



## Operators with Gaussian Kernel Bounds on Mixed Morrey Spaces

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**Abstract.** Let  $\mathcal{L}$  be an analytic semigroup on  $L^2(\mathbb{R}^n)$  with Gaussian kernel bound, and let  $\mathcal{L}^{-\frac{\alpha}{2}}$  be the fractional operator associated to  $\mathcal{L}$  for  $0 < \alpha < n$ . In this paper, we prove some boundedness properties for the commutator  $[b, \mathcal{L}^{-\frac{\alpha}{2}}]$  on Mixed Morrey spaces  $L^{q,\mu}(0, T, L^{p,\lambda}(\mathbb{R}^n))$ , when  $b$  belongs to  $BMO(\mathbb{R}^n)$  or to suitable homogeneous Lipschitz spaces.

### 1. Introduction

Let us consider the infinitesimal generator  $\mathcal{L}$  of an analytic semigroup  $\{e^{-t\mathcal{L}}\}_{t>0}$  on  $L^2(\mathbb{R}^n)$ , with a kernel  $p_t(x, y)$  satisfying a Gaussian upper bound. That is, there exist two positive constants  $A$  and  $C$  such that

$$|p_t(x, y)| \leq \frac{C}{t^{n/2}} e^{-A \frac{|x-y|^2}{t}} \quad \text{for every } x, y \in \mathbb{R}^n, t > 0. \quad (1)$$

In this paper we are concerned with the study of the fractional integral operator  $\mathcal{L}^{-\alpha/2}$  associated to  $\mathcal{L}$  defined for any  $0 < \alpha < n$  as

$$\mathcal{L}^{-\alpha/2} f(x, t) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-s\mathcal{L}}(f(x, t)) s^{\alpha/2-1} ds. \quad (2)$$

This property is satisfied by a large class of differential operators. As a first example, let us consider a real vector potential  $a = (a_1, a_2, \dots, a_n)$  and an electric potential  $V$ . We assume that  $a_k \in L^2_{\text{loc}}(\mathbb{R}^n)$  for any  $k = 1, 2, \dots, n$  and  $V \geq 0, V \in L^1_{\text{loc}}(\mathbb{R}^n)$ . We say that

$$A = -(\nabla - ia)^2 + V$$

is the *magnetic Schrödinger operator*. In the paper [20] by Simon was proved a diamagnetic inequality, which implies the following pointwise estimate

$$|e^{-tA} f| \leq e^{-t\Delta} |f|$$

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for any  $t > 0$  and  $f \in L^2_{\text{loc}}(\mathbb{R}^n)$ . This estimates ensures that the semigroup  $e^{-tA}$  has a kernel with Gaussian upper bound of the type (1).

As a further example, let us consider the following divergence form operator

$$L = -\text{div}(A\nabla),$$

with  $A = (a_{ij}(x))_{i,j=1,\dots,n}$  be a  $n \times n$  matrix of complex entries  $a_{ij} \in L^\infty(\mathbb{R}^n)$ . Moreover, there exists  $\lambda > 0$  such that

$$\text{Re} \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2$$

for every  $x \in \mathbb{R}^n$ ,  $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ . We remark that the operator  $\mathcal{L}$  needs to be considered in the weak sense. When the dimension of the space  $n = 1, 2$ , the kernel of the semigroup  $e^{-tL}$  has Gaussian upper bound of the type (1). In the particular case of real entries, the Gaussian bound holds true for any  $n \in \mathbb{N}$  (see [1]).

The aim of this paper is to obtain boundedness results for a class of fractional integral operators on Mixed Morrey spaces. The idea behind the proof of these kind of mixed Morrey estimates is to begin with classic Morrey estimates for the operator  $\mathcal{L}$ . Thus, following the idea of Piccinini [14], we consider domains of type A.

**DOMAIN OF TYPE A.** Let  $T > 0$  and  $\Omega$  be a bounded open subset of  $\mathbb{R}^n$ , such that there exists a constant  $A > 0$  such that

$$|Q(x, \rho) \cap \Omega| \geq A\rho^n \quad \text{for every } x \in \Omega, 0 \leq \rho \leq \text{diam}(\Omega),$$

with  $Q(x, \rho)$  a cube centered in  $x$  having edges parallel to the coordinate axes and length  $2\rho$ .

Moreover, we recall the definition of classic Morrey spaces, introduced by Morrey in 1938 in [12].

**Definition 1.1.** Let  $1 < p < +\infty$ ,  $0 < \lambda < n$ . We define the Morrey space  $L^{p,\lambda}(\Omega)$ , with  $\Omega \subset \mathbb{R}^n$ , as the class of functions  $f \in L^p(\Omega)$  such that the norm

$$\|f\|_{L^{p,\lambda}(\Omega)} := \left( \sup_{\substack{x \in \Omega, \\ \rho > 0}} \frac{1}{\rho^\lambda} \int_{\Omega \cap B_\rho(x)} |f(y)|^p dy \right)^{\frac{1}{p}}$$

is finite, with obvious modifications if  $\Omega = \mathbb{R}^n$ .

The exponent  $\lambda$  can also take values not belonging to  $]0, n[$ , but the unique cases of real interest are the ones for which  $\lambda \in ]0, n[$ . In literature, there exist various extensions of the concept of Morrey spaces. In this paper we are concerned with some kind of anisotropic Morrey spaces, namely Morrey spaces with mixed norm.

