed to the whole classroom and not to a specific student. |
| | - → | Questions | |
| | ⇒ | Revoiced utterances | |
| | → | Gestures | |
| | ⇒ | Imitated gestures in the context of a semiotic game | In case of semiotic games, the imitated gesture is generally synchronous with a specific utterance. |
| | - → | Writing gestures | |
| | ····→ | Gazes | |
| UTTERANCES | T ⇓ | Semiotic game played by the teacher | |
| | T--→ | Teacher's questions | |
| GESTURES |  | Redundant gestures | Gestures which are redundant with respect to the information conveyed in the related utterance. |
| |  | Non-redundant gestures | Gestures which are non-redundant with respect to the accompanying speech. |
| |  | Nature and function of gestures | Gestures are analyzed according to the classification of McNeill (1992). He identifies four types of gestures: deictic, iconic or representational, metaphoric and beats. Moreover, we considered writing gestures (Shein, 2012). Gestures may assume three different functions: narrative, interactive or grounding. |
| |  | Nature and function of gazes | We considered some significant gazes, underlining their features and functions (sensing or signaling) according to the Interpersonal Gaze Processing Model developed by Cañigueral and Hamilton (2019). |
| ARTEFACTS INTERACTIONS | X → | Ascending control on the artefact | The student uses the tool with an explorative approach, looking for relationships and mathematical properties. |
| | → X | Descending control on the artefact | The student already has a conjecture in mind and uses the tool to verify it, searching for data supporting his hypothesis. |
| | X←→ Y | Conflict between artefacts | |
| | X § Y | Synergy between artefacts | |

**Table 1** - Symbology adopted for the encoding and analysis in the Timeline

## 8.4 METHODS FOR DATA ANALYSIS

Before starting to analyze data, we produced a prospective analysis of the possible results of the teaching experiments based on: the background of the students in mathematics, some retrospective interviews to the teacher asking her what her aim from a specific intervention was and also, in the case of 2019 and 2020 teaching experiments, the results emerged from the previous experimentations.

Then, we analyzed the data in two phases and using two complementary lenses. In the first phase, we performed a *macroanalysis*: the video was gone over repeatedly so to identify those crucial episodes clearly revealing covariational reasoning processes and then transcribed verbatim. At this macro-level, we began to develop some hypotheses that might explain relations among the observed aspects in the classroom environment, but this process is ongoing and interactive as we continue to add data and analyze them in a wider and more global perspective. In the second phase, the *microanalysis*, we applied a range of qualitative data analysis techniques in tune with the theoretical lenses to address our research questions. One of the most important aspects of interactions in the classroom are the mutual relationships between its different components in different snapshots of their evolution (synchronic analysis) and in their evolution in time (diachronic analysis). The Timeline tool enabled us to capture the complex structure of interactions in the classroom and how the goal of the teaching-experiment is achieved. The microanalysis provided us access to the joint point of view of the teacher and the students within our analysis, rather than only those of the teacher or of the students (Vygotsky, 1978; van der Veer & Valsiner, 1991), and a focus on the semiotic resources involved.

After a detailed analysis of the single episodes based on the components collected in the Timeline, we produced a more transversal analysis of the episodes spread over four main layers:

- Layer (a): Covariational reasoning – This layer of analysis is focused on the identification of the different levels of COV 1 according to Thompson and Carlson's taxonomy and on the manifestation of higher orders of covariational reasoning and the semiotic forms in which they emerge. Particular attention is given to the elements denoting a transition from COV 1 toward COV 2. This analysis is the result of both a synchronic and a diachronic use of the Timeline and of a process of descriptive coding (Saldana, 2015) that enabled us to describe and label the emerging covariational reasoning. This process went through several cycles until we obtained a definitive coding;

- Layer (b): Linguistic analysis – In this layer we analyzed students' utterances revealing covariational aspects with the purpose of identifying some linguistic markers connoting the different levels of covariational reasoning. After having classified the levels of COV and thanks to a synchronic use of the Timeline, these markers were identified mainly paying attention to the *syntactic* and *lexical* dimension. The former brought the focus on the use of comparative structures, the presence of coordinated or subordinated clauses, all relevant features of syntactic complexity in students' language production (Abedi, 2006; Prediger & Sahin-Gür, 2019), and the predominance of unary, binary, or ternary relations when co-varying two magnitudes, as done in the study of Chesnais (2018) about teachers' articulation of relations of symmetry. Concerning the latter, the lexical dimension, the attention was directed towards the identification of recurring adverbs or other terms denoting aspects of locality or globality in students' utterances, the use of qualitative or quantitative terms with respect to the evolution of the levels of COV, and the presence of subjective versus objective sentences (e.g., the use of personal pronouns). Moreover, when needed we conducted a more punctual analysis of the terminology used so to identify the input mental spaces referring to the multiple representations involved;
- Layer (c): Discourse levels - Freeing us from any attempt to adopt a historical or ontogenetic approach, in the identification of some possible levels of the discourse about mathematical modelling of real phenomena we focused on a main distinctive characteristic of the discourse according to the commognition theoretical framework that is the narratives interlacing during the discussions, i.e. descriptions of objects, or of relations between objects, and the purpose connoting these narrative e.g., reporting, explaining meanings, or interpreting, in relation to the various steps of the modelling process. We also focused on the emergence of possible blends, and on those words that denote an informal and intuitive language with respect to a more formal and scientific one. This analysis was the result of a mainly diachronic use of the Timeline;
- Layer (d): Adaptive teaching strategies – This analysis is the result of a disentangling work of all the teaching strategies adopted by the teacher so to identify those that really help in fostering covariation, i.e., those that enable students to better grasp, and therefore express, their covariational reasoning. To identify these strategies, we used a descriptive coding (Saldana, 2015) to code each intervention of the teacher according to its goal and to the presence of elements of responsive guidance. For each intervention, we composed a description of the particular goals that had prompted the teacher to intervene and to

enhance the trustworthiness of our analysis, we interviewed the teacher retrospectively, watching the episodes with her and asking her to explain her motives from the intervention. For example, if the teacher imitates students' gesture or words, we coded this intervention, after the teacher's confirmation, as a semiotic game. Then, we revised the codes focusing on strategies that were recurrent, i.e., happened more than one time during the discussion and that displayed elements of similarity. We observed that strategies rarely manifest singularly, but throughout all the episodes they manifest interlaced with other strategies.

All these results and their analysis will be compared to the initial working hypotheses and the prospective analysis so to produce a retrospective analysis and elaborate some preliminary concluding remarks that in the end of the study will be revised and combined in a global and coherent framework.

# 9 THE TEACHER AND THE SCHOOL

Silvia is the mathematics and physics teacher who deeply contributed to the planning and design of all the teaching experiments analyzed in this study and involved her students in these activities. She widely uses an adaptive teaching approach during her lessons: to better support this claim, in this chapter we are going to provide more details concerning her background formation and her school practices and didactic methodology. Moreover, a few details about the environment of the secondary school where the teaching experiments were carried out are given.

## 9.1 THE TEACHER: BACKGROUND FORMATION

After the four-year degree in Mathematics, in 2000 Silvia began her career as teacher. At the beginning she made some substitutions and worked in a private school; since 2007 she got tenure and worked for five years in a professional school and then moved to a scientific-oriented school in province of Turin and she is still teaching in this school where all our teaching experiments took place. Throughout all her teaching career, she always remained in contact with Professor Arzarello, her thesis advisor, and the research groups on Mathematics Education at Turin University Department of Mathematics. Her thesis had an experimental footprint: roughly speaking, she proposed to analyze data from a test (PM5) administered in the 5$^{th}$ grade of a primary school. Concluded the thesis, she continued working with an association of teachers mainly from primary school dealing with formation and assessment, which collaborated with the design and administration of the PM5 test. Nowadays, Silvia is a founding member of that association. Since 2010, Silvia has joined the group of INVALSI, Istituto nazionale per la valutazione del sistema educativo di istruzione e di formazione[12]. Within this group, Silvia contributes to the writing of the tests employed in the assessment of students' knowledge and competencies on a national scale. Lately Silvia joined the subgroup of INVALSI which specifically focuses on tests for grade 13. In addition to the huge amount of work required for the design of the tests and the development of a reliable assessment method, a long period of formation and training is required to all those who are involved in the INVALSI group, and this formation typically translates into annual meeting addressed to the creation, reading and use of those tests. Her professional career is also connoted by a constant content and pedagogical formation: this

[12] National Institute for the Evaluation of the Education and Training System.

opportunity is offered by the University of Turin that starting from 2015 initiated a group of teacher-researchers whose professional development is taken care by the research group in Mathematics Education of the University itself. The group of teachers regularly meets on Monday afternoon every two weeks and works on specific thematic issues. In the meanwhile, Silvia became tutor of m@t.abel, a national project addressed to teachers and that, with the collaboration of disciplinary experts and employing an e-learning and blended modality, provides teachers with some tools to bring students closer to Mathematics in an engaging and practical way, proposing activities that facilitate the understanding of the close relationship between theoretical abstraction and daily life events. Finally, Silvia collaborated with the Unione Matematica Italiana[13] association to design a didactical path on the new Indicazioni Nazionali, Italian National Curricula for high school; she was actively engaged in several courses for the Piano Lauree Scientifiche[14] and attended a two-year master addressed to teachers' trainers in Mathematics Education. All these elements and experiences connoting Silvia's background reveal her strong pedagogical and content knowledge which are essential to develop suitable adaptive teaching strategies. Her wise approach to education will also emerge from her school practices better described in the following section.

## 9.2 TEACHING PRINCIPLES AND METHODS

The approach of Silvia to teaching well reflects not only her wide background but also her vision of the mathematics classroom. The information presented in this paragraph comes from some informal interviews to Silvia and an attentive observation of her lessons.

Mathematics classroom, in Silvia's ideal vision of teaching, is intended as a *mathematics laboratory* according to the definition presented in *Matematica 2003. La matematica per il cittadino* (MIUR, UMI & SIS, 2003), where we can read:

> Il laboratorio di matematica non è un luogo fisico diverso dalla classe, è piuttosto un insieme strutturato di attività volte alla costruzione di significati degli oggetti matematici. Il laboratorio, quindi, coinvolge persone (studenti e insegnanti), strutture (aule, strumenti, organizzazione degli spazi e dei tempi), idee (progetti, piani di attività didattiche, sperimentazioni). L'ambiente del laboratorio di matematica è in qualche modo assimilabile a quello della bottega rinascimentale, nella quale gli apprendisti imparavano facendo e vedendo fare, comunicando fra loro e con gli esperti. La costruzione di significati, nel laboratorio di matematica, è strettamente legata, da una parte, all'uso degli strumenti

[13] Italian Mathematical Union.
[14] Scientific Degree Program.

> utilizzati nelle varie attività, dall'altra, alle interazioni tra le persone che si sviluppano durante l'esercizio di tali attività[15]. (MIUR, UMI & SIS, 2003, p. 26)

Thereby, all the proposed activities have the common goal to construct shared meanings of mathematical objects and this goal is reached through different didactical practices. First, a method deeply used by Silvia in her lessons is that of Varied Inquiry (MVI), a multi-layered spiral approach that guides students to become engaged in inquiry-based learning. Starting from the observation of a certain situation, students are asked to formulate questions and give them answers; then modifying the situation, and as a consequence of what is observed, new observations, questions and answers arise. MVI thus allows the construction of mathematical competencies, in which knowledge is intertwined with the students' argumentative skills in situations where they are involved as mathematician investigators to solve and pose themselves some problems. It is the teacher's responsibility to foster the transition from one layer of the MVI spiral to the other and to advance the learning within each layer through inquiry-based learning. These activities mainly consist of an alternation between small working group phases and collective discussions with the whole classroom. The mathematics discussions orchestrated by the teacher are "una polifonia di voci articolate su un oggetto matematico (concetto, problema, procedura, ecc.), che costituisce un motivo dell'attività di insegnamento apprendimento"[16] (Bartolini Bussi et al., 1995, p. 7). The role of the teacher within a mathematics discussion is that of a guide that inserts a specific discussion in the flow of classroom activities and influences in a determinant way the development of the discussion through her interventions. Not only, throughout all the classroom activities her role is that of a formalizer of the mathematical contents and of a mediator who directs students' attention to what is relevant: the teacher cannot inject knowledge into students, but she can guide them in the process of domestication of the eye (Radford, 2010) through an intense recourse to semiotic resources like gestures, words, and rhythm. The semiotic game is one of the main strategies naturally used by Silvia to support students' internalization processes and improve their achievements and this

[15] The mathematics laboratory is not a physical place different from the class, it is rather a structured set of activities aimed at constructing the meanings of mathematical objects. The laboratory, therefore, involves people (students and teachers), structures (classrooms, tools, organization of spaces and timings), ideas (projects, educational activity plans, experiments). The environment of the mathematics laboratory is somehow similar to that of the Renaissance workshop, in which the apprentices learned by doing and seeing others do, communicating with each other and with the experts. The construction of meanings, in the mathematics laboratory, is closely linked, on the one hand, to the use of the tools used in the various activities, on the other, to the interactions between people developing during the exercise of such activities.

[16] A polyphony of voices articulated on a mathematical object (concept, problem, procedure, etc.) which constitutes a reason of the teaching-learning activity.

strategy is used when the non-verbal resources used by the students reveal that they are in a ZPD.

Pulling the string, we can conclude that knowledge is intended as a social construction according to a Vygostkijan perspective and the students construct this knowledge not only in the interactions with others, both the teacher and other students, but also thanks to a varied inquiry approach in which students have the possibility to explore and investigate on their own what doing mathematic means. The teacher does not pour her knowledge into students, but she creates the conditions of possibility for students to gain consciousness about mathematical objects.

Looking at Silvia's lessons through a commognitive lens, we can notice how mathematical knowledge is generated by discursive actions of the individuals and the teacher enters as a facilitator, but her leadership is recognized by the whole classroom and thanks to her knowledge, she can guide students through the resolution of commognitive conflicts arising during the mathematical activity.

Among all the artefacts adopted in a mathematics classroom, all the technological supports introduced in the various activities help in making the classroom a digital-rich environment, but a non-technological artefact widely used by Silvia assumes a central role: the blackboard. Silvia writes on it all the relevant information emerging during the collective discussions and students are conscious that what is written on the blackboard is worthy to be remembered.

Other two aspects connote in an original way Silvia's approach to teaching. First, in a perspective of vertical curriculum, concepts are introduced gradually and sometimes anticipated with respect to what is reported in Indicazioni Nazionali. One powerful tool not listed in the Indicazioni Nazionali but used in Silvia's classroom, is that of finite differences[17] (f.d.) that have the double advantage of initiating students to differential calculus ever since the first years of secondary school and of being easily implementable through didactic softwares. A second aspect is a wide mesh a priori analysis of the activities: while designing the activities Silvia identifies a wide field of possible contents to work on, but the goals of the task are not strictly defined; they become clearer during the retrospective analysis that constitutes an opportunity of learning for

[17] Considering a certain variable $z$, the first finite differences of z are all the differences between two consecutives values of that variable: $z_2 - z_1$, $z_3 - z_2$, ..., $z_n - z_{n-1}$. Second finite differences are the first differences of the first differences and so the definition can be extended to n-th finite differences.

the teacher herself. Hence the design of didactic activities is conceived as a dynamic process that can be reshaped after the retrospective analysis.

## 9.3 The environment of the secondary school [18]

At a geographical-territorial level, the high school where we held the teaching experiments locates in province of Turin and has a wide catchment area: the enrolled students come from about 20 different municipalities, distributed on a geographic area that is particularly lively from the cultural point of view. The school building dates back to 1989 and is located in an environment free of disturbing elements. The school has laboratories (science, physics, computer science, languages), drawing rooms, an equipped auditorium, a library, gym, sports fields and outdoor green spaces, interactive whiteboards (IW) and PC in each classroom. This upper secondary school offers many branches and among this there is also the scientific one, where our three teaching experiments were conducted. In general, the social, economic, and cultural level of the students' families is medium-high, such to guarantee to at least 95% of students the continuation of studies at the University. There are no particular social, economic, and cultural disadvantages. In the last years there has been an increase in students with special educational needs (SEN). School students acquire satisfactory levels of social and civic skills, learning to learn, digital skills, and a spirit of initiative and entrepreneurship. The school adopts common indicators and evaluation criteria for conduct and certifies the key competences and citizenship acquired in the two-year period. The Disciplinary Departments draw up a vertical programming in which the skills and abilities that students should achieve are specified for each school year.

[18] The information reported in this paragraph comes from the *Rapporto di Auto Valutazione* (RAV) [Self-assessment report] of the secondary school in question.

# 10 GALILEO TEACHING EXPERIMENT (2017)[19]

This first teaching experiment took place in 2017 in Turin (Italy), where Prof. Arzarello (University of Turin - Italy) and Dr Swidan (visiting fellow from Ben-Gurion University - Israel) developed and designed a teaching session based on simulations of a real-world phenomenon, namely, a ball rolling on an inclined plane, the so-called Galileo experiment. They observed the teacher, Silvia, as she led two 1.5-hour lessons. At the beginning of each lesson, students were required to work in small groups, sharing within each group a worksheet (containing the tasks) and a computer. While the students worked in small groups, the teacher walked around the classroom and periodically interacted with those students who had questions or needed help. After the small working group session, the teacher held a discussion with all the students. She initiated the discussion by asking the students to share with the whole classroom the mathematical ideas that emerged during their participation in the group work.

Concerning my involvement in this teaching experiment, I started to deal with this research problem only in 2019, nearly at the end of the first year of my Ph.D. program. Hence, I did not contribute to the design phase of this teaching experiment, but I was deeply engaged in the phase of data analysis. The prospective analysis presented in the following was made by me basing on the description of the tasks and the considerations of the researchers and the teacher involved. Some details about this experimentation and a first analysis can be found in Arzarello (2019).

*Participants*

The participants in this case study comprised an entire 9th grade classroom of 20 students. At the time the study took place, the participants had already learned the concepts associated with linear functions, which had been taught to them based on their high school textbook, but not yet those of the quadratic function. The students were familiar with the concept of finite differences and its representation: in particular they had previously studied in their math course that a function with n-th differences constant and the previous n-1 not constant, is a n-th-degree polynomial function. They already had reasoned on properties of functions starting from numerical data, specifically values of finite differences for functions represented in tables.

[19] Some of the findings presented in this chapter are the result of a collaboration with Dr O. Swidan, Prof. F. Arzarello and the teacher S. Beltramino. The findings here described will be partially presented in a paper entitled “Adaptive Instruction Strategies to Foster Covariational Reasoning in a Digital Rich Environment” that will be published on a Special Issue on the Journal of Mathematical Behavior.

Nonetheless, they had no formal physics background on quantities like time, space, velocity, acceleration, and their mutual relationships; the students were already familiar with the concept of parabola just from the lower secondary school. In addition, the students were also familiar with conventional function graph software (e.g., GeoGebra), which they had already used in the framework of their school's formal mathematics curriculum.

*Data collection*

During the original teaching experiment, the sessions were video recorded in their entirety, including the general discussion led by the teacher. During the group work, all the student groups (and their computer screens) were filmed as they worked together to solve the task, including teacher's intervention with a dedicated camera. All the written worksheets adopted during lessons were in Italian and here they are integrally reported translated into English. The data analyzed in the following come from the first of the two teacher-led discussions, the only research material in my possession. The video recording was watched multiple times and those episodes revealing the emergence of covariational reasoning were transcribed and then deeply analyzed according to the methods presented in Section 8.4.

The parents of the students and the school consented to the use of the produced multimedia material for research purposes: the original version of the used consent form is contained in **Appendix A**.

## 10.1 OVERVIEW OF THE TASKS AND PROSPECTIVE ANALYSIS

A sequence of two tasks was designed based on the assumption that the exploration of different characteristics of the same phenomenon adopting multiple representations may lead students to construct the mathematical meaning of the quadratic function in its different aspects. Below we describe the tasks and the artefacts related to them.

### 10.1.1 Task 1

In the first task students watched a short video reproducing the well-known Galileo experiment, a ball rolling on an inclined plane, and then students were asked to share their observations about the motion of the ball. The video (**Figure 10**, available here) shows a pendulum always marking the same unit of time, the distance traversed by the ball in that interval, and the total amount of distance covered until that moment, while the inclination angle of the plane is kept fixed. Insofar as it demonstrates the rolling of a ball on an inclined plane, the video is effectively an artefact with an important role in the process of the students' construction of mathematical meaning. Indeed, it works as a mediator between the students' general understanding of the physical phenomenon of a rolling ball on an inclined plane and its mathematical model, which is described by a quadratic function.

**First Worksheet (Task 1)**

*Task 1*

Your task is to watch **Galileo Inclined Plane Experiment** clip and to answer the following questions.

A) What caught your attention while watching the clip? Write as many observations as you can.

B) Can you make a conjecture on which one of the observations will change if the plane inclination changes? Why?

**Corresponding computer screen**

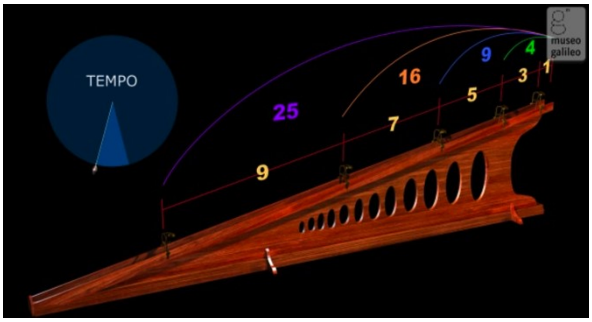



**Figure 10** - Galileo experiment video (Galileo Museum, Florence)

### 10.1.2 Task 2

The second task entailed the interactive exploration of the same situation analyzed in the first task. In this case the artefact was a dynamic digital environment, made using the GeoGebra software, that allowed students to simulate the rolling of a ball while simultaneously observing the values of the distances traversed by the ball while it is in motion via a numerical representation of the distance-time relationship. The applet interface consisted of two parts (**Figure 11**). On the left, it displayed an inclined plane simulating the one of Galileo's experiment shown in the video and students could vary the inclination of the plane by dragging the blue point at the end of the inclined plane; on the right, there was a table with two columns containing data related to time and distance traversed. In this task, the students were requested to think of an equation describing the motion of the ball on the inclined plane.

**Second Worksheet (Task 2)**

*Task 2*

Your task is to explore how the change of the plane's inclination may affect the movement of the ball. Please, open **Galileo 2 applet** and change the plane inclination by dragging the blue point in the applet.

A) Can you make a conjecture on how the change of the inclination may affect the ball movement?

B) Change the plane inclination by dragging the blue point in the applet to verify or refute the conjectures you raised in (A). Do your conjectures change? If yes, why? If not, prove your conjectures.

C) Can you find an equation that describes the ball movement? Why or why not? Justify your answer.

**Corresponding computer screen**

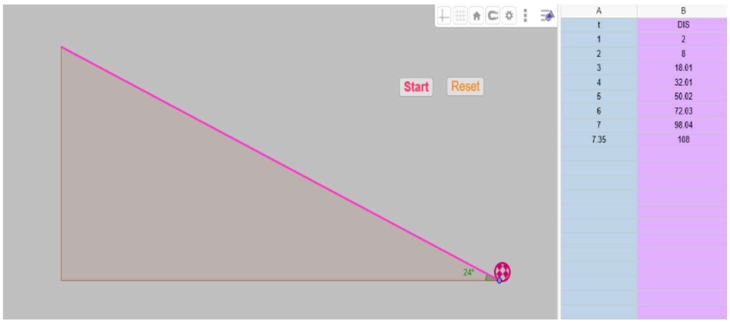



**Figure 11** - The GeoGebra applet interface

Given the description above, it is clear that the two representations of the Galileo experiment, the video and the GeoGebra applet, assume a complementary function (Ainsworth, 2006): while the former provides values of time and distance for a fixed angle and shows all the physical details of the experiment, the latter enables student to grasp the dependence of distance traversed on the angle of inclination of the plane. We can consider how much the interaction between the two artefacts occurs in a consonant or dissonant way speaking of synergy or conflict respectively between the effects of such interactions. A similar approach will be used with respect to the other artefacts present in our story: the blackboard used by the teacher to underline some crucial moments in students' conceptualization, the tables of values made by students to compute first finite differences of time and distance, and the mathematical formulas (on the screen or on the blackboard) supporting students in their processes. In addition to these artefacts, the students were free to use other artefacts, such as GeoGebra, Excel sheets, calculators, sheets of paper, etc.

### 10.1.3 Prospective analysis

In the following chapter we are going to analyze some relevant episodes from a teacher-led discussion which was conducted after the working group sessions on Task 1 and 2. Given the complementary function of the two main representations of the physical phenomenon, the video and the applet, we expect students to use the information provided by the two artefacts in a synergic way. We assume that the numerical information contained in the video and the table of numerical values will lead students to reason and communicate in a mainly quantitative perspective. Students may not be able to guess the trend of *s*-*t* graph given their little background and we do not expect them to elaborate any physical interpretation of the phenomenon. Second-order covariation will probably emerge gradually. Moreover, the teacher will deeply contribute to enhance higher order covariational reasoning. Indeed, when interviewing the teacher, it resulted that she had very clear in her mind which was the goal of the teaching experiment, namely that students could fully grasp the motion law of the ball, particularly its dependence on the inclination angle of the plane, and that they could represent it with a suitable algebraic expression, involving distance, time, and the inclination angle.

## 10.2 DATA ANALYSIS

Below we illustrate in detail five episodes of the discussion examined using the double lens of the macro and micro analysis. The reader can see all the details concerning the analysis of the episodes in the Timelines reported at the end of each episode. During the tasks, the students worked divided in five groups that we are going to denote with the letters A-B-C-D-E. The letter accompanying the name of the student in the transcript denotes the group to which the student belonged. Throughout the whole analysis, four different artefacts will make their appearance: the video (V), the GeoGebra applet (G), the formulas (F) and the blackboard (B).

### 10.2.1 Episode 1 (Discussion, 23:40-23:56)

In this short episode (23:40-23:56), Andrea answers the teacher's question "What did you observe while watching the video?" [1] describing the distance between the metal bells that are on the inclined plane.

| | Timing | Who | Utterances | Gestures |
|---|---|---|---|---|
| 6[20] | 00:23:40 | Andrea$_B$ | The distance between the doors was always two. | |
| 7 | 00:23:44 | Teacher | It was always? | |
| 8 | 00:23:45 | Andrea$_B$ | Two | |
| 9 | 00:23:46 | Virginia$_B$ | Increased by two | |

[20] Lines from 1 to 5 are only present in the Timeline.

| | | | | |
|---|---|---|---|---|
| 10 | 00:23:48 | Teacher | It was always two, or it is increased by two? | |
| 11 | 00:23:50 | Andrea$_B$ | Increased always by two between a door and the other. | *Gesture with his left hand (**Figure 12**, left side)* |
| 12 | 00:23:56 | Teacher | The distance between the doors always increased by two. | *The teacher moves her right hand rightward as she utters (**Figure 12**, right side)* |

At [6], Andrea describes the distance between two consecutive metal bells that are on the inclined plane shown in the video. The teacher, who is listening to him carefully, uses his words to formulate a question, "It was always?". In this case, the teacher, conscious that his answer is incorrect, decides to use the same semiotic register as the students do to clarify what the students mean [7] and encourage him to better reflect on his statement. Right after, Virginia introduces the word "increase", trying to correct Andrea's statement. There is an evident confusion between the value of a quantity and the value of its increment. At this moment, it seems that the teacher is not sure what the students mean. Do they mean the distance increased by two or the distance is a constant and equal two? To clarify Andrea's intention, the teacher plays the semiotic game again, but this time she integrates the two utterances articulated by Andrea and Virginia [10] and adds also a non-verbal component. The teacher also reproduces Andrea's metaphoric gesture [12], namely opening thumb and forefinger, that simulates the distance between the doors. The teacher grasps that Andrea is leaning on the gestural component to better express his thought and so she includes it into the semiotic game thereby generating common semiotic resources that can be shared within the classroom discussion (**Figure 12**).

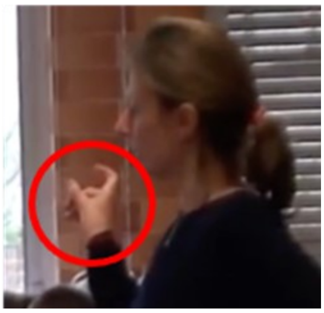


**Figure 12** - On the left, the fingers gesture made by Andrea; on the right the same gesture reproduced by the teacher

The interaction flowchart (**Figure 13**) referred to this episode presents a back-and-forth structure revealing the ability of the teacher to foster the dialogue. This goal is achieved through the questioning (dashed arrows) and the revoicing of the students' words (double arrows).

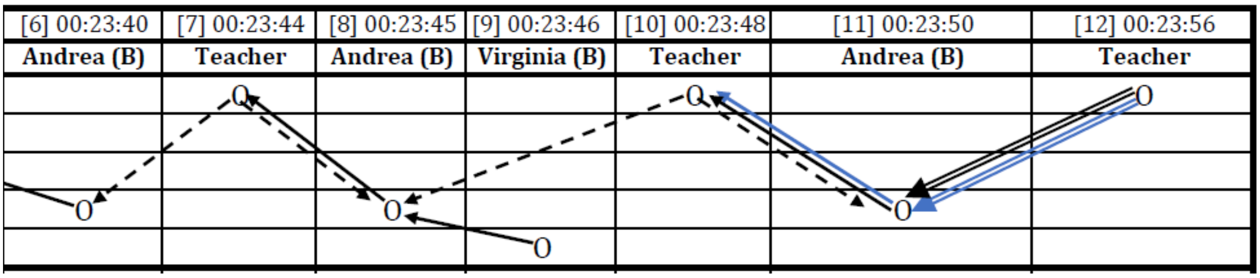



**Figure 13** – The interaction flowchart from Episode 1

In this episode the students are trying to describe what they observed in the video and the language they adopt is the everyday one, not yet scientific, according to Vygotsky 's (1986) distinction: we are going to label this level of the discourse as *descriptive*; indeed, the gestures adopted by both the teacher and the students have mainly a narrative function because they help to better describe the experiment shown in the video simulating door distancing. The video-artefact is what most influences this episode: the references to it are numerous and clear. In terms of covariational reasoning, in the first part of the discussion many variables are highlighted by the terms used by the students: speed, door spacing, time, inclination. They continue to constitute an indistinct jumble, in which we begin to capture traces of concealment in a not so clear way. Even if a first sketch of covariation appears, the language reduces it to the variations of the values of the same quantities (L2) (**Figure 14** - COV row). In response to this confusion, the teacher plays a semiotic game which seems to help Andrea to describe the distance between two consecutive metal bells correctly.

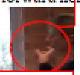
| | | [1] 00:22:37 | [2] 00:22:45 | [3] 00:22:57 | [4] 00:23:10 | [5] 00:23:30 | [6] 00:23:40 | [7] 00:23:44 | [8] 00:23:45 | [9] 00:23:46 | [10] 00:23:48 | [11] 00:23:50 | [12] 00:23:56 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Teacher | Lorenzo (A) | Teacher | Teacher | Marta (D) | Andrea (B) | Teacher | Andrea (B) | Virginia (B) | Teacher | Andrea (B) | Teacher |
| INTERACTION FLOWCHART | T | O | | O | O | | | O | | | O | | O |
| | L | | O | | | | | | | | | | |
| | M | | | | | O | | | | | | | |
| | A | | | | | | O | | O | | | O | |
| | Vir | | | | | | | | | O | | | |
| UTTERANCES | Teacher | What did you observe while watching the video? T--▸ | | *Ball increases speed during the downhill.... At the beginning it goes slowlier and then it increased the speed.* T⇓ | Someone else wrote something different or that completes Lorenzo's sentence? T--▸ | | | It was always? T--▸ | | | It was always two or it increased by two? T--▸ | | *The distance between the doors always increased by two.* T⇓ |
| | Students | | Speed increases gradually as it goes down. At the beginning it is slower... Then later it increases... | | | The time is always the same, the speed changes. | The distance between the doors was always two. | | Two. | Increased by two. | | Increased always by two between a door and the other. | |
| LINGUISTIC ANALYSIS | | | *A aumenta gradualmente mentre [la palla] scende.* A increases gradually as [the ball] goes down. | | | *A è sempre la stessa, B cambia.* A is always the same, B changes. | *A era sempre di 2.* A was always 2. | | 2. | *A aumentava di 2.* A increased by 2. | | *A aumentava sempre di 2 tra una porta e l'altra.* A always increased by 2 between a door and the other. | |
| DISCOURSE LEVEL | | | Descriptive | | | Descriptive | Descriptive | | Descriptive | Descriptive | | Descriptive | |
| GESTURES | Teacher | T looks at the whole classroom and then directs her gaze on group A.  gaze, signaling function | | The teacher looks at Lorenzo and introduces a gesture to denote space traversed.  metaphoric gesture, narrative function | | | | | | | | | The teacher nods and imitates Andrea's gesture moving forward her fingers.  metaphoric gesture, narrative function |
| | Students | | Lorenzo's hand gesture reproducing ball taking velocity and space traversed.  iconic gesture, narrative function | | | | | | | | | Andrea's hand gesture reproducing distance between the doors.  metaphoric gesture, narrative function | |
| INSCRIPTIONS | | | | | | | | | | | | | |
| ARTEFACTS INTERACTIONS | Teacher | Reference to V | | | | | | | | | | | |
| | Students | | | | | | Reference to V | | | | | Reference to V | Reference to V |
| COV | Order - Level | | COV 1 - L2 (speed – distance traversed) | | | COV 1 - L1 (time – speed) | V - L1 (door spacing) | | | V - L2 (door spacing) | | V - L2 (door spacing) | |

**Figure 14 –** Timeline 1 (Galileo 2017)

### 10.2.2 Episode 2 (Discussion, 27:00-28:45)

In the following excerpt (27:00-28:45), after the students have recognized the variables that are involved in the phenomenon, the motion of a ball along an inclined plane, the teacher requests them to find an equation that describes the movement of the ball.

| | Timing | Who | Utterances | Gestures |
|---|---|---|---|---|
| 13 | 00:27:00 | Teacher | Let's go to the next question: "Can you find an equation, a formula which describes the movement of the ball?" | |
| 14 | 00:27:27 | Caterina$_A$ | We have to find the *x* and the *y*; for the *x*-axis, something that increases and for the *y*-axis something that decreases... because actually, the line in the applet was a decreasing line. | |
| 15 | 00:27:46 | Teacher | Which is the decreasing line? | *The teacher comes closer to group A to see their screen.* |
| 16 | 00:27:55 | Caterina$_A$ | Imagining this is a Cartesian plane, the pink line which denotes the movement of the ball [on the applet it has the same color of the trajectory of the ball] is decreasing… on the *y*-axis, there must necessarily be something that is decreasing. | *Caterina makes a pointing gesture, following with her finger the trend of the pink line on the screen (see* ***Figure 15****).* |
| 17 | 00:28:17 | Teacher | Did you hear what Caterina said? Caterina says we have to find an *x* that is increasing and a *y*, which is decreasing because the pink line represents the movement of the ball... but they [group A] are not able to understand what put on the axes. | |
| 18 | 00:28:42 | Teacher | What are the "something," the variables essentially, that help us describing the motion? | |
| 19 | 00:28:45 | Different voices | Time and space; [someone says] the inclination. | |

To foster the discussion in the classroom the teacher asks questions – addressing one to the whole class and one to a specific student (**Figure 17** - interaction flowchart row). In [13] the teacher addresses a question to the whole classroom that is to find the equation that represents the ball movement. Caterina argues that to find the equation, they should consider *x* and *y*, and that *x* should be something increasing and *y* something decreasing. Simultaneously with her utterance, she performs a pointing gesture and then a metaphoric one. The former, which has a grounding function, is used to point at the inclined plane in the applet. The latter, which has narrative function, is used to reproduce the decreasing line. Caterina comes to these insights because of the shape of the simulated apparatus in the app, where the inclined plane is displayed as a decreasing linear function (**Figure 15**).

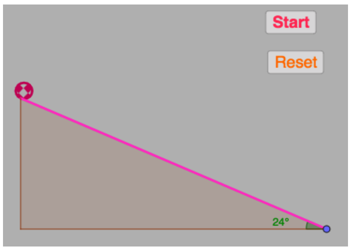



**Figure 15** - The screen of the GeoGebra applet in which the ball trajectory is marked by a pink line

The teacher does not judge Caterina's answer [15]. On the contrary, the teacher approaches Caterina's laptop and looks at her screen (**Figure 16**). The teacher's gaze, which has a sensing function, suggests that she does not understand Caterina's claim and that by gazing at her screen she wants to understand it better. In [16] Caterina elaborates her determination keeping on the same level of discourse, that we are going to refer to as *analytical*: students try to express in mathematical terms and to generalize the specific physical situation proposed. In a similar way as in [14], Caterina simultaneously performs a pointing gesture as she articulates her utterance. This time the pointing gesture functions as narrative and grounding. By addressing a question to the whole classroom, the teacher shifts the discussion from one-to-one to a general discussion involving the whole classroom [17]. In doing so, the teacher again plays the semiotic game as she repeats Caterina's words and redundantly performs a metaphoric gesture, which its function is grounding (**Figure 17** - gesture row). Although the idea of Caterina's group is based on the analogy between a physical trail and the graph representing a motion on the trail, a well-known misconception in the literature (Clement, 1985), the teacher values the group's work and moves forward with the discussion by addressing a new question to the whole classroom [18]. The covariational reasoning in the excerpt above is characterized as L2 of first-order covariational reasoning (**Figure 17** - COV row), in which the students and the teacher describe qualitatively the relationship between *x* and *y* using the verbs decrease/increase.

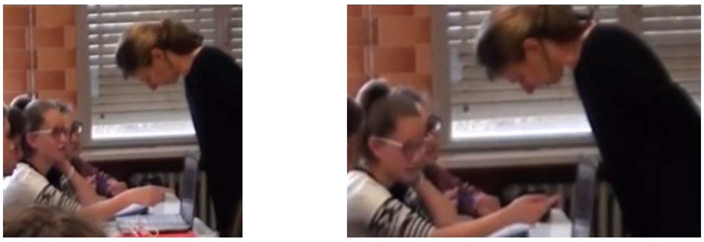


**Figure 16** - The teacher comes closer to group A to see the screen of their laptop

| | | [13] 00:27:00 Teacher | [14] 00:27:27 Caterina (A) | [15] 00:27:46 Teacher | [16] 00:27:55 Caterina (A) | [17] 00:28.17 Teacher | [18] 00:28:42 Teacher |
|---|---|---|---|---|---|---|---|
| INTERACTION FLOWCHART | T | | | | | | |
| | C | | | | | | |
| UTTERANCES | Teacher | Let's go to next question: "Can you find an equation, a formula which describes the movement of the ball?" T--▸ | | Which is the decreasing line? T--▸ | | *Did you hear what C said? C says we have to find an x that is increasing and a y which is decreasing because the pink line represents the movement of the ball... but they [group A] are not able to understand what put on the axes.* T⇓ | What are the "something", the variables essentially, that help us describing the motion? T--▸ |
| | Students | | We have to find the x and the y; for the x-axis something that increases and for the y-axis something that decreases... because actually, the line in the applet was a decreasing line. | | Imagining this is a Cartesian plane, the pink line which denotes the movement of the ball [on the applet it has the same color of the trajectory of the ball] is decreasing... on the y-axis, there must necessarily be something that is decreasing. | | |
| LINGUISTIC ANALYSIS | | | *Per l'asse x qualcosa che cresce e per l'asse y qualcosa che decresce.* For the x-axis something increasing and for the y-axis something decreasing. | | *Sull'asse y deve esserci per forza qualcosa che decresce.* On the y-axis, there must necessarily be something that is decreasing. | *Una x che cresce e una y che decresce.* An x that is increasing and a y which is decreasing. | |
| DISCOURSE LEVEL | | | Analytical | | Analytical | Analytical | |
| GESTURES | Teacher | | | T comes closer to C and looks at her screen. gaze, sensing function | | Hand gesture reproducing the axis of the cartesian plane. metaphoric gesture, grounding function | |
| | Students | | C moves her hand reproducing a decreasing line. pointing gesture, grounding function; metaphoric gesture, narrative function | | C follows with her finger the pink line in the applet. pointing gesture, narrative and grounding function | | |
| INSCRIPTIONS | | | | | C refers to the pink line in the applet. | | |
| ARTEFACTS INTERACTIONS | Teacher | | | | | Reference to the cartesian plane and G | |
| | Students | | Reference to the cartesian plane and G | | Reference to the cartesian plane and G | | |
| COV | Order - Level | | COV 1 - L2 (generic magnitudes x and y) | | COV 1 - L2 (generic magnitudes x and y) | COV 1 - L2 (generic magnitudes x and y) | |

**Figure 17 –** Timeline 2 (Galileo 2017)

### 10.2.3 Episode 3 (Discussion, 30:58-31:49)

In this episode (30:58-31:49) Alessandro criticizes Caterina's answer and argues that both the variables – time and distance – should increase. The teacher asks Alessandro to explain why the two variables are increasing.

| | Timing | Who | Utterances | Gestures |
|---|---|---|---|---|
| 20 | 00:30:58 | Teacher | Why do they [time and space] both increase? | |
| 21 | 00:31:05 | AlessandroE | In my opinion, when you increase the length, the ball takes longer to do it, hence, if you tilt the plane, the ball is faster… | |
| 22 | 00:31:11 | Teacher | I don't understand… | |
| 23 | 00:31:21 | AlessandroE | According to the inclination of the plane, the time increases and… it takes less time... the ball takes less time to travel that distance… | |
| 24 | 00:31:29 | Teacher | The ball takes less time to travel that distance. | |
| 25 | 00:31:35 | AlessandroE | The same distance, but since the plane is more inclined, the ball has a higher speed, it is faster. | |
| 26 | 00:31:41 | Teacher | Ok, at the same distance… | |
| 27 | 00:31:45 | AlessandroE | Less time | |
| 28 | 00:31:48 | Teacher | And at the same time? | |
| 29 | 00:31:49 | AlessandroE | Greater distance. | |

When the teacher asks Alessandro [20], she uses a metaphoric gesture, the function of which is interactive, and looks at Alessandro as she invites him to explain why both the variables should increase. Redundantly with respect to his utterances, he makes metaphoric and iconic gestures which have narrative function to describe why the two variables increase (**Figure 20** – gestures row). In effect, Alessandro introduces two new variables into the discourse with the teacher, the length of the plane and the speed of the ball. It seems that he associates the length of the plane with its inclination. To encourage Alessandro to provide more description, the teacher indicates that she does not understand his description in [21]. Using simultaneous gestures and utterances (**Figure 18**), Alessandro describes his idea again by referring this time explicitly to the inclination of plane [23]. It seems that, based on the experiments he did with the applet and on the values in the table, he concludes that as the inclination increases, the ball goes faster.

**Figure 18** - The metaphoric gestures performed by Alessandro while describing the relationship between the motion of the ball and the length of the plane

This type of reasoning is corresponding to L2 of the first-order covariational reasoning, in that Alessandro draws a qualitatively connection between the two variables – inclination of the plane and the velocity (**Figure 20** – COV row). In [24] the teacher plays the semiotic game by revoicing with an approval tone to support his idea. In [26] the teacher plays again the semiotic game, this time focusing her question only on one term used by Alessandro – the distance or the time. The teacher asks him what would happen to the time if the distance is kept fixed [26], and what would happen to the distance if the time fixed [28]. In both cases the teacher supports her utterances with simultaneous metaphoric gestures, which have narrative function (**Figure 19**).

**Figure 19** - On the left, the gesture performed by the teacher and on the right, the same gesture reproduced by Alessandro

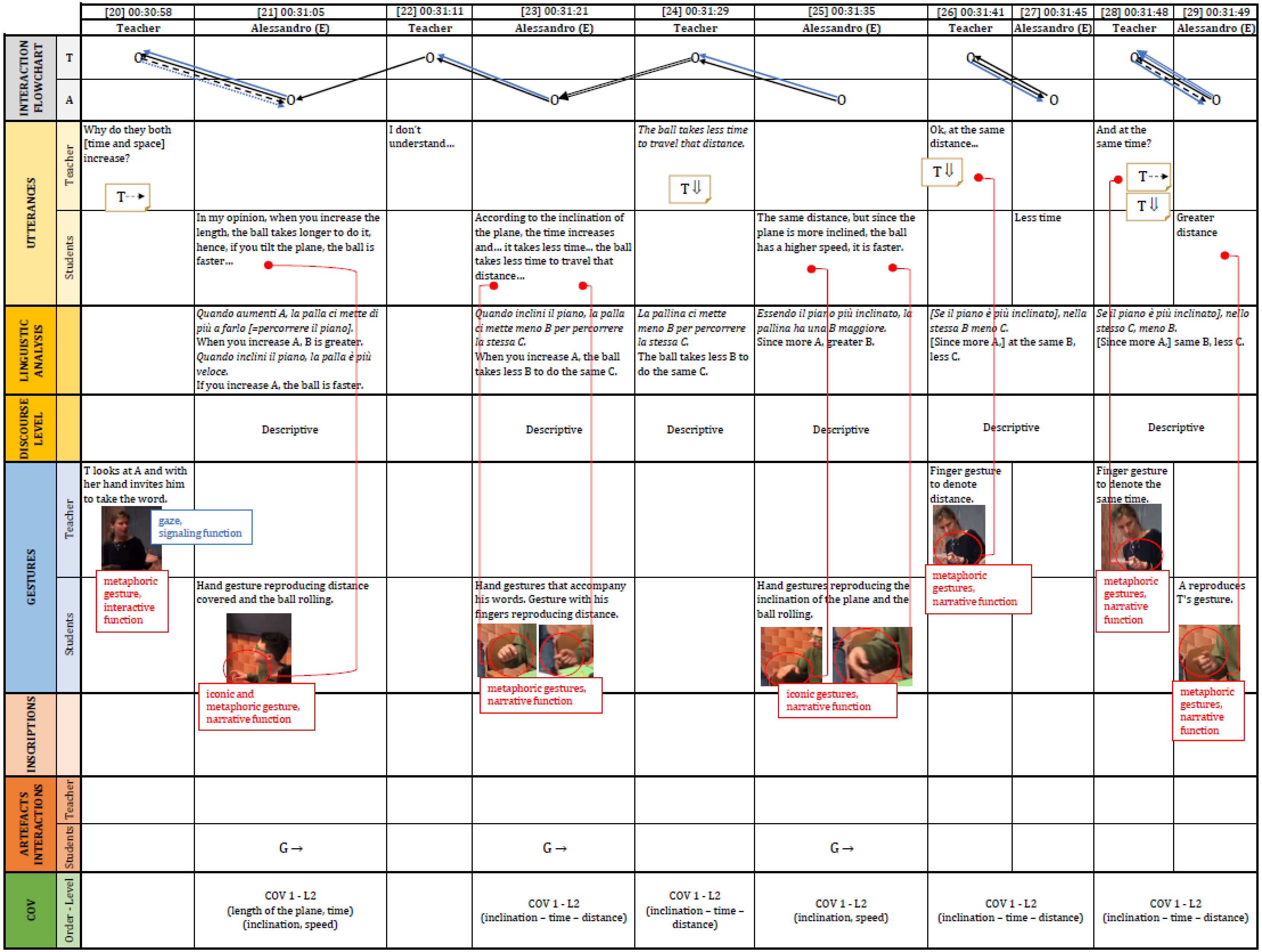



**Figure 20** – Timeline 3 (Galileo 2017)

### 10.2.4 Episode 4[21] (Discussion, 37:00-41:19)

To help the students construct the meaning of the quadratic function describing the law of the motion, the teacher turns to the various artefacts that are available in the environment. Through her play of the semiotic game, the teacher draws the students' attention to the several artefacts, each time focusing on one of them. The following episode (37:00-41:19) illustrates the influence of three main artefacts, the video, the GeoGebra applet and the blackboard: the teacher resorts first to the video, then, thanks to Alessandro's response, she introduces the GeoGebra artefact.

| | Timing | Who | Utterances | Gestures |
|---|---|---|---|---|
| 30 | 00:37:00 | Chiara$_C$ | If it is a parabola, shouldn't be there something to the second power? (→F) | |
| 31 | 00:37:05 | Teacher | If it is a parabola, there will be something to the second power. Is there something to the second power? (T→) | |
| 32 | 00:37:10 | Ada$_A$ | In the top of the video it is written $s: t^2$(V→) | |
| 33 | 00:37:15 | Teacher | | *The teacher writes the formula from the video on the blackboard.* |
| 34 | 00:37:20 | Teacher | | *The teacher looks around waiting for a reaction.* |
| 35 | 00:37:34 | Teacher | Then? (T→) | |
| 36 | | | [noise] | |
| 37 | 00:38:10 | Virginia$_B$ | In the video there were the sums of all the routes, for example, when the space was 16, time was 4. (V→) | |
| 38 | 00:38:15 | Teacher | | *The teacher draws a sketch of the inclined plane on the blackboard (B→)* |
| 39 | 00:38:26 | Virginia$_B$ | If we count the time, we can see that $s$ is $t^2$; $4^2$ = 16. (V→) | |
| | | | [...] | |

[21] In this section we also introduce the notations with arrows to highlight the different dynamics of artefacts and the role of the teacher. Lines 40-41-42 are present only in the Timeline.

| | | | | |
|---|---|---|---|---|
| 43 | 00:40:11 | AlessandroE | It is correct, but here it is different: if we make $2^2$ it doesn't come out for us… | |
| 44 | 00:40:40 | Teacher | *The teacher rephrases Alessandro's words.* | |
| 45 | 00:40:52 | Teacher | *The teacher rephrases Virginia's words.* | |
| 46 | 00:41:15 | Teacher | Alessandro says "it's ok" even though ... | |
| 47 | 00:41:16 | AlessandroE | But it doesn't come in our examples that we made on GeoGebra | *He checks the values on the sheet of paper.* |
| 48 | 00:41:19 | Teacher | It is correct, although, not so correct according to Alessandro, because in the examples it didn't come out in this way. | |

This episode can be translated into the following narrative: previously the students had discussed whether the graph describing the motion could be a straight line, and discarded it basing on what they had seen in the GeoGebra spreadsheet: the parabola appears first evoked by them as "something to the second power" ([30], →F). The evocation is further reinforced by the teacher ([31] T⇓ and T→), by the reference to V ([32], V→) and by the further actions of the teacher ([33-34] T⇓; [35] T→): see **Figure 21**.

**Figure 21** - Ada, on the left, writes the formula from the video in the air using her pen. The teacher, performing a writing gesture, notes that formula (s:t²) on the blackboard

Still referring to the video, Virginia evokes the numbers and formulas that appear there [30] and the teacher reinforces Virginia's observations by drawing on the blackboard ([38], B→; **Figure 23** – inscriptions row). Finally, Virginia explains the formula of the video ([39], V→).

From [43] to [48] a conflict appears, created by the interventions of Alessandro, who disagrees with Virginia. While Virginia bases her computations on what is in the video, Alessandro supports his claim with the computations his group made in GeoGebra: we indicate this conflict with V←→G. The conflict is apparent, and the teacher revoices both Alessandro's and Virginia's utterances [44-45] thus pushing the conflict forward to the class. The implicit meaning of her repeated interventions makes clear to the students that the differences between the two computations are not the effect of a trivial mistake, but that the conflict is more subtle and worthy of consideration.

We can make three observations about this issue: they highlight the relevance of this point in the development of teaching-learning processes in the classroom. First, this conflict can be interpreted as a commognitive conflict: the two students' computations are incommensurable and only apparently incompatible. As Sfard (2008) points out, the only way to overcome a conflict is to make apparent that there are two incommensurable discourses, while no factual incompatibility is concerned. This result is got through teacher's revoicing: as observed above, with her implicit claim that there is no calculation error she avoids the incompatibility issue and makes possible the emergence of incommensurability in the classroom. Second, the conflict concerns a turning point in the way students grasp the covariation among the different quantities involved in the ball motion phenomenon: the discrepancy between the Virginia's and Alessandro's calculations is the starting point for entering into second-order covariation. It is again the teacher who makes apparent this with her written revoicing of the two students' utterances (**Figure 22**).


**Figure 22** - The inscriptions made by the teacher on the blackboard while revoicing Alessandro's and Virginia's words

Third, the conflict emerges through the apparently contradictory results produced by the two artefacts, the video and the GeoGebra applet: it is an instrumented conflict. The teacher does not mediate at all but supports only the emergence of the conflict avoiding the trivial incompatibility interpretation, as pointed above. This way of acting by the teacher shows one of the main features of her approach to managing the class. As a further general comment, we can say that in this part of the video the artefacts are present at different times and are used, sometimes spontaneously by students, sometimes at the teacher's request, to carry out inquiries, and to formulate or validate hypotheses. Therefore, a thread of relations between V, G, and B, that goes from one to the other and that are generally discontinuous, is created through the actions and comments of the students. However, the actions that students perform with such artefacts are often inconsistent, so showing a conflict between the different uses. [continue on p. 90]

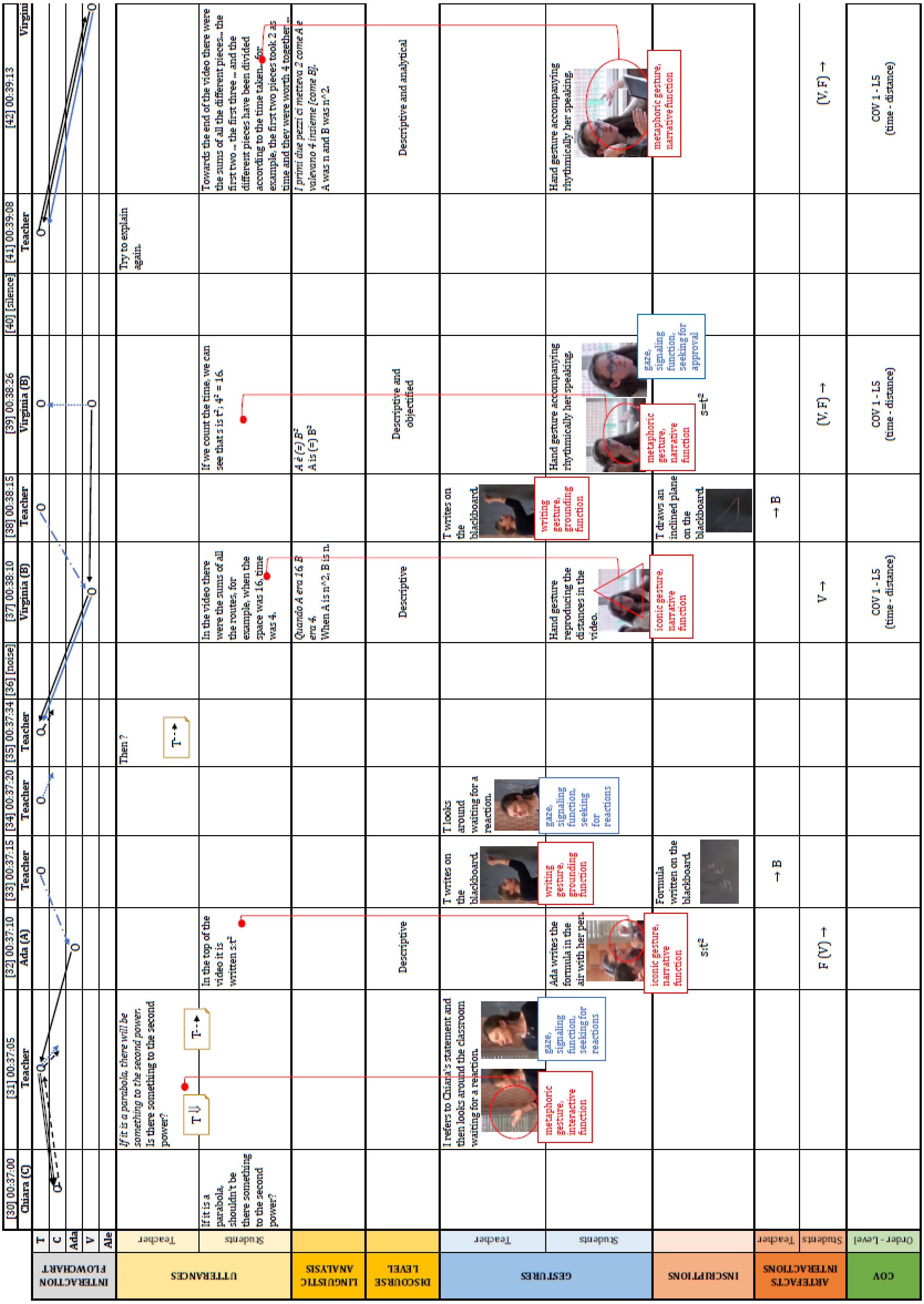



**Figure 23 –** Timeline 4 – Part I (Galileo 2017)

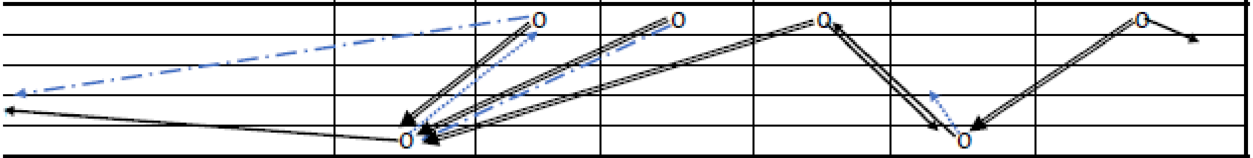
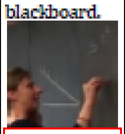
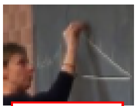
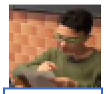
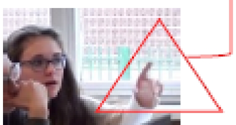
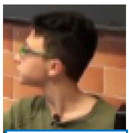
| [42] 00:39:13 | [43] 00:40:11 | [44] 00:40:40 | [45] 00:40:52 | [46] 00:41:15 | [47] 00:41:16 | [48] 00:41:19 |
|---|---|---|---|---|---|---|
| ia (B) | Alessandro (E) | Teacher | Teacher | Teacher | Alessandro (E) | Teacher |
|  | | | | | | |
| | | [T rephrases Ale's words] T ⇓ | [T rephrases V's words] T ⇓ | Alessandro says "it's ok" even though ... | | It is correct, although, not so correct according to Alessandro, because in the examples it didn't come out in this way. |
| So if one looks at that division and if we denote the space as the y and time as the x ... time in that case was 2 and the space was 4 ... Therefore, as it says here, time to second power gives space because 2 to the second is equal to 4 ... hence $y = x^2$ | It is correct, but here it is different: if we make $2^2$ it doesn't come out for us ... | | | | But it doesn't come in our examples that we made on GeoGebra. | T ⇓ |
| *Il tempo alla seconda dà lo spazio... quindi* $y= x^2$ Time to second power gives space... hence $y= x^2$ | | | | | | |
| Analytical and objectified | | | | | | |
| | | T writes on the blackboard.  writing gesture, grounding function | T draws on the blackboard.  writing gesture, grounding function | | Ale checks the values on the sheet of paper.  gaze, sensing function | |
| Pointing gesture addressed to the formula on the blackboard.  pointing gesture, grounding function | Ale looks at T.  gaze, signaling function, seeking for approval | | | | | |
| $y=x^2$ | | T writes $y = x^2$ on the blackboard.  | T draws the doors on the inclined plane like in the video.  | | | |
| | | → (B, F) | → B | | | |
| (V, F) → | V ←→ G | | | | V ←→ G | |
| COV 1 - L5 (time - distance) | | | | | | |

**Figure 24** – Timeline 4 – Part II (Galileo 2017)

These culminate with the final conflict discussed above. In this part, according to the hierarchy of the first order covariation, we have (until [39]) a smooth continuous covariation (L5) with detail (**Figure 24** – COV row).

