Transition semigroups associated to nonlinear stochastic equations

Università di Modena e Reggio Emilia
Dipartimento di Scienze Fisiche, Informatiche e Matematiche
Dottorato di ricerca in Matematica e Informatica
In convenzione con
l'Università degli Studi di Ferrara e l'Università degli Studi di Parma
XXXIV Ciclo

Relatore:
Prof. Alessandra Lunardi

Tesi di:
Davide Augusto Bignamini
Matricola n. 134075

Coordinatore del Dottorato:
Prof. Cristian Giardinà

Riassunto

L'argomento principale di questa tesi è lo studio di semigruppi di transizione di una classe di equazioni differenziali stocastiche non lineari in spazi di Hilbert separabili e di dimensione infinita. Più precisamente consideriamo semigruppi di transizione associati alla soluzione mild generalizzata di equazioni di Kolmogorov stocastiche con dato inziale in uno spazio di Hilbert $\mathcal{X}$ separabile di dimensione infinita e con drift perturbato da una funzione non lineare definita su un sottoinsieme di $\mathcal{X}$. La teoria dei semigruppi di transizione associati a queste equazioni differenziali stocastiche si è sviluppata a partire dagli anni ottanta e le sue basi sono esposte in tre libri di G. Da Prato e J. Zabczyk.

Nel primo capitolo di questa tesi richiamiamo preliminari di analisi funzionale, analisi infinito dimensionale, probabilità, teoria dei semigruppi, processi Markoviani di Wiener e di Ornstein-Uhlenbeck necessari per definire il contesto in cui lavoreremo. Nel capitolo due definiamo l'equazione differenziale stocastica e il semigruppo di transizione associato, che sono i due principali protagonisti di questa tesi. In condizioni molto generali, studiamo l'esistenza e l'unicità della soluzione mild generalizzata dell'equazione differenziale stocastica. Nel capitolo tre trattiamo alcune proprietà di regolarizzazione del semigruppo. Nel capitolo quattro dimostriamo una disuguaglianza logaritmica di Harnack e alcune sue conseguenze. Nel capitolo cinque mostriamo l'esistenza e l'unicità di una misura invariante di probabilità ν per il semigruppo di transizione. Inoltre dimostriamo che il semigruppo di transizione è unicamente estendibile ad un semigruppo fortemente continuo nello spazio $L^2(\mathcal{X},\nu)$, e che il suo generatore infinitesimale N_2 è la chiusura di un operatore di tipo Ornstein-Uhlenbeck perturbato. Nel capitolo sei studiamo la regolarità di Sobolev del dominio di N_2, dimostriamo alcune disuguaglianze di Sobolev logaritmiche e di Poincaré e un risultato di ipercontrattività per il semigruppo. Nel capitolo sette consideriamo sia problemi stazionari che di evoluzione in un aperto $\mathcal{O}$ di $\mathcal{X}$, definendo il corrispondente semigruppo arrestato o semigruppo di Dirichlet, di cui studiamo il generatore infinitesimale nello spazio $L^2(\mathcal{O},\nu)$.

Abstract

The main topic of this thesis is the study of transition semigroups of a class of nonlinear stochastic evolutin equations in an infinite dimensional separable Hilbert space. More precisely, we consider transition semigroups associated to the generalized mild solutions of stochastic Kolmogorov equations with initial data in an infinite dimensional separable Hilbert space $\mathcal{X}$ and with the drift perturbed by a nonlinear function defined on a subset of $\mathcal{X}$. The theory of transition semigroups associated to such stochastic differential equations was developed starting from 1980s, an account of this theory is presented in three books by G. Da Prato and J. Zabczyk. In the first chapter of this thesis we recall some preliminaries about functional analysis, infinite dimensional analysis, probability, semigroup theory, Wiener, Ornstein-Uhlenbeck and Markovian processes necessary to define the framework in which we work. In chapter two we define the stochastic differential equation and its transition semigroup, which are the main objects studied in this thesis. Under rather general conditions, we study existence and uniqueness of the generalized mild solution of the stochastic differential equation. In chapter three we discuss some smoothing properties of the semigroup. In chapter four we prove a logarithmic Harnack inequality and some of its consequences. In chapter five we show existence and uniqueness of a probability invariant measure ν for the transition semigroup. We also show that the transition semigroup is uniquely extendable to a strongly continuous semigroup in the space $L^2(\mathcal{X}, \nu)$, whose infinitesimal generator N_2 is the closure, in this space, of a perturbed Ornstein-Uhlenbeck type operator. In chapter six we study the Sobolev regularity of the domain of N_2, we prove some logarithmic Sobolev and Poincaré inequalities, and a hypercontractivity result for the transition semigroup. In chapter seven we consider stationary and evolution equations in an open set $\mathcal{O}$ of $\mathcal{X}$, defining the stopped semigroup or Dirichlet semigroup associated to it, and studying the infinitesimal generator of such semigroup in the space $L^2(\mathcal{O}, \nu)$.

Contents

Introduction

This thesis is devoted to Markov transition semigroups associated to stochastic evolution equations on infinite dimensional Hilbert spaces.

The history of Markov transition semigroups began in the early twentieth century when, on one side, M. Smoluchowski, A. Einstein and P. Langevin studied the Brownian particle velocity and, on the other side, A. A. Markov attempted to describe mathematically the phenomenon of the Brownian motion. Markov focused on a property of Brownian motion that informally can be described as follows: the past and the future are conditionally independent given present, for every possible value of the present. Einstein and Langevin used two different approaches to study the Brownian particle velocity. Einstein started from a Fokker–Planck equation describing the time evolution of the probability density function of the position of a particle. Langevin had the idea to describe the velocity of a Brownian particle by a process which is a solution of the first prototype of a stochastic equation. In the 1930's this process was studied in detail by the two physicists from whom it took its name: L. Ornstein and G. E. Uhlenbeck.

Einstein and Langevin had not the mathematical theory developed years later by K. Itô and A. N. Kolmogorov, however they were able to describe the Brownian motion law and to conclude that its trajectories are not functions of bounded variation. In 1923 N. Wiener gave the first correct mathematical construction of the Brownian motion which was also the first construction of a Markov process with continuous trajectories.

Markov processes and stochastic differential equations theories were developed between the 1930s and 1950s by Doeblin, Doob, Feller, Itô, Lévy, Kolmogorov and many other ones. These theories are the connection between Einstein's and Langevin's approaches and, more in generally, between stochastic equations and parabolic partial differential equations.

In the 1960's the first studies about infinite dimensional stochastic equations began. L. Gross , Yu. L. Daleckĭi and P. Malliavin introduced Hilbert space valued Wiener and Ornstein–Uhlenbeck processes and investigated some classes of deterministic parabolic equations for functions of infinitely many variables.

We now present a modern mathematical formalization of the problems discussed above.

Let $\mathcal{X}$ be a separable Hilbert space equipped with the norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $A : \mathrm{Dom}(A) \subseteq \mathcal{X} \to \mathcal{X}$ be the infinitesimal generator of a strongly continuous semigroup e^{tA} (see Subsection 1.4.1) and let $R : \mathcal{X} \to \mathcal{X}$ be a bounded linear operator. Let $\{W(t)\}_{t\geq 0}$ be a $\mathcal{X}$-cylindrical Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (see Subsection 1.7.3). We consider the following infinite dimensional version of the Langevin equation

$$\begin{cases} dX(t,x) = AX(t,x)dt + RdW(t), & t > 0; \\ X(0,x) = x. \end{cases} \tag{1}$$

Under suitable hypotheses on A and R, for any $x \in \mathcal{X}$, (1) has unique mild solution, namely a process $\{X(t,x)\}_{t\geq 0}$ such that for any $t \geq 0$

$$X(t,x) = e^{tA}x + W_A(t), \quad W_A(t) := \left(\int_0^t e^{(t-s)A} RdW(s)\right), \quad \mathbb{P}\text{-a.s.}, \tag{2}$$

see Section 1.8. $\{X(t,x)\}_{t\geq 0}$ is a $\mathcal{X}$ valued Ornstein-Uhlenbeck process.

Let $f : \mathcal{X} \to \mathcal{X}$ be a smooth enough function, we consider the function $u : [0,+\infty) \times \mathcal{X} \to \mathcal{X}$ defined by

$$u(t,x) := \mathbb{E}[f(X(t,x))] := \int_\Omega f(X(t,x)(\omega))\mathbb{P}(d\omega), \quad x \in \mathcal{X},\ t \geq 0.$$

Under suitable hypotheses (e.g. [42, Chapter 6] and [43, Section 9.3]), u solves the following infinite dimensional version of the Fokker-Planck equation considered by Einstein

$$\begin{cases} \dfrac{du}{dt}(t,x) = \frac{1}{2}\mathrm{Tr}[R^2\nabla^2 u(t,x)] + \langle Ax, \nabla u(t,x)\rangle, & x \in \mathrm{Dom}(A),\ t \geq 0, \\ u(0,x) = f(x), \end{cases} \tag{3}$$

where $\nabla^2 u(t,x)$ and $\nabla u(t,x)$ are the Fréchet Hessian and Fréchet Gradient of the function $u(t,\cdot) : \mathcal{X} \to \mathcal{X}$.

For $f \in B_b(\mathcal{X})$ (the space of bounded and Borel measurable functions from $\mathcal{X}$ to $\mathbb{R}$), the family of operators defined by

$$(T(t)f)(x) := u(t,x) = \mathbb{E}[f(X(t,x))], \quad x \in \mathcal{X},\ t \geq 0, \tag{4}$$

is a semigroup and it is called Markov transition semigroup associated to (1) (see Section 1.9), it is known as the Ornstein-Uhlenbeck semigroup.

It is possible to prove that, for any $t \geq 0$, the law μ_t of $W_A(t)$ is a Gaussian measure of mean 0 and covariance operator

$$Q_t := \int_0^t e^{sA}RR^* e^{sA^*} ds, \tag{5}$$

see Subsection 1.6.2 for the definition of Gaussian measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. Via change of variables, one sees that the Ornstein-Uhlenbeck semigroup (4) has the Mehler representation

$$(T(t)f)(x) := \int_{\mathcal{X}} f(e^{tA}x + y)\mu_t(dy), \quad x \in \mathcal{X},\ t \geq 0. \tag{6}$$

We note that, if $\mathcal{X} = \mathbb{R}^n$ with $n \in \mathbb{N}$, $A \equiv 0$ and $R = \sqrt{2}\mathrm{I}_{\mathcal{X}}$ in (5) and (6), then $T(t)$ is the heat semigroup and (3) reads as

$$\begin{cases} \dfrac{du}{dt}(t,x) = \Delta u(t,x), \quad x \in \mathbb{R}^n,\ t \geq 0, \\ u(0,x) = \varphi(x), \quad x \in \mathbb{R}^n, \end{cases}$$

where $\Delta u(t,x)$ is the Laplacian of the function $u(t,\cdot) : \mathcal{X} \to \mathcal{X}$.

In this thesis we will study the transition semigroup of a nonlinear version of the Langevin equation (1). More precisely let $F : \mathrm{Dom}(F) \subseteq \mathcal{X} \to \mathcal{X}$ be a smooth enough function. We will consider the transition semigroup $P(t)$ associated to the nonlinear stochastic differential equation

$$\begin{cases} dX(t,x) = (AX(t,x) + F(X(t,x)))dt + RdW(t), \quad t > 0; \\ X(0,x) = x. \end{cases} \tag{7}$$

Due to the nonlinearity of (7), $P(t)$ does not have a Mehler formula similar to (6).

The study of stochastic partial differential equations (SPDEs) was devoloped from 1960s (see [26]). The stochastic versions of many classic PDEs that had been studied are: reaction–diffusion, wave, beam, Burgers, Musiela, Navier–Stokes, Kardar–Parisi–Zhang, Kuramoto–Sivashinsky, Cahn–Hilliard, Landau–Lifshitz–Gilbert, etc. These SPDEs were studied with different approaches: the semigroups approach (Da Prato and Zabczyk, see [43]), the variational approach (Pardoux, Krylov and Rozovskii, see [71]) and the random field approach (Walsh, see [98]). In my thesis I study some stochastic reaction–diffusion type equations in a separable Hilbert space and theirs corresponding transition semigroups, using the Da Prato and Zabczyk approach presented in the books [20, 41, 42, 43].

Now we describe in detail the contents of the thesis. Except for *Chapter 1.*, the final section of all other chapters is devoted to bibliographic comments and examples.

Chapter 1. This chapter is devoted to the preliminary results that we will use in the next chapters. In Section 1.3 we recall some basic definitions and results about dissipative mappings. Section 1.4 is devoted to the semigroups theory, in Section 1.5 we state some basic definitions and results about spectral theory of compact operators. Sections 1.6 and 1.7 are devoted to Gaussian measures and Wiener processes in Hilbert spaces, respectively. In Section 1.8 we define an integration with respect to a Wiener process and we state some properties of the stochastic convolution process $\{W_A(t)\}_{t\geq 0}$ defined in (2). Section 1.9 is devoted to the theory of Markov processes and in Section 1.10 we recall some results about the Ornstein-Uhlenbeck semigroup given by (6). Finally in Section 1.11 we present a regularizing sequence for dissipative functions from X into itself.

Chapter 2. In this chapter we study the solution of (7) via the approach introduced in [20, Chapters 6 and 7] and [43, Section 7.2]. In particular we investigate the case where $F : \mathrm{Dom}(F) \subseteq \mathcal{X} \to \mathcal{X}$ and $A : \mathrm{Dom}(A) \subseteq \mathcal{X} \to \mathcal{X}$ satisfy some dissipativity conditions. This framework covers a large class of reaction diffusion systems (see [20, Chapters 6 and 7]). If $\mathrm{Dom}(F) = \mathcal{X}$, for any $x \in \mathcal{X}$ it is possible to consider the mild solution of (7), namely a process $\{X(t,x)\}_{t\geq 0}$ that satisfies

$$X(t,x) = e^{tA}x + \int_0^t e^{(t-s)A}F(X(s,x))ds + \int_0^t e^{(t-s)A}RdW(s), \quad \mathbb{P}\text{-a.s.} \tag{8}$$

However, if $\mathrm{Dom}(F) \subset \mathcal{X}$ is a proper subset of $\mathcal{X}$, (8) may not make sense for every $x \in \mathcal{X}$. Hence we need a more general notion of solution to avoid the problem of $\mathrm{Dom}(F)$. We follow idea of [20, Chapters 6 and 7] and [43, Section 7.2] assuming that there exists a Banach space $E \subseteq \mathrm{Dom}(F)$ densely and continuously embedded in $\mathcal{X}$ such that $F_{|E} : E \to E$ is locally Lipschitz continuous. Then, under suitable hypotheses, it is possible to prove that for any $x \in E$, the SPDE (7) has a unique mild solution $\{X(t,x)\}_{t\geq 0}$ such that its trajectories take values in E. Then, exploiting the density of E, one proves that for any $x \in \mathcal{X}$ there exists a process $\{X(t,x)\}_{t\geq 0}$, such that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|X(\cdot,x_n) - X(\cdot,x)\| = 0, \quad \forall\, T > 0,\ \mathbb{P}\text{-a.s.} \tag{9}$$

for any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ converging to x and $X(t,x_n)$ being the unique mild solution of (7) with initial datum x_n. We call the limit $\{X(t,x)\}_{t\geq 0}$ in (9) generalized mild solution of (7). Moreover we provide some useful estimates for the moments of the generalized mild solution $\{X(t,x)\}_{t\geq 0}$. Later we will prove that the family of operators $\{P(t)\}_{t\geq 0}$ defined by

$$(P(t)\varphi)(x) := \mathbb{E}[\varphi(X(t,x))], \quad \varphi \in B_b(\mathcal{X}),\ x \in \mathcal{X},\ t \geq 0 \tag{10}$$

is the transition semigroup associated to (2.0.1), we simply denote it by $P(t)$. In this chapter we work in the general framework of [43, Sections 7.2], and in such general framework we provide several useful estimates, generalizing the ones of [20, Chapters 6-7] and [30, Chapters 4].