**Definition 1.2 ([19]).** Let  $1 < p, q < +\infty$ ,  $0 < \lambda < n$ ,  $0 < \mu < 1$ . We define the Mixed Morrey space  $L^{q,\mu}(0, T, L^{p,\lambda}(\Omega))$  as the class of functions  $f$  such that

$$\|f\|_{L^{q,\mu}(0, T, L^{p,\lambda}(\Omega))} := \left( \sup_{\substack{t_0 \in (0, T), \\ \rho > 0}} \frac{1}{\rho^\mu} \int_{(0, T) \cap (t_0 - \rho, t_0 + \rho)} \left( \sup_{\substack{x \in \Omega, \\ \rho > 0}} \frac{1}{\rho^\lambda} \int_{\Omega \cap B_\rho(x)} |f(y, t)|^p dy \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}$$

is finite, with obvious modifications if  $\Omega = \mathbb{R}^n$ .

We remark that in the above definition the exponent  $\mu$  belongs to the interval  $]0, 1[$ , because  $t \in \mathbb{R}$  and the mixed norm is constructed by the iteration of the classic Morrey norm.

One of the advantages of Mixed Morrey spaces is that they allow us to treat separately time and space. This property could be useful in the study of evolution operators such as Kolmogorov operator and ultraparabolic equations treated in detail in [15–18].

On the other hand, the study of the boundedness of integral operators in suitable function spaces has an intrinsic interest in Harmonic Analysis. So, in this context, boundedness problems in Morrey-type spaces are actual and the Morrey spaces with mixed norm could be useful in the study of the boundedness of fractional integral operators with rough kernel and Schrödinger operators, extensively studied in the context of several generalized Morrey-type spaces in [5–7].

In [19] the author obtains regularity results for linear parabolic Partial Differential Equations (PDEs for short) with discontinuous coefficients by means of preliminary estimates of the following operator

$$I_\alpha f(x, t) = \frac{\Gamma(\frac{n-\alpha}{2})}{2^\alpha \pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})} \int_{\mathbb{R}^n} \frac{f(y, t)}{|x - y|^{n-\alpha}} dy, \quad \text{a. e. in } \mathbb{R}^n,$$

for which the author establishes the following theorem.

**Theorem 1.3 ([19]).** *Let  $0 < \alpha < n$ ,  $1 < p < n/\alpha$ ,  $0 < \lambda < n - \alpha p$ ,  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$ ,  $1 < q < +\infty$ ,  $0 < \mu' < 1$  and  $f \in L^{q', \mu'}(0, T, L^{p, \lambda}(\Omega))$ . Then*

$$\| I_\alpha f \|_{L^{q, \mu}(0, T, L^{q, \lambda}(\Omega))} \leq C \| f \|_{L^{q', \mu'}(0, T, L^{p, \lambda}(\Omega))} .$$

**Remark 1.4.** *We remark that, if  $\mathcal{L} = -\Delta$  is the Laplacian on  $\mathbb{R}^n$ , then the operator  $\mathcal{L}^{-\alpha/2}$  is the classical fractional integral operator  $I_\alpha$  (see, for instance, [21, 22]). For this reason, every result presented in this paper is an extension of the boundedness results contained in [19].*

Moreover, let  $b \in L^1_{\text{loc}}(\mathbb{R}^n \times (0, T))$  we define the commutator between  $b$  and  $\mathcal{L}^{-\alpha/2}$  as follows

$$[b, \mathcal{L}^{-\alpha/2}](f)(x, t) = b(x)\mathcal{L}^{-\alpha/2}(f)(x, t) - \mathcal{L}^{-\alpha/2}(bf)(x, t), \tag{3}$$

where for the sake of simplicity we assume that for  $b$  the time  $t$  is fixed.

In order to state some classical results and our main ones we need of the definition of the class of functions with bounded mean oscillation and the definition of homogeneous Lipschitz space.

**Definition 1.5 (John, Nirenberg [8]).** *Let  $b$  be a locally integrable function defined on  $\mathbb{R}^n$ . The function  $b$  is in the space  $BMO(\mathbb{R}^n)$  if the BMO norm*

$$\| b \|_* := \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |b(x) - b_B| dx$$

is finite, where  $B$  runs over the class of all balls in  $\mathbb{R}^n$  and  $b_B = \frac{1}{|B|} \int_B b(y) dy$ .

For this class of functions, we recall the following result (see [8]), which is going to play a role in the proof of one of the main results of this paper (see Theorem 1.9).