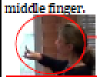
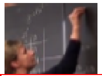
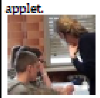
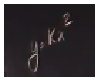
### 10.2.5 Episode 5 (Discussion, 47:10-48:45)

In the last part of the discussion (47:10-48:45), the role of the artefacts changes dramatically: from the initial conflict between them to a final synergy in their instrumented use, "where each activity enhances the potential of the others" (Faggiano, Montone & Mariotti, 2018, p.1). The teacher makes possible the positive evolution of the V←→G conflict, which concluded the previous episode, into the synergy of three artefacts V § G § B [49]. The teacher's requests and interventions direct students' attention to the numerical data written on the blackboard and the students' actions, generated by the synergic interweaving between V § G § B and these data, lead to the construction of a general formula F describing mathematically the physical phenomenon. Moreover, the elaboration of this general formula, including a parameter which depends on the plane inclination angle, reveals not only a deeper understanding of the *s*-*t* covariation, but also the grasping of a new and more complex order of covariation.

| | Timing | Who | Utterances | Gestures |
|---|---|---|---|---|
| 49 | 00:47:10 | Giorgia$_E$ | In this case it might be $y = 2{,}13 \cdot x^2$, in this case with 25°, because it is in this case that a constant value of 2.13 is reached... The values can vary. | |
| 50 | 00:47:24 | Teacher | *The teacher rephrases Giorgia's words.* | |
| 51 | 00:47:45 | Teacher | And if the angle varies? | |
| 52 | 00:47:48 | Giorgia$_E$ | The constant value can vary. | |
| 53 | | | *The teacher suggests trying different values of the inclination and students try using GeoGebra.* | |
| 54 | 00:48:20 | Andrea$_B$ | [incomprehensible] The constant number depends on the angle. | |
| 55 | 00:48:28 | Teacher | I'll repeat because he speaks in a low voice. He said: if we change the angle of 28°, the constant changes and becomes 2.36, but it is always 2.36. | |

| | | | | |
|---|---|---|---|---|
| 56 | 00:48:45 | Giorgia E | It might be $y = k \cdot x^2$, where $k$ is a constant varying with the inclination. | *The teacher writes the formula on the blackboard.* |

In the episode reported above (47:10-48:45), the teacher allows the transition from the first- to the second-order covariation, made explicit through students' utterances and denoted in Timeline 5 (**Figure 25**) as intermediate order, and the full achievement of the COV 2 [56]. Giorgia is the student who first better succeeds in the formalization through an explicit formula describing the motion of the ball. She elaborates a formula (F→) for a specific value of the angle ([49], $y = 2{,}13 \cdot x^2$): this achievement is reached starting from the support of the numerical data in the table, provided by the applet in GeoGebra, and the computations shown on the blackboard that the ratios between distance and time to the second power always provides the same constant (specifically 2.13; see inscriptions row - **Figure 25**). Thanks to the teacher's questioning and directions, shown also in the interaction flowchart in **Figure 25**, the dependence of the constant on the inclination angle has become explicit and the property is experimented in GeoGebra with different values of the angle (→G): several students find that the constant depends on the value of the plane inclination angle. The transition from the first to the second-order covariation (intermediate order) deserves particular attention in this analysis: it happens gradually, and a clear cognitive pivot of this transition can be detected in Giorgia's utterance "the constant value can vary" [52]: this apparently contradictory statement makes explicit the epistemic conflict between the parameter and the variables in the formula. The student refers to the unknown concept of parameter in an intuitive and natural way, adopting an expression that is already documented in the literature (Bloedy-Vinner, 2001). The use of antithetical expressions is also typical of some common locutions of the algebraic language, e.g., arbitrary constant (Bernardi, 1994). Moreover, this idea of varying quantity represents one of the main facets of the concept of variable that we recalled in Section 3.1. When a variable is conceived as a varying quantity, it does not stand for a single unknown value but for a domain of possible values. Despite the focus remaining on the covariation of the dependent and independent variable, the presence of the variable $k$ introduces an underlying idea of motion and dynamicity. Just in a second moment the students understand that the value of that quantity determines the situation as a whole. Referring to our example, like the students well say, changing the value of the angle of inclination of the plane, not only determines a change in the value of the constant $k$ but also of the traversed space. In this perspective, the parameter can be

| | [49] 00:47:10 Giorgia (E) | [50] 00:47:24 Teacher | [51] 00:47:45 Teacher | [52] 00:47:48 Giorgia (E) | [53] | [54] 00:48:20 Andrea (B) | [55] 00:48:28 Teacher | [56] 00:48:45 Giorgia (E) |
|---|---|---|---|---|---|---|---|---|
| INTERACTION FLOWCHART – T | | | | | | | | |
| INTERACTION FLOWCHART – G | | | | | | | | |
| INTERACTION FLOWCHART – A | | | | | | | | |
| UTTERANCES – Teacher | | *[T rephrases Giorgia's words]* T⇓ | And if the angle varies? T--> | | [The teacher suggests trying different values of the inclination and students try using GeoGebra] | | I'll repeat because he speaks in a low voice. He said: *if we change the angle of 28°, the constant changes and becomes 2.36, but it is always 2.36.* T⇓ | |
| UTTERANCES – Students | In this case it might be $y=2.13\cdot x^2$, in this case with 25°, because it is in this case that a constant value of 2.13 is reached... The values can vary. | | | The constant value can vary. | | The constant number depends on the angle. | | It might be $y = k\cdot x^2$, where k is a constant varying with the inclination. |
| LINGUISTIC ANALYSIS | *$y = 2{,}13 \cdot x^2$ in questo caso con 25°* $y = 2{,}13 \cdot x^2$ in this case with 25° | | | *Il valore della costante potrebbe cambiare.* The constant value can vary. | | *Il numero della costante dipende dall'angolo.* The constant number depends on the angle. | | *$y = k \cdot x^2$ dove k è una costante che varia con l'inclinazione.* $y = k \cdot x^2$ where k is a constant varying with the inclination. |
| DISCOURSE LEVEL | Objectified | | | Objectified | | Objectified | Objectified | Objectified |
| GESTURES – Teacher | | T writes the formula on the blackboard. writing gesture, grounding function | | | T comes closer to group A. | | T simulates an enlargement of the angle with the index and middle finger.  iconic gesture, narrative function | T writes the formula on the blackboard.  writing gesture, grounding function |
| GESTURES – Students | Pointing gesture addressed to the computations on the blackboard.  pointing gesture, narrative function | | | | Group A works with the GG applet.  | | | metaphoric gesture, interactive function  |
| INSCRIPTIONS |  | Formula written on the blackboard.  | | | | | $y = 2{,}13 \cdot x^2$ | Formula written on the blackboard.  |
| ARTEFACTS INTERACTIONS – Teacher | | → (B, F) | | | | | F → | → (B, F) |
| ARTEFACTS INTERACTIONS – Students | V § G § B | | | F → | → G | F → | | F → |
| COV – Order - Level | Intermediate order | | T enhances COV with the angle of inclination of the plane. | Intermediate order | | Intermediate order | Intermediate order | COV 2 |

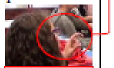
| | [57] 00:48:53 Teacher | [58] 00:49:00 Giorgia (E) | [59] Teacher | [60] 00:49:12 Giorgia (E) | [61] Teacher | [62] 00:49:19 Giorgia (E) | [63] 00:49:26 Teacher |
|---|---|---|---|---|---|---|---|
| INTERACTION FLOWCHART – T | | | | | | | |
| INTERACTION FLOWCHART – G | | | | | | | |
| INTERACTION FLOWCHART – A | | | | | | | |
| UTTERANCES – Teacher | Giorgia says it might be something like that, $y = k\cdot x^2$, where k...? T--> | | *[T repeats Giorgia's words]* T⇓ | | *[T repeats Giorgia's her words]* T⇓ | | That might be correct. |
| UTTERANCES – Students | | Varies based on the inclination. | | As a consequence, space varies. | | Modifying all of the coefficients. | |
| LINGUISTIC ANALYSIS | | | | | | | |
| DISCOURSE LEVEL | | | | | | | |
| GESTURES – Teacher | | | | | | | |
| GESTURES – Students | | | | Hand gesture reproducing space traversed.  iconic and metaphoric gesture, narrative function | | Hands move forward underlining the idea of consequence.  metaphoric gesture, interactive function | |
| INSCRIPTIONS | | | | | | | |
| ARTEFACTS INTERACTIONS – Teacher | | | | | | | |
| ARTEFACTS INTERACTIONS – Students | | | | | | | |
| COV – Order - Level | T enhances COV 2. | | | | | | |

**Figure 25** – Timeline 5 (Galileo 2017)

intended as a *higher order variable* in the sense of Arcavi et al. (2016) and this idea well fits with our notion of second-order covariation. According to a commognitive perspective, the expression “the constant value can vary” is a clear sign of a commognitive conflict due to a double interpretation of the word “constant” on two different semantic levels (Sfard, 2008). The transitional order of covariational reasoning represents the coming to the surface of the incommensurability between the mathematical narratives and it culminates when the process of evolution to second-order covariation is completed, and students grasp the role of the parameter within the formula of the motion of the ball. Eventually, Giorgia is able to develop a generalized formula ([56], $y=k \cdot x^2$), in which the dependence on the inclination angle is encapsulated within the parameter $k$. The latter students’ interventions express their ability to read the formula $s = k \cdot t^2$ on a "double order": at a first-order the covariation $s$-$t$; at a second-order the covariation between the $(s,t)$ function and the parameter $k$.

The artefacts interactions row in the Timeline (**Figure 25**) collects all the different modalities in which the artefacts intervene in the discussion, highlighting the synergy between their purpose of use. The teacher constantly uses the blackboard in order to take notes of the emerging formulas (→B) and these writing gestures have a grounding function: the blackboard B represents the artefact which allows the creation of common resources shared by the whole classroom for use in the discussion. Moreover, the writing gestures of the teacher implicitly reveal to the students the relevance of what is written on the blackboard. Contrary to our expectations, the analysis of the gestures produced by the students does not reveal many non-redundant gestures with respect to speech. We expected to find them especially during cognitive transitional phases, but they are not so evident: one possible explanation for this could be the previous working group session during which students already reasoned on the tasks proposed and elaborated their own reasoning. The level of the mathematical discourse is initially analytical and then flows into a higher level that we are going to refer to as objectified ([49], [56]) when a mathematical formula is clearly elaborated (**Figure 25** – discourse level row).

## 10.3 DISCUSSION

In this section we are going to analyze the five episodes previously described according to the four layers of analysis presented in Section 8.4.

### 10.3.1 Layer (a): Covariational reasoning

The episodes analyzed in the five Timelines reveal the existence of a transitional phase between the two orders of covariation in which the focus of reasoning remains on the covariation of the dependent and independent variable, but the presence of a parameter introduces an underlying idea of motion and dynamicity. As described in Arcavi et al. (2016), the parameter is conceived as a varying quantity: it does not stand for a single unknown value but for a domain of possible values. This intermediate order requires a further investigation: data from other T.E.s will help us to better understand whether it is a standalone order or rather a low level of COV 2 that contributes to a cognitive theorization of this second-order construct. COV 2 is fully reached at the end of the discussion when Giorgia conceptualizes that using *k* is possible to condense in one formula an entire family of relationship.

The tasks performed during the teaching experiment show that the technological supports allow an instrumentation not only of the first-order of covariation but also of the second one: specifically, in the GeoGebra applet, dragging the blue point located at the end of the plane, students can modify the angle of inclination, observe the related values of time and distance traversed contained in the table and deduce the properties of the *s*-*t* function. This kind of instrumentation of COV 2 is a clear example of metavariation (Hoffkamp, 2009). Indeed, varying the inclination of the plane enables to investigate covariation in several scenarios.

### 10.3.2 Layer (b): Linguistic analysis

This layer of analysis is the result of a synchronic use of the Timeline: after having identified the levels of covariational reasoning emerging in students' utterances, we analyzed the linguistic structure of their statements. We are going to refer to the magnitudes involved denoting them with a capital letter (A, B…) and not mentioning them explicitly: this approach helped us to focus specifically on the structure of the sentences and not on the specific magnitudes. Here we collect in a table the emerging examples and the related observations.

| *COV 1* | *Description of the level* | *Examples from Galileo 2017 T.E.* | *Syntactical and lexical analysis* |
|---|---|---|---|
| L1 | The students perceive a change in both the magnitudes, but the way in which this change happens is not described and in particular they do not explain how a change in a quantity affects the other. | *[5] A è sempre la stessa, B cambia.*<br>A is always the same, B changes. | From the **syntactical** point of view, we can recognize this structure:<br>*Sentence about A*, *sentence about B* where the two sentences are not correlated.<br>From the **lexical** point of view, we can observe the use of the adverb ***always***, sign of a qualitative description. |

| | | | |
|---|---|---|---|
| L2 | Students observe that an increase/decrease in one variable produces an increase/decrease in the other variable involved. | *[2] A aumenta gradualmente mentre [la palla] scende.*<br>A increases gradually as [the ball] goes down.<br><br>*[14] Per l'asse x qualcosa che cresce e per l'asse y qualcosa che decresce.*<br>For the *x*-axis something increasing and for the *y*-axis something decreasing.<br><br>*[16] Quando aumenti A, la palla ci mette di più a farlo [=percorrere il piano].*<br>When you increase A, B is greater.<br><br>*[21] Quando inclini il piano, la palla è più veloce.*<br>When you increase A, the ball is faster.<br><br>*[23] Quando inclini il piano, la palla ci mette meno B per percorrere la stessa C.*<br>When you increase A, the ball takes less B to do the same C.<br><br>*[25] Essendo il piano più inclinato, la pallina ha una B maggiore.*<br>Since more A, greater B.<br><br>*[27] [Se il piano è più inclinato], nella stessa B meno C.*<br>[Since more A,] same B, less C.<br><br>*[29] [Se il piano è più inclinato], nello stesso C, meno B.* | The **syntactical** structures that can be identified are the following:<br>*Sentence about A* ***as/while*** *sentence about B.*<br>*Since* ***more/less*** *A,* ***more/less*** *B to do the* ***same*** *C. [3 magnitudes]*<br>***When*** *you increase A****,*** *B is greater.*<br>*Since* ***more*** *A,* ***greater*** *B.*<br>The relations are mainly binary or ternary and clauses are mainly subordinate. It also emerges the use of comparatives.<br><br>From the **lexical** standpoint, students use temporal linking words: ***as***, ***while***, ***when*** (often followed by a subjective sentence in which 'you' is the subject: when *you* increase A). In all these examples we can observe a qualitative description of the variation of the two or three magnitudes involved, and it is clear that a change in one quantity produces a change in the others. |

| | | | |
|---|---|---|---|
| | | [Since more A,] same C, less B. | |
| L5 | At this level of covariational reasoning, students overcome the simpler coordination of the numerical values of the quantities and arrive at a more global vision of the relationship between the two magnitudes. | *[37] Quando A era 16, B era 4.*<br>When A is n^2, B is n.<br><br>*[39] A è (=) $B^2$*<br>A is (=) $B^2$<br><br>*[43] Il tempo alla seconda dà lo spazio... quindi $y= x^2$*<br>Time to second power gives space… hence $y= x^2$ | From the **syntactical** standpoint, the relation between A and B is condensed adopting the verb ***to be*** or the ***equal sign.***<br>From the **lexical** point of view, the continuous covariation between magnitudes A and B is expressed through the description in formal terms of their mathematical relationship or the use of a formula with an independent ($x$) and a dependent variable ($y$). Sentences are objective and relations expressed are true globally. The verb *to be* is used with the same meaning of the equal sign.<br>Sentence [37] distinguishes for the presence of a subordinate clause in which the relation is expressed in a local form adopting numerical values. |

| *Higher COV* | *Description of the level* | *Examples from Galileo 2017 T.E.* | *Syntactical and lexical analysis* |
|---|---|---|---|
| Intermediate COV | Transitional cognitive order toward a full achievement of COV 2. Students try to describe in an intuitive way the concept of parameter. | *[49] $y = 2{,}13 \cdot x^2$ in questo caso con 25°*<br>$y = 2{,}13 \cdot x^2$ in this case with 25°<br><br>*[52] Il valore della costante potrebbe cambiare.*<br>The constant value can vary.<br><br>*[54] Il numero della costante dipende dall'angolo.*<br>The constant number depends on the angle. | From the **syntactical** standpoint, we do not observe recurring structures.<br>From the **lexical** standpoint, we can notice a quantitative approach [49] with the elaboration of a mathematical formula for a fixed value of the angle or the use of some antithetic expressions [52-54] in which students state that the value of the so-called constant can vary. |
| COV 2 | Students succeed in the elaboration of a general formula containing the parameter: $k$ | *[56] $y = k \cdot x^2$ dove $k$ è una costante che varia con l'inclinazione.* | From the **syntactical** standpoint, we do not observe relevant structures.<br>The **lexical** analysis reveals that this order of covariation |

| | | | |
|---|---|---|---|
| | encapsulates the dependence on a specific magnitude and allows to describe an entire class of phenomena. | $y = k \cdot x^2$ where $k$ is a constant varying with the inclination. | manifests in a quantitative way, with a formula containing the two variables and a parameter.<br>The parameter is still identified with an antithetic expression i.e., a constant varying with the inclination, probably due to the absence of a more specific mathematical background. |

Finally, a few examples of variational reasoning can be recognized: at [6], referring to door spacing, Andrea says that it "was always 2" [6] (L1 – discrete variation); only in a second moment, thanks to teacher's and Virginia's intervention, he elaborates that it "increased by 2" [10] (L2 – gross variation).

### 10.3.3 Layer (c): Discourse levels

A diachronic use of the Timelines enabled us to appreciate the evolution of the mathematical discussion developing throughout the five episodes. In particular, we identified three main levels of the mathematical discourse:

- *Descriptive*: students' utterances are simple descriptions of the inputs from different external representations. There are explicit references to the artefacts and to the specific physical context presented. The language adopted is the everyday one, not yet scientific, according to Vygotsky 's (1986) distinction.

  For instance, in [2] Lorenzo, while describing what he observed in the video, says "speed increases gradually as it goes down"; in [32] Ada says "In the top of the video it is written $s: t^2$" and in [37] Virginia states "In the video there were the sum of all routes [...]". Utterances are objective. Something slightly different can be observed in [21] where Alessandro, starting from the inputs provided by the video, claims that "if you tilt the plane, the ball is faster". He imagines modifying the physical situation presented in the video and expresses it in a subjective way: he uses the personal pronoun "you" followed by the verb "tilt".
- *Analytical*: at this level, the use of an analytical language starts making its appearance in the mathematical discourse. Students conceptualize the magnitudes involved with a mathematical symbol, a variable, and reason in terms of numerical relationships or in terms of $x$ and $y$ in the Cartesian plane. The sentences are more "isolated" from the

specific physical contest and describe some possible variations of the physical phenomenon.

Some clear examples of this level can be recognized in Timeline 2 (

**Figure 16** - The teacher comes closer to group A to see the screen of their laptop

**Figure *17***) when Caterina starts thinking of the *s*-*t* graph and says "for the *x*-axis something that increase and for the *y*-axis something that decreases…" [14] and again reinforces at [16] saying "On the *y*-axis there must necessarily be something that is decreasing". Or at [42], Virginia reflecting on the data presented in the video observes that "the first two pieces took 2 as time and they were worth 4 together".

- *Objectified*: the process of modelling leads to the elaboration of a mathematical relationship, a function or a formula that has a general validity and does not fit only with the specific physical contest presented at the beginning. The mathematical discourse is objective and formal. At [39], Virginia states "$s$ is $t^2$", and again at [42]: "time to the second power gives space […] hence $y=x^2$". In the last episode, other formulas appear: "$y=2.13 \cdot x^2$" ([49]) and "$y=k \cdot x^2$" ([56]). The formula allows to describe mathematically the real situation with a global approach.

### 10.3.4 Layer (d): Adaptive teaching strategies

The process of descriptive coding helped us to identify four main strategies that the teacher used to adapt her instruction to the students in the classroom. It is worthy to note that the teacher did not follow the strategies separately; on the contrary, we observed several strategies interlaced in the same episode. The strategies were named based on the teacher aim from performing the strategy. We retrospectively interviewed the teacher asking her what her aim from a specific intervention was. The strategies were recurring but the episodes we chose exemplified them at best. Strategies are listed below in order of appearance in time and not in order of relevance:

1 - *Semiotic game*: through this game, the teacher revoices students' words and reproduces students' gestures to ascribe them with mathematical meanings. In the first episode (10.2.1), the semiotic game played by the teacher helps Andrea to describe the distance between two constitutive metal bells correctly.

2 - *Fostering the discussion in the classroom and facilitating its flow:* the teacher encourages the discussion in the classroom and makes it run through questioning. The teacher never judges students' argument as it is correct or incorrect, but she always lets the students try to respond by formulating a new question, with the aim of pushing them to deepen the problem at stake. This strategy is evident in Episode 2 (10.2.2).

3 - *Exploring students' actions and thinking*: adopting an investigative approach, the teacher tries to better understand what the students are really thinking and referring to. In Episode 3 (10.2.3), we have observed that the teacher addresses many questions, sometimes to a specific student, other times to the whole classroom and she pretends to have not understood Alessandro's statement, so to help him dig into his thoughts.

4 - *Drawing students' attention to the information provided by the different artefacts* used in the learning process: the teacher brings the students' attention to the mathematical relationships displayed by the digital artefacts and uses the blackboard to create shared semiotic resources with the whole classroom; the aim is to look at a situation from a fresh standpoint. This strategy is predominant in the last two episodes: in Episode 4 (10.2.4), the role of the teacher is essential in bringing to the surface the conflict emerging in the classroom and to better analyze the apparently contradictory information provided by the artefacts involved; then in Episode 5 (10.2.5) she successfully guides students toward the resolution of the conflict helping them in using artefacts in a synergic way.

## 10.4 CONCLUDING REMARKS

The analysis of the evolution of students' reasoning when dealing with the conceptualization of ball motion shows two important features concerning how they can learn functions. From the one side, their learning processes are fostered by teaching situations, where they are asked to model real phenomena through an inquiry methodology. As we have seen, many of the features of modelling tasks we recalled in Section 8.2.1 are observable in our teaching experiment, so confirming that this inquiry-based approach to a modelling task reveals useful for triggering students' understanding of a complex mathematical construct like second-order covariation of magnitudes. From the other side, continuous covariational reasoning, both of first- and second-order, reveals important for their conceptual understanding of functions. We will now shortly discuss the two items. For this issue, it is interesting to recall the explanation given by Bloedy-Vinner (2001) about the difficulties that students meet when facing situations where they must manage ‘second order functions’, namely functions whose argument is a parameter, and their corresponding values are equations or functions. This mathematical definition explains why students run into a cognitive conflict when working with parameters and they could perceive them as constants that vary. Embracing a commognitive perspective, this conflict could be interpreted as something more than just psychological: on a semantic level it reflects in the use of an oxymoronic expression; on a discourse level it reveals the encounter between incompatible narratives and the necessity to make evolve the mathematical discourse so to resolve the incommensurability of the previous discourse accepting the possibility of introducing new mathematical notions, e.g. the parameter in our case, in order to solve that emerging conflict. The data analysis also revealed the emergence of a transitional phase in between the two orders of covariation in which the parameter $k$ makes its appearance and introduces dynamicity. Finally, the analysis of Episode 4 (10.2.4) underlines the important role played by the teacher in non-trivializing the emergence of the commognitive conflict and in contributing to its resolution adopting suitable teaching strategies, like directing students’ attention to the most suitable information, questioning, and enhancing unexplored ways of reasoning.

# 11 GALILEO TEACHING EXPERIMENT (2019)

This second teaching experiment took place in September/October 2019, and it was substantially a replication of the one conducted in 2017 with some slightly modifications. The design of the tasks was revised by the teacher, Silvia, and me under the supervision of Prof. Arzarello. The activity proposed took at least 3 weeks of work for a total amount of nearly 16 hours including a few hours of homework. Students worked divided in 5 small group-works (specifically 3 groups of 4 students and 2 groups of 5 students). Groups remained always the same throughout all the activities.

During the group-work, students conducted mainly exploratory activities and the teacher supervised the work, resolving possible difficulties with the tools and answering students' questions with suggestions on how to face the activity. All the group-work sessions were followed by classroom discussion mediated by the teacher. She always started from the different answers of the groups, underlining similarities and differences, and enhancing an argumentative approach in order to justify their assumptions.

As a researcher, I was present in class during most of the activities proposed and I personally took care of the video-recordings.

*Participants*

The 10$^{th}$ grade classroom involved was made of 22 students. Thanks to their previous studies in mathematics, students knew the meaning of finite differences and that a function with constant first differences is a line, a function with constant second differences (and first differences not constant) is a second-degree polynomial function and in general a function with n-th differences constant and the previous n-1 not constant, is a n-th-degree polynomial function. Students were used to work with technology and in particular with GeoGebra applets. They already had reasoned on properties of functions starting from numerical data that is with values of finite differences for functions represented in tables.

According to the physics school program, students were also familiar with the notions of distance, velocity, and the decomposition of forces along an inclined plane. Students had not yet studied the scientific concept of acceleration and used that term from their everyday experiences of situations related to motion (e.g., cars).

*Data collection*

The entire teaching experiment was video-recorded adopting various disposals to film the working groups and a dedicated camera to film the teacher. In some cases, thanks to the use of the software Camtasia, the computer screens were also recorded. All the written worksheets used during the lessons were in Italian and here they are integrally reported translated into English. The written protocols of the students were collected and analyzed. The salient parts of the lessons revealing the emergence of covariational reasoning were transcribed and then deeply analyzed.

The parents of the students and the school consented to the use of the multimedia material produced; the original version of the used consent form is contained in **Appendix A**.

## 11.1 OVERVIEW OF THE TASKS AND PROSPECTIVE ANALYSIS

The main purpose of this activity was that of exploring the physical phenomenon of the motion of a ball running along an inclined plane. This was done through different representations provided by the vision of a video, some applets in the GeoGebra environment, the reading of Galileo's original text on falling bodies and the reproduction of the experiment in the laboratory. The final goal was to obtain and deeply understand the law of the motion of the ball.

All the tasks are reported below in eight different worksheets with the corresponding computer screen when present. We are going to present all the details and the prospective analysis, based mainly on the teaching experiment of 2017, merely of those tasks that we are going to analyze in the following. The tasks which are not object of analysis will be presented briefly, just to provide an overall view of the experimentation.

### 11.1.1 Task 1

In Task 1, the students had first to watch on the computer shared by the whole group the video from the website of Galileo Museum (Florence) describing the motion of a ball along an inclined plane (the same of 2017 T.E.). After the vision of the video, students were asked to formulate some observations about the motion of the ball.

| |
|---|
| **First Worksheet (Task 1)**<br>*Task 1*<br>Look at the video about the experiment of the Inclined Plane of Galileo (Galileo Museum, Florence). |

| |
|---|
| What caught your attention in the video concerning the movement of the ball? Write down all the observations that come up to your mind. |
| **Corresponding computer screen (see Figure 10)** |

This request can be classified as a formulation task since students are asked to write some observations arising from the vision of the video.

### 11.1.2 Task 2

In Task 2, students had to explore a GeoGebra applet showing on the left the ball running along an inclined plane in which the angle of inclination of the plane is highlighted in green color and on the right a table displays in two different columns values of time and distance covered by the ball at each time. In particular, time intervals are equal to 1 second. Moving the blue point at the end of the inclined plane, it is possible to change its inclination. Resetting the simulation and making it run again, different values of time and space are obtained. Hence students had the possibility to explore how those values change when the inclination of the plane varies.

| |
|---|
| **Second Worksheet (Task 2)**<br>*Task 2*<br>Explore the GeoGebra file Galileo1.ggb.<br>You can move the blue point in order to modify the inclination of the plane.<br>a) What happens varying the inclination of the plane? Why?<br>b) According to you, are the conjectures you have made after the vision of the video verified? How can you prove them?<br>If otherwise they are not verified, how would you change them?<br>c) Can you find an equation describing the motion of the ball? Which one? Justify your answer. |
| **Corresponding GeoGebra screen (see Figure 11)** |

Task 2 initially consists in an explorational phase within the GeoGebra environment: changing the inclination of the plane, the table provides the values of time and distance traversed. Point (b) of the task is more a situation of validation. Students are asked to combine both the information provided by the initial video and the applet, validate their previous assumptions, and support them with an argumentative approach. The last question (c) requests to provide a reasonable equation describing the motion of the ball. The video of Galileo Museum shows the expression "$s: t^2$" where $s$ and $t$ are recognized by the students since they are familiar with the standard notations adopted in physics to denote distance and time. Even if the video does not give information on the meaning of the formula, it could direct the students to consider the right variables to formulate the equation of the motion law.

Before moving to the following tasks, the teacher introduced a mathematical discussion (Discussion 1) focused on Task 1 and Task 2. The main goal was that of sharing the possible conjectures and equations formulated by the groups and comment on them with an argumentative approach.

### 11.1.3 Task 3

Task 3 was a sort of complement to the previous activities. Students were involved in justifying their previous conjectures comparing them with the extra information provided by a new GeoGebra file. The table on the right of the applet presented two extra columns containing the numerical values of the first finite differences of distance and in the middle of the screen there was a cartesian plane showing a discrete graph of the distance-time relationship. Changing the inclination of the plane shown on the left, students could observe how the shape of the graph and the finite differences are related to the angle of inclination of the plane.

**Third Worksheet (Task 3)**

*Task 3*

In this activity you are asked to explain and justify your answers using the graphs and their numerical representations.

Open the file Galileo2.ggb and observe how the distance traversed varies with the angle of inclination of the plane.

a) Observe the shapes of the curves related to the variation of the inclination of the plane. How can you justify them?
b) How do the finite differences of the values of the $x$ and the $y$ of the points of the graph change with the variation of the inclination of the plane? Motivate your statements.
c) Do you have other observations? Which ones?

**Corresponding GeoGebra screen (Galileo2.ggb)**

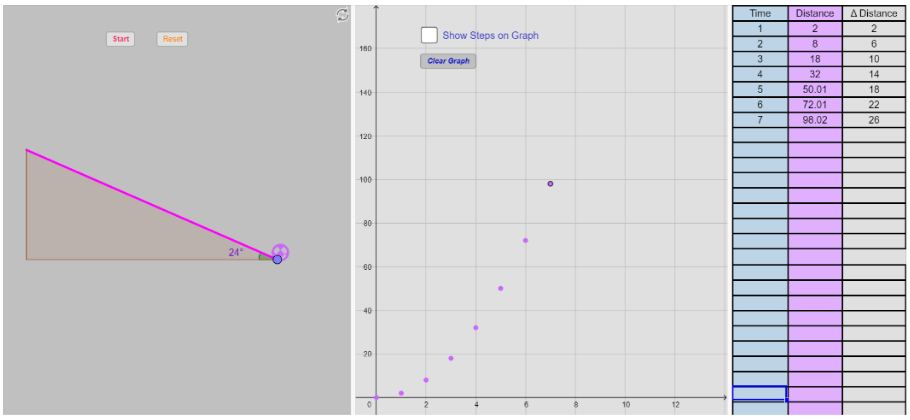



**Figure 26 –** The GeoGebra applet interface (Galileo2.ggb)

This activity was followed by a teacher-led discussion (Discussion 2) aiming at deeply investing the relationship between the distance-time graph and the angle of inclination of the plane, so enhancing second-order covariational reasoning processes.

### 11.1.4 Task 4

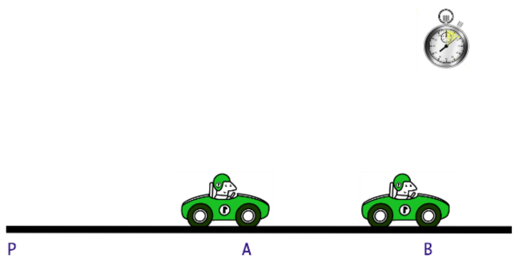
A new concept is introduced in Task 4. Firstly, students faced the concept of relative distance with its definition and an example referring to a car and distance traversed. Students were asked to make assumptions on how relative distances change in time and then, using a GeoGebra applet similar to the one of Task 3 with in addition a column reporting relative increments in distance, students were invited to verify their assumptions. The same approach was proposed to investigate the relationship between relative distances and the inclination of the plane. At the end of the group-work session, a mathematical discussion was introduced by the teacher with the aim of reasoning on the conjectures elaborated by the different groups. This task will not be object of analysis.

**Fourth Worksheet (Task 4)**

*Task 4*

A relative distance is defined as the ratio between the distance traversed by a body in a specific unit of time and the distance covered by the same body until that moment.

For example, if you look at the figure



we can observe a car left from point P, that in the unit of time Δt, has travelled the distance $\overline{AB}$ from point A to point B. The relative distance (RC) is the ratio between the distance $\overline{AB}$ and the distance $\overline{PB}$ covered by the car until that moment: $RC = \frac{\overline{AB}}{\overline{PB}} = \frac{\Delta s}{s}$.

a) Make some assumptions on how the relative distance changes in time. Justify your hypotheses.
b) Open the applet Galileo3.ggb, run the simulation and observe how the relative spaces of the ball change. Are your observations in accordance with the conjectures previously formulated? If not, how would you change your conjectures?
c) Vary the inclination of the plane. How do the relative distances of the ball change?
d) Verify your hypothesis with the applet: what can you observe? Motivate your answers. Looking at the applet, do other conjectures come to your mind? How do you justify them? Which of them can you prove?
e) Provide some examples inspired by real life in which talking about relative increments makes sense.

**Corresponding GeoGebra screen**

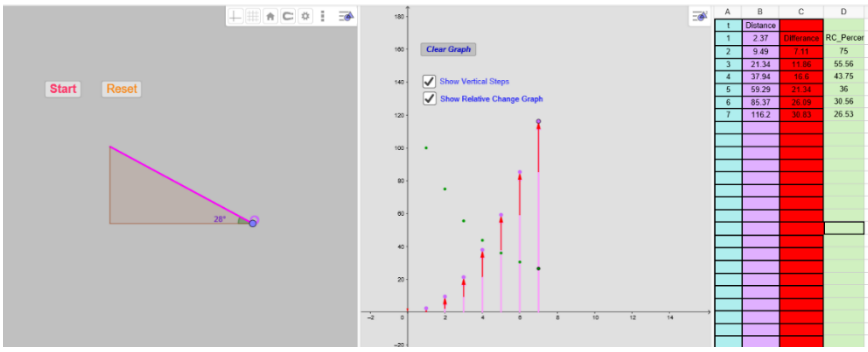



**Figure 27** - The GeoGebra applet interface (Galileo3.ggb)

## 11.1.5 Task 5

Task 5 was an activity of re-elaboration. Neither exploration was required, nor extra information was introduced. Students were involved in making the point on all the knowledge acquired in the previous activities and were invited to communicate it in a written and mathematical form. A few hours were devoted to this writing activity. These elaborates were assessed through the method of comparative judgement: all the details entailing this research study will be provided in Chapter 15.

**Fifth Worksheet (Task 5)**

*Task 5*

Thinking back to the work carried out on the inclined plane, write to schoolmates of another class to outline the work itself and, specifically, the relationship that describes and explains mathematically the motion of the ball along the inclined plane. This report should be a theoretical support for you and your schoolmates.

### 11.1.6 Task 6

Task 6 entailed the reading of the original text of Galileo describing the experiment on the inclined plane. Students read the text in their groups and then, with the help of the teacher, they made sure to have caught the meaning of all the words used by the author and how Galileo made the experiment with the tools available at his time. Moreover, a brief video, from Galileo Museum, provided some information on the cultural context of that historical period. Then students were involved in reproducing themselves the experiment of Galileo. In the physics laboratory they had the chance to verify directly the assumptions made by Galileo, detecting with a stopwatch the time taken by the ball to cover the whole plane and having the possibility to change the inclination of the plane. This task will not be object of analysis.

**Sixth Worksheet**

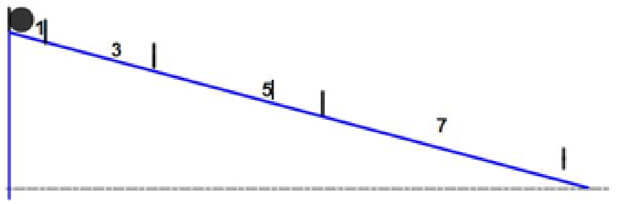
*Galileo and the Falling Bodies.*

Taken from "Discorsi e dimostrazioni matematiche intorno a due nuove scienze attinenti alla meccanica e ai movimenti locali".

[The text here reported is in Italian, its original language]

*“In un regolo, o vogliam dir corrente, di legno, lungo circa 12 braccia, e largo per un verso mezzo braccio e per l’altro 3 dita, si era in questa minor larghezza incavato un canaletto, poco più largo di un dito; tiratilo drittissimo, e, per averlo ben pulito e liscio, incollatovi dentro una carta pecora zannata e lustrata al possibile, si faceva in esso scendere una palla di bronzo durissimo, ben rotondata e pulita;*

*costituito che si era il detto regolo pendente, elevando sopra il piano orizzontale una delle sue estremità un braccio o due ad arbitrio, si lasciava (come dico) scendere per il detto canale la palla, notando, nel modo che appresso dirò, il tempo che consumava nello scorrerlo tutto, replicando il medesimo atto molte volte per assicurarsi bene della quantità del tempo, nel quale non si trovava mai differenza né anco della decima parte d’una battuta di polso. Fatta e stabilita precisamente tale operazione, facemmo scender la medesima palla solamente per la quarta parte della lunghezza di esso canale, e misurato il tempo della sua discesa, si trovava sempre puntualissimamente esser la metà dell’altro; e facendo poi l’esperienza di altre parti, esaminando ora il tempo di tutta la lunghezza col tempo della metà, o con quello delli due terzi o dei 3/4, o in conclusione con qualunque altra divisione, per esperienze ben cento volte replicate sempre s’incontrava, gli spazii passati esser tra loro come i quadrati de i tempi, e questo in tutte le inclinazioni del piano, cioè del canale nel quale si faceva scender la palla; dove osservammo ancora i tempi delle scese per diverse inclinazioni mantener esquisitamente tra loro quella proporzione che più troveremo essergli assegnata e dimostrata dall’autore. Quanto poi alla misura del tempo, si teneva una gran secchia piena d’acqua, attaccata in alto, la quale per un sottil cannellino, saldatogli nel fondo, versava un sottil filo d’acqua, che s’andava ricevendo con un piccol bicchiero per tutto ‘l tempo che guisa raccolte, s’andavano di volta in volta con esattissima bilancia pesando, dandoci le differenze e la palla scendeva nel canale e nelle sue parti: le particelle poi dell’acqua, in tal proporzioni de i pesi loro le*

| |
|---|
| *differenze e proporzioni de i tempi; e questo con tal giustezza, che, come ho detto, tali operazioni, molte e molte volte replicate, già mai non differivano d'un notabil momento."*  |
| **Seventh Worksheet (Task 6)**<br>*Task 6*<br>a) To reflect:<br>- Galileo hypothesizes that when two bodies fall from the same height and are not subjected to friction forces (but only to weight force) then they will reach the soil in equal times even if they have very different masses.<br>- How to verify this assumption? Galileo studied the motion of bodies with different masses making them roll along an inclined plane. He eliminated the friction force as much as he could and so he succeeded in studying the motion of free falling of bodies even without having at disposal sophisticated instruments to measure time and distance. The inclination of the plane makes the motion slower, hence more easily observable.<br>- Watch the video https://catalogo.museogalileo.it/multimedia/CadutaGravi.html<br>b) Let us reconstruct the plane of Galileo in the laboratory. |
| **Corresponding screenshot of the video by Galileo Museum**<br><br>**Figure 28** - Screenshot of the video by Galileo Museum |

### 11.1.7 Task 7

Last task consisted in a questionnaire conceived for investigating the general perception of the students about the activity. Question 1 was formulated in order to obtain feedback about the structure of the activity and the use of multiple artefacts aimed at investigating the same problem from different perspectives; question 2 aimed at collecting some insights on the potentiality of GeoGebra applets according to students' point of view and in the end question 3 left carte blanche for comments and suggestions to improve the teaching experiment.

**Eight Worksheet (Questionnaire)**

*Questionnaire*

1. The law of the inclined plane was obtained adopting different tools (video, GeoGebra, Galileo's text, experiment). Do you think it was worthy exploring the problem with these different tools? Why?
2. According to you, which are the advantages in the use of GeoGebra applets? Which of their functionalities did you take advantage of in order to verify the conjectures made during the vision of the video and the formulation of the equation relating time and distance?
3. If this experience was proposed to your 10th grade schoolmates next year, which modifications would you suggest making? Do you have any other inputs to give or comments to share?

### 11.1.8 Prospective analysis

The expected results of this T.E. are in line with the prospective analysis already outlined in 10.1.3 so what we are going to expound in the following are the substantial differences and improvements with respect to the 2017 T.E.. Firstly, a comment about the representations involved: the video and the first GeoGebra applet have a complementary function has already outlined in 10.1.2. The second GeoGebra applet has instead a construct deeper understanding role because it provides not only numerical additional information, first finite differences of distance, but also a discrete representation of the values of time and distance that was completely absent in the previous artefacts and that constitutes a mediator between the simulation of the inclined plane on the left and the table of values on the right. Hence, in this T.E., the graphical representation component is much more advanced. Since in Task 3 students could envision a discrete version of the *s*-*t* graph and work on it during an entire working group session, we expect students to better succeed in the elaboration of a mathematical formula thanks also to the support of the second GeoGebra applet, to explore more in detail the relation between the shape of the graph and the angle of inclination of the plane and to constantly establish a connection between the mathematical model and its physical interpretation given their wider knowledge. Given the findings emerged from the 2017 T.E., we expect students to express in a descriptive way when referring to the video artefacts reasoning both in qualitative and quantitative terms and to succeed in envisioning some possible variations of the situation proposed when changing the angle. We also expect some references to velocity to emerge given their more extensive preparation in physics. When referring to the GeoGebra applet, we expect students to move their discourse on an analytical or even objectified level and to achieve more easily the elaboration of a general mathematical formula, maybe also adopting their knowledge about the decomposition of forces along the inclined plane. Finally, after having worked with the second GeoGebra applet, we hypothesize students would describe with a holistic approach the distance-time graph highlighting some properties of its trend and relating them to finite differences. The experiment and the text of Galileo constitute the final steps in this T.E. hence their influence in students' reasoning will not emerge in the episodes analyzed in the following. Concerning the teacher, we assume she will manage the whole T.E. with great ability since it is a replication of the previous one and we expect to recognize the same adaptive teaching strategies identified in the 2017 T.E. analysis.

## 11.2 DATA ANALYSIS

In this section we illustrate in detail seven episodes examined adopting the Timeline tool and using the double lens of the macro and micro analysis. In particular:

- Episode 1 (11.2.1) and Episode 2 (11.2.2) are excerpts from the first classroom discussion (Discussion 1) which was introduced after the first two tasks. Students had worked in small groups on the video and the first GeoGebra applet. These episodes correspond to the episodes analyzed during 2017 T.E. and so they constitute a fertile ground for a strict comparison of the results of the discussion, reflecting on how the different background of the students influenced their approach to second-order covariation;
- Episode 3 (11.2.3) and Episode 4 (11.2.4) come from the working group session during which students worked on the second GeoGebra applet displaying the distance-time graph and the first finite differences of distance. These are the only two episodes referring to work-group activity and in particular they refer to two different groups. They will help us to dig into students' reasoning when specifically dealing with the covariation of the parameter (the angle of inclination of the plane) and the shape of the distance-time graph. In Section 11.2.5 we will comment briefly on the results emerging from the other three groups, focusing exclusively on their written productions;
- Episode 5 (11.2.5), Episode 6 (11.2.7), and Episode 7 (11.2.8) are excerpts from the second teacher-led discussion (Discussion 2) which took place after the group-work session involving the second GeoGebra applet.

During the tasks, the students worked divided in five groups that we are going to denote with the letters A-B-C-D-E. The letter accompanying the name of the student in the transcript denotes the group to which the student belonged. Throughout the whole analysis, seven different artefacts will make their appearance: the video (V), the GeoGebra applet (G), the formulas (F), the table of numerical values (T), the finite differences computed by the students (FD), the interactive whiteboard (IW) and the blackboard (B).

### 11.2.1 Episode 1 (Discussion 1, 09:15-12:32)

This episode is an excerpt from the first teacher-led discussion: the teacher is asking her students about the observation they wrote watching the video and then working with the GeoGebra applet. Group A starts sharing with the whole classroom some considerations about the video and then Silvia turns to group B for other claims.

| | **Timing** | **Who** | **Utterances** | **Gestures** |
|---|---|---|---|---|
| 64 | 00:09:15 | Teacher | Then? | |
| 65 | 00:09:20 | Fabio$_B$ | The more the angle of inclination of the plane comes closer to 90°, the more the speed and so the descent time increases because the ball reaches before its maximum velocity and its maximum acceleration… | |
| | | | […] | |
| 66 | 00:10:25 | Teacher | According to you, in which situation will we reach the maximum velocity? | |
| 67 | 00:10:30 | Fabio$_B$ | During the last section… […] | |
| 68 | 00:10:50 | Teacher<br>Fabio$_B$<br>Teacher | Ok, hence if we have the inclined plane, the maximum velocity is in the last section,<br>The last section [simultaneously with the teacher]<br>but before you said "the more the angle of inclination of the plane comes closer to 90°, the more the speed increases until it reaches its maximum velocity"… | |
| 69 | 00:11:03 | Fabio$_B$ | Maximum for that angle… | |

| | | | | |
|---|---|---|---|---|
| 70 | 00:11:05 | Teacher | Maximum for that angle… And the maximum speed by far? | |
| 71 | 00:11:12 | Fabio$_B$ | It depends on the force you apply. In this case I apply only the weight force but if I push it the force changes […] | |
| 72 | 00:11:36 | Teacher | No no, we just have the weight force. Will we have a maximum speed by far? And a minimum speed by far or not? | |
| 73 | 00:11:50 | Fabio$_B$ | The minimum speed [is reached] with an angle of 0,0000001. It can't be 0° otherwise there won't be the motion… and so the speed would be 0. | |
| 74 | 00:12:10 | Teacher | If we take an angle of 0,00001 we have a certain velocity and then it increases until… | |
| 75 | 00:12:25 | Fabio$_B$ | Until the angle is of 90° […] | |
| 76 | 00:12:32 | Teacher | | *[The teacher writes on the IW]* |

With the aim of making the conversation flow, the teacher clearly addresses group B [64] (**Figure 29** – interaction flowchart): Fabio takes the word and shares some considerations trying to relate the angle of inclination of the plane to the speed of the ball. The student states that "the more the angle comes to 90°, the more the descent speed of the ball increases" [65]: while Fabio speaks, he makes some iconic and metaphoric gestures simulating the angle of inclination and the ball descending the plane. Fast-talking, Fabio claims, incorrectly, that also the descent time increases, but it seems that he intended "decreases" because immediately after he tries to support his statement attributing it to the fact that the ball reaches first its maximum velocity and acceleration. The claim about acceleration is incorrect too but we recall that students had not studied it yet during their physics course, so they use this term in an intuitive way and as something related to velocity. What is emerging in Fabio's reasoning is covariation between the angle and the speed (**Figure 29** – L2, COV row); the teacher wisely grasps what is emerging and values it writing on the IW a sentence that condenses it: "the more the angle comes closer to 90°, the more the speed increases" (**Figure 29** – inscriptions row); now this claim is under the eyes of the whole classroom. At [66], the teacher tries to enhance covariation between the angle and speed asking Fabio in which situation the ball will reach the maximum velocity. The student

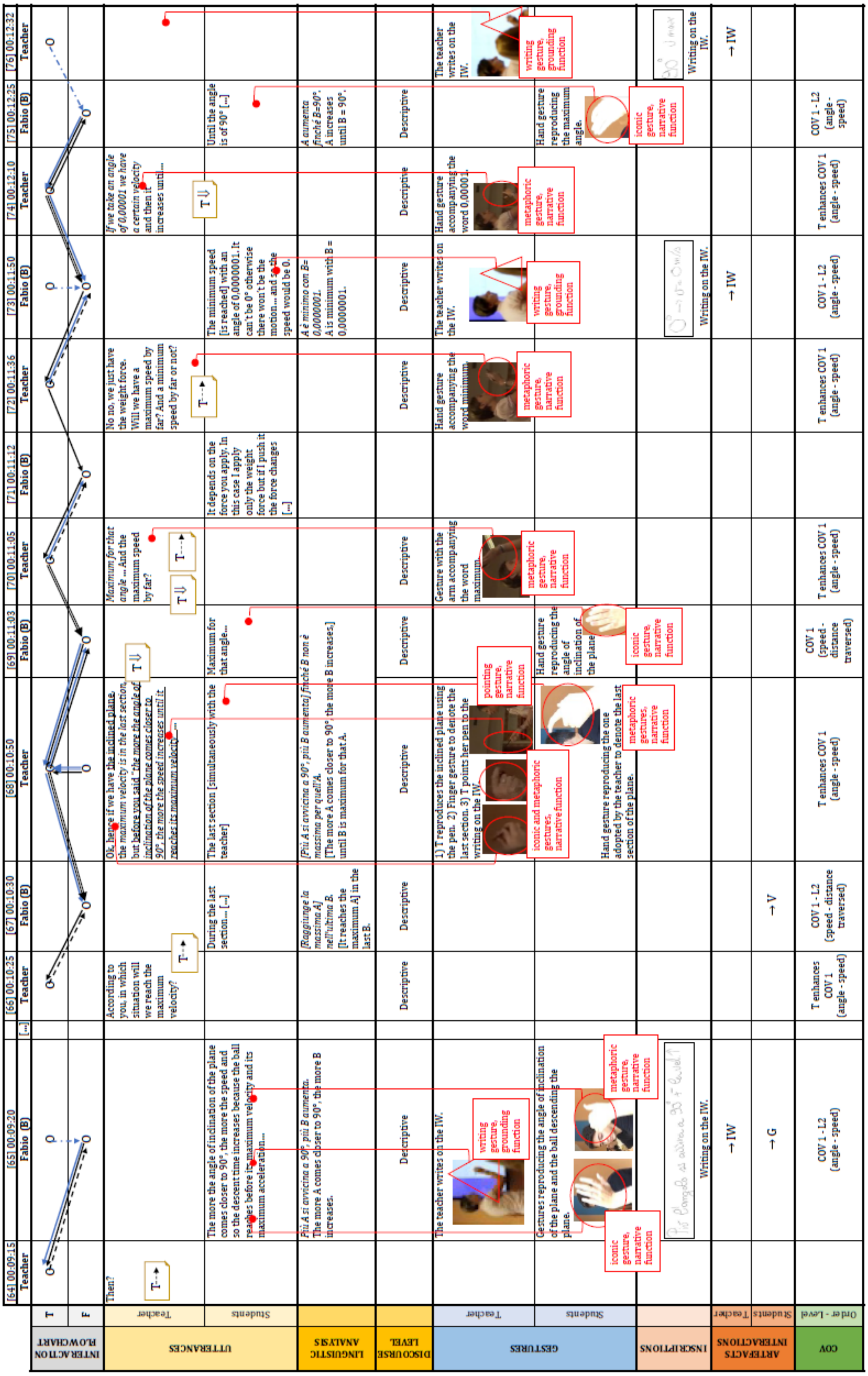


**Figure 29 –** Timeline 1 (Galileo 2019)

answers that the ball reaches its maximum speed in the last section [of the inclined plane; he refers to the sections shown in the video] so he reduces his reasoning into a covariation between speed and distance traversed (**Figure 29** – COV row). The teacher does not abandon her purpose to consolidate covariational reasoning with the angle of inclination: playing the semiotic game, she recalls what Fabio said with an approval tone and simulates the situation described by the student with a pen; then she asks again clarification, but Fabio still crushes his reasoning onto a covariation between speed and distance; Silvia tries to explicitly ask for a situation in which an "absolute speed by far" [70] is reached accompanying her words with a metaphoric arm gesture (**Figure 29** – gesture row). Initially Fabio responds that it depends on the force applied [71], but the teacher precises that only the weight force is acting and again reformulates her question [72]. Finally, Fabio introduces a covariational reasoning between the angle and the speed: the student states that the minimum speed is reached with an angle of 0,0000001 (very small, but different than 0). Silvia rephrases his words with an approval and supportive tone and then relaunches the word to Fabio who says that the velocity increases until an angle of 90° is reached [75]. In both cases, [73] and [76], the teacher writes on the IW the values of the limit angles and the associated velocities (**Figure 30**). The level of covariational reasoning is low (L2) in this first part of the discussion: the student does not seem ready yet to covary the angle of inclination with the speed; it is essential the role of the teacher in highlighting those elements of Fabio's reasoning that are relevant for the mathematical discussion and her wise questioning in transitioning toward covariational reasoning between the angle and the speed. The level of the discourse can be identified as descriptive (**Figure 29** – discourse level row).

