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Octahedral, dicyclic and special linear solutions of some Hamilton-Waterloo problems

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Abstract

We give a sharply-vertex-transitive solution of each of the nine Hamilton-Waterloo problems left open by Danziger, Quattrocchi and Stevens.

Keywords: Hamilton-Waterloo problem, group action, octahedral binary group, dicyclic group, special linear group.

Math. Subj. Class.: 05C70, 05E18, 05B10

1 Introduction

A cycle decomposition of a simple graph $\Gamma = (V, E)$ is a set $\mathcal{D}$ of cycles whose edges partition E. A partition $\mathcal{F}$ of $\mathcal{D}$ into classes (2-*factors*) each of which covers all V exactly once is said to be a 2-*factorization* of Γ. The *type* of a 2-factor F is the partition $\pi = [\ell_1^{n_1}, \ldots, \ell_t^{n_t}]$ (written in exponential notation) of the integer $|V|$ into the lengths of the cycles of F.

A 2-factorization $\mathcal{F}$ of K_v (the complete graph of order v) or $K_v - I$ (the cocktail party graph of order v) whose 2-factors are all of the same type π is a solution of the so-called Oberwolfach Problem $\mathrm{OP}(v; \pi)$. If instead the 2-factors of $\mathcal{F}$ are of two different types π and ψ, then $\mathcal{F}$ is a solution of the so-called Hamilton-Waterloo Problem $\mathrm{HWP}(v; \pi, \psi; r, s)$ where r and s denote the number of 2-factors of $\mathcal{F}$ of type π and ψ, respectively.

A complete solution of the OPs whose 2-factors are uniform, namely of the form $\mathrm{OP}(\ell n; [\ell^n])$, has been given in [1] and [12]. Other important classes of OPs has been

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solved in [4, 15]. For the time being, to look for a solution to all possible OPs and, above all, HWPs is too ambitious. Anyway it is reasonable to believe that we are not so far from a complete solution of the HWPs whose 2-factors are uniform, namely of the form $\mathrm{HWP}(v; [h^{v/h}], [w^{v/w}]; r, s)$. We can say this especially because of the big progress recently done in [10].

Danziger, Quattrocchi and Stevens [11] treated the HWPs whose 2-factors are either triangle-factors or quadrangle-factors, they namely studied $\mathrm{HWP}(12n; [3^{4n}], [4^{3n}]; r, s)$. In the following such an HWP will be denoted, more simply, by $\mathrm{HWP}(12n; 3, 4; r, s)$. They solved this problem for all possible triples (n, r, s) except the following ones:

(i) $(4, r, 23 - r)$ with $r \in \{5, 7, 9, 13, 15, 17\}$;

(ii) $(2, r, 11 - r)$ with $r \in \{5, 7, 9\}$.

Six of the nine above problems have been recently solved in [14] where it was pointed out that all nine problems were also solved in a work still in preparation [2] by the authors of the present paper. Meanwhile, a solution for each of the remaining three problems not considered in [14] have been given in [16]. Notwithstanding, in the present paper we want to present our solutions to the nine HWPs left open by Danziger, Quattrocchi and Stevens in detail. These solutions, differently from those of [14, 16], are full of symmetries since they are *G-regular* for a suitable group G. We recall that a cycle decomposition (or 2-factorization) of a graph Γ is said to be G-regular when it admits G as an automorphism group acting sharply transitively on all vertices. Here is explicitly our main result:

Theorem 1.1. *There exists a $\overline{O}$-regular 2-factorization of $K_{48} - I$ having r triangle-factors and $23 - r$ quadrangle-factors where $\overline{O}$ is the binary octahedral group and $r \in \{5, 7, 9, 13, 15, 17\}$.*

There exists a Q_{24}-regular 2-factorization of $K_{24} - I$ having r triangle-factors and $11 - r$ quadrangle-factors where Q_{24} is the dicyclic group of order 24 and $r \in \{7, 9\}$.

There exists a $SL_2(3)$-regular 2-factorization of $K_{24} - I$ having six triangle-factors and five quadrangle-factors where $SL_2(3)$ is the 2-dimensional special linear group over $\mathbb{Z}_3$.

2 Some preliminaries

The use of the *classic* method of differences allowed to get cyclic (namely Z_v-regular) solutions of some HWPs in [8, 9, 13]. Now we summarize, in the shortest possible way, the method of *partial differences*. This method, explained in [7] and successfully applied in many papers (see, especially, [6]), has been also useful for the investigation of G-regular 2-factorizations of a complete graph of odd order [9]. The G-regular 2-factorizations of a cocktail party graph can be treated similarly.

Throughout this paper any group G will be assumed to be written multiplicatively and its identity element will be denoted by 1. Let Ω be a *symmetric* subset of a group G; this means that $1 \notin \Omega$ and that $\omega \in \Omega$ if and only if $\omega^{-1} \in \Omega$. The *Cayley graph* on G with *connection-set* Ω, denoted by $\mathrm{Cay}[G : \Omega]$, is the simple graph whose vertices are the elements of G and whose edges are all 2-subsets of G of the form $\{g, \omega g\}$ with $(g, \omega) \in G \times \Omega$.

Remark 2.1. If λ is an involution of a group G, then $\mathrm{Cay}[G : G \setminus \{1, \lambda\}]$ is isomorphic to $K_{|G|} - I$. So, in the following, such a Cayley graph will be always identified with the cocktail party graph of order $|G|$.