**Theorem 1.6.** *Assume that  $b \in BMO(\mathbb{R}^n)$ . Then, for any  $1 \leq p < \infty$ , we have that*

$$\sup_{B \subset \mathbb{R}^n} \left( \frac{1}{|B|} \int_B |b(x) - b_B|^p dx \right)^{\frac{1}{p}} \leq C \| b \|_* .$$

Useful in the sequel is the function space  $BMO_L$  that is a  $BMO$ -type space associated to an operator  $L$ . Let us assume that the kernel  $p_t(x, y)$  of  $\{e^{-tL}\}_{t>0}$  satisfies an upper bound of this type:

$$|p_t(x, y)| \leq t^{-\frac{n}{2}} g\left(\frac{|x - y|}{\sqrt{t}}\right),$$

for all  $x, y \in \mathbb{R}^n$  and all  $t > 0$ . Here we assume that  $g$  is a positive, bounded, decreasing function satisfying

$$\lim_{r \rightarrow \infty} r^{n+\varepsilon} g(r) = 0, \quad \text{for some } \varepsilon > 0. \tag{4}$$

Let  $\varepsilon$  be the constant in (4) and  $0 < \beta < \varepsilon$ . A function  $f \in L^p_{loc}(\mathbb{R}^n)$  is said to be of type  $(p, \beta)$  if it satisfies

$$\left( \int_{\mathbb{R}^n} \frac{|f(x)|^p}{(1 + |x|)^{n+\beta}} dx \right)^{\frac{1}{p}} \leq c < \infty. \tag{5}$$

We denote by  $\mathcal{M}_{(p,\beta)}$  the set of all functions of type  $(p, \beta)$ . If  $f \in \mathcal{M}_{(p,\beta)}$ , then we define the norm of  $f$  in  $\mathcal{M}_{(p,\beta)}$  as

$$\|f\|_{\mathcal{M}_{(p,\beta)}} = \inf\{c \geq 0 : (5) \text{ holds}\}.$$

We would like to point out that  $\mathcal{M}_{(p,\beta)}$  is a Banach space under the norm  $\|f\|_{\mathcal{M}_{(p,\beta)}}$ . We set

$$\mathcal{M}_p = \bigcup_{\beta: 0 < \beta < \varepsilon} \mathcal{M}_{(p,\beta)}.$$

For any  $f \in L^p(\mathbb{R}^n)$ ,  $1 \leq p < \infty$ , in [10] the author defined a kind of sharp maximal function  $M_L^\sharp f$  associated with the semigroup  $\{e^{-tL}\}_{t>0}$  by

$$M_L^\sharp f(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y) - e^{-t_B L} f(y)| dy,$$

where  $t_B = r_B^2$  and  $r_B$  is the radius of the ball  $B$ .

Let  $f \in \mathcal{M}_p$ , with  $1 < p < \infty$ . We say that  $f \in BMO_L$  if  $M_L^\sharp f \in L^\infty(\mathbb{R}^n)$  and we define

$$\|f\|_{BMO_L} = \|M_L^\sharp f\|_{L^\infty}.$$

For further details about  $BMO_L$  spaces we refer the reader to [4].

Moreover, we recall the following definition of homogeneous Lipschitz space.

**Definition 1.7.** Let  $0 < \beta \leq 1$ . The homogeneous Lipschitz space  $\dot{\Lambda}_\beta(\mathbb{R}^n)$  is the space of all locally integrable functions  $b$  such that

$$\|b\|_{\dot{\Lambda}_\beta} := \sup_{x, h \in \mathbb{R}^n, h \neq 0} \frac{|b(x+h) - b(x)|}{|h|^\beta}$$

is finite.

Estimates for the commutator (3) in the framework of Lebesgue spaces can be found in [2], where Auscher and Martell prove the following result.

**Theorem 1.8.** Let  $0 < \alpha < n$ ,  $1 < p < n/\alpha$  and  $1/q = 1/p - \alpha/n$  and  $\omega \in A_{p,q}$ . If  $b \in BMO(\mathbb{R}^n)$ , then the commutator  $[b, \mathcal{L}^{-\alpha/2}]$  is bounded from  $L^p(\omega^p)$  to  $L^q(\omega^q)$ .

Now, we state our main results regarding the boundedness of the commutator  $[b, \mathcal{L}^{-\alpha/2}]$  in the framework of mixed Morrey space, when  $b \in BMO(\mathbb{R}^n \times (0, T))$  or  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n \times (0, T))$ .

**Theorem 1.9.** *Let  $0 < \alpha < n$ ,  $1 < p < n/\alpha$  and  $1/q = 1/p - \alpha/n$ ,  $0 < \lambda/n < p/q$ ,  $0 < \mu < 1$ . Let us suppose  $b \in BMO(\mathbb{R}^n \times (0, T))$ . Then*

$$\| [b, \mathcal{L}^{-\alpha/2}] f \|_{L^{q,\mu} \left(0,T,L^{\frac{\lambda q}{np}}(\mathbb{R}^n)\right)} \leq C \| f \|_{L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))} .$$

**Theorem 1.10.** *Let  $0 < \beta < n$ ,  $0 < \alpha + \beta < n$ ,  $1 < p < n/(\alpha + \beta)$ ,  $1/s = 1/p - (\alpha + \beta)/n$ ,  $0 < \mu < 1$ ,  $1 < q < +\infty$ . Let us suppose  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n \times (0, T))$  and  $0 < \lambda/n < p/s$ . Then*

$$\| [b, \mathcal{L}^{-\alpha/2}] f \|_{L^{q,\mu} \left(0,T,L^{\frac{\lambda s}{np}}(\mathbb{R}^n)\right)} \leq C \| f \|_{L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))} .$$