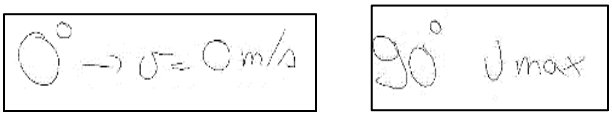



**Figure 30** – Inscriptions of the teacher on the IW

### 11.2.2 Episode 2 (Discussion 1, 28:34-31:32)[22]

The teacher is concluding the roundup during which she asked to the five groups their observations. Now it is the time of the last group, group E, and Valeria, as a representative of the group, takes the word. We will notice that during this first discussion the students do not succeed so easily in the elaboration of a mathematical formula but given their more extensive background, they correctly intuit the shape of the distance-time graph and are able to explain the dependence on the inclination angle from a representational point of view.

[22] The analysis of this episode has been partially presented in Bagossi (2021a).

| | Timing | Who | Utterances | Gestures |
|---|---|---|---|---|
| 77 | 00:28:34 | Valeria$_E$ | Hence, we noticed that first finite differences of time were always of 1 second, except for the last one, while in one second the first finite differences of distance increased more and more so we noticed that there was an acceleration. | |
| 78 | 00:28:54 | Teacher | Ok, in one second it covered more distance and so there was acceleration. And then you noticed that the second finite differences of distance…. | |
| 79 | 00:29:06 | Valeria$_E$ | They were constant and the third were equal to 0. | |
| 80 | 00:29:13 | Teacher | The second were constant and the third, always of distance, were null. | |
| 81 | 00:29:24 | Valeria$_E$ | Then, we thought that the graph could be of second degree since second finite differences are constant and also because we knew that the formula of the acceleration is $s/t^2$. | |
| | | | […] | |
| 82 | 00:30:25 | Teacher | And you assumed that the graph could be… | |
| 83 | 00:30:29 | Valeria$_E$ | A curve that before had an inclination almost horizontal and then became always more vertical. | |
| 84 | 00:30:37 | Teacher | Could you draw it? | |
| 85 | 00:30:49 | Valeria$_E$ | We divided the horizontal axis that was the one of time in various sections representing one second and then we noticed that in the time of one second the inclination was always more vertical. | *She draws the graph on the interactive whiteboard.* |
| 86 | 00:31:17 | Teacher | Ok, because in one second it covered always more distance. | |

| | | | | |
|---|---|---|---|---|
| | | | And if the angle changes, what happens according to you? | |
| 87 | 00:31:20 | Valeria$_E$ | If the angle changes, the uphill is faster. | |
| 88 | 00:31:28 | Teacher | And if you should make another graph changing the angle? | |
| 89 | 00:31:32 | Valeria$_E$ | | *She draws another curve, more inclined.* |

Valeria takes the word and begins describing what they did during they work-group session. She immediately starts to expose how they worked with finite differences: they noticed that first finite differences (f.d.) of time were always of one second while first f.d. of distance increased more and more and they, incorrectly, related this increase to acceleration [77]. Silvia does not focus anymore on the misconceptions emerging during the mathematical discussion, but instead starts taking notes on the IW about f.d.. While pointing with the pen to the IW (**Figure 33** – gesture row), the teacher revoices Valeria's word and asks for the order of f.d.. The student claims that the second f.d. of distance are constant and the third are null [79]. While writing on the IW, the teacher repeats Valeria's words [80] and then the student continues exposing and says that "the graph could be of second degree because second f.d. are constant and also because we knew the formula of acceleration is $s/t^2$" (they make this assumption about the formula as a generalization of the formula about velocity) [81]. Silvia continues taking notes on the IW revealing to students the relevance of those conjectures. If until this moment the level of the discourse could be identified as analytical given the strong rely on finite differences, now something different is revealing: f.d. are related to the degree of the graph, and as a consequence to its shape, and the argumentation is also supported by the presumed formula of acceleration containing time to the second power. The language adopted seems to reveal a blend of elements coming from the various representations (algebraic, numerical, and graphical) that now are connected in a coherent sentence: we are going to denote this level of the discourse as *interpretative* (**Figure 33** – discourse level row). At [82], Silvia asks more details about the graph so to facilitate the flow of the discussion; Valeria replies describing not only with her words but also adopting some metaphoric gestures (**Figure 31**) "a curve that before had an inclination almost horizontal and then became always more vertical" [83].

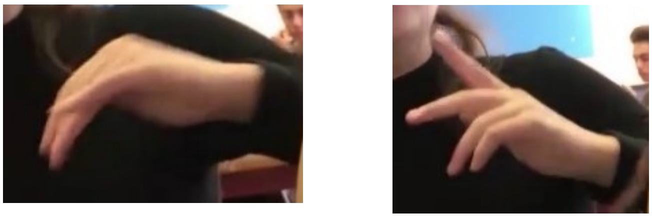


**Figure 31** – Hand gestures performed by Valeria while describing the graph of the function

The teacher invites Valeria to draw the curve on the IW [84]: the student comes closer to the IW and starts narrating how they reasoned to elaborate the shape of the graph (see **Figure 32**, left side). They did not use a global approach but instead reasoned in terms of fixed intervals of time [85]. The teacher revoices her words with an approval tone and then asks what happens when the angle changes [86]. This question represents the opportunity to enhance higher order covariational reasoning, modifying the angle, that is the parameter, and observing how it affects the situation as a whole. At [87], Valeria clearly answers that "If the angle changes, the uphill [of the graph] is faster" and then, under request of the teacher, draws on the IW a second curve, more inclined (see **Figure 32**, right side).

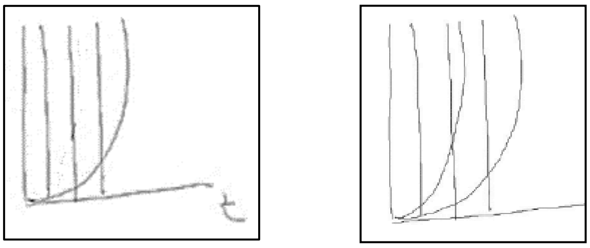



**Figure 32** – On the left, the first curve drawn on the IW after having divided the horizontal axis in intervals of 1 second. On the right, a second curve, more inclined, is added by the student

In the last part of this episode the level of the discourse turns to objectified: the student is able to envision the distance-time relationship in a smooth continuous way (**Figure 33** – COV row) and to represent it in a graphical form. Once again, it is the clever questioning of the teacher that gives the right direction to the discussion and allows the transition toward second-order covariational reasoning. Specifically, in this episode, COV 2 manifests in a qualitative and representational form: even if the students do not elaborate a mathematical formula, they grasp the distance-time relationship and express it in a graphical form [85; 89]. The dependence on the inclination angle manifests in the properties of the graph function and specifically in a change of the graph slope.

Finite differences of time and distance constitute the algebraic tool which mainly leads the students to the inference of the properties of the *s*-*t* relationship [77; 79], not only algebraically

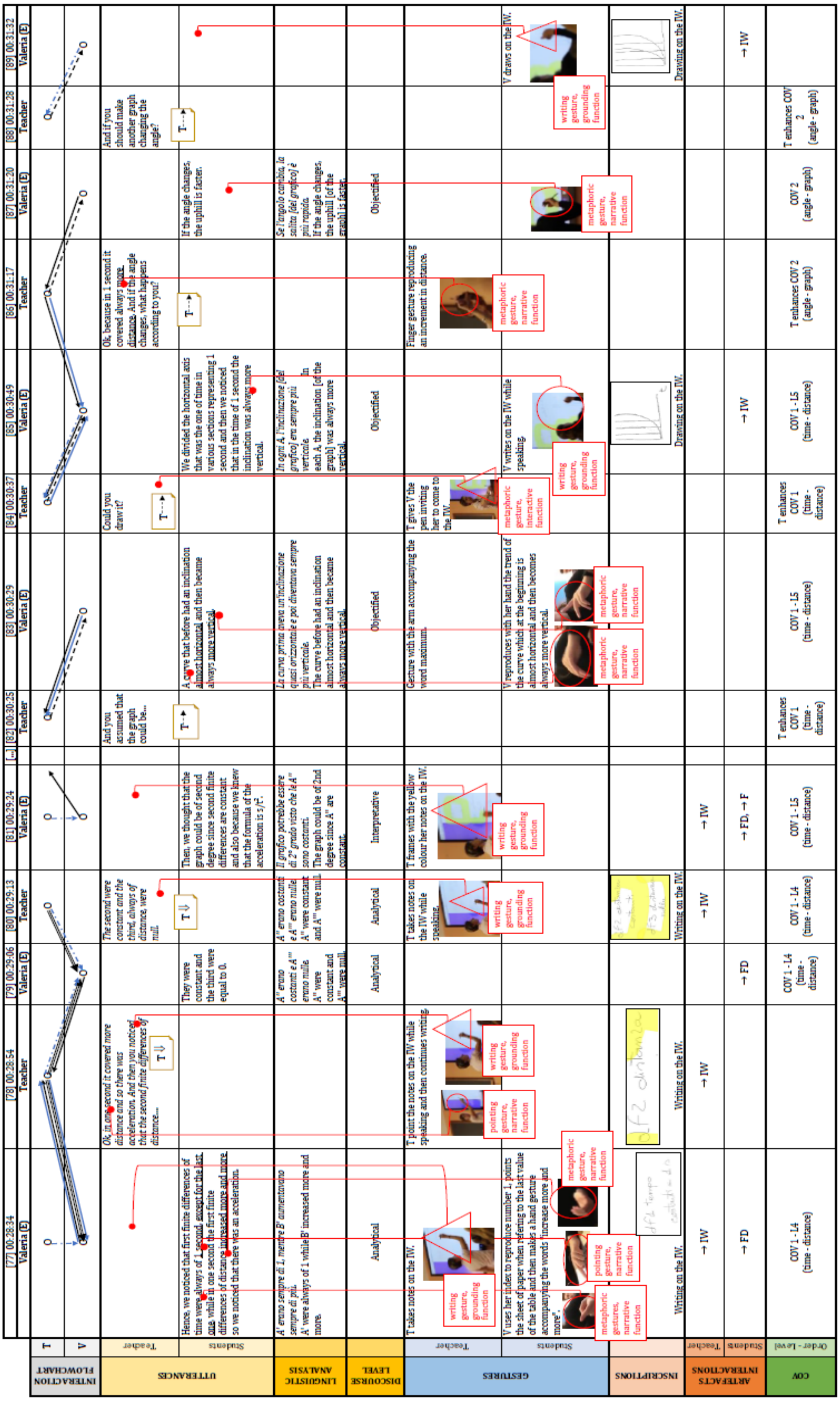


**Figure 33** – Timeline 2 (Galileo 2019)

but also graphically. Actually, it is the possibility of changing the inclination of the plane, a metavariation (Hoffkamp, 2009; 2011), that better allows to explore the dependence on the inclination angle, so changing the situation as a whole and enhancing COV 2. The two following episodes are excerpts from the working-group session which followed immediately the previous discussion: the students specifically worked on the second GeoGebra applet and tried to answer to the questions presented on the third worksheet (see 11.1.3). The two episodes refer to two different groups, B and E. In 11.2.5 we are going to comment briefly on the data from worksheets of groups A, C and D.

### 11.2.3 Episode 3 (Group-work B, 01:02:15-01:03:00)

This episode refers to group B: they are at the beginning of their working group session. Matilde reads out loud question a) on their worksheet and then students start sharing their observations.

| | **Timing** | **Who** | **Utterances** | **Gestures** |
|---|---|---|---|---|
| 90 | 01:02:15 | $\text{Matilde}_B$ | *Matilde reads out loud question (a) on the worksheet.* | |
| 91 | 01:02:22 | $\text{Matteo}_B$ | Practically, the more the angle increases, the more the inclination [of the graph] increases; the smaller is the angle, the more comes like this… | *Matteo describes with his pen the trend of the graph on the screen.* |
| 92 | 01:02:28 | $\text{Fabio}_B$ | But why? She [the teacher] wants that we find a function… | |
| 93 | 01:02:36 | $\text{Matteo}_B$ | Wait, this is a table with *x*, *y* and the differences. It [the graph] is of second degree hence this [pointing second f.d. on their sheet] is the half…no, the double of the coefficient… I seem to remember from the last year… | |
| 94 | 01:02:55 | $\text{Fabio}_B$ | Yes, yes. | |
| 95 | 01:03:00 | $\text{Matteo}_B$ | So, 3.51 is the coefficient. | |

Immediately after Matilde’s reading, Matteo takes the word and tries to share his observations: at [91] he correctly notices that “the more the angle increases, the more the inclination [of the graph] increases” and he accompanies his words with some gestures with a narrative function through which using his pen he follows the trend of the discrete curve displayed on the GeoGebra applet (**Figure 34** – gesture row). Despite the correctness of Matteo's statement, his mate Fabio insists that the teacher is not requesting only qualitative observations but more than that "a function" [92]. Then Matteo observes with deeper attention the various information provided by the GeoGebra applet [93]: the table of finite differences and the graph that is of second degree; hence remembering from the past year, Matteo recalls that second f.d. (he points at their computations on their sheet, **Figure 34** – inscriptions row) are the double of the coefficient of the function. So, they divide by two the number they obtained as value of second f.d. (7.02) and Matteo writes 3.51 as coefficient of their function [95] ($y= 3.51 \cdot x^2$ – inscriptions row, **Figure 34**). The level of the discourse is initially descriptive when Matteo limits himself to elaborate in a qualitative way the inputs provided by the GeoGebra applets, but then turns to objectified when the distance-time graph is related to its mathematical properties and in particular to a mathematical formula. Matteo succeeds in elaborating that generally the coefficient of a second-degree function is the half of second f.d. and then translates it into a numerical value for the specific angle they have chosen on their applet which returns a coefficient of 3.51. What is actually missing in students' reasoning is a physical interpretation of the phenomenon explored: their remarks are based on a property-oriented view of function thanks to which they relate finite differences to the shape and degree of the polynomial function. The level of covariational reasoning is clearly of an order higher than the first: Matteo's statement [91] locates on the meta-variation level because the student is able to envision how a changing in the parameter affects the situation as a whole. At [93] Matteo still reasons at second-order covariation (**Figure 34** – COV row), but this time he mainly focuses on a general mathematical formula describing the distance-time curve globally and in this formula the dependence on the parameters affecting the real phenomenon is enclosed in the specific value of second f.d. At [95], reasoning crushes onto an intermediate order since Matteo tries to find the coefficient for a specific situation, i.e., the angle shown in the applet.

Finally, the interaction flowchart reveals the strong role of Matteo in moving forward the discussion and in contributing with punctual and relevant observations, thanks also to Fabio's provocative question (**Figure 34** – interaction flowchart).

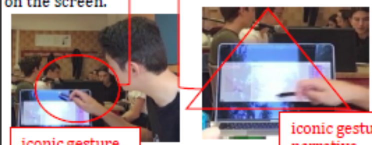
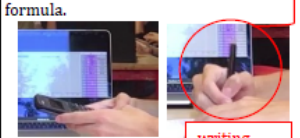
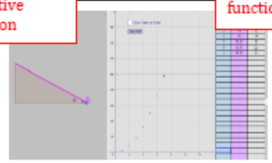
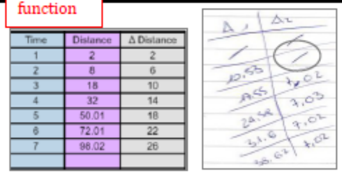
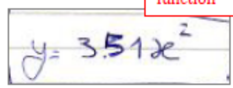
| | | [90] 01:02:15 Matilde (B) | [91] 01:02:22 Matteo (B) | [92] 01:02:28 Fabio (B) | [93] 01:02:36 Matteo (B) | [94] 01:02:55 Fabio (B) | [95] 01:03:00 Matteo (B) |
|---|---|---|---|---|---|---|---|
| INTERACTION FLOWCHART | Mati / Matt / F | O | O | O | O | O | O |
| UTTERANCES | | *Matilde reads out loud question a) on the worksheet.* | Practically, the more the angle increases, the more the inclination [of the graph] increases; the smaller is the angle, the more comes like this… | But why? She [the teacher] wants that we find a function… | Wait, this is a table with x, y and the differences. It [the graph] is of second degree hence this [pointing the second f.d. on their sheet] is the half…no, the double of the coefficient… I seem to remember from the last year… | Yes, yes. | So, 3.51 is the coefficient. |
| LINGUISTIC ANALYSIS | | | *Più l'angolo aumenta, più l'inclinazione [del grafico] aumenta.* The more A increases, the more the inclination [of the graph] increases. | | *Il grafico è di secondo grado quindi le A'' sono il doppio del coefficiente.* The graph is of second degree hence A'' are the double of the coefficient. | | |
| DISCOURSE LEVEL | | | Descriptive | Objectified | Objectified | | Objectified |
| GESTURES | | | Matteo describes with his pen the trend of the graph on the screen.  iconic gesture, narrative function; iconic gesture, narrative function | | Matteo points at the table of f.d. and then at the value of 2nd f.d. on their worksheets.  pointing gesture, narrative function; pointing gesture, narrative function | | Matteo computes the half of 7.02 with the calculator and then writes it in the formula.  writing gesture, grounding function |
| INSCRIPTIONS | | |  GeoGebra applet | |   | |   Matteo writes the 3.51 coefficient in the formula. |
| ARTEFACTS INTERACTIONS | | | → G | | G → | | |
| COV | Order - Level | | COV 2 | | COV 2 | | Intermediate order |

**Figure 34 –** Timeline 3 (Galileo 2019)

### 11.2.4 Episode 4 (Group-work E, 47:12-48:06)

This episode refers to group E, the same group that firstly succeeded in relating numerical values of time and distance to the shape of the graph (11.2.2). In the following excerpt, they try to answer to question (a) on worksheet 3 (11.1.3).

| | Timing | Who | Utterances | Gestures |
|---|---|---|---|---|
| 96 | 00:47:12 | Valeria$_E$ | The distance traversed in one second increases while increasing the inclination of the plane. | |
| 97 | 00:47:21 | Arianna$_E$ | In this way it seems you speak of the single point… | |
| 98 | 00:47:55 | Arianna$_E$ | I'd say… The greater is the angle, I'd say the angle so that we don't have problems of definition, the greater is the angle, the faster the parable will grow… | |
| 99 | 00:48:06 | Valeria$_E$ | The greater is the angle, the greater is the distance traversed in one second and the faster the parable will grow. | |

Looking at the GeoGebra applet on their computer screen (sensing gaze, gesture row – **Figure 35**) Valeria tries to answer to question (a): she observes that "the distance traversed in one second increases while increasing the inclination of the plane" [96]. She relates the inclination of the plane with both distance and time, condensing them in the distance traversed in one second (i.e., first f.d.). Arianna immediately reacts to her mate's claim contesting the formulation of her utterance: firstly, Arianna states that in that way the utterance seems referred to a "single point" [97], then rephrases Valeria's claim saying that "the greater is the angle, the faster the parable will grow" [98]. Her reformulation suggests that instead of focusing on what happens on a fixed interval of time, one second, she favors a description that relates the variation of the angle to the function, the parable, as a unicum and while speaking Arianna follows with her finger the trend of the curve displayed on the screen (**Figure 35** – gesture row). Finally, Valeria welcomes Arianna's suggestion and tries to formulate an answer that could be written on their worksheet and that keeps into account both points of view: "the greater is the angle, the greater is the distance traversed in one second and the faster will grow the parable" [99]. In this excerpt, even if the order of covariational reasoning is COV 2, two different approaches emerge. Initially Valeria focuses on a covariation between finite differences and the angle implicitly interpreted as a parameter: her discourse is objectified and mainly insists not only on a quantitative aspect, but also on a local point of view rather than a global one. Then Arianna moves the attention on the distance-time graph conceived as a whole and co-varies, or more precisely meta-varies, it with the angle of inclination. The mathematical discourse, already objectified in its initial stage, becomes interpretative when the inputs from the table of values and the graph, are coordinated in a unique sentence in which the physical phenomenon, i.e., the inclined plane, is related to its mathematical representation, i.e., the parable. Finally, the modality in which the students of group E use the GeoGebra applet is ascending because their approach is mainly explorative (artefacts interaction row – **Figure 35**).

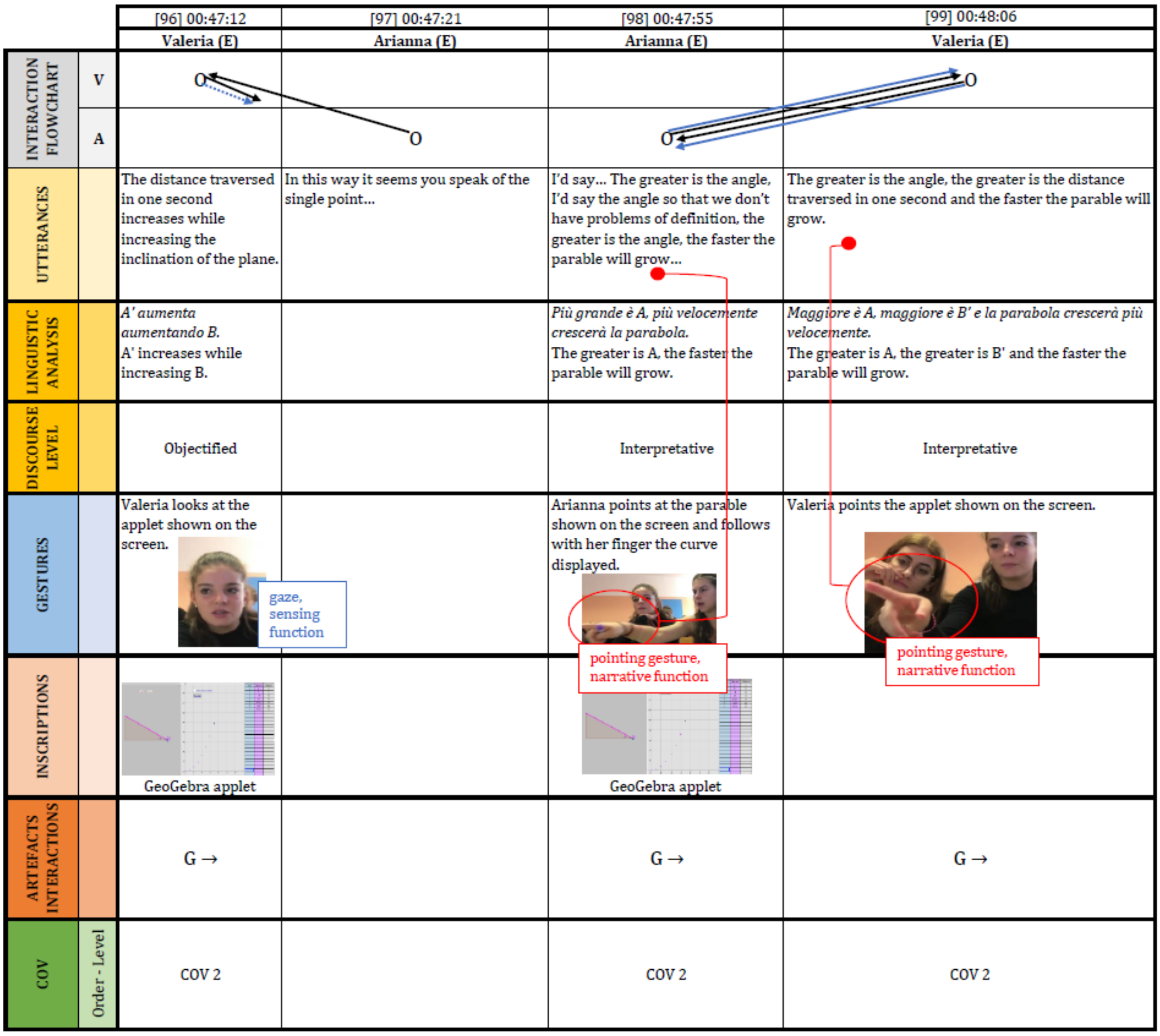



**Figure 35** – Timeline 4 (Galileo 2019)

### 11.2.5 Data from Task 3 of group-work A-C-D

In this paragraph we briefly report also the relevant considerations produced by the other three groups on their worksheets. Group A answered to question (a) saying that “As the angle increases, the curve comes always closer to *y*-axis, because thanks to acceleration it covers the same distance but faster” [100] (**Figure 36**).

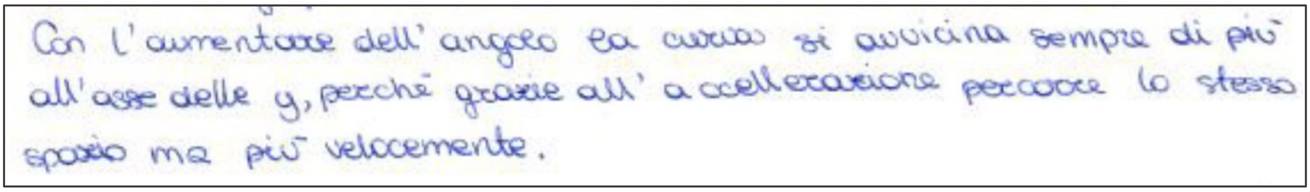
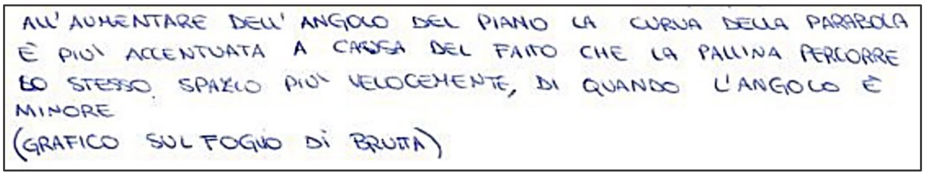

Con l'aumentare dell'angolo la curva si avvicina sempre di più all'asse delle y, perché grazie all'accellerazione percorre lo stesso spazio ma più velocemente.

**Figure 36 –** Group A’s answer to question (a)

Group C instead accompanied their answer also with a graphical representation to support their statement. Answer to question (a) reports: “As the angle of the plane increases, the curve of the parable is more accentuated because the ball travels the same distance faster respect to when the angle is smaller (graph on the nasty sheet)” [101].


ALL'AUMENTARE DELL'ANGOLO DEL PIANO LA CURVA DELLA PARABOLA È PIÙ ACCENTUATA A CAUSA DEL FATTO CHE LA PALLINA PERCORRE LO STESSO SPAZIO PIÙ VELOCEMENTE, DI QUANDO L'ANGOLO È MINORE
(GRAFICO SUL FOGLIO DI BRUTTA)

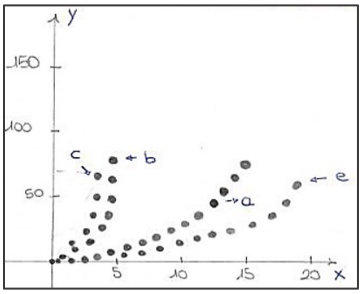



**Figure 37** - Group C’s answer to question (a)

**Figure 37** shows on the left the answer of group C and on the right their graphical representation in which they drew four different curves corresponding to four different values of the angle of inclination of the plane. Moreover, their answer to question (b), the one about finite differences, says that “when the angle is greater, first finite differences are greater, because they are proportional to the angle of inclination of the plane” [102] (**Figure 38**).

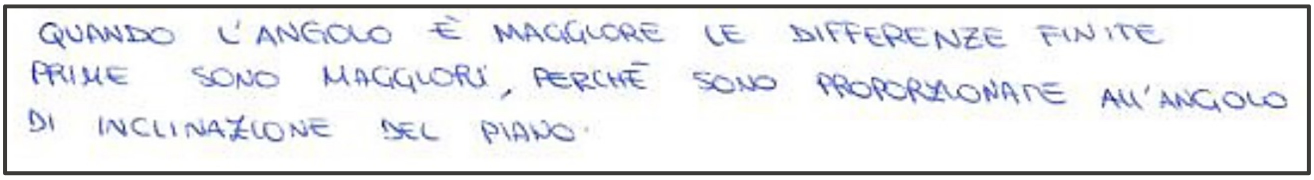

QUANDO L'ANGOLO È MAGGIORE LE DIFFERENZE FINITE PRIME SONO MAGGIORI, PERCHÉ SONO PROPORZIONATE ALL'ANGOLO DI INCLINAZIONE DEL PIANO.

**Figure 38** - Group C’s answer to question (b)

Finally, group D stated that “increasing the angle the curves are more inclined. And so, decreasing it the curves will be less inclined. This because the curve of the graph represents the acceleration of the ball as α increases” [103]. **Figure 39** shows on the left their written answer and on the right a drawing in which they showed two different possible curves for two different angles of inclination where in particular α is supposed smaller than β.

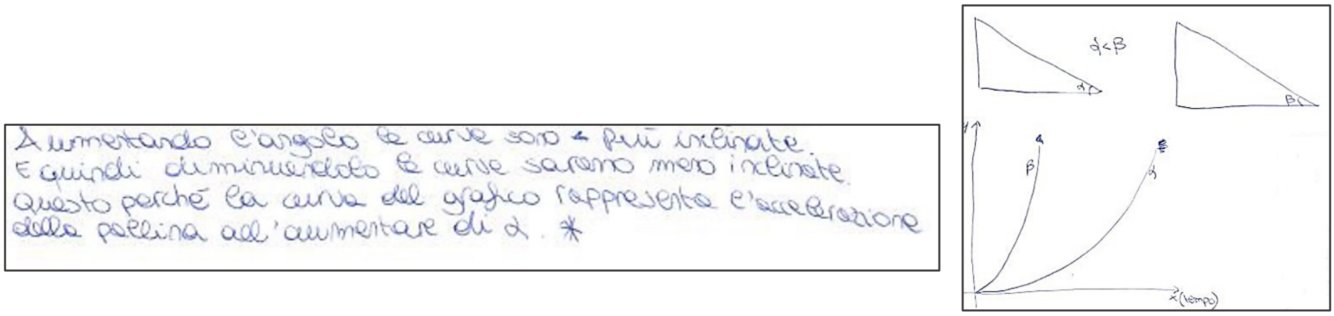



**Figure 39** - Group D’s answer to question (a)

The following episodes are excerpts from the discussion following the working group session about Task 3 (Discussion 2). This discussion was conducted at the beginning of a new lesson, and it took nearly one hour and a half. The teacher had the possibility to read the groups’ answers before starting the classroom discussion.

### 11.2.6 Episode 5 (Discussion 2, 07:35-08:50)

This episode locates at the beginning of the classroom discussion: the teacher opens the discussion asking clarifications about the answers they wrote on worksheets. Her aim seems to relate the increase in distance traversed to the numerical representation, meaning first f.d. provided in the table. Students initially refer to the graphical representation, but then Silvia directs their attention elsewhere.

| | Timing | Who | Utterances | Gestures |
|---|---|---|---|---|
| 104 | 00:07:35 | Teacher | What does it mean that in equal times the distances that we have increase? | |
| 105 | 00:07:41 | Chiara$_B$ | The ball to cover a larger distance must accelerate. | |
| 106 | 00:07:50 | Teacher | The ball must change its velocity, increase its velocity. In some way it has to accelerate.<br>How do we know from here that our ball in equal times does not traverse equal distances? | |

| | | | | |
|---|---|---|---|---|
| 107 | 00:08:11 | Giulia$_C$ | The curve starts slower and then goes higher and higher. | |
| 108 | 00:08:22 | Teacher | Ok, for sure it stars slow and then it goes up and up. | |
| 109 | 00:08:28 | Fabio$_B$ | From the first finite differences equal…in the table. | |
| 110 | 00:08:32 | Teacher | There would be equal first finite differences if… | |
| 111 | 00:08:38 | Fabio$_B$ | If it always goes at the same speed. | |
| 112 | 00:08:45 | Teacher | Ok, if we had first finite differences equal, it would have always the same speed, but she [Giulia] says that at a certain point it drives up so… what does it mean? | |
| 113 | 00:08:48 | Giulia$_C$ | It goes faster. | |
| 114 | 00:08:50 | Teacher | It goes faster. | |

At [104] the teacher asks the whole classroom what it means that "in equal times the distances that we have increase". Chiara, from group D, replies saying that it is due to acceleration [105] and Silvia, playing the semiotic game, rephrases her words with an approval tone [106] and introduces some metaphoric gestures indicating a variation in speed (gesture row – **Figure 40**). Immediately after she poses a new question in order to deeply investigate that issue and asks where they can deduct from the GeoGebra applet (she points at the IW displaying it; inscriptions row - **Figure 40**) that information [106]. Giulia answers referring to the graph: "The curve starts slower and then goes higher and higher" [107]. Her claim unveils a blended language: the term "slower" that could be used to characterize the speed of the ball at the initial stage of motion, is used also to describe the trend of the curve; in this sense the ball rolling down the inclined plane and the distance-time curve are blended in one sentence revealing a physical interpretation of the graph itself (interpretative level – **Figure 40**). Silvia repeats Giulia's sentence underlining its correctness, but the tone of her voice reveals that she expected a different answer. Indeed, Fabio grasps that and raises his voice replying "From the first finite differences… equal in the table" [109]. Fabio's sentence is not well formulated in the contest of that discussion, but Silvia catches that Fabio is on the right way and helps him to better explain himself. Posing herself as a springboard, Silvia initiates a sentence and invites Fabio to complete it. Thanks to the joint labor of the teacher and Fabio, they arrive at the conclusion that "there would be first finite

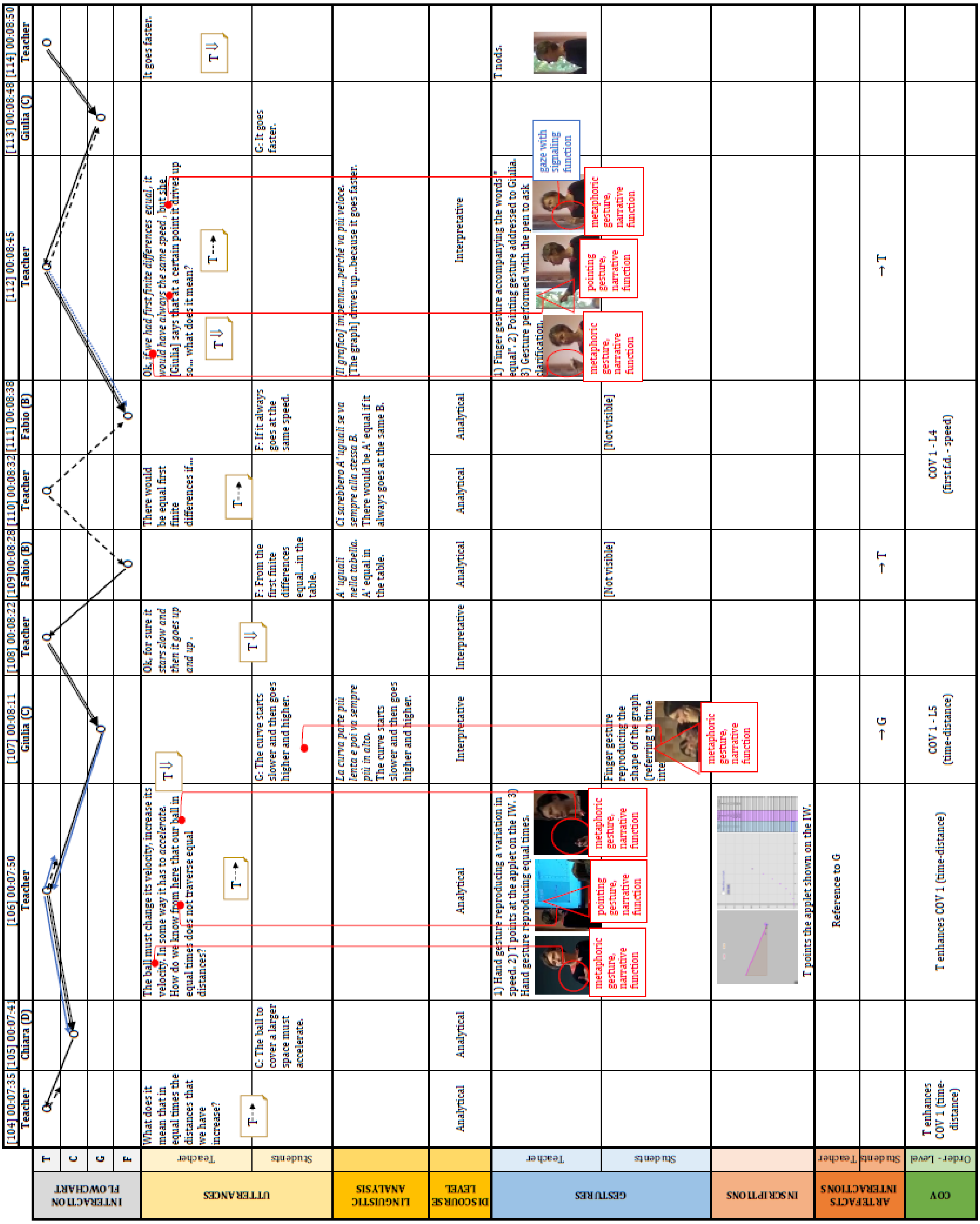



**Figure 40** – Timeline 5 (Galileo 2019)

differences equal if it [the ball] goes always at the same speed" [110-111]. Playing the semiotic game, Silvia revoices the whole sentence and then reconnects to the utterance initially elaborated by Giulia [107] asking what it means that a certain point "it [the curve] drives up" [112]. This time Giulia replies with a physical interpretation that is an increase in velocity, "it goes faster" [113]. The teacher repeats her words with an approval tone [114] and nodding (gesture row – **Figure 40**). An overview of the episode in its entirety reveals that: Giulia's reasoning evolves from the distance-time graph [107] to the motion of the ball [113]; Fabio's approach instead evolves from the table of numerical values [109] to an interpretation of the physical phenomenon [111]; Silvia suggests to use a certain interpretative key of the phenomenon: not only the interaction flowchart again reveals the determinant role of the teacher in giving the right direction to the discussion, but Silvia also enhances a blended approach in which the graphical/numerical and kinematics aspects interlace. The level of the discourse is mainly analytical and becomes interpretative when students are able to connect the distance-time graph to the motion of the ball [107, 113]. The artefact specifically appearing in this episode is the GeoGebra applet and in particular the table of values contained it which are used by the students with a descending control that is to answer to teacher's questions, and so in continuity of purpose (artefacts interactions row – **Figure 40**).

### 11.2.7 Episode 6 (Discussion 2, 16:26 -18:38)

During this short episode (16:26-18:38), the teacher is asking her students about the formula describing the motion of the ball and in particular she is enhancing second-order covariation inviting students to better explicate the dependence of the function on the angle of inclination. A student from group B, Matteo, tries to answer. During the discussion, the applet is shown on the interactive whiteboard (IW) and the teacher writes on the blackboard (B) the relevant considerations emerging from the students.

| | **Timing** | **Who** | **Utterances** | **Gestures** |
|---|---|---|---|---|
| 115 | 00:16:26 | Teacher | And how did the equation result? | |
| 116 | 00:16:30 | Matteo$_B$ | Practically it varies according to the angle. | |
| 117 | 00:16:33 | Teacher | The function varies according to the angle. | |
| 118 | 00:16:35 | Matteo$_B$ | The coefficient varies. | |
| 119 | 00:16:38 | Teacher | The coefficient of the function in which sense? | |
| 120 | | Teacher | *T suggests choosing on the applet another value for the angle, hence now the applet* | |

| | | | | |
|---|---|---|---|---|
| | | | *shows two different graphs for two different values of the angle, 20° and 90°.* | |
| 121 | 00:17:16 | Teacher | What do you tell me about this function? | |
| 122 | 00:17:20 | Matteo$_B$ | The inclination [of the graph] changes according to the angle because with a minor angle the inclination is minor, with a greater angle the function goes up first. […] $y$ is the distance, $x$ is the time and the coefficient, to find the coefficient you have to divide by two the second finite differences | *The teacher writes the function on the blackboard.* |
| 123 | 00:18:00 | Teacher | So, you say that according to you the equation is this: $y$=*coefficient* $\cdot$ $x^2$ | |
| 124 | 00:18:10 | Matteo$_B$ | $y$ is the distance, the coefficient can be found dividing by 2 the second finite differences and $x^2$ is the time. | |
| 125 | 00:18:18 | Fabio$_B$ | $x$ and $y$ must be greater or equal than zero otherwise it doesn't exist. | |

After the working-group session, the teacher asks her students about the equation describing the motion of ball [115]. She clearly looks at Matteo inviting him to take the word (gesture row – **Figure 41**). Matteo answers that "the function varies according to the angle" [116] and with his left hand performs a metaphoric gesture reproducing the angle of inclination of the plane (gesture row – **Figure 41**). The teacher revoices Matteo's words [117] playing a semiotic game and immediately after Matteo clarifies that in particular is the coefficient of the function that varies [118]. The level of the mathematical discourse is objectified (discourse level row – **Figure 41**) because the student already shows to possess mastery of the function describing the law of the motion and he is explicating a further dependence of the coefficient of the function on the angle of inclination [118]. The covariational reasoning is not fully second-order but in a transitional phase (COV row – **Figure 41**) because the coefficient is treated as a varying quantity but not yet as a parameter within a family of function. The focus remains on the single distance-time function. The teacher asks clarification about that dependence [119] and suggests working with the GeoGebra applet shown on the IW so to visualize simultaneously two different

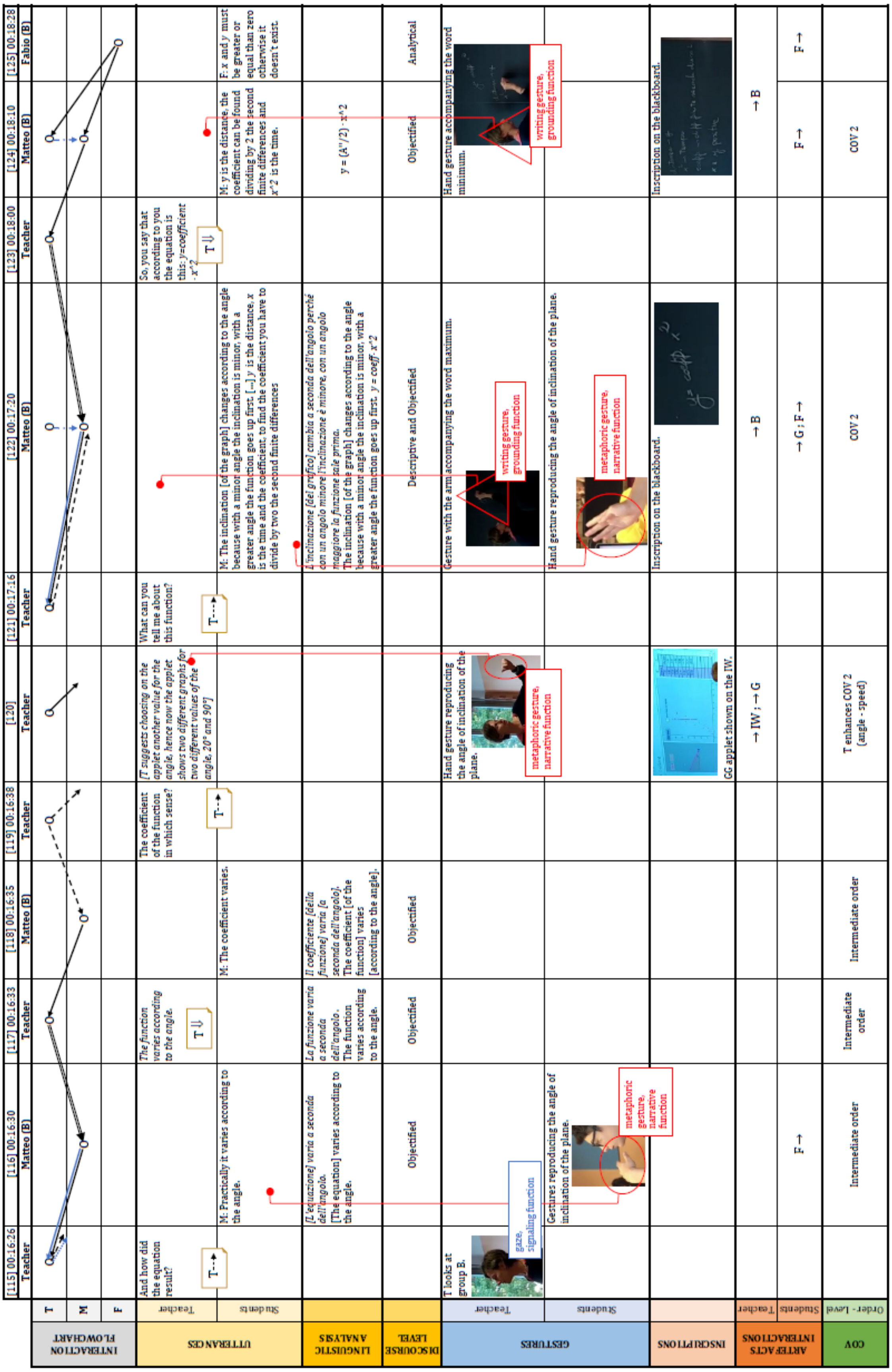



**Figure 41** – Timeline 6 (Galileo 2019)

graphs for two different values of the angle of inclination [120]. Through this proactive request (see interaction flowchart – **Figure 41**), the teacher triggers and supports second-order covariational reasoning, encouraging to investigate the relationship between the graph of the function and the angle. The teacher asks again for some considerations [121] and finally Matteo, while performing a metaphoric gesture (gesture row – **Figure 41**), makes explicit how the trend of the function changes according to the angle and formulates the expression of the function [122]. In the meanwhile, the teacher writes on the blackboard the formula ($y{=}coeff{\cdot}x^2$ - inscriptions row – **Figure 41**): this gesture is for sure non-redundant with respect to speech since the teacher is not speaking and it has a grounding function (gesture row – **Figure 41**) because clearly reveals to her students the importance of that formula and in fact it deserves to be written on the blackboard, visible to the whole classroom. Matteo's statement [122] reveals a second-order covariational reasoning: starting from a description of what was displayed on the applet (discourse level row – **Figure 41**), the discourse turns into objectified when he elaborates a formula condensing that dependence. The artefacts appearing in this episode, meaning the GeoGebra applet, the sought formula, the blackboard, and the interactive whiteboard, are all used in continuity of purpose (artefacts interactions row – **Figure 41**) that is deeply investigate the angle-graph relationship; both the modalities of control (ascending and descending) can be observed.

### 11.2.8 Episode 7 (Discussion 2, 57:00-59:56)

This episode locates towards the end of the discussion: the relevant mathematical elements to describe the phenomenon proposed have already emerged during the discussion but not all the groups have fully grasped it. In the following three minutes (57:00-59:56), Silvia tries to pull the strings of the discussion clarifying the role of the parameter and its mathematical interpretation.

| | Timing | Who | Utterances | Gestures |
|---|---|---|---|---|
| 126 | 00:57:00 | Teacher | We said that 0.88 is a constant and then we tried to understand what it represents. Will it still be 0.88 the constant if we change the angle? | |
| 127 | 00:57:30 | Many voices | No. | |
| | | Matteo$_B$ | It changes according to the angle. | |
| 128 | 00:57:33 | Teacher | It changes according to the angle… are we sure? Let's try | |

| | | | | |
|---|---|---|---|---|
| | | | to change the angle… What could be the coefficient with an angle of 26°? | |
| 129 | 00:58:05 | Matteo$_B$ | 2.21 | |
| 130 | 00:58:07 | Teacher | Why did you find it immediately without using the calculator? | |
| 131 | 00:58:10 | Matteo$_B$ | Because dividing by $1^2$ is like dividing by 1. | |
| 132 | 00:58:16 | Teacher | Ok, so you say I can just take one value because I'm sure it is constant. | |
| | | | […] | |
| 133 | 00:58:44 | Teacher | Hence what does this number represent? | |
| 134 | 00:58:55 | Adele$_C$ | The half of first finite differences… | |
| 135 | 00:58:58 | Teacher | *[T starts making the computation out loud, but they are incorrect]* | |
| 136 | 00:59:07 | Adele$_C$ | Of second finite differences… yes, the half of second finite differences. | |
| 137 | 00:59:16 | Teacher | *[T makes the computation out loud and this time it is correct]* | |
| 138 | 00:59:56 | Teacher | So, this could be the half of second finite differences… | |

Reacting to some student's claim, the teacher recalls that they have concluded that "0.88 is constant" [126] and simultaneously points at the blackboard where it is written the formula $\frac{y}{x^2} = 0.88$ (**Figure 42** – inscriptions row). Immediately after she asks if 0.88 would still be the constant if they "change the angle" [126] and Matteo, as done in Episode 6, replies that "it changes according to the angle" [127]. Despite the correctness of the answer, Silvia wants to deepen this point and asks to the students if they are sure. She suggests changing the angle, in particular to choose randomly an angle of 26°, and asks "[w]hat could be the coefficient" in this case [128]. While formulating her request, the teacher addresses to group E with a hand gesture (gesture row – **Figure 42**) because their computer screen is connected to the IW and so visible by the whole classroom (inscriptions row – **Figure 42**). Matteo immediately replies "2.21" [129] and Silvia, surprised by the rapidity of his answer, asks him how he could find that value

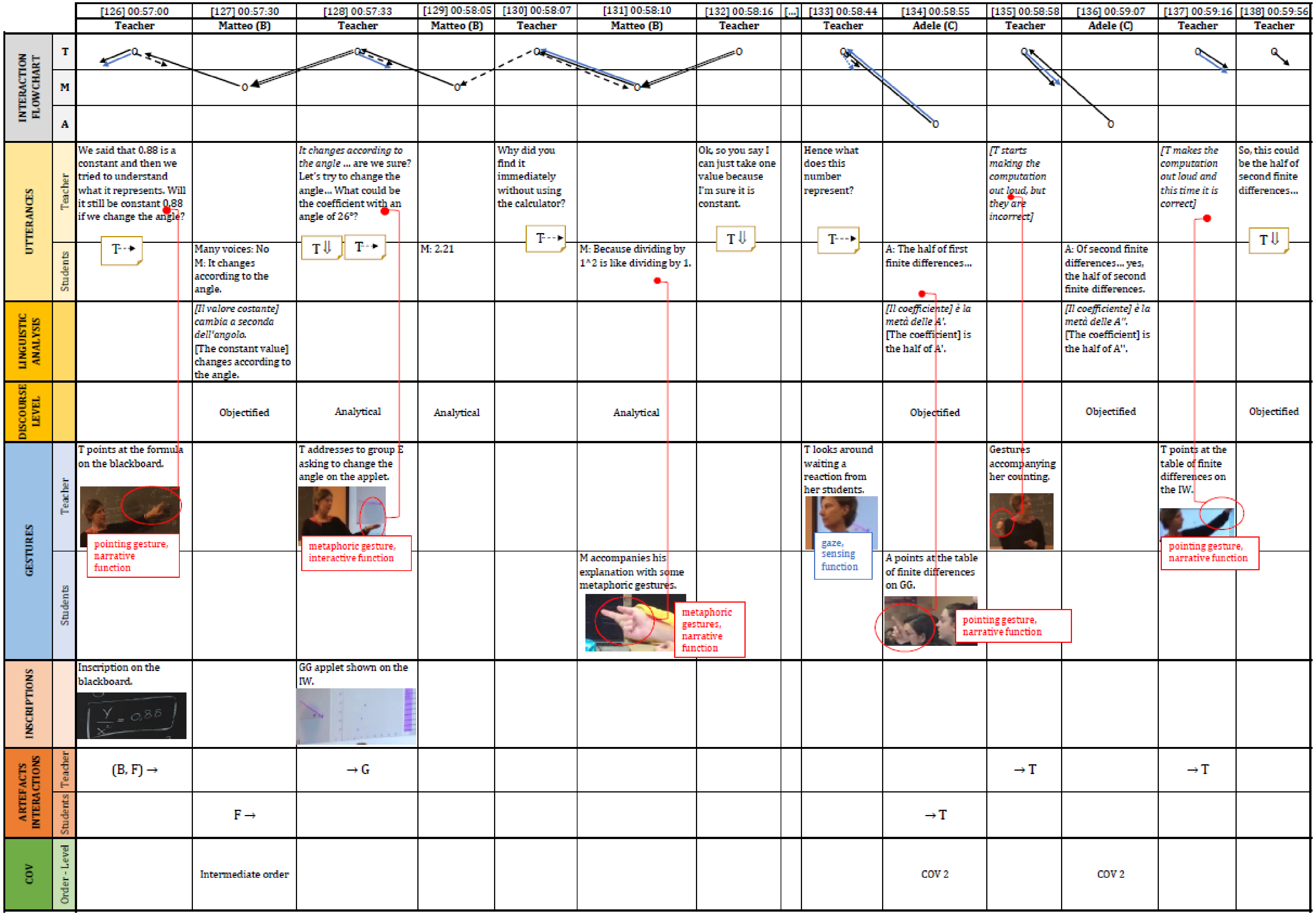



**Figure 42** – Timeline 7 (Galileo 2019)

"immediately without using the calculator" [130]. Accompanying his words with some metaphoric gestures, Matteo replies that he simply divided by 1, that is equal to $1^2$ [131]. Silvia tries to better explain that claim so concise implying that the computation is possible just because the coefficient is assumed to be constant [132] and so the ratio provides the same value of the time and distance traversed chosen. After having clarified how to compute the value of the coefficient she moves to investigate the meaning of that coefficient, what that number represents [133] and looks around waiting for a rection from her students (gestures row – **Figure 42**). Adele replies "the half of first f.d." [134] and points at the table of f.d. shown on the IW; Silvia is conscious of the incorrectness of her answer and so starts making the computation out loud [135]. Adele realizes that she expressed wrongly and so corrects herself saying "the half of second finite differences" [136]. Silvia restarts making the computation out loud, pointing at the table of f.d. on the IW (gesture row – **Figure 42**) and this time they are correct [137]. Finally, with an approval tone, Silvia states that the coefficient could be the half of second f.d. [138]. The class is not much talkative, and the answers provided by students are often synthetic: the interaction flowchart reveals the ability of the teacher in enhancing the discussion and valuing and amplifying the content of students' claims. The level of the discourse alternates between analytical and objectified: it is analytical when students refer to a specific angle of inclination and the related numerical coefficient [128-129-131] and becomes objectified when the coefficient is interpreted in terms of a general expression, the half of second finite differences [136-138]. Artefacts are mainly used with a descending control (artefacts interactions – **Figure 42**).

## 11.3 DISCUSSION

This section contains a transversal analysis of the seven Episodes presented previously according to the four different layers of analysis.

### 11.3.1 Layer (a): Covariational reasoning

The intermediate order of covariational reasoning identified during the 2017 T.E. emerges naturally also in this T.E.: it consists in the conceptualization that the coefficient of the function depends on a certain magnitude, that in this case is the angle of inclination of the plane [118]. The term 'parameter' is not used by the students; they refer to it in a more intuitive way, a constant value that changes depending on the angle [127], adopting an antithetic expression. In this case finite differences constitute the pre-analytic tool that mainly enhances the instrumentation of second-order covariation: they allow students to use a more mathematical language and to deduce the analytical and graphical property of the distance-time graph which is of second degree and depending on second finite differences of distance. Second-order covariation is achieved from two different mathematic standpoints: the elaboration of a general mathematical formula in which the coefficient explicitly depends on second f.d., and a level supported by metavariation in which students, exploiting the representations provided by the various technological tools, envision how the distance-time graph is affected by a change in the angle of inclination. Therefore, two different levels can be distinguished: a quantitative one (formula) and a qualitative one (metavariation).