Chapter 3. In this chapter, under suitable hypotheses, we study some regularizing properties of the transition semigroup $P(t)$ given by (10). In [14, 61] the authors consider equation (7) with $F = RG$, for some $G : \mathcal{X} \to \mathcal{X}$ Lipschitz continuous. Let $\mathrm{Lip}_b(\mathcal{X})$ be the space of bounded and Lipschitz continuous functions from $\mathcal{X}$ to $\mathbb{R}$. They prove that

$$P(t)(B_b(\mathcal{X})) \subseteq \mathrm{Lip}_b(\mathcal{X}), \quad t > 0.$$

whenever

$$e^{tA}(\mathcal{X}) \subseteq Q_t^{1/2}(\mathcal{X}), \quad t > 0, \tag{11}$$

where Q_t is defined in (5). The main novelty of this chapter is to study the same SPDE considered in [14, 61] in the case where (11) is not verified. Specifically, we assume that the part of A in the separable Hilbert space

$$H_R := (R(\mathcal{X}), \langle R^{-1}(\cdot), R^{-1}(\cdot)\rangle) \tag{12}$$

(see Subsection 1.2.5) generates a strongly continuous semigroup and we prove that for any $t > 0$, $x \in \mathcal{X}$, $h \in H_R$ and $\varphi \in B_b(\mathcal{X})$, we have

$$|P(t)\varphi(x+h) - P(t)\varphi(x)| \leq K(t)\|R^{-1}h\|.$$

where $K(\cdot) : (0, +\infty) \to (0, +\infty)$ is continuous and for small t

$$K(t) \approx \frac{k}{\sqrt{t}},$$

for some $k > 0$.

Chapter 4. This chapter is devoted to prove a logarithmic Harnack-type inequality for the transition semigroup $P(t)$ defined in (10). The first formulation of the Harnack inequality dates back to 1887, it can be found in his seminal paper [66], and concerns positive harmonic functions. Over the years, these types of inequalities have been studied by many other authors, such as J. Moser, J. Serrin, N.S. Trudinger, J. Hadamard, B. Pini, D. G. Aronson. E. De Giorgi and J. Nash. A possibility to get the Harnack-type inequality in an infinite dimensional setting consists in replacing a classical formulation to the dimension-free logarithmic Harnack Inequality (LHI) first introduced by F.-Y. Wang in [99] for the study of diffusion semigroups on Riemannian manifolds. In this chapter we generalize the results of [38, 93]. To be more precise, in the same spirit as *Chapter 3.*, we prove a (LHI) type inequality along H_R (see (12)), namely

$$|P(t)\varphi(x+h)|^p \leq P(t)|\varphi(x)|^p e^{c(t)\|R^{-1}h\|^2}, \qquad t > 0,\ x \in \mathcal{X},\ h \in H_R; \tag{13}$$

for any bounded and Borel measurable function $\varphi : \mathcal{X} \to \mathbb{R}$, any $p > 1$ and some continuous function $c : (0, +\infty) \to \mathbb{R}$. In [38] the authors assume that the operator R in (7) has bounded pseudo-inverse. Instead in [93] the perturbation F in the SPDE (7) is assumed to be Lipschitz continuous and dissipative along H_R. In this chapter R^{-1} is not assumed bounded. In Section 4.1 we prove (13) in the case where the perturbation F is Lipschitz continuous. In Section 4.2 we will prove (13) in the case where $F : \mathrm{Dom}(F) \subseteq \mathcal{X} \to \mathcal{X}$ satisfies a dissipativity hypothesis along H_R.

Chapter 5. In this chapter we study existence and uniqueness of a invariant measure for $P(t)$, and a core for the extension to $L^2(\mathcal{X}, \nu)$ of the transition semigroup $P(t)$ defined in (10). Adding some assumptions to those considered in *Chapter 2.*, we prove that $P(t)$ has a unique invariant measure ν and we show that it is uniquely extendable to a strongly continuous semigroup $P_p(t)$ in $L^p(\mathcal{X}, \nu)$, for any $p \geq 1$. We denote by N_2 the infinitesimal generator of $P_2(t)$. We prove in a more general setting the results previously proved in some specific cases in [11, Section 3], [31], [30, Sections 3.5 and 4.6] and [42, Section 11.2.2]. More precisely, we prove that N_2 is the closure in $L^2(\mathcal{X}, \nu)$ of the following second order Kolmogorov operator

$$N_0\varphi(x) := \frac{1}{2}\mathrm{Tr}[R^2\nabla\varphi(x)] + \langle Ax + F_0(x), \nabla\varphi(x)\rangle, \quad \varphi \in \xi_A(\mathcal{X}),\ x \in \mathrm{Dom}(A),$$

where

$$\xi_A(\mathcal{K}) := \mathrm{span}\{\text{real and imaginary parts of the functions } x \mapsto e^{i\langle x,h\rangle_{\mathcal{K}}} \mid h \in \mathrm{Dom}(A^*)\}.$$

and

$$F_0(x) = \begin{cases} F(x) & x \in E, \\ 0 & x \in \mathcal{X}\backslash E, \end{cases}$$

where E is the separable Banach space used in Chapter 2 to construct the generalized mild solution. To do this the fact that $\nu(E) = 1$ is essential, and to prove such equality we use the estimates of the moments of the mild geralized solution given in Chapter 2.

Chapter 6. In this chapter we describe the domain of N_2. As expected, we shall prove that the domain of N_2 is embedded in suitable Sobolev spaces. In [30, Section 3.6.1] and [32] the authors assume that R in (7) has a continuous pseudo-inverse (see Subsection 1.2.5) and work with the Sobolev space $W^{1,2}(\mathcal{X}, \nu)$ defined as the domain of the closure in $L^2(\mathcal{X}, \nu)$ of the Fréchet gradient operator $\nabla : \xi_A(\mathcal{X}) \subseteq L^2(\mathcal{X}, \nu) \to L^2(\mathcal{X}, \nu; \mathcal{X})$. We emphasize that this case presents no significant differences in defining and studying Sobolev spaces compared to the case when $R = \mathrm{I}_{\mathcal{X}}$. In [5, 16, 17, 59] the authors assume that $R = Q^{1/2}$, where Q is a positive, self-adjoint and trace class operator, and $F = -Q\nabla U$ where $U : \mathcal{X} \to \mathbb{R}$ is a Fréchet differentiable and convex function, such that ∇U is Lipschitz continuous. They consider the Sobolev space $W^{1,2}_{Q^{1/2}}(\mathcal{X}, \nu)$ defined as the closure in $L^2(\mathcal{X}, \nu)$ of the operator $Q^{1/2}\nabla : \xi_A(\mathcal{X}) \subseteq L^2(\mathcal{X}, \nu) \to L^2(\mathcal{X}, \nu; \mathcal{X})$. We underline that, if $F = -Q\nabla U$, then the invariant measure ν is a weighted Gaussian measure and N_2 is the self-adjoint operator associated to the quadratic form

$$G(\varphi, \psi) = \int_{\mathcal{X}} \langle Q^{1/2}\nabla\varphi, Q^{1/2}\nabla\psi\rangle d\nu, \qquad \varphi, \psi \in W^{1,2}_{Q^{1/2}}(\mathcal{X}, \nu).$$

Conversely, if F is not of that form, then N_2 is not necessarily associated to a quadratic form. We stress that, in the infinite dimensional case, the Sobolev spaces $W^{1,2}(\mathcal{X}, \nu)$ and $W^{1,2}_{Q^{1/2}}(\mathcal{X}, \nu)$ have not equivalent norms.

In this chapter we consider the same framework of *Chapter 3.*, namely the perturbation F in (7) is equal to RG for some Lipschitz continuous and Fréchet differentiable $G : \mathcal{X} \to \mathcal{X}$, and $R \in \mathcal{L}(\mathcal{X})$ is non-negative. We prove that $\mathrm{Dom}(N_2)$ is contained in the Sobolev space $W^{1,2}_R(\mathcal{X}, \nu)$ defined as the domain of the closure of the operator

$$R\nabla : \xi_A \subseteq L^2(\mathcal{X}, \nu) \to L^2(\mathcal{X}, \nu, \mathcal{X}).$$

Moreover we prove that the transition semigroup $P(t)$ and its invariant measure ν satisfy a logarithmic Sobolev and Poincaré inequalities, and a hypercontractivity property.

Chapter 7. Let $\mathcal{O}$ be an open set of $\mathcal{X}$ and let $B_b(\mathcal{O})$ be the space of bounded and Borel measurable functions from $\mathcal{O}$ to $\mathbb{R}$. In Section 7 we consider the Dirichlet semigroup

$$P^{\mathcal{O}}(t)\varphi(x) := \mathbb{E}\left[\varphi(X(t,x))\mathbb{I}_{\{\omega\in\Omega\,:\,\tau_x(\omega)>t\}}\right], \quad \varphi \in B_b(\mathcal{O}),\ x \in \mathcal{O},\ t > 0$$

where $\{X(t,x)\}_{t\geq 0}$ is the generalized mild solution of (7) studied in *Chapter 2.*, and τ_x is the stopping time defined by

$$\tau_x = \inf\{s > 0 \ : \ X(s,x) \in \mathcal{O}^c\}.$$

We prove that ν is sub-invariant for $P^{\mathcal{O}}(t)$; therefore $P^{\mathcal{O}}(t)$ is uniquely extendable to a strongly continuous semigroup $P_p^{\mathcal{O}}(t)$ in $L^p(\mathcal{O},\nu)$, for any $p \geq 1$. We denote by M_2 the infinitesimal generator of $P_2^{\mathcal{O}}(t)$. In this chapter we extend to a nonlinear case some results proved in [33] in the case where $F = 0$ in (7). We consider only the case where F is a gradient perturbation, namely it has a potential. In this case the invariant measure ν is a weighted Gaussian measure and it is possible to associate a quadratic form $\mathcal{Q}_2$ to N_2. Under some additional hypotheses there exists a quadratic form $\mathcal{Q}_2$ on $W_R^{1,2}(\mathcal{X},\nu)$ such that

$$\int_{\mathcal{X}} (N_2\varphi)\psi d\nu = \mathcal{Q}_2(\varphi,\psi) = -\frac{1}{2}\int \langle R\nabla\varphi, R\nabla\psi\rangle d\nu, \quad \forall \varphi \in \mathrm{Dom}(N_2),\ \psi \in W_R^{1,2}(\mathcal{X},\nu).$$

After we consider the Sobolev space $\mathring{W}_R^{1,2}(\mathcal{X},\nu)$ of the functions $u : \mathcal{O} \to \mathbb{R}$ such that their null extension $\widehat{u}$ belongs to $W_R^{1,2}(\mathcal{X},\nu)$, and the quadratic form $\mathcal{Q}_2^{\mathcal{O}}$ on $\mathring{W}_R^{1,2}(\mathcal{O},\nu)$ defined by

$$\mathcal{Q}_2^{\mathcal{O}}(\varphi,\psi) = \mathcal{Q}_2(\widehat{\varphi},\widehat{\psi}), \quad \forall \varphi,\psi \in \mathring{W}_R^{1,2}(\mathcal{O},\nu).$$

In this chapter we prove that the infinitesimal generator M_2 of $P_2^{\mathcal{O}}(t)$ is the operator $N_2^{\mathcal{O}}$ associated with $\mathcal{Q}_2^{\mathcal{O}}$, namely

$$\mathrm{Dom}(N_2^{\mathcal{O}}) := \{\varphi \in \mathring{W}_R^{1,2}(\mathcal{O},\nu) \ : \ \exists \beta \in L^2(\mathcal{O},\nu) \text{ s.t.} \int_{\mathcal{O}} \beta\psi d\nu = \mathcal{Q}_2^{\mathcal{O}}(\beta,\psi) \ \ \forall \psi \in \mathring{W}_R^{1,2}(\mathcal{X},\nu)\}$$

$$N_2^{\mathcal{O}}\varphi = \beta, \quad \varphi \in \mathrm{Dom}(N_2^{\mathcal{O}}).$$

Chapter 1

Preliminaries

1.1 Notations

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two real Banach spaces equipped with the norm $\|\cdot\|_{\mathcal{K}_1}$ and $\|\cdot\|_{\mathcal{K}_2}$ respectively. We denote by $\mathcal{B}(\mathcal{K}_1)$ the family of the Borel subsets of $\mathcal{K}_1$ and by $B_b(\mathcal{K}_1;\mathcal{K}_2)$ the set of the bounded and Borel measurable functions from $\mathcal{K}_1$ to $\mathcal{K}_2$. When $\mathcal{K}_2 = \mathbb{R}$ we simply write $B_b(\mathcal{K}_1)$. We denote by $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ the space of the linear bounded operators from $\mathcal{K}_1$ to $\mathcal{K}_2$, if $\mathcal{K}_1 = \mathcal{K}_2$ we simply write $\mathcal{L}(\mathcal{K}_1)$. We denote by $\mathrm{I}_{\mathcal{K}_1}$ the identity operator on $\mathcal{K}_1$. Let $A : \mathrm{Dom}(A) \subset \mathcal{K}_1 \to \mathcal{K}_1$ be a linear operator and let $E \subset \mathcal{K}_1$ be another Banach space. The part A_E of A in E is defined as

$$\mathrm{Dom}(A_E) := \{x \in \mathrm{Dom}(A) \cap E \ : \ Ax \in E\}, \quad A_E x := Ax, \ x \in \mathrm{Dom}(A_E).$$

We denote by $C_b(\mathcal{K}_1;\mathcal{K}_2)$ the set of the continuous bounded functions from $\mathcal{K}_1$ to $\mathcal{K}_2$. If $\mathcal{K}_2 = \mathbb{R}$ we simply write $C_b(\mathcal{K}_1)$.

Let $f : \mathcal{K}_1 \to \mathcal{K}_2$, we say that f is Fréchet differentiable at the point $x \in \mathcal{K}_1$, if there exists $L_x \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ such that

$$\lim_{\|h\|_{\mathcal{K}_1} \to 0} \frac{\|f(x+h) - f(x) - L_x h\|_{\mathcal{K}_2}}{\|h\|_{\mathcal{K}_1}} = 0.$$

When it exists, the operator L_x is unique and it is called Fréchet derivative of f at the point $x \in \mathcal{K}_1$. We set $\mathcal{D}f(x) := L_x$. We say that f is twice Fréchet differentiable at the point $x \in \mathcal{K}_1$ if the map $\mathcal{D}f : \mathcal{K}_1 \to \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ is Fréchet differentiable at the point $x \in \mathcal{K}_1$, hence the second order Fréchet derivative of f at the point $x \in \mathcal{K}_1$ is the Fréchet derivative $\mathcal{D}(\mathcal{D}f)(x)$ of $\mathcal{D}f$ at the point $x \in \mathcal{K}_1$. We set $\mathcal{D}^2 f(x) = \mathcal{D}(\mathcal{D}f)(x)$ For any $x \in \mathcal{K}_1$, $\mathcal{D}(\mathcal{D}f)(x)$ is a linear bounded operator from $\mathcal{K}_1$ to $\mathcal{L}(\mathcal{K}_1, \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2))$ and there exists a unique bilinear form $b_x : \mathcal{K}_1 \times \mathcal{K}_1 \to \mathcal{K}_2$ such that

$$(\mathcal{D}^2 f(x)h)k = b_x(h,k), \quad h, k \in \mathcal{K}_1,$$

we still denote b_x by $\mathcal{D}^2 f(x)$. For any $k \in \mathbb{N}$, in analogous way, we can define the notion of k-Fréchet differentiability of a function f and we denote by $\mathcal{D}^k f(x)$ its k-Fréchet derivative at the point $x \in \mathcal{K}_1$. We denote by $C_b^k(\mathcal{K}_1;\mathcal{K}_2)$, $k \in \mathbb{N} \cup \{\infty\}$ the set of the k-times Fréchet differentiable functions from $\mathcal{K}_1$ to $\mathcal{K}_2$ with bounded derivatives up to the order k. If $\mathcal{K}_2 = \mathbb{R}$ we simply write $C_b^k(\mathcal{K}_1)$.