Let $Cycle(G)$ be the set of all cycles with vertices in G and consider the natural right action of G on $Cycle(G)$ defined by $(c_1, c_2, \ldots, c_n)^g = (c_1 g, c_2 g, \ldots, c_n g)$ for every $C = (c_1, c_2, \ldots, c_n) \in Cycle(G)$ and every $g \in G$. The stabilizer and the orbit of any $C \in Cycle(G)$ under this action will be denoted by $Stab(C)$ and $Orb(C)$, respectively. The *list of differences* of $C \in Cycle(G)$ is the multiset ΔC of all possible quotients xy^{-1} with (x, y) an ordered pair of adjacent vertices of C. One can see that the multiplicity $m_{\Delta C}(g)$ of any element $g \in G$ in ΔC is a multiple of the order of $Stab(C)$. Thus it makes sense to speak of the *list of partial differences* of C as the multiset ∂C on G in which the multiplicity of any $g \in G$ is defined by

$$m_{\partial C}(g) := \frac{m_{\Delta C}(g)}{|Stab(C)|}.$$

We underline the fact that ∂C is, in general, a multiset. Note that if ∂C is a set, namely without repeated elements, then it is symmetric so that it makes sense to speak of the Cayley graph $\mathrm{Cay}[G : \partial C]$. The following elementary but crucial result holds.

Lemma 2.2. *If $C \in Cycle(G)$ and ∂C does not have repeated elements, then $Orb(C)$ is a G-regular cycle-decomposition of* $\mathrm{Cay}[G : \partial C]$.

By Remark 2.1, as an immediate consequence of the above lemma we can state the following result.

Theorem 2.3. *Let λ be an involution of a group G. If $\{C_1, \ldots, C_t\}$ is a subset of $Cycle(G)$ such that $\bigcup_{i=1}^{t} \partial C_i = G \setminus \{1, \lambda\}$, then $\bigcup_{i=1}^{t} Orb(C_i)$ is a G-regular cycle-decomposition of $K_{|G|} - I$.*

We need, as last ingredient, the following easy remarks.

Remark 2.4. If $C \in Cycle(G)$ and $V(C)$ is a subgroup of G, then $Orb(C)$ is a 2-factor of the complete graph on G whose stabilizer is the whole G.

If $C_1, \ldots, C_t$ are cycles of $Cycle(G)$ and $\bigcup_{i=1}^{t} V(C_i)$ is a complete system of representatives for the left cosets of a subgroup S of G, then $\bigcup_{i=1}^{t} Orb_S(C_i)$ is a 2-factor of the complete graph on G whose stabilizer is S.

3 Octahedral solutions of six Hamilton-Waterloo problems

Throughout this section G will denote the so-called *binary octahedral group* which is usually denoted by $\overline{O}$. This group, up to isomorphism, can be viewed as a group of units of the skew-field $\mathbb{H}$ of *quaternions* introduced by Hamilton, that is an extension of the complex field $\mathbb{C}$. We recall the basic facts regarding $\mathbb{H}$. Its elements are all real linear combinations of 1, i, j and k. The sum and the product of two quaternions are defined in the natural way under the rules that

$$i^2 = j^2 = k^2 = ijk = -1.$$

If $q = a + bi + cj + dk \neq 0$, then the inverse of q is given by

$$q^{-1} = \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2}.$$

The 48 elements of the multiplicative group G are the following:

$$\begin{array}{l} \pm 1, \pm i, \pm j, \pm k; \\ \frac{1}{2}(\pm 1 \pm i \pm j \pm k); \\ \frac{1}{\sqrt{2}}(\pm x \pm y), \quad \{x, y\} \in \binom{\{1,i,j,k\}}{2}. \end{array}$$

The use of the octahedral group G was crucial in [3] to get a Steiner triple system of any order $v = 96n + 49$ with an automorphism group acting sharply transitively an all but one point. Here G will be used to get a G-regular solution of each of the six Hamilton-Waterloo problems of order 48 left open in [11]. We will need to consider the following subgroups of G of order 16 and 12, respectively:

- $K = \langle k,\ \frac{1}{\sqrt{2}}(j - k)\rangle$;
- $L = \langle \frac{1}{\sqrt{2}}(j - k),\ \frac{1}{2}(-1 - i + j + k)\rangle$.

3.1 An octahedral solution of HWP$(48; 3, 4; 5, 18)$

Consider the nine cycles of $Cycle(G)$ defined as follows.

$$\begin{aligned}
C_1 &= \big(1,\ -\tfrac{1}{\sqrt{2}}(1-k),\ \tfrac{1}{2}(1-i-j-k)\big)\\
C_2 &= \big(1,\ \tfrac{1}{2}(-1-i+j+k),\ \tfrac{1}{2}(-1+i-j-k)\big)\\
C_3 &= \big(1,\ \tfrac{1}{2}(-1+i+j-k),\ \tfrac{1}{2}(-1-i-j+k)\big)\\
C_4 &= \big(1,\ k,\ -1,\ -k\big)\\
C_5 &= \big(1,\ j,\ -1,\ -j\big)\\
C_6 &= \big(1,\ \tfrac{1}{\sqrt{2}}(-i+k),\ -\tfrac{1}{2}(1+i+j+k),\ -\tfrac{1}{\sqrt{2}}(j+k)\big)\\
C_7 &= \big(1,\ \tfrac{1}{\sqrt{2}}(i-j),\ \tfrac{1}{\sqrt{2}}(1+i),\ \tfrac{1}{2}(1-i-j+k)\big)\\
C_8 &= \big(1,\ \tfrac{1}{2}(1-i+j-k),\ k,\ -\tfrac{1}{\sqrt{2}}(1+j)\big)\\
C_9 &= \big(1,\ \tfrac{1}{\sqrt{2}}(1-i),\ -\tfrac{1}{\sqrt{2}}(1+i),\ \tfrac{1}{2}(-1-i+j-k)\big)
\end{aligned}$$