### 11.3.2 Layer (b): Linguistic analysis

In continuity with the analysis presented in 10.3.2, we analyze the various levels and orders of students' covariational reasoning according to their syntactical and lexical features.

| *COV 1* | *Description of the level* | *Examples from Galileo 2019 T.E.* | *Syntactical and lexical analysis* |
|---|---|---|---|
| L2 | In these examples we can observe a qualitative description of the covariation, also referring to some benchmark numerical values, of two or more magnitudes: the level of reasoning is not yet fully quantitative because students are not able to express clearly what happens between the limit values.<br>Students are able to co-vary the two magnitudes underlining the values reached in limit situations. | *[65] Più A si avvicina a 90°, più B aumenta.*<br>The more A comes closer to 90°, the more B increases.<br><br>*[66-67] [Raggiunge la massima A] nell'ultima B.*<br>[It reaches the maximum A] in the last B.<br><br>[68-69] *[Più A si avvicina a 90°, più B aumenta] finché B non è massima per quell'A.*<br>[The more A comes closer to 90°, the more B increases,] until B is maximum for that A.<br><br>[73] *A è minimo con B= 0,0000001.*<br>A is minimum with B = 0,0000001. | From the **syntactical** standpoint, the linguistic structures that can be identified are the following:<br>***The more*** A, ***the more*** B<br>A is ***maximum/minimum*** for B=n.<br>We can observe some binary relations and the use of comparatives. Both coordinated and subordinated clauses are used.<br><br>From the **lexical** standpoint, language adopted is mainly qualitative and objective. The adjectives ***maximum*** and ***minimum*** are employed to refer to the limit values assumed by the magnitudes. |

| | | | |
|---|---|---|---|
| | | [74-75] *A aumenta finché B=90°.*<br>A increases until B=90°. | |
| L4 | The use of f.d. displayed in students' reasoning denotes a higher level of covariation since f.d. reveal not only how a magnitude varies in time but also its rate of variation. | *[77] A' erano sempre di 1, mentre B' aumentavano sempre di più.*<br>A' were always of 1 while B' increased more and more.<br><br>*[79] A'' erano costanti e A''' erano nulle.*<br>A'' were constant and A''' were null.<br><br>*[110-111] Ci sarebbero A' uguali se va sempre alla stessa B.*<br>There would be A' equal if it always goes at the same B. | The f.d. of A and B (A' and B' respectively) make their appearance as magnitudes: in [77] there is a covariation between A' and B' which condenses the rate of change of the two magnitudes; in [79] both second and third f.d. (A'' and A''') are considered and in [110-111] first f.d. are related to a physical interpretation.<br><br>From the **syntactical** standpoint, we can identify the following structures:<br>A' were always the same ***while*** B' increased.<br>A' equal ***if*** B is always the same.<br>A'' were constant ***and*** A''' were null.<br>Relations are mainly binary, no comparatives, and both coordinated and subordinated clauses can be detected.<br><br>From the **lexical** standpoint, we can observe a recurrent use of the adverb ***always,*** objective and quantitative sentences. |
| L5 | In this T.E. the level of smooth continuous covariation manifests in the full conceptualization of the distance-time graph that is conceived as a multiplicative object: students do not refer anymore explicitly to the starting magnitudes but mainly focus on the description of the trend of the graph. | *[81] Il grafico potrebbe essere di secondo grado visto che le A'' sono costanti.*<br>The graph could be of second degree since A'' are constant.<br><br>*[83] La curva prima aveva un'inclinazione quasi orizzontale e poi diventava sempre più verticale.*<br>The curve before had an inclination almost horizontal and then | From the **lexical** standpoint, a recurrent use of the adverb ***always*** [83-85-107] can be interpreted as an indicator of globality. The description of the trend of the graph is expressed in qualitative terms: in [83] and [85] by reasoning in terms of fixed intervals of time; in [107] and [112-113] a more global approach emerges, related also to a physical interpretation of the curve's behavior. In [81] instead, the degree of the curve, and so its shape, is explained by a reference to second f.d., i.e., a pre-analytical property. |

| | | | |
|---|---|---|---|
| | | became always more vertical.<br><br>*[85] In ogni A, l'inclinazione [del grafico] era sempre più verticale.*<br>In each A, the inclination [of the graph] was always more vertical.<br><br>*[107] La curva parte più lenta e poi va sempre più in alto.*<br>The curve starts slower and then goes higher and higher.<br><br>*[112-113] [Il grafico] impenna…perché va più veloce.*<br>[The graph] drives up…because it goes faster. | ***Horizontal*** and ***vertical*** are recurrent adjectives used to connote the behavior of the curve. In [107] we can observe the use of the adjective ***slower***, a kinematic term attributable to the motion of the ball and instead used to refer to the graph.<br><br>No recurring **syntactical** structures were detected. |

| *Higher COV* | *Description of the level* | *Examples from Galileo 2019 T.E.* | *Syntactical and lexical analysis* |
|---|---|---|---|
| Intermediate order | This transitional order mainly manifests in the elaboration that the coefficient of the function depends on the angle of inclination of the plane. Even in this case, students do not refer to it as a parameter but more intuitively as a constant value that changes according to the angle. | *[118] Il coefficiente [della funzione] varia [a seconda dell'angolo].*<br>The coefficient [of the function] varies [according to the angle].<br><br>*[127] [Il valore costante] cambia a seconda dell'angolo.*<br>[The constant value] changes according to the angle. | From the **syntactical** standpoint, we do not observe relevant structures.<br>From the **lexical** standpoint, we can notice a qualitative approach through which students express the dependence of the coefficient on the angle of inclination. |

| | | | |
|---|---|---|---|
| COV 2<br>(i) | Students' forms of reasoning revealing a full achievement of COV 2 in this T.E. can be actually divided in two sublevels: the first one is a covariation between the angle of inclination and the distance-time graph i.e., changing the angle which effects produces on the trend of the graph: this way of reasoning is an example of meta-variation supported by the GeoGebra applet enabling to explore it. | *[87] Se l'angolo cambia, la salita [del grafico] è più rapida.*<br>If A changes, the uphill [of the graph] is faster.<br><br>*[91] Più l'angolo aumenta, più l'inclinazione [del grafico] aumenta.*<br>The more A increases, the more the inclination [of the graph] increases.<br><br>*[93] Il grafico è di secondo grado quindi le A'' sono il doppio del coefficiente.*<br>The graph is of second degree hence A'' are the double of the coefficient.<br><br>*[96] A' aumenta aumentando B.*<br>A' increases, increasing B.<br><br>*[98] Più grande è A, più velocemente crescerà la parabola.*<br>The greater is A, the faster the parable will grow.<br><br>*[99] Maggiore è A, maggiore è B' e la parabola crescerà più velocemente.*<br>The greater is A, the greater is B' and the faster the parable will grow.<br><br>*[100] Con l'aumentare dell'angolo la curva si avvicina sempre di più all'asse delle y, perché* | From the **syntactical** standpoint, we can identify the following recurring structures:<br>If A changes, ***the more*** the inclination increases.<br>***The more*** A increases, ***the more*** the inclination [of the graph] increases.<br>A' increases, increasing B.<br>***The greater*** is A, ***the greater*** is B' and ***the faster*** will grow the parable.<br>***As*** A increases, the curve comes always closer to y-axis.<br>***As*** A of the plane increases, the curve of the parable is ***more*** accentuated.<br>Increasing A**,** the curves are ***more*** inclined.<br>Relations are mainly binary, and the second magnitude is typically the inclination of the graph or the graph itself. Comparative structures are recurrent, and clauses are both coordinated or subordinated.<br><br>From the **lexical** standpoint, the adverb ***more*** is extremely used, and the linking word ***as*** is also recurrent. The sentences are objective and show a qualitative approach to describe globally how the trend of the graph evolves. |

| | | | |
|---|---|---|---|
| | | *grazie all'accelerazione percorre lo stesso spazio ma più velocemente.*<br>As A increases, the curve comes always closer to *y*-axis, because thanks to acceleration it covers the same distance but faster.<br><br>*[101] All'aumentare dell'angolo del piano la curva della parabola è più accentuata a causa del fatto che la pallina percorre lo stesso spazio più velocemente, di quando l'angolo è minore.*<br>As A of the plane increases, the curve of the parable is more accentuated because the ball the same distance faster respect to when the angle is smaller.<br><br>*[103] Aumentando A, le curve sono più inclinate. Diminuendolo, le curve sono meno inclinate. Questo perché la curva del grafico rappresenta l'accelerazione della pallina all'aumentare di* $\alpha$.<br>Increasing A, the curves are more inclined. Hence, decreasing it, the curves are less inclined. This because | |

| | | | |
|---|---|---|---|
| | | the curve of the graph represents the acceleration of the ball as $\alpha$ increases. | |
| COV 2 (ii) | The second level seems to consist in the elaboration of a general formula containing a coefficient which sometimes is clearly interpreted in terms of second f.d.. The way in which the coefficient affects the graph is made explicit. | *[122] L'inclinazione [del grafico] cambia a seconda dell'angolo perché con un angolo minore l'inclinazione è minore, con un angolo maggiore la funzione sale prima. $y = coeff \cdot x^2$* The inclination [of the graph] changes according to the angle because with a minor angle the inclination is minor, with a greater angle the function goes up first. $y = coeff \cdot x^2$<br><br>*[124]* $y = (A''/2) \cdot x^2$<br><br>*[136] [Il coefficiente] è la metà delle A''.* [The coefficient] is the half of A''. | From the **syntactical** standpoint, we do not observe relevant or recurring structures.<br><br>The **lexical** analysis reveals that this second level of COV 2 mainly lies on a quantitative and analytical approach rather than on a graphical interpretation, but still reveals a sense of globality. |

### 11.3.3 Layer (c): Discourse levels

This layer of analysis is conducted thanks to a diachronic analysis of the Timeline and is focused on the evolution of the mathematical discourse throughout all the episodes previously described. We recognized the same three main levels of the mathematical discourse identified during the Galileo 2017 T.E. and a fourth additional level:

- *Descriptive*: Episode 1 (11.2.1), at the beginning of the first classroom discussion, is the one in which students adopt a descriptive approach reporting and interpreting the inputs provided specifically by the GeoGebra applet and the video and relating them. For instance, Fabio describes how the angle of inclination of the plane, shown in the applet, affects the descent speed of the ball, deducible from the video [65-68].
- *Analytical*: the level of the discourse is analytical when students mainly refer to f.d. as in [77] where Valeria observes that "first finite differences of distance increased more and more" while "[second] were constant and the third were equal to 0" [79]; or as in 2017

Galileo T.E., this level manifests when they refer to the numerical value of the coefficient of the function for a specific angle (e.g., 2.21 for an angle of 26° [129]). This second level of discourse lasts less in this teaching experiment with respect to the 2017 one: the students possess more mathematical instruments to move from an analytical approach to the objectification and so to a mathematical model describing globally the phenomenon.

- *Objectified*: this level already manifests at the end of the first discussion (11.2.2), arising from the contribution of group E who succeeded during the working-group session in envisioning and representing the distance-time graph and then reproduced it on the IW during the classroom discussion. Valeria, as group E spokesman, described the trend of the curve [83] and was also able to envision how that trend could change with a variation of the angle of inclination [87]. The level of the discourse is clearly objectified also during the second working group session (11.2.3, 11.2.4, 11.2.5) where thanks to the functionality of the GeoGebra applet, enabling to change the inclination of the plane, students could observe which changes it globally produced on the distance-time graph. This level manifests also in Episodes 6 and 7 (11.2.7, 11.2.8) when Silvia led students towards the elaboration of a general mathematical formula describing globally the situation.
- *Interpretative*: this additional fourth level of discourse does not differ much from the objectified one concerning the globality of the approach and the mathematical objects involved, but the language adopted reveals something deeper than the mathematical conceptualization. The emerging narratives reveal a blend of elements related to the various representations (algebraic, numerical, and graphical) connected in a coherent sentence ( "the graph could be of second degree because second f.d. are constant and also because we knew the formula of acceleration is $s/t^2$" [81], "the greater is the angle, the faster the parable will grow" [98], "The curve starts slower and then goes higher and higher" [107], "[The graph] drives up… because it goes faster" [112-113]). The words used by students also are detectors of a physical interpretation blended with the mathematical description: in [81] Valeria refers to the formula of acceleration, in [112-113] the trend of the graph is explained by referring to the speed of the ball; in [107] the term "slower" used to characterize the trend of the curve is actually an adjective that could also be used to describe the motion of the ball at the beginning of its descent on the inclined plane. The graphic and kinematic aspects are blended as if the graph and the

phenomenon were the same thing. Looking at the mathematical discourse with the lens of the modelling cycle, this step could be interpreted as a return to the real phenomenon after having mastered the mathematical interpretation.

### 11.3.4 Layer (d): Adaptive teaching strategies

The four strategies identified in the 2017 Galileo T.E. are clearly recognizable also in this teaching experiment. Some examples follow:

1 - *Semiotic game*: this strategy often manifests only in its oral components with the revoicing or readjustment of the students' claims by the teacher [78-80]; an example of semiotic game involving also the gestural component can be recognized at [70], where Silvia not only repeats Fabio's words [69] and uses them to formulate a new question, but also imitates the gesture previously performed by Fabio with her arm.

2 - *Fostering the discussion in the classroom and facilitating its flow:* in this T.E. this second strategy emerges even more predominantly because the class is not much talkative and making questions is a strategy widely used by the teacher to enhance the conversation and to dig deeply into students' reasoning [66-70-115-119]. Moreover, at [135] we have a typical example of the non-judging behavior of Silvia who instead of underlining the student's mistake, brings it to the attention of the whole classroom so arousing students' spontaneous reactions: at [135] Silvia shows that the coefficient of the function is not the half of first finite differences making the computations out loud and so leading students to realize that they are not correct; at [136] Adele corrects herself spontaneously. Another feature emerging from the analysis of this T.E. is the *springboard role* assumed by the teacher: her questions are often the starting point of a wider discussion, or she starts an open sentence leaving to students the possibility to complete it: "There would be equal first finite differences if…" [110].

3 - *Exploring students' actions and thinking*: questions have often the goal to promote an argumentative approach: "[…] what does it mean?" [112], or to obtain a deeper explanation of the answer provided by the student: "Why did you find it immediately without using the calculator?" [130], "The coefficient of the function in which sense?" [135].

4 - *Drawing students' attention to the information provided by the different artefacts* used in the learning process: an example of this strategy can be observed in Episode 5 (11.2.6); at this point of the discussion, the teacher would like students to relate the increase in speed of the ball to the numerical values of f.d. At [106] the teacher directs the attention of the students to the applet

shown on the IW: "How do we know from *here* […]?" also using a pointing gesture. Giulia initially refers to the graph; only in a second moment thanks to Silvia's revoicing leaking out she expected a different answer, Fabio moves his attention to the table of f.d. and then Silvia insists on this point and on its physical interpretation.

## 11.4 CONCLUDING REMARKS

One of the main conjectures formulated during the prospective analysis and confirmed by the data analysis is the numerous references to a physical interpretation of the real phenomenon given the students' more extensive background in that field. Even when referring to the video reproducing the Galileo experiment, they refer to the speed of descent of the ball [65] and are able to envision a variation of the angle of inclination of the plane [65]. It is just the clever coaching of the discussion managed by the teacher that determines the direction of the discussion and establishes some parameters as fixed. The speed of the ball is intended as a magnitude itself and covaried with the angle of inclination [73-75]. The discussion does not reveal much insistence on the covariation between the numerical values of the magnitudes involved: the analytical phase of mathematical discourse lasts less and already at the end of the first discussion some groups reach an objectified level, that is a global view of the distance-time graph. They do not succeed immediately in the elaboration of a mathematical formula, but they conceptualize the graph as a second-degree function, deducing it from the numerical values of finite differences [81], and manage to represent it in a graphical form [85]. Second-order covariation does not emerge spontaneously, but the teacher enhances it thanks to her wise questioning.

What does not emerge is the misconception between the trajectory and the law of the motion of the ball. Students instead refer to acceleration, a notion that they have not explored yet during their physics lessons and so they call it into question in an intuitive way, adopting an everyday language and as a generalization of the concept of velocity. Silvia does not insist much on the incorrect way in which students refer to it but many times during the discussion underlines that the formula of acceleration is their deduction and not something explicitly provided by the video or the applet.

During the working group session on the second applet, the groups succeed in elaborating a general formula describing the situation [95] in which the coefficient depends on the second finite differences and the description of the graph is often related to a physical interpretation of the motion of the ball [99-100]. Students describe in a qualitative way how the angle affects the trend of the graph [101-103] and grasp the dependence of finite differences on the angle of inclination [102].

The presence of a constant reference to the real phenomenon is also made manifest in the use of a blended language, i.e., an interpretative level of the discourse that was absent in the 2017 T.E. and that denotes a wiser mastery of the steps of the modelling cycle. During the second discussion, the teacher leads the students in sharing with the whole classroom the different considerations that they have elaborated during the working-group session: even if the class is not very talkative and Silvia must stimulate a lot the discussion to make students intervene, the objectified and interpretative levels of the mathematical discourse that emerge are a clear symptom of the greater advance in the conceptualization of the phenomenon both from a mathematical and physical standpoint.

### 11.4.1 Toward second-order covariation: comparing the two teaching experiments

Comparing the two T.E.s (2017 and 2019), the following considerations can be elaborated:

- COV 2 emerges less during the 2017 T.E. because students possess a narrower mathematical background: they favour an everyday language, while 10th grade students possess the mathematical knowledge to succeed faster in a mathematical formalization. Thanks to this wider background, during 2019 T.E. the analytical phase lasts less, and students can move faster to the level of objectification and a higher order of covariation;
- In 2019 T.E., finite differences are the artefact that mainly support the instrumentation of covariation and allow the use of a pre-analytic language that will foster during schooling the introduction to the concept of incremental ratio and then of derivative. Finite differences are considered a magnitude themselves, even if encapsulating the rate of change of a magnitude in time, and they are covaried with both the graph [107] and the speed of the ball [111];
- The intermediate order of covariation manifests in both the T.E.s and it can be interpreted as a progression toward second-order covariation: it mainly manifests as a conceptualization of the dependence of the coefficient of the function on something, in this case the angle of inclination from the physical point of view [118-127] and lately on second finite differences on the mathematical point of view [93-136]. A

provisional interpretation, requiring deeper knowledge, leads us to attribute to this intermediate order a cognitive connotation rather than a mathematical one: it can be intended as first level of COV 2 in which the student mentally envisions that one of the magnitude changes in a different way with respect to the others involved and influencing the whole mathematical scenario: at this level students can use an antithetic expression more than the rigorous term "parameter" to describe this covariation;

- In the two classroom discussions, second-order covariation is reached with two different approaches: in the 9th grade classroom, COV 2 emerges from the initial magma of the magnitudes at stake; in the 10th grade classroom, COV 2 is often flattened into COV 1 between distance and time: this kind of COV 1 absorbs what 9th grade students couldn't express because they had no notion of speed; moreover the term "acceleration" is used in an incorrect way to denote something unknown but that they perceive as necessary to describe the situation, i.e. a change in velocity. Hence, in this case physical notions support the reasoning process and allow a more compact way of reasoning. Using a metaphoric image, we could say that the main difference lays in the different initial approach to the discussion: in 2017 it is from below, ascending from the magma of magnitudes; in 2019 it is from above, descending from students' wider mathematical background.

### 11.4.2 Students' feedback (from Task 7)

In this paragraph, we are going to report and briefly comment on some of the answers provided by students to Task 7 (11.1.7). We are not going to provide a methodologically rigorous analysis, but reading their inputs represented for us the chance to verify if the goals of the experimentation and methodology proposed were achieved and to make some reflections before starting planning the following T.E.. Indeed, we are going to report some of their answers.

To question 1), the totality of students agreed that obtaining the law of the inclined plane using different tools, "enabled to study the same topic, phenomenon in different modalities and standpoints" [T7-S5] or again "various aspects of the same thematic" [T7-S12]. Some students motivated in synthesis the peculiarities of each of the adopted tools. The *video* provided a "general framework, it showed what was the phenomenon to be observed and which aspects we had to pay attention to" [T7-S12]; the standpoint was that of some "observers, since the experiment was conducted by another person, without many theorical connections" [T7-S22]. The *GeoGebra applet* enabled to "see all the aspects of the phenomenon" [T7-S12], "without showing the limits of reality" [T7-S22]. The *text of Galileo* allowed them to "understand the goal and real ideas of who created it and gave solutions to the problem" [T7-S22] and finally the *experiment*, "even if it resulted imprecise, was useful to show that what [they] had hypothesised and studied from the theoretical point of view was real" [T7-S12]. Other students instead, underlined that the experiment "was not much helpful because [they] did not succeed in fully proving the law of the inclined plane, but doing it, [they] realized the complexity and difficulties that surely Galileo had in proving his theory" [T7-S9].

Question 2) specifically invited to focus on the potentialities of the GeoGebra applet and students stated that it was useful for many reasons: "using the GeoGebra applet was like doing an experiment and see physically the ball rolling down the plane" [T7-S5]; it "allows you to avoid making many calculations" [T7-S14]; it enables to "see all the diverse options derived from the problem all on the same plane and to compare the various values" [T7-S14]; "the possibility of changing easily the inclination of the plane was very beneficial" [T7-S5], and "through the data obtained in the applet [they] could verify if the formulas found were valid or not" [T7-S1].

Finally, many students filling in their questionnaire, stated that "the method of study of the motion of the ball on the inclined plane was efficient because interactive and satisfying (for the conclusions found)" [T7-S13], "the mind is more stimulated in the search for a solution" [T7-S15], in fact, "finding a formula by analysing only videos and graphs is more satisfying than finding a formula already given" [T7-S7]. Concerning the use of different technological tools, "it made the activity more engaging and to some extent fun" [T7-S2].

# 12 DEW POINT TEACHING EXPERIMENT (2020)

This third and last teaching experiment took place in September/October 2020: the design of the tasks and planning of the T.E. was conducted in summer 2020 by the teacher, Silvia, and me under the supervision of Prof. Arzarello. The T.E. took nearly 3 weeks of work for a total amount of nearly 12 hours including some hours of homework. The methodology of work was nearly the same of the previous T.E.: group-work sessions followed by classroom discussion mediated by the teacher. She always started from the different answers of the groups, underlining similarities and differences, and enhancing an argumentative approach to justify their assumptions.

*Participants*

The 11th grade classroom involved was the same of the 2019 T.E.: it was still made of 22 students and during the working group sessions, they worked divided in the same 5 small group-works of the previous experimentation. The mathematical background of these students has been already described in Chapter 10; moreover, students added to their background an entire experimentation devoted to covariation (even if this term was never explicitly used by Silvia with her students) and they increased their mastery in working with functions. Students had not studied yet the exponential functional: this teaching experiment was the pretext to introduce this function to her students at the end of it, but these lessons are not part of our study.

During this T.E. concerning the investigation of the relation between temperature and humidity, many notions of science were required; students had already studied them in their science lessons in the previous years, but a general review of these concepts was planned as integral part of the activities during the design phase of the T.E.

*Data collection*

This teaching experiment was conducted during the period of the Coronavirus pandemic, hence it was held in a mixed modality: partially in presence and partially online through Google Meet platform. All the lessons were recorded through the recording function offered by the Meet platform and, the teacher, on her own, also positioned some devices within the classroom to record the lessons. All the materials produced were collected and shared on a Google Classroom platform. As a researcher, I participated to most of the lessons but connecting on the Meet platform, because I was not allowed to be present in the classroom.

The parents of the students and the school consented to the use of the multimedia material produced and the original version of the used consent form is contained in **Appendix A**.

## 12.1 OVERVIEW OF THE TASKS AND PROSPECTIVE ANALYSIS

As presented in 8.2.3, this T.E. had three main aims: (a) investigating the relationship between humidity and temperature; (b) reading and interpreting the *psychrometric chart* in order to explain real phenomena; (c) distinguishing the role of variables and parameters in reading charts. These goals were achieved through the exploitation of different representations of the same phenomenon: a classroom experiment, a real psychrometric chart, and two GeoGebra applets. This time, the classroom experiment constituted the starting point of the whole modelling activity and, as the data analysis revealed, it was a solid reference point throughout the whole T.E.: it represented the element which enabled students to interpret from a physical point of view the mathematical representations. In this T.E. a different competence of second-order covariation is required. In the 2017 and 2019 T.E.s, being able to reason covariationally was intended as the ability to construct and interpret graphs of the type $y=f(m,x)$ representing a real phenomenon. This time it is more complicated because, in a nutshell, second-order covariation lays in the ability to interpret a situation that mathematically is described by the same formula $f(x,y,z)=0$ in which once the parameter is $y$ and once is $z$. In our specific case, $y$ and $z$ are connected to each other by the word humidity (absolute/relative).
All the tasks of the T.E. are reported below in different worksheets: we are going to present all the details and the prospective analysis, merely of those tasks that we are going to analyze in the following. The tasks which are not object of analysis will be presented briefly, just to provide an overall view of the experimentation.

### 12.1.1 Task 1

As homework, students were assigned the reading of a newspaper article[23] published on *la Repubblica* which dealt with the topic of hot temperature in summer and in its title contained the term “perceived temperature”. Then students had to answer the two following questions:

1) Have you ever heard of relative humidity? When? On which occasions?
2) Have you ever heard of perceived temperature? On which occasions?

[23] The newspaper article (in Italian) is available online at this page.

Students wrote their answers directly on the Classroom platform. The following lesson the teacher introduced a classroom discussion on the answers provided and then they searched on Internet the definitions of perceived temperature, absolute humidity, relative humidity, their units of measure and the definition of dew point. During the lesson, some concrete examples accompanying the introduced notions were quoted (condensation on the can, the steam on the mirror after the shower, sweating system of human body…). In the last part of the lesson, Silvia displayed on the IW a GeoGebra applet (Sole.ggb, see **Figure 43**) in which a table contained the values of temperature and relative humidity collected during a sunny day and two graphs represented those same sets of data with respect to time. The teacher explained how the data were collected and represented; in particular, since the magnitudes represented on *x*- and *y*-axis have different units of measure, some suitable translations and dilatations were introduced to make them more readable and comparable.

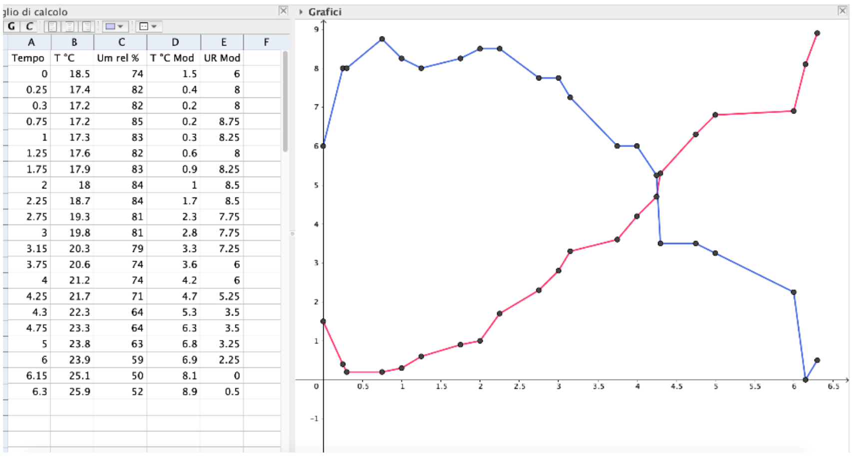



**Figure 43** – Screenshot of the GeoGebra applet, Sole.ggb

### 12.1.2 Task 2

After the previous lesson, students were assigned as homework to read a theoretical worksheet containing all the principal notions related to temperature and humidity, which had already been recalled, and to watch a video[24] containing the same explanation provided during the previous lesson; after that, students were asked to answer the following question: 3) Is there a relation between temperature and relative humidity?

Students uploaded their answers on the Classroom platform. The following lesson, the teacher started a classroom discussion commenting on their answers. At the end of the discussion, Silvia made the following experiment with her students: given a metal pot full of water at room

[24] The video (in Italian) made by the teacher is available here.

temperature, they gradually added some cubes of ice; every time they registered time passed and the lower temperature in the pot. When they saw the outside surface of the pot fogging, that temperature corresponded to the dew point. All the data were collected on the blackboard (**Figure 44**).

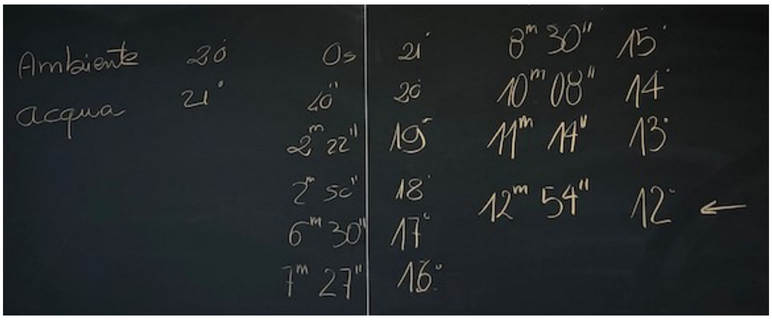



**Figure 44 –** Data of the experiment reported on the blackboard

As homework, students were assigned to watch a video[25] made by me reproducing the dew point experiment, an expedient that we adopted to allow students to work in a subsequent moment on a common data set. The language used in the video to present the experiment, which is reported in the First Worksheet about Task 3 (12.1.3), is deliberately of Galilean memory, both to create a link with the experimentation in which the students were involved during the previous year and to underline the importance of sensible experiences for the experimental scientific method. After viewing the video, students had to replicate the experiment on their own.

### 12.1.3 Task 3

Students faced a working group session of 1-hour on Google Meet divided in the same 5 groups (A-B-C-D-E) of the 2019 T.E. and worked on the worksheet reported below translated into English. During the session, students started from the data of the experiment presented in the video, and they were asked for a possible relation between the starting temperature and the dew point temperature. Then they were guided through the reading of a real psychrometric chart reported in their worksheet, and subsequently using a GeoGebra applet simulating it, students were asked to find the coordinates of the point of intersections between the green curves (indicating a different percentage of relative humidity) and the horizontal line $y=y_{DEW\ POINT}$. Finally, students were asked again for a possible relationship between temperature and humidity. The sessions were recorder through Meet, but they will not be analyzed in the following. The approach to this task was mainly explorative but given the complexity of the

[25] The video is available here.

psychrometric chart and the huge amount of information contained in it, it took times to the students to answer the questions on the worksheet.

This task was followed by 1-hour discussion in presence (Discussion 1) led by the teacher during which students' answers were shared and discussed. Moreover, guided by the teacher, students tried to retrace the steps of the pot experiment on the psychrometric diagram. One episode from this discussion will be analyzed in the following section.

**First Worksheet (Task 3)**

*Rinse in class with your classmates and then take a bucket with water and ice, a pot full for 3/4 of water, a thermometer, and a syringe with which you will gradually inject the frozen water into the pot, which you will mix with care. You will measure and record the temperature away. You will diligently observe the outside of the pot: when you see the air fogging on it, record the measured temperature. That will be the dew point.*

---

Which is the temperature of the water at the beginning of the experiment? ________________

Does it correspond to the temperature of the air in the room? ______________________

Which temperature does the dew point correspond to (see theoretical worksheet)? ________

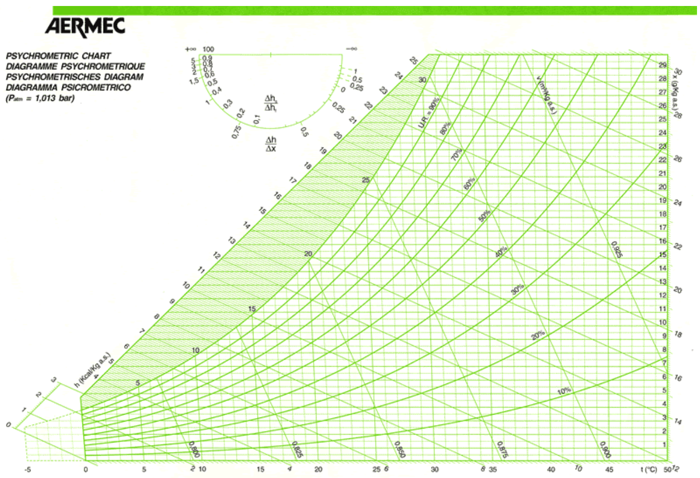
| Time passed (hh:mm:ss) | Temperature (°C) |
|---|---|
| 0:00:00 | 21,6 |
| 0:01:51 | 21 |
| 0:02:25 | 20,5 |
| 0:03:10 | 20 |
| 0:04:33 | 19,5 |
| 0:05:36 | 19 |
| 0:07:04 | 18,5 |
| 0:08:51 | 18 |
| 0:10:18 | 17,5 |
| 0:12:00 | 17 |
| 0:14:56 | 16,5 |
| 0:17:33 | 16 |
| 0:20:03 | 15,5 |
| 0:23:35 | 15 |
| 0:27:11 | 14,5 |
| 0:28:24 | 14,3 |

According to you, does the pressure change during the experiment? Why?

Thinking about the data of the video and of your homemade experiment, do you think there is a relationship between the initial temperature and the dew point?

We will learn to read the *psychrometric chart* or *Carrier diagram*, useful to determine the properties of a water-to-air mixture at constant pressure. On the *x*-axis the temperatures of the dry air are reported, while on the *y*-axis the absolute humidity is indicated. The graph is the one below but, as you can see, it's challenging to be read. For this reason, we will proceed step by step.

Open the file psicrometrica.ggb (**Figure 45**); the green curves indicate the relative humidity of the mixture. The 100% relative humidity curve is the saturation curve, which is the dew temperature curve. Move the rh (relative humidity) and temperature sliders and locate the point P on the saturation curve (the one with 100% relative humidity) which corresponds to the dew temperature obtained in the experiment.
Write here its coordinates: P=(     ,     )
Draw the horizontal line $y=y_p$ and look for the curve of relative humidity that passes through the point Q that has ordinate equal to $y_p$ and as abscissa the room temperature, the initial one of the water. What is the relative humidity at this point?
Mark points P and Q on the diagram below.




Using the psicrometrica.ggb file again, move the temperature and rh (relative humidity) sliders and find the coordinates of the points of intersection between the horizontal line *y=yp* and the green curves of the relative humidity. Complete the following table.

| | **Temperature** | **Absolute humidity** | **Relative humidity** |
|---|---|---|---|
| P_1 | | | 10% |
| P_2 | | | 20% |
| P_3 | | | 30% |
| P_4 | | | 40% |
| P_5 | | | 50% |
| P_6 | | | 60% |
| P_7 | | | 70% |

| | | | |
|---|---|---|---|
| P_8 | | | 80% |
| P_9 | | | 90% |
| P | | | 100% |

Read the values in the table. What do you notice?
How do temperature and humidity vary? Can you find a relationship that describes their variation?

**Corresponding computer screen**

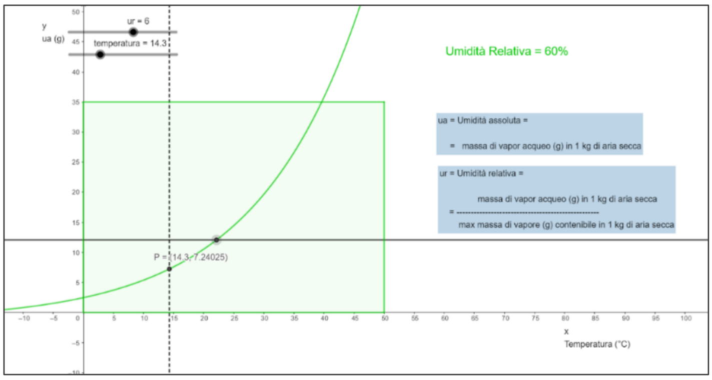



**Figure 45** – Screenshot of the psicrometrica.ggb applet interface

### 12.1.4 Task 4

At the end of the previous lesson, the following homework was assigned to students:

**Second Worksheet (Task 4)**

What do you think will be the trend of the graph that represents the values of relative humidity as a function of temperature? Try to trace (freehand or with GeoGebra, you choose) a likely chart **justifying** your choices **adequately.**

Students uploaded their works on Classroom the day before the lesson. At the beginning of the following lesson, the teacher devoted half an hour to a classroom discussion (Discussion 2) in which she commented on students' answers: she showed on the IW the various answers provided by the students underlining in particular the different approaches in drawing the graph and asking them to motivate their choices.

### 12.1.5 Task 5

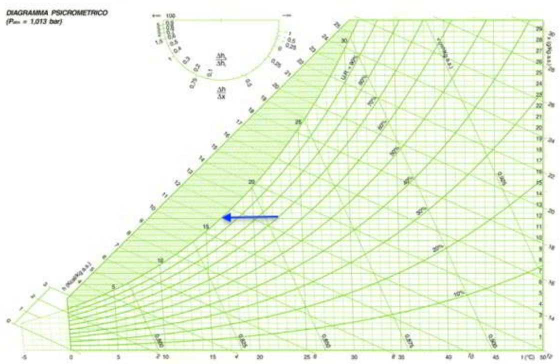
During the second part of the previous lesson, students divided in small groups, 2 or 3 people so to respect social distancing, worked on the worksheet reported below translated into English. The worksheet invited students to open a new GeoGebra applet, Nuovo_psicro (**Figure 46**), showing this time the relationship between relative humidity, on the $y$-axis, and temperature, on the *x*-axis. A few questions guided students in observing which magnitudes were represented in the new reference system with respect to the old one. Then a table recalled each step of the pot experiment, which magnitudes varied and how, and how each step of the experiment could be represented on the Carrier diagram. The result was a cycle on the chart. Finally, a question asked to reproduce the same cycle in the new reference system. At the end of the lesson the groups gave their worksheets to the teacher.

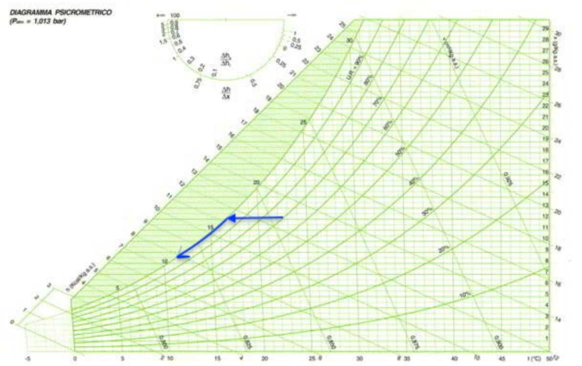
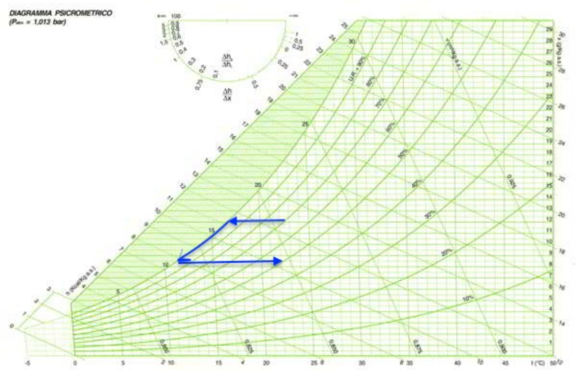
**Third Worksheet (Task 5)**

Open the GeoGebra file Nuovo_psicro.ggb.
Which are the magnitudes represented on the x-axis and y-axis of the new reference system? Which were the ones in the old system?
When switching from one reference system to another, what is no longer represented by the coordinates? Can you identify the magnitude in question in the new reference system? How?

Search and mark on the new graph the points P and Q, where:
- In the old reference system, the coordinates of point P are (14.3; 12.06708). It represents the point on the saturation curve where the temperature coincides with the dew temperature;
- In the old reference system, the coordinates of point Q are (22.1355; 12.06708). It represents the point with room temperature and initial specific humidity: in other words, it represents the initial situation regarding the specific humidity (no additional water vapor is inserted or removed) and initial room temperature.

Is it correct to say that if the temperature increases by 8°C, then the relative humidity increases by 8%?
Thinking back to the experiment of the pot, it is possible to describe it in this way:

| What happens | How magnitudes change | How to move on the graph |
|---|---|---|
| Starting from a certain initial temperature and with a given specific humidity the pot gets cooler. | Absolute humidity = constant<br>Specific humidity = constant<br>Relative humidity = increase<br>Temperature = decrease |  |

| | | |
|---|---|---|
| While continuing to lower the temperature in the pot, some drops of water form on the wall: water vapor content in the air condenses. | Absolute humidity = decrease<br>Specific humidity = decrease<br>Relative humidity = constant<br>Temperature = decrease |  |
| Let us take off the pot with the fogging from the classroom and leave that the ambient temperature returns to the initial value. | Absolute humidity = constant<br>Specific humidity = constant<br>Relative humidity = decrease<br>Temperature = increase |  |

Can you repeat the same cycle on the new graph?

**Corresponding computer screen**

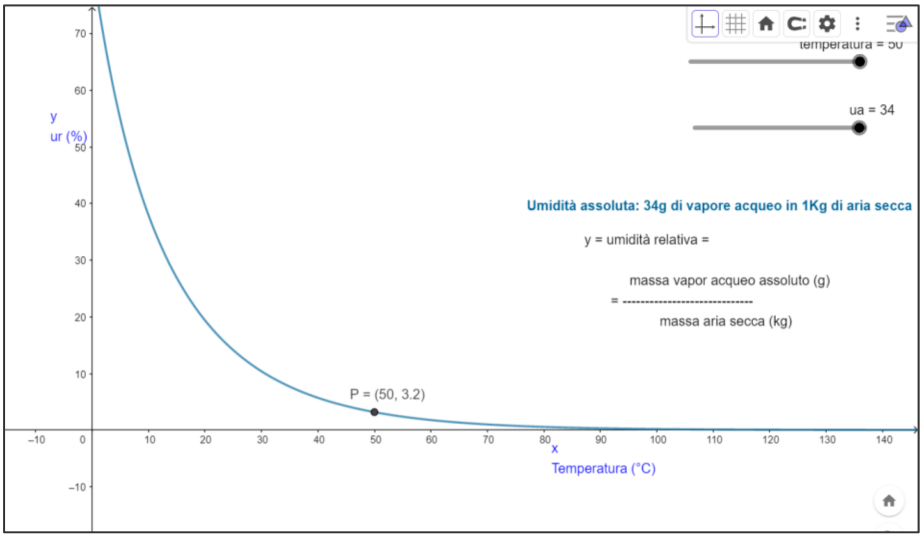



**Figure 46 –** Screenshot of the Nuovo_psicro.ggb applet interface

This task was followed by two hours of classroom discussion, both of 1 hour and about a week apart. During the first of the two discussion (Discussion 3), the teacher directed students' attention towards the idea that the two graphs, contained in the two GeoGebra applets, describe the same physical situation from two different mathematical points of view. In particular, the teacher introduced a new GeoGebra applet (psicro.5, see **Figure 47**), displaying it on the IW, in which the two graphs were displayed side by side. These representations constitute two different descriptions of the same phenomenon from two different standpoints. Adopting the lens of conceptual blending, they can be interpreted as two different knowledge input spaces that students are required to blend in a new knowledge space so to have a global and coherent understanding of the phenomenon. Hence, the theoretical lens of conceptual blending will help us in detecting the emerging cognitive mechanisms when different mathematical representations are involved.

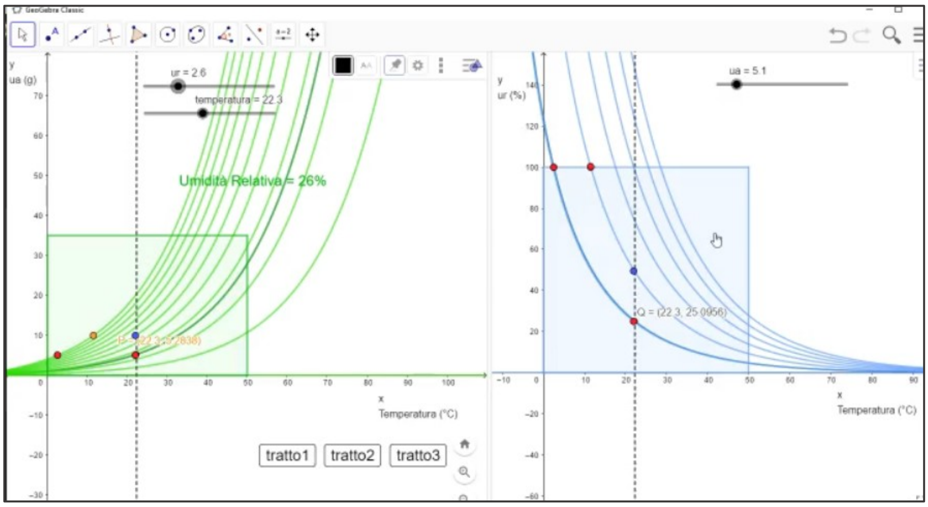



**Figure 47** – GeoGebra applet showing both the diagrams

Finally, during the last 1-hour discussion, all the concepts emerged during the previous lessons were recalled: Silvia guided the students in doing it by posing suitable questions. In the end, the notion of *Humidex index*[26], an index elaborated to measure the degree of wellness according to different thermodynamic conditions, was introduced and explained. This notion enabled students to better understand the concept of perceived temperature which they had read in the newspaper article at the beginning the T.E.: one of the purposes of these activities was to provide students with some tools to interpret and explain real phenomena.

### 12.1.6 Prospective analysis

This T.E. differs considerably from the others two. The first point to be underlined is that this time the initial step of the mathematical modelling process is a classroom experiment and not a simulation. We expect the students to largely use the physical interpretation of the described phenomenon to be able to read such a complicated chart as the psychrometric chart is. The

[26] Some information about the Humidex index can be found here.

external representation that we assume will emerge the most in students' reasoning is exactly the experiment and so we expect to recognize a wide presence of a blended language in which the mathematical and physical knowledges are merged. Concerning covariational reasoning, we expect it to be even harder to apply the taxonomy so far adopted; indeed, we predict that the instrumentation supporting this T.E. will mainly facilitate a holistic and global approach, rather than a more local one focused on the coordination of numerical values of the magnitudes involved. Hence, second-order covariation should be predominant with respect to COV 1, and it will probably manifest in a different form because the level of metavariation, that is present in the GeoGebra applet, is not the real focus of classroom discussions that privilege the ability to establish relations between different representations. This point is a clear example of what we called wide mesh a priori analysis of the activities (Section 9.2), typical of Silvia's teaching method. Instead of focusing on COV 2 as it emerged in the Galileo teaching experiment, we tried to give space to students' emerging conceptualizations, and this resulted in a redesign on the spot of the activities previously planned.

Even if the Carrier diagram is of complicated reading, we expect students to benefit from the previous experimentation they participated in, specifically in the agile use of the technological supports and in relating the graphical representations with some mathematical properties (degree of the function, finite differences, and a general formula describing it).

Silvia will presumably adopt the typical adaptive strategies that connote her teaching, but we expect her to play a determinant role in mediating between the several representations involved.

## 12.2 DATA ANALYSIS

In this section we are going to illustrate in detail six episodes mainly examined adopting the Timeline tool and the data from students' answers to Task 2.

- Data from Task 2 analyzed here (12.1.1) are students' answers to the question "Is there a relation between temperature and relative humidity?". Some of the most relevant answers revealing covariational reasoning will be commented qualitatively and later on analyzed from a lexical and syntactical point of view .
- Episode 1 (12.2.2) is an excerpt from the first classroom discussion (Discussion 1) which was conducted after the working group session on Task 3. Students had worked in small groups on the data from their classroom experiment and the reading of a real psychrometric chart guided by some instruction provided on a worksheet. During that session, the students worked divided in the same five groups of the previous T.E..

- Episode 2 (12.2.3) comes from the second teacher-led discussion (Discussion 2) conducted in presence after that, as homework, students had worked on the trend of the graph that represents the values of relative humidity as a function of temperature. This episode specifically refers to the answer provided by a student, Matteo, who tried to identify a precise mathematical formula.
- Episode 3 (12.2.4), Episode 4 (12.2.5), Episode 5 (12.2.6) and Episode 6 (12.2.7) are four excerpts from the third teacher-led discussion (Discussion 3) which took place in presence after that students had worked on Task 5, the last one, during which students were asked to trace the experiment cycle on the graph showing the relationship between relative humidity, on the *y*-axis, and temperature, on the *x*-axis.

Throughout the whole analysis, four different artefacts will make their appearance: the classroom experiment (E), the GeoGebra applets (G), the formulas (F), and the interactive whiteboard (IW). In the Timelines students' gestures are not reported because they were not visible due to the poor quality of the video recordings.

### 12.2.1 Data from Task 2 (students' answers to question 3)

Concerning question 3 to Task 2, only 18 students (over 22) uploaded their answers on the Classroom platform. Here we are going to comment qualitatively on the different approaches adopted by students when trying to describe the relationship between temperature and relative humidity: students had to reflect using the data provided in **Figure 43** or the same class of data they collected during the experiment they did on their own at home. All the 18 students replied affirmatively, i.e., that there is a link or a relationship between temperature and relative humidity even if some of them specified that the relation "varies according to different factors" [139 - S11] or again that they "are not totally connected to each other because even with different temperatures the humidity is equal" [140 - S4]. Focusing on the sentences formulated by students to describe this possible relationship, they claim that (we report just some examples as a reference, both in Italian and translated into English):

| | |
|---|---|
| [141 – S21] | **When** the temperature decreases, the relative humidity increases and vice versa.<br>*Quando la temperatura diminuisce, l'umidità relativa aumenta e viceversa.* |
| [142 - S4] | **If** the temperature increases, the humidity decreases, **while if** the temperature decreases, the humidity increases.<br>*Se la temperatura aumenta, l'umidità diminuisce, mentre se la temperatura diminuisce, l'umidità aumenta.* |

| | |
|---|---|
| [143 – S1] | The piecewise line representing the temperature [red] initially has rather low values, while the blue one [relative humidity] has higher values. Then the piecewise red line begins to grow **while** the other begins to decrease.<br>*La linea spezzata rappresentante la temperatura inizialmente ha valori abbastanza bassi, mentre quella blu ha valori più alti. Successivamente la linea spezzata rossa inizia a crescere mentre l'altra comincia a decrescere.* |
| [144 – S19] | The data, in the two open piecewise lines, are approximately <u>inversely proportional</u>, **when** one grows, the other decreases and vice versa.<br>*I dati, nelle due linee aperte spezzate, sono approssimativamente inversamente proporzionali, quando una cresce, l'altra diminuisce e viceversa.* |
| [145 – S15] | The two graphs, after they have been modified to facilitate the reading, seem <u>almost mirrored</u>, that is with the increase of the temperature the relative humidity decreases, and at 17. 2°, the minimum temperature recorded, the highest is relative humidity.<br>*I due grafici, dopo che sono stati modificati per facilitarne la lettura, sembrano quasi specchiati, cioè con l'aumento della temperatura l'umidità relativa diminuisce, e ai 17. 2°, la temperatura minima registrata, si ha l'umidità relativa maggiore.* |
| [146 – S12] | When the temperature drops, the relative humidity tends to rise and vice versa. This fact has, in my opinion, a simple physical explanation: when it is warmer the water tends to evaporate more and the air, consequently, to get drier; when the temperature is lower, the water vapor present in the air tends not to rise, and the air is consequently wetter.<br>*Quando la temperatura scende, l'umidità relativa tende a salire e viceversa. Questo fatto ha, secondo me, una semplice spiegazione fisica: quando fa più caldo l'acqua tende a evaporare maggiormente e l'aria, di conseguenza, a farsi più secca; quando la temperatura è invece più bassa il vapore acqueo presente nell'aria tende a non salire, e l'aria risulta, di conseguenza, più umida.* |
| [147 – S16] | As the temperature increases, the water vapor particles will decrease and, conversely, as the temperature decreases, they will be present in the air in greater quantity.<br>*All'aumentare della temperatura le particelle di vapore acqueo diminuiranno e al contrario queste al diminuire della temperatura, saranno presenti nell'aria in quantità maggiore.* |

We can observe that despite the numerical values provided by the experiment, the students mainly express the possible temperature-relative humidity relationship in qualitative terms. The covariational reasoning emerging is COV 1 – L2 (coordination of values) and the syntactical structures recurring are the same already identified for this level of reasoning i.e., binary relations expressed as "A increases *while* B decreases" or again "*when* A increases, B decreases". Only a few students tried to describe globally this relationship and speak of inverse proportionality [144] or using an informal language, the two graphs "seem almost mirrored" [145]. Finally, a few students tried also to motivate that relationship from the standpoint of a physical interpretation. Indeed, they refer to the process of evaporation of the water vapor, "when it is warmer the water tends to evaporate more and the air, consequently, to get drier" [146] or "As the temperature increases, the water vapor particles will decrease" [147]. The level of the discourse emerging from students' claims is still at a descriptive stage and the references to the physical domain are not blended with the mathematical knowledge but simply juxtaposed to it.

### 12.2.2 Episode 1 (Discussion 1, 39:54-43:23)

This is an excerpt from the 1-hour discussion in presence (Discussion 1) led by the teacher after the working group session on Task 3. During this episode, students, guided by the teacher, tried to retrace what happened during the pot experiment on the psychrometric diagram. The applet reproducing the psychrometric chart is shown on the IW and students have already identified point P (14.3; 12.06) that represents the point on the saturation line in which the temperature coincides with the temperature of the dew point. Point Q instead has coordinates Q (22.14; 12.06) and represents the point that has the same ordinate as P and the ambient temperature as abscissa (see **Figure 48**).

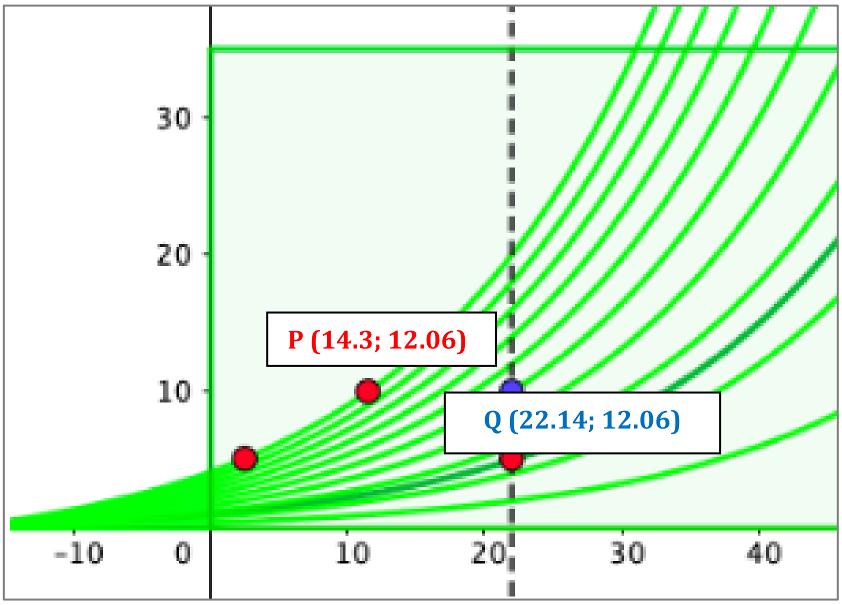



**Figure 48 –** Applet shown on the IW. We added the coordinates of points P and Q to facilitate the reading of the transcript

| | Timing | Who | Utterances | Gestures |
|---|---|---|---|---|

| | | | | |
|---|---|---|---|---|
| 148 | 00:39:54 | Teacher | On the graph, how can I read these passages? We said this is the starting point because we said we do not go from P to Q, but we start from Q. Starting from Q, where did we go? | *The teacher reproduces with a finger the cycle on the graph and then points at point Q on the IW.* |
| 149 | 00:40:25 | Giorgia | We decreased the temperature hence we moved to the left. | |
| 150 | 00:40:28 | Teacher | We decreased the temperature hence we moved to the left. In which way? Did you just decrease the temperature or not? We are during the moment in which you continued to pour and pour [the ice]. | |
| 151 | 00:40:46 | Emanuele | Only the temperature decreases. | |
| 152 | 00:40:50 | Teacher | Only the temperature decreases. And so, on the graph, how do you move? | |
| 153 | 00:40:55 | Emanuele | Horizontally. | |
| 154 | 00:40:56 | Teacher | Horizontally. We have point Q and we move horizontally to decrease the temperature. Until when do we move horizontally? | |
| 155 | 00:41:11 | Emanuele | Until the dew point. | |
| 156 | 00:41:13 | Teacher | Until the dew point that is until when we find on which of these green curves? | |
| 157 | 00:41:24 | Emanuele | Until that of 100%. | |
| 158 | 00:41:39 | Teacher | […] And then? What did we do after we reached the saturation of 100%? Did we stop immediately? […] | |
| 159 | 00:42:05 | Giorgia | No, we waited until it condensed well, and, in the meanwhile, we continued to add ice. | |
| 160 | 00:42:13 | Teacher | So, what did you do? | |
| 161 | 00:42:14 | Giorgia | I continued to decrease the temperature. | |
| 162 | 00:42:18 | Teacher | Hence on the graph, where do you move? | |
| 163 | 00:42:19 | Giorgia | To the left. | |
| 164 | 00:42:20 | Teacher | To the left. Horizontally? | |

| | | | | |
|---|---|---|---|---|
| 165 | 00:42:21 | Giorgia | No… [not really convinced]. If you have reached the dew point yes… only the temperature changes. | |
| | | | […] | |
| 166 | 00:42:50 | Emanuele | If we already have the dew point, humidity is decreasing. | |
| 167 | 00:42:56 | Teacher | If we already have the dew point, humidity is decreasing. And so? | |
| 168 | 00:43:03 | Valeria | It tends toward the $x$-axis | |
| 169 | 00:43:05 | Teacher | Not only the temperature decreases, and it tends toward the $y$-axis but also toward the $x$-axis. In which way do we move on this graph? | |
| 170 | 00:43:23 | Valeria | Following the curve. | |

The teacher opens the discussion underlining that the starting point of the experiment is represented by point Q, and not P [148], the one that has coordinates referred to the initial conditions of the experiment, and with her finger Silvia points at Q on the applet displayed on the IW (**Figure 49** – gesture row). Then, Silvia asks her students where they would move on the graph. Giorgia takes the word and referring to experiment states that since they "decreased the temperature, we moved to the left [on the graph]" [149]. The teacher revoices Giorgia's words with an approval tone and accompanies her words with a horizontal movement of the hand in the air (**Figure 49** – gesture row). Then, she asks if only temperature was decreasing and recalls that they are at that step of the experiment during which they continued to pour ice into the pot and she simulates the gesture of pouring, an iconic gesture with a narrative function (**Figure 49** – gesture row). Emanuele intervenes remarking that "only the temperature decreases" [151] and so they move "horizontally" [ 153]. Silvia again repeats Emanuele's words with an approval tone, she retraces the shift to the left on the graph on the IW, so attributing her gesture a grounding function (**Figure 49** – gesture row), and again asks to the whole classroom (dotted short arrow in the interaction flowchart – **Figure 49**) until when they move horizontally [154]. Emanuele replies "until the dew point" [155] and Silvia, after having revoiced his words, asks for the green curve on which they should stop [156], simulating the trend of the curve with her hand (gesture row – **Figure 49**) and Emanuele adds "until that of 100%" [157]. The teacher facilitates again the flow of discussion asking what happened after they reached the saturation [158] and looks around the classroom with a signalizing gaze looking for students' reactions. Giorgia observes that during the experiment they continued to pour ice waiting for the condensation to be more visible [159] and when the teacher asks her what she did [160], Giorgia adds that she "continued to decrease the temperature" [161]. Silvia repeats her word with an approval tone and invites to translate that decrease on the graph [162]. Giorgia replies that they move "to the left" [163] and then Silvia asks if that movement is horizontal [164]. Giorgia initially states "no", but she is not confident with her answer. This uncertainty reveals her difficulty in getting into the blend. The tone of her voice denotes that she is making a cognitive effort to overcome the difficulty in amalgamating the information provided by different representations. Indeed, she changes her mind in a "yes" because after having reached the dew point "only the temperature changes" [165]. After a while, Emanuele claims that after the dew point, "humidity is decreasing" [166]. Silvia revoices his words and asks for more explanations [167]. Then, Valeria takes the word and specifies that it also "tends toward the x-axis" [168]. Hence, Silvia remarks that the movement on the graph is not only toward the $y$-axis but also toward the $x$-axis and reproduces the trend with a metaphoric gesture of the hand (gesture row – **Figure 50**). Then she asks in which way do they move on the graph [169] and Valeria adds that they move following the [green] curve [170].