Let $f : \mathcal{K}_1 \to \mathcal{K}_2$. We say that f is Gateaux differentiable at the point $x \in \mathcal{K}_1$ if there exists $T_x \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ such that for any $h \in \mathcal{K}_1$ we have

$$\lim_{t \to 0} \frac{\|f(x+th) - f(x) - tT_x h\|_{\mathcal{K}_2}}{t} = 0, \tag{1.1.1}$$

$T_x h$ is called Gateaux derivative of f at the point $x \in \mathcal{K}_1$ along $h \in \mathcal{K}_1$ and we denote it by $\mathcal{D}^G \Phi(x)h$.

Proposition 1.1.1 (Fact 1.13(b), p. 8 [86])**.** *If the limit in* (1.1.1) *exists uniformly for $h \in \mathcal{K}_1$ such that $\|h\|_{\mathcal{K}_2} \leq 1$ then f is Fréchet differentiable at $x \in \mathcal{K}_1$.*

Let H be a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_H$, if $f \in C_b^1(H)$ then, for every $x \in H$ there exists a unique $k_x \in H$ such that for every $h \in H$

$$\mathcal{D}f(x)(h) = \langle h, k_x \rangle_H,$$

k_x is called Fréchet gradient of f at the point $x \in H$ and we denote it by $\nabla f(x)$. Moreover if $f \in C_b^2(H)$ then, for any $x \in H$, there exists a unique $Q_x \in \mathcal{L}(H)$ such that, for any $h, k \in H$ we have

$$\mathcal{D}^2 f(x)(h,k) = \langle Q_x h_1, h_2 \rangle_H,$$

Q_x is called Fréchet Hessian of f at the point $x \in H$ and we denote it by $\nabla^2 f(x)$.

1.2 Linear operators

In this section we recall some basic definitions and results about the theory of linear operators. Let $\mathcal{K}$ be a Banach space equipped with the norm $\|\cdot\|_{\mathcal{K}}$.

1.2.1 Closed operators

Let $A : \mathrm{Dom}(A) \subset \mathcal{K} \to \mathcal{K}$ be a linear operator. We say that A is closed if for any sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathrm{Dom}(A)$ such that $x_n \to x$ and $Ax_n \to y$ we have $x \in \mathrm{Dom}(A)$ and $y = Ax$. We say that A is closable if for any $\{x_n\}_{n \in \mathbb{N}} \subset \mathrm{Dom}(A)$ such that $x_n \to 0$ and $Ax_n \to y$ we have $y = 0$. In this case we define the closure of A in the following way

$$\mathrm{Dom}(\overline{A}) := \{x \in \mathcal{X} \mid \exists \{x_n\}_{n \in \mathbb{N}} \subset \mathrm{Dom}(A) \mid x_n \to x,\ \{Ax_n\}_{n \in \mathbb{N}} \text{ converges in } \mathcal{K}\},$$
$$\overline{A}x = \lim_{n \to +\infty} Ax_n, \quad x \in \mathrm{Dom}(\overline{A}).$$

Let $T > 0$, $A : \mathrm{Dom}(A) \subset \mathcal{X} \to \mathcal{X}$ be a closed operator. Let $f : [0,T] \to \mathrm{Dom}(A)$, if both f

and Af are Bochner integrable in $[0,T]$, then

$$\int_0^T f(t)dt \in \mathrm{Dom}(A), \quad A\int_0^T f(t)dt = \int_0^T Af(t)dt.$$

1.2.2 Adjoint

Let $A : \mathrm{Dom}(A) \subset \mathcal{K} \to \mathcal{K}$ be a linear operator such that $\mathrm{Dom}(A)$ is dense in $\mathcal{K}$. The adjoint of A is defined as the operator $A^* : \mathrm{Dom}(A^*) \subset \mathcal{K}^* \to \mathcal{K}^*$ such that

$$\mathrm{Dom}(A^*) := \{l \in \mathcal{K}^* \;:\; \exists k > 0 \;:\; |l(Ax)| \leq k\|x\|_{\mathcal{K}},\ \forall x \in \mathrm{Dom}(A)\},$$

$$l(Ax) = (A^*l)(x), \quad \forall x \in \mathrm{Dom}(A),\ \forall l \in \mathrm{Dom}(A^*).$$

If $\mathcal{K}$ is a Hilbert space and after the canonical identification of $\mathcal{K}$ and K^*, we say that A is a self-adjoint operator when $A = A^*$.

1.2.3 Resolvent

We still denote by $\mathcal{K}$ the complexification of $\mathcal{K}$. Let $A : \mathrm{Dom}(A) \subset \mathcal{K} \to \mathcal{K}$ be a linear operator. We define the resolvent set of A as

$$\rho(A) := \{\lambda \in \mathbb{C} \;:\; (A - \lambda \mathrm{I}) : \mathrm{Dom}(A) \to \mathcal{K} \text{ is bijective and its inverse is bounded}\}.$$

Instead $\sigma(A) := \mathbb{C}\backslash\rho(A)$ is called spectrum of A. The elements $\lambda \in \sigma(A)$ such that $(A - \lambda\mathrm{I})$ is not injective are called eigenvalues and the elements $x \in \mathrm{Dom}(A)$ such that $Ax = \lambda x$ are called eigenvectors associated to the eigenvalues. We denote by $\sigma_p(A)$ the set of eigenvalues of A and it is called point (or punctual) spectrum. Moreover for any $\lambda \in \rho(A)$ we can define the resolvent of A in the following way

$$R(\lambda, A) = (A - \lambda\mathrm{I})^{-1}.$$

1.2.4 Square root and positive operators

Let $A : \mathrm{Dom}(A) \subset \mathcal{K} \to \mathcal{K}$ be a linear operator. If there exists a unique operator $B : \mathrm{Dom}(B) \subset \mathcal{X} \to \mathcal{X}$ such that $\mathrm{Dom}(B^2) = \mathrm{Dom}(A)$ and $A = B^2$ then we call B square root of A and we denote it by $\sqrt{A}$. Clearly a general linear operator may not have a square root.

We assume that $\mathcal{K}$ is a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{K}}$. Let $T \in \mathcal{L}(\mathcal{K})$. We say that T is *non-negative* (*positive*) if for every $x \in \mathrm{Dom}(T) \setminus \{0\}$

$$\langle Tx, x\rangle \geq 0 \ (> 0).$$

In an anologous way we define the non-positive (negative) operators. We have the following basic results about non-negative and bounded operators.

Proposition 1.2.1. *Let $T \in \mathcal{L}(\mathcal{K})$ be a non-negative operator.*

1. *T is a self-adjoint operator.*
2. *If $G \in \mathcal{L}(\mathcal{K})$ is non-negative and G and T commute then GT is a non-negative operator.*
3. *T has a unique square root $\sqrt{T}$. Moreover*
   (a) *if G commutes with T then G commutes with $\sqrt{T}$,*
   (b) *if T is bjective then $\sqrt{T}$ is bjective.*

Let $T \in \mathcal{L}(\mathcal{K})$. It easy to see that T^*T is a non-negative operator, so we can define the absolute value of T in the following way,

$$|T| := \sqrt{T^*T} \tag{1.2.1}$$

1.2.5 Pseudo-inverse

We conclude this section defining the notion of pseudo-inverse for an operator $T \in \mathcal{L}(\mathcal{K})$. We refer to [71, Appendix C].

Definition 1.2.2. *Let $T \in \mathcal{L}(\mathcal{K})$. The pseudo inverse of T is defined as*

$$T^{-1} := \left(T_{|Ker(T)^{\perp}}\right)^{-1}$$

where $Ker(T)^{\perp}$ is the orthogonal of $Ker(T)$. In an equivalent way we can define the pseudo inverse of T as the operator that associates to $x \in T(\mathcal{K})$ the element of minimum norm in $T^{-1}(\{x\})$.

It is easy to see that the pseudo-inverse $T^{-1} : T(\mathcal{K}) \to \mathrm{Ker}(T)^{\perp}$ is a linear operator and the space $H_T := T(\mathcal{K})$ is an Hilbert space with the inner product

$$\langle x, y\rangle_T := \left\langle T^{-1}x, T^{-1}y\right\rangle_{\mathcal{K}}, \quad x, y \in T(\mathcal{K}).$$

Moreover if $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $\mathrm{Ker}(T)^{\perp}$ then $\{Te_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $H_T := T(\mathcal{K})$.

1.3 Dissipative mappings

We recall some basic results about subdifferentials and dissipative maps. We refer to [20, Appendix A], [27] and [43, Appendix D] for the results of this section. Let $\mathcal{K}$ be a separable Banach space. For any $x \in \mathcal{K}$, we define the subdifferential $\partial\|x\|_{\mathcal{K}}$ of $\|\cdot\|_{\mathcal{K}}$ at $x \in \mathcal{K}$ as

$$\partial\|x\| := \{x^* \in \mathcal{K}^* \mid \|x+y\|_{\mathcal{K}} \geq \|x\|_{\mathcal{K}} + x^*(y),\ \forall\, y \in \mathcal{K}\}.$$

Moreover $\partial\|x\|_{\mathcal{K}}$ is close and convex and for any $x \neq 0$ we have

$$\partial\|x\| = \{x^* \in \mathcal{K}^* \mid {}_{\mathcal{K}}\langle x, x^*\rangle_{\mathcal{K}^*} = \|x\|_{\mathcal{K}},\ \|x^*\|_{\mathcal{K}^*} = 1\}.$$

Let $[t_0, t_1] \subset [0, +\infty)$ and let $u : [t_0, t_1] \to \mathcal{K}$ be a differentiable function. Then the function $\gamma := \|u\|_{\mathcal{K}} : [t_0, t_1] \to [0, +\infty)$ is left-differentiable at any $t_0 \in [t_0, t_1]$ and

$$\frac{d^-\gamma}{dt}(t_0) := \lim_{h \to 0^-} \frac{\gamma(t_0 + h) - \gamma(t_0)}{h} = \min\{{}_E\langle u'(t_0), x^* \rangle_{E^*} \ : \ x^* \in \partial\|u(t_0)\|_{\mathcal{K}}\}. \tag{1.3.1}$$

Moreover, let $b \in \mathbb{R}$ and let $g : [t_0, t_1] \to [0, +\infty)$ be a continuous function. If

$$\frac{d^-\gamma}{dt}(t) \leq b\gamma(t) + g(t),$$

then, for any $t \in [t_0, t_1]$, we have

$$\gamma(t) \leq e^{b(t-t_0)}\gamma(t_0) + \int_{t_0}^{t} e^{b(t-s)} g(s)ds, \quad t \in [t_0, t_1]. \tag{1.3.2}$$

Definition 1.3.1. *A map $f : \mathrm{Dom}(f) \subseteq \mathcal{K} \to \mathcal{K}$ is said to be dissipative if, for any $\alpha > 0$ and $x, y \in \mathrm{Dom}(f)$, we have*

$$\|x - y - \alpha(f(x) - f(y))\|_{\mathcal{K}} \geq \|x - y\|_{\mathcal{K}} \tag{1.3.3}$$

If f is a linear operator, $f(x) = Ax$ (1.3.3) reads as

$$\|(\lambda \mathrm{I} - A)x\|_{\mathcal{K}} \geq \lambda\|x\|_{\mathcal{K}}, \quad \forall \lambda > 0, \ x \in \mathrm{Dom}(A)$$

We say that f is m-dissipative if the range of $\lambda \mathrm{I} - f$ is the whole space $\mathcal{K}$ for some $\lambda > 0$ (and so for all $\lambda > 0$).

Using the notion of subdifferential we have the following useful charaterization for the dissipative maps.

Proposition 1.3.2. *Let $f : \mathrm{Dom}(f) \subseteq \mathcal{K} \to \mathcal{K}$. f is dissipative if and only if, for any $x, y \in \mathrm{Dom}(f)$ there exists $z^* \in \partial\|x - y\|$ such that*

$${}_{\mathcal{K}}\langle f(x) - f(y), z^* \rangle_{\mathcal{K}^*} \leq 0. \tag{1.3.4}$$

If $\mathcal{K}$ is a Hilbert space (1.3.4) becomes

$$\langle f(x) - f(y), x - y \rangle_{\mathcal{K}} \leq 0.$$

1.4 Semigroups theory

In this subsection we recall some basic definitions and results of the semigroups theory, we refer to [53, 54, 73]. Let $\mathcal{K}$ be a Banach space. From here on we will use the notation $T(t)$ to denote a semigroup of linear bounded operators $\{T(t)\}_{t>0}$, namely a family $\{T(t) \ : \ t \geq 0\} \subseteq \mathcal{L}(\mathcal{K})$ such that

$$T(0) = \mathrm{I}_{\mathcal{K}}, \quad T(t + s) = T(t)T(s), \quad \forall t, s \geq 0.$$

1.4.1 Strongly continuous semigroups

Let $T(t)$ be a semigroup on $\mathcal{K}$. We say that $T(t)$ is strongly continuous if for any $x \in \mathcal{K}$ the function $T(\cdot)x : [0, +\infty) \to \mathcal{K}$ is continuous. In this case we define the infinitesimal generator of $T(t)$ as the operator $A : \mathrm{Dom}(A) \subset \mathcal{K} \to \mathcal{K}$ defined by

$$\mathrm{Dom}(A) := \left\{ x \in \mathcal{K} \ : \ \exists \lim_{t\to 0} \frac{T(t)x - x}{t} \right\}, \quad Ax = \lim_{t\to 0} \frac{T(t)x - x}{t}.$$

Sometimes we denote by e^{tA} the semigroup $T(t)$. Let $M \geq 1$ and $w \in \mathbb{R}$. We denote by $\mathcal{G}(M, w)$ the set of the strongly continuous semigroups $T(t)$ such that

$$\|T(t)\|_{\mathcal{L}(\mathcal{K})} \leq Me^{wt}, \quad t \geq 0.$$

Moreover if $M = 1$ and $w \leq 0$ we say that $T(t)$ is a contraction semigroup.

We state one of the most important results about strongly continuous semigroups.

Theorem 1.4.1 (Hille-Yosida)**.** *Let $A : \mathrm{Dom}(A) \subset \mathcal{K} \to \mathcal{K}$ be a linear operator. A is the infinitesimal generator of a strongly continuous semigroup belonging to $\mathcal{G}(M, w)$ if and only if the following conditions are verified:*

1. $\mathrm{Dom}(A)$ *is dense.*

2. $\{\lambda \in \mathbb{R} \ : \ \lambda > w\} \subset \rho(A)$.

3. $\|R(\lambda, A)\|_{\mathcal{L}(\mathcal{K})} \leq \frac{M}{(\lambda - w)^k}, \quad k \in \mathbb{N}, \ \lambda > w.$

Remark 1.4.2. *If $\mathrm{Dom}(A)$ is not dense in $\mathcal{K}$ and the points 2-3 of the Hille–Yosida Theorem hold true, then we can consider the space $\mathcal{K}_0 = \overline{\mathrm{Dom}(A)}$. By the Hille–Yosida theorem, the part of A in $\mathcal{K}_0$ generates a strongly continuous semigroup on $\mathcal{K}_0$.*

We state some useful properties of the strongly continuous semigroup.