**Theorem 1.11.** *Let  $0 < \beta < 1$ ,  $0 < \alpha + \beta < n$ ,  $1 < p < n/(\alpha + \beta)$ ,  $1/s = 1/p - (\alpha + \beta)/n$ ,  $0 < \mu < 1$ ,  $1 < q < +\infty$ . Let us suppose  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n \times (0, T))$ , then*

$$\sup_{\substack{t_0 \in (0,T), \\ \rho > 0}} \left( \frac{1}{\rho^\mu} \int_{(0,T) \cap (t_0-\rho, t_0+\rho)} \| [b, \mathcal{L}^{-\alpha/2}] f(x, t) \|_{BMO_L}^q dt \right) \leq C \| b \|_{\dot{\Lambda}_\beta} \| f \|_{L^{q,\mu} \left(0,T,L^{p,\frac{\rho}{s}}(\mathbb{R}^n)\right)} .$$

Throughout the paper,  $B = B(x_0, r_B)$  denotes the ball with the center  $x_0$  and radius  $r_B$ . Given a ball  $B$  and  $\lambda > 0$ ,  $\lambda B$  denotes the ball with the same center as  $B$  whose radius is  $\lambda$  times that of  $B$ . Moreover, we denote the Lebesgue measure of  $B$  by  $|B|$ . Also, we will use  $C$  to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. Furthermore, by  $A \sim B$ , we mean that there exists a constant  $C > 1$  such that

$$\frac{1}{C} \leq \frac{A}{B} \leq C$$

and we denote the conjugate exponent of  $q > 1$  by  $q' = \frac{q}{q-1}$ .

## 2. Proofs of the main results

*Proof.* [Proof of Theorem 1.9] Let us fix a ball  $B = B_{r_B}(x_0) \subseteq \mathbb{R}^n$  and write  $f = f_1 + f_2$ , where  $f_1 = f\chi_{2B}$ . We remark that  $\chi_{2B}$  denotes the characteristic function of  $2B$ . Since the commutator  $[b, \mathcal{L}^{-\alpha/2}]$  is a linear operator, we have that

$$\begin{aligned} & \frac{1}{|B|^{\frac{q\lambda}{np}}} \left( \int_B |[b, \mathcal{L}^{-\alpha/2}] f(x, t)|^q dx \right)^{\frac{1}{q}} \\ & \leq \frac{1}{|B|^{\frac{q\lambda}{np}}} \left( \int_B |[b, \mathcal{L}^{-\alpha/2}] f_1(x, t)|^q dx \right)^{\frac{1}{q}} + \frac{1}{|B|^{\frac{q\lambda}{np}}} \left( \int_B |[b, \mathcal{L}^{-\alpha/2}] f_2(x, t)|^q dx \right)^{\frac{1}{q}} \\ & \equiv I_1 + I_2 \end{aligned}$$

We start by estimating the term  $I_1$ . Thus, by applying Theorem 1.8 and considering that by definition  $f_1 = f\chi_{2B}$  we get the following inequality

$$I_1 \leq \frac{C}{|B|^{\frac{q\lambda}{np}}} \| b \|_* \left( \int_{2B} |f(x, t)|^p dx \right)^{\frac{1}{p}} \leq C \| b \|_* \| f \|_{L^{p,\frac{\lambda}{n}}(\mathbb{R}^n)} . \tag{6}$$

Now, we proceed by estimating the term  $I_2$ . Let us denote by  $K_\alpha(x, y)$  the kernel of  $\mathcal{L}^{-\alpha/2}$ . Then, for any  $x \in B, t \in (0, T)$ , we can write

$$\begin{aligned} |[b, \mathcal{L}^{-\alpha/2}] f_2(x, t)| &\leq |b(x) - b_B| \int_{(2B)^c} |K_\alpha(x, y)| |f(y, t)| dy + \\ &\quad + \int_{(2B)^c} |b(x) - b_B| |K_\alpha(x, y)| |f(y, t)| dy \\ &\equiv I + II. \end{aligned}$$

Since the kernel of  $e^{-tL}$  is  $p_t(x, y)$ , then it follows from (2) that (see [11])

$$K_\alpha(x, y) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty p_t(x, y) t^{\frac{\alpha}{2}-1} dt. \tag{7}$$

Thus, by consider the Gaussian upper bound (1) and the expression (7), we can deduce (see [3, 11])

$$|K_\alpha(x, y)| \leq C \int_0^\infty e^{-A \frac{|x-y|^2}{t}} t^{\frac{\alpha}{2}-\frac{n}{2}-1} dt \leq \frac{C}{|x-y|^{n-\alpha}}. \tag{8}$$

Thus, we get

$$I \leq |b(x) - b_B| \sum_{k=1}^\infty \frac{1}{|2^{k+1}B|^{1-\frac{\alpha}{n}}} \int_{2^{k+1}B} |f(y, t)| dy.$$

By using Hölder’s inequality, we get

$$\begin{aligned} \int_{2^{k+1}B} |f(y, t)| dy &\leq \left( \int_{2^{k+1}B} |f(y, t)|^p dy \right)^{\frac{1}{p}} \left( \int_{2^{k+1}B} 1 dy \right)^{\frac{1}{p'}} \\ &\leq C \|f\|_{L^{p, \frac{\lambda}{n}}(\mathbb{R}^n)} \frac{|2^{k+1}B|^{1-\frac{\alpha}{n}}}{|2^{k+1}B|^{\frac{1}{q}-\frac{\lambda}{np}}}. \end{aligned} \tag{9}$$