The teacher supports a lot towards understanding the phenomenon. Indeed, the difficulty of the faced topic emerges in the short interventions of the students: they answer with concise claims and only after a continuous stimulation by the teacher that constantly asks them how they would move on the graph, so inviting them to relate the experiment to GeoGebra. These artefacts, the experiment (E) and the GeoGebra applet (G) shown on the IW, are the two artefacts that mainly influence the episode and are used by both the teacher and the students with a descending control (artefacts interactions row – **Figure 50**). The teacher enhances a synergic use of the two artefacts that emerges with evidence in some of students' claims such as [149] or [165]. Indeed, all the episode is centered on a game of displacement between the graph and the experiment that produces a cognitive and interpretative effort in the students and results in the blending of the knowledge from three different input spaces clearly revealed by the lexical analysis: (i) *notions of change and dynamicity* (decreased [149]; decreases [151]; until that of 100% [157]; decrease [161]; is decreasing [166]); (ii) *spatial references connected to the graphical representation* (left [149]; horizontally [153]; left [163]; it tends toward the x-axis [168]; following the curve [170]); (iii) *physical interpretation referred to the classroom experiment* (until it condensed [159]; continued to pour ice [159]). A strong example of blending can be recognized

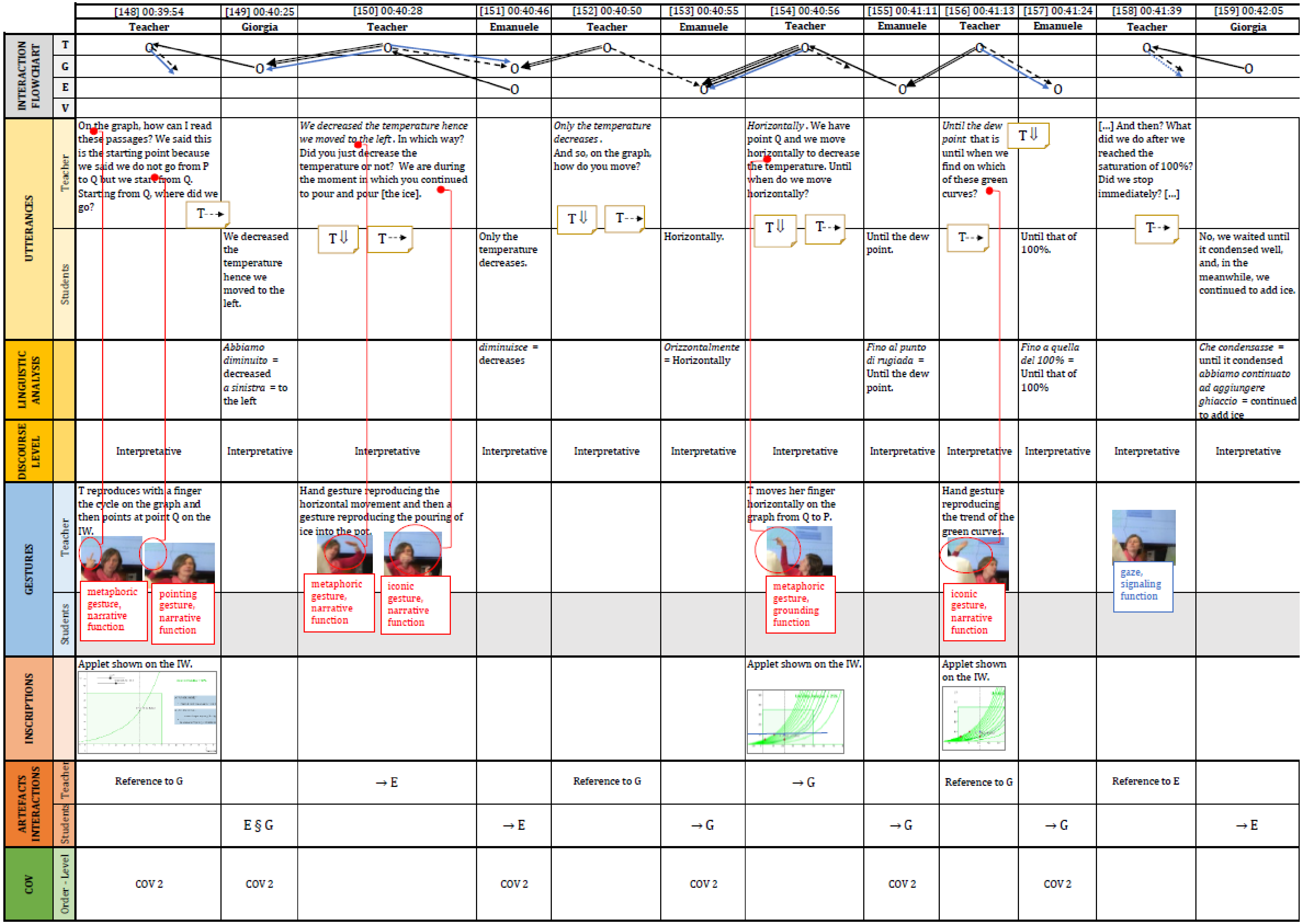



**Figure 49** – Timeline 1 – Part I (Dew point 2020)

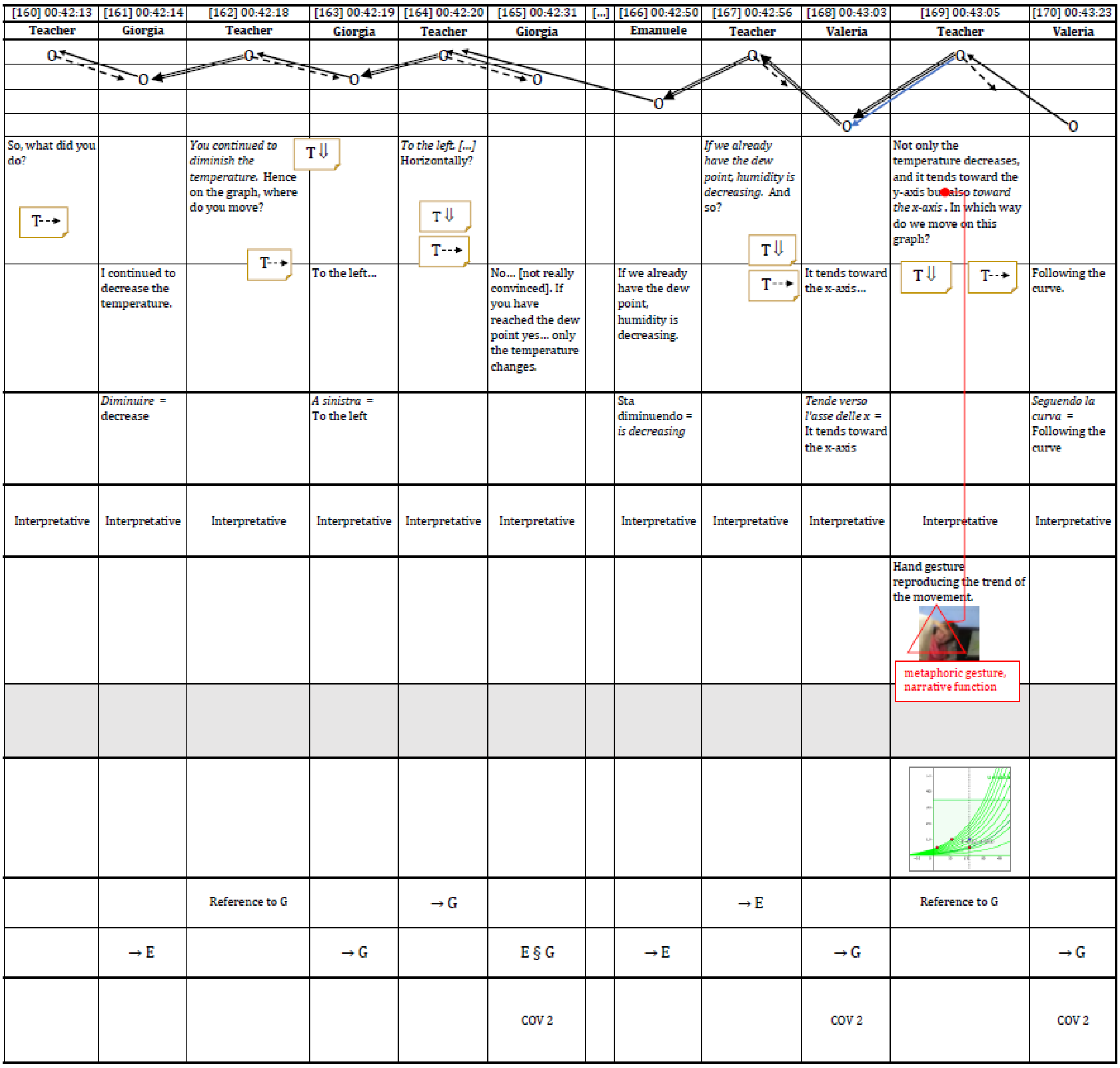



**Figure 50** – Timeline 1 – Part II (Dew point 2020)

in Giorgia's claim "If you have reached the dew point yes… only the temperature changes" [165] that despite its incorrectness reveals her ability to blend various information provided by different sources. From the discourse standpoint, we can notice the interlacing of two different narratives: a qualitative one, used to describe what happened during the experiment and in the narration the use of personal pronouns emerges predominantly (e.g., "*we* decreased the temperature" [149], "*we* waited until it condensed well" [159], "*I* continued to decrease the temperature" [161]); a quantitative one used to describe how the magnitudes involved are changing (e.g., "Only the temperature decreases" [151], "humidity is decreasing" [166]). Concerning covariational reasoning, in this episode it seems difficult to apply the levels classification used until this moment. The students already possess the psychrometric diagram representing the covariation of magnitudes involved and they are moving on this representation: we can for sure recognize the enhancement of a global approach supported by the representations involved; hence, this reveals a different characterization of second-order covariation that still needs to be clarified in light of the analysis of the other episodes.

### 12.2.3 Episode 2 (Discussion 2, 23:07-25:28)

During Task 4 (12.1.4) students were asked to sketch individually the trend of the graph of relative humidity with respect to temperature. Silvia opens the lesson with a classroom discussion in which she comments on students' answers. In particular, in this episode (23:07-25:28) we analyze the excerpt during which the teacher shows on the IW the solution elaborated by Matteo and asks him to explain how he found it. This episode has been chosen because Matteo is the only students who not only sketched the graph but also looked for an algebraic expression.

| | Timing | Who | Utterances | Gestures |
|---|---|---|---|---|
| 171 | 00:23:07 | Teacher | What did you do? | |
| 172 | 00:23:10 | Matteo | First I tried to look for a function… I located all the points and then through a system I tried to look for a function passing through all the points, but it came a little higher or a little lower. Then I tried to play with the sliders and… [unclear] | |
| 173 | 00:24:00 | Teacher | What kind of function did you think of? | |
| 174 | 00:24:03 | Matteo | I thought that relative humidity was a number *a* over *b* times the temperature plus *c*… […] but it came nothing good… | *The teacher writes the formula on the IW.* |
| 175 | 00:24:35 | Teacher | In which sense nothing good? | |
| 176 | 00:24:40 | Matteo | The function didn't touch all the points… | |
| 177 | 00:24:41 | Teacher | The function didn't touch all the points… | |
| | | | […] | |
| 178 | 00:25:16 | Teacher | And so, what can we conclude? | |
| 179 | 00:25:20 | Matteo | [unclear] | |

| | | | | |
|---|---|---|---|---|
| 180 | 00:25:28 | Teacher | The function is not that one or the data do not perfectly fit the function… but it can be that the function probably is not a hyperbole… | |

The teacher asks Matteo to explain what he did to identify the function shown on the IW (**Figure 51**) and looks at him inviting him to react (signaling gaze, gesture row – **Figure 52**). Matteo replies that he located all the points on the Cartesian plane [using GeoGebra] and then he tried to look for a function passing for all the points, but the function "came a little higher or a little lower" [172]. He adds that then he tried to play with the sliders, but the final part of his sentence is not understandable [172]. The teacher asks him to state which function he thought of [173] and Matteo describes in words a formula in which he relates relative humidity and temperature, and such formula contains three parameters that he calls with the letters *a*, *b*, and *c* [174].

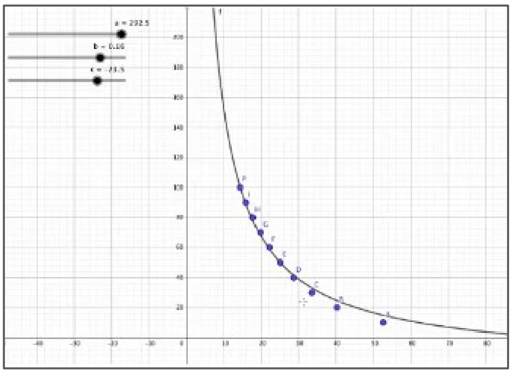


**Figure 51 –** Possible relative humidity-temperature graph proposed by Matteo

Meanwhile he speaks, the teacher writes on the IW the formula (inscriptions row – **Figure 52**): the writing gesture has a grounding function and reveals the relevance of that algebraic expression. Matte concludes his intervention stating that nothing good came and the teacher asks him clarifications [175]. The student clarifies that "[t]he function didn't touch all the points" [176]: this claim can be interpreted as a commognitive conflict: students, convinced that all the functions with a certain shape are hyperboles, collide with the software that instead uses a more advanced language and questions their conviction. Then the teacher asks what they can conclude from those observations [178]. Even though Matteo's answer is not understandable, the teacher sums up his reasoning and presents two possibilities: the function may be wrong, or data are imprecise and do not perfectly fit the function, but she remarks that the sought function probably is not a hyperbole [180]. We just recall that students had not studied yet the exponential function.

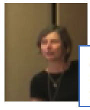
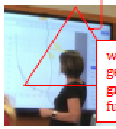
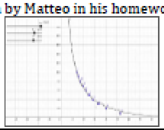
| | | [171] 00:23:07 Teacher | [172] 00:23:10 Matteo | [173] 00:24:00 Teacher | [174] 00:24:03 Matteo | [175] 00:24:35 Teacher | [176] 00:24:40 Matteo | [177] 00:24:41 Teacher | [...] | [178] 00:25:16 Teacher | [179] 00:25:20 Matteo | [180] 00:25:28 Teacher |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| INTERACTION FLOWCHART | T | O | | O | O | O | | O | | O | | O |
| | G | | O | | O | | O | | | | O | |
| UTTERANCES | Teacher | What did you do? T--→ | | What kind of function did you think of? T--→ | | In which sense *nothing good*? T ⇓ T--→ | | *The function didn't touch all the points ...* T ⇓ | | And so, what can we conclude? T--→ | | The function is not that one or the data do not perfectly fit the function... but it can be that the function probably is not a hyperbole... |
| | Students | | First I tried to look for a function... I located all the points and then through a system I tried to look for a function passing through all the points, but it came a little higher or a little lower. Then I tried to play with the sliders and... [unclear] | | I thought that relative humidity was a number $a$ over $b$ times the temperature plus $c$ ... [...] but it came nothing good... | | The function didn't touch all the points... | | | | [unclear] | |
| LINGUISTIC ANALYSIS | | | *Cercare una funzione* = look for a function | | $r.h. = \frac{a}{bt+c}$ | | | *La funzione non toccava tutti i punti.* The function didn't touch all the points... | | | | *Può essere che la funzione probabilmente non sia un'iperbole* = it can be that the function probably is not a hyperbole |
| DISCOURSE LEVEL | | | Objectified | | Interpretative | | Objectified | | | | | |
| GESTURES | Teacher | T looks at Matteo inviting him to reply.  gaze, signaling function | | | T writes the formula on the IW.  writing gesture, grounding function | | | | | | | |
| | Students | | | | | | | | | | | |
| INSCRIPTIONS | | T shows on the IW the answer given by Matteo in his homework.  | | | Formula written by the teacher on the IW. $\frac{a}{bt+c}$ | | | | | | | |
| ARTEFACTS INTERACTIONS | Teacher | G → | | | → IW | | | | | | | Reference to F and G |
| | Students | | → (G, F) | | → F | | → G | | | | | |
| COV | Order - Level | | COV 2 | | COV 2 | | | | | | | |

**Figure 52** – Timeline 2 (Dew point 2020)

The approach followed by Matteo is that of, given some points, looking for the formula of a function that passes for all the points and so discover the graphical-functional relationship. The level of the discourse is objectified because Matteo already possesses a global image of the function, but the formula stated at [174] is in a blended form since Matteo does not refer to some generic variables $x$ and $y$ but to the specific magnitudes involved, temperature and relative humidity, and the formula also contains three letters $a$, $b$, $c$ that denote the dependence also on other undefined factors. Matteo explicitly states that he played with the sliders to adjust the trend of the function [172]. In this episode, second-order covariation fully manifests in a quantitative form, i.e., in the elaboration of a generic algebraic formula aiming at describing globally the situation. The intervention of the teacher does not reveal explicitly the reason why the graph does not perfectly fit the data, but Silvia helps in bringing out some possibilities and in addressing students on the right path [180].

### 12.2.4 Episode 3 (Discussion 3, 3:43-4:12)

This is the first of four episodes extrapolated from Discussion 3, the classroom discussion that followed a working group session during which students worked on a new applet showing the graph of absolute humidity with respect to temperature, and relative humidity instrumented through a slider. Students were asked to retrace on this new graph the steps of the pot experiment. In this excerpt (3:43–4:12) the teacher is commenting on the algebraic expression of the function (**Figure 53**) and is guiding students in reflecting on how the slider associated to the numerical value of the numerator affects the graph of the function.

$$f(x) = \frac{10}{0.6\left(e^{\frac{0,2+x}{17,5}} - 0,4\right)}$$

**Figure 53** – Analytic expression of the function shown in the GeoGebra applet

This episode has been chosen because the questioning of the teacher invites to reflect globally on the properties of the function influenced by a specific parameter and this kind of reasoning is essential to develop second-order covariation.

| | Timing | Who | Utterances | Gestures |
|---|---|---|---|---|
| 181 | 00:03:43 | Teacher | To have a numerator equal to 10 or 15, what does it mean from the point of view of the function? | |

| | | | | |
|---|---|---|---|---|
| 182 | 00:03:48 | Girls | G1: It is dilatated<br>G2: Exactly… | |
| 183 | 00:03:50 | Teacher | We have a dilatation, for sure, that is we have a variation of our function. Do we have the same function? | |
| 184 | 00:04:00 | Teacher | This and this are the same function? | *The teacher changes the value of the slider related to the absolute humidity and shows two different graphs on the IW.* |
| 185 | 00:04:05 | Student | No | |
| 186 | 00:04:06 | Teacher | No, but are they so much different? | |
| 187 | 00:04:10 | Many voices | No, they are similar. | |
| 188 | 00:33:02 | Teacher | T: No, they are not the same function because they have two different analytic expression, they have two different graphs […], but the form, the structure of the function is always the same, it does not change much. | *The teacher changes the value of absolute humidity slider and shows the function with a.h.=10 and a.h.=15.* |

At [181], the teacher asks to her students how a different value of the numerator affects the function. Some girls answer that the function is dilatated [182] and the teacher revoices the girls' claim [183] and uses her hands to simulate the dilatation of the function (gesture row – **Figure 54**). Then Silvia asks her students if they have the same function [183], but students do not react immediately. Hence, the teacher changes the value of the slider of absolute humidity and, showing consecutively two different graphs, Silvia asks if the two are the same function [184]. A non-identifiable student whispers no [185]; the teacher remarks his answer with an approval tone and asks if the two functions are so much different [186]. Many students reply that they are similar [187], and Silvia adds that "the form, the structure of the function is always the same, it does not change much" [188]. While saying this, Silvia again shows on the IW the function for two different values of absolute humidity. The teacher uses the GeoGebra applet with a descending control because she is conscious that it contains information that can help students to answer her questions; the students use it with an ascending control because they look at it so to obtain suitable information (artefacts interaction row – **Figure 54**). Concerning covariational reasoning, the teacher clearly enhances COV 2 inviting students to reflect on how the parameter affects the trend of the function and enriches their findings, since they are not much talkative,

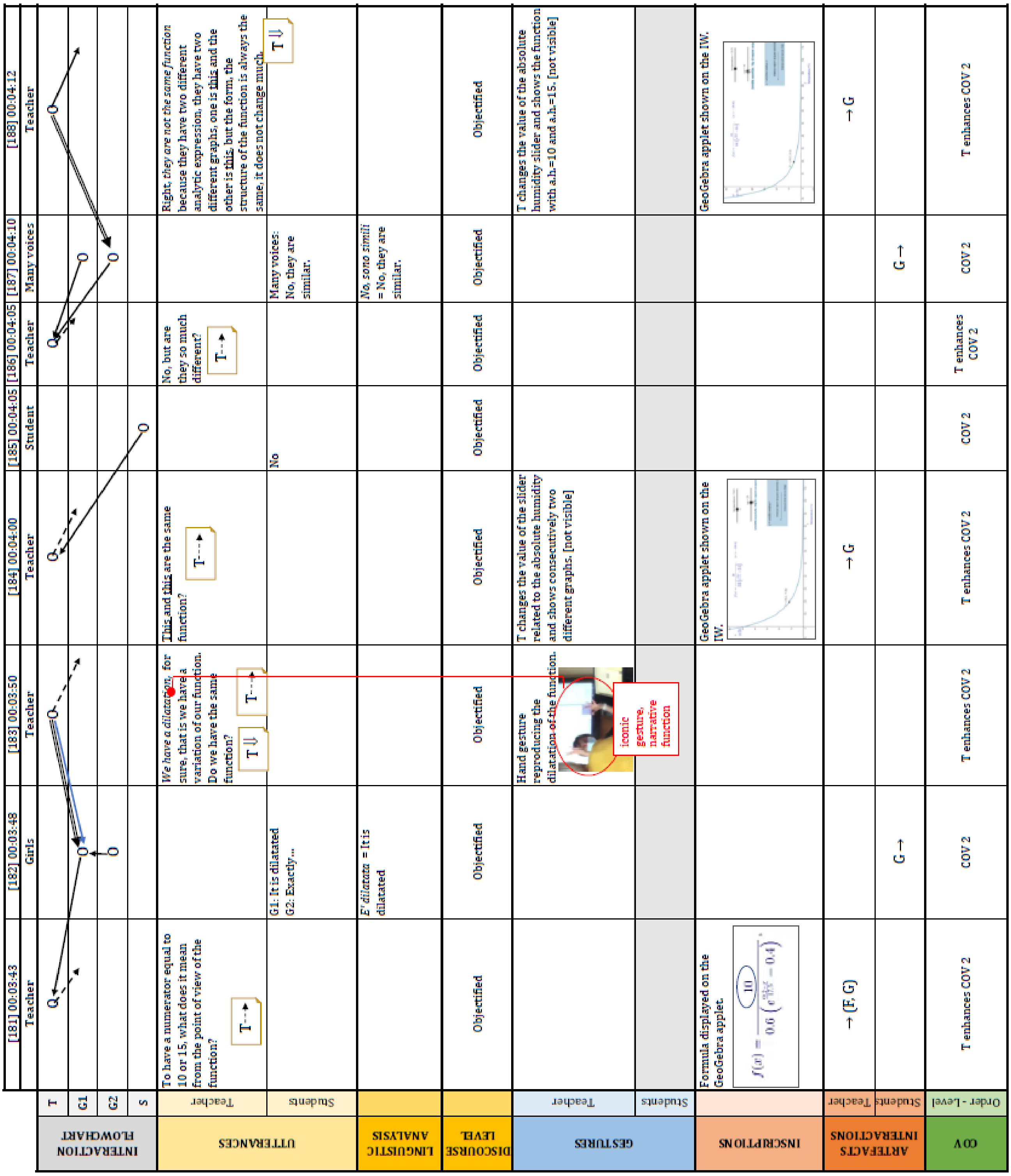



**Figure 54** – Timeline 3 (Dew point 2020)

stating that even if the two functions have different analytical expression, their form does not change much. The level of the discourse is objectified because students have already worked on the absolute humidity – temperature function and so they fully conceive it as a mathematical object. The linguistic analysis instead reveals the use of terms such as “dilatated” [182] and “similar” [187] that are used by students to describe qualitatively the relation between the trend of the graphs of the function obtained for different values of the absolute humidity i.e., the parameter.

### 12.2.5 Episode 4 (Discussion 3, 18:00-22:41)

At this point of the discussion, the teacher is focusing on reconstructing the cycle of the pot experiment on the new chart, the one describing the relationship between absolute humidity and temperature. In this episode (18:00-22:41), the teacher is discussing about the second step, the one during which they continued to decrease the temperature in the pot by adding ice cubes and then some drops of water formed on the wall. Silvia starts a classroom discussion reacting to the graphs made by the students as homework and asking them to explain their choices. In particular, she observes that some students drew a horizontal trait and others a vertical one.

| | **Timing** | **Who** | **Utterances** | **Gestures** |
|---|---|---|---|---|
| 189 | 00:18:00 | Teacher | Second trait. Will it be rectilinear vertical or horizontal like it happens here? And why? Let’s start from the first. Which are the groups that made the second trait rectilinear vertical? Why? | |
| 190 | 00:18:30 | Matteo | Because we decreased the absolute humidity hence the point went down… | |
| 191 | 00:18:33 | Teacher | Ok, you have decreased the absolute humidity. If you go down vertically, does the absolute humidity decrease? Yes. Does the relative humidity decrease? Yes. Is there something that remains constant? | |
| 192 | 00:18:55 | Matteo | The temperature. | |
| | | | […] | |
| 193 | 00:20:00 | Teacher | What is happening instead on the horizontal trait? | |
| 194 | 00:20:03 | Arianna | The relative humidity maintains constant. | |
| 195 | 00:20:05 | Teacher | The relative humidity maintains constant. | |
| 196 | 00:20:08 | Arianna | And the temperature decreases. | |

| | | | | |
|---|---|---|---|---|
| 197 | 00:20:11 | Teacher | The temperature decreases. The absolute humidity? Does it decrease or remain constant? | |
| 198 | 00:20:20 | Adele | Decreases. | |
| 199 | 00:20:22 | Teacher | Decreases. Why? | |
| 200 | 00:20:24 | Emanuele | You have the condensation. | |
| 201 | 00:20:26 | Teacher | Ok, you have the condensation, and this is what happens practically. But on the graph why? […] | |
| 202 | 00:20:36 | Adele | The curve changes. | |
| | | | […] | |
| 203 | 00:21:16 | Teacher | What does the second trait tell us? You spoke of condensation, what happened during the second trait? | |
| 204 | 00:21:26 | Emanuele | It happened that we arrived at the dew point. | |
| 205 | 00:21:30 | Teacher | We arrived at the dew point, but the dew point is the beginning, the end, or the half of the second trait? | |
| 206 | 00:21:38 | Emanuele | The end. | |
| 207 | 00:21:39 | Teacher | The end? Do you agree? | |
| 208 | 00:21:45 | Many voices: | The beginning! | |
| 209 | | | *T recalls what happened during the experiment, which magnitudes changed and which others remained constant.* | |
| 210 | 00:22:34 | Teacher | Which graph will be the correct one? The horizontal segment or the vertical segment? | |
| 211 | 00:22:40 | Many voices: | Horizontal! | |
| 212 | 00:22:41 | Teacher | Horizontal, oh yeah! | |

As usual, Silvia begins the discussion with a question: she asks if the second trait is linear vertical or horizontal and in the meanwhile colors in yellow the two traits in the solutions proposed by two different groups and shown on the IW (inscriptions row – **Figure 57**). In particular, she starts asking clarification to the students of those groups who chose to represent the second trait of the cycle with a vertical line (**Figure 55**).

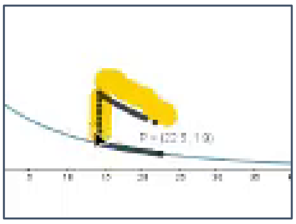


**Figure 55** – Graph of the cycle with a vertical second trait

Matteo replies that they did so because the absolute humidity is decreasing hence the point goes down [190]. The teacher revoices Matteo's words and then rhetorically asks if relative humidity decreases and she replies yes and then again asks if there is something that remains constant [191] and Matteo answers "the temperature" [192].

After a while, the teacher moves the attention to the other graph that she projects on the IW (**Figure 56**) and asks to describe what happens in this case [193].

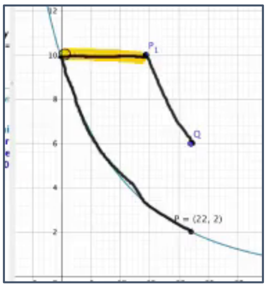



**Figure 56** - Graph of the cycle with a horizontal second trait

Arianna takes the word and says that "the relative humidity remains constant" [194] and "the temperature decreases" [196]. The teacher revoices her words both the times and then asks about the absolute humidity, if it remains constant or decreases [197], and Adele again replies that it decreases [198]. The teacher asks to motivate why it is decreasing [199]. Initially, Emanuele states that it is due to condensation [200], but Silvia hopes for an explanation based on the graphical representation [201] and while demanding for a different interpretation she points at the applet on the IW (gesture row – **Figure 58**). Adele then replies that absolute humidity is decreasing because "the [blue] curve changes" [202]. After a while, Silvia relaunches the discussion asking for a physical interpretation of the second trait of the cycle and recalls that Emanuele was speaking of condensation [203] and with a hand gesture addresses him (gesture row – **Figure 58**). Hence, Emanuele adds that they arrived at the dew point [204] and Silvia intervenes demanding where the dew point is located on the second trait [205] and Emanuele replies at the end [206]. Silvia, conscious that the answer is incorrect, asks to other students if they agree [207] and a chorus of voices answers “the beginning” [208]. To conclude the discussion and finally declare which of the two traits is the correct one, Silvia recalls which magnitudes changed and which remained constant during that phase of the experiment [209], and eventually asks which of the two traits is the correct one [210] reproducing with her hands the trend of the two segments (gesture row – **Figure 58**). The classroom replies the horizontal one [211] and the teacher revoices it with an exclamation of approval [212].

We can recognize how the teacher masterly manages a synergic use of the two artefacts, the chart and the experiment, which helps students in reflecting on which magnitudes change and how they change during the different steps of the pot experiment and the traits of the chart. As in Episode 1 (12.2.2), the inputs from the classroom experiment and the graph shown in the GeoGebra applet are blended in students’ reasoning and this data strongly emerges from the lexical analysis of students’ claims. The same three input spaces identified in Episode 1 recur in this excerpt: (i) *notions of change and dynamicity* (decreased [190]; constant [194]; decreases [196]; decreases [198]; the curve changes [202]); (ii) *spatial references connected to the graphical representation* (point went down [190]; arrived [204]; horizontal [211]); (iii) *physical interpretation referred to the classroom experiment* (condensation [200]). It’s Silvia who enhances the blend of those information thanks to her interventions and questioning, hence the level of the discourse is interpretative. Moreover, even in this case we can observe the enhancement of a global approach supported by the representations involved. Both the teacher and the students use the artefacts involved with an ascending control (artefacts interactions row – **Figure 57** and **Figure 58**).

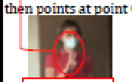
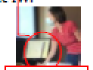
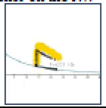
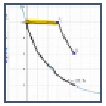
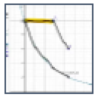
| | | [189] 00:18:00 | [190] 00:18:30 | [191] 00:18:33 | [192] 00:18:55 | […] | [193] 00:20:00 | [194] 00:20:03 | [195] 00:20:05 | [196] 00:20:08 | [197] 00:20:11 | [198] 00:20:20 | [199] 00:20:22 | [200] 00:20:24 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Teacher | Matteo | Teacher | Matteo | | Teacher | Arianna | Teacher | Arianna | Teacher | Adele | Teacher | Emanuele |
| INTERACTION FLOWCHART | T | | | | | | | | | | | | | |
| | M | | | | | | | | | | | | | |
| | Ar | | | | | | | | | | | | | |
| | Ad | | | | | | | | | | | | | |
| | E | | | | | | | | | | | | | |
| UTTERANCES | Teacher | Second trait. Will it be rectilinear vertical or horizontal like it happens here? And why? Let's start from the first. Which are the groups that made the second trait rectilinear vertical? Why? T⇢ | | *Ok, you have decreased the absolute humidity.* If you go down vertically, does the absolute humidity decrease? Yes. Does the relative humidity decrease? Yes. Is there something that remains constant? T⇓ | | | What happens instead on the horizontal trait? T⇢ | | *The relative humidity maintains constant.* T⇓ | | *The temperature decreases.* The absolute humidity? Does it decrease or remain constant? T⇓ | | *Decreases.* Why? T⇓ T⇢ | |
| | Students | | Because we decreased the absolute humidity hence the point went down... | T⇢ | M: The temperature. | | | The relative humidity maintains constant. | | And the temperature decreases. | T⇢ | Decreases. | | You have the condensation. |
| LINGUISTIC ANALYSIS | | | *Abbiamo diminuito* = decreased *il punto è sceso* = the point went down | | | | | *costante* = constant | | *diminuisce* = decreases | | *Diminuisce* = decreases | | *Hai la condensazione* = You have the condensation |
| DISCOURSE LEVEL | | Interpretative | Interpretative | Interpretative | Interpretative | | Interpretative | Interpretative | Interpretative | Interpretative | Interpretative | Interpretative | Interpretative | Interpretative |
| GESTURES | Teacher | T reproduces with her fingers the number 2 referring to the second trait. Then she colours in yellow the traits on the IW which she is referring to on the graph and then points at point Q on the IW.   metaphoric gesture, narrative function; writing gesture, narrative function | | | | | | | | | | | | |
| | Students | | | | | | | | | | | | | |
| INSCRIPTIONS | | The traits coloured in yellow by the teacher on the IW.   | | | | | Graph shown on the IW.  | | | | | | | |
| ARTEFACTS INTERACTIONS | Teacher | → G | | → G | | | → G | | → G | | → G | | | |
| | Students | | E § G | | | | | → G | | → G | | → G | | → E |
| COV | Order - Level | | COV 2 | | COV 2 | | | COV 2 | | COV 2 | | | | |

**Figure 57** – Timeline 4 – Part I (Dew point 2020)

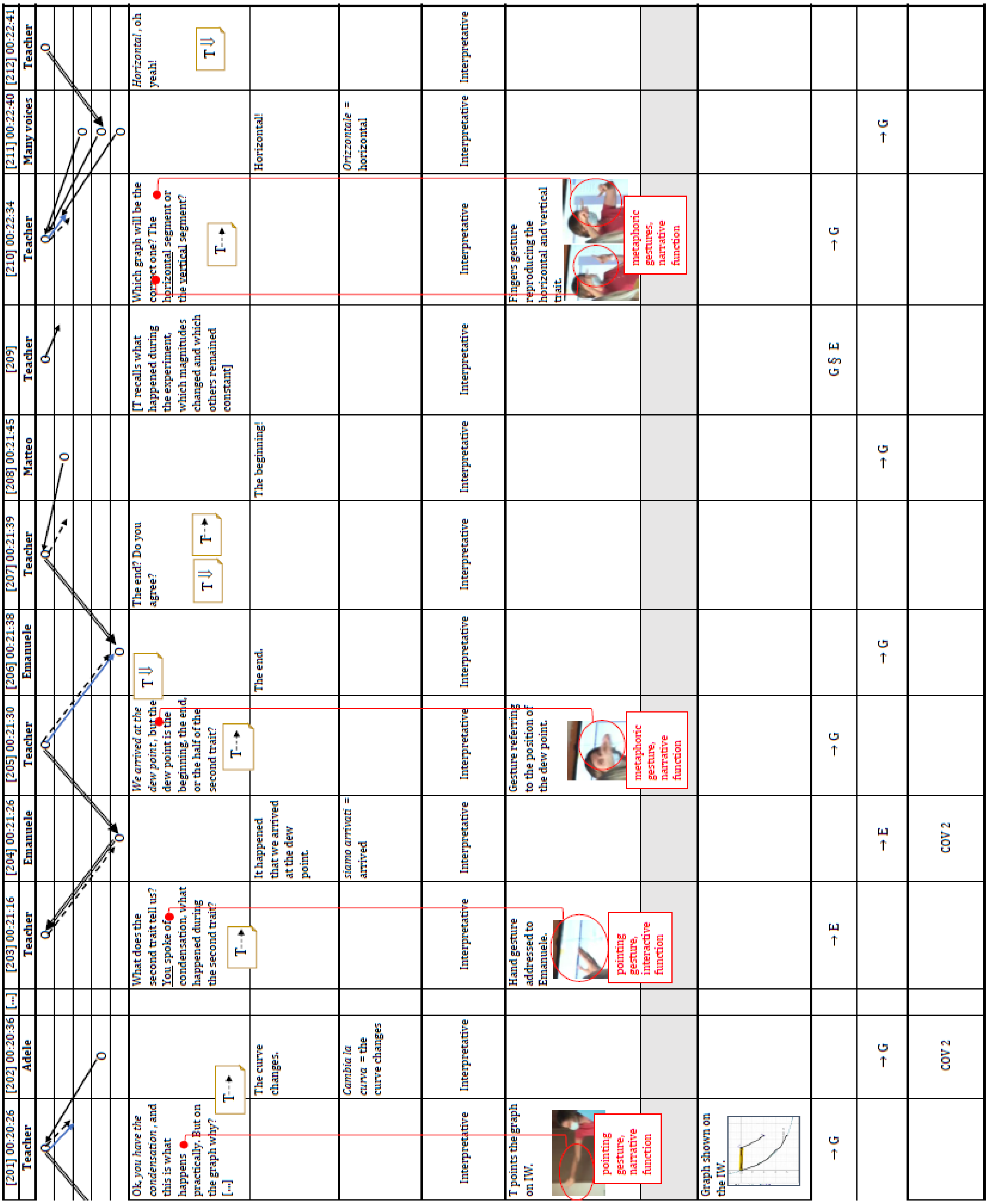


**Figure 58** – Timeline 4 – Part II (Dew point 2020)

### 12.2.6 Episode 5 (Discussion 3, 31:45-33:03)

This excerpt (31:45-33:03) belongs to Discussion 3 and it is analyzed just in its verbal component because no significant gestures or interactions were performed during this episode.

By the way, the episode is significant because shows the approach used by the teacher to reflect on the similarities and differences between the two psychrometric charts and it can be intended as a form of conceptualization of the different role of variables and parameters.

| | **Timing** | **Who** | **Utterances** | **Gestures** |
|---|---|---|---|---|
| 213 | 00:31:45 | Teacher | Are they two different situations/scenarios? | |
| 214 | 00:23:10 | Matteo | No. | |
| 215 | 00:31:50 | Teacher | No. Why do the two graphs are different if they are not two different situations? | |
| 216 | 00:32:00 | Matteo | The value represented on the $y$-axis is different. | |
| 217 | 00:32:02 | Teacher | The value represented on the $y$-axis is different. If you should make a comparison with something that is not mathematical but concerns real life… We have the same situation/scenario, but the value represented on the $y$-axis is different… If you should make an analogy… | |
| 218 | 00:32:35 | Giorgia | From the physical point of view, they represent the same thing but from the graphical point of view no… because they are two different values. | |
| 219 | 00:32:42 | Teacher | Oh! From the physical point of view, they represent the same thing but from the graphical point of view no because they are two different values. […] Two different situations depending on what? | |
| 220 | 00:33:02 | Matteo | A different point of view. | |
| 221 | 00:33:03 | Teacher | A different point of view. What we are doing is a different point of view. | |

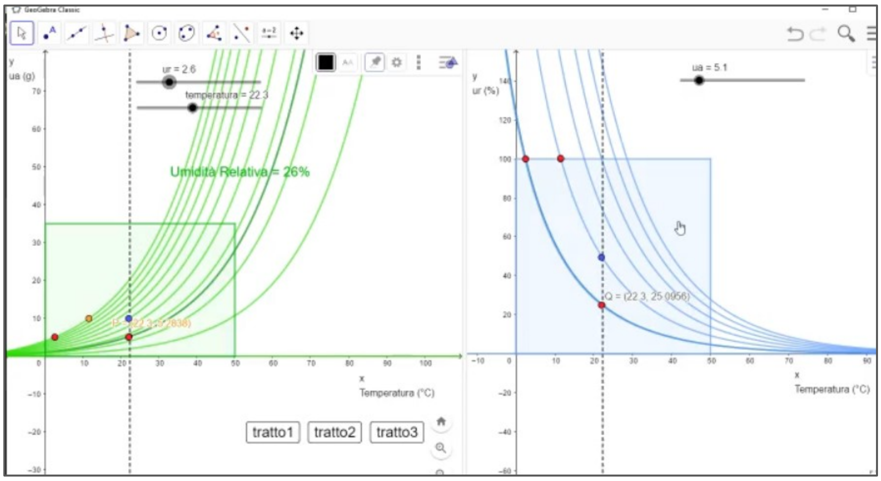



**Figure 59** – Applet containing the two charts simultaneously projected on the IW

At this point of the discussion, the teacher projects on the IW a new applet showing simultaneously the relationship of absolute humidity versus temperature (green chart) and of relative humidity versus temperature (blue chart). The teacher asks to the classroom if the graphs represent two different situations or scenarios [213]. Matteo obviously replies no [214], so Silvia asks why the graphs are different if they are not representing different things [215]. Matteo adds that the value represented on the $y$-axis is different [216], but Silvia invites students to provide a holistic interpretation, an analogy with something non-strictly mathematical [217]. Giorgia suggests that from the physical standpoint the situation is the same, what differs is the graphical representation [218]. Silvia revoices Giorgia's words with an approval tone and asks what the difference between the two situations depends on [219], and Matteo replies "a different point of view" [220] with the approval of the teacher [221]. With this interpretation, Silvia is suggesting that the different role assumed by variables and parameters does not determine a different mathematical situation but a change in standpoint that can be grasped in a different graphical representation. What is emerging in this episode is a kind of blending that is "backwards" with respect to the blending of Fauconnier and Turner: the students already possess the blend and what they are trying to do is to deconstruct it into its input spaces. Initially Matteo notices that the value represented on the $y$-axis is different [216], and then Giorgia adds that the diagrams differ only for their representational aspects but not the physical interpretation [218]. In the end Matteo remarks that their difference can be translated into a different standpoint (of representation) [220]. All these elements suggest that the level of the discourse is the interpretative one and the lexical analysis denotes a global and holistic approach. Concerning covariation, the reasoning emerging is of second-order and supported by the cognitive mechanism of blending.

### 12.2.7 Episode 6 (Discussion 3, 36:30-39:10)

This episode (36:30-39:10) locates towards the end of the Discussion 3: Silvia is still showing the applet with the two graphs on the IW and this time she is showing also the two *analytic* representations associated to the two graphs (**Figure 60**) and she is asking her students how the passage from one formula to the other one can happen.

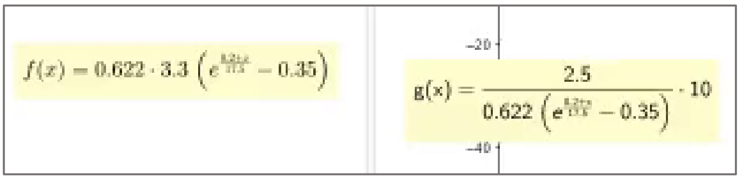


**Figure 60 –** The two formulas associated to the two graphs contained in the applet

| | Timing | Who | Utterances | Gestures |
|---|---|---|---|---|
| 222 | 00:36:30 | Teacher | How can I pass from this function to this function? What can I do? Try… | |
| 223 | 00:36:55 | Fabio | I have to put two parameters in place of the value that I want to put on the *y*-axis hence I have to put a parameter both for absolute humidity and relative humidity | |
| | | | […] | |
| 224 | 00:37:30 | Fabio | Instead of writing the value of the absolute humidity and relative humidity I put *a* and *b*… | |
| 225 | 00:37:33 | Teacher | Can you dictate me what happens here? | |
| | | | […] | |
| 226 | 00:37:53 | Fabio | In the second [case] I put… *[he dictates the formula]* | *The teacher writes the formula on the IW.* |
| 227 | 00:38:33 | Fabio | And then I make that *b* goes in place of *a* and *a* in place of *b*... | |
| 228 | 00:38:48 | Teacher | So, you say we make *a* times 0,66 times the parenthesis equal *b* times 10 | *The teacher writes the formula on the IW.* |
| 229 | 00:38:56 | Fabio | Then I divide by 10 both sides | |
| 230 | 00:39:00 | Teacher | Then I divide by 10 both sides | *The teacher writes the formula on the IW.* |
| 231 | 00:39:10 | Fabio | Then I can substitute *a* with the value I want, and I find *b* | |

The teacher addresses the classroom asking how they can pass from one formula to the other one and with a supportive tone she encourages students to try to guess some passages [222]. Fabio replies that two parameters are needed, to be put in place of the two values represented on the *y*-axis, i.e., absolute humidity and relative humidity respectively [223]. A while later he adds that in place of absolute humidity and relative humidity he should put letters *a* and *b* [224]. Then Silvia asks him to dictate her the formula so that she can write the passages on the IW [225]. Indeed, referring to the second graph/function, Fabio dictates to the teacher what he is thinking of and Silvia writes the formula on the IW (see **Figure 61** and **Figure 62** – inscriptions row). The non-redundant gesture of writing has a grounding function and reveals to the classroom the relevance of that claim (gesture row – **Figure 62**).

$$a = \frac{b}{0{,}66(-)} \cdot 10$$

**Figure 61 –** Inscription referred to the second function

Fabio then adds that in the other case, the role of *a* and *b* should change: *a* should go in place of *b* and *b* in place of *a* [227]. The teacher tries to better explicate the passages introduced by Fabio; she clearly states the formula exchanging the roles of *a* and *b* and writes it on the IW [228]. Then Fabio adds that they should divide both sides of the expression by 10 [229], Silvia repeats his words and does that passage on the IW [230] and finally Fabio concludes that to find *b* is enough to substitute any value of *a* [231]. In this episode, the level of the discourse is objectified: even if Fabio is referring to the algebraic expression of the function, it is already conceived as an object. What Fabio is trying to describe is how to pass from one analytical representation to the other one. He uses the term parameter, previously introduced by their teacher, to refer to both relative and absolute humidity, leaving untouched the independent variable that is temperature. What emerges from this episode, except for the correctness of the algebraic passages, is that the interpretation previously enhanced by their teacher that a parameter determines a certain representation of the mathematical situation and changing it means assuming a different standpoint. This time the representations involved are not graphical but analytical: Fabio shows the ability not only to switch from the graphical to the analytical representation, but also to reproduce that mechanism of change in standpoint between the analytical representations. The lexical analysis reveals two main input spaces: the *real phenomenon* with references to the magnitudes involved (absolute humidity and relative humidity [222]; absolute humidity and

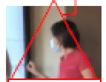
| | | [222] 00:36:30 | [223] 00:36:55 | [...] | [224] 00:37:30 | [225] 00:37:33 | [...] | [226] 00:37:53 | [227] 00:38:33 | [228] 00:38:48 | [229] 00:38:56 | [230] 00:39:00 | [231] 00:39:10 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Teacher | Fabio | | Fabio | Teacher | | Fabio | Fabio | Teacher | Fabio | Teacher | Fabio |
| INTERACTION FLOWCHART | T | | | | | | | | | | | | |
| | F | | | | | | | | | | | | |
| UTTERANCES | Teacher | How can I pass from this function to this function? What can I do? Try... T ⇢ | | | | Can you dictate me what happens here? T ⇢ | | | | So, you say we make *a* times 0,66 times the parenthesis equal *b* times 10... | | *Then we divide by 10 both sides.* T ⇓ | |
| | Students | | I have to put two parameters in place of the value that I want to put on the y-axis hence I have to put a parameter both for absolute humidity and relative humidity. | | Instead of writing the value of the absolute humidity and relative humidity I put *a* and *b* ... | | | In the second [case] I put... [he dictates the formula] | And then I make that *b* goes in place of *a* and *a* in place of *b* ... | | Then I divide by 10 both sides. | | Then I can substitute *a* with the value I want, and I find *b*. |
| LINGUISTIC ANALYSIS | | | *umidità assoluta e umidità relativa* = absolute humidity and relative humidity *asse y* = y-axis *parametro* = parameter | | *umidità assoluta e umidità relativa* = absolute humidity and relative humidity | | | | *b va al posto di a e a va al posto di b* = *b* goes in place of *a* and *a* in place of *b* | | *divido entrambi i lati per 10* = divide by 10 both sides | | *sostituisco a con il valore che voglio e trovo b* = substitute *a* with the value I want, and I find *b* |
| DISCOURSE LEVEL | | Objectified | Objectified | | Objectified | Objectified | | Objectified | Objectified | Objectified | Objectified | Objectified | Objectified |
| GESTURES | Teacher | | | | | | | T writes the formula on the IW. [gesture not visible] writing gesture, grounding function | | T writes the formula on the IW.  writing gesture, grounding function | | T writes the formula on the IW.  writing gesture, grounding function | |
| | Students | | | | | | | | | | | | |
| INSCRIPTIONS | | Formulas of the GG applet shown on the IW.  | | | | | | Formula written by the teacher on the IW. $a = \frac{b}{0{,}66(-)} \cdot 10$ | | Formula written by the teacher on the IW. $a \cdot 0{,}66(-) = b \cdot 10$ | | Formula written by the teacher on the IW. $\frac{a \cdot 0{,}66(-)}{10} = b \cdot \cancel{10}$ | |
| ARTEFACTS INTERACTIONS | Teacher | Reference to F | | | | | | → IW | | → IW , → (G, F) | | → IW | |
| | Students | | → (G, F) | | → (G, F) | | | → (G, F) | | | | → (G, F) | |
| COV | Order - Level | | COV 2 | | COV 2 | | | | COV 2 | | COV 2 | | COV 2 |

Figure 62 – Timeline 5 (Dew point 2020)

relative humidity [224]) and the second input space is the *analytical and formal representation* ($y$-axis [223]; parameter [223]; $a$ and $b$ [224]; $b$ goes in place of $a$ and $a$ in place of $b$ [227]; divide by 10 both sides [229]; substitute $a$ with the value I want, and I find $b$ [231]). From the discourse point of view, we can recognize a narrative in which the presence of personal pronouns is predominant: *I* have to put [223]; *I* want to put [223]; *I* have to put [223]; *I* put $a$ and $b$… [224]; *I* divide [229]; *I* can substitute [231]; *I* find [231]. A strong presence of verbs can be identified: verbs are used to describe the algebraic passages to move from one analytical representation to the other one. The level of the discourse is objectified because students masterly possess the functions as mathematical objects, they have already worked on their graphical representation and now they are translating them into an analytical representation. Concerning covariational reasoning, it seems always more evident that the episodes of these T.E. have revealed a different manifestation or at least shade of this second-order covariation construct and it differs from both the level of metavariation or the construction of a formula containing parameters characteristic of that specific mathematical model. This new level seems to be characterized by a global approach strongly supported by the mechanism of conceptual blending and that has its cognitive counterpart in a change in standpoint. The need of a more coherent mathematical interpretation is emerging.

## 12.3 DISCUSSION

### 12.3.1 Layer (a): Covariational reasoning[27]

Which characterization does second-order covariational reasoning assume when interpreting charts like the psychrometric one where at least three magnitudes are involved and one of these can be mathematically interpreted as a parameter? If we think of some typical examples of first-order covariational reasoning such as "A increases, while B decreases", they appear only in the answers to Task 2 (12.2.1), where students tried to describe qualitatively the relation between temperature and humidity. In the following steps of the T.E., the representations involved enhanced a different approach to second-order covariation. Indeed, the psychrometric chart at disposal of the students synthetizes in a unique diagram the relations between the magnitudes involved and flattens in two dimensions the relations between three different magnitudes

[27] The findings of this paragraph will be partially presented in a paper contribution in CERME12:
Bagossi, S. (accepted). Second-order covariation: it is all about standpoints, In *Twelfth Congress of the European Society for Research in Mathematics Education* (CERME 12).

(temperature, absolute humidity, and relative humidity). The students implement reasoning revealing a global approach supported by the adopted representations. In particular, in the physical interpretation of the relations described in the chart, students are deeply supported by the pot experiment: all the classroom discussions develop through an interpretation of the diagram with respect to the various steps of the experiment. This results in the presence of blends in which the knowledges from the experiment, the graphical representation, and notions of change and dynamicity are merged. These blends can be recognized also thanks to the lexical analysis of the sentences from both the teacher and the students which display the adoption of a blended language with terms that relate to different input spaces. What the students observe in the last episode (12.2.7) is that considering relative humidity as a variable or a parameter does not change the situation, but the perspective with which you look at the situation: the students remark that the relationship is always the same, only expressed in different terms. In this perspective, second-order covariation is the construct that enables to read the same mathematical situation from two different standpoints: in the real psychrometric chart the parameter is the relative humidity, and this magnitude is the one of second order; in the blue graph (**Figure 47**) absolute humidity becomes the parameter and so it is the second order variable determining a different standpoint. Students succeed in elaborating this change of point of view not only with the graphical representations but also with the analytical ones. In 12.2.7, Fabio is able to condense the passage from the green graph to the blue one, elaborating a formula that for the first case is approximately $y=a\cdot b^x$, and that in the second case becomes $a=y/b^x$. Second-order covariation means being able to interpret this formula choosing to interpret one of the variables involved as a parameter. Cognitively, this requires blending the input spaces, represented by the two different representations, into a unique blend, which, on its side, allows to map it back onto the two paired counterparts in the two input spaces (as it happens in 12.2.6) and so to operate what we call a *backwards blending*. In both cases, it is a change in standpoint that sustains the cognitive mechanism of blending. This new emerging characterization of COV 2 led us to explore the possibility of using the universal language of category theory to describe these mathematical processes. This issue will be object of analysis in Chapter 14.