We note that $Stab(C_i) = V(C_i)$ for $2 \le i \le 5$ while all other C_i's have trivial stabilizer. Thus, by Lemma 2.2, one can check that $Orb(C_i)$ is a ℓ_i-cycle decomposition of $\mathrm{Cay}[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i's are the symmetric subsets of G listed below.

$$\begin{aligned}
\Omega_1 &= \{-\tfrac{1}{\sqrt{2}}(1-k), \tfrac{1}{2}(1-i-j-k), -\tfrac{1}{\sqrt{2}}(1+i)\}^{\pm 1}\\
\Omega_2 &= \{\tfrac{1}{2}(-1-i+j+k)\}^{\pm 1}\\
\Omega_3 &= \{\tfrac{1}{2}(-1+i+j-k)\}^{\pm 1}\\
\Omega_4 &= \{k\}^{\pm 1}\\
\Omega_5 &= \{j\}^{\pm 1}\\
\Omega_6 &= \{\tfrac{1}{\sqrt{2}}(-i+k), \tfrac{1}{\sqrt{2}}(j-k), \tfrac{1}{\sqrt{2}}(1-k), -\tfrac{1}{\sqrt{2}}(j+k)\}^{\pm 1}
\end{aligned}$$

$$\Omega_7 = \{\tfrac{1}{\sqrt{2}}(i-j), \tfrac{1}{2}(1+i-j-k), \tfrac{1}{\sqrt{2}}(i+j), \tfrac{1}{2}(1-i-j+k)\}^{\pm 1}$$
$$\Omega_8 = \{\tfrac{1}{2}(1-i+j-k), -\tfrac{1}{2}(1+i+j+k), -\tfrac{1}{\sqrt{2}}(i+k), -\tfrac{1}{\sqrt{2}}(1+j)\}^{\pm 1}$$
$$\Omega_9 = \{\tfrac{1}{\sqrt{2}}(1-i), i, \tfrac{1}{\sqrt{2}}(1+j), \tfrac{1}{2}(-1-i+j-k)\}^{\pm 1}$$

One can see that the Ω_i's partition $G \setminus \{1,-1\}$. Thus, by Theorem 2.3 we can say that $\mathcal{C} := \bigcup_{i=1}^{9} Orb_G(C_i)$ is a G-regular cycle-decomposition of $K_{48} - I$. Now set $F_i = Orb_{S_i}(C_i)$ where

$$S_i = \begin{cases} K & \text{for } i = 1; \\ G & \text{for } 2 \le i \le 5; \\ L & \text{for } 6 \le i \le 9. \end{cases}$$

By Remark 2.4, each F_i is a 2-factor of $K_{48} - I$ with $Stab(F_i) = S_i$, hence $Orb(F_i)$ has length 3 or 1 or 4 according to whether $i = 1$, or $2 \le i \le 5$, or $6 \le i \le 9$, respectively. The cycles of F_i are triangles or quadrangles according to whether or not $i \le 3$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{9} Orb(F_i)$ is a G-regular 2-factorization of $K_{48} - I$ with 5 triangle-factors and 18 quadrangle-factors, namely a G-regular solution of HWP$(48; 3, 4; 5, 18)$.

3.2 An octahedral solution of HWP$(48; 3, 4; 7, 16)$

Consider the seven cycles of $Cycle(G)$ defined as follows.

$$C_1 = \big(1,\ -\tfrac{1}{\sqrt{2}}(i+j),\ \tfrac{1}{2}(1-i+j+k)\big)$$
$$C_2 = \big(1,\ \tfrac{1}{2}(-1-i+j+k),\ \tfrac{1}{2}(1-i-j-k)\big)$$
$$C_3 = \big(1,\ \tfrac{1}{2}(-1+i+j-k),\ \tfrac{1}{2}(-1-i-j+k)\big)$$
$$C_4 = \big(1,\ \tfrac{1}{\sqrt{2}}(-i+k),\ \tfrac{1}{2}(1+i+j-k),\ -\tfrac{1}{\sqrt{2}}(j+k)\big)$$
$$C_5 = \big(1,\ \tfrac{1}{\sqrt{2}}(i-j),\ \tfrac{1}{\sqrt{2}}(1-k),\ \tfrac{1}{\sqrt{2}}(1+i)\big)$$
$$C_6 = \big(1,\ \tfrac{1}{\sqrt{2}}(1+k),\ -\tfrac{1}{2}(1+i+j+k),\ \tfrac{1}{\sqrt{2}}(1+j)\big)$$
$$C_7 = \big(1,\ -\tfrac{1}{2}(1+i+j+k),\ \tfrac{1}{2}(1-i+j-k),\ \tfrac{1}{2}(1-i-j+k)\big)$$

We note that $Stab(C_3) = V(C_3)$ while all other C_i's have trivial stabilizer. Thus, by Lemma 2.2, one can check that $Orb(C_i)$ is a ℓ_i-cycle decomposition of Cay$[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i's are the symmetric subsets of G listed below.

$$\Omega_1 = \{-\tfrac{1}{\sqrt{2}}(i+j), \tfrac{1}{2}(1-i+j+k), \tfrac{1}{\sqrt{2}}(-j+k)\}^{\pm 1}$$
$$\Omega_2 = \{\tfrac{1}{2}(-1-i+j+k), \tfrac{1}{2}(1-i-j-k), \tfrac{1}{2}(-1-i+j-k)\}^{\pm 1}$$
$$\Omega_3 = \{\tfrac{1}{2}(-1+i+j-k)\}^{\pm 1}$$
$$\Omega_4 = \{\tfrac{1}{\sqrt{2}}(-i+k), -\tfrac{1}{\sqrt{2}}(1-k), \tfrac{1}{\sqrt{2}}(i+k), -\tfrac{1}{\sqrt{2}}(j+k)\}^{\pm 1}$$
$$\Omega_5 = \{\tfrac{1}{\sqrt{2}}(i-j), -j, \tfrac{1}{2}(1-i+j-k), \tfrac{1}{\sqrt{2}}(1+i)\}^{\pm 1}$$
$$\Omega_6 = \{\tfrac{1}{\sqrt{2}}(1+k), \tfrac{1}{\sqrt{2}}(-1+j), -\tfrac{1}{\sqrt{2}}(1+i), \tfrac{1}{\sqrt{2}}(1+j)\}^{\pm 1}$$
$$\Omega_7 = \{-\tfrac{1}{2}(1+i+j+k), -i, -k, \tfrac{1}{2}(1-i-j+k)\}^{\pm 1}$$