Proposition 1.4.3. *Let $T(t)$ be a strongly continuous semigroup on $\mathcal{K}$ and let $A : \mathrm{Dom}(A) \subset \mathcal{K} \to \mathcal{K}$ be its infinitesimal generator.*

1. *For any $x \in \mathrm{Dom}(A)$ and $t \geq 0$ we have $AT(t)x = T(t)Ax$.*

2. *For any $x \in \mathrm{Dom}(A)$ the function $T(\cdot)x : [0 + \infty) \to \mathcal{K}$ is differentiable and $\dfrac{dT(t)x}{dt} = AT(t)x$, for any $t > 0$.*

3. *For any $x \in \mathcal{X}$ we have $\lim_{n\to+\infty} nR(n, A)x = x$.*

Now we recall some results in the case where the operator A is dissipative. The first one is the the Lumer-Phillips theorem.

Proposition 1.4.4.

1. *Let $A : \mathrm{Dom}(A) \subset \mathcal{K} \to \mathcal{K}$ be a linear and dissipative operator such that $\mathrm{Dom}(A)$ is dense. The closure $\overline{A}$ of A is the infinitesimal generator of a strongly continuous and contraction semigroup if and only if $(\lambda \mathrm{I} - A)(\mathcal{K})$ is dense in $\mathcal{K}$ for some $\lambda > 0$.*

2. *Let $A : \mathrm{Dom}(A) \subset \mathcal{K} \to \mathcal{K}$ be the infinitesimal generator of a strongly continuous and contraction semigroup. Then A is dissipative.*

1.4.2 Analytic semigroups

Now we introduce the analytic semigroups. We consider the complexification of $\mathcal{K}$, and we still denote it by $\mathcal{K}$.

Let $A : \mathrm{Dom}(A) \subseteq \mathcal{K} \to \mathcal{K}$ be a linear operator. We say that A is a sectorial operator if there exist $M > 0$, $\eta_0 \in \mathbb{R}$ and $\theta_0 \in (\pi/2, \pi]$ such that

$$S_0 := \{\lambda \in \mathbb{C} \mid \lambda \neq \eta_0,\ |\arg(\lambda - \eta_0)| < \theta_0\} \subseteq \rho(A);$$

$$\|R(\lambda, A)\|_{\mathcal{L}(\mathcal{K})} \leq \frac{M}{|\lambda - \eta_0|}, \quad \forall \lambda \in S_0. \tag{1.4.1}$$

We call analytic semigroup generated by A the semigroup e^{tA} defined by

$$e^{tA} := \int_{\gamma_{r,\eta}+w} e^{t\lambda} R(\lambda, A) d\lambda, \tag{1.4.2}$$

where $r > 0$, $\eta \in (\frac{\pi}{2}, \eta_0)$ and

$$\gamma := \{\lambda \in \mathbb{C}\ :\ |\arg \lambda| = \eta,\ |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C}\ :\ |\arg \lambda| \geq \eta,\ |\lambda| = r\}.$$

Due to the analyticity of $e^{t(\cdot)}R(\cdot, A)$, (1.4.2) is independent of r and η.

We state some properties of analytic semigroups.

Proposition 1.4.5. *Let $A : \mathrm{Dom}(A) \subseteq \mathcal{K} \to \mathcal{K}$ be sectorial operator. Then the family of operators defined in* (1.4.2) *is a semigroup on $\mathcal{K}$, that satisfies the following properties.*

1. *There exists $M_0 > 0$ such that for any $t > 0$*

$$\left\|e^{tA}\right\|_{\mathcal{L}(\mathcal{K})} \leq M_0 e^{\eta_0 t}, \tag{1.4.3}$$

where η_0 is the constant in (1.4.1).

2. *For any $t > 0$ and $k \in \mathbb{N}$*

$$e^{tA}(\mathcal{K}) \subseteq \mathrm{Dom}(A^k). \tag{1.4.4}$$

Moreover for every $\epsilon > 0$ there exists $C_{\epsilon,k} > 0$ such that

$$\left\|t^k A^k e^{tA}\right\|_{\mathcal{K}} \leq C_{\epsilon,k} e^{(\eta_0+\epsilon)t}, \quad t > 0. \tag{1.4.5}$$

3. *For any $x \in \overline{\mathrm{Dom}(A)}$*

$$\lim_{n \to +\infty} nR(n, A)x = x. \tag{1.4.6}$$

4. *Setting $f(t) = e^{tA}$, we have*

$$f \in C^\infty((0, +\infty), \mathcal{L}(\mathcal{K})). \tag{1.4.7}$$

Moreover $f^{(k)}(t) = A^k f(t)$, for any $t > 0$ and $k \in \mathbb{N}$.

Remark 1.4.6. *If A is non positive, then, by* (1.4.5) *for any $\alpha \geq 0$ and $\epsilon > 0$ there exists $C_{\epsilon,\alpha} > 0$ such that*

$$\left\|t^\alpha(-A)^\alpha e^{tA}\right\|_{\mathcal{K}} \leq C_{\epsilon,\alpha} e^{(\eta_0+\epsilon)t}, \quad t > 0.$$

1.4.3 Semigroups on $B_b(\mathcal{K})$

We recall some basic definitions about the semigroups defined on $B_b(\mathcal{K})$. Let $T(t)$ be a semigroup on $B_b(\mathcal{K})$.

1. We say that $T(t)$ is non-negative if for any non-negative valued $\varphi \in B_b(\mathcal{K})$ and for any $t \geq 0$, $T(t)\varphi$ has non-negative values.

2. We say that $T(t)$ is Feller, if for any $t \geq 0$ we have
$$T(t)\left(C_b(\mathcal{K})\right) \subseteq C_b(\mathcal{K}).$$

3. We say that $T(t)$ is Strong Feller, if for any $t > 0$ we have
$$T(t)\left(B_b(\mathcal{K})\right) \subseteq C_b(\mathcal{K}).$$

4. We say that $T(t)$ is contractive, if for any $t \geq 0$ and $\varphi \in B_b(\mathcal{K})$ we have
$$\|T(t)\varphi\|_\infty \leq \|\varphi\|_\infty.$$

5. Let μ be a probability measure on $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$. We say that μ is invariant for $T(t)$ if, for any $\varphi \in C_b(\mathcal{K})$ and $t \geq 0$, we have
$$\int_{\mathcal{K}} T(t)\varphi(x)\nu(dx) = \int_{\mathcal{K}} \varphi(x)\nu(dx).$$

1.5 Compact operators

Compact operators are very important tools when working in infinite dimension. In this thesis we will apply it to the study of Gaussian measures on infinite dimensional Hilbert spaces. We refer to the following books for a more extensive treatment of the theory of compact operators and its application [51, 89, 90].

Let $\mathcal{K}$ and $\mathcal{Y}$ be two separable Hilbert spaces equipped with inner products $\langle\cdot,\cdot\rangle_{\mathcal{K}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{Y}}$ respectively. We denote by $\|\cdot\|_{\mathcal{K}}$ and $\|\cdot\|_{\mathcal{Y}}$ the norms induced by $\langle\cdot,\cdot\rangle_{\mathcal{K}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{Y}}$ respectively.

Definition 1.5.1. *Let $T \in \mathcal{L}(\mathcal{K},\mathcal{Y})$. We say that T is compact if, for any bounded $M \subseteq \mathcal{X}$, the subset $T(M) \subset \mathcal{Y}$ is relatively compact. Equivalently, for any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathcal{X}$, the sequence $\{Tx_n\}_{n\in\mathbb{N}} \subset \mathcal{Y}$ has a converging subsequence. We denote by $\mathcal{L}_\infty(\mathcal{K},\mathcal{Y})$ the space of compact operators from $\mathcal{K}$ to $\mathcal{Y}$, if $\mathcal{K} = \mathcal{Y}$ we simply write $\mathcal{L}_\infty(\mathcal{K})$.*

We have the following basic results about compact operators.

Proposition 1.5.2. *Let $T \in \mathcal{L}(\mathcal{K})$.*

1. *T is compact if and only if T^* is compact.*

2. *T is compact if and only if it maps weakly converging sequences into norm converging sequences.*

Now we recall one of the most important theorem for self-adjoint and compact operators.

Theorem 1.5.3. *Let $T \in \mathcal{L}_\infty(\mathcal{K})$ be a self-adjoint operator. We denote by $\sigma_p(T)$ the set of eigenvalues of T. The following statements hold.*

1. *For any $\lambda \in \sigma_p(T)$ such that $\lambda \neq 0$, the eigenspace $\mathcal{K}_\lambda$ with eigenvalue λ is finite dimensional.*

2. *$\sigma_p(T) \subseteq \mathbb{R}$ is not empty and it is at most numerable.*

3. *$\sigma_p(T)$ has at most one accumulation point which can only be 0.*

4. *$\|T\|_{\mathcal{L}(\mathcal{K})} = \sup\{|\lambda| \,:\, \lambda \in \sigma_p(T)\}$.*

5. *There exists an orthonormal basis of $\mathcal{K}$ consisting of eigenvectors of T.*

Proposition 1.5.4. *Let $T \in \mathcal{L}(\mathcal{K})$ be a self-adjoint operator. T is compact if and only if there exist an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ of $\mathcal{K}$ and a sequence $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ converging to 0 such that*

$$Tx = \sum_{k\in\mathbb{N}} \lambda_k \langle x, e_k\rangle e_k, \quad x \in \mathcal{K},$$

where the series converging in $\mathcal{L}(\mathcal{K})$. In this case $\{e_k\}_{k\in\mathbb{N}}$ and $\{\lambda_k\}_{k\in\mathbb{N}}$ are the eigenvectors and the eigenvalues of T respectively.

By Theorem 1.5.3, $\sigma_p(T)$ is finite if and only if the range of T is a finite dimensional space. In general the range of T is not dense in $\mathcal{K}$.

Corollary 1.5.5. *Let $T \in \mathcal{L}_\infty(\mathcal{K})$ be a self-adjoint and injective operator. Then the range of T is dense in $\mathcal{K}$.*

1.5.1 Hilbert–Schmidt operators

Now we introduce the Hilbert–Schmidt operators.

Definition 1.5.6. *Let $T \in \mathcal{L}_\infty(\mathcal{K}, \mathcal{Y})$ and let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{K}$. We say that T is a Hilbert–Schmidt operator if*

$$\|T\|_{\mathcal{L}_2(\mathcal{K},\mathcal{Y})} := \sqrt{\sum_{k\in\mathbb{N}} \|Te_k\|^2_{\mathcal{Y}}} < +\infty.$$

The norm $\|\cdot\|_{\mathcal{L}_2(\mathcal{K},\mathcal{Y})}$ does not depend on the choice of the basis of X. We denote by $\mathcal{L}_2(\mathcal{K}, \mathcal{Y})$ the space of the Hilbert–Schmidt operators from $\mathcal{K}$ to $\mathcal{Y}$, if $\mathcal{Y} = \mathcal{K}$ we set $\mathcal{L}_2(\mathcal{K}, \mathcal{K} = \mathcal{L}_2(\mathcal{K})$

In the case where T is self-adjoint we have the following characterization for the Hilbert–Schmidt operators.

Proposition 1.5.7. *Let $T \in \mathcal{L}_\infty(\mathcal{K})$ be a self-adjoint operator. $T \in \mathcal{L}_2(\mathcal{K})$ if and only if*

$$\sum_{k\in\mathbb{N}} \lambda_k^2 < +\infty,$$

where $\{\lambda_k\}_{k\in\mathbb{N}}$ are the eigenvalues of T. Moreover

$$\|T\|_{\mathcal{L}_2(\mathcal{K},\mathcal{Y})} = \sqrt{\sum_{k\in\mathbb{N}} \lambda_k^2}.$$

1.5.2 Trace class operators

Finally we define the trace class (or nuclear) operators.

Definition 1.5.8. *Let $T \in \mathcal{L}_\infty(\mathcal{K})$. We say that T is a trace class (or nuclear) operator if one of the following equivalent conditions are verified.*

1. *There exists an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ of $\mathcal{K}$ such that*

$$\sum_{k\in\mathbb{N}} \langle |T| e_k, e_k\rangle < +\infty. \tag{1.5.1}$$

In this case the sum in (1.5.1) *does not depend on the choice of the basis, and it is denoted by* $\mathrm{Tr}[T]$ *(Trace of T).*

2. *The operator $\sqrt{|T|}$ defined in* (1.2.1) *is a Hilbert–Schmidt operator.*

Moreover

$$\mathrm{Tr}[T] = \left\| \sqrt{|T|} \right\|^2_{\mathcal{L}_2(\mathcal{K})}.$$

Also for the trace class operator we have a useful charaterization in the self-adjoint case.

Proposition 1.5.9. *Let $T \in \mathcal{L}_\infty(\mathcal{K})$ be a self-adjoint operator. T is a trace class operator if and only if*

$$\sum_{k\in\mathbb{N}} |\lambda_k| < +\infty,$$

where $\{\lambda_k\}_{k\in\mathbb{N}}$ are the eigenvalues of T. Moreover

$$\mathrm{Tr}[T] = \sum_{k\in\mathbb{N}} |\lambda_k| \in \mathbb{R}.$$

1.6 Gaussian measures and Random Gaussian variables

In this section we introduce the notion of Gaussian measure on a infinite dimensional separable Banach space. We refer to [12] for a detailed overview of this topic. Before introducing the Gaussian measures we need to recall some standard notations from probability theory.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $\mathcal{K}$ be a separable Banach space equipped with the norm $\|\cdot\|_{\mathcal{K}}$. Let $\xi : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathcal{K}, \mathcal{B}(\mathcal{K}))$ be a random variable, namely a measurable function. We call law of ξ the probability measure on $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$ defined by

$$\mathscr{L}(\xi)(B) := \mathbb{P} \circ \xi^{-1}(A) := \mathbb{P}(\xi^{-1}(A)) = \mathbb{P}(\{\omega \in \Omega \ : \ \xi(\omega) \in A\}), \quad A \in \mathcal{B}(\mathcal{K}).$$

Sometimes we will use the notation $\xi \sim \gamma$ to indicate that the random variable ξ has law γ. We denote

$$\mathbb{E}[\xi] := \int_\Omega \xi(w)\ \mathbb{P}(d\omega) = \int_{\mathcal{K}} x\ \mathscr{L}(\xi)(dx)$$

the expectation of ξ with respect to $\mathbb{P}$. If γ is a probability measure on $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$ we denote by $\widehat{\gamma}$ its Fourier transform, namely

$$\widehat{\gamma}(l) := \int_{\mathcal{K}} e^{il(x)}\gamma(dx), \quad l \in \mathcal{K}^*.$$

We define the support of γ as the closed subset $S \subset \mathcal{K}$ such that $S \subset V$ for any $V \in \mathcal{B}(\mathcal{K})$ such that $\gamma(V) = 1$.

Let γ_1 and γ_2 be two probability measures on $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$. We say that γ_1 is absolutely continuous with respect to γ_2 if $\gamma_2(B) = 0$ implies that $\gamma_1(B) = 0$, for any $B \in \mathcal{B}(\mathcal{K})$. If γ_1 is absolutely continuous with respect to γ_2 and viceversa we say that γ_1 and γ_2 are equivalent. We say that γ_1 and γ_2 are singular if there exists $B \in \mathcal{B}(\mathcal{K})$ such that $\gamma_1(B) = 0$ and $\gamma_2(B) = 1$.