### 12.3.2 Layer (b): Linguistic analysis

| *COV 1* | *Description of the level* | *Examples from Dew point 2020 T.E.* | *Syntactical and lexical analysis* |
|---|---|---|---|
| L2 | In these examples we can observe a qualitative description | [141] *Quando A diminuisce, B aumenta e viceversa.* | From the **syntactical** standpoint, we recognize the same structures |

| | | | |
|---|---|---|---|
| | of the covariation between the magnitudes: the level of reasoning is not yet fully quantitative because students are not able to express clearly what happens between the limit values.<br>Students are able to covary the two magnitudes underlining the values reached in limit situations. | When A decreases, B increases and vice versa.<br><br>[142] *Se A aumenta, B diminuisce, mentre se la A diminuisce, B aumenta.*<br>If A increases, B decreases, while if A decreases, B increases.<br><br>[143] A *inizia a crescere mentre B comincia a decrescere.*<br>A begins to grow while B begins to decrease.<br><br>[144] *I dati […] sono approssimativamente inversamente proporzionali, quando una cresce, l'altra diminuisce e viceversa.*<br>The data […] are approximately inversely proportional, when A grows, B decreases and vice versa.<br><br>[145] *Con l'aumento di A, B diminuisce, e ai 17. 2°, la A minima registrata, si ha la B maggiore.*<br>With the increase of A, B decreases, and at 17.2°, the minimum A recorded, the highest is B. | already identified for this level of reasoning:<br>***When*** *A decreases, B increases*.<br>***If*** *A increases, B decreases.*<br>*A begins to grow* ***while*** *B begins to decrease.*<br>***When*** *A grows, B decreases.*<br>*The* ***minimum*** *A, the* ***highest*** *B.*<br>Relations are predominantly binary, and clauses are subordinated.<br>An example of unary relation is [144] *Data are inversely proportional,* whereas [145] displays the use of superlatives.<br><br>From the **lexical** standpoint, language adopted is mainly qualitative and objective. The verbs ***to increase/to decrease*** are the most used to describe the behavior of the magnitudes involved. Recurring linking words are ***when***, ***if***, ***while***.<br>A sign of global description can be detected in the expression ***inversely proportional*** [144]. |

Concerning the sublevels of second-order covariation we could distinguish, we did not observe recurring or relevant structures from the syntactical point of view, hence hereafter we will mainly focus on the lexical analysis that provides useful findings for our research purposes.

| *Higher COV* | *Description of the level* | *Examples from Dew point 2020 T.E.* | *Lexical analysis* |
|---|---|---|---|
| COV 2 (i) | The first sublevel of COV 2 in this T.E. consists of a covariation between the parameter and the graph describing the absolute humidity-temperature relation: again, this way of reasoning is an example of meta-variation supported by the GeoGebra applet. | *[182] È dilatata*<br>It is dilatated<br><br>*[187] Sono simili*<br>They are similar | From the **lexical** standpoint, the sentences are objective and contain qualitative terms (***dilatated***, ***similar***) used to describe globally how the trend of the graph evolves. |
| COV 2 (ii) | The second level of COV 2 assumes a quantitative connotation: students look for a function describing the real phenomenon. | [172] *Cercare una funzione*<br>Look for a function<br><br>[174] *Un numero a fratto b per la temperatura più c*<br>A number *a* over *b* times the temperature plus *c* | From the **lexical** standpoint, we can notice the references to a mathematical formula which contains more than one parameter: this is the result of a greater awareness by students in the use of variables and parameters. |
| COV 2 (iii) | A third level of COV 2 makes its appearance in this T.E. and it is supported by the cognitive mechanism of blending. It enables to switch between the various MERs at stake and to apply a change in standpoint within the same kind of representations. Not only, given the resulting blends, students succeed in reconstructing the input spaces and this is what we call a backwards blending. | **Input space 1:**<br>[149] *diminuiva*<br>decreased<br>[151] *diminuisce*<br>decreases<br>[157] *fino a quella del 100%*<br>until that of 100%<br>[161] *diminuisce*<br>decrease<br>[166] *sta diminuendo*<br>is decreasing<br>[190] *diminuiva*<br>decreased<br>[194] *costante*<br>constant<br>[196] *diminuisce*<br>decreases<br>[198] *diminuisce*<br>decreases<br>[202] *cambia la curva*<br>the curve changes | Throughout all the episodes, the **lexical** analysis allows to identify four main input spaces:<br>1) notions of change and dynamicity;<br>2) spatial references connected to the graphical representation;<br>3) physical interpretation referred to the classroom experiment;<br>4) analytical representation.<br>These four spaces clearly refer to the various MERs used in the T.E. (the psychrometric chart, the experiment, the graphs contained in the GeoGebra applets and the analytical representations). They constitute the input spaces used by students to create some blends that help them to switch from one |

| | | | |
|---|---|---|---|
| | | [222] *umidità assoluta e umidità relativa*<br>absolute humidity and relative humidity<br>[224] *umidità assoluta e relativa*<br>absolute humidity and relative humidity<br><br>**Input space 2:**<br>[149] *sinistra*<br>left<br>[153] *orizzontalmente*<br>horizontally<br>[163] *sinistra*<br>left<br>[168] *tende verso l'asse delle x*<br>it tends toward the *x*-axis<br>[170] *seguendo la curva*<br>following the curve<br>[190] *il punto è sceso*<br>the point went down<br>[204] *siamo arrivati*<br>arrived<br>[211] *orizzontale*<br>horizontal<br><br>**Input space 3**:<br>[159] *che condensasse*<br>until it condensed<br>[159] *abbiamo continuato ad aggiungere ghiaccio*<br>continued to add ice<br>[200] *condensazione*<br>condensation<br><br>**Input space 4:**<br>[223] *asse y*<br>*y*-axis<br>[223] *parametro*<br>parameter<br>[227] *b va al posto di a e a va al posto di b*<br>*b* goes in place of *a* and *a* in place of *b* | representation to another one. The terms used are mainly verbs and adjectives: they are often holistic and refer to movements on the psychrometric chart. |

| | | | |
|---|---|---|---|
| | | [229] *divido per 10 entrambi i membri* divide by 10 both sides [231] *sostituisco a con il valore che voglio e trovo b* substitute *a* with the value I want, and I find *b* | |

### 12.3.3 Layer (c): Discourse level

In this T.E., we recognized three of the four levels of the mathematical discourse identified in the previous T.E.s. What is missing is the analytical level that does not emerge in the classroom discussions and various assumptions can be made about this point. First, this step of the modelling discourse probably emerged during the working group sessions that we chose not to analyze; second, the more advanced background of the students who have already been involved in mathematical modelling activities possess more confidence in the elaboration of a mathematical formula describing a certain situation and finally, the representations involved support a more holistic approach that overcomes the analytical interpretation. Concerning the other three levels:

- *Descriptive*: this level can be recognized in students' answers presented in 12.2.1. Students describe qualitatively the graphs referring to the trend of temperature and humidity. The narrative is objective and the level of covariational reasoning still low. The references to the physical domain are still not blended with the mathematical knowledge.
- *Objectified:* this level is predominant in Episode 2 (12.2.3) when Matteo tries to elaborate a formula to describe mathematically the absolute humidity-temperature relation, Episode 3 (12.2.4) where students reflect on how the coefficient affects the trend of the graphs, and Episode 6 (12.2.7) in which Fabio is explaining how to switch from one analytical representation to another. At this level students already possess the notion of function as mathematical object and works on it both in its graphical and analytical representation. Narratives emerging are both subjective e.g., in Episode 6 when Fabio describe the algebraic passages to turn one formula into another one and he often uses the personal pronoun *I*, or in [172] when Matteo tells how he tried to look for a suitable function, and objective as in Episode 3.

- *Interpretative:* this level is the most present in Episodes 1 (12.2.2) and 4 (12.2.5) where the teacher strongly enhances the interpretation of the psychrometric chart with the pot experiment and vice versa. This level has been highlighted also thanks to the lexical analysis: it has revealed the presence of several input spaces addressing the various MERs involved and constitute the solid ground for students' blends such as "If you have reached the dew point yes… only the temperature changes" [165] or "You spoke of condensation, what happened during the second trait? It happened that we arrived at the dew point." [203-204]. This level of the discourse also corresponds to a high level of second-order covariational reasoning that is strongly supported by the cognitive mechanism of conceptual blending, but this point needs a greater deepening.

### 12.3.4 Layer (d): Adaptive teaching strategies

Before moving on to analyse in detail the four recurring strategies, let us start with a general comment: the difficulty that working with a chart like the psychrometric one entails is evident and it emerges also from the classroom discussions. The students, who are not much talkative themselves, generally intervene with dry answers or short sentences. It is the teacher that dictates the rhythm of the discussion and enhances it through a tight questioning and wise interventions. Concerning the teaching strategies, the ones we could identify are the same of the other two T.E.s and in particular:

1 - *Semiotic game*: given the poor quality of video recordings, we are not able to conclude much about the gestural component. The revoicing is a strategy widely used by Silvia as we can observe at [150], [152] or [199], just to quote a few examples. And in all these examples, as in others, the revoicing of students' words is followed by a question to enhance the discussion.

2 - *Fostering the discussion in the classroom and facilitating its flow:* this strategy is strongly used by the teacher in all the episodes to enhance the discussion. Some examples are: "So, what did you do?" [160], "And so? [167], "What happens instead on the horizontal trait?" [193]. In Episode 4 (12.2.5) we have another example of the non-judging behavior of the teacher who leaves the students the possibility to explain their choices of a horizontal rather than a vertical trait and helps them to find the right answer reflecting on the behavior of the magnitudes involved along those traits. At [207], instead of reacting to the wrong answer provided by Emanuele, the teacher relaunches the issue to the whole classroom with a question: "The end? Do you agree?".

3 - *Exploring students' actions and thinking*: some examples of this strategy are: "What kind of function did you think of?" [173], "In which sense nothing good?" [175], "Why?" [199]. This kind of questions help students to better explicate their reasoning and to provide more detailed explanations. This approach enhances students' argumentative skills.

4 - *Drawing students' attention to the information provided by the different artefacts:* given the wide variety of MERs adopted in this T.E. and their complexity of reading and interpretation, the role of the teacher reveals essential in mediating between the various representations and in directing students' attention to the relevant information provided by them. In Episode 1, Silvia masterly helps students to switch from the experiment to the chart and vice versa: "And so, on the graph, how do you move?" [152], "What did we do after we reached the saturation of 100%? Did we stop immediately?" [158], "This and this are the same function?" [184].

## 12.4 CONCLUDING REMARKS

As stated in the prospective analysis, this T.E. differs considerably from the others two and this fact is also confirmed by the diversity of the results obtained. The strong influence of the pot experiment is evident throughout most of the episodes analyzed. It constitutes the benchmark measure with which interpret the other representations also attributing a physical interpretation. As expected in the prospective analysis, we recognized a wide presence of a blended language in which the mathematical and physical knowledges are merged, and these linguistic markers are an externalization of the cognitive processes of the conceptual blending. Concerning covariational reasoning, we identified a new level of COV 2, a level characterized by a dynamic and holistic approach that differs from the levels of COV 2 displayed in 2017 and 2019 T.E.s. This different connotation reveals the necessity of a more punctual investigation and definition of this level. The mathematical discourse does not display a consistent analytical phase in favor of an objectified and interpretative level, characterized by narratives that reveal a sense of globality and that are supported by the representations involved. The wider mathematical background of students is evident: a relevant example can be recognized at [174] where Matteo elaborates a mathematical formula containing three different parameters: his choice reflects the greater confidence with which they handle the analytical representations and the awareness of the different role of variables and parameters. The teaching strategies used by Silvia are those we had already identified (12.2.3).

# 13 DISCUSSION AND CONCLUSIONS

In this chapter we are going to provide a detailed answer to our four research questions, and then discuss the didactical implications of our study, its limitations, and some possible future lines of research.

## 13.1 ANSWER TO RESEARCH QUESTIONS

### 13.1.1 Answer to research question 1

How can the Thompson and Carlson's theoretical framework about covariation be enlarged so to encompass second-order covariation in a unique and coherent construct?

Enlarging the theoretical framework about covariation, first requires a general definition of covariational reasoning, the form of reasoning we investigated in this study, that is:

> **Definition (Covariational reasoning)** A person reasons covariationally when she is able to suitably envision the relationships between two or more mathematical objects.

This approach considers covariational reasoning as a form of reasoning with a larger epistemological and cognitive value as already quoted in Arzarello (2019). To move to higher-order covariational reasoning, students should already have conceptualized function as mathematical object and this entails an increased complexity in the mathematical conceptualization that leads us to speak of an epistemological enlargement. Not only, we can also speak of an ontological enlargement if we consider that the above definition addresses mathematical objects rather than variables as done in the framework by Thompson and Carlson. Before moving to a detailed description of our actual vision of the covariation construct, some other two definitions are required; they underline the distinction between order and level and will enable us to adopt a proper and coherent terminology in the following:

> **Definition (Order)** Form of covariational reasoning connoted by specific mathematical objects and their mutual relations.

> **Definition (Level)** Class of behaviors or characteristic of a person's capacity to reason covariationally (Thompson & Carlson, 2017, p. 435)

From these definitions emerges that while the definition of order of covariational reasoning has an epistemological and ontological connotation, the definition of level, the same adopted by

Thompson and Carlson, is purely cognitive and addresses the different steps in the process of conceptualization. This clarification will help us to answer to research question 2 and say whether the facets of COV 2 we distinguished are truly levels of COV 2 according to the definition previously set out.

Within this enlarged framework and at the end of this preliminary study, the construct of covariation seems to assume the configuration sketched in the following diagram:

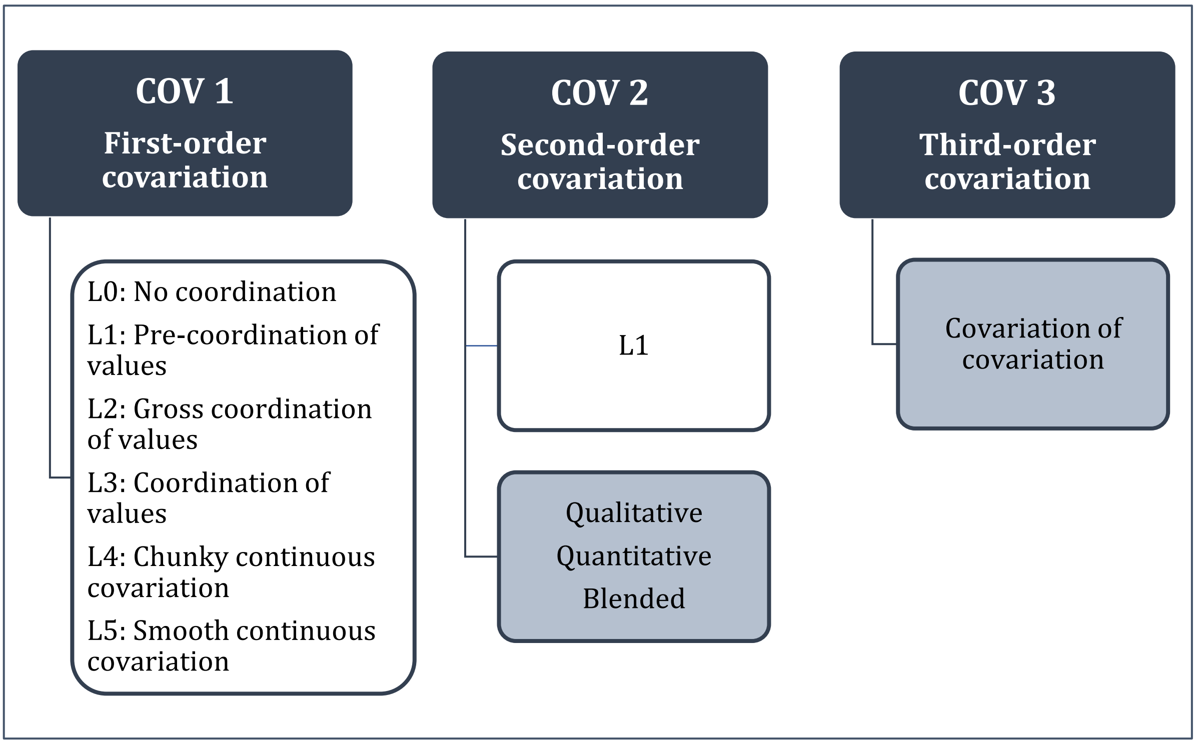



**Figure 63** – Enlarged framework about covariation.

As stated at the beginning, in this enlarged framework mathematical objects are considered jointly with their mutual relations and what connotes the different orders is the increasing complexity of the mathematical objects involved and as a consequence of the reasoning required. Concerning first-order covariation (COV 1), i.e., the taxonomy of the six levels of covariational reasoning elaborated by Thompson and Carlson (2017), it is defined as:

> **Definition (First-order covariation)** The ability to envision the values of two varying quantities and to envision them as they vary simultaneously.

Its mathematical objects are the variables representing varying quantities and how they can covary. At the higher levels of COV 1 students are able to envision the two or more magnitudes varying smoothly and continuously hence they succeed in conceiving the resulting function which cognitively is a multiplicative object. Two other orders of covariation are included.

Second-order covariation (COV 2) is the construct deeply analyzed through the whole study and that we initially defined as (Chapter 5):

> **Definition (Second-order covariation)** The ability to suitably envision a further relationship in a family of invariant relations among two or more varying quantities, where this family is characterized by the presence of one or more parameters.

In particular, parameters allow to represent classes of real phenomena as families of relations between variables characterized, from the point of view of the mathematical representation, by specific parameters that determine the peculiarities of the mathematical model. Furthermore, the label 'second-order covariation' seems particularly suitable to underline the role played by parameters: indeed, Bloedy-Vinner (2001) already used the expression second order function to address those functions whose argument is a parameter and whose output is a function or an equation depending on a specific parameter value. COV 2 is an order of reasoning that involves not only the variables but also the functions as mathematical objects and their mutual relations. While empirically validating the above definition throughout the data, new facets of this kind of reasoning emerged: how a specific variable chosen as parameter affects graphically the trend of the graph and analytically the family of functions; how the choice of a certain variable as parameter produces a specific graphical representation and how this representation changes when the variable elected as parameter is another one, that cognitively is a change in standpoint. The issue of definition and classification of these aspects of COV 2 will be deeply explored in the answer to research question 2. Indeed, the taxonomy of COV 1 is a cognitive one and levels are supposed to be developmental, whereas in our study we did not conduct a capillary investigation of students' forms of reasoning; hence, the cognitive connotation of the other orders should deserve a deeper investigation.

The last order included in the diagram is that of covariation of covariation, called third-order covariation (COV 3):

> **Definition (Third-order covariation)** The ability to consider functions globally and focus on how the changes in one graph are linked to the changes in another graph related to the first one, and conversely (Swidan, Sabena & Arzarello, 2020).

Even if this order covariation has not been explored in this study, it seems a suitable extension of the first two orders for the following reason. Third-order covariation, COV 3, involves two or more functions connected by a certain relation and this order of reasoning means envisioning how the behavior of one function affects the trend of the other one related to it but a more in-depth study is required. We introduced the term third-order covariation to denote a form of reasoning that seems a reasonable extension of the other two orders, but it has never been investigated from a covariational reasoning standpoint. This interpretation leaves the door open to the possibility of identification of other orders of covariation

What are these orders of covariational reasoning important for? Concerning COV 1, a huge body of literature shows how it supports the understanding of proportion, rate of change and linearity, variables, trigonometry, exponential growth, functions of one and two variables and detailed references for all these topics can be found in Thompson and colleagues (2017). About COV 2, the experimentations analyzed in this research allow us to point out the relevance of this order of reasoning for the modelling of classes of real phenomena, dynamic situations, and parametric functions. Finally, starting from the topic analyzed in Osama, Sabena and Arzarello (2020), COV 3 seems the suitable form of reasoning to conceptualize the relation between a function and its derivative, a function and its antiderivative, or introducing a physical connotation a velocity-time graph and the graph of rate of change of velocity with respect to time. These are only a few examples to give concreteness to this form of reasoning.

We hope that this enlarged framework truly respects the original nature of covariational reasoning and the motivations that led so many researchers to study it theoretically. Once Thompson wrote us:

> My sense is that there is a misperception of covariational reasoning as I have always meant it —the misperception is that covariational reasoning is fundamentally about graphs. I've never meant this. I've always meant that covariational reasoning is rooted in quantitative reasoning. Graphs are convenient for researchers as a context for investigating covariational reasoning and a natural medium for learners/teachers to express it, but I've always thought of covariational reasoning in terms of conceptualizing situations. (Thompson, email, July 7th, 2020)

In our opinion, the context of our investigation well reflects that covariation is neither about reading graphs nor about interpreting formulas, but it is a matter of conceptualization.

Our research also sheds light on many other relevant factors that deeply support covariational reasoning in the classroom environment: we focused on the role of teachers and some possible strategies that can foster students' understanding of complex mathematical concepts; in addition, we observed how the instrumentation of these processes through suitable tools produces numerous benefits and finally the design of tasks should enable to explore different orders of covariation.

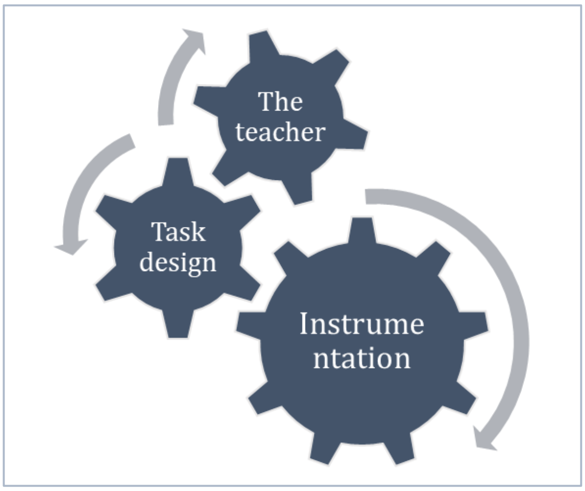



**Figure 64 –** Didactical connotation of the covariation construct

We do believe that all these elements strongly support the introduction of a covariational approach within the classroom environment, hence they constitute a characterization of the construct on the didactical standpoint and for this reason we retain appropriate include them in our theoretical framework. Finally, the data analysis of 2020 T.E. highlighted that a more structured and global interpretation of the theoretical construct of covariation seems to be required so to fully embrace all the connotations of this construct and the universal language of category theory could provide useful insights. This issue will be deeply explored in Chapter 14, but for now we include in our framework an additional dimension that aims at providing a mathematical interpretation of these orders of covariation.

To conclude, we characterize the theoretical construct of covariation on three different dimensions: cognitive, didactical, and mathematical. Keeping into account all these elements, the final aspect of the covariation construct could be the one pictured in **Figure 65**.

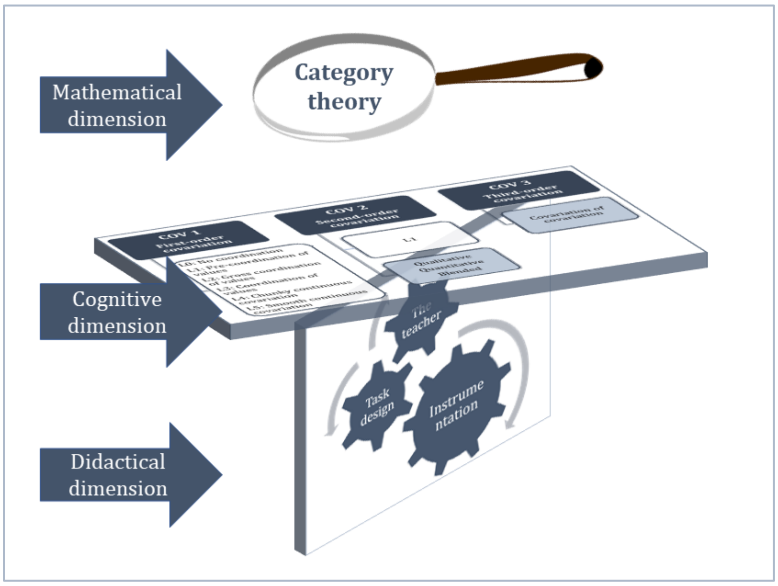



**Figure 65** – Final version of the theoretical construct of covariation

### 13.1.2 Answer to research question 2

Is it possible to identify some levels connoting second-order covariation?

Concerning second-order covariation, the analysis of the data from the three teaching experiments conducted throughout this study, revealed the following different ways in which students conceptualize situations requiring this form of reasoning:

- A *qualitative conceptualization*: students co-vary a varying quantity, mathematically a parameter, and a family of functions depending on it describing qualitatively how a change in the parameter affects the trend of the graph. This way of reasoning can strongly be supported by the design of suitable dynamic environments that enable that level of exploration that in the literature is called *metavariation* (Hoffkamp, 2011).
- A *quantitative conceptualization:* the students succeed in the elaboration of a general formula describing the real phenomenon containing one or more parameters that are characteristic of the mathematical model; students are aware of the way in which the parameters affect the analytical representation of the model and which magnitudes they depend on.
- A *blended conceptualization:* students are able to read the same mathematical situation from two different standpoints and to recognize a correspondence between the involved representations. The approach to the representations involved is global and dynamic and supported by the cognitive mechanism of blending. Students are able to switch between the various MERs at stake and to apply a change in standpoint within the same kind of

representation: analytically, it means being able to read a certain formula choosing to interpret one of the variables involved as a second-order variable, i.e., a parameter; graphically, it means being able to identify correspondences between the related graphs. Not only, given the resulting blends, students enact a *backwards blending* when they succeed in reconstructing the input spaces. This blended level seems to assume a qualitative connotation but on a more abstract level that concerns representations. The categorical discussion in the following part could shed new light on these aspects.

Given the definitions provided in 13.1.1, these different conceptualizations can be interpreted neither as orders nor as levels. Hence we are going to intend them as a mathematical characterization of COV 2 and we are going to refer to them as *facets* of COV 2.

**Definition (Facet)** Mathematical aspects connoting the orders of covariational reasoning.

The choice of the term facets, between various possible choices, wants to exclude a hierarchy between the various aspects and include instead a possible overlap between them. We identified three facets of COV 2 (qualitative, quantitative and blended) which remark the complex epistemological nature of this form of reasoning.

However, our research question 2 asks for some levels of this second-order construct and, coherently with our enlarged framework, they should be intended as cognitive levels. Our data analysis revealed the existence of a transition phase between COV 1 and COV 2 that we often referred to as intermediate order, but the entity of the object involved is the same of COV 2 and it consists of a cognitive meaning hence we could define it as a first level of COV 2:

**Definition (COV 2 – L1)** At this level, the student would express saying that the coefficient of the function varies depending on a certain magnitude.

At this level, students' focus of reasoning remains on the covariation between the dependent and independent variable, but the presence of the "parameter" makes it appearance. Adopting the classification elaborated by Arcavi et al. (2016) that "parameter" is conceived as a varying quantity: it does not stand for a single unknown value but for a domain of possible values and introduces an underlying idea of motion and dynamicity. Depending on students' background, they can refer to it without using this formal term but in a more intuitive way, e.g., a constant value that changes, adopting an antithetic expression. In a commognitive perspective, the use of oxymoronic expression could be interpreted as a commognitive conflict generated by the encounter of incompatible narratives. Our methodology of research did not enable us to identify other levels of COV 2: a more detailed and individual investigation of students' conceptualization is required. If we could advance an a-priori hypothesis or a possible evolution of the COV 2 levels starting from L1, we imagine them as a transition from specific to generic[28]. In line with Mason and Pimm's interpretation (1984), the deeper conceptualization of a varying quantity as a parameter, a second-order variable, could be intended as the ability to see the general in the particular, i.e., to disclose that such function does not represent a single real phenomenon but a class of real phenomena with some common properties (generality) and each of them different depending on the specific value assumed by the characteristic parameter (specificity). This evolution is possible because also the way in which that varying quantity is interpreted changes.

### 13.1.3 Answer to research question 3

Which linguistic markers connote specifically each of the levels of students' covariational reasoning?

All the results concerning the linguistic analysis (syntactical and lexical) in relation to the various levels of covariational reasoning are collected in the following tables:

| *COV 1* | *Examples from T.E.s* | *Syntactical analysis* | *Lexical analysis* |
|---|---|---|---|
| L1 | *A is always the same, B changes.* | *Sentence about A**,** sentence about B* where the two sentences are not correlated. | We can observe the use of the adverb ***always***, sign of a qualitative description. |
| L2 | *Sentence about A* ***as/while*** *sentence about B.*<br>*A begins to grow* ***while*** *B begins to decrease.*<br>*Since* ***more*** *A,* ***greater*** *B.*<br>*Since* ***more/less*** *A,* ***more/less*** *B to do the* ***same*** *C. [3 magnitudes]*<br>***The more*** A, ***the more*** B<br>A is ***maximum/ minimum*** for B=n.<br>***When*** *you increase A**,** B is greater.*<br>***When*** *A decreases, B increases.* | We can observe some binary relations and the use of comparatives. Both coordinated and subordinated clauses are used. | Language adopted is mainly qualitative and objective. The verbs ***to increase/to decrease*** are the most used to describe the behavior of the magnitudes involved. Recurring linking words are ***as***, ***if***, ***while***, ***when*** (sometimes followed by a subjective sentence in which *you* is the subject). The adjectives ***maximum*** and ***minimum*** are employed to refer to the limit values assumed by the magnitudes. |

[28] We would like to thank Prof. Samuele Antonini for sharing with us this interesting interpretation.

| | | | |
|---|---|---|---|
| | ***When*** *A grows, B decreases.*<br>***If*** *A increases, B decreases.*<br>*The* ***minimum*** *A, the* ***highest*** *B.* | | |
| L3 | No examples in the analyzed episodes. | | |
| L4 | *A' were always the same* ***while*** *B' increased.*<br>*A' equal* ***if*** *B is always the same.*<br>*A'' were constant* ***and*** *A''' were null.* | Relations are mainly binary, no comparatives, and both coordinated and subordinated clauses can be detected. | We can observe a recurrent use of the adverb ***always***, objective, and quantitative sentences. Finite differences are often at stake. |
| L5 | *A* ***is*** *$B^2$*<br>*Time to second power gives space… hence* $y= x^2$<br>*The graph could be of 2nd degree since A'' are constant.*<br>*The curve before had an inclination almost* ***horizontal*** *and then became always more* ***vertical***. | The continuous covariation between magnitudes A and B is expressed through the description in formal terms of their mathematical relationship or the use of a formula with an independent ($x$) and a dependent variable ($y$). | Sentences are objective and relations expressed are true globally. The verb ***to be*** is used with the same meaning of the equal sign.<br>A recurrent use of the adverb ***always*** can be interpreted as an indicator of globality. The description of the trend of the graph is expressed in qualitative terms. ***Horizontal*** and ***vertical*** are recurrent adjectives used to connote its behavior. |

| *COV 2* | *Examples from T.E.s* | *Syntactical analysis* | *Lexical analysis* |
|---|---|---|---|
| L1 | *$y = 2{,}13 \cdot x^2$ in this case with 25°*<br>*The constant value can vary.*<br>*The constant number depends on the angle.*<br>*The coefficient [of the function] varies [according to the angle].*<br>*[The constant value] changes according to the angle.* | We do not observe recurring syntactical structures. | We can notice a quantitative approach with the elaboration of a mathematical formula for a fixed value of the angle or a qualitative approach through which students express the dependence of the coefficient on the angle of inclination. Students use some antithetic expressions stating that the value of the so-called constant can vary. |
| Qualitative | *If A changes,* ***the more*** *the inclination increases.* | Relations are mainly binary, and the second object involved is | The adverb ***more*** is extremely used, and the linking word ***as*** is also recurrent. The sentences |

| | | | |
|---|---|---|---|
| | ***The more*** *A increases,* ***the more*** *the inclination [of the graph] increases.*<br>*A' increases, increasing B.*<br>***The greater*** *is A,* ***the greater*** *is B' and* ***the faster*** *will grow the parable.*<br>***As*** *A increases, the curve comes always closer to y-axis.*<br>***As*** *A of the plane increases, the curve of the parable is* ***more*** *accentuated.*<br>*Increasing A****,*** *the curves are* ***more*** *inclined.* | typically the inclination of the graph or the graph itself.<br>Comparative structures are recurrent, and clauses are both coordinated and subordinated. | are objective and show a qualitative approach to the global description of how the trend of the graph evolves. |
| Quantitative | $y = k \cdot x^2$ where *k* is a constant varying with the inclination.<br>The inclination [of the graph] changes according to the angle because with a minor angle the inclination is minor, with a greater angle the function goes up first. $y = coeff \cdot x^2$<br>$y = (A''/2) \cdot x^2$<br>[The coefficient] is the half of A''. | We observe the presence of a formula, i.e., an analytical representation, containing the two variables and a parameter. | This mathematical characterization of COV 2 mainly lies on a quantitative and analytical approach rather than on a graphical interpretation, but still reveals a sense of globality. The parameter is still identified with an antithetic expression i.e., a constant varying with the inclination, probably due to the absence of a more specific mathematical background. |
| Blended | Input spaces:<br>1) notions of change and dynamicity;<br>2) spatial references connected to the graphical representation;<br>3) physical interpretation referred to the classroom experiment; | No recurring syntactical structures. | The lexical analysis allowed to identify different input spaces referring to the various MERs used in the T.E.. They constitute the input spaces used by students to create some blends that help them to switch from one representation to another one. The terms used are mainly ***verbs*** and ***adjectives***: they are often holistic and involve movement in the psychrometric chart. |

| | | | |
|---|---|---|---|
| | 4) analytical representation. | | |

These results show that language has a role of mediator between the representations involved and students' mental images that is a common element in several of the already cited studies such as Janvier (1978), Hoffkamp (2009, 2011) and Lisarelli (2019).

### 13.1.4 Answer to research question 4

Which levels characterizing the discourse about modelling of real phenomena can be distinguished and how do they relate to covariation?

The analysis of the discourse about modelling a real phenomenon in a commognitive perspective, i.e., focusing on narratives, led us to distinguish four different levels of the discourse that can be characterized as follows:

| *Level of the discourse* | *Description* |
|---|---|
| Descriptive | It consists of a description of the inputs from the different representations of the real phenomenon. There are explicit references to the artefacts and to the specific physical contest presented, but the latter are still not blended with the mathematical knowledge. The narrative is objective and levels of covariational reasoning are low. The language adopted is the everyday one, not yet scientific. |
| Analytical | At this level, the use of an analytical language starts making its appearance in the mathematical discourse. Students represent the magnitudes involved with a mathematical symbol, a variable, and reason in terms of numerical relationships or in terms of $x$ and $y$ in the Cartesian plane. The sentences are more extrapolated from the specific physical contest and describe some possible variations of the real phenomenon. The mathematical discourse shows different degrees of abstraction. The levels of covariational reasoning are low but show a quantitative approach. |
| Objectified | The process of modelling leads to the elaboration of a mathematical relationship, a formula that has a general validity and does not fit only with the specific physical contest presented at the beginning. The formula allows to describe mathematically the real situation with a global approach. At this level students possess the notion of function as mathematical object and masters it both in its graphical and analytical representation. Language is scientific and narratives encountered are both objective or subjective (presence of personal pronouns and verbs to describe algebraic passages). COV 1 is fully achieved while COV 2 emerges in its qualitative or quantitative characterization. |

| | |
|---|---|
| Interpretative | This level of discourse does not differ much from the objectified one concerning the globality of the approach and the mathematical objects involved, but the language adopted reveals something deeper than the mathematical interpretation. The emerging narratives reveal a blend of elements from the various representations (algebraic, numerical, and graphical) connected in coherent sentences. The terms used by students are often detectors of a physical interpretation blended with the mathematical description. Sometimes, the graphic and kinematic aspects are blended as if the graph and the phenomenon were the same thing. Looking at the mathematical discourse with the lens of the modelling cycle, this step could be interpreted as a return to the real phenomenon after having mastered the mathematical interpretation. COV 2 mainly manifests in its blended characterization. |

These four levels of the mathematical discourse about the modelling of a real phenomenon could be interpreted with the following perspective:

| *Discourse level* | *Approach* |
|---|---|
| Observational | Local |
| Analytical | |
| Objectified | Global |
| Interpretative | |

**Table 2 –** Classification of the discourse levels.

The first two levels, the descriptive and analytical, refer to a local contraction and are prevalent in the 2017 T.E.; the objectified and interpretative levels instead reveal a global interpretation of the phenomenon proposed and higher orders of covariational reasoning; they mainly emerge in the 2019 T.E. and 2020 T.E..

### 13.1.5 Answer to research question 5

Which adaptive teaching strategies does the teacher use to responsively guide the students to engage in covariational reasoning within classroom activities?

In the teaching situation and in the learning processes characterizing the three teaching experiments, we have pointed out four main recurring strategies, through which the teacher is able to responsively guide the students to engage in covariational reasoning:

1) managing the semiotic game: the teacher repeats students' semiotic productions (words, gestures) to ascribe them with mathematical meanings;

2) fostering the discussion in the classroom and facilitating its flow, through suitable questioning in order to make students to deepen the problem at stake;
3) exploring students' actions and thinking: the teacher investigates what students are thinking;
4) drawing students' attention to the information provided by the different artefacts used in the learning process and mediating between the various representations.

These strategies globally show some of the main features of adaptive instruction pointed out in literature:

- The carefully designed teaching situation shows a strong pedagogical and content knowledge. The basic idea of covariation between quantities is introduced through a nice modelling experience, where mathematics is used to suitably give reason of a physical phenomenon, whose appeal has both a perceptual and a historical flavor. The students are given a variety of organized artefacts, whose productive use and interplay can trigger and support them towards the idea of covariation.
- The teacher shows a deep understanding of and familiarity with her students: suitably alternating/coupling her strategies, she is able to tune with their processes and to support them towards the understanding of second-order covariation, in particular with an expert use of the semiotic resources produced by the artefacts. In this way the teacher is able to transform "the routine into full-fledged exploration" (Sfard, 2008, p. 813). She shows to be able to continually learn from her students and to develop on the spot consequent ways and strategies to teach them, gradually using the available semiotic resources.
- Adaptive instruction goes somehow beyond the commognitive perspective: "discursive rules of the mathematics classroom […] are an evolving product of the teacher's and students' collaborative efforts" (Sfard, 2008, p. 589), but adaptive teaching requires an extra effort for the teacher. In order to adapt her teaching strategies, she has to do some extra work: first of deep understanding of her students and their strengths and weaknesses; second of adaptation of her teaching strategies according to the situations that develop within the classroom, mastery that Silvia has gained over the years with constant formation and experience in the classroom. It is therefore an effort made not only on the field, during lessons, but also before entering the classroom. Through the macro and micro analysis and the Timeline we have been able to illustrate and appreciate the work of a teacher who cleverly knows how to adapt her modality of teaching without

dictating the discursive rules of the mathematics classroom only through her own discursive actions but valuing what emerges from her students.

## 13.2 DIDACTICAL IMPLICATIONS

Our analysis can refine the general pedagogical idea of adaptive instruction pointing out how teacher's content knowledge plays a crucial role in her coaching of the teaching experiments: without a deep knowledge of the taught topic, it would not be possible to take decisions on the spot to face the unexpected students' productions. On the contrary, we have seen that in all moments the teacher is able to suitably manage the signs produced by the artefacts and by the students in order to coach an instrumentation of the mathematical content to be taught towards the main goal of the teaching activity. In this sense we can speak of an *instrumented adaptive instruction*. We can define it as an adaptive instruction in which the teacher develops ways and strategies on the spot to promote her students learning of the content to be taught through a suitable instrumentation supported by the available artefacts. This requires teacher's deep knowledge of the affordances of the used artefacts with respect to the specific piece of knowledge to be taught in that moment and to her students' competences with the used artefact. Theoretically, the study sheds new light on the role of teachers on how they can adapt their instruction to teach specific mathematic content that is covariational reasoning. Pedagogically, the study represents a valuable example of how an expert teacher can use her skills and expertise, demonstrating the specific practices that promote the covariational reasoning among students.

Our research findings show evidence that certain types of instructional strategies such as semiotic games can be particularly effective to promote adaptive instruction and to reach high levels of covariational reasoning. As well, we have shown that a suitable use of instruments can properly foster such processes. We believe that the findings of this study can be the basis for a professional development course for teachers aiming at adapting their instruction and reaching a high level of mathematical thinking (Stein et al., 2008). Of course, both kinds of results, the strategies, and the semiotic analysis, may be significant for the discussion within a professional development course. Discussing with the teachers the strategies that have been found in this study may enrich their ways for teaching mathematics with digital tools and discussing the semiotic analysis with the teachers may help them to become aware of the multiple semiotic resources that a teacher can use to help students to construct mathematical meanings. Paying

attention to the students and the teacher semiotic resources, we hope that the communication between the teacher and the students moves beyond "show and tell" (Stein et al., 2008), and can include what the students and the teacher do and say. To trigger and foster adaptive instruction, we hypothesize that the results of this study can set principles to support teachers in designing both digital tasks and the interaction with students, to prompt productive mathematical learning and engage students in high order reasoning, such as: posing questions, generalizing, developing argumentation, and communicating their thinking processes. The professional development of teachers who will attend a course organized according to the principles set from this study, entails a further direction of research that we already started exploring (Section 13.5).
Finally, our teacher, Silvia, is familiar with the semiotic games' principles, and the students are familiar with the teaching style of the teacher, who emphasizes, among other things, the classroom discussion and pays special attention to different kinds of signs and strategies. These characteristics of the teacher and the students are peculiar of this study. To extend our findings, especially about the instructional strategies that prompt the evolution of mathematical meanings, more research is needed.

## 13.3 Results in summary

In this section we would like to summarize the main results of this study about the theoretical framework of covariation in a concise and orderly way for the convenience of the reader.

The term covariational reasoning is redefined as follows:

> **Definition (Covariational reasoning)** A person reasons covariationally when she is able to suitably envision the relationships between two or more mathematical objects.

This new definition attributes to covariational reasoning a wider epistemological and cognitive value because, with respect to the definition by Thompson and Carlson (2017), not only variables are involved but more generally mathematical objects and this leads to an increased complexity in the mathematical conceptualization.

Covariation is a multidimensional construct, and by its dimensions we mean the different aspects of covariational reasoning. In our study we focus on three different dimensions: cognitive, didactical, and mathematical.

Moreover, the nature of the mathematical objects involved in the covariational reasoning determines the order of covariation:

**Definition (Order)** Form of covariational reasoning connoted by specific mathematical objects and their mutual relations.

If we consider the entity of the mathematical objects involved in the different orders of covariation we can also speak of an ontological enlargement of the theoretical framework. In this study we could distinguish three orders of covariation:

**Definition (First-order covariation – COV 1)** The ability to envision the values of two varying quantities and to envision them as they vary simultaneously. (Thompson & Carlson, 2017)

**Definition (Second-order covariation – COV 2)** The ability to suitably envision a further relationship in a family of invariant relations among two or more varying quantities, where this family is characterized by the presence of one or more parameters.

**Definition (Third-order covariation – COV 3)** The ability to consider functions globally and focus on how the changes in one graph are linked to the changes in another graph related to the first one, and conversely. (Swidan, Sabena & Arzarello, 2020)

Cognitively, these orders can be characterized by some levels:

**Definition (Level)** Class of behaviors or characteristic of a person's capacity to reason covariationally (Thompson & Carlson, 2017, p. 435)

Thompson and Carlson (2017) identified six levels of first-order covariational reasoning that are outlined in Section 2. In this study we identified also one level of second-order covariation:

**Definition (COV 2 – L1)** At this level, the student would express saying that the coefficient of the function varies depending on a certain magnitude.

At this level, students' focus of reasoning remains on the covariation between the dependent and independent variable, but the presence of the "parameter" makes its appearance.

Moreover, the orders of covariation may present various facets:

**Definition (Facet)** Mathematical aspects connoting the orders of covariational reasoning.

This term denotes an absence of hierarchy between the various aspects and include instead a possible overlap between them. We identified three facets of COV 2 (qualitative, quantitative, and blended) which remark the complex epistemological nature of this form of reasoning.

**Definition (Qualitative facet)** Students co-vary a varying quantity, mathematically a parameter, and a family of functions depending on it describing qualitatively how a change in the parameter affects the trend of the graph.

**Definition (Quantitative facet)** The students succeed in the elaboration of a general formula describing the real phenomenon containing one or more parameters that are characteristic of the mathematical model; students are aware of the way in which the parameters affect the analytical representation of the model and from which magnitudes they depend on.

**Definition (Blended facet)** Students are able to read the same mathematical situation from two different standpoints and to recognize a correspondence between the involved representations. The approach to the involved representations is global and dynamic and supported by the cognitive mechanism of blending.

We remark again that this form of reasoning is essential when dealing with the conceptualization of mathematical situations and as, the design of our research shows, it is not merely about reading graphs or about interpreting formulas.

Concerning the didactical dimension, throughout this study we explored the relevance of role of the teacher, instrumentation, and task design to enhance and support covariational reasoning processes.

About the mathematical dimension, a deeper interpretation of the construct based on category theory is investigated in the following chapter.

## 13.4 LIMITATIONS OF OUR STUDY

In Section 8.1 we have already outlined some constraints of our research that are specific features of the design of our research. Indeed, our investigations are limited to covariational reasoning involved in tasks about *mathematical modelling* of real phenomena: this topic includes a wide range of activities relevant not only in mathematics, but more in general in the STEM field, but they are characterized by specific connotations. Many other mathematical concepts require covariational reasoning to be fully understood and they have not been explored in this study.
In this section we would like to delineate some other features that constitute a limitation for our study and that could be interpreted as some points to which work on to improve the results or possible future directions of research. To start, just a *small sample* of students is involved in our teaching experiments: the participants of our study belong to the same institution hence they

share a common social and also mathematical background; in particular the second class, the one involved in 2019 and 2020 T.E., during the third experiment has greatly benefited from the work done during the previous academic year on the law of the inclined plane and they handle with greater safety and awareness various mathematical representations. Moreover, the *teacher*, always the same throughout the whole study, has a strong preparation not only concerning adaptiveness intended as responsive guidance of her teaching, but also about covariational reasoning and has been deeply involved in the design of our research. Her mathematical and pedagogical background has certainly facilitated the introduction in her teaching practices of activities on covariation. Moreover, our findings come from *classroom discussions* and written elaborates of the students but not from individual interviews of students: this methodology of investigation did not allow us to elaborate a complete cognitive taxonomy of the levels of COV 2 as done by Thompson and Carlson (2017) for COV 1. A more detailed examination of this point, both through individual sessions with students and involving a greater number of subjects, could reinforce the qualitative results of our research. To conclude, the linguistic and discursive analysis of the classroom discussions inevitably calls into play a *cultural issue*: our students are Italian students. As a consequence, the features of the discourse and lexical and syntactical structures identified are strongly influenced by the cultural heritage and the translation into English, although as literal and faithful as possible with regard to our possibilities, necessarily implies that certain nuances of meaning and characteristics get lost or modified. It would be interesting to get data also from different cultural contexts.

## 13.5 FURTHER DIRECTIONS OF RESEARCH AND OPEN QUESTIONS

At the beginning of this research, we ventured into an uncharted ground. Despite our preliminary definition of second-order covariation and conscious that we were theorizing about others' cognitions, that is the delicate nature of a reflexive research itself, the results we obtained overcome both our a priori hypotheses and our prospective analysis. Many times, we had to revise and enlarge our theoretical framework so to embrace in a coherent and complete way all the results emerging and at the end of this challenging investigation we can state that questions in our minds are probably more than the answers and second-order covariation revealed much more complicated than expected. At this point, we would like to outline some final questions and issues that engage our minds.

1. Do other orders of covariation exist?

We started our research investigating second-order covariation and we concluded it, at least for now, theorizing the existence of three different orders of covariation that differ because of the increasing complexity of the mathematical objects involved. The question arises: do other orders of covariation exist? What would happen if we could explore other branches of Mathematics? For instance, what would happen if we focused on differential equations? The question is not random since in our T.E.s, students already worked with finite differences, a powerful and preliminary tool to the introduction of the concept of derivative, and they approached the notion of rate of change which includes in itself how a quantity varies in time and so are a mathematical concept showing a greater degree of complexity.

## 2. The issue of continuity

In Thompson and Carlson's framework (2017) about covariation, we could recognize several instances that refer to the concept of continuity. Starting from the well-known bottle problem, the task itself states the bottle is filled with a continuous flux of water. The description of the highest level of first-order covariation, smooth continuous covariation, reads "the person envisions both variables varying smoothly and *continuously*" (Thompson & Carlson, 2017, p. 435). Moreover, the authors propose a covariational definition of function that we reported in Chapter 3. This definition makes us wonder if somehow hides the notion of continuity which is an essential property of functions but does not connote neither the totality of possible functions nor the whole variety of real phenomena. Let us consider a variation of the Galileo experiment: what would have happened if the inclined plane was placed on a table and the rolling ball at the end of the plane fell on the floor? This is just an example of a real situation that introduces a non-continuous function. The question that we are asking is if the covariational approach fits also for the introduction to the function concept in general, including those cases of non-continuity. Another remarkable example that came to our mind while reflecting on this issue is the Dirichlet function, known to be non-continuous in every point of the domain.

## 3. The link between cognition and category theory

The Dew point teaching experiment has strongly questioned our preliminary definition of second-order covariation and led us to introduce a new mathematical characterization of the COV 2 interpretation that we have clarified and condensed in the blended connotation. In analytical terms corresponds to the study of a function $z=f(x, y)$ first considering $z=f(x,m)$ with $m$

parameter (e.g., absolute humidity) and then *z=f(n,y)* with *n* parameter (e.g., relative humidity). These findings led us to consider the necessity of a more rigorous mathematical interpretation of these covariational processes supported by the cognitive mechanism of conceptual blending. We thought of the language of category theory that for its property of universality could help us to enlighten this issue. Some preliminary findings will be presented in Chapter 14.

## 4. Teachers' training on covariation

Investigating teachers' knowledge and practices about covariation is a new, and apparently necessary avenue of research in Mathematics Education and the positive findings of our study show that it is really worth it to introduce the students to this form of reasoning, but teachers need to be prepared. Recently in Italy, a preliminary survey, elaborated by a research group[29] in Mathematics Education which I am part of, has been administered to Italian teachers to explore their background on covariation, their perception of its relevance and their didactical practices involving this kind of reasoning. The design of the survey was outlined based on the existing literature on covariation as a theoretical construct and the results of our teaching experiments. A special focus was reserved for the distinction between variables and parameters, which is essential to fostering second-order covariation. This questionnaire was just the prior phase of a teachers' development course that aims to improve teachers' knowledge of teachers on this subject and to encourage them to introduce it in their classes by providing them with appropriate tools. The course is entitled *Varia tu che covario anch'io* [You vary and I covary too] and the poster of the course is shown in **Figure 66**: it is an online course started in mid-November to which applied about 70 teachers from the whole national territory and nearly 40 of these have also been involved in a phase of didactic design and experimentation in class. The mini website of this course, where all the presentations and slides are upload is available here. We are looking forward to collecting and analyzing the results of this project.

[29] This research group which coincides with the group of trainers of the course is formed of: Prof. F. Arzarello, four researchers in Mathematics Education: Dr F. Ferretti, Dr C. Giberti, Dr G. Lisarelli and Dr E. Taranto, a teacher, Silvia Beltramino, and me.


**Figure 66** - Poster of the teachers' development course "Varia tu che covario anch'io"

# 14 A MATHEMATICAL INTERPRETATION OF THE COVARIATION CONSTRUCT BASED ON CATEGORY THEORY AND CONCEPTUAL BLENDING

In this chapter[30] we are going to use the category theory to interpret the cognitive processes related to covariational reasoning that we observed in our teaching experiments. This unusual application of categories outside mathematics in the years has become more and more diffuse. Categories, and specifically the Yoneda lemma, are now employed to enter and explain different cognitive problems and theories. For example, the so-called integrated information theory of consciousness (Tononi, 2012) proposes that a certain specific conceptual structure called MICS, maximally irreducible conceptual structure, is identical to conscious experience (Oizumi et al., 2014). Roughly speaking, using the mathematical formalism of category theory it is possible to show that exists a proper translation between the domain of conscious experience and that of the MICS, so that very difficult questions in the domain of consciousness can be resolved in the domain of mathematics (Tsuchiya et al., 2016; Phillips, 2018, 2020). The Yoneda lemma is also one of the basic ideas in the Synthetic Philosophy of Contemporary Mathematics (Zalamea, 2012; 2021), from where it was transposed into Mathematics Education, in order to provide a categorical definition of mathematical objects specific to Mathematics Education (Asenova, 2021).

This chapter has been written in the spirit suggested by Spivak in the Introduction to his book: "to create a bridge between the vast array of mathematical concepts that are used daily by mathematicians to describe all manner of phenomena that arise in our studies and the models and frameworks of scientific disciplines such as physics, computation, and neuroscience" (Spivak, 2014, p. 5).

## 14.1 A CATEGORICAL INTERPRETATION OF COV 2 – QUANTITATIVE CHARACTERIZATION

In this section, we are going to explore a categorical definition of the first level of COV 2 and of its quantitative characterization. In the first paragraph we are going to introduce the categorical

[30] The preliminary findings presented in this chapter are the result of an ongoing study in collaboration with Dr M. Asenova and Prof. F. Arzarello.

notions that we are going to use to introduce the adjunction and in the second paragraph we are going to expose our categorical interpretation supporting it with some examples referred to our experimentations.

### 14.1.1 Notions of category theory

A category $C$ is essentially a universe of the mathematical discourse thought in a very abstract way: it is determined by specifying 'objects' and 'relations' among these (Goldblatt, 1984). The sets themselves are an example of category (objects being the sets themselves and relations the usual functions among them), as well as other mathematical universes, from vector spaces to topos, to sheaves, etc. in which objects and relations assume specific characteristics each time.

**Definition 1 (Category)** A *category* $C$ generally consists of:

- a class of $C$-objects denoted as $Ob(C)$;
- a class of $C$-relations, called *morphisms* or *arrows*;
- operations assigning to each $C$-morphism $f$ a $C$-$dom(f)$ object, the domain of $f$, and a $C$-$cod(f)$ object, the codomain of $f$, such that $f: dom(f) \to cod(f)$;
- an operation associating to each pair $<f,g>$ of $C$-morphism with $dom(g)=cod(f)$ a $C$-morphism $g \circ f$, the composition of f and g, with $dom(g \circ f)=dom(f)$ and $cod(g \circ f)=cod(g)$, such that the following properties are true:
- associativity: $h \circ (g \circ f) = (h \circ g) \circ f$;
- existence for each $X \in Ob(C)$ a $C$-morphism $id_X: X \to X$, the *identity* on $X$, such that: $id_X \circ f = f$ and $g \circ id_X = f$.

In the examples that we will discuss in this chapter the basic category will be that of *Set*, whose objects are the sets, and the morphisms are the functions between sets. Hence, we introduce the following notation.

**Notation** Given sets $A$ and $B$, we can form in *Set* the collection $B^A$ of all morphisms that have domain $A$ and codomain $B$, i.e., $B^A = \{f: A \to B\}$.

**Definition 2 (Evaluation function)** To characterize $B^A$ by arrows we observe that associated with $B^A$ is a special morphism called *evaluation function*: $ev: B^A \times A \to B$, given by the rule $ev(f, x) = f(x)$. Its inputs are pairs of the form $(-, x)$ where $x \in A$.

The categorical description of $Y^X$ comes from the fact that $ev$ enjoys a universal property amongst all set morphisms of the form $g: AxX \to Y$. Given any such $g$, there is one and only one function $g^\wedge: A \to Y^X$ such that the diagram

$$
\begin{tikzcd}
B^A \times A \arrow[dr, "ev"] & \\
& B \\
C \times A \arrow[uu, dashed, "\hat{g} \times id_A"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

commutes, where $g^\wedge x\, id_A$ is the product function that for every input $(c,a) \in CxA$ gives as output $(g(c), id_A(a)) = (g(c), a)$.

The idea behind the definition of $g^\wedge$ is that the action of $g^\wedge$ causes any particular $c$ to determine a function $A \rightarrow B$ by fixing the first elements of arguments of $g$ at $c$, and allowing the second elements to range over $A$. In other words, for a given $c \in C$, we define $g_C$: $A \rightarrow B$ by the rule $g_c(a) = g(<c, a>)$, for each $a \in A$. Now $g^\wedge: C \rightarrow B^A$ can be defined by $g^\wedge(c) = g_c$ for all $c \in C$.