Hence, by applying Theorem 1.6, we get

$$\begin{aligned} \frac{1}{|B|^{\frac{\lambda}{np}}} \left( \int_B I^q dx \right)^{\frac{1}{q}} &\leq C \|f\|_{L^{p, \frac{\lambda}{np}}(\mathbb{R}^n)} \sum_{k=1}^\infty \frac{|B|^{q(\frac{1}{q}-\frac{\lambda}{np})}}{|2^{k+1}B|^{q(\frac{1}{q}-\frac{\lambda}{np})}} \left( \frac{1}{|B|^q} \int_B |b(x) - b_B|^q dx \right)^{\frac{1}{q}} \\ &\leq C \|f\|_{L^{p, \frac{\lambda}{np}}(\mathbb{R}^n)} \|b\|_* \sum_{k=1}^\infty \left( \frac{1}{2^{(k+1)n}} \right)^{q(\frac{1}{q}-\frac{\lambda}{np})} \\ &\leq C \|f\|_{L^{p, \frac{\lambda}{np}}(\mathbb{R}^n)} \|b\|_* \end{aligned} \tag{10}$$

where the last series is convergent since  $q(\frac{1}{q} - \frac{\lambda}{np}) > 0$ .

On the other hand

$$\begin{aligned}
 II &\leq \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |b(y) - b_B| |K_\alpha(x, y)| |f(y, t)| dy \\
 &\leq \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |b(y) - b_{2^{k+1}B}| |K_\alpha(x, y)| |f(y, t)| dy + \\
 &\quad + \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |b_{2^{k+1}B} - b_B| |K_\alpha(x, y)| |f(y, t)| dy \equiv III + IV.
 \end{aligned}$$

In order to estimate terms III and IV, we observe that

$$|y - x| \sim |y - x_0| \quad \text{for } x \in B, y \in (2B)^c.$$

Hence, from (8) it follows that

$$III \leq C \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|^{1-\frac{\alpha}{n}}} \int_{2^{k+1}B \setminus 2^k B} |b(y) - b_{2^{k+1}B}| |f(y, t)| dy.$$

By Hölder’s inequality it follows that

$$\begin{aligned}
 \int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}| |f(y, t)| dy &\leq \left( \int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}|^{p'} dy \right)^{\frac{1}{p'}} \left( \int_{2^{k+1}B} |f(y, t)|^p dy \right)^{\frac{1}{p}} \\
 &\leq \|f\|_{L^{p, \frac{\lambda}{n}}(\mathbb{R}^n)} |2^{k+1}B|^{\frac{q\lambda}{np}} \left( \int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}|^{p'} dy \right)^{\frac{1}{p'}}
 \end{aligned}$$

From Theorem 1.6 we obtain

$$\left( \int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}|^{p'} dy \right)^{\frac{1}{p'}} \leq C \|b\|_* \frac{|2^{k+1}B|^{\frac{1}{q} + \frac{1}{p'}}}{|2^{k+1}B|^{\frac{q}{q}}}. \tag{11}$$

Note that  $\frac{1}{q} + \frac{1}{p'} = 1 - \frac{\alpha}{n}$ . Hence, considering (11) we have

$$\begin{aligned}
 \frac{1}{|B|^{\frac{q\lambda}{np}}} \left( \int_B III^q dx \right)^{\frac{1}{q}} &\leq C \|b\|_* \|f\|_{L^{p, \frac{\lambda}{n}}(\mathbb{R}^n)} \sum_{k=1}^{\infty} \frac{|B|^{q(\frac{1}{q} - \frac{\lambda}{np})}}{|2^{k+1}B|^{q(\frac{1}{q} - \frac{\lambda}{np})}} \\
 &\leq C \|b\|_* \|f\|_{L^{p, \frac{\lambda}{n}}(\mathbb{R}^n)}. \tag{12}
 \end{aligned}$$

Since  $b \in BMO(\mathbb{R}^n)$ , a simple calculation gives

$$|b_{2^{k+1}B} - b_B| \leq CK \|b\|_*.$$

Thus, by (8) and (9), we get

$$\begin{aligned}
 IV &\leq C \|b\|_* \sum_{k=1}^{\infty} \frac{K}{|2^{k+1}B|^{1-\frac{\alpha}{n}}} \int_{2^{k+1}B} |f(y, t)| dy \\
 &\leq C \|b\|_* \|f\|_{L^{p, \frac{\lambda}{n}}(\mathbb{R}^n)} \sum_{k=1}^{\infty} \frac{K}{|2^{k+1}B|^{q(\frac{1}{q} - \frac{\lambda}{np})}}.
 \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{|B|^{\frac{q\lambda}{np}}} \left( \int_B IV^q dx \right)^{\frac{1}{q}} &\leq C \|b\|_* \|f\|_{L^{p, \frac{\lambda}{n}}(\mathbb{R}^n)} \sum_{k=1}^{\infty} \frac{|B|^{q(\frac{1}{q} - \frac{\lambda}{np})}}{|2^{k+1}B|^{q(\frac{1}{q} - \frac{\lambda}{np})}} \\ &\leq C \|b\|_* \|f\|_{L^{p, \frac{\lambda}{n}}(\mathbb{R}^n)}. \end{aligned} \tag{13}$$