One can see that the Ω_i's partition $G \setminus \{1,-1\}$. Thus, by Theorem 2.3 we can say that $\mathcal{C} := \bigcup_{i=1}^{7} Orb_G(C_i)$ is a G-regular cycle-decomposition of $K_{48} - I$. Now set $F_i = Orb_{S_i}(C_i)$ where

$$S_i = \begin{cases} K & \text{for } i = 1, 2; \\ G & \text{for } i = 3; \\ L & \text{for } 4 \le i \le 7. \end{cases}$$

By Remark 2.4, each F_i is a 2-factor of $K_{48} - I$ with $Stab_G(F_i) = S_i$, hence $Orb_G(F_i)$ has length 3 or 1 or 4 according to whether $i = 1, 2$ or $i = 3$ or $4 \le i \le 7$, respectively.

The cycles of F_i are triangles or quadrangles according to whether or not $i \le 3$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{7} Orb_G(F_i)$ is a G-regular 2-factorization of $K_{48} - I$ with 7 triangle-factors and 16 quadrangle-factors, namely a G-regular solution of HWP$(48; 3, 4; 7, 16)$.

3.3 An octahedral solution of HWP$(48; 3, 4; 9, 14)$

Consider the eight cycles of $Cycle(G)$ defined as follows.

$$\begin{aligned}
C_1 &= \big(1,\ \tfrac{1}{\sqrt{2}}(i+j),\ \tfrac{1}{2}(1-i-j-k)\big)\\
C_2 &= \big(1,\ -\tfrac{1}{\sqrt{2}}(1-k),\ \tfrac{1}{\sqrt{2}}(1+j)\big)\\
C_3 &= \big(1,\ \tfrac{1}{2}(-1-i+j+k),\ \tfrac{1}{2}(1+i-j+k)\big)\\
C_4 &= \big(1,\ \tfrac{1}{\sqrt{2}}(-i+k),\ \tfrac{1}{\sqrt{2}}(1-i),\ \tfrac{1}{2}(-1-i+j-k)\big)\\
C_5 &= \big(1,\ \tfrac{1}{\sqrt{2}}(i-j),\ \tfrac{1}{2}(-1+i+j+k),\ -\tfrac{1}{\sqrt{2}}(j+k)\big)\\
C_6 &= \big(1,\ \tfrac{1}{\sqrt{2}}(1+i),\ \tfrac{1}{\sqrt{2}}(1-i),\ \tfrac{1}{2}(1-i-j+k)\big)\\
C_7 &= \big(1,\ k,\ -1,\ -k\big)\\
C_8 &= \big(1,\ j,\ -1,\ -j\big)
\end{aligned}$$

We note that $Stab(C_i) = V(C_i)$ for $i = 7, 8$ while all other C_i's have trivial stabilizer. By Lemma 2.2, one can check that $Orb(C_i)$ is a ℓ_i-cycle decomposition of Cay$[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i's are the symmetric subsets of G listed below.

$$\begin{aligned}
\Omega_1 &= \{\tfrac{1}{\sqrt{2}}(i+j), \tfrac{1}{2}(1-i-j-k), \tfrac{1}{\sqrt{2}}(-1+i)\}^{\pm 1}\\
\Omega_2 &= \{-\tfrac{1}{\sqrt{2}}(1-k), \tfrac{1}{\sqrt{2}}(1+j), \tfrac{1}{2}(-1+i+j+k)\}^{\pm 1}\\
\Omega_3 &= \{\tfrac{1}{2}(-1-i+j+k), \tfrac{1}{2}(1+i-j+k), \tfrac{1}{2}(-1-i-j+k)\}^{\pm 1}\\
\Omega_4 &= \{\tfrac{1}{\sqrt{2}}(-i+k), \tfrac{1}{2}(1-i+j+k), \tfrac{1}{\sqrt{2}}(i+k), \tfrac{1}{2}(-1-i+j-k)\}^{\pm 1}\\
\Omega_5 &= \{\tfrac{1}{\sqrt{2}}(i-j), \tfrac{1}{\sqrt{2}}(j-k), -\tfrac{1}{\sqrt{2}}(1+j), -\tfrac{1}{\sqrt{2}}(j+k)\}^{\pm 1}\\
\Omega_6 &= \{\tfrac{1}{\sqrt{2}}(1+i), i, \tfrac{1}{\sqrt{2}}(1-k), \tfrac{1}{2}(1-i-j+k)\}^{\pm 1}\\
\Omega_7 &= \{k\}^{\pm 1}\\
\Omega_8 &= \{j\}^{\pm 1}
\end{aligned}$$

Now note that the Ω_i's partition $G \setminus \{1,-1\}$. Thus, by Theorem 2.3 we can say that $\mathcal{C} := \bigcup_{i=1}^{8} Orb(C_i)$ is a G-regular cycle-decomposition of $K_{48} - I$. Now set $F_i = Orb_{S_i}(C_i)$

where

$$S_i = \begin{cases} K & \text{for } 1 \leq i \leq 3; \\ L & \text{for } 4 \leq i \leq 6; \\ G & \text{for } i = 7, 8. \end{cases}$$

By Remark 2.4, each F_i is a 2-factor of $K_{48} - I$ with $Stab_G(F_i) = S_i$, hence $Orb_G(F_i)$ has length 3 or 4 or 1 according to whether $1 \leq i \leq 3$ or $4 \leq i \leq 6$ or $i = 7, 8$, respectively. The cycles of F_i are triangles or quadrangles according to whether or not $i \leq 3$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{8} Orb_G(F_i)$ is a G-regular 2-factorization of $K_{48} - I$ with 9 triangle-factors and 14 quadrangle-factors, namely a G-regular solution of HWP$(48; 3, 4; 9, 14)$.