Let γ_1 and γ_2 be two probability measures on $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$, if γ_1 is absolutely continuous with respect to γ_2, then, by the Radon–Nykodym, theorem there exists a unique non-negative $\rho : \mathcal{K} \to \mathbb{R}$ non-negative such that

$$\gamma_1(dx) = \rho(x)\gamma_2(dx).$$

The function ρ is called density (or Radon–Nykodym derivative) of γ_1 with respect to γ_2. If also γ_2 is absolutely continuous with respect to γ_1, then ρ is positive and

$$\gamma_2(dx) = \frac{1}{\rho(x)}\gamma_1(dx).$$

Now we introduce the notion of Gaussian measure on $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$. We recall that a probability measure γ is a nondegenerate Gaussian measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ if there exist $m \in \mathbb{R}$ and $\sigma > 0$ such that

$$\gamma(A) = \frac{1}{\sqrt{2\pi}} \int_A e^{\frac{|x-m|^2}{2\sigma^2}} dx, \quad A \in \mathcal{B}(R).$$

Instead we say that γ is a degenerate Gaussian measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ if there exist $m \in \mathbb{R}$ such that

$$\gamma(A) = \delta_m(A), \quad A \in \mathcal{B}(R),$$

where $\delta_m(\cdot)$ is the Dirac measure defined by

$$\delta_m(A) = \begin{cases} 1, & m \in A \\ 0, & m \notin A. \end{cases}$$

Definition 1.6.1. *Let γ be a probability measure on $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$. We say that γ is a Gaussian measure if for any $l \in \mathcal{K}^*$ the probability measure $\gamma \circ l^{-1}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a Gaussian measure. We say that γ is a nondegenerate Gaussian measure if for any $l \in \mathcal{K}^*$ such that $l \neq 0$ the probability measure $\gamma \circ l^{-1}$ is a nondegenerate Gaussian measure.*

Definition 1.6.2. *Let $\xi : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathcal{K}, \mathcal{B}(\mathcal{K}))$ be a random variable. We say that ξ is a Gaussian random variable if for any $l \in \mathcal{K}^*$ the random variable $l(\xi)$ is a real Gaussian random variable. We say that ξ is a nondegenerate Gaussian random variable if for any $l \in \mathcal{K}^*$ such that $l \neq 0$ the random variable $l(\xi)$ is a real nondegenerate Gaussian random variable.*

Now we define the notions of mean and covariance for a Gaussian measure.

Proposition 1.6.3. *Let γ be a Gaussian measure on $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$. We consider the applications $a_\gamma : \mathcal{K}^* \to \mathbb{R}$ and $B_\gamma : \mathcal{K}^* \times \mathcal{K}^* \to \mathbb{R}$ defined by*

$$a_\gamma(l) := \int_{\mathbb{R}} l(x)\gamma(dx), \quad l \in \mathcal{K}^*,$$

$$B_\gamma(l_1, l_2) := \int_R (l_1(x) - a_\gamma(l_1))(l_2(x) - a_\gamma(l_2))\gamma(dx) \quad l_1, l_2 \in \mathcal{K}^*.$$

a_γ is linear and continuous and it is called mean of γ. B_γ is bilinear, symmetric, non-negative and continuous and it is called covariance of γ. We say that γ is centered if $a_\gamma(\cdot) \equiv 0$.

Remark 1.6.4. *Let $\xi : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathcal{K}, \mathcal{B}(\mathcal{K}))$ be a Gaussian random variable such that $\xi \sim \gamma$. For any $l \in \mathcal{K}^*$, we have that $l(\xi)$ is a real gaussian random variable with mean $a_\gamma(l)$ and variance $B_\gamma(l, l)$. In particular ξ is a nondegenerate Gaussian random variable if and only if $B_\gamma(l, l) \neq 0$ for any $l \in \mathcal{K}^*$.*

Since $\mathcal{K}$ is separable, as in finite dimensional case we have a characterization for the Gaussian measures via their Fourier transforms.

Proposition 1.6.5. *Let γ be a probability measure on $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$. γ is Gaussian if and only if there exist $a_\gamma : \mathcal{K}^* \to \mathbb{R}$ linear and continuous and $B_\gamma : \mathcal{K}^* \times \mathcal{K}^* \to \mathbb{R}$ bilinear, symmetric, non-negative and continuous such that*

$$\widehat{\gamma}(l) = e^{ia_\gamma(l) - \frac{1}{2}B_\gamma(l,l)}, \quad l \in \mathcal{K}^*.$$

Now we state the following theorem which ensures that even in infinite dimension the Gaussian measures have finite moments of every order.

Theorem 1.6.6 (Fernique)**.** *Let γ be a centered Gaussian measure on $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$. Then there exists $\alpha > 0$ such that*

$$\int_{\mathcal{K}} e^{\alpha \|x\|^2_{\mathcal{K}}} \gamma(dx) < +\infty. \tag{1.6.1}$$

1.6.1 The reproducing Kernel and the Cameron–Martin theorem

Let γ be a Gaussian measure on $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$. It is easy to see that $\mathcal{K}^*$ is contained in $L^2(\mathcal{K}, \gamma)$ and that the map $j : \mathcal{K}^* \to L^2(\mathcal{K}, \gamma)$ defined by

$$j(l) = l - a_\gamma(l), \quad l \in \mathcal{K}^*,$$

is continuous.

Definition 1.6.7. *The reproducing kernel of γ is the space defined by*

$$\mathcal{K}^*_\gamma := \overline{j(\mathcal{K}^*)}^{L^2(\mathcal{K},\gamma)}.$$

It is not difficult to prove that a_γ and B_γ can be continuously extended to $\mathcal{K}^*_\gamma$. Moreover for any $l \in \mathcal{K}^*_\gamma$ we have

$$\widehat{\gamma} = e^{-\frac{1}{2}\|l\|^2_{L^2(\mathcal{K},\gamma)}}.$$

Now we define the operator $R_\gamma : \mathcal{K}^*_\gamma \to (\mathcal{K}^*)'$ by

$$R_\gamma(l)(g) = \langle l, g - a_\gamma(g)\rangle, \quad l \in \mathcal{K}^*_\gamma,\ g \in \mathcal{K}^*.$$

Using the fact that $\mathcal{K}$ is separable, it is possible to prove that for any $l \in \mathcal{K}^*_\gamma$ there exists a unique $x_l \in \mathcal{K}$ such that

$$R_\gamma(l)(g) = g(x_l), \quad g \in \mathcal{K}^*.$$

Hence, from here on, we identify $R_\gamma(l)$ with the element $x_l \in \mathcal{K}$ and we write

$$R_\gamma(l)(g) = g(R_\gamma(l)), \quad g \in \mathcal{K}^*.$$

Definition 1.6.8. *The Cameron–Martin space of the measure γ is the space defined by*

$$H_\gamma := \{h \in \mathcal{K} \ :\ \|h\|_{H_\gamma} < +\infty\}, \quad \|h\|_{H_\gamma} := \sup\{l(h) : l \in \mathcal{K}^*,\ \|j(l)\|_{L^2(\mathcal{K},\gamma)} \le 1\}.$$

Proposition 1.6.9. *Let $h \in \mathcal{K}$. Then $h \in H_\gamma$ if and only if there exists $\widehat{h} \in \mathcal{K}^*_\gamma$ such that $h = R_\gamma(\widehat{h})$, and in this case*

$$\|h\|_{H_\gamma} = \|\widehat{h}\|_{L^2(\mathcal{K},\gamma)}.$$

Therefore R_γ is an isometry from $\mathcal{K}^*_\gamma$ to H_γ and H_γ is a Hilbert space equipped with the inner product

$$\langle h, k\rangle_{H_\gamma} := \left\langle \widehat{h}, \widehat{k}\right\rangle_{L^2(\mathcal{K},\gamma)}.$$

The space H_γ is called Cameron–Martin space of γ.

Let $h \in \mathcal{K}$. We define the probability measure

$$\gamma_h(B) = \gamma(B - h), \quad B \in \mathcal{B}(\mathcal{K}).$$

In the finite dimensional case it is easy to see that γ_h and γ are equivalent. However in the infinite dimensional case this is not true, in general. The Cameron-Martin theorem shows exactly for which $h \in \mathcal{K}$ this fact occurs.

Theorem 1.6.10. *If $h \in H_\gamma$, then γ and γ_h are equivalent, and we have*

$$\gamma_h = e^{\widehat{h} - \frac{1}{2}\|h\|^2_{H_\gamma}}\gamma.$$

If $h \notin H_\gamma$, then γ and γ_h are singular.

1.6.2 The Hilbert case

In this subsection we focus on the particular case where $\mathcal{K}$ is a separable Hilbert space. We refer to the books [43, 71] for a more detailed overview of this topic.

Let $\mathcal{K}$ be a separable Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{K}}$. Let γ be a Gaussian measure on $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$ with mean a_γ and covariance B_γ. By the Riesz representation theorem we can identify $\mathcal{K}$ with $\mathcal{K}^*$. There exists $a \in \mathcal{K}$ and a non-negative operator $Q \in \mathcal{L}(\mathcal{K})$ such that

$$a_\gamma(l) = \langle l, a \rangle, \quad l \in \mathcal{K},$$

$$B_\gamma(l_1, l_2) = \langle Ql_1, l_2 \rangle, \quad l_1, l_2 \in \mathcal{K}.$$

We have the following characterization for the Gaussian measures and for their Cameron-Martin spaces.

Proposition 1.6.11. *Let γ be a probability measure on $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$. γ is Gaussian if and only if there exist $a \in \mathbb{R}$ and a linear, non-negative and trace class operator $Q : \mathcal{K} \to \mathcal{K}$ such that*

$$\widehat{\gamma}(x) = e^{\langle x, a \rangle - \frac{1}{2}\left\|Q^{1/2}x\right\|^2}, \quad x \in \mathcal{K}.$$

In this case we use the notation $\gamma \sim \mathcal{N}(a, Q)$.

We fix $a \in \mathcal{K}$ and a non-negative and trace class operator Q. By Theorem 1.5.3, there exists an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ of $\mathcal{K}$ such that

$$\lambda_k e_k = Qe_k, \quad k \in \mathbb{N},$$

where $\{\lambda_k\}_{k\in\mathbb{N}}$ are the eigenvalues of Q. In particular, since Q is non-negative then $\lambda_k \geq 0$ for any $k \in \mathbb{N}$. So we have a useful characterization for the random Gaussian variables, as follows.

Proposition 1.6.12. *Let $\xi : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathcal{K}, \mathcal{B}(\mathcal{K}))$ be a random variable. $\xi \sim \mathcal{N}(a, Q)$ if and only if*

$$\xi = \sum_{k\in\mathbb{N}} \beta_k \sqrt{\lambda_k} e_k + a, \tag{1.6.2}$$

where $\{\beta_k\}_{k\in\mathbb{N}}$ are independent real Gaussian random variables with mean 0 *and variance* 1. *The series in* (1.6.2) *converges in $L^2((\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{K}, \mathcal{B}(\mathcal{K})))$. Moreover we have*

$$\mathbb{E}[\langle \xi, u \rangle] = \langle m, u \rangle, \quad u \in \mathcal{K},$$

$$\mathbb{E}[\langle \xi - a, u\rangle\langle \xi - a, v\rangle] = \langle Qu, v\rangle, \quad u, v \in \mathcal{K},$$
$$\mathbb{E}[\|\xi - a\|^2_{\mathcal{K}}] = \mathrm{Tr}[Q].$$

We also have a characterization of the Cameron–Martin space.

Proposition 1.6.13. *Let $\gamma \sim \mathcal{N}(a, Q)$. Then the Cameron-Martin space of γ has the following characterization*

$$H_\gamma = Q^{1/2}(\mathcal{K}), \quad \langle h, k\rangle_{H_\gamma} = \left\langle Q^{-\frac{1}{2}}h, Q^{-\frac{1}{2}}k\right\rangle, \quad k, h \in H_\gamma,$$

where $Q^{-1/2}$ is the pseudo-inversa of $Q^{-1/2}$, see Subsection 1.2.5.

Let $\gamma \sim \mathcal{N}(a, Q)$. The next properties follow from the spectral decomposition of Q.

1. For any $\alpha < \inf\{\frac{1}{2\lambda_k} \ : \ k \in \mathbb{N}\}$, (1.6.1) is verified.
2. $\{\lambda_k^{1/2} e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of the Cameron-Martin space $H_\gamma = Q^{1/2}(\mathcal{K})$.
3. The Cameron-Martin space H_γ has finite dimension $\Leftrightarrow$ the set $\sigma_p(Q)$ of the eigenvalues of Q is finite.
4. The Cameron-Martin space H_γ is dense in $\mathcal{X}$ $\Leftrightarrow$ $\lambda_k > 0$ for any $k \in \mathbb{N}$ $\Leftrightarrow$ Q is positive $\Leftrightarrow$ γ is a nondegenerate Gaussian measure.
5. If Q is positive then $\mathrm{supp}\,(\gamma) = \mathcal{K}$ and

$$\mathcal{K}^*_\gamma := \left\{ f : \mathcal{K} \to \mathbb{R} \ : \ f(x) = \sum_{k\in\mathbb{N}} \langle x - m, e_k\rangle \lambda_k^{1/2} \right\},$$

where the series $\sum_{k\in\mathbb{N}} \langle x - m, e_k\rangle\lambda_k^{1/2}$ converges in $L^p(\mathcal{K}, \gamma)$, for any $p \geq 1$. Moreover

$$H_\gamma := \left\{ x \in \mathcal{K} \ : \ \sum_{k\in\mathbb{N}} \langle x, e_k\rangle \lambda_k^{-1} < +\infty \right\}.$$

The next result is a generalization of the Cameron-Martin theorem.

Theorem 1.6.14 (Feldman-Hajek)**.** *Let $\gamma_1 = \mathcal{N}(m_1, Q_1)$ and $\gamma_2 = \mathcal{N}(m_2, Q_2)$ be two Gaussian measures on $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$. γ_1 and γ_2 are equivalent if and only if the following conditions are verified.*

1. $H := Q_1^{1/2}(\mathcal{K}) = Q_2^{1/2}(\mathcal{K})$.
2. $m_1 - m_2 \in H$
3. $(Q_1^{-1/2}Q_2^{1/2})(Q_1^{-1/2}Q_2^{1/2})^* - \mathrm{I}$ *is a Hilbert–Schmidt operator on* $\overline{H}^{\mathcal{K}}$.

1.6.3 Integration by part formula

In this section we will show that it is possible to associate an integration by part formula to any Gaussian measure on a separable infinite dimensional Hilbert space $\mathcal{K}$. We refer to [42, Chapters 9-10].

Let $\mathcal{K}$ be a separable Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{K}}$. Let $Q \in \mathcal{L}(\mathcal{K})$ be a positive and trace class operator. Let $\{e_k\}_{k\in\mathbb{N}}$ be orthonormal basis of $\mathcal{K}$ consisting of eigenvectors of Q, and let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the corresponding eigenvalues. Let μ be the Gaussian measure on $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$ with mean 0 and covariance operator Q. Let $\xi(\mathcal{K})$ be the subspace of $C_b^\infty(\mathcal{K})$ spanned by the exponential functions of the form

$$\varphi(x) = e^{\langle x, h\rangle_{\mathcal{K}}}, \quad h \in \mathcal{K}.$$

Proposition 1.6.15. *Let $\varphi, \psi \in \xi(\mathcal{K})$. Then for any $k \in \mathbb{N}$ and $x \in \mathcal{K}$ we have*

$$\int_{\mathcal{K}} \left(\frac{\partial \varphi}{\partial e_k}\right)(x)\psi(x)\mu(dx) + \int_{\mathcal{K}} \varphi(x)\left(\frac{\partial \psi}{\partial e_k}\right)(x)\mu(dx) = \frac{1}{\lambda_k}\int_{\mathcal{K}} x_k \varphi(x)\psi(x)\mu(dx), \tag{1.6.3}$$

where $\dfrac{\partial \varphi}{\partial e_k}(x) = \langle \nabla\varphi(x), e_k\rangle_{\mathcal{K}}$, $\dfrac{\partial \psi}{\partial e_k}(x) = \langle \nabla\psi(x), e_k\rangle_{\mathcal{K}}$ and $x_k = \langle x, e_k\rangle_{\mathcal{K}}$.