Summating (12) and (13), we thus obtain

$$\frac{1}{|B|^{\frac{q\lambda}{np}}} \left( \int_B II^q dx \right)^{\frac{1}{q}} \leq C \|b\|_* \|f\|_{L^{p, \frac{\lambda}{n}}(\mathbb{R}^n)}. \tag{14}$$

By combining inequalities (6) and (10) with the above inequality (14) and taking the supremum over all balls  $B \subseteq \mathbb{R}^n$ , we complete the proof of the classical Morrey estimate obtaining the following classical Morrey estimate

$$\left( \sup_{x \in \mathbb{R}^n, \rho > 0} \frac{1}{\rho^{\frac{\lambda q}{np}}} \int_{B_\rho(x)} |[b, \mathcal{L}^{-\alpha/2}]f(y, t)|^q dy \right)^{\frac{1}{q}} \leq C \left( \sup_{x \in \mathbb{R}^n, \rho > 0} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} |f(y, t)|^p dy \right)^{\frac{1}{p}}.$$

Now we deal with the mixed norm estimate. Let us elevate to  $q$  and integrate over  $(0, T) \cap (t_0 - \rho, t_0 + \rho)$  the previous inequality, it follows that

$$\begin{aligned} \int_{(0, T) \cap (t_0 - \rho, t_0 + \rho)} \left[ \left( \sup_{x \in \mathbb{R}^n, \rho > 0} \frac{1}{\rho^{\frac{\lambda q}{np}}} \int_{B_\rho(x)} |[b, \mathcal{L}^{-\alpha/2}]f(y, t)|^q dy \right)^{\frac{1}{q}} \right]^q dt \\ \leq C \int_{(0, T) \cap (t_0 - \rho, t_0 + \rho)} \left( \sup_{x \in \mathbb{R}^n, \rho > 0} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} |f(y, t)|^p dy \right)^{\frac{q}{p}} dt. \end{aligned}$$

Multiplying the above inequality, elevating to  $\frac{1}{q}$  and finally taking the suprema of both sides, we obtain

$$\begin{aligned} \sup_{t_0 \in (0, T), \rho > 0} \left( \frac{1}{\rho^\mu} \int_{(0, T) \cap (t_0 - \rho, t_0 + \rho)} \left[ \left( \sup_{x \in \mathbb{R}^n, \rho > 0} \frac{1}{\rho^{\frac{\lambda q}{np}}} \int_{B_\rho(x)} |[b, \mathcal{L}^{-\alpha/2}]f(y, t)|^q dy \right)^{\frac{1}{q}} \right]^q dt \right)^{\frac{1}{q}} \leq \\ C \sup_{t_0 \in (0, T), \rho > 0} \left( \frac{1}{\rho^\mu} \int_{(0, T) \cap (t_0 - \rho, t_0 + \rho)} \left( \sup_{x \in \mathbb{R}^n, \rho > 0} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} |f(y, t)|^p dy \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}. \end{aligned}$$

Thus, the proof is complete.  $\square$

*Proof.* [Proof of Theorem 1.10] From (8) we have the following pointwise inequality

$$|\mathcal{L}^{-\alpha/2}(f)(x, t)| \leq CI_\alpha(|f|)(x, t) \quad \text{for every } x \in \mathbb{R}^n, t \in (0, T).$$



Furthermore, by the definition of  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$  and (8), we deduce that

$$\begin{aligned}
 |[b, \mathcal{L}^{-\alpha/2}](f)(x, t)| &\leq \int_{\mathbb{R}^n} |b(x) - b(y)| |K_\alpha(x, y)| |f(y, t)| dy \\
 &\leq C \|b\|_{\dot{\Lambda}_\beta} \int_{\mathbb{R}^n} \frac{|f(y, t)|}{|x - y|^{n-\alpha-\beta}} dy \\
 &\leq C \|b\|_{\dot{\Lambda}_\beta} I_{\alpha+\beta}(f)(x).
 \end{aligned}
 \tag{15}$$

Hence we obtain the desired result applying the results contained in [9].  $\square$

In order to prove Theorem 1.11, we need the following technical Lemma (see, for instance, [3]).

**Lemma 2.1.** For  $0 < \alpha < n$ , the difference operator  $(I - e^{-t\mathcal{L}})\mathcal{L}^{-\alpha/2}$  has an associated kernel  $\widetilde{K}_{\alpha,\xi}(y, z)$  which satisfies the following estimate

$$|\widetilde{K}_{\alpha,\xi}(y, z)| \leq \frac{C}{|y - z|^{n-\alpha}} \frac{t}{|y - z|^2}.
 \tag{16}$$

*Proof.* [Proof of Theorem 1.11]

For any given  $x \in \mathbb{R}^n$ , let us fix a ball  $B = B_{r_0}(x_0) \subseteq \mathbb{R}^n$  containing  $x$ . We write  $f = f_1 + f_2$ , where  $f_1 = f\chi_{2B}$  and we set  $t_B = r_B^2$ . Then

$$\begin{aligned}
 \frac{1}{|B|} \int_B |[b, \mathcal{L}^{-\alpha/2}]f(y, t) - e^{-t_B\mathcal{L}}[b, \mathcal{L}^{-\alpha/2}]f(y, t)| dy &\leq \\
 \frac{1}{|B|} \int_B |[b, \mathcal{L}^{-\alpha/2}]f_1(y, t)| dy + \frac{1}{|B|} \int_B |e^{-t_B\mathcal{L}}[b, \mathcal{L}^{-\alpha/2}]f_1(y, t)| dy + \\
 + \frac{1}{|B|} \int_B |[b, \mathcal{L}^{-\alpha/2}]f_2(y, t) - e^{-t_B\mathcal{L}}[b, \mathcal{L}^{-\alpha/2}]f_2(y, t)| dy &\equiv J_1 + J_2 + J_3.
 \end{aligned}$$