3.4 An octahedral solution of HWP$(48; 3, 4; 13, 10)$

Consider the nine cycles of $Cycle(G)$ defined as follows.

$$\begin{aligned}
C_1 &= \big(1,\ -\tfrac{1}{\sqrt{2}}(i+j),\ -\tfrac{1}{\sqrt{2}}(1+j)\big) \\
C_2 &= \big(1,\ \tfrac{1}{2}(1-i+j-k),\ -\tfrac{1}{\sqrt{2}}(i+k)\big) \\
C_3 &= \big(1,\ \tfrac{1}{\sqrt{2}}(-i+j),\ \tfrac{1}{2}(1-i-j-k)\big) \\
C_4 &= \big(1,\ \tfrac{1}{2}(-1+i-j+k),\ \tfrac{1}{\sqrt{2}}(i-k)\big) \\
C_5 &= \big(1,\ \tfrac{1}{2}(-1-i+j+k),\ \tfrac{1}{2}(-1+i-j-k)\big) \\
C_6 &= \big(1,\ k,\ -1,\ -k\big) \\
C_7 &= \big(1,\ j,\ -1,\ -j\big) \\
C_8 &= \big(1,\ -\tfrac{1}{2}(1+i+j+k),\ \tfrac{1}{2}(-1+i-j+k),\ \tfrac{1}{\sqrt{2}}(1+j)\big) \\
C_9 &= \big(1,\ -\tfrac{1}{\sqrt{2}}(1+k),\ -k,\ \tfrac{1}{2}(-1+i+j-k)\big)
\end{aligned}$$

We note that $Stab(C_i) = V(C_i)$ for $5 \leq i \leq 7$ while all other C_i's have trivial G-stabilizer. Thus, by Lemma 2.2, one can check that $Orb(C_i)$ is a ℓ_i-cycle decomposition of Cay$[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i's are the symmetric subsets of G listed below.

$$\begin{aligned}
\Omega_1 &= \{-\tfrac{1}{\sqrt{2}}(i+j), -\tfrac{1}{\sqrt{2}}(1+j), \tfrac{1}{2}(1+i+j-k)\}^{\pm 1} \\
\Omega_2 &= \{\tfrac{1}{2}(1-i+j-k), -\tfrac{1}{\sqrt{2}}(i+k), \tfrac{1}{\sqrt{2}}(1+i)\}^{\pm 1} \\
\Omega_3 &= \{\tfrac{1}{\sqrt{2}}(-i+j), \tfrac{1}{2}(1-i-j-k), \tfrac{1}{\sqrt{2}}(j-k)\}^{\pm 1} \\
\Omega_4 &= \{\tfrac{1}{2}(-1+i-j+k), \tfrac{1}{\sqrt{2}}(i-k), -\tfrac{1}{\sqrt{2}}(j+k)\}^{\pm 1} \\
\Omega_5 &= \{\tfrac{1}{2}(-1-i+j+k)\}^{\pm 1} \\
\Omega_6 &= \{k\}^{\pm 1} \\
\Omega_7 &= \{j\}^{\pm 1} \\
\Omega_8 &= \{-\tfrac{1}{2}(1+i+j+k), i, \tfrac{1}{\sqrt{2}}(-1+i), \tfrac{1}{\sqrt{2}}(1+j)\}^{\pm 1} \\
\Omega_9 &= \{-\tfrac{1}{\sqrt{2}}(1+k), \tfrac{1}{\sqrt{2}}(1-k), \tfrac{1}{2}(1-i+j+k), \tfrac{1}{2}(-1+i+j-k)\}^{\pm 1}
\end{aligned}$$

Now note that the Ω_i's partition $G\backslash\{1,-1\}$. Thus, by Theorem 2.3 we can say that $\mathcal{C} := \bigcup_{i=1}^{9} Orb(C_i)$ is a G-regular cycle-decomposition of $K_{48} - I$. Now set $F_i = Orb_{S_i}(C_i)$ where

$$S_i = \begin{cases} K & \text{for } 1 \leq i \leq 4; \\ G & \text{for } 5 \leq i \leq 7; \\ L & \text{for } i = 8, 9. \end{cases}$$

By Remark 2.4, each F_i is a 2-factor of $K_{48} - I$ with $Stab_G(F_i) = S_i$, hence $Orb_G(F_i)$ has length 3 or 1 or 4 according to whether $1 \leq i \leq 4$ or $5 \leq i \leq 7$ or $i = 8, 9$, respectively.

The cycles of F_i are triangles or quadrangles according to whether or not $i \leq 5$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{9} Orb_G(F_i)$ is a G-regular 2-factorization of $K_{48} - I$ with 13 triangle-factors and 10 quadrangle-factors, namely a G-regular solution of HWP$(48; 3, 4; 13, 10)$.