For any $(c,a) \in CxA$ we then get $ev(g^\wedge(c), a)) = g_c(a) = g(<c, a>)$ and so the above diagram commutes. But the requirement that the diagram commutes, i.e. that $ev((g^\wedge(c), a)) = g(<c, a>)$, means that $g^\wedge(c)$ must be the function that for input $a$ gives output $g(<c, a>)$, i.e. $g^\wedge(c)$ must be $g_c$ as above. By abstraction then we say that:

> **Definition 3 (Exponentiation)** A category $C$ has *exponentiation* if it has a product for any two $C$-objects, and if for any given $C$-objects $a$ and $b$ there is a $C$-object $b^a$ and a $C$-arrow $ev: b^a\ x\ a \rightarrow b$, called an *evaluation arrow*, such that for any $C$-object $c$ and $C$-arrow $g: c\ x\ a \rightarrow b$, there is a unique $C$-arrow $g^\wedge: c \rightarrow b^a$ making the diagram
>
> $$
\begin{tikzcd}
b^a \times a \arrow[dr] & \\
& b \\
c \times a \arrow[uu, dashed, "\hat{g} \times id_a"] \arrow[ur] &
\end{tikzcd}
$$
>
> commute i.e., a unique $g^\wedge$ such that $ev \circ (g^\wedge\ x\ id_a) = g$. The assignment of $g^\wedge$ to $g$ establishes a bijection $C(cxa,b) \cong C(c,b^a)$ where the symbol $\cong$ denotes an equivalence of categories (Goldblatt, 1984).

Two morphisms ($g$ and $g^\wedge$) that correspond to each other under this bijection will be called *exponential adjoints* of each other.

Given a function $g: AxX \rightarrow Y$ we can consider the function $g^\wedge: A \rightarrow Y^X$ defined as $g^\wedge(a) = f$, where $f$ is the function in $Y^X$ such that $f(x) = g(a,x) \in Y$. The function $g^\wedge$ is a typical example of adjunction, a pervasive concept in category theory; Mac Lane (1978) writes: "[...] adjoint functors occur almost everywhere in many branches of Mathematics. [...] a systematic use of all these adjunctions illuminates and clarifies these subjects" (p. 107).

> **Definition 4 (Right adjoint)** If $C$ has exponentials, then there is a bijection $C\,(c \times a, b) \to C\,(c, b^a)$ for all objects $a$, $b$, $c$, indicating the presence of an adjunction. Let $F\colon C \to C$ be the right product functor $- \times a$ taking any $c$ to $c \times a$. Then $F$ has as *right adjoint* the functor $(\ )^a : C \to C$ taking any $b$ to $b^a$ and any arrow $f\colon c \to b$ to $f^a : c^a \to b^a$, which is the exponential adjoint to the composite $f \circ ev'\colon c^a \times a \to c \to b$, i.e. the unique arrow for which the diagram
>
> $$\begin{tikzcd} b^a \times a \arrow[r, "ev"] & b \\ c^a \times a \arrow[u, dashed, "f^a \times id_a"] \arrow[ur, "f \circ ev'"'] & \end{tikzcd}$$
>
> commutes. The adjoint situation is $\dfrac{c \to b^a}{c \times a \to b}$.

Thus, $C$ has exponentials if and only if the functor $- \times a$ has a right adjoint for each $C$-object $a$.

## 14.1.2 Categorical interpretation

The processes of covariational reasoning develop in students' mind through different steps that we are going to present and discuss using some examples, referring also to the mathematical situations proposed in our T.E.s.

### Example 1: a family of lines

*Order 1*. Given a function like $y = 3x$, the students would say it is a straight line (passing for $O$) and they would capture the covariation between $x$ and $y$ according to the levels of COV 1.

*Order 2 – L1*. Students consider <u>the</u> function $y=mx$: they say that IT is a <u>generic</u> line (for $O$) while $m$ is a parameter, a sign that can be replaced with a number and then returns a precise line. The main focus is always the covariation between $x$ and $y$, but the parameter appears, and the language extends with the addition of the generic word or similar, which marks a difference from the first order.

*Order 2 - Quantitative*. Students consider <u>the family</u> of functions $y=mx$: it is studied with the variation of the parameter $m$; then they catch a covariation at the two orders, that between $m$ and the function $y=mx$ (COV 2 – quantitative), and considering $m$ fixed, that between $x$ and $y$ (COV 1). Language can be a marker of this phenomenon (e.g., from a generic line to a family of lines). The difference between COV 1 and COV 2 consists of this:

- at the <u>first-order</u>, we have the function $g$: $(m,x) \longmapsto mx$ for $m$ <u>fixed number</u>;

- then at <u>second-order – L1</u>, $m$ becomes a parameter and the language changes: <u>generic line</u>;
- at the <u>second-order - quantitative</u>, we have the function $g^\wedge: m \longmapsto f$, where $f: X \rightarrow Y$ is defined as $f: x \longmapsto mx$: <u>family of functions</u>.

Then the second-order covariation is obtained by passing from a function ($g$) to the new function $g^\wedge$. The category (objects and functions) in which we work is extended by considering the function $g^\wedge$ next to the function $g$. This second aspect is not formalized by students but can be grasped mainly in the way of reading the expression $y=mx$.

Example 2: a family of parables

*Order 1*. Given a function like $y = 3x^2$, the students would say it is a parable or second-degree function and they would capture the covariation between $x$ and $y$ according to the taxonomy of COV 1.

*Order 2 – L1*. Students consider <u>the</u> function $y=ax^2$: they grasp the covariation between $x$ and $y$, and they read $a$ as a parameter, a form of sign variation that makes them consider the generic parable $y=ax^2$.

*Order 2 – Quantitative.* Students consider <u>the family</u> of functions $y= ax^2$: it is studied with the variation of the parameter $a$; then they catch a covariation at the two orders, that between $a$ and the function $y= ax^2$ (COV 2 – quantitative level), and considering $a$ fixed, that between $x$ and $y$ (COV 1). Language can be a marker of this phenomenon (e.g., from a generic parable to a family of parables).

At the first-order, we have the function $g: (a,x) \longmapsto ax^2$ (generic line); at the second order, we have the function $g^\wedge: a \longmapsto f$, where $f: X \rightarrow Y$ is defined as $f: x \longmapsto ax^2$. Even in this case this second aspect is not formalized by students but can be grasped mainly in the way of reading the expression $y= ax^2$.

Example 3: law of the inclined plane

In the 2017-2019 T.E.s, the situation under investigation was a little bit more complicated: we have a function $f$ that associates to an angle $\gamma$ and to time $t$ the traversed distance but such function results in turn from the composition of function $k(\gamma)= \frac{1}{2}\, g \cdot \sin(\gamma)$ (where $g$ is the constant of gravity) with the quadratic function that to the couple $(k, t)$ associates the value $k \cdot t^2$:

$$k: \gamma \longmapsto \tfrac{1}{2}\, g \cdot \sin(\gamma);$$
$$f: (k, t) \longmapsto k \cdot t^2.$$

In 2017 T.E., the function $k$ did not appear in an explicit form: the students observed just towards the end of the classroom discussion that the constant $k$ that appears in the formula $y = kx^2$ varies with the inclination but they do not go further.

To better explain this case with the categorical interpretation, we simplify the situation neglecting at first the function $k(\gamma)$. We do not lose anything in substance, as we will show in a while. We consider a function $f$ that associates to a couple of values $k$ and $t$ a value $s$ (referring to the situation of the inclined plane, $k$ is our $\gamma$ above, at the moment released from the angle, $t$ is the time and $s$ the space traversed on the inclined plane) and we consider $f^\wedge$, according to the definition given above. Let us now consider $K$ the set of $k$, $T$ that of the times, $S$ that of the spaces (on the inclined plane). It will result $f: K \, x \, T \rightarrow S$ and $f^\wedge: K \rightarrow S^T$ defined as $f^\wedge(k) = h$, where $h$ is the function in $S^T$ such that $h(t) = f(k,t) \in S$.

Even in this case $f$ is the function that represents the covariation at order 1, while $f^\wedge$ represents the one at order 2.

If we consider $k$ as the explicit function of $\gamma$, nothing changes in the substance. It is sufficient to replace the $f^\wedge$ function to $g$ and proceed as before as shown in the following diagram:

$$\begin{tikzcd} \Gamma \times T \arrow[r, "\langle k, id_T \rangle"] \arrow[rr, bend right, "f^\wedge"'] & K \times T \arrow[r, "f"] & S \end{tikzcd}$$

The exponential adjunctions $f: C \, x \, A \rightarrow B$ and $f^\wedge : C \rightarrow B^A$ enable to see their mutual relations in two modalities: from category $C\ (CxA,B)$ to the category $C\ (C,B^A)$ meaning, from functions like $f$ to functions as $f^\wedge$ (right adjunction), or vice versa from $f^\wedge$ to $f$ (left adjunction). COV 2 (Quantitative) expresses a form of adjunction that enables to see the same phenomenon in two different ways (through $f$ and $f^\wedge$), but deeply connected. As shown in this study, didactically the phenomenon concerns the understanding of a formula with parameters, but it is part of a form of conceptualization (abstraction) that involves the whole mathematics. Indeed, "[t]he isolation and explication of the notion of adjointness is perhaps the most profound contribution that category theory has made to the history of general mathematical ideas" (Goldblatt, 1984, p. 138).

## 14.2 A CATEGORICAL INTERPRETATION OF COV 2 – BLENDED CHARACTERIZATION

In this section we are going to explore a possible interpretation of the blended characterization of COV 2. Second-order covariation is a construct that here is interpreted and studied with a synthetic approach, i.e., through its relations with the context, and in this chapter the context is given by functions. In the first paragraph, we are going to introduce the mathematical tools needed to enunciate the Yoneda lemma and then in the second paragraph to explore the how this lemma can be helpful in interpreting cognitive covariational processes.

### 14.2.1 Notions of category theory and the Yoneda lemma

**Definition 5 (Dual category)** Given a category $C$, its *dual category* $C^{op}$ is such that:

(i) $Ob(C^{op}) = Ob(C)$,

(ii) $(id_X)C^{op} = (id_X)C$,

(iii) $C^{op}(X, Y) = C(Y, X)$,

(iv) $g \circ_{C^{op}} f = f \circ_C g$.

In category theory the so-called principle of duality applies, according to which a proposition formulated in the category $C$ gives rise to a dual proposition in the dual category of $C$, $C^{op}$.

**Definition 6 (Functor)** A *functor F* is a map between two categories that sends objects into objects and morphisms into morphisms.

**Definition 7 (Covariant functor)** If $A$ and $B$ are two categories, a *functor* $F$: $A \rightarrow B$ *covariant* between $A$ and $B$ is a function from $Ob(A)$ to $Ob(B)$ such that, for each $X, Y \in Ob(A)$, is defined the application $F_{X;Y}$: $A(X,Y) \rightarrow B(F(X),F(Y))$, that maps each morphism $f$: $X \rightarrow Y$ of $A$ into a morphism $F(f)$: $F(X) \rightarrow F(Y)$ of $B$ and such that: $F(g \circ f) = F(g) \circ F(f)$ and $F(id_X) = id_{F(X)}$.

**Definition 8 (Contravariant functor)** If $A$ and $B$ are two categories, a functor $F$: $A \rightarrow B$ *contravariant* between $A$ and $B$ is a function from $Ob(A)$ to $Ob(B)$ such that, for all $X, Y \in Ob(A)$ is defined the application $F_{X;Y}$: $A(X,Y) \rightarrow B(F(Y),F(X))$, that maps each morphism $f$: $X \rightarrow Y$ of $A$ into a morphism $F(f)$: $F(Y) \rightarrow F(X)$ of $B$ and such that: $F(g \circ f) = F(f) \circ F(g)$ e $F(id_X) = id_{F(X)}$.

**Definition 9 (Embedding)** A functor $F$: $C \rightarrow D$ is said *embedding* if, given $f_1, f_2 \in C(X,Y)$, from $F(f_1) = F(f_2)$ follows $f_1 = f_2$.

**Definition 10 (Natural transformation)** Given two functors $F, G$: $C \rightarrow D$, a *natural transformation* $\tau$: $F \Rightarrow G$ is a class $N \subseteq D(X;Y)$ such that $N=\{\tau_X: F(X) \rightarrow G(X) \mid X \in Ob(C)\}$. $\tau$ is "natural" in the sense that induces a commutative diagram such that $\tau_X \circ G(f)=F(f) \circ \tau_Y$:

$$\begin{tikzcd} F(X) \arrow[r, "\tau_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\tau_Y"'] & G(Y) \end{tikzcd}$$

If $\forall\ X \in Ob(A)\ \tau_X$ is an isomorphism in $D$, then $\tau$ is a *natural isomorphism*. Moreover $id\tau$: $F \Rightarrow F$.

The knowledge of a mathematical object can be distinguished into an *analytical knowledge* that is the study of the object itself through the analysis of its components, and a *synthetic knowledge* expressed through its knowledge in context, that is when the object is studied though its interactions with the environment (Zalamea, 2012). The synthetic knowledge of an object can be exemplified as follows: "the object $A$, interpreted in the analytical sense, as object itself, is considered as being part of a category jointly with other objects that constitute its context ($C$); expressing the relations in $C$ in function of $A$ through a functor $h_A$ between $C$ and *Set*, the object intended in the analytical sense 'disappears' as object of knowledge (A) and leaves space to the morphisms that express its relations with the environment. The set of such morphisms is the object of knowledge intended in a synthetic way" (Asenova, 2021, pp. 406-407).

"The concept of representable functor ($h_A$) allows to express the fact that a mathematical object can be interpreted through its relations with the context, where the context is constituted by the relationships with the objects of a category to which the object belongs to. Hence, let us introduce the definitions of covariant hom-functor and representable functor" (Asenova, 2021, p. 405).

> **Definition 11 (Covariant/Contravariant hom-functor)** Let $C$ be a category. A *covariant hom-functor* represented by an object $A \in Ob(C)$ is the functor: $C(A,–)$: $C \rightarrow$ *Set*, defined as: $C(A,–)(X) = C(A, X)$ for all $X \in Ob(C)$; if $f \in C(X, Y)$ we put $C(A, \mathrm{f}) = C(A,–)(f) = C(A, X) \rightarrow C(A, Y)$ the application $C(A, f)$ $(g$: $A \rightarrow X) = f \circ g : A \rightarrow Y$. In an analogous manner, it can be defined the concept of *contravariant hom-functor* $C(–, A)$: $C^{op} \rightarrow Set^C$.

The category whose objects are the contravariant hom-functor from $C^{op}$ to $Set^C$ is called *category of presheaves on C with values in Set* and such functors are called *presheaves*.

> **Definition 12 (Representable functor)** A *functor* $F$: $C \rightarrow Set$ is said *representable* if exists an object $A \in Ob(C)$ and a natural isomorphism $\tau$: $F \Rightarrow C(A,–)$. The pair $(A, \tau)$ is said *representation of F*.

If $h_A$ is a covariant functor representable by an object $A$, then the contravariant functor representable by $A$ is the functor $h^A$: $C^{op} \rightarrow Set^C$.

The synthetic knowability of an object is assured by the Yoneda Lemma and by the Yoneda embedding.

> **Proposition 1 (Yoneda lemma)** Let $F: C \to Set$ be a (covariant) functor and $A \in Ob(C)$. There is a bijective correspondence $y: Nat(h_A, F) \to F(A)$ between the set of natural transformations $h_A \Rightarrow F$ and the set $F(A)$.

In other words, Yoneda lemma states that there exists a bijective correspondence between (i) the set of all natural transformations between a generic functor $F$ that originates the representable functor $h_A$ and the functor $F$ itself and (ii) the images of the object $A$ according to functor F. A detailed demonstration of Yoneda lemma can be found in Mac Lane (1978, pp. 59-62).

Yoneda embedding is an embedding in the dual category of a small category $C$ into the category $Set^C$ of the contravariant functors (presheaves) from $C^{op}$ to $Set^C$, through a contravariant representable functor $h^A$ that creates a copy of $C^{op}$ in $Set^C$, expressing $C^{op}$ in function of an object $A$.

The functor $h^A$ is *fully faithful* since the functors from $C^{op}$ to $Set^C$ are both surjective and injective and therefore it guarantees the embedding of $C^{op}$ into $Set^C$.

Using a suitable categorical framework (essentially the lemma of Yoneda) allows to interpret categorically the cognitive processes, which mark the transition from first- to second-order covariation and the development of the blended characterization of second-order covariation that we have seen in the dew point teaching experiment. Moreover, it will be possible to give reason also of the first cognitive level of COV 2 we already described (13.1.2). For this interpretation, we base on the just published Ph.D. dissertation of Dr M. Asenova (Asenova, 2021).

We will sketch the main ideas of this categorical interpretation, illustrating it for very simple situations, which however catch its main mathematical features. The situation we described for L1 of COV 2, requires that we start from a category, let us say $M$, where functions like $y=m_i x$ live as objects and with morphisms between them: they simply transform the coefficients $m_i$, one into another. Now let us consider the category $Set^M$, whose objects are the contravariant functors (presheaves). Now, we fix an object y= $m_i x$, and, using the Yoneda lemma, embed all $M$ into $Set^M$ through representable functors $h_{m_i}$ (they represent the whole $M$ through the $m_i$). This operation can be made for each y= $m_i x$. Hence the presheaves represent $M$, each with respect to an $m_i$.

Now in *Set*$^M$ there will be natural transformations between the different $h_{m_i}$: they translate the way how $h_{m_i}$ represents $M$, into the way how $h_{m_j}$ represents $M$. In this way, through the natural transformations one gets what we have called the generic function $y=mx$. So, this is the categorical counterpart of the first level of COV 2. Now something new is necessary to have fully COV 2. What we need is:

(i) a non-representable functor $F$ from $M$ to *Set*$^M$: it translates all $M$ in *Set*$^M$ in a unique way;

(ii) natural transformations between $F$ and the $h_{m_i}$: these allow to connect the 'compact' representation of $M$ through $F$ to the previous representations $h_{m_i}$.

In this way, the natural transformations between $F$ and the $h_{m_i}$ generate the family of functions y= $m_i x$ as a unique object, that is as described in second-order covariation: $m \longmapsto$ y= $mx$. This situation is pictured in **Figure 67**.

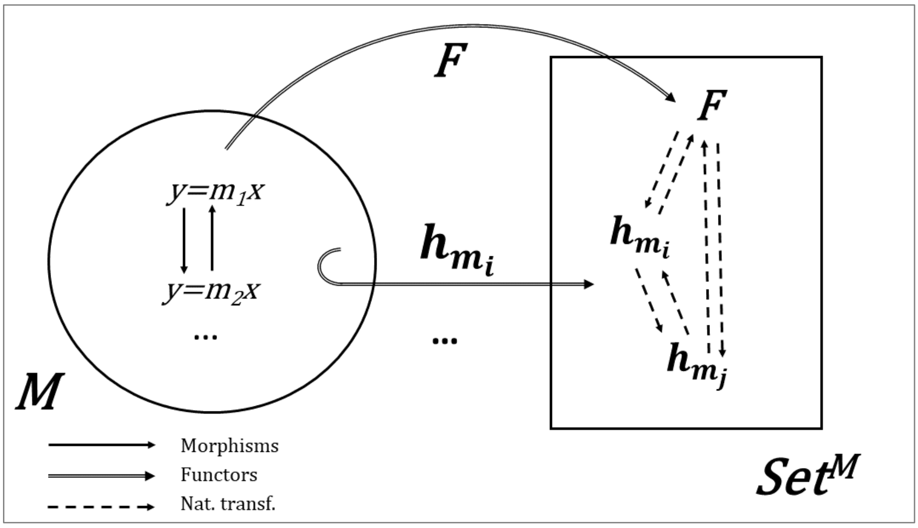



**Figure 67 –** Schema representing how the natural transformations between $F$ and the $h_{m_i}$ generate the family of functions $y= m_i x$ as a unique object

It is interesting to observe that the usual way in which adjunction is described, that is as a *currying* (Spivak, 2014, pp. 90-91), does not give reason of the full complex structure that can be behind it. The analysis of students' cognitive processes has disclosed it, while the language of categories has allowed to systematize it within a rigorous mathematical frame.

An analogous phenomenon has been detected considering another aspect of students' cognitive processes while entering second-order covariation. We will sketch briefly also this part, since we think to dedicate our future research to issues like these, which have grown up only in the last weeks of our reflection about covariation.

We have seen that in the dew point experiment students enter more deeply into COV2: this more elaborated aspect of second-order covariation is emblematically illustrated by Matteo's productions (12.2.2). Indeed, Matteo is able to manage mathematically a formula containing more than one parameter. In Chapter 12 we have used the blending model of Fauconnier and Turner (2002) to comment that. This can be done also through the above categorical framework: we will sketch shortly how to do it. We have just seen how the production of a family of functions (*y=mx*) is got analyzing the functorial relationships between and within the categories *M* and *Set^M^* (**Figure 67**).

Now, let us adapt the situation represented in that picture to a new more general situation. Let $0 = f(m, n, x)$ be a function in three variables: passing from an implicit formulation to an explicit one, it is possible to generate a first family of functions, let us indicate it as $n = f_1(\bar{m}, x)$, considering $n$ as the dependent variable and $m$ as a parameter; similarly we can get another family of functions, let us indicate it as $m = f_2(\bar{n}, x)$, considering $m$ as the dependent variable and $n$ as a parameter. **Figure 68** illustrates how to do that.

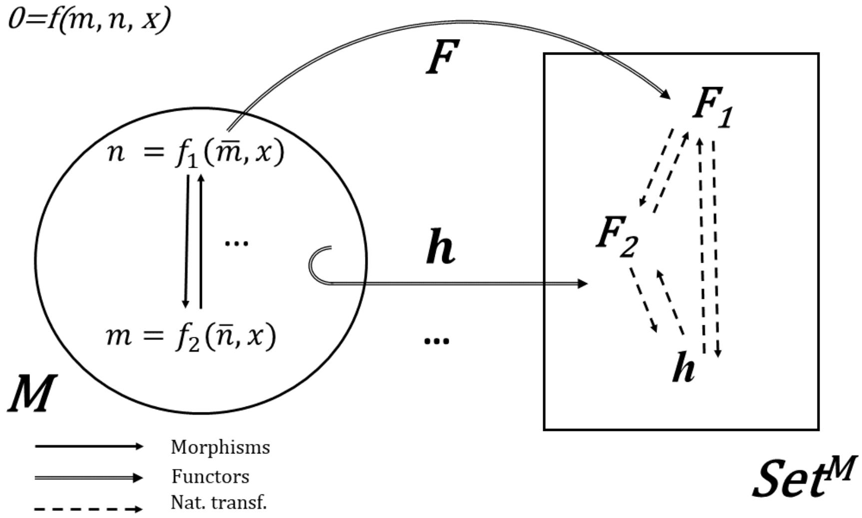



**Figure 68 –** Schematic representation of functorial relationships between and within the categories *M* and $Set^M$ considering functions of three variables

The representable functors indicated with $h$, whose meaning is analogous to the $h_{m_i}$ used in the previous discussion, allow to represent the different functions $f_i$ within the category $Set^M$ and to get first the generic functions $f_i$ because of the actions of natural transformations between the different $h$; and then the two distinct families of functions $f_1$ and $f_2$, because of the natural transformations between suitably non-representable functors $F_1$ and $F_2$ in $Set^M$. A further step is now possible, namely because of natural transformations between $F_1$ and $F_2$, which relate them to each other and consequently we can obtain the two families of functions $y_1, y_2$. It is exactly this situation to picture mathematically the cognitive processes that appeared in the pot experiment and allowed students to go back and forth through the different parameters and manage the two representations of the psychrometric chart (**Figure 47**).

The wider mathematical background of students is evident: a meaningful example can be recognized at [174], where Matteo elaborates a mathematical formula containing three different parameters: his choice reflects the greater confidence with which he handles the analytical representations and the awareness of the different role of variables and parameters.

### 14.2.2 Blending and categories

In Chapter 12, it is shown that the way COV 2 is elaborated by students in many cases reveal the cognitive phenomenon of conceptual blending (Fauconnier & Turner, 2002). Again, we can translate it into a categorical framework, and once more the categorical framework reveals as a mathematical counterpart of the cognitive processes of our students.

The mathematical interpretation of the blending mechanism is longstanding in the literature, but we only recently learned about it through in-depth studies on the state of the art of conceptual blending. One of the forerunners of the modelling of conceptual blending using the theory of categories is Goguen (1999) who mainly applied it to computation. In this dissertation, we are going to follow the approach by Schorlemmer and Plaza (2021) who strongly rely on Goguen's ideas and base their interpretation on the construct of *amalgam*. As the authors remark, "[b]y framing blending in category theory, we will take the focus off the mental spaces and its structures and put it on the mappings and the projections between spaces, and how these mappings relate to each other" (p. 8). This section is devoted to capture the meaning of blending through a categorical approach: at the end we will show how the amalgam allows to embed blending into some general kinds of categories, called *realms*.

Roughly speaking, blending mechanism consists of two pairs of arrows from the two input spaces ($I$ and $J$) to the blend ($B$), and of the two from the generic space ($G$) to the inputs. It is their combined action that produces the structure of a blending (**Figure 69**).

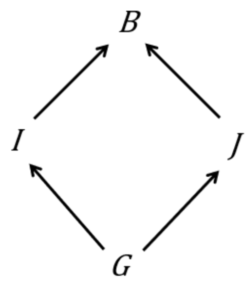



**Figure 69** - Diagram showing the functioning of conceptual network integration

The arrows from the generic space $G$ to the input spaces allow to determine what part of the input spaces is fused in the blend $B$, and this happens because of the commutativity of the diagram of **Figure 69**. It is so required that all the arrows from $G$ to $B$ obtained by composition through $I$ and $J$ are equal. The two technical categorical tools that allow to represent this situation are the *span* and the *co-cone*, which in their turn base on the construct of *diagram* and *cone*.

> **Definition 13 (Diagram)** A *diagram D* in a category $C$ is a functor $D$: $J \to C$, where $J$ is a directed graph (where nodes are $C$-objects and edges are $C$-arrows in $J$). The graph $J$ is called the shape of $D$.

In our case we will limit to consider finite diagrams (with a finite number of nodes and edges). To simplify, we will directly represent the structure of a diagram $J$ upon the objects and arrows of the category $C$. We will be particularly interested in two types of shapes (**Figure 70**): following Schorlemmer and Plaza (2021), we will call them *v-shapes* (or *spans*) and *w-shapes*.

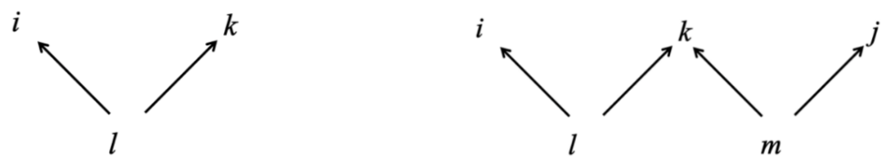



**Figure 70** – On the left a v-span or span and on the right a w-shape

> **Definition 14 (Cone)** Let $D$: $J \to C$ be a diagram of shape $J$ in a category $C$. A *cone* to $D$ is an object $N$ of $C$ together with a family $\phi_X$: $N \to D(X)$ of morphisms indexed by the objects $X$ of $J$, such that for every morphism $f$: $X \to Y$ in $J$, we have $D(f) \circ \phi_X = \phi_Y$ as shown in the following diagram:

$$
\begin{tikzcd}
 & N \arrow[dl, "\phi_X"'] \arrow[dr, "\phi_Y"] & \\
D(X) \arrow[rr, "D(f)"'] & & D(Y)
\end{tikzcd}
$$

**Definition 15 (Limit)** A *limit* of the diagram $D: J \to C$ is a cone $(L, \psi)$ for $D$ such that any cone $(N, \phi)$ for $D$ factors through it, that is for any cone $(N, \phi)$ for $D$, there exists, unique, a morphism $\lambda: N \to L$ so that the following equality $\psi_X \circ \lambda = \phi_X$ holds — i.e., the following triangles commute for all $X$ in $J$:

$$
\begin{tikzcd}
 & N \arrow[d, dashed, "\lambda"] \arrow[dl, bend right, "\phi_X"'] \\
 & L \arrow[dl, "\psi_X"] \\
D(X) &
\end{tikzcd}
$$

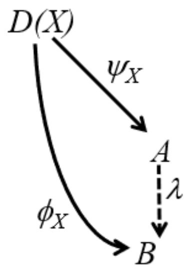
The dual notion of cone is that of co-cone:

**Definition 16 (Co-cone)** Let $D: J \to C$ be a diagram of shape $J$ in a category $C$. A *co-cone* $(A, \phi)$ for $D$ is a $C$-object $A$, called *apex* of the co-cone, together with a family of $C$-arrows $\phi_X: D(X) \to A$ (one for each object $X$ of $J$) such that, for each morphism $f: X \to Y$ in $J$, the equality $\phi_Y \circ D(f) = \phi_X$ holds — i.e., the following triangles commute:

$$
\begin{tikzcd}
D(X) \arrow[rr, "D(f)"] \arrow[dr, "\phi_X"'] & & D(Y) \arrow[dl, "\phi_Y"] \\
 & A &
\end{tikzcd}
$$

A dual consideration holds for the notion of colimits as universal co-cones.

**Definition 17 (Colimit)** A co-cone $(A, \psi)$ for $D$ is said to be a *colimit* for $D$ if, for each other co-cone $(B, \phi)$ for $D$, there exists a unique $C$-arrow $\lambda: A \to B$ such that, for each object $X$ in $J$, the following equality $\lambda \circ \psi_X = \phi_X$ holds — i.e., the following triangles commute for all $X$ in $J$:




When the diagram D is a span, colimits are called *pushouts.*

Co-cones seem to be the natural candidates to model blends; however, as observed by Schorlemmer and Plaza (2021), in a blend the arrows from $I$ and $J$ (**Figure 69**) may not be defined for all structure (objects and arrows) of the inputs, but only for that meaningful in the blend; conversely some more structure might be necessary in the blend; moreover, from the same inputs more than one blend might be originated, while a pushout is unique up to isomorphisms. All this makes the categorical picture above too 'strict' for modelling blends. To overcome this difficulty, Goguen (1999) and Schorlemmer and Plaza (2021) propose two different solutions, which however are coherent each other. We will now sketch the one elaborated by Schorlemmer and Plaza. It is based on a categorical notion of partial functions and allows to define amalgams, which, in their turn, can model conceptual blends.

In a categorical environment, a partial function $A \rightarrow B$ can be represented as a span of two arrows, one of which is a monomorphism (indicated with $\rightarrowtail$); $A_0$ is the subobject of $A$, domain of the function (**Figure 71**).

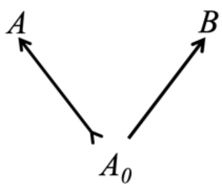



**Figure 71 –** Representation of a partial function

In order that partial arrows are a category, their composition must be defined (Jay, 1991). To do that we need preliminary the notion of *pullback* (p.b.).

> **Definition 18 (Pullback)** Let $f: X \rightarrow Z$ and $g: Y \rightarrow Z$ two morphisms in a category $C$. Their pullback is given by an object $P$ and two morphisms $p_1 : P \rightarrow X$ and $p_2 : P \rightarrow Y$, for which the following diagram commutes:

$$
\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}
$$

and moreover, it is universal, that is for any other object Q and morphisms $q_1$: $Q \rightarrow X$ and $q_2$: $Q \rightarrow Y$ in $C$, there exists a unique morphism $u$: $Q \rightarrow P$, such that the following diagram also commutes:

$$
\begin{tikzcd}
Q \arrow[dr, dashed, "u"] \arrow[drr, "q_2"] \arrow[ddr, "q_1"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}
$$

For example, in case of *Set* the pullback of functions $f$: $X \rightarrow Z$ and $g$: $Y \rightarrow Z$ gives $X \times_Z Y$ as apex:

$$X \times_Z Y = \{(x, y) \in X \times Y : f(x) = g(y)\} = \bigcup_{z \in f[X] \cap g[Y]} f^{-1}(\{z\}) \times g^{-1}(\{z\})$$

together with the restrictions of the projection maps $\pi_1$ and $\pi_2$ to $X \times_Z Y$.

Having defined the p.b. construct, we can now define the composition of partial arrows as span composition through their p.b., as indicated in **Figure 72**. $A_1$, the apex of the pullback, can be thought of as the largest subobject of $A$ on which $f$ is defined and takes values in the domain of $g$.

$$
\begin{tikzcd}
A & & B & & C \\
& A_0 \arrow[ul, tail, "m"] \arrow[ur, "f"'] \arrow[rr, phantom, "\textbf{p.b.}"] & & B_0 \arrow[ul, tail, "n"] \arrow[ur, "g"'] & \\
& & A_1 \arrow[ul, tail, "n_1"] \arrow[ur, "f_1"'] & &
\end{tikzcd}
$$

**Figure 72 –** Representation of a pullback

In the particular case of a pullback of two monomorphisms, the apex of the pullback is also referred to as the intersection of the subobjects represented by the monomorphisms, and given two subobjects $A_1$ and $A_2$ of an object $A$, we will write $A_1 \cap A_2$ for their intersection.

Hence in order that composition of partial maps is well-defined, the class of monomorphisms that represents them must be closed under pullbacks. Having them, one also gets the inverse morphisms as pullbacks of monomorphisms, because of the universality of pullbacks.

Schorlemmer and Plaza (2021) restrict the monomorphisms to so called *realms*, that is the class of monomorphisms that are closed under pullbacks, composition, and isomorphisms and moreover such that their subobjects do have *finite unions* ([31]). The corresponding subobjects are called *admissible* (Jay, 1991).

To give the definition of amalgam it is still necessary to generalize the notion of *v*-diagram within realms (Schorlemmer & Plaza, 2021).

> **Definition 18 (Generalization of a *v*-diagram)** Let $C$ be a category with realm $M$. Let V be a *v*-diagram made by the two morphisms $f: G \to I$ *and* $g: G \to J$ in $C$. A *generalization* of $V$ is the *w*-diagram in $C$
>
> $$I_0 \leftarrow f^{-1}(I_0) \rightarrowtail f^{-1}(I_0) \cup g^{-1}(J_0) \leftarrowtail g^{-1}(J_0) \rightarrow J_0$$
>
> such that $m: I_0 \rightarrowtail I$ and $n: J_0 \rightarrowtail J$ are monomorphisms in $M$ (see **Figure 73**).

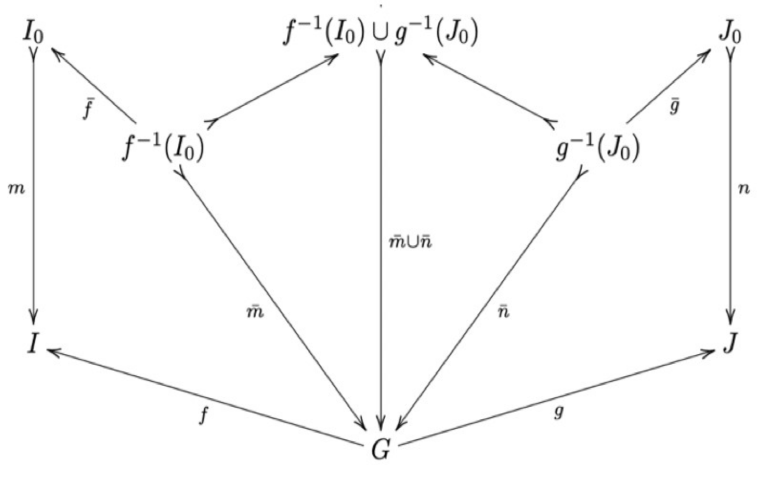



**Figure 73** – Representation of the generalization of a v-diagram taken from Schorlemmer and Plaza (2021)

[31] The union of a family $\{A_i\}_{i \in I}$ of subobjects of A is defined as the subobject $A'$ of $A$, denoted by $\cup_{i \in I} A_i$ (with each $A_i$ also subobject of $A'$) such that, if, for an arrow $f: A \to B$, each $A_i$ is carried into some subobject $B'$ of $B$ by $f$, then $A'$ is also carried into $B'$ by $f$.

The generalization of a given $v$-diagram in a realm $M$ that has finite unions, together with the pairs $< v, w >: < m, n > \rightarrow < m', n' >$ of monomorphisms such that $m = v \circ m'$ $and$ $n = w \circ n'$, form a category, which has finite intersections.

In **Figure 73** the $M$-monomorphisms $m$ and $n$, which represent the subobjects $I_0$, $J_0$ of $I$ and $J$ respectively, determine the generalization of the $v$-diagram; arrows $\bar{m}, \bar{n}$ are the pullbacks of $m$ and $n$ along $f$ and $g$, respectively; and $\bar{m} \cup \bar{n}$ is the inclusion monomorphism of the union $f^{-1}(I_0) \cup g^{-1}(J_0)$ in $G$.

Now let us complete the generalization diagram adding its colimit in order to get the amalgam structure.

> **Definition 19 (Amalgam)** Let $C$ be a category with realm $M$. An amalgam (with apex $A$) for a $v$-diagram V in $C$ is the colimit for a generalization $W$ of $V$ (see **Figure 74**).

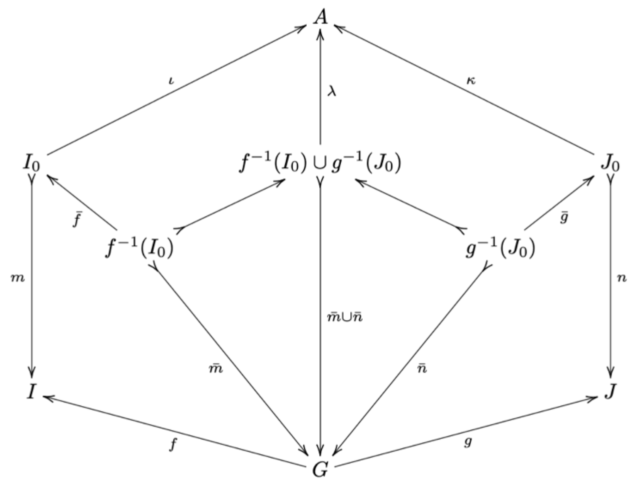



**Figure 74 –** Representation of an amalgam taken from Schorlemmer and Plaza (2021)

Comparing **Figure 73** and **Figure 74**, one can see the relationship of the amalgam with the generalization diagram W: in fact the arrows $\iota, \lambda, \kappa$ are the injections of colimit of $W$ into its apex $A$.

Using the amalgam structure is possible to interpret properly the conceptual blend within a categorical context. With reference to **Figure 74**, $I$ and $J$ are the input spaces, $G$ is the generic space, $A$ is the blend, $I_0$ and $J_0$ are the parts (subspaces) of $I$ and $J$ that are used in the production of the blend. All the story develops with arrows in a realm $M$, which makes possible the realization of suitable technicalities: the arrows $f, g$ from the generic subspace are total, but the contribution to the blend of the inputs $I, J$ may use only a part of them, and this is possible thanks to the intermediation within *M* of the subobjects $I_0$ and $J_0$, which through the inverses $f^{-1}$, $g^{-1}$ can be combined into the *M*-object $f^{-1}(I_0) \cup g^{-1}(J_0)$. On the other side, the injection of the colimit through $\iota, \lambda, \kappa$ into $A$ makes possible the combination of the ingredients of $I_0$ and $J_0$ into the blend *A*.

In an ongoing research we are investigating the link between our previous categorical discussion about *M*, $Set^M$ and their relationships through functors in order to connect the categorical discussion of 14.2.1 within the amalgamation framework described above. A further line of investigation will concern third-order covariation through the introduction of functors between presheaves.

# 15 ASSESSING COVARIATION AS A FORM OF CONCEPTUAL UNDERSTANDING THROUGH COMPARATIVE JUDGEMENT[32]

This chapter focuses on the importance of covariational reasoning within the processes of mathematics teaching and learning. Despite the internationally recognized relevance of covariation, research shows that only a small percentage of students and teachers is able to adopt covariational reasoning and the majority of mathematics curricula do not contain explicit references to covariational skills. In particular, when covariational reasoning manifests as conceptual knowledge, it becomes challenging to assess, and the need for innovative methods of assessment emerges; there is a need for suitable assessment to highlight the characteristics of covariation and capture the various features that characterize conceptual understanding. Comparative judgement, a method that does not require detailed scoring rubrics, is particularly appropriate for assessing complex mathematical competencies adopting a holistic approach. In line with these streams of thought, this study aims to investigate the way *comparative judgement* can help in the assessment of covariational reasoning skills underlying a less structured modelling task and, indirectly, the perception and relevance attributed by mathematics teachers to covariation as a theoretical construct.

## 15.1 RATIONALE

Most mathematics curricula worldwide, including Italian syllabi, highlight the relevance of modelling in students' activities in order to develop up-to-date mathematical literacy. Among the areas of mathematical content in which mathematical literacy is applied, and which are particularly linked to modelling activities, PISA names *Change and relationships*. It claims: "Being more literate about change and relationships involves understanding fundamental types of change and recognizing when they occur in order to use suitable mathematical models to describe and predict change" (OECD-PISA, 2022). One of the crucial aspects in such processes is the ability to develop covariational forms of reasoning, which can help to model "the change and

[32] The results presented in this chapter come from a study in collaboration with Dr F. Ferretti and Prof. F. Arzarello. These findings have been presented in *Incontri con la Matematica XXV* and have been published in the proceedings of the conference:
Bagossi, S. (2021c). Valutare la conoscenza concettuale con il Comparative Judgement [Assessing conceptual understanding with Comparative Judgement]. In B. D'Amore (Ed.), *Atti del Convegno Nazionale "Incontri con la matematica" nr. 35* (pp. 173-174). Bologna: Pitagora.

the relationships with appropriate functions and equations, as well as creating, interpreting and translating among symbolic and graphical representations of relationships" (OECD-PISA, 2022). However, notwithstanding the didactical relevance of covariational reasoning, there are some obstacles in adopting it as a regular and effective practice in Italian classrooms. Firstly, covariational reasoning is not explicitly addressed in the Italian National Curricula; this absence is reflected in school practices and the majority of mathematics textbooks, so most Italian teachers are not aware of covariation and therefore do not foster its use in their classrooms. Secondly, covariational reasoning, insofar as it represents a form of conceptual understanding, is difficult to assess (Bisson et al., 2016). These findings, illustrating the absence of guidelines addressing covariational reasoning in classroom mathematical modelling activities and on its assessment, led us to engage in this research aimed at sowing some seeds to counteract this situation.

The object of this study is the assessment of some open-ended written tasks concerning mathematical modelling of a real phenomenon, which required a strong conceptual understanding of covariation and usage of suitable related reasoning skills, aimed at exploring and describing interconnections between the mathematical model and the physical phenomenon under investigation. We identified in comparative judgement (CJ) a valuable tool for assessing students' conceptual understanding involving covariational skills, and making its assessment easier for teachers less confident with covariation. Open-ended tests, the object of this study, were administered to students of a 10$^{th}$ grade class at the end of a teaching experiment conducted in 2019. Students were asked to prepare a written report on an activity regarding the well-known Galileo experiment (Galileo, 1638) of a ball rolling along an inclined plane to represent the law of motion in action.

The specific goals of this research are twofold: firstly, they investigate how CJ can help in assessment of an open-ended test concerning a modelling task which requires covariational reasoning skills; secondly, they verify if teachers can (implicitly or explicitly) recognize features of covariational reasoning as important in grading students' works; as a by-product, they allowed to gain some insight on Italian teachers' reactions to the CJ method. Thus, in addition to performing a CJ session that adopts an online tool, a post-judging questionnaire was designed and distributed to investigate teachers' comparative processes and the features which influenced them.

This chapter is organized as follows. After a review of the literature (15.2), we highlight existing research gaps concerning the assessment of covariation (15.2.1). The method of CJ is introduced in section 15.2.2 and all the details about the methodology and the design of our research are provided in section 15.3. The data obtained are reported in section 15.4 and the contribution of this study is discussed in section 15.5 to provide answers to the research questions. Finally, the limitations of this research are outlined, together with further research purposes.

## 15.2 THEORETICAL BACKGROUND

Covariational reasoning consists of cognitively demanding activities, which require the development of "multiple levels of sophistication" (Thompson & Carlson, 2017, p. 436). Indeed, it is an example of what many researchers call "conceptual knowledge" (Skemp, 1976), insofar as it involves not only a deep understanding of the principles and theories that govern a certain domain of knowledge (Rittle-Johnson et al., 2001), but also of the relationships between various information (Hiebert & Lefevre, 1986*)*; above all, it does not reduce this to the mere application of mechanical procedures. Covariation is proved to be essential in activities concerning mathematical modelling (Thompson, 2011), specifically when involving motion or dynamic situations.

One important didactical problem related to students' achievement of covariation in modelling activities lies in its concrete assessment and this issue involves three main difficulties.

First, covariational reasoning deals with a typical conceptual knowledge construct, which is more complex than a content area or a procedure to be assessed because of the variety and complexity of students' reasoning. Bisson et al. (2016) highlight the difficulties of using standard assessment practices when conceptual understanding is under the lens because "conceptual understanding is an important but nebulous construct which experts can recognize examples of, but which is difficult to specify comprehensively and accurately in scoring rubrics" (p. 143).

Second, this opaque situation for assessing covariation manifests also with regard to activities using mathematical modelling. Studies on this topic have mainly focused on the theoretical definition of mathematical models (Niss et al., 2007; Niss, 1989) or on a valid definition of mathematical modelling competence (Niss & Højgaard, 2019), but there is not usually a systematic exploration of the issue of its assessment which should reflect not only the aims of applications and appropriate modelling (Blum, 2015) but also students' ability to reason in a covariational manner.

The third issue concerns some institutional and didactical aspects. As pointed out by Thompson et al. (2017), covariation is not a regular feature in mathematics curricula, except for a few Eastern countries, and therefore is not explicitly considered in students' assessment surveys. In addition, teachers themselves struggle when teaching covariation and one of the reasons is that they are neither able to master the use of covariational reasoning (Thompson et al., 2017), nor to include it in their school practices.

### 15.2.1 The assessment of covariation

Given the peculiarity and complexity of covariation as a theoretical and cognitive construct, the variety of fields and topics where covariation can be applied, its occasional and only implicit presence in school curricula and practices, and, above all, the limited knowledge of teachers in this regard, any assessment of this form of reasoning turns out to be challenging, and a non-homogeneity can be detected both in terms of achievements and assessment practices. Here we provide some examples of existing modalities in which covariational reasoning has been assessed, and we use those findings to help identify the benefits that the comparative judgement method can bring.

The six-level taxonomy developed by Thompson and Carlson's (2017), elaborated through individual interviews, aims at classifying the different ways in which one person can reason in a covariational way. Their framework levels should be interpreted as "descriptors of a class of behaviors" or as "characteristics of a person's capacity to reason covariationally" (Thompson & Carlson, 2017, p. 435) and there are several qualitative studies that have used these descriptions of mental actions as a coding scheme (e.g., Carlson et al., 2002). Another significant contribution is the diagnostic assessment of teachers' mathematical meanings called Mathematical Meanings for Teaching secondary mathematics (MMTsm; Thompson, 2015). This assessment has a very detailed rubric to score students' quantitative and covariational reasoning and has been used with both US and Korean teachers (Yoon & Thompson, 2020). In Thompson et al. (2017) on the other hand, the investigation of US teachers' attitude to reason in a covariational manner was conducted through the development of a specific scoring rubric based on the features of graphs that teachers were asked to sketch. As they underline, the scoring categories were not defined theoretically because scorers would not have had the theoretical background to understand the rubric.

What seems to be missing, and proves even more challenging, is the assessment of covariation intended as a form of conceptual understanding. In this sense, covariation "is defined by how it is perceived, understood and used by the relevant community of expert practitioners" (Bisson et al., 2016, p. 143), and in a strict rubric of items it risks reducing to a "rigid definition that fails to capture the full meaning and usage that exists in practice" (p. 143). Moreover, the limited pedagogical and mathematical knowledge of teachers about this theoretical construct, makes it hard to elaborate suitable items of a scoring rubric whose meaning is shared by the expert mathematicians' community. Finally, "conceptual understanding is best assessed using open-ended and relatively unstructured tasks" (p. 143), but assessing open-ended tests is difficult because of the variety and unpredictability of students' responses, and also the huge amount of time required to obtain both a valid and reliable outcome.

In recent years, an alternative approach to the assessment and enhancement of conceptual understanding has been proposed (Jones & Karadeniz, 2016) based on the technique of comparative judgement. In this study, instead of developing a specific content-based scoring rubric, we use the holistic method of CJ to assess some students' written productions involving covariational reasoning. Although CJ is usually adopted as a time-saving technique to assess a huge number of tests, we used it to assess just a small sample of tests. One first research aim can be the understanding of whether CJ can be valuable in assessment of open-ended tasks involving a complex construct as covariation, specifically when the experts who judge students' works are expert mathematicians not fully aware of the potentialities of covariational reasoning; a second purpose of the research is the investigation of the features considered relevant in the assessment of an open-ended mathematical text, in order to examine whether these characteristics correspond to those relevant in a covariational perspective, also gaining insight on this innovative assessment technique from the perspective of a varied sample of Italian teachers.

### 15.2.2 Comparative judgement: an assessment method

One explanatory factor of the difficulties that accompany assessment practices, in Italy as in other contexts, is linked to changes in the way of understanding the teaching and learning of mathematics nowadays. Teaching does not consist merely in the transmission of disciplinary concepts but must also enable students to autonomously build "significant learning" (in the sense of Ausubel, 1960). Learning becomes meaningful when it does not arise from the accumulation of notions and information, but when the learner becomes able to use these in order to tackle complex problems, identifying paths and tools that help him/her to act effectively and competently. Summing up, school education should enhance conceptual understanding rather than mere procedural knowledge, but all the issues concerning assessment are amplified when it comes to assessing complex forms of conceptual knowledge such as covariational reasoning. All this has obviously led to the need to search for new methods and tools in the assessment field. It is within this perspective that the use of comparative judgement in the educational field was born. Its application has been successful in a various of educational assessment contexts (Tarricone & Newhouse, 2016) and, as we will argue in this study, it may be helpful also in the assessment of unstructured tasks involving covariational reasoning.

Even though CJ may be innovative in the Italian context, internationally it has received increased attention in the last two decades. It constitutes an efficient alternative approach to traditional marking (Pollitt, 2012), works particularly well in absence of restrictive indications or scoring rubrics, and allows the assessment of forms of complex knowledge such as covariation, with a holistic approach (Jones et al., 2019). It is rooted in the psychological principle that people are more accurate in making a comparison between two objects rather than when expressing an isolated judgement (Thurstone, 1927; Laming, 1984). Some experts are asked to make pairwise comparisons of students' works, choosing the one they consider better according to a global construct. The results are fitted to the Bradley-Terry statistical model (1952) which returns a unique score for each student, i.e., a scaled rank order of the works. The research literature supports both the validity and reliability of comparative judgement. It is proven to be even more reliable than traditional marking in open-ended assessment not only in mathematics (Jones & Alcock, 2014; Steedle & Ferrara, 2016). This assessment technique has been used with several mathematical topics: problem solving (Jones & Inglis, 2015), conceptual understanding (Bisson et al., 2016, Jones & Karadeniz, 2016; Jones & Alcock, 2014), mathematical proof (Davies, Alcock, & Jones, 2020) and statistical knowledge (Marshall et al., 2020, Bisson et al., 2016). As far as we know, CJ has never been used to assess tests requiring covariational reasoning. Typically, CJ is used to assess tests on a national scale in countries like the UK or New Zealand but is little-known in Italy. Its advantages reside in the non-problematic nature of the variety of possible students' responses, the non-necessity of a scoring rubric to grade the productions, and the resulting reduction in time required to assess a huge number of tests.

Suitable statistical methods allow us to verify the reliability and validity of CJ outcomes. All the details useful for the purposes of our research are provided in section 15.4.1.

## 15.3 METHODOLOGY

### 15.3.1 The context of the CJ experimentation

The written productions analyzed in this chapter refer to Task 5 presented in section 11.1.5.

We simply recall that after the working group sessions with the video and GeoGebra applets and a few classroom discussions, the students were asked to write a text in which they described (to students of another class) the activity completed and the law of the obtained motion, producing a form of theoretical support for the study. The task of the essay is integrally reported below translated into English:

> *Task 5*
> Thinking back to the work carried out on the inclined plane, write to schoolmates of another class to outline the work itself and, specifically, the relationship that describes and explains mathematically the motion of the ball along the inclined plane. This report should be a theoretical support for you and your schoolmates.

The expedient of a narrative addressed to schoolmates from a different class and that of the theoretical support for study have been chosen to invite students to adopt a formalization approach and not to take details for granted. Students were accustomed to this kind of task and were familiar with the terminology used in the prompt.

*Participants*

The students, who are accustomed to written tasks in mathematics and physics, were involved in this activity without prior notice, had two hours of time to complete the task, and received no specific instruction on the formal structure of the essay. The texts were from two to four pages long; students could freely report graphs, formulas, and numerical tables in their written work.

*Judges*

The research group recruited 13 judges from pre-existing contacts. Only three of the judges were males and all are mathematicians and experts in the field: specifically, 2 of them were middle school teachers, 7 were high school teachers of mathematics and physics and 4 were university professors specialized in Mathematical Physics. The group of judges was heterogeneous, to collect different points of view on this evaluation experience. Based on informal interviews, we can state that none of the judges had ever heard about the CJ assessment technique and only half of them were aware of the meaning of the construct of covariation, having attended some research seminars held by the authors. The mathematical competence of the judges is sufficient to ensure reliability of the outcomes (Bisson et al., 2016; Jones & Alcock, 2014), particularly as they were asked to identify the better test from a mathematical point of view, rather than that which displayed more covariational reasoning. All the judges volunteered for the case study and received no financial compensation.

### 15.3.2 The research design

*Phase prior to comparative judgement*

A few weeks before starting the CJ session, judges were introduced to comparative judgement through e-mail correspondence which explained the peculiarity of this assessment approach and the main differences from traditional marking. On the first day of the project, the judges received an instruction sheet containing all the information related to the teaching experiment on the law of the inclined plane, how the activity was structured, its main objectives and the task given to the students. Moreover, the instruction sheet showed specifically how the comparisons platform worked. The judges could keep the instruction sheet to hand throughout the whole CJ session. The judges did not undergo any trial CJ session because one of the research purposes was to capture their first impressions when facing this new kind of assessment approach.

*Comparative judgement procedure*

The 22 open-ended tests were anonymized for a matter of privacy, scanned, and uploaded to an online comparative judgement platform (www.nomoremarking.com), freely available for research purposes. When starting comparisons, judges saw a screen recalling the main aims of the Galileo teaching experiment and were asked to "choose the best mathematical text". Judges made 17 judgements each, for a total amount of 221 judgements, so that each text was compared at least 10 times according to literature standards (Jones & Alcock, 2014). The judges worked in their free time and had 20 days of time to complete their comparisons.

The CJ website displayed the two tests to be compared side by side and the judges just had to click the left or right button to express their preference. An example of the display screen is shown in **Figure 75**.

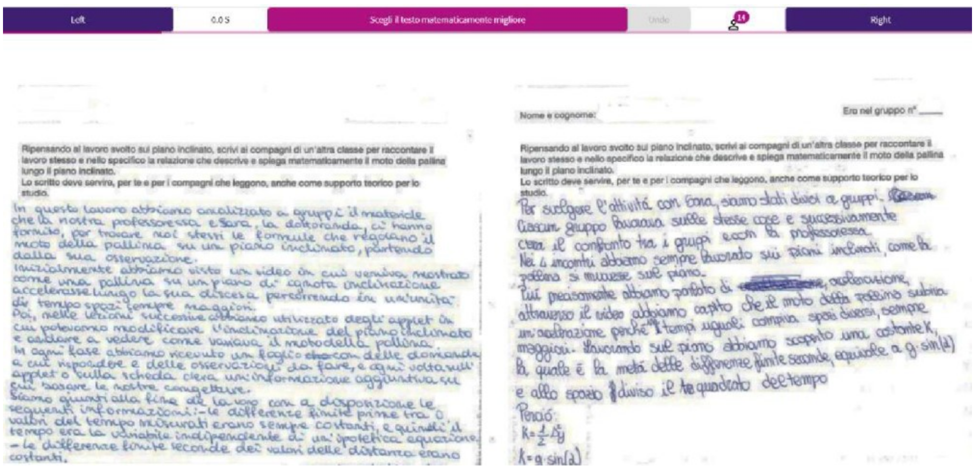



**Figure 75** - Screen for the comparisons displayed on the online engine

*Post-judging questionnaire*

One week after completing the CJ procedure, the judges received by email new instructions to fill in an online questionnaire. The judges were sent two texts chosen by the researchers and based on the following criteria: the two tests had obtained a good and close final score, but they also presented many dissimilarities. They were written by students of different gender, a boy and a girl, had different lengths, and one was clearly written but presented misspelling, while the other was untidier. A picture of the first page of two tests is contained in **Appendix B** along with a translation into English of the content.