Our aim is to estimate each term  $J_i$ , with  $i = 1, 2, 3$ . It is known that  $I_{\alpha+\beta}$  is bounded from  $L^p(\mathbb{R}^n)$  to  $L^s(\mathbb{R}^n)$  (see [13]). Let us consider the first term  $J_1$ . Applying Hölder’s inequality and considering inequality (15), we get

$$\begin{aligned}
 J_1 &\leq \frac{1}{|B|} \left( \int_B |[b, \mathcal{L}^{-\alpha/2}]f_1(y, t)|^s dy \right)^{\frac{1}{s}} \left( \int_B 1 dy \right)^{\frac{1}{s'}} \\
 &\leq C \|b\|_* \frac{1}{|B|} \left( \int_{2B} |f(y, t)|^p dy \right)^{\frac{1}{p}} \left( \int_B 1 dy \right)^{\frac{1}{s'}} \\
 &\leq C \|b\|_* \|f\|_{L^{p, \frac{p}{s}}(\mathbb{R}^n)} \frac{|2B|^{\frac{s\alpha}{pm}}}{|B|^{\frac{s}{s'}}} \\
 &\leq C \|b\|_* \|f\|_{L^{p, \frac{p}{s}}(\mathbb{R}^n)}.
 \end{aligned}$$

As far as we are concerned with the term  $J_2$ , since the kernel of  $e^{-t_B \mathcal{L}}$  is  $p_{t_B}(y, z)$ , we can write

$$\begin{aligned} J_2 &\leq \frac{1}{|B|} \int_B \int_{\mathbb{R}^n} |p_{t_B}(y, z)| |[b, \mathcal{L}^{-\alpha/2}] f_1(z, t)| dz dy \\ &\leq \frac{1}{|B|} \int_B \int_{2B} |p_{t_B}(y, z)| |[b, \mathcal{L}^{-\alpha/2}] f_1(z, t)| dz dy + \\ &\quad + \sum_{k=1}^{\infty} \frac{1}{|B|} \int_B \int_{2^{k+1}B \setminus 2^k B} |p_{t_B}(y, z)| |[b, \mathcal{L}^{-\alpha/2}] f_1(z, t)| dz dy \equiv J'_2 + J''_2. \end{aligned}$$

For any  $y \in B$  and  $z \in 2B$ , by (1) we have

$$|p_{t_B}(y, z)| \leq C(t_B)^{-n/2}.$$

Thus

$$J'_2 \leq C \frac{1}{|B|} \int_B \int_{2B} \frac{1}{(t_B)^{n/2}} |[b, \mathcal{L}^{-\alpha/2}] f_1(z, t)| dz dy \leq C \frac{1}{|2B|} \int_{2B} |[b, \mathcal{L}^{-\alpha/2}] f_1(z, t)| dz.$$

By applying the same arguments as in the estimate of the term  $J_1$ , we can also deduce

$$J'_2 \leq C \|b\|_* \|f\|_{L^{p, \frac{p}{5}}(\mathbb{R}^n)}.$$

As we have already done before, we remark that for any  $y \in B, z \in (2B)^c$  we have that

$$|z - y| \sim |z - z_0|.$$

Thus, by applying one more time (1) we get

$$|p_{t_B}(y, z)| \leq C \frac{(t_B)^{n/2}}{|y - z|^{2n}}.$$

Hence

$$\begin{aligned} J''_2 &\leq C \sum_{k=1}^{\infty} \frac{1}{|B|} \int_B \int_{2^{k+1}B \setminus 2^k B} \frac{(t_B)^{n/2}}{|y - z|^{2n}} |[b, \mathcal{L}^{-\alpha/2}] f_1(z, t)| dz dy \\ &\leq C \sum_{k=1}^{\infty} \frac{1}{2^{kn}} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |[b, \mathcal{L}^{-\alpha/2}] f_1(z, t)| dz. \end{aligned}$$

Following the same lines as for the estimate of the term  $J'_2$ , we can also show that

$$\begin{aligned} \int_{2^{k+1}B} |[b, \mathcal{L}^{-\alpha/2}] f_1(z, t)| dz &\leq C \left( \int_{2B} |f(z, t)|^p dz \right)^{\frac{1}{p}} \left( \int_{2^{k+1}B} 1 dz \right)^{\frac{1}{p'}} \\ &\leq C \|b\|_* \|f\|_{L^{p, \frac{p}{5}}(\mathbb{R}^n)} \frac{|2B|^{\frac{5}{p}}}{|2^{k+1}B|^{\frac{5}{p}}} |2^{k+1}B|. \end{aligned}$$