3.5 An octahedral solution of HWP$(48; 3, 4; 15, 8)$

Consider the seven cycles of $Cycle(G)$ defined as follows.

$$\begin{aligned}
C_1 &= \left(1,\ \tfrac{1}{2}(-1-i+j+k),\ \tfrac{1}{\sqrt{2}}(i+k)\right)\\
C_2 &= \left(1,\ -\tfrac{1}{\sqrt{2}}(i+j),\ -\tfrac{1}{\sqrt{2}}(1+j)\right)\\
C_3 &= \left(1,\ \tfrac{1}{2}(-1+i+j-k),\ \tfrac{1}{2}(1-i+j+k)\right)\\
C_4 &= \left(1,\ \tfrac{1}{2}(1+i+j+k),\ \tfrac{1}{\sqrt{2}}(1+j)\right)\\
C_5 &= \left(1,\ \tfrac{1}{2}(1-i+j-k),\ \tfrac{1}{\sqrt{2}}(i-k)\right)\\
C_6 &= \left(1,\ -j,\ k,\ -\tfrac{1}{\sqrt{2}}(1-k)\right)\\
C_7 &= \left(1,\ \tfrac{1}{\sqrt{2}}(i-j),\ \tfrac{1}{2}(-1-i+j-k),\ \tfrac{1}{2}(-1+i+j+k)\right)
\end{aligned}$$

Here, every C_i has trivial stabilizer. Thus, by Lemma 2.2, one can check that $Orb(C_i)$ is a ℓ_i-cycle decomposition of Cay$[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i's are the symmetric subsets of G listed below.

$$\begin{aligned}
\Omega_1 &= \{\tfrac{1}{2}(-1-i+j+k), \tfrac{1}{\sqrt{2}}(i+k), \tfrac{1}{\sqrt{2}}(-j+k)\}^{\pm 1}\\
\Omega_2 &= \{-\tfrac{1}{\sqrt{2}}(i+j), -\tfrac{1}{\sqrt{2}}(1+j), \tfrac{1}{2}(1+i+j-k)\}^{\pm 1}\\
\Omega_3 &= \{\tfrac{1}{2}(-1+i+j-k), \tfrac{1}{2}(1-i+j+k), \tfrac{1}{2}(-1-i+j-k)\}^{\pm 1}\\
\Omega_4 &= \{\tfrac{1}{2}(1+i+j+k), \tfrac{1}{\sqrt{2}}(1+j), \tfrac{1}{\sqrt{2}}(1+i)\}^{\pm 1}\\
\Omega_5 &= \{\tfrac{1}{2}(1-i+j-k), \tfrac{1}{\sqrt{2}}(i-k), \tfrac{1}{\sqrt{2}}(j+k)\}^{\pm 1}\\
\Omega_6 &= \{-j, +i, \tfrac{1}{\sqrt{2}}(1-k), -\tfrac{1}{\sqrt{2}}(1-k)\}^{\pm 1}\\
\Omega_7 &= \{\tfrac{1}{\sqrt{2}}(i-j), -\tfrac{1}{\sqrt{2}}(1+i), +k, \tfrac{1}{2}(-1+i+j+k)\}^{\pm 1}
\end{aligned}$$

Now note that the Ω_i's partition $G \setminus \{1, -1\}$. Thus, by Theorem 2.3 we can say that $\mathcal{C} := \bigcup_{i=1}^{7} Orb(C_i)$ is a G-regular cycle-decomposition of $K_{48} - I$. Set $F_i = Orb_{S_i}(C_i)$ where

$$S_i = \begin{cases} K & \text{for } 1 \leq i \leq 5; \\ L & \text{for } i = 6, 7. \end{cases}$$

By Remark 2.4, each F_i is a 2-factor of $K_{48} - I$ with $Stab_G(F_i) = S_i$, hence $Orb_G(F_i)$ has length 3 or 4 according to whether $1 \leq i \leq 5$ or $i = 6, 7$, respectively. The cycles of F_i are triangles or quadrangles according to whether or not $i \leq 5$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{7} Orb_G(F_i)$ is a G-regular 2-factorization of $K_{48} - I$ with 15 triangle-factors and 8 quadrangle-factors, namely a G-regular solution of HWP$(48; 3, 4; 15, 8)$.

3.6 An octahedral solution of HWP$(48; 3, 4; 17, 6)$

Consider the ten cycles of $Cycle(G)$ defined as follows.

$$\begin{aligned}
C_1 &= \big(1,\ -\tfrac{1}{\sqrt{2}}(1-k),\ -\tfrac{1}{\sqrt{2}}(i+k)\big)\\
C_2 &= \big(1,\ -\tfrac{1}{\sqrt{2}}(i+j),\ \tfrac{1}{2}(-1+i+j+k)\big)\\
C_3 &= \big(1,\ \tfrac{1}{2}(1+i-j-k),\ -\tfrac{1}{\sqrt{2}}(1+j)\big)\\
C_4 &= \big(1,\ \tfrac{1}{\sqrt{2}}(-i+j),\ \tfrac{1}{\sqrt{2}}(-i+k)\big)\\
C_5 &= \big(1,\ \tfrac{1}{2}(1-i+j-k),\ \tfrac{1}{\sqrt{2}}(1-j)\big)\\
C_6 &= \big(1,\ \tfrac{1}{2}(-1-i+j+k),\ \tfrac{1}{2}(-1+i-j-k)\big)\\
C_7 &= \big(1,\ \tfrac{1}{2}(-1+i+j-k),\ \tfrac{1}{2}(-1-i-j+k)\big)\\
C_8 &= \big(1,\ k,\ -1,\ -k\big)\\
C_9 &= \big(1,\ j,\ -1,\ -j\big)\\
C_{10} &= \big(1,\ \tfrac{1}{\sqrt{2}}(1+i),\ \tfrac{1}{\sqrt{2}}(1-i), \tfrac{1}{2}(1-i-j+k)\big)
\end{aligned}$$

We note that $Stab(C_i) = V(C_i)$ for $6 \leq i \leq 9$ while all other C_i's have trivial stabilizer. Thus, by Lemma 2.2, one can check that $Orb(C_i)$ is a ℓ_i-cycle decomposition of Cay$[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i's are the symmetric subsets of G listed below.