Definition 1.6.16. *Let $\{g_k\}_{k\in\mathbb{N}}$ be an arbitrary orthonormal basis of $\mathcal{K}$. For any $k \in \mathbb{N}$, we denote by $\mathcal{FC}_b^k(\mathcal{K})$ the set of functions $f : \mathcal{K} \to \mathbb{R}$ such that, for some $n \in \mathbb{N}$, there exists a function $\varphi \in C_b^k(\mathbb{R}^n)$ such that for all $x \in \mathcal{K}$*

$$f(x) = \varphi(\langle x, g_1\rangle, \dots, \langle x, g_n\rangle).$$

We call maps of this type cylindrical functions. We denote by $\mathcal{FC}_b^k(\mathcal{K};\mathcal{K})$ the linear span of the functions $x \to v(x)y$ with $v \in \mathcal{FC}_b^k(\mathcal{K})$ and $y \in \mathcal{K}$.

It is also possible to define the integration by part formula (1.6.3) on the space $\mathcal{FC}_b^1(\mathcal{K})$ (see, for example, [34, 35]). It is easy to prove that $\xi(\mathcal{K})$ is dense in $L^p(\mathcal{K}, \mu)$ for any $p \geq 1$, so we have the following result.

Proposition 1.6.17. *Let $R \in \mathcal{L}(\mathcal{K})$ such that $Ker(R) = \{0\}$ and $Q^{1/2}(\mathcal{K}) \subset R(\mathcal{K})$. Then the operator*

$$R\nabla : \xi(\mathcal{K}) \subseteq L^2(\mathcal{K}, \mu) \to L^2(\mathcal{K}, \mu, \mathcal{K}).$$

is closable. The Sobolev $W_R^{1,2}(\mathcal{K}, \mu)$ is defined as domain of its closure.

In the next chapters of this thesis we will see that it is possible to define Sobolev spaces even with respect to non-Gaussian measures.

1.7 Wiener Processes

In this section we will introduce the notion of Wiener process with values in a infinite dimensional Hilbert space. We refer to [41, 71] for the results in this section.

Before we recall some basic notions about the theory of stochastic process.

1.7.1 Basic definitions

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a separable Banach space $\mathcal{K}$ equipped with the norm $\|\cdot\|_{\mathcal{K}}$ and an interval I of $\mathbb{R}$.

A $\mathcal{K}$-valued stochastic process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is a family $\{Y(t)\}_{t\in I}$ of random variables $Y(t) : (\Omega, \mathcal{F}, \mathbb{P}) \to (K, \mathcal{B}(\mathcal{K}))$. Now we are going to define what we mean by the law of a process. We consider a $\mathcal{K}$-valued stochastic process $\{Y(t)\}_{t\in I}$. Let $\mathcal{K}^I$ be the space of all functions from I to $\mathcal{K}$. Let $n \in \mathbb{N}$ and let S_n be the set of $\tau := (t_1, ..., t_n) \in I^n$ such that $t_i \neq t_j$ for any $i \neq j$. Let $\mathcal{G}_n$ be the σ-field generated by the sets $B_1 \times ... \times B_n$ where $B_1, ..., B_n \in \mathcal{B}(\mathcal{K})$. We denote by $\mathcal{C}_I$ the σ-field generated by the cylindrical sets $C(\tau, B)$ defined by

$$C(\tau, B) := \{f \in \mathcal{K}^T \ : \ (f(t_1), ..., f(t_n)) \in B\},$$

where $n \in \mathbb{N}$, $\tau := (t_1, ..., t_n) \in S_n$ and $B \in \mathcal{G}_n$. It is possible to prove that the map $\mathscr{T} : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathcal{K}^I, \mathcal{C}_I)$ defined by

$$\mathscr{T}(\omega) := Y(\cdot)(\omega).$$

is measurable, so

$$\mathbb{P} \circ \mathscr{T}^{-1}(C(\tau, B)) := \mathbb{P}(\{\omega \in \Omega \ : \ (Y(t_1)(\omega), ..., Y(t_n)(\omega)) \in B\}), \quad C(\tau, B) \in \mathcal{C}_I,$$

is a probability measure on $(\mathcal{K}^I, \mathcal{C}_T)$. The measure $\mathbb{P} \circ \mathscr{T}^{-1}$ is called law of the process $\{Y(t)\}_{t\in I}$.

Let $\{Y(t)\}_{t\in I}$ be a $\mathcal{K}$-valued process on $(\Omega, \mathcal{F}, \mathbb{P})$ we say that $\{Y(t)\}_{t\in I}$ is continuous (right continuous) if the map $Y(\cdot) : [0, +\infty) \to \mathcal{K}$ is $\mathbb{P}$-a.s. continuous (right continuous). For $p \geq 1$, we say that $\{Y(t)\}_{t\in I}$ is p-integrable if, for any $t \in I$, we have

$$\mathbb{E}[\|Y(t)\|^p_{\mathcal{K}}] < +\infty.$$

A filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ is a family of σ-field $\{\mathcal{F}\}_{t\in I}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for any $0 \leq s \leq t$, and we call $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\in I}, \mathbb{P})$ filtered probability space. We say that a filtration $\{\mathcal{F}\}_{t\in I}$ is complete if

- $\mathscr{N} \subset \mathcal{F}_0$, where $\mathscr{N}$ is the set of the elements $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$.

We say that a complete filtration $\{\mathcal{F}\}_{t\in I}$ is normal if

- for any $t > 0$, we have $\mathcal{F}_t = \cap_{s>t} \mathcal{F}_s$.

A filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\in I}, \mathbb{P})$ is normal (complete) if $\{\mathcal{F}\}_{t\in I}$ is normal (complete).

Let $\{\mathcal{F}\}_{t\in I}$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\{Y(t)\}_{t\in I}$ be a $\mathcal{K}$-valued stochastic process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ we say that $\{Y(t)\}_{t\in I}$ is adapted to $\{\mathcal{F}\}_{t\in I}$ if

- $Y(t)$ is $\mathcal{F}_t$-measurable for any $t \in I$,

in this case we say that $\{Y(t)\}_{t\in I}$ is a $\mathcal{K}$-valued stochastic process defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\in I}, \mathbb{P})$. We say that $\{Y(t)\}_{t\in I}$ is predictable in I if $Y(\cdot)(\cdot) : (I \times \Omega, \mathscr{F}_I) \to$

$(\mathcal{K}, \mathcal{B}(\mathcal{K}))$ is measurable where $\mathscr{F}_I$ is the σ-field generated by the sets

$$(s,t] \times F, \quad s,t \in I,\ s < t,\ F \in \mathcal{F}_s.$$

Let $\{Y(t)\}_{t\in I}$ be a $\mathcal{K}$-valued stochastic process defined on $(\Omega, \mathcal{F}, \mathbb{P})$, then there always exists a complete filtration $\{\mathcal{F}_t\}_{t\in I}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\{Y(t)\}_{t\in I}$ is adapted to $\{\mathcal{F}_t\}_{t\in I}$. Indeed it is sufficient to consider the filtration $\{\mathcal{F}_t\}_{t\in I}$ defined by

$$\mathcal{F}_t := \sigma(\mathscr{N} \cup \mathcal{F}_t^0), \quad \mathcal{F}_t^0 := \sigma(Y(s) \ : \ s \leq t), \tag{1.7.1}$$

this filtration is called natural filtration of $\{Y(t)\}_{t\geq 0}$. Let $\{Y_1(t)\}_{t\in I}$ e $\{Y_2(t)\}_{t\in I}$ be two $\mathcal{K}$-valued processes defined on the complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\in I}, \mathbb{P})$. We say that $\{Y_1(t)\}_{t\in I}$ is a version (or a modification) of $\{Y_2(t)\}_{t\in I}$ if, for any $t \in I$ we have

$$Y_1(t) = Y_2(t), \quad \mathbb{P}\text{-a.s.}$$

1.7.2 Q-Wiener process

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a separable Hilbert space $\mathcal{K}$ equipped with the inner product $\langle \cdot, \cdot \rangle_K$. Let $Q \in \mathcal{L}(\mathcal{K})$ be a non negative and trace operator.

Definition 1.7.1. *A $\mathcal{K}$-valued Q-Wiener process $\{W(t)\}_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is a $\mathcal{K}$-valued process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ that verifies the following conditions.*

(i) $W(0) = 0$

(ii) *For any $n \in \mathbb{N}$ and $0 \leq t_1 < t_2 < \cdots < t_n$ the random variables $W(t_1)$, $W(t_2) - W(t_1), \ldots,$ $W(t_n) - W(t_{n-1})$ are independent.*

(iii) *$W(t) - W(s) \sim \mathcal{N}(0, (t-s)Q)$ for any $0 \leq s \leq t$.*

(iv) *$\{W(t)\}_{t\geq 0}$ is continuous.*

We show a useful characterization of $\mathcal{K}$-valued Q-Wiener process.

Proposition 1.7.2. *Let $Q \in \mathcal{L}(\mathcal{K})$ be a non-negative trace class operator. A $\mathcal{K}$-valued process $\{W(t)\}_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is a $\mathcal{K}$-valued Q-Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ if and only if*

$$\{W(t)\}_{t\geq 0} = \left\{ \sum_{k\in\mathbb{N}} \sqrt{\lambda_k} \beta_k(t) e_k \right\}_{t\geq 0}, \tag{1.7.2}$$

where e_k and λ_k are the eigenvectors and eigenvalues of Q respectively, and $\{\beta_k(t)\}_{t\geq 0}$ are independent real Brownian motions. The series in (1.7.2) *converges in*
$L^2((\Omega, \mathcal{F}, \mathbb{P}), C([0,T], \mathcal{K}))$, for any $T > 0$.

We say that $\{W(t)\}_{t\geq 0}$ is a $\mathcal{K}$-valued Q-Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ if $\{W(t)\}_{t\geq 0}$ is a $\mathcal{K}$-valued Q-Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and

1. $\{W(t)\}_{t\geq 0}$ is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$;

2. $\sigma(W(t)-W(s))$ is independent of $\mathcal{F}_s$ for any $0 \le s \le t$.

It is not difficult to prove the following proposition.

Proposition 1.7.3. *Let $\{W(t)\}_{t\ge0}$ be a $\mathcal{K}$-valued Q-Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Then the natural filtration of $\{W(t)\}_{t\ge0}$ defined as in* (1.7.1) *is normal and $\{W(t)\}_{t\ge0}$ is a $\mathcal{K}$-valued Q-Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$.*

We stress that the series in (1.7.2) converges in $L^2((\Omega, \mathcal{F}, \mathbb{P}), C([0,T], \mathcal{K}))$ since the inclusion $Q^{1/2}(\mathcal{K}) \subset \mathcal{K}$ defines a Hilbert–Schmidt embedding from $Q^{1/2}(\mathcal{K})$ to $\mathcal{K}$. In the next subsection we will be interested in the case where Q is not a trace class operator and therefore $Q^{1/2}$ is not a Hilbert–Schmidt operator.

1.7.3 Cylindrical Wiener processes

Let $Q \in \mathcal{L}(\mathcal{K})$ be a non-negative operator. We have seen in Subsection 1.2.5 that the pseudo inverse $Q^{-1/2}$ of $Q^{1/2}$ defines a Hilbert space $(\mathcal{K}_0, \langle , \rangle_0)$ in the following way

$$\mathcal{K}_0 = Q^{1/2}(\mathcal{K}), \quad \langle h, k\rangle_0 = \left\langle Q^{-1/2}h, Q^{-1/2}k\right\rangle, \quad h, k \in \mathcal{K}_0.$$

Moreover let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathrm{Ker}(Q^{1/2})^{\perp}$.
Then $\{g_k\}_{k\in\mathbb{N}} = \{Q^{1/2}e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{K}_0$. Let $\{\alpha_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ be a sequence of positive numbers such that $\sum_{k\in\mathbb{N}} \alpha_k^2 < +\infty$. Then the operator $J : K_0 \to \mathcal{K}$ defined by

$$Jx := \sum_{k\in\mathbb{N}} \alpha_k \langle x, g_k\rangle_0 g_k, \quad x \in \mathcal{K}_0, \tag{1.7.3}$$

is a Hilbert–Schmidt operator. Let J^* be the adjoint of J and let J' be the operator defined by J^* identifying K and K_0 with their dual spaces. We consider operator $Q_1 = JJ' : \mathcal{K} \to \mathcal{K}$. By the definition, Q_1 is a non-negative and a trace class operator. As before we consider the Hilbert space defined by the pseudo inverse of $Q_1^{1/2}$

$$\mathcal{K}_1 := Q_1^{1/2}(\mathcal{K}), \quad \left\langle Q_1^{-1/2}h, Q_1^{-1/2}k\right\rangle_1, \quad h, k \in \mathcal{K}_1.$$

Proposition 1.7.4. *The operator J is an isometry from $(\mathcal{K}_0, \langle , \rangle_0)$ to $(\mathcal{K}_1, \langle , \rangle_1)$. Moreover, let $\{\{\beta_k(t)\}_{t\ge0} : k \in \mathbb{N}\}$ be independent real Brownian motions. The process $\{W(t)\}_{t\ge0}$ defined by*

$$W(t) := \sum_{k\in\mathbb{N}} \beta_k(t) J g_k,$$

is a $\mathcal{K}$-valued Q_1-Wiener process. The series converges in $L^2((\Omega, \mathcal{F}, \mathbb{P}), C([0,T], \mathcal{K}))$, for any $T > 0$.

Remark 1.7.5. *Equivalently, Proposition* (1.7.4) *states that the series*

$$\left\{\sum_{k\in\mathbb{N}} \beta_k(t) Q^{1/2} e_k\right\}_{t\ge0},$$

converges in $L^2((\Omega, \mathcal{F}, \mathbb{P}), C([0,T], \mathcal{K}'))$,
for any $T > 0$, *where* $\mathcal{K}'$ *is the Hilbert space* $(\mathcal{K}, \|J(\cdot)\|_{\mathcal{K}})$. *Hence* $\{W(t)\}_{t\geq 0}$ *is a* $\mathcal{K}'$*-valued* Q*-Wiener process.*

Definition 1.7.6. *We call* $\mathcal{K}$*-valued generalized* Q*-Wiener process the* $\mathcal{K}$*-valued* Q_1*-Wiener process of Proposition 1.7.4. If* $Q = \mathrm{I}_{\mathcal{K}}$ *we use the expression "*$\mathcal{K}$*-cylindrical Wiener process". Clearly if* $\{W(t)\}_{t\geq 0}$ *is a* $\mathcal{K}$*-valued generalized* Q*-Wiener process such that* Q *is trace class then* $\{W(t)\}_{t\geq 0}$ *is a* $\mathcal{K}$*-valued* Q*-Wiener process, since we can take* $J = \mathrm{I}_{\mathcal{K}}$.

Remark 1.7.7. *Some authors use the expression "*$\mathcal{K}$*-cylindrical* Q*-Wiener process", also for the* $\mathcal{K}$*-valued generalized* Q*-Wiener process, see for example [71]. We refer to [6, 91] for an overview of cylindrical processes.*

1.8 Integration with respect to a Wiener process

We fix a separable Hilbert space $(\mathcal{K}, \|\cdot\|_{\mathcal{K}}, \langle , \rangle_K)$ and a $\mathcal{K}$-valued generalized Q-Wiener process $\{W(t)\}_{t\geq 0}$ defined on a normal filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. In this section we define an integration with respect to $\{W(t)\}_{t\geq 0}$ first in the case where $Q \in \mathcal{L}(\mathcal{K})$ is non-negative and trace class and after in the case where Q is not a trace class operator, we recall that

$$\mathcal{K}_0 := Q^{1/2}(\mathcal{K}), \quad \langle k, h \rangle_0 := \left\langle Q^{-1/2}k, Q^{-1/2}h \right\rangle, \quad h, k \in \mathcal{K}_0 \tag{1.8.1}$$

First of all we have to recall some basic notions from the probability theory.