The first part of the questionnaire aimed at investigating which factors influenced the outcome of the CJ procedure. After asking judges to state which of the two texts they considered better in mathematical terms, we elaborated a four-point set of items representing an ordinal scale from 1 (little influence) to 4 (strong influence) and the judges were presented with 10 features of students' work:

1) Presence of errors
2) Use of formal notations
3) Untidy presentation
4) Structure of the presentation
5) Use of graphics and images
6) Ability to synthesize
7) Use of a formal mathematical vocabulary
8) Modelling capability
9) Ability to describe *exhaustively* the relationships between distance, time, and the angle (of inclination of the plane)
10) Ability to describe *formally* the relationships between distance, time, and the angle.

This list of features, and also the following questions, were mainly inspired by the work on judgement processes by Jones and Inglis (2015): we adapted them according to the characteristics displayed by our written texts and added some features specifically related to the content of the task proposed. Feature (8), "modelling capability", explicitly refers to the modelling competence stressed in National Curricula and intended as the ability to represent classes of real phenomena. Features (9) and (10) instead were designed to grasp specifically the relevance of covariation. The tasks proposed during the experimentation (see section11.1) allowed us not only to investigate the distance-time relationship, but also to explore in depth how the angle affects the distance-time graph (Arzarello, 2019). Specifically, feature (9) refers to the ability to describe that relationship in a natural language, while feature (10) focuses on the ability to condense it into a suitable formula.

The questionnaire concluded with three open-ended questions: 1. Please list any other features you think may have influenced your judgement when comparing texts. 2. Please comment on this overall experience and state your feelings during the comparative process. 3. In his theory of quantitative reasoning, P. Thompson states that a person reasons in a covariational manner when able to envision the values of two or more quantities as varying simultaneously. In which of the two texts do you think the presence of covariation between physical magnitudes can be better captured and why?

The judges again worked in their free time and had 2 weeks to complete their task.

## 15.4 DATA ANALYSIS AND RESULTS

### 15.4.1 Outcome of the CJ procedure

The CJ method allows positioning a set of complex objects on a unidimensional scale (Davies et al., 2020). The CJ website fitted the 221 judgements with the statistical Bradley-Terry model (1952) and using a maximum likelihood estimation procedure (Pollitt, 2012). It produced a final parameter estimate for each student ($M$=0, $SD$=1.7). The distribution of the class's scaled scores is shown in **Figure 76**.

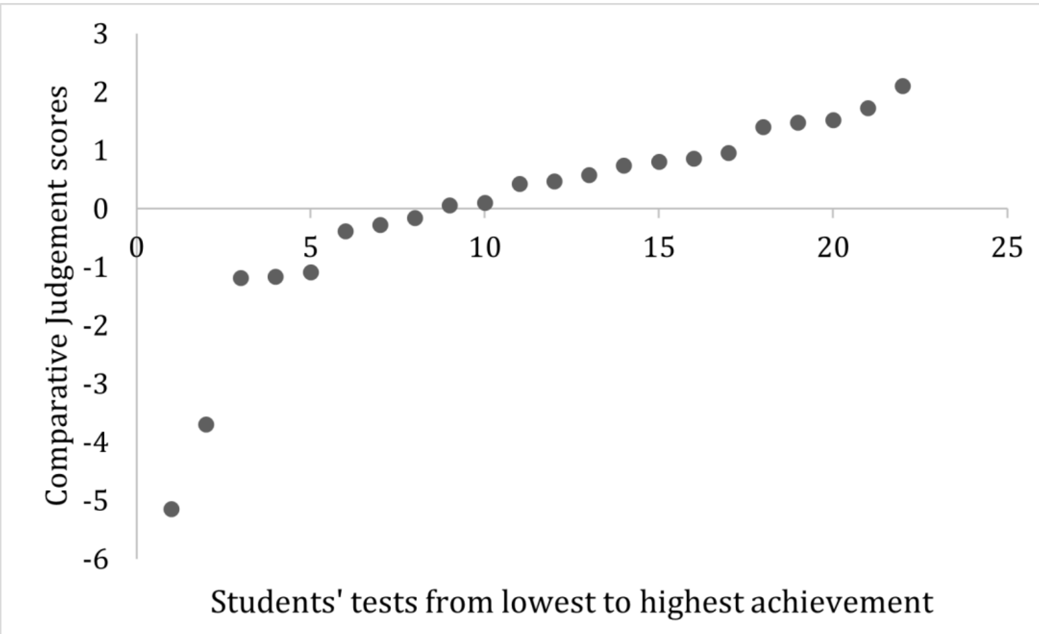



**Figure 76** - Distribution of CJ scores as per number of students reported on the vertical axis (N=22)

*Reliability*

The reliability of an assessment procedure refers to the consistency of its outcomes. Internal consistency can be measured with the Scale Separation Reliability (SSR), analogous to Cronbach's alpha. The outcome showed high internal consistency since $SSR \geq 0.7$ ($SSR = 0.81$). Moreover, looking for 'misfitting' judges, we computed an infit statistic for every judge and compared these data to a threshold value, i.e., two standard deviations above the mean of the infit (Marshall et al., 2020; Pollitt, 2012). Only one judge resulted as misfitting because slightly above the threshold value (0.03 above), but we did not consider it necessary to adapt the CJ scores by removing the misfitting judge.

*Criterion validity*

In order to evaluate the validity of the outcome of the CJ procedure, we considered the *criterion validity* by computing the Pearson Product-Moment correlations between the parameter estimate of the CJ procedure and some benchmark measures. We would expect positive correlations, meaning that those students who are more successful on these other covariation-related performances were also more successful in the assigned task.

We correlated the CJ outcome with the marks assigned by their teacher to a physics test concerning problems on accelerated motion and the motion of bodies along the inclined plane ($r$=0.57) (**Figure 77**). The test was specifically designed by their teacher to enhance covariational reasoning. We also correlated the parameter estimate with the marks assigned by their mathematics teacher, who has a solid background in covariation, to her own students' written productions specifically focusing on covariational skills in the assessment phase ($r$=0.51). These results, while modest, are in alignment with the results of other research in literature on conceptual understanding in secondary and tertiary mathematics (Bisson et al., 2016), which reported correlation coefficients between 0.35 and 0.56. Finally, we correlated CJ scores with the final course scores (i.e., the evaluation obtained at the end of the school year) in mathematics ($r$=0.48): this positive and significant correlation can be interpreted as a sign that covariational reasoning is a transversal competence in mathematics.

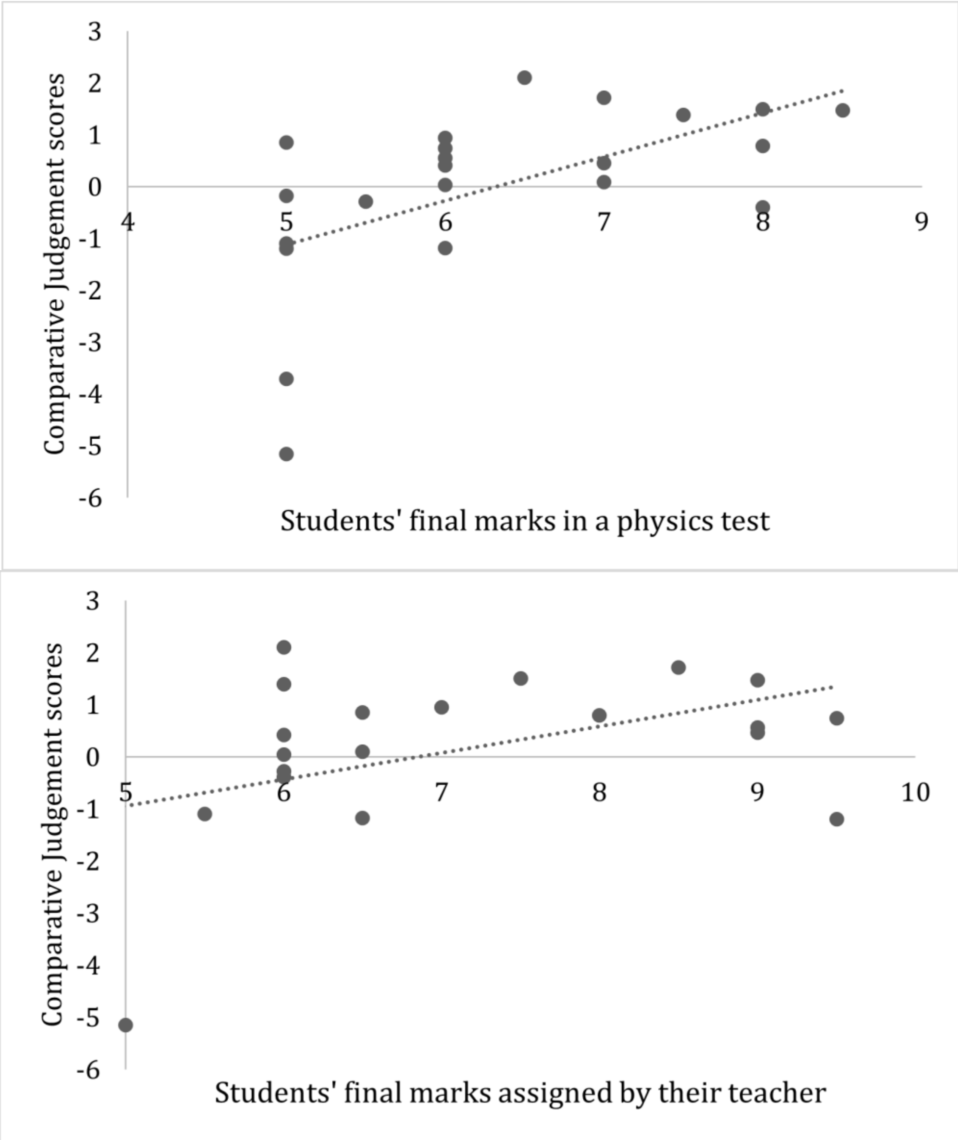



**Figure 77** - Scatter plots of the relationship between CJ scores, those on a physics test on the same topic, and those assigned by their teacher on the same test

### 15.4.2 How much does covariation matter?

The last open-ended question (3) of the questionnaire asked the judges to state which of the two tests displayed a stronger covariational reasoning and why. Seven judges preferred Test 2, 4 preferred Test 1 but two judges expressed a preference without justifying it. Moreover, one explicitly stated that he didn't know, and another said neither of the two because he considered "the two tests very confusing" [J4]. Globally, 10 out of 13 answers were in agreement over which answer was mathematically better, with 5 of these opinions expressed by judges who were familiar with the construct of covariational reasoning.

Those judges who expressed a preference for Test 2 provided the following reasons:

- “In the supported conclusion, the covariation between the two magnitudes involved is perfectly grasped and the formula simply assumes the role of symbolic expression of that covariation, so that it is not even mentioned anymore” [J1]. In fact, the student concluded with a sentence summarizing her considerations: “The distances traversed by the ball are directly proportional to the squares of times”;
- “In several steps the relationships of dependence of the mutual growth are underlined” [J2];
- “It expresses which magnitudes depend on the others” [J8];
- “Not only did they report formulas, but they also tried to explain variations. They also referred to dependent and independent variables” [J7];
- “The conclusive section summed up the different hypotheses and discussed each one in depth, eventually arriving at formulas in correct terms of the law of ‘falling’ bodies on an inclined plane (*s* as a function of time *t*) free of friction” [J11];
- Another answer, more difficult to interpret, attributed her decision to the “more schematic structure of the discourse” of the study [J6].

Instead, the judges who preferred Test 1 justified their choice by saying:

- It clearly expresses how “the ratio between space and time is constant and this ratio is equal to the half of acceleration” [J3];
- It “highlights a greater understanding of the subject in its overall aspect” [J10];
- “It addresses finite differences” [J13], referring to a table made by the student (see **Figure 78**) in which he reported distances and first finite differences of distance in function of time and the parameter *k*, i.e., providing a general expression and not numerical values.

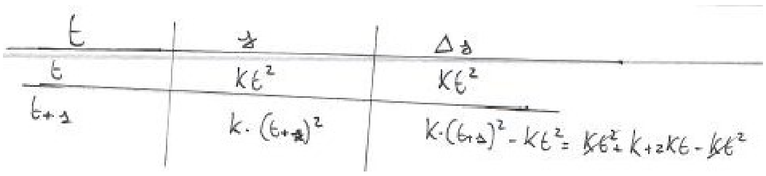



**Figure 78** - Table of finite differences present in Test 1

The conceptual aspects of covariation are highlighted, considering the role that some judges award to formulas in the statements above. We will provide two examples. In the first comment, the judge who chose Test 2 states that “the formula just assumes the role of symbolic expression of that covariation” [J1] and is “condensed” into the sentence “the distances traversed by the ball are directly proportional to the squares of times”. So, one crucial point for the judge is that formulas address conceptual knowledge, but formulas alone are not always enough; this is stressed also in another statement: “Not only did they report formulas, but they also tried to explain variations” [J7]. In another stream of thought, the judge who chose Test 1 considers formulas positive when judging, if they allow the formulation of “general expressions” and not simply numerical ones, relying on a conceptual representation such as that of finite differences. In this case, students use their existing knowledge (the finite differences method) to elaborate methods for solving the problem. In both cases, covariation is thought of as an “implicit or explicit understanding of the principles that govern a domain and of the interrelations between units of knowledge in a domain” (Rittle-Johnson et al., 2001, p. 347), namely of a key feature of conceptual knowledge, according to the definition provided by Rittle-Johnson and colleagues.

### 15.4.3 Insights into the judging process

Teachers are not completely objective in assessment of their students’ work and CJ is a tool that could help in this sense, given that judges do not know the students’ they are assessing, and the texts are anonymized. Parenthetically, we were interested in investigating which features could influence the judging process; this was the main purpose of the questionnaire. The analysis of the results of the post-judgement survey revealed that all the features had a positive influence on the judging process. The mean and standard deviation judge ratings (N=13) for each feature presented in the questionnaire are reported in **Table 3**. The table is organized into three sections (non-mathematical features, mathematical features, and covariational features) and then within each section, features are arranged by increasing mean.

| Quantitative questions (1=little influence, 4=strong influence) | Mean | Standard deviation |
|---|---|---|
| Untidy presentation | 2.69 | 0.91 |
| Structure of the presentation | 2.85 | 0.95 |
| Ability to synthesize | 3 | 0.88 |
| Presence of errors | 2.30 | 1.07 |
| Use of graphics and images | 2.46 | 0.84 |
| Use of formal notations | 2.69 | 0.61 |
| Use of a formal mathematical vocabulary | 2.92 | 0.62 |
| Modelling capability | 3.08 | 0.73 |
| Ability to describe exhaustively the relationships between distance, time, and the angle | 2.92 | 0.73 |

| | | |
|---|---|---|
| Ability to describe formally the relationships between distance, time, and the angle | 2.92 | 0.83 |

**Table 3** - Table containing the features of the survey divided into three sections: non-mathematical features (white), mathematical features (grey), covariational features (dark grey).

The statistical results show very similar means and standard deviations, so we cannot make significant claims about differences, particularly considering the small sample size. Even though results are not generalizable from a statistical point of view, they still provide a preliminary overview of the features considered relevant by judges. A few comments follow.

Since we are dealing with open-ended productions on a mathematical topic, in line with Jones and Karadeniz (2016), we consider it appropriate to investigate the influence of written communication skills on the CJ outcome. Concerning the non-mathematical features, the results are as follows: "untidy presentation" ($M$=2.69, $SD$=0.91), "structure of the presentation" ($M$=2.84, $SD$=0.25), "ability to synthesize" ($M$=3, $SD$=0.88). In the open-ended question concerning other factors that may have influenced the comparisons, three judges remarked how handwriting and misspellings strongly influenced their judging.

The least influential features were "presence of errors" ($M$=2.30, $SD$=1.07) and "use of graphs and images" ($M$=2.46, $SD$=0.84).

The other strictly mathematical features were "use of formal notation" ($M$=2.69, $SD$=0.16), "use of formal mathematical vocabulary" ($M$=2.92, $SD$= 0.62), and "modelling capability" ($M$=3.08, $SD$=0.20).

The post-judging questionnaire also contained two features (9-10) concerning specifically the grasp of the relationship between the parameter and variables involved and "ability to describe exhaustively/formally the relationships between distance, time, and the angle" ($M$=2.92, $SD$=0.73 and $SD$=0.83 respectively) and both were quoted as relevant by the judges. Eleven judges attributed equal importance to both features; in the case of the other two judges, one gave more weight to the discursive aspect and the other to the formal aspect (this is reflected in the slightly different values of standard deviation).

In the open-ended question 1, another feature emerged: the ability to formulate hypotheses and to justify them in a rigorous way [J11].

An overview of the statistical results reveals all features to have quite similar averages. The non-prevalence of a specific feature over the others can be read as a sign that when covariation manifests as conceptual understanding, it is difficult to isolate, and it results in an intertwining of many factors. This consideration supports our point of view that in this case a holistic approach such as CJ is required.

### 15.4.4 Opinions on CJ as an assessment technique

The comments of the judges on this experience express different opinions. Five judges stated the experience was “positive” or at least “interesting”. It was defined “optimal from the point of view of accurate assessment because it diverts attention from the error of the single student” [J1]; “direct comparison between the two tests enables (from the beginning) different points of view without valuing or penalizing too much a single student” [J2]. Another judge said that “the assessment is quite fast, and the comparison facilitates the judgement” [J13]. The median time of judgement for the judges ranked from to 2 to 11 minutes for a median time of 297 s (nearly 5 minutes per comparison). These time data are an overestimate because the timer of the online platform does not stop when judges move away from the computer leaving the screen open. The time required for the comparisons in our experimentation does not differ much from the estimate reported in Marshall and colleagues (2020) and, in our opinion, it is reasonable when taking into consideration the length of the texts, their nature and difficulties linked to deciphering the handwriting. However, the time is less than that usually required to grade a single test by a traditional marking method.

A negative aspect underlined even by those judges who positively evaluated the experience, lay in “bad handwriting, errors of misspelling, lack of linguistic correctness and untidiness”.

Those judges who considered the experience as “complicated” and “hard”, supported their statement by saying that “most of the texts presented both strengths and weaknesses and it was not always evident which test was the best” [J3]. The tests showed “different modalities of reporting” [J7], “some students used mainly natural language, others formal, but eventually reached the same conclusion” [J8]. The presence of “descriptive elements related to an experience of the students” made the assessment “challenging” compared with other forms like “tests in the state examination” [J10].

Another judge stated that to “really focus on the mathematical aspects of the content”, the tests should be “visually comparable”. Only two judges clearly stated that they “prefer” or “need an evaluation grid” [J9 - J10].

## 15.5 DISCUSSION

Data emerging from our analysis seem to positively support the assumption that an unstructured test and the CJ technique may be helpful in the assessment of covariation. We provide arguments for this claim. The measures of correlations described in section 15.4.1 with other benchmark measures specifically focused on covariation provided a positive and significative correlation. In the post-judging survey, 77% of the judges claimed that the test displaying a greater ability to reason in a covariational manner was also the best mathematical test, at least regarding the two texts analyzed during the online survey. Moreover, the features to which the judges attributed greater importance are also those that better reflect the design principles underlying this teaching experiment. The modelling capability was definitely the most influential element and actually also the central core of all tasks proposed to the students, together with the use of formal mathematical vocabulary, followed by the ability to describe the relationship among time, distance and the angle of inclination of the plane. The method of CJ seems to have provided a reliable assessment of the task even though judges were neither looking for covariation nor were specifically trained in it, a fact which was facilitated by the global approach characterizing CJ.

Our investigation allowed us to obtain implicitly some insights into our judges' perception of the construct of covariation and the relevance attributed to it. Irrespective of the higher or lower level of awareness concerning covariational reasoning of the judges, they identified the relationship among the magnitudes involved in the physical phenomenon as strongly relevant for their judgements. The answers to the open-ended question clearly reveal that a full grasp of this relationship does not necessarily translate into elaboration of a formula condensing all quantities involved, but certainly results in a complete description of the relationships of dependence and mutual growth. One judge interestingly referred to the table of finite differences, a powerful tool but barely used in Italian school practices, which can be considered a forerunner instrument to the study of rate of change and can surely help students in grasping how variables vary and co-vary.

While the validity of CJ as an assessment method in general, and specifically concerning conceptual understanding, is not in question, it is an unknown and rarely used tool in Italy since in our country there is a long tradition of the use of scoring rubrics in the assessment field. Hence, even though it was not a primary purpose of research, we were curious about gaining some insight into how Italian teachers could approach CJ. Although none of them had ever experimented with CJ, no judge expressed any discomfort in using it and only two judges clearly referred to the lack of a scoring rubric. In most cases, the positive factors of this assessment method were highlighted. Most of the judges involved in our research recognized that CJ allows a more holistic approach to assessment of conceptual understanding. Some judges stated that CJ avoids awarding too much importance to single errors that are often irrelevant in mathematical solution processes.

This study also presents many limitations: the sample of students involved is small, especially compared to other studies about CJ (e.g., 250 responses for Jones & Inglis (2015); 200 responses for Steedle & Ferrera (2016)). Moreover, these students were accustomed with this kind of unstructured tasks, which is not so common in Italy, and their teacher has a strong background concerning covariation, highlighting this reasoning in her lessons. Although the number of judges involved in this experimentation is in agreement with the standard literature on CJ (Jones & Alcock, 2014), it is still a small number with which to generalize the questionnaire conclusions: they simply provide interesting insights into which elements capture attention when reading an open-ended test, especially concerning a physical-mathematical topic, and offer some preliminary information on the attitude of Italian teachers toward covariation. We hope our study may act as evidence that larger investigation on the assessment of covariation in the form of conceptual understanding is required and of value. We chose to undertake this line of research adopting comparative judgement since it allows us to tackle the challenges and difficulties we outlined in the literature review, but this does not exclude the fact that other methods may be valid or that more teacher training on covariational reasoning is needed.

# 16 VARIABLES AND PARAMETERS: A COMPUTER SCIENCE ANALOGY

One of the educational problems which students struggle with since the first years of upper secondary school is the distinction between variables and parameters. After the short excursus concerning the role of variables within algebraic language we already outlined in Section 3.1, in this chapter[33], we are going to provide a rigorous definition of variables and parameters according to the logic language and to explore a computer science analogy based on C programming language. Finally, some proposals of 'instrumentation' are described, using Dynamic Geometry Software or Computer Algebra System: the purpose is that of a simpler comprehension of the distinction between variables and parameters.

## 16.1 FROM A RIGOROUS DEFINITION TO A COMPUTER SCIENCE ANALOGY

In the world of today's research in Mathematics Education, different approaches exist trying to define the meaning of variables and parameters. For example, a discursive definition of these concepts can be found in the work by Thompson and Carlson (2017). The authors state that a symbol can be used to represent the values assumed by a quantity with three different meanings. It is used the term *constant* to denote the value of a quantity that never varies; if the values of a quantity change from scenario to scenario but they do not vary within the same scenario then it is a *parameter*; eventually, if the values of a quantity vary within the same scenario, then it assumes the meaning of *variable*.

For sure, the concepts of variable and parameter are located on two different levels from the logical point of view and this distinction essentially resides in the use of the universal quantifiers. We report here the example present in Bernardi (1994): the writing $y = mx$ denotes the bundle of lines with center the origin and adopting a formal notation it can be represented as $\{\{(x,y) \mid y = mx\} \mid m \in \mathbb{R}\}$. We notice that firstly are quantified the variables $x$ and $y$ and in a second moment

[33] The content of this chapter is an adaptation of the study presented in:
Bagossi, S. (2021). Variabili e parametri: un'analogia informatica [Variables and parameters: a computer science analogy], *L'insegnamento della matematica e delle scienze integrate*, Vol 44 B, 74–88.
We thank Prof. F. Arzarello for overseeing the work and the colleague I. E. Stan for the valuable IT advice.

the parameter *m*. This is exactly the connoting characteristic of the parameter: it is the variables which is quantified last, that is the variable on which acts the more external quantifier. The same situation reoccurs in the case of equations or problems with a parameter: when the text of an exercise says "*for each value of m find the values of x such that…*", *x* is for sure the unknown and m is the parameter, not because of the conventional use of the chosen letters, but because *m* is quantified more externally. This kind of formulation is defined in Bloedy-Vinner (2001) as *dynamic* or again as *ordered and quantified structure* because it corresponds to a potential order of substitutions that is typical of parametric expressions.

The logical characterization just described offers the possibility of an interesting analogy based on the computer science and specifically with programming languages. In order to propose such analogy, we will adopt typical syntax of the programming language C, known for its generality of application that sometimes makes it more useful and efficient of other languages more powerful. The basic notions here reported are mostly taken from Kernighan and Ritchie (2007).

A generic program in C consists of *functions* and *variables*. The functions contain the instructions that specify which operations have to be carried out, while the variables store the values used during the execution. Every program contains a principal function called *main*, that marks the beginning of the program. The function *main* can call other functions by their name and the arguments of such functions are explicated between round brackets.

The program described in **Figure 79** allows to visualize the first 10 terms of an arithmetic progression starting from *a* and with a common difference of 1. In particular, the instruction *for* is a cycle or iteration that requires three arguments: an initial value, a condition (in this case a superior limit) that controls the iteration of the cycle and if the condition is true the body of the cycle is executed and finally the increment of the index of the cycle. In our program the index of the cycle is *i*: it starts from 0 and until it is minor than 10 it is increased by 1 (*i*++) at the successive iteration. The function *printf* allows to "print" i.e., visualize the result; *%d* specifies that the argument of the function is an integer in decimal notation (*d* stands for digit), while *n* is considered a special character in the computer science language that is interpreted as break lines (*n* stands for new line) and hence it prints the values one below the other. Finally, the instruction *return 0* returns the value 0, an exit-code, to signal that the program was successful. The double slash // allows to insert comments, that is notes not read by the program, on a same line, while /* */ allows to insert comments on more lines.

```
#include <stdio.h>  //It inserts information on the standard library
int a=5;

main( )  {              /*Principal function main. The curly brackets
                          enclose the instruction of main*/
  int i;
  for (i = 0; i < 10; i++)      /*Cycle for makes slide the values of the
                                  parameter i from 0 to 10*/
    printf ("%d\n", i+a);       /*The function printf allows to visualize
                                  the result*/
  return 0;
}
```

**Figure 79** - Example of elementary program in C

Trying to execute the program using an online compilator, for instance OnlineGDB available at this link, it is obtained the progression shown in **Figure 80**.

```
5
6
7
8
9
10
11
12
13
14


...Program finished with exit code 0
Press ENTER to exit console.
```

**Figure 80** - Values printed by the program

In language C, all the variables have to be declared before being used, usually at the beginning of the function. A declaration makes explicit the properties of the variables and it is constituted by a type followed by a list of variables. In our example *int* denotes that the variables a and I are integers, but other choices are possible: for example, *float* denotes a floating-point number with single precision (depending on the machine), while *double* stands for a floating-point number with double precision (depending on the machine).

In addition to the typology of variables, there exists another distinction concerning their visibility. The variables are said *local* or *internal* when they are declared within the function using them. They are born when the function enters in execution and die at the end of the execution of the program, for this reason they are called *automatic*. Alternatively, it is possible to define variables *global* or *external* to all the functions by means of their name. Since the global variables have permanent duration, instead of appearing and disappearing basing on the calls of the functions, they maintain their values also after that the functions that have modified them have stopped operating. The external variables can be defined once and only once outside every function.

In our program (**Figure 81**), *a* is global variable because it is external to all the functions and maintains a value equal to 5 for all the duration of the program. An example of local variable is *i* because it is declared within the function *main* and dies externally to this function.

```
#include <stdio.h>
int a=5;                          //Global variable

main( )  {

  int i;                          //Local variable
  for (i = 0; i < 10; i++)

    printf (“%d\n”, i+a);

  return 0;
}
```

**Figure 81** - Global and local variables in our program

Local and global variables differ for their *visibility scope*, that is the part of the source text C in which the declaration is active. The local variables have a *block scope* where block means a series of instruction delimited by {}; the global variables have a *file scope* instead and hence they are visible from the point in which they are declared until the end of the program.

These brief premises describing the characteristics of local and global variables in C allow us to explicate the analogy between variables and parameters. The parameters, as the global variables (*a* in our code), are defined more externally and their values remain unchanged for all the duration of the program if we refer to computer variables, for the scenario or the problem if we refer to mathematical parameters. The (mathematical) variables instead, exactly like local variables (*i* in our code), are defined more internally and their value changes from block to block, from scenario to scenario, from problem to problem.

## 16.2 PROPOSALS OF INSTRUMENTATION

In the previous paragraph were presented some different approaches through which it is possible introduce in a rigorous way the distinction between a variable and a parameter, nevertheless the knowledge required is not trivial. However, the concept of parameter, precisely for its large presence in numerous topics, needs to be faced since the first years of secondary school. As the study by Chiarugi et al. (1995) shows, a non-full understanding of this concept entails non only conceptual difficulties but also of manipulative kind in the execution of computations.

A possible path, and surely declinable in various modalities according to the school level, is that of *instrumentation* (see Chapter 4). One of the DGS largely used in the Italian school is GeoGebra. The *slider* command allows the instrumentation of the concept of parameter. Introducing such slider, the settings allow to define its nature choosing between the options of number (possibly integer) or angle and to set the range in which this parameter varies.

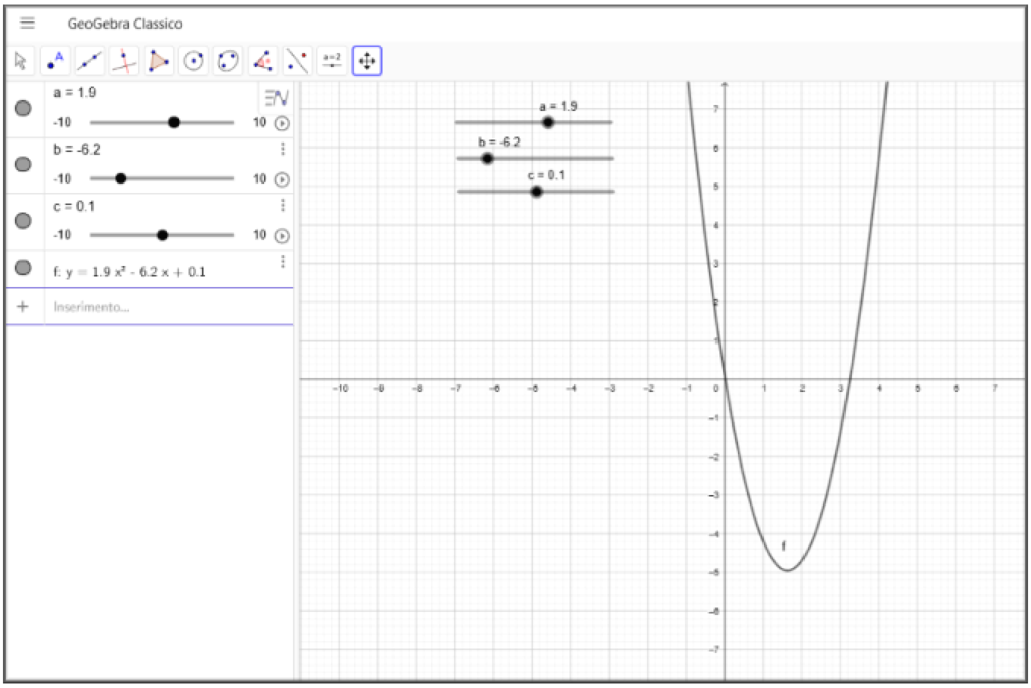



**Figure 82** - Instrumentation of the parable on GeoGebra

The example reported in **Figure 82** shows an instrumentation of the equation of the parable. Each of the coefficient of the equation is represented by a slider: varying the values assumed by the different coefficients it is possible to observe how the graph of the parable changes. This exploration enables the students to conjecture how each of the three parameters is responsible of the properties of the resulting graph. Moreover, the students have the possibility to experiment how the quantification of the parameter is antecedent that of the variables *x* and *y*. In fact, it is necessary to fix the value of the parameters to obtain the values of the coordinates in each point of the graph: these are defined for each real *x*, but they vary according to the chosen values of the parameters.

An example of instrumentation similar to the one described above concerning the study of the bundles of parables, but making use of the software Derive, is presented in Reggiani (2000).

Another form of instrumentation always inherent to the study of second-degree functions is described in the contribution by Arzarello contained in Hanna & de Villiers (2012). The students of a 9th degree classroom in a scientific-oriented school were involved in the study of functions using the tables of finite differences. Students had already learnt that first-degree functions have the first finite differences constant. The teacher asked the students to make conjectures on the functions whose first finite differences change linearly and to use a spreadsheet made within a CAS environment. As shown in **Figure 83**, students had at disposal a first spreadsheet showing the numerical values of *x*, *f(x)*, of first and second finite differences and of the parameters *a*, *b*, *c* of the function $y = ax^2 + bx + c$; the second spreadsheet contained the same data but expressed in an algebraic form that is in function of the step *h* of increment of the variable *x* and of the coefficients *a*, *b*, *c*.

**a**

| | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| • | | | | | | | | | |
| 1 | x | f(x) | df(x) | ddf(x) | x0 | a | b | c | h |
| 2 | 0 | 3 | -1 | -4 | 0 | -2 | 1 | 3 | 1 |
| 3 | 1 | 2 | -5 | -4 | | | | | |
| 4 | 2 | -3 | -9 | -4 | | | | | |
| 5 | 3 | -12 | -13 | -4 | | | | | |
| 6 | 4 | -25 | -17 | -4 | | | | | |
| 7 | 5 | -42 | -21 | -4 | | | | | |
| 8 | 6 | -63 | -25 | -4 | | | | | |
| 9 | 7 | -88 | -29 | -4 | | | | | |
| 10 | 8 | -117 | -33 | -4 | | | | | |
| 11 | 9 | -150 | -37 | -4 | | | | | |
| 12 | 10 | -187 | -41 | -4 | | | | | |
| 13 | 11 | -228 | -45 | -4 | | | | | |
| 14 | 12 | -273 | -49 | -4 | | | | | |
| 15 | 13 | -322 | -53 | -4 | | | | | |
| 16 | 14 | -375 | -57 | | | | | | |
| 17 | 15 | -432 | | | | | | | |

**b**

| | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| • | | | | | | | | | |
| 1 | x | f(x) | df(x) | ddf(x) | x0 | a | b | c | h |
| 2 | x0 | a*x0^2+b*x0... | a*(h^2+2*h*x0)+ ... | 2*a*h^2 | x0 | a | b | c | h |
| 3 | h+x0 | a*(h+x0)^2+ ... | a*(3*h^2+2*h*x0... | 2*a*h^2 | | | | | |
| 4 | 2*h+x0 | a*(2*h+x0)^2... | a*(5*h^2+2*h*x0... | 2*a*h^2 | | | | | |
| 5 | 3*h+x0 | a*(3*h+x0)^2... | a*(7*h^2+2*h*x0... | 2*a*h^2 | | | | | |
| 6 | 4*h+x0 | a*(4*h+x0)^2... | a*(9*h^2+2*h*x0... | 2*a*h^2 | | | | | |
| 7 | 5*h+x0 | a*(5*h+x0)^2... | a*(11*h^2+2*h*x... | 2*a*h^2 | | | | | |
| 8 | 6*h+x0 | a*(6*h+x0)^2... | a*(13*h^2+2*h*x... | 2*a*h^2 | | | | | |
| 9 | 7*h+x0 | a*(7*h+x0)^2... | a*(15*h^2+2*h*x... | 2*a*h^2 | | | | | |
| 10 | 8*h+x0 | a*(8*h+x0)^2... | a*(17*h^2+2*h*x... | 2*a*h^2 | | | | | |
| 11 | 9*h+x0 | a*(9*h+x0)^2... | a*(19*h^2+2*h*x... | 2*a*h^2 | | | | | |
| 12 | 10*h... | a*(10*h+x0)^... | a*(21*h^2+2*h*x... | 2*a*h^2 | | | | | |
| 13 | 11*h... | a*(11*h+x0)^... | a*(23*h^2+2*h*x... | 2*a*h^2 | | | | | |
| 14 | 12*h... | a*(12*h+x0)^... | a*(25*h^2+2*h*x... | 2*a*h^2 | | | | | |
| 15 | 13*h... | a*(13*h+x0)^... | a*(27*h^2+2*h*x... | 2*a*h^2 | | | | | |
| 16 | 14*h... | a*(14*h+x0)^... | a*(29*h^2+2*h*x... | | | | | | |
| 17 | 15*h... | a*(15*h+x0)^... | | | | | | | |

**Figure 83** - Table (a) contains the numerical values of finite differences; table (b) contains the algebraic expression of finite differences

In a certain sense, we could define this kind of instrumentation as algebraic instead of geometric: in fact, the table of finite differences are those which enable to deduce the properties of the parable and in particular on how to make conjectures on how the parameters $a$, $b$, $c$ are responsible of the specific properties of the parable.

Finally, another form of instrumentation has been presented in teaching experiment conducted in 2017 (see Section 10.1). One of the tasks proposed had the purpose to obtain the law of the ball running along an inclined plane: specifically, through a GeoGebra applet, shown in **Figure 84**, students could visualize in the central part the parabolic graph describing the *s*-*t* relationship; the values of time, distance and first finite differences collected in the table on the right and on the left could modify the inclination angle of the plane and so visualize how it influenced the *s*-*t* graph and the numerical values contained in the table. Hence the students had the chance to explore the dependence of the law of the ball motion on the inclination angle of the plane, dependence encapsulated in the parameter describing the law itself: $s = k \cdot t^2$.

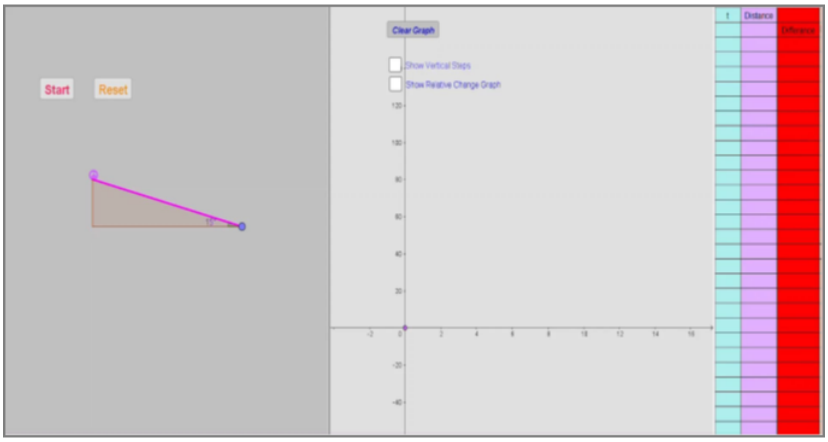



**Figure 84** - Screen of the GeoGebra applet

## 16.3 Conclusions

The manipulation of algebraic expressions and the introduction to the algebraic language are topics that have engaged the researchers in Mathematics Education for a long time. This study proposes a computer science-based analogy, alternatively to the rigorous language of logic, so to explore the distinction between variables and parameters and some proposals that we define instrumented in order to translate them in various way on a didactical level. Remains of strong interest understand how Italian teachers face this theme in their school practice and which instruments could offer a valid support for a better approach to algebraic language.

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# APPENDIX A

Original version of the consent form used during 2017 teaching experiment:

Liceo Scientifico Statale [redacted]

a.s. 2016 – 2017

Ai genitori degli studenti della classe 1° [redacted]

**Oggetto: Richiesta di autorizzazione per effettuare audio/videoregistrazioni di alcune attività svolte in classe**

Dal mese di gennaio, come comunicato al Consiglio di Classe, interverrà in classe un dottorando in Didattica della Matematica attivo presso il dipartimento di Matematica dell'Università degli Studi di Torino, Osama Swidan.
L'intervento del dottorando verterà sull'introduzione al concetto di funzione, proponendo alcune attività didattiche, pensate, organizzate e progettate dalla docente di classe insieme a Swidan e al Nucleo di Ricerca in Didattica della Matematica dell'Università degli Studi di Torino, coordinato dal professor F. Arzarello.
Durante gli interventi in classe il dottorando, oltre ad apportare il proprio contributo, filmerà alcune lezioni, altre verranno registrate e altre ancora fotografate in modo che tali materiali possano costituire per il gruppo di ricerca didattica occasione di studio, riflessione e analisi critica delle attività svolte, ovviamente nell'ottica di portare un valore aggiunto anche alla didattica nella classe stessa.
Le riprese e le fotografie costituiranno materiale di studio e di indagine per la ricerca didattica e per la stesura della tesi. Nelle riprese saranno preferite le mani e i gesti ai volti.
L'insegnante e il dottorando, si impegnano a utilizzare tale materiale solamente a tal fine e a modificare i nomi dei ragazzi, nel caso in cui le immagini debbano essere mostrate.

L'insegnante si rende disponibile per ulteriori informazioni e precisazioni.

Si coglie l'occasione per porgerVi i più cordiali.

L'insegnante | Il Dirigente Scolastico

Silvia [redacted] | [redacted]

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Il sottoscritto ______________________________________________

genitore di _______________________________________

☐ **autorizzo** l'effettuazione di video-registrazioni durante le attività svolte nelle lezioni di matematica della classe 1° [redacted] e l'utilizzo del materiale (video e schede di lavoro) ai fini didattici e di ricerca

☐ **non autorizzo** l'effettuazione di video-registrazioni durante le attività svolte nelle lezioni di matematica della classe 1° [redacted] e l'utilizzo del materiale (video e schede di lavoro) ai fini didattici e di ricerca.

[redacted],

In fede

Original version of the consent form used during 2019 teaching experiment:

Liceo Scientifico Statale [redacted]

a.s. 2019 – 2020

Ai genitori degli studenti della classe 2° [redacted]

**Oggetto: Richiesta di autorizzazione per effettuare audio/videoregistrazioni di alcune attività svolte in classe**

Come già comunicato ai vostri figli, nei mesi di ottobre e aprile durante le ore di Matematica e Fisica potrà essere presente in aula la dott.[ssa] Sara Bagossi, impegnata in un dottorando di ricerca in Matematica presso il dipartimento di Matematica dell'Università degli Studi di Modena e Reggio Emilia.
L'intervento preparato con la dottoranda verterà sulla modellizzazione di fenomeni naturali e sul legame tra i vari linguaggi della matematica.
Gli obiettivi della ricerca (e della sua tesi di dottorato) sono individuare le potenzialità didattiche di una specifica tecnica di insegnamento denominata "Metodo della ricerca variata" dal Gruppo di Ricerca in Didattica della Matematica coordinato del prof. F. Arzarello del Dipartimento di Matematica "G. Peano" dell'Università degli Studi di Torino di cui io faccio parte. In particolare ci si pone l'obiettivo di aumentare la consapevolezza negli studenti delle proprie conoscenze disciplinari e di sistematizzarle attraverso il lavoro in classe.
Tutti gli interventi verranno progettati prima dalla docente di classe insieme alla dott.[ssa] Bagossi e al prof. Arzarello, impegnato da anni nella ricerca in Didattica della Matematica e relatore della tesi di dottorato.

Durante gli interventi in classe la dottoranda, oltre ad apportare il proprio contributo, filmerà alcune lezioni e fotograferà dei momenti significativi, in modo che tali materiali possano costituire occasione di studio, riflessione e analisi critica delle attività svolte, ovviamente nell'ottica di portare un valore aggiunto anche alla didattica nella classe stessa.

Tutte le riprese e le fotografie costituiranno esclusivamente materiale di studio e di indagine per la ricerca didattica e per la stesura della tesi. Tendenzialmente nelle riprese sarà privilegiata l'inquadratura di mani e gesti rispetto ai volti. A tal fine, nel caso in cui le immagini debbano essere mostrate, i nomi di battesimo degli studenti saranno modificati.
Si garantisce che le immagini e le riprese saranno comunque utilizzate solo in contesti che non pregiudicheranno in alcun modo la dignità personale e il decoro dei minori.

L'insegnante si rende disponibile per ulteriori informazioni e precisazioni.
Si coglie l'occasione per porgerVi i più cordiali.

L'insegnante — Il Dirigente Scolastico

Silvia [redacted] — [redacted]

✂ - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

Il sottoscritto ______________________________________________

genitore di ____________________________________________

☐ **autorizzo** l'effettuazione di video-registrazioni durante le attività svolte nelle lezioni di Matematica e Fisica della classe 2° [redacted] e l'utilizzo del materiale (video e schede di lavoro) ai fini didattici e di ricerca

☐ **non autorizzo** l'effettuazione di video-registrazioni durante le attività svolte nelle lezioni di Matematica e Fisica della classe 2° [redacted] e l'utilizzo del materiale (video e schede di lavoro) ai fini didattici e di ricerca.

[redacted], In fede

Original version of the consent form used during 2020 teaching experiment:

Liceo Scientifico Statale [redacted]

a.s. 2020 – 2021

Ai genitori degli studenti della classe 3° [redacted]

**Oggetto: Richiesta di autorizzazione per effettuare audio/videoregistrazioni di alcune attività svolte in classe**

Come già comunicato ai vostri figli, nei mesi di settembre e ottobre durante le ore di Matematica si svolgerà una seconda attività di modellizzazione di fenomeni naturali, in continuità con la sperimentazione proposta lo scorso anno scolastico. Tale intervento si inserisce nel progetto di ricerca della dott.ssa Sara Bagossi, impegnata in un dottorato di ricerca in Matematica presso il dipartimento di Matematica dell'Università degli Studi di Modena e Reggio Emilia.
Gli obiettivi della ricerca (e della sua tesi di dottorato) sono individuare le potenzialità didattiche di una specifica tecnica di insegnamento denominata "Metodo della ricerca variata" dal Gruppo di Ricerca in Didattica della Matematica coordinato del prof. F. Arzarello del Dipartimento di Matematica "G. Peano" dell'Università degli Studi di Torino di cui io faccio parte. In particolare ci si pone l'obiettivo di aumentare la consapevolezza negli studenti delle proprie conoscenze disciplinari e di sistematizzarle attraverso il lavoro in classe.
Tutti gli interventi verranno progettati prima dalla docente di classe insieme alla dott.ssa Bagossi e al prof. Arzarello, impegnato da anni nella ricerca in Didattica della Matematica e relatore della tesi di dottorato. Inoltre, la dottoranda parteciperà alla sperimentazione collegandosi a distanza.

Durante lo svolgimento della sperimentazione, la docente filmerà alcune lezioni e/o videoregistrerà le attività a distanza in modo che tali materiali possano costituire occasione di studio, riflessione e analisi critica delle attività svolte, ovviamente nell'ottica di portare un valore aggiunto anche alla didattica nella classe stessa.

Tutte le riprese e le fotografie costituiranno esclusivamente materiale di studio e di indagine per la ricerca didattica e per la stesura della tesi. Tendenzialmente nelle riprese sarà privilegiata l'inquadratura di mani e gesti rispetto ai volti. A tal fine, nel caso in cui le immagini debbano essere mostrate, i nomi di battesimo degli studenti saranno modificati.
Si garantisce che le immagini e le riprese saranno comunque utilizzate solo in contesti che non pregiudicheranno in alcun modo la dignità personale e il decoro dei minori.

L'insegnante si rende disponibile per ulteriori informazioni e precisazioni.

Si coglie l'occasione per porgerVi i più cordiali.

L'insegnante — Il Dirigente Scolastico

Silvia [redacted] — [redacted]

✂ - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

Il sottoscritto ____________________________________________

genitore di _______________________________________

☐ **autorizzo** l'effettuazione di video-registrazioni durante le attività svolte nelle lezioni di Matematica della classe 3° [redacted] e l'utilizzo del materiale (video e schede di lavoro) ai fini didattici e di ricerca

☐ **non autorizzo** l'effettuazione di video-registrazioni durante le attività svolte nelle lezioni di Matematica della classe 3° [redacted] e l'utilizzo del materiale (video e schede di lavoro) ai fini didattici e di ricerca.

[redacted], In fede

# APPENDIX B

Original version of Test 1, page 1:

Ripensando al lavoro svolto sul piano inclinato, scrivi ai compagni di un'altra classe per raccontare il lavoro stesso e nello specifico la relazione che descrive e spiega matematicamente il moto della pallina lungo il piano inclinato.
Lo scritto deve servire, per te e per i compagni che leggono, anche come supporto teorico per lo studio.

In questo periodo scolastico abbiamo, come classe, svolto un'attività in collaborazione con la nostra professoressa di matematica e fisica, Silvia Beltramino, e con una dottoranda in matematica, Sara. Questa attività si è suddivisa in quattro incontri; durante i quali siamo andati a "scoprire" un argomento nuovo, cioè l'accellerazione e lo spazio relativo.

Durante il primo incontro, durante il quale, come in tutti gli altri siamo stati filmati e registrati, abbiamo visionato un video, all'interno del quale veniva illustrato il moto di una pallina lungo un piano inclinato; la pallina percorreva distanze diverse, indicate in successione da numeri dispari consecutivi, ~~in un~~ nello stesso lasso di tempo, cioè un'oscillazione di un pendolo. Ciò accadeva grazie all'accellerazione, cioè l'aumento della velocità della pallina, del suo moto di percorrimento del piano inclinato, in un certo tempo. L'accellerazione si trova fa applicando la seguente formula:
$\frac{P_x}{m}$. $P_x$ è la componente della forza peso della pallina parallela al piano che si trova facendo ~~l'angolo di inclinazione~~ il seno dell'angolo di inclinazione del piano de per la forza peso totale; quindi ~~P~~ che si scompone in $m \cdot g$. Il divisore massa totale invece è la massa della pallina. La formula finale sarà:
$\frac{m \cdot g \cdot \sin(\alpha)}{m}$ Le due masse si semplificano e rimane: $g \cdot \sin(\alpha)$.
Durante lo studio abbiamo notato una formula, cioè $\frac{s \to \text{spazio}}{t^2 \to \text{tempo al quadrato}}$.
Andando avanti con lo studio e con le spiegazioni abbiamo scoperto che questa formula indica k, un valore che poi useremo tra poco. ~~la distanza percorsa dalla pallina in base al tempo~~
Nell'attività abbiamo lavorato principalmente su GeoGebra, un programma che permette di costruire un piano con ~~il tempo~~ un grafico ed una tabella ad esso relativi. Noi potevamo far partire la pallina da in cima al piano e ne sendo

Original version of Test 2, page 1:

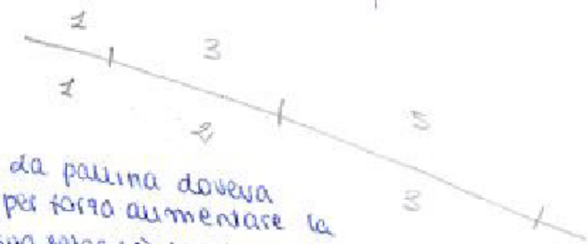
Ripensando al lavoro svolto sul piano inclinato, scrivi ai compagni di un'altra classe per raccontare il lavoro stesso e nello specifico la relazione che descrive e spiega matematicamente il moto della pallina lungo il piano inclinato.

Lo scritto deve servire, per te e per i compagni che leggono, anche come supporto teorico per lo studio.

Attraverso il progetto, intrapreso il 29 settembre in cui a coinvolgerci sono state una ~~[illegible]~~ dottoranda, Sara e la nostra professoressa di matematica e fisica, Silvia Beltramino, abbiamo avuto l'opportunità di apprendere molteplici nozioni sul moto di una pallina su un piano inclinato.

Il primo incontro si è sviluppato nell'arco di un ora...
abbiamo inizialmente preso visione di un filmato (durato poco più di 1 min) e successivamente (divisi in gruppi) abbiamo descritto secondo fondamentalmente 2 aspetti. (quello visivo e quello comunicativo)
Ci è stata consegnata una scheda, dove appunto abbiamo raccontato "fisicamente" quello che ci aveva visivamente colpito (quindi in pratica il piano inclinato, il pendolo posizionato sul lato opposto di $\alpha$ ed il tempo indicato in un cerchio blu in alto a sinistra... aspetti poco rilevanti), ma soprattutto "ciò" che di fisico stava a significare il moto della pallina.
Sono molte le ipotesi che in gruppo abbiamo discusso, qualcuna sicuramente mi sfugge... ma principalmente eravamo arrivati alla conclusione che il piano era suddiviso in porzioni di misure pari a (1;3;5;7...) (una successione di numeri dispari) che la pallina percorreva durante 1 oscillazione del pendolo...

| | |
|---|---|
| 1 | 1 |
| 2 | 3 |
| 3 | 5 |
| 4 | 7 |




da pallina doveva per forza aumentare la sua velocità in quanto nello stesso tempo (1t) percorreva spazi sempre più ampi
1; 3; 5...

Circa all'inizio del video era poi comparsa una formula $\boxed{\frac{s}{t^2}}$; questa era implicita in quanto appunto non contiene l'elemento che si voleva andare a trovare.
Ricordo che tra le successive osservazioni avevamo dedotto si trattasse dell'accelerazione (ci è venuto poi smentito in seguito, in quanto è = alla distanza)

Translation into Italian of Test 1, page 1:

*Thinking back to the work carried out on the inclined plane, write to schoolmates of another class to outline the work itself and, specifically, the relationship that describes and explains mathematically the motion of the ball along the inclined plane. This report should be a theoretical support for you and your schoolmates.*

In these schooldays, we have carried out an activity as class in collaboration with our math and physics teacher, S.B., and a PhD student in Math, S.. This activity divided into four meetings during which we "discovered" a new topic, that is accelleration and relative distance.

During the first meeting, during which, as in all the others, we were video-taped and recorded, we saw a video in which was illustrated the motion of a ball along an inclined plane, the ball traversed different distances, denoted in sequence by odd consecutive numbers, in the same period of time, that is a ripple of a pendulum. That was due to acceleration, that is the increasing of the speed of the ball, of its motion of the inclined plane, in a certain time. The acceleration can be found ~~fr~~ applying the following formula: $P_x/m$. $P_x$ is the component of the weight force parallel to the inclined plane, that can be found doing ~~the inclination angle~~ the sin of the inclination angle of the plane $\alpha$ times the total weight force ~~hence P~~ which decomposes in m [mass] · g [constant]. The divisor instead is the mass of the ball. The final formula will be: $\frac{\cancel{m} \cdot g \cdot sin(\alpha)}{\cancel{m}}$. The two masses are simplified and remains $g \cdot sin(\alpha)$.

During the study we noticed a formula, that is $\frac{s\ [distance]}{t^2\ [time\ squared]}$.

Going on with the study and with the explanation, we discovered that this formula ~~denotes the distance traversed by the ball according to time~~ $K_{//}$ a value that then we will use in a while. In the activity we worked mainly on GeoGebra, a software that allows to construct a plane with ~~the time~~ a graph and a table related to it. We could make the ball start from the top of the plane and it illustrated the motion.

Translation into Italian of Test 2, page 1:

*Thinking back to the work carried out on the inclined plane, write to schoolmates of another class to outline the work itself and, specifically, the relationship that describes and explains mathematically the motion of the ball along the inclined plane. This report should be a theoretical support for you and your schoolmates.*

Through the project undertaken on September 29th in which we were involved by a PhD student, S., and our math and physics teacher, S.B., we had the opportunity to learn many notions on the motion of the ball along an inclined plane. The first meeting developed within an hour… we initially saw a video (lasted a little over a minute) and later (divided in groups) we described it according to basically two aspects (the visual one and the communicative one).
We were given a worksheet, where precisely we described "physically" what visually impressed us (that is practically the inclined plane, the pendulum positioned on the opposite side of $\alpha$ and the time denoted in a blue circle on the top left…not so relevant aspects), but above all "what" of physical meant the motion of the ball.
Many are the hypotheses that we discussed in group, I'm definitely missing something… but mainly we arrived at the conclusion that the plane was divided in portions of measures equal to (1; 3; 5; 7…) (a sequence of odd numbers) that the ball covered during one swing of the pendulum….

The ball increased for sure its speed since in the same time (1 s) it traversed always larger distances 1; 3; 5…

Nearly at the beginning of the video a formula appeared $\frac{s}{t^2}$; this was implicit since does not contain the element that we wanted to find. I remember that among the successive observations we deduced it was the acceleration (that was later denied, since it is equal to the distance).

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