$$\begin{aligned}
\Omega_1 &= \{-\tfrac{1}{\sqrt{2}}(1-k), -\tfrac{1}{\sqrt{2}}(i+k), \tfrac{1}{2}(-1-i+j-k)\}^{\pm 1}\\
\Omega_2 &= \{-\tfrac{1}{\sqrt{2}}(i+j), \tfrac{1}{2}(-1+i+j+k), \tfrac{1}{\sqrt{2}}(-1+i)\}^{\pm 1}\\
\Omega_3 &= \{\tfrac{1}{2}(1+i-j-k), -\tfrac{1}{\sqrt{2}}(1+j), \tfrac{1}{\sqrt{2}}(j+k)\}^{\pm 1}\\
\Omega_4 &= \{\tfrac{1}{\sqrt{2}}(-i+j), \tfrac{1}{\sqrt{2}}(-i+k), \tfrac{1}{2}(1-i-j-k)\}^{\pm 1}\\
\Omega_5 &= \{\tfrac{1}{2}(1-i+j-k), \tfrac{1}{\sqrt{2}}(1-j), \tfrac{1}{\sqrt{2}}(j-k)\}^{\pm 1}\\
\Omega_6 &= \{\tfrac{1}{2}(-1-i+j+k)\}^{\pm 1}\\
\Omega_7 &= \{\tfrac{1}{2}(-1+i+j-k)\}^{\pm 1}\\
\Omega_8 &= \{k\}^{\pm 1}\\
\Omega_9 &= \{j\}^{\pm 1}\\
\Omega_{10} &= \{\tfrac{1}{\sqrt{2}}(1+i), i, \tfrac{1}{\sqrt{2}}(1-k), \tfrac{1}{2}(1-i-j+k)\}^{\pm 1}
\end{aligned}$$

Now note that the Ω_i's partition $G \setminus \{1, -1\}$. Thus, by Lemma 2.2 we can say that $\mathcal{C} := \bigcup_{i=1}^{10} Orb(C_i)$ is a G-regular cycle-decomposition of $K_{48} - I$. Set $F_i = Orb_{S_i}(C_i)$

where

$$S_i = \begin{cases} K & \text{for } 1 \le i \le 5; \\ G & \text{for } 6 \le i \le 9; \\ L & \text{for } i = 10. \end{cases}$$

By Remark 2.4, each F_i is a 2-factor of K_{48} with $Stab_G(F_i) = S_i$, hence $Orb_G(F_i)$ has length 3 or 1 or 4 according to whether $1 \le i \le 5$ or $6 \le i \le 9$ or $i = 10$, respectively. The cycles of F_i are triangles or quadrangles according to whether or not $i \le 7$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{10} Orb_G(F_i)$ is a G-regular 2-factorization of $K_{48} - I$ with 17 triangle-factors and 6 quadrangle-factors, namely a G-regular solution of HWP$(48; 3, 4; 17, 6)$.

4 Dicyclic solutions of two Hamilton-Waterloo problems

In this section G will denote the dicyclic group of order 24 which is usually denoted by Q_{24}. Thus G has the following presentation:

$$G = \langle a, b \,|\, a^{12} = 1,\ b^2 = a^6,\ b^{-1}ab = a^{-1} \rangle$$

Note that the elements of G can be written in the form $a^i b^j$ with $0 \le i \le 11$ and $j = 0, 1$. The group G has a unique involution which is a^6 and we will need to consider the following subgroups of G:

- $H = \langle b \rangle = \{1, b, a^6, a^6 b\}$;
- $K = \langle a^2 \rangle = \{1, a^2, a^4, a^6, a^8, a^{10}\}$;
- $L = \langle a^2 b, a^3 \rangle = \{1, a^3, a^6, a^9, a^2 b, a^8 b, a^5 b, a^{11} b\}$.

4.1 A dicyclic solution of HWP(24; 3, 4; 7, 4)

Consider the four cycles of $Cycle(G)$ defined as follows.

$$\begin{aligned} C_1 &= (1,\ a^3 b,\ a^5) \\ C_2 &= (1,\ a^{10},\ a^7 b) \\ C_3 &= (1,\ a^4,\ a^8) \\ C_4 &= (1,\ b,\ a^3 b,\ a) \end{aligned}$$

We note that the $Stab(C_3) = V(C_3)$ while all other C_i's have trivial stabilizer. Thus, by Lemma 2.2, one can check that $Orb(C_i)$ is a ℓ_i-cycle decomposition of Cay$[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i's are the symmetric subsets of G listed below.

$$\begin{aligned} \Omega_1 &= \{a^3 b, a^5, a^2 b\}^{\pm 1} \\ \Omega_2 &= \{a^2, ab, a^5 b\}^{\pm 1} \\ \Omega_3 &= \{a^4\}^{\pm 1} \\ \Omega_4 &= \{b, a^3, a^4 b, a\}^{\pm 1} \end{aligned}$$

Now note that the Ω_i's partition $G \backslash \{1, a^6\}$. Thus, by Theorem 2.3 we can say that $\mathcal{C} := \bigcup_{i=1}^{4} Orb(C_i)$ is a G-regular cycle-decomposition of $K_{24} - I$. Now set $F_i = Orb_{S_i}(C_i)$

where

$$S_i = \begin{cases} L & \text{for } i = 1, 2; \\ G & \text{for } i = 3; \\ K & \text{for } i = 4. \end{cases}$$

By Remark 2.4, each F_i is a 2-factor of $K_{24} - I$ with $Stab_G(F_i) = S_i$, hence $Orb_G(F_i)$ has length 3 or 1 or 4 according to whether $i = 1, 2$ or $i = 3$ or $i = 4$, respectively.