Proposition 1.8.1. *Let* $\xi \in L^1((\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{K}, \mathcal{B}(\mathcal{K})))$ *be a random variable and let* $\mathcal{G}$ *be a* σ*-field contained in* $\mathcal{F}$*. Then there exists a unique (up to modifications) random variable* $Z \in L^1((\Omega, \mathcal{G}, \mathbb{P}), (\mathcal{K}, \mathcal{B}(\mathcal{K})))$ *such that*

$$\int_A \xi(\omega)\mathbb{P}(d\omega) = \int_A Z(\omega)\mathbb{P}(d\omega), \quad \forall\, A \in \mathcal{G}.$$

The random variable Z *is called conditional expectation of* ξ *with respect to* $\mathcal{G}$ *an it is denoted by* $\mathbb{E}[\xi|\mathcal{G}]$.

Now we define the notion of martingale.

Definition 1.8.2. *Let* $\{Y(t)\}_{t\geq 0}$ *be a* $\mathcal{K}$*-valued stochastic process defined on* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ *(so* $\{Y(t)\}_{t\geq 0}$ *is adapted to the filtration). We say that* $\{Y(t)\}_{t\geq 0}$ *is a* $\mathcal{K}$*-valued martingale (with respect to* $\{\mathcal{F}(t)\}_{t\geq 0}$*) if the following two conditions are verified.*

1. $\mathbb{E}[\|Y(t)\|_{\mathcal{K}}] < +\infty$, *for any* $t \geq 0$.
2. *For any* $0 \leq s \leq t$ *we have*
$$\mathbb{E}[Y(t)|\mathcal{F}_s] = Y(s).$$

We define a space that will be fundamental in the construction of stochastic integral

Definition 1.8.3. *Let $T > 0$. We denote by $(M_T^2, \|\cdot\|_{M_T^2})$ the Banach space of the continuous and square integrable $\mathcal{K}$-martingales $Y := \{Y(t)\}_{t\in[0,T]}$, such that*

$$\|Y\|_{M_T^2} : \mathbb{E}[\sup_{t\in[0,T]} \|Y(t)\|_{\mathcal{K}}^2]^{1/2}.$$

It is possible to prove that $\{W(t)\}_{t\geq 0}$ belongs to M_T^2. Now we have all the tools to define the stochastic integral.

1.8.1 Stochastic integral when Q is a trace class operator

We assume that the operator Q is non-negative and trace class. We refer to [41, Section 4.2] and [71, Section 2.3] for a detailed study about this argument. We define a set of processes for which the stochastic integral is well defined.

Definition 1.8.4. *Let $T > 0$. We denote by Ξ_T the set of $\mathcal{L}(\mathcal{K})$-valued processes $\{\phi(t)\}_{t\geq 0}$ of the form*

$$\phi(t) = \phi_i, \quad t \in (t_i, t_{i+1}], \quad i = 1, ..., n.$$

where $n \in \mathbb{N}$, $0 = t_1 < ... < t_n = T$ and ϕ_i is a $\mathcal{L}(\mathcal{K})$-valued random variable $\mathcal{F}_{t_i}$-measurable, for any $i = 1, ..., n$.

We fix $T > 0$. For any $\{\phi(t)\}_{t\in[0,T]} \in \Xi_T$ we consider the process $\{\mathcal{I}(\phi)(t)\}_{t\in[0,T]}$ defined by

$$\mathcal{I}(\phi)(t) := \int_0^t \phi(s)dW(s) := \sum_{k=0}^{n-1} \phi_i \left(W(\min(t_{i+1}, t)) - W(\min(t_i, t))\right), \quad t \in [0, T]. \tag{1.8.2}$$

Proposition 1.8.5. *For any $\{\phi(t)\}_{t\in[0,T]} \in \Xi_T$ the process $\mathcal{I}(\phi) := \{\mathcal{I}(\phi)(t)\}_{t\in[0,T]}$ defined in (1.8.2) belongs to M_T^2 and*

$$\|\mathcal{I}(\phi)\|_{M_T^2} = \|\mathcal{I}(\phi)\|_T := \mathbb{E}\left[\int_0^T \|\phi(s)\|^2_{\mathcal{L}_2^0(\mathcal{K})} ds\right]^{\frac{1}{2}} = \mathbb{E}\left[\int_0^T \mathrm{Tr}[\phi(s)Q\phi(s)^*]ds\right]^{\frac{1}{2}},$$

where $\mathcal{L}_2^0(\mathcal{K}) := \mathcal{L}_2(\mathcal{K}_0, \mathcal{K})$ and $\mathcal{K}_0$ is the Hilbert space defined in (1.8.1). Moreover the map $\mathcal{I} : \Xi \to M_T^2$ is an isometry.

$\mathcal{I}$ can be extended to the closure $\overline{\Xi}_T$ of Ξ_T. The extension of $\mathcal{I}$ is an isometry and it is the unique continuous extension of $\mathcal{I}$. In the next proposition we characterize $\overline{\Xi}$.

Proposition 1.8.6. *We denote by $\mathcal{N}_W^2([0, T], L_2^0(\mathcal{K}))$ the closure of Ξ, namely the set of processes $\phi := \{\phi(t)\}_{t\in[0,T]}$ such that $\phi := \{\phi(t)\}_{t\in[0,T]}$ is predictable and*

$$\|\phi\|_T < +\infty.$$

For any $\phi \in \mathcal{N}_W^2([0, T], \mathcal{L}_2^0(\mathcal{K}))$ the stochastic integral

$$\int_0^t \phi(s)dW(s),$$

is well defined and it belongs to M_T^2.

1.8.2 Stochastic integral when Q is not a trace class operator

We assume that the operator Q is non-negative. We refer to [41, Section 4.2] and [71, Section 2.5] for a detailed study about this argument.

We fix a sequence of positive numbers $\{\alpha_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ such that $\sum_{k\in\mathbb{N}} \alpha_k^2 < +\infty$. Let J be the isometry defined in (1.7.3) using $\{\alpha_k\}_{k\in\mathbb{N}}$. In Subsection 1.7.3 we have defined $\{W(t)\}_{t\geq 0}$ as the $\mathcal{K}$-valued Q_1-Wiener process $\{W_1(t)\}_{t\geq 0}$, where $Q_1 = JJ'$. Let $\mathcal{K}_0 := Q^{1/2}(\mathcal{K})$ and $\mathcal{K}_1 := Q_1^{1/2}(\mathcal{K})$ be the Hilbert spaces defined in subsection 1.7.3. It is easy to prove that

$$L \in \mathcal{L}_2^0(\mathcal{K}) := \mathcal{L}_2(\mathcal{K}_0, \mathcal{K}) \Leftrightarrow L \circ J^{-1} \in \mathcal{L}_2^1(\mathcal{K}) := \mathcal{L}_2(\mathcal{K}_1, \mathcal{K}).$$

Hence, let $T > 0$, for any $\phi \in \mathcal{N}_W^2([0,T], \mathcal{L}_2^0(\mathcal{K}))$ we define the stochastic integral with respect to $\{W(t)\}_{t\geq 0}$ by

$$\int_0^t \phi(s)dW(s) = \int_0^t \phi(s) \circ J^{-1} dW_1(s), \quad t \in [0,T]. \tag{1.8.3}$$

It is possible to prove that the definition of stochastic integral does not depend on the choice of $\{\alpha_k\}_{k\in\mathbb{N}}$. Indeed it is possible to prove that the right hand side of (1.8.3) does not depend on the choice of $\{\alpha_k\}_{k\in N}$. Moreover the stochastic integral (1.8.3) belongs to M_T^2.

1.8.3 Properties of stochastic integral

We assume that the operator Q is non-negative. We summarize some useful properties of stochastic integral. We refer to [41, Chapter 4] and [71, Chapter 2] for a detailed study about this argument.

Proposition 1.8.7. *Let $T > 0$.*

1. *For any $\phi \in \mathcal{N}_W^2([0,T], \mathcal{L}_2^0(\mathcal{K}))$ we have*

$$\begin{aligned}\mathbb{E}\left[\left\|\int_0^t \phi(s)dW(s)\right\|_{\mathcal{K}}^2\right] &= \mathbb{E}\left[\int_0^t \|\phi(s)\|_{\mathcal{L}_2^0(\mathcal{K})}^2 ds\right] \\ &= \mathbb{E}\left[\int_0^t \mathrm{Tr}[\phi(s)Q\phi(s)^*]ds\right], \quad t \in [0,T].\end{aligned}$$

2. *For any $\phi \in \mathcal{N}_W^2([0,T], \mathcal{L}_2^0(\mathcal{K}))$, and $p > 0$ there exists $c_p > 0$ such that we*

$$\begin{aligned}\mathbb{E}\left[\sup_{r\in[0,t]}\left\|\int_0^r \phi(s)dW(s)\right\|_{\mathcal{K}}^p\right] &\leq c_p\mathbb{E}\left[\int_0^t \|\phi(s)\|_{\mathcal{L}_2^0(\mathcal{K})}^2 ds\right]^{p/2} \\ &= c_p\mathbb{E}\left[\int_0^t \mathrm{Tr}[\phi(s)Q\phi(s)^*]ds\right]^{p/2}, \quad t \in [0,T].\end{aligned}$$

It is also possible to prove a Itô formula.

Theorem 1.8.8. *For $T > 0$ and $x \in \mathcal{K}$, consider the process $\{X(t,x)\}_{t\in[0,T]}$ defined by*

$$X(t,x) := x + \int_0^t \varphi(s)ds + \int_0^t \phi(s)dW(s), \quad t \in [0,T],$$

where $\phi \in \mathcal{N}^2_W([0,T], \mathcal{L}^0_2(\mathcal{K}))$ and $\{\varphi(t)\}_{t\in[0,T]}$ is a predictable $\mathcal{K}$-valued process such that $\varphi(t) \in L^1((\Omega, \mathcal{F}_t, \mathbb{P}), (\mathcal{K}, \mathcal{B}(\mathcal{K})))$.

If $F : [0,T] \times \mathcal{K} \to \mathcal{K}$ is twice Fréchet differentiable then

$$\begin{aligned} F(t, X(t,x)) = F(0,x) &+ \int_0^t \langle F_x(s, X(s,x)), \phi(s)dW(s)\rangle + \\ &+ \int_0^t \left[F_t(s, X(s)) + \langle F_x(s, X(s)), \varphi(s)\rangle\right] ds \\ &+ \frac{1}{2}\int_0^t \mathrm{Tr}[F_{xx}(s, X(s,x))\phi(s)Q\phi(s)^*]ds \end{aligned}$$

where $F_t = \dfrac{\partial F}{\partial t}$, $F_x = \dfrac{\partial F}{\partial x}$ and $F_{xx} = \dfrac{\partial^2 F}{\partial x^2}$.

By Propositions 1.7.2 and 1.7.4, for any $t \geq 0$ we know that

$$W(t) = \left\{\sum_{k\in\mathbb{N}} \beta_k(t) Q^{1/2} e_k\right\}_{t\geq 0}, \quad \mathbb{P}\text{-a.s.} \tag{1.8.4}$$

where $\{\{\beta_k(t)\}_{t\geq 0} \,:\, k \in \mathbb{N}\}$ are independent real Brownian motions, $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{K}$. The series in (1.8.4) converges in $L^2((\Omega, \mathcal{F}, \mathbb{P}), C([0,T], U))$, with $U = \mathcal{K}$ if Q is trace class or $U = \mathcal{K}'$ (see Remark 1.7.5) if Q is not trace class. Let $T > 0$. For any $\phi \in \mathcal{N}^2_W([0,T], \mathcal{L}^0_2(\mathcal{K}))$ and $h \in \mathcal{K}$ we denote by

$$\int_0^t \langle \phi(s)h, dW(s)\rangle := \sum_{k\in\mathbb{N}} \int_0^t \left\langle \phi(s)h, Q^{1/2}e_k\right\rangle d\beta_k(s), \quad t \geq 0.$$

1.8.4 The stochastic convolution

We assume that the operator Q is non-negative. We refer to [41, Chapter 5].

We fix one and for all $R \in \mathcal{L}(\mathcal{K})$. Let A be the infinitesimal generator of a strongly continuous semigroup e^{tA} on $\mathcal{K}$. We consider the stochastic convolution process $\{W_A(t)\}_{t\geq 0}$ defined by

$$W_A(t) := \int_0^t e^{(t-s)A} R dW(s), \quad t \geq 0.$$

For $t > 0$ set

$$\{\phi_t(s)\}_{s\in[0,t]} := \{e^{(t-s)A}R\}_{s\in[0,t]},$$

so the stochastic convolution is well defined if for any $t \geq 0$ we have

$$\mathbb{E}\left[\int_0^t \|\phi_t(s)\|^2_{\mathcal{L}^0_2(\mathcal{K})}\right] = \int_0^t \left\|e^{sA}R\right\|^2_{L^0_2(\mathcal{K})} ds = \int_0^t \mathrm{Tr}[e^{sA}RQR^*e^{sA^*}]ds < +\infty.$$

Since $\phi_t(s)$ depends on t we can not claim that the process $\{W_A(t)\}_{t\geq 0}$ is a martingale with continuous trajectories. However the process $\{W_A(t)\}_{t\geq 0}$ has some useful features, as the next propositions shows.

Proposition 1.8.9. *Let $T > 0$ and assume that*

$$\int_0^t \mathrm{Tr}[e^{sA}RQR^*e^{tA^*}]ds < +\infty.$$

Then the following statements are verified.

(i) *The process $\{W_A(t)\}_{t\in[0,T]}$ is Gaussian, predictable and continuous in mean square, namely for any $t_0 \in [0,T]$ and for any sequence $\{t_n\}_{n\in\mathbb{N}}$ converging to t_0 we have*

$$\lim_{n\to\infty} \mathbb{E}[\|W_A(t_n) - W_A(t_0\|_K^2] = 0.$$

(ii) *For any $t \in [0,T]$, the random variable $W_A(t)$ is Gaussian with mean 0 and covariance operator Q_t, where*

$$Q_t := \int_0^t e^{sA}RQR^*e^{sA^*}ds.$$

(iii) *The trajectories of the process $\{W_A(t)\}_{t\in[0,T]}$ belongs to $L^2([0,T],\mathcal{K})$ $\mathbb{P}$-a.s.. Moreover the law of the process $\{W_A(t)\}_{t\in[0,T]}$ is the Gaussian measure on $L^2([0,T],\mathcal{K})$ with mean 0 and covariance operator $\mathcal{Q}_T$, where for any $\varphi \in L^2([0,T],\mathcal{K})$*

$$Q_T\varphi(t) := \int_0^T \left(\int_0^{\min(s,t)} e^{(t-r)A}RQR^*e^{(s-r)A^*}dr\right) ds, \quad t \in [0,T].$$

(iv) *For any $p > 1$ there exists c_p such that*

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|W_A(t)\|_{\mathcal{K}}^p\right] \leq c_p \left(\int_0^T \mathrm{Tr}[e^{sA}RQR^*e^{sA^*}]ds\right)^{p/2} < +\infty.$$

In some cases, for every $T > 0$, $\{W_A(t)\}_{t\in[0,T]}$ has Gaussian law even on $C([0,T],\mathcal{K})$.

Proposition 1.8.10. *Let $T > 0$ and assume that*

$$\int_0^T \mathrm{Tr}[e^{sA}RQR^*e^{tA^*}]ds < +\infty.$$

Let $M \subseteq \mathcal{K}$ be a separable Banach space densely and continuously embedded in $\mathcal{K}$ such that for $\mathbb{P}$-a.a. $\omega \in \Omega$ the function $W_A(\cdot)(\omega)$ belongs to $C([0,T],M)$. Then the trajectories of the process $\{W_A(t)\}_{t\in[0,T]}$ belongs to $C([0,T],M)$ $\mathbb{P}$-a.s. and $\{W_A(t)\}_{t\in[0,T]}$ has Gaussian law on $C([0,T],M)$. Moreover

$$\mathbb{E}[\sup_{t\in[0,T]} \|W_A(t)\|_E^p] < +\infty, \quad \forall p \geq 1.$$

We refer to [78, Remark 3.4] for a proof of Proposition 1.8.10. In the case where $M = \mathcal{K}$ we have a sufficient condition to ensure the continuity of $\{W_A(t)\}_{t\geq 0}$.