Consequently, we have that

$$\begin{aligned} J_2'' &\leq C \|b\|_* \|f\|_{L^{p, \frac{p}{s}}(\mathbb{R}^n)} \sum_{k=1}^{\infty} \frac{1}{2^{kn}} \left( \frac{|2B|^s}{|2^{k+1}B|^s} \right)^{\frac{1}{s}} \\ &\leq C \|b\|_* \|f\|_{L^{p, \frac{p}{s}}(\mathbb{R}^n)} \sum_{k=1}^{\infty} \frac{1}{2^{kn}} \frac{|2B|}{|2^{k+1}B|} \\ &\leq C \|b\|_* \|f\|_{L^{p, \frac{p}{s}}(\mathbb{R}^n)} \sum_{k=1}^{\infty} \frac{1}{2^{kn}} \frac{1}{2^{kn}} \\ &\leq C \|b\|_* \|f\|_{L^{p, \frac{p}{s}}(\mathbb{R}^n)}. \end{aligned}$$

Hence, by considering both equation (16) and the definition of  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ , we have

$$\begin{aligned} J_3 &= \frac{1}{|B|} \int_B |(I - e^{-tB \cdot \mathcal{L}}) \mathcal{L}^{-\alpha/2} ([b(y) - b(\cdot)] f_2)(y, t)| dy \\ &\leq \frac{1}{|B|} \int_B \int_{(2B)^c} |\tilde{K}_{\alpha, \xi}(y, z)| |b(y) - b(z)| |f(z, t)| dz \\ &\leq C \|b\|_{\dot{\Lambda}_\beta} \frac{1}{|B|} \int_B \int_{(2B)^c} \frac{1}{|y - z|^{n-\alpha-\beta}} \frac{r_B^2}{|y - z|^2} |f(z, t)| dz dy \\ &\leq C \|b\|_{\dot{\Lambda}_\beta} \sum_{k=1}^{\infty} \frac{1}{2^{kn}} \frac{1}{|2^{k+1}B|^{1-\frac{\alpha+\beta}{n}}} \int_{2^{k+1}B} |f(z, t)| dz. \end{aligned}$$

Moreover, we have that

$$\int_{2^{k+1}B} |f(z, t)| dz \leq C \|f\|_{L^{p, \frac{p}{s}}(\mathbb{R}^n)} |2^{k+1}B|^{1-\frac{\alpha+\beta}{n}}.$$

Therefore,

$$J_3 \leq C \|b\|_{\dot{\Lambda}_\beta} \|f\|_{L^{p, \frac{p}{s}}(\mathbb{R}^n)} \sum_{k=1}^{\infty} \frac{1}{2^{2k}} \leq C \|b\|_{\dot{\Lambda}_\beta} \|f\|_{L^{p, \frac{p}{s}}(\mathbb{R}^n)}.$$

Combining the above estimates we've obtained for the terms  $J_1, J_2, J_3$  and taking the supremum over all balls  $B \subseteq \mathbb{R}^n$  we finally obtain the following estimate:

$$\| [b, \mathcal{L}^{-\alpha/2}] f(x, t) \|_{BMO_L} \leq C \|b\|_{\dot{\Lambda}_\beta} \|f\|_{L^{p, \frac{p}{s}}(\mathbb{R}^n)} = C \|b\|_{\dot{\Lambda}_\beta} \left( \sup_{\substack{x \in \mathbb{R}^n, \\ \rho > 0}} \frac{1}{\rho^{\frac{p}{s}}} \int_{B_\rho(x)} |f(y, t)|^p dy \right)^{\frac{1}{p}}.$$

Elevating to  $q$ , integrating on  $(0, T) \cap (t_0 - \rho, t_0 + \rho)$ , it follows that

$$\begin{aligned} &\int_{(0, T) \cap (t_0 - \rho, t_0 + \rho)} \| [b, \mathcal{L}^{-\alpha/2}] f(x, t) \|_{BMO_L}^q dt \leq \\ &C \|b\|_{\dot{\Lambda}_\beta} \int_{(0, T) \cap (t_0 - \rho, t_0 + \rho)} \left( \sup_{\substack{x \in \mathbb{R}^n, \\ \rho > 0}} \frac{1}{\rho^{\frac{p}{s}}} \int_{B_\rho(x)} |f(y, t)|^p dy \right)^{\frac{q}{p}} dt. \end{aligned}$$

Multiplying the above inequality by  $\frac{1}{\rho^\mu}$ , taking the supremum of both sides and, finally, elevating to  $\frac{1}{q}$ , we obtain

$$\sup_{\substack{t_0 \in (0, T), \\ \rho > 0}} \left( \frac{1}{\rho^\mu} \int_{(0, T) \cap (t_0 - \rho, t_0 + \rho)} \| [b, \mathcal{L}^{-\alpha/2}] f(x, t) \|_{BMO_L}^q dt \right)^{\frac{1}{q}} \leq C \| b \|_{\dot{\Lambda}_\beta} \sup_{\substack{t_0 \in (0, T), \\ \rho > 0}} \left( \frac{1}{\rho^\mu} \int_{(0, T) \cap (t_0 - \rho, t_0 + \rho)} \left( \sup_{\substack{x \in \mathbb{R}^n, \\ \rho > 0}} \frac{1}{\rho^{\frac{p}{s}}} \int_{B_\rho(x)} |f(y, t)|^p dy \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}$$

and the proof is complete.  $\square$

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