The cycles of F_i are triangles or quadrangles according to whether or not $i \leq 3$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{4} Orb_G(F_i)$ is a G-regular 2-factorization of $K_{24} - I$ with 7 triangle-factors and 4 quadrangle-factors, namely a G-regular solution of HWP$(24; 3, 4; 7, 4)$.

4.2 A dicyclic solution of HWP$(24; 3, 4; 9, 2)$

Consider the four cycles of $Cycle(G)$ defined as follows.

$$\begin{aligned} C_1 &= (1, b, a^6, a^6 b) \\ C_2 &= (1, a^4 b, a^6, a^{10} b) \\ C_3 &= (1, a^4, a^7 b) \\ C_4 &= (1, a^3 b, a^8 b) \\ C_5 &= (a^4, a^7, a^5) \end{aligned}$$

We note that $Stab(C_i) = V(C_i)$ for $i = 1, 2$ while all other C_i's have trivial stabilizer. By Lemma 2.2, one can check that $Orb(C_i)$ is a ℓ_i-cycle decomposition of Cay$[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i's are the symmetric subsets of G listed below.

$$\begin{aligned} \Omega_1 &= \{b\}^{\pm 1} \\ \Omega_2 &= \{a^4 b\}^{\pm 1} \\ \Omega_3 &= \{a^4, ab, a^5 b\}^{\pm 1} \\ \Omega_4 &= \{a^3 b, a^2 b, a^5\}^{\pm 1} \\ \Omega_5 &= \{a^1, a^2, a^3\}^{\pm 1} \end{aligned}$$

Also here the Ω_i's partition $G \setminus \{1, a^6\}$, hence $\mathcal{C} := \bigcup_{i=1}^{5} Orb_G(C_i)$ is a G-regular cycle-decomposition of $K_{24} - I$ by Theorem 2.3. Now set:

$$\begin{aligned} F_1 &= Orb_G(C_1), \quad F_2 = Orb_G(C_2), \\ F_3 &= Orb_L(C_3), \quad F_4 = Orb_H(C_4) \cup Orb_H(C_5). \end{aligned}$$

By Remark 2.4, each F_i is a 2-factor of $K_{24} - I$ and we have

$$Stab_G(F_1) = Stab_G(F_2) = G; \quad Stab_G(F_3) = L; \quad Stab_G(F_4) = H$$

so that the lengths of the G-orbits of $F_1, \ldots, F_4$ are 1, 1, 3 and 6, respectively. The cycles of F_i are triangles or quadrangles according to whether or not $i \geq 3$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{5} Orb_G(F_i)$ is a G-regular 2-factorization of $K_{24} - I$ with 9 triangle-factors and 2 quadrangle-factors, namely a G-regular solution of HWP$(24; 3, 4; 9, 2)$.

5 A special linear solution of HWP$(24; 3, 4; 5, 6)$

In this section G will denote the 2-dimensional special linear group over $\mathbb{Z}_3$, usually denoted by $SL_2(3)$, namely the group of 2×2 matrices with elements in $\mathbb{Z}_3$ and determinant one. The only involution of G is $2E$ where E is the identity matrix of G. The 2-Sylow subgroup Q of G, isomorphic to the group of quaternions, is the following:

$$Q = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

We will also need to consider the subgroup H of G of order 6 generated by the matrix $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$. Hence we have:

$$H = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \right\}.$$

The use of the special linear group G was crucial in [5] to get a Steiner triple system of any order $v = 144n + 25$ with an automorphism group acting sharply transitively an all but one point. Here G will be used to get a G-regular solution of the last Hamilton-Waterloo problem left open in [11].

Consider the six cycles of $Cycle(G)$ defined as follows.

$$C_1 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \right)$$
$$C_2 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} \right)$$
$$C_3 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \right)$$
$$C_4 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \right)$$
$$C_5 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} \right)$$
$$C_6 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right)$$

Here the stabilizer of C_i is trivial for $i = 1, 6$ while it coincides with $V(C_i)$ for $2 \leq i \leq 5$. By Lemma 2.2, one can check that $Orb(C_i)$ is a ℓ_i-cycle decomposition of Cay$[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i's are the symmetric subsets of G listed below.

$$\Omega_1 = \left\{ \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \right\}^{\pm 1}$$
$$\Omega_2 = \left\{ \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix} \right\}^{\pm 1} \quad \Omega_3 = \left\{ \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} \right\}^{\pm 1}$$
$$\Omega_4 = \left\{ \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}^{\pm 1} \quad \Omega_5 = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \right\}^{\pm 1}$$
$$\Omega_6 = \left\{ \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}^{\pm 1}$$

Once again we see that the Ω_i's partition $G \setminus \{E, 2E\}$, therefore $\mathcal{C} := \bigcup_{i=1}^{6} Orb(C_i)$ is a G-regular cycle-decomposition of $K_{24} - I$. Now set $F_i = Orb_{S_i}(C_i)$ with

$$S_i = \begin{cases} Q & \text{for } i = 1; \\ G & \text{for } 2 \leq i \leq 5; \\ H & \text{for } i = 6. \end{cases}$$

By Remark 2.4, each F_i is a 2-factor of $K_{24} - I$ and we have $Stab_G(F_i) = S_i$ so that the lengths of the G-orbits of $F_1, \dots, F_6$ are 3, 1, 1, 1, 1 and 4, respectively.

The cycles of F_i have length 3 or 4 according to whether or not $i \leq 3$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{24} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{6} Orb_G(F_i)$ is a G-regular 2-factorization of $K_{24} - I$ with 5 triangle-factors and 6 quadrangle-factors, namely a G-regular solution of $\mathrm{HWP}(24; 3, 4; 5, 6)$.

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