Proposition 1.8.11. *Let $T > 0$. If there exists $\eta \in (0,1)$ such that*

$$\int_0^T \frac{1}{s^\eta} \mathrm{Tr}[e^{sA} RQR^* e^{tA^*}] ds < +\infty,$$

then for $\mathbb{P}$-a.a. $\omega \in \Omega$ the function $W_A(\cdot)(\omega)$ belongs to $C([0,T], \mathcal{K})$.

In [20, Chapter 6] and [41, Chapter 5] the authors present some explicit cases where $\{W_A(t)\}_{t\geq 0}$ is continuous.

1.9 The Markov processes

In this section we recall some basic definitions and results about the Markov processes. We refer to this books [42, 41, 43, 48, 49, 95] for a complete overview.

Let $\mathcal{K}$ be a separable Banach space. We denote by $\mathscr{P}(\mathcal{K})$ the set of all Borel probability measures on $\mathcal{K}$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a normal filtered probability space, throughout this section we will consider processes defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

Definition 1.9.1. *We call transition function on $\mathcal{K}$ a function $\mathcal{P}$ on $[0, +\infty) \times \mathcal{K} \times \mathcal{B}(E)$ such that*

1. *$\mathcal{P}(t, x, \cdot) \in \mathscr{P}(\mathcal{K})$, for any $t \geq 0$, $x \in E$;*
2. *$\mathcal{P}(t, \cdot, \Gamma) \in B_b(\mathcal{K})$, for any $t \geq 0$ and $\Gamma \in \mathcal{B}(E)$;*
3. *for any $t, s \geq 0$, $x \in E$ and $\Gamma \in \mathcal{B}(\mathcal{K})$ we have*
$$\mathcal{P}(t+s, x, \Gamma) = \int_\Gamma \mathcal{P}(t, y, \Gamma)\mathcal{P}(s, x, dy); \tag{1.9.1}$$
4. *$\mathcal{P}(0, x, \Gamma) = \mathbb{I}_\Gamma(x)$, for any $x \in E$ and $\Gamma \in \mathcal{B}(\mathcal{K})$.*

We call $\mathcal{P}(t, x, \cdot)$ transition probabilities and (1.9.1) *Chapman–Kolmogorov equation.*

Let $\mathcal{P}$ be a transition function. Using $\mathcal{P}$ we define a family of linear operators $\{P(t)\}_{t\geq 0}$ acting on $B_b(\mathcal{K})$ in the following way

$$P(t)f(x) := \int_{\mathcal{K}} f(y)\mathcal{P}(t, x, dy), \tag{1.9.2}$$

by (1.9.1), $\{P(t)\}_{t\geq 0}$ is a semigroup on $B_b(\mathcal{K})$.

Definition 1.9.2. *$\{P(t)\}_{t\geq 0}$ is called Markov transition semigroup (or simply transition semigroup) associated to the Markov function $\mathcal{P}$*

Theorem 1.9.3 (Theorem 2.1 of [48])**.** *Let $T(t)$ be a contraction semigroup on $B_b(\mathcal{K})$. This two statements are equivalent.*

- *There exists a Markovian transition function $\mathcal{T}$ on $\mathcal{K}$ such that $T(t)$ is the transition semigroup of $\mathcal{T}$.*

- *$T(t)f \geq 0$ is a non-negative values function for any non-negative values function f and $t \geq 0$.*

In this thesis we consider the following definition of Markov process

Definition 1.9.4. *Let $\{X(t)\}_{t\geq 0}$ be a $\mathcal{K}$-valued process, let $\mathcal{P}$ be a transition function and let $\{P(t)\}_{t\geq 0}$ be the semigroup associated to $\mathcal{P}$ as in* (1.9.2). *We say $\{X(t)\}_{t\geq 0}$ is the homogeneous Markov process with transition function $\mathcal{P}$ if, for any $t, s \geq 0$ and $f \in B_b(\mathcal{K})$ we have*

$$\mathbb{E}[f(X(t+s))|\mathcal{F}_s] = P(t)(f(X(s))), \quad \mathbb{P}\text{-}a.s.$$

In this case $P(t)$ is called transition semigroup associated to $\{X(t)\}_{t\geq 0}$. When the process $\{X(t)\}_{t\geq 0}$ is the solution of a stochastic equation we say that $P(t)$ is the transition semigroup associated to the equation.

Remark 1.9.5. *The process $\{X(t)\}_{t\geq 0}$ of Definition 1.9.4 satisfies the following Markov property: for any $t, s \geq 0$ and $\Gamma \in \mathcal{B}(\mathcal{K})$ we have*

$$\mathbb{E}\left[\mathbb{I}_{\{\omega\in\Omega\,:\,X(t+s)(\omega)\in\Gamma\}}|\mathcal{F}_s\right] = \mathbb{E}\left[\mathbb{I}_{\{\omega\in\Omega\,:\,X(t+s)(\omega)\in\Gamma\}}|\sigma(X(s))\right].$$

Now we state a useful result to prove the existence and uniqueness of an invariant measure for a transition semigroup. First of all we need to define a notion of convergence for measures.

Definition 1.9.6. *Let $\{\mu_n\}_{n\in\mathbb{N}} \subseteq \mathscr{P}(\mathcal{K})$ and $\mu \in \mathscr{P}(\mathcal{K})$. We say that $\{\mu_n\}_{n\in\mathbb{N}}$ narrow converges to μ ($\mu_n \to_* \mu$) if, for any $\varphi \in C_b(\mathcal{K})$, we have*

$$\lim_{n\to+\infty} |\langle\varphi, \mu_n\rangle_* - \langle\varphi, \mu\rangle_*| = 0.$$

where

$$\langle\varphi, \mu\rangle_* := \int_{\mathcal{K}} \varphi(x)\mu(dx).$$

Moreover we call Narrow topology ($\tau^*(\mathcal{K})$) the coarsest topology on the space $\mathscr{P}(\mathcal{K})$ such that, for any $f \in C_b(\mathcal{K})$, the function $\mu \to \int_{\mathcal{K}} f d\mu$ is continuous.

Before proceeding we make some remarks on narrow topology.

Remark 1.9.7.

1. *If $\mathcal{K}$ has infinite dimension then $C_b(\mathcal{K})^*$ does not coincide with $\mathscr{P}(\mathcal{K})$, however $\mathscr{P}(\mathcal{K})$ can be identified with a convex subset of the unitary ball of $C_b(\mathcal{K})^*$. In particular the narrow topology $\tau^*(\mathcal{K})$ coincides with the weak* topology on $C_b(\mathcal{K})^*$. We stress that sometimes the weak* topology is just called weak topology.*

2. *Let D be a countable and dense subset of $\mathcal{K}$. The space of all the convex combinations of Dirac measures δ_x such that $x \in D$ is narrow dense in $\mathscr{P}(\mathcal{K})$.*

We refer to [3, Section 5.1] for a more detailed overview about the Narrow topology. Let $\mathcal{P}$ a homogeneous transition function on $\mathcal{K}$. As we defined the transition semigroup $P(t)$ on $B_b(\mathcal{K})$, we also define a semigroup on $\mathscr{P}(\mathcal{K})$.

Definition 1.9.8. *For any $t \geq 0$ and $\mu \in \mathscr{P}(\mathcal{K})$, we set*

$$U(t)\mu(\Gamma) := \int_{\mathcal{K}} \mathcal{P}_t(y,\Gamma)\mu(dy), \quad \Gamma \in \mathcal{B}(\mathcal{K}),$$

The semigroups $P(t)$ and $U(t)$ are related by

$$\langle P(t)\varphi, \mu\rangle_* = \langle \varphi, U(t)\mu\rangle_*, \quad t \geq 0, \varphi \in B_b(\mathcal{K}), \mu \in \mathscr{P}(\mathcal{K}).$$

Definition 1.9.9. *Let $\mu \in \mathscr{P}(\mathcal{K})$. We say that μ is invariant for $P(t)$ if for any $t \geq 0$ we have*

$$U(t)\mu = \mu,$$

namely

$$\int_{\mathcal{K}} P(t)\varphi(x)\mu(dx) = \int_{\mathcal{K}} \varphi(x)\mu(dx), \quad \varphi \in C_b(\mathcal{K}).$$

We have the following useful result about existence and uniqueness of an invariant measure for $P(t)$ (see [43, Proposition 11.4 and Remark 11.6]).

Proposition 1.9.10. *Assume that $P(t)$ is a Feller semigroup. Let $\nu \in \mathscr{P}(\mathcal{K})$ such that for any $x \in \mathcal{K}$ we have*

$$U(t)\delta_x \to_* \nu, \quad \text{as } t \to +\infty,$$

where δ_x is the Dirac measure in x. Then ν is the unique invariant measure of $P(t)$.

A very significant case is when a transition function is associated with a process that is a solution of a particular stochastic differential equation. In the next section we will present a basic example.

1.10 The Ornstein–Uhlenbeck case

In this section we will introduce the Ornstein–Uhlenbeck semigroup. We refer to [74] for an overview in the finite dimensional case and we refer to [75] for an overview in the infinite dimensional case.

1.10.1 The Ornstein–Uhlenbeck semigroup in spaces of continuous functions

Let $\mathcal{K}$ be a separable Hilbert space. Let $\{W(t)\}_{t\geq 0}$ be a $\mathcal{K}$-cylindrical Wiener process defined on a normal filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Let A be the infinitesimal generator of a strongly continuous semigroup e^{tA} on $\mathcal{K}$ and let $R \in \mathcal{L}(\mathcal{K})$ be a positive operator, such that for any $t > 0$ we have

$$\int_0^t \mathrm{Tr}[e^{tA}R^2e^{tA^*}dt] < +\infty. \tag{1.10.1}$$

We consider the stochastic differential equation

$$\begin{cases} dX(t,x) = AX(t,x)dt + RdW(t), & t > 0; \\ X(0,x) = x \in \mathcal{K}. \end{cases} \tag{1.10.2}$$

It is well known that, for any $x \in \mathcal{K}$, (1.10.2) has a unique mild solution, namely a process $\{X(t,x)\}_{t\geq 0}$ such that, for any $t > 0$,it verifies

$$X(t,x) = e^{tA}x + W_A(t), \quad \mathbb{P}\text{-a.s.}$$

where $\{W_A(t)\}_{t\geq 0}$ is the stochastic convolution process defined by

$$W_A(t) := \int_0^t e^{sA} RdW(s), \quad t > 0.$$

Moreover, by the uniqueness of $\{X(t,x)\}_{t\geq 0}$, it is possible to prove that the random variables $X(t,x)$ define a Markov transition function (see Definition 1.9.1). We call Ornstein–Uhlenbeck semigroup the transition semigroup associated to $\{X(t,x)\}_{t\geq 0}$, namely

$$\begin{aligned} T(t)f(x) = \mathbb{E}[f(X(t,x))] &= \int_\Omega f(X(t,x)(\omega))\mathbb{P}(d\omega) \\ &= \int_{\mathcal{K}} f(y)\mathscr{L}(X(t,x))(dy), \quad t \geq 0,\ x \in \mathcal{K},\ f \in B_b(\mathcal{K}), \end{aligned} \tag{1.10.3}$$

Since, for any $t \geq 0$, $W_A(t)$ is Gaussian measure of mean 0 and covariance operator

$$Q_t x := \int_0^t e^{sA} R^2 e^{sA^*} x,$$

then, via a change of variable, the semigroup (1.10.3) has the following Mehler representation

$$T(t)f(x) = \int_{\mathcal{K}} f(e^{tA}x + y)\mathcal{N}(0, Q_t)(dy), \quad f \in B_b(\mathcal{K}),\ x \in \mathcal{K}. \tag{1.10.4}$$

We remark that due to the Mehler representation, it is possible to define the semigroup (1.10.4) also on the space $B_b(\mathcal{K};\mathcal{K})$.
It is well known in literature that $T(t)$ is not strongly continuous in $B_b(\mathcal{K})$ and even in the space of bounded and uniformly continuous functions. For a detailed study of the semigroup $T(t)$ in spaces of continuous functions with weighted sup-norms, we refer to [18, 19], [30, Section 2.8.3] and [40, Section 2]. We are more interested in its behaviour with respect to the mixed topology. For an in-depth study of the mixed topology we refer to [62]. In this section we list the results [62] that will be useful to our aims. Consider the Banach space

$$C_{b,2}(\mathcal{K}) := \left\{ f : \mathcal{K} \to \mathbb{R} \;\middle|\; x \mapsto \frac{f(x)}{1 + \|x\|^2_{\mathcal{K}}} \text{ belongs to } C_b(\mathcal{K}) \right\}.$$

endowed with the norm

$$\|f\|_{b,2} := \sup_{x\in\mathcal{K}} \left(\frac{|f(x)|}{1+\|x\|_{\mathcal{K}}^2} \right), \qquad f \in C_{b,2}(\mathcal{K}).$$

Theorem 1.10.1.

(i) *A sequence* $\{\phi_n\}_{n\in\mathbb{N}} \subseteq C_{b,2}(\mathcal{K})$ *converges with respect to the mixed topology to* $\phi \in C_{b,2}(\mathcal{K})$ *if, and only if,*

$$\sup_{n\in\mathbb{N}} \|\phi_n\|_{b,2} < +\infty,$$

and, for any compact set $K \subseteq \mathcal{K}$,

$$\lim_{n\to+\infty} \sup_{x\in K} (|\phi_n(x) - \phi(x)|) = 0.$$

(ii) *The semigroup* $T(t)$ *defined in* (1.10.3) *is strongly continuous on* $C_{b,2}(\mathcal{K})$ *with respect to the mixed topology.*

We also state a charaterization for the uniform convergence on compact sets, that will be useful next.

Proposition 1.10.2. *A sequence* $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq C_b(\mathcal{K})$ *is uniformly convergent on every compact subset of* $\mathcal{K}$ *to a function* $\varphi \in C_b(\mathcal{K})$ *if, and only if,* $\{\varphi_n\}_{n\in\mathbb{N}}$ *is pointwise convergent to* φ *and the family* $\{\varphi_n \,|\, n \in \mathbb{N}\}$ *is such that, for any* $x_0 \in \mathcal{K}$ *and* $\epsilon > 0$ *there exists* $\delta := \delta(x_0, \varepsilon) > 0$ *such that, for any* $n \in \mathbb{N}$ *and* $x \in \mathcal{K}$ *with* $\|x - x_0\|_{\mathcal{K}} \le \delta$ *we have* $|\varphi_n(x) - \varphi_n(x_0)| \le \epsilon$.

Definition 1.10.3. *We denote by* $L_{b,2}$ *the infinitesimal generator, with respect to the mixed topology, of the semigroup* $T(t)$ *in* $C_{b,2}(\mathcal{K})$.

Therefore the domain of $L_{b,2}$ is defined by

$$\mathrm{Dom}(L_{b,2}) := \left\{ \varphi \in C_{b,2}(\mathcal{K}) \,\middle|\, \exists \lim_{t\to 0} \frac{T(t)\varphi - \varphi}{t} \text{ with respect to the mixed topology} \right\}.$$

By Theorem 1.10.1, we obtain the following charaterization of $\mathrm{Dom}(L_{b,2})$.

Proposition 1.10.4. *A function* $\varphi \in C_{b,2}(\mathcal{K})$ *belongs to* $\mathrm{Dom}(L_{b,2})$ *if, and only if, there exists* $\psi \in C_{b,2}(\mathcal{K})$ *such that*

(i) *for any compact subset* K *of* $\mathcal{K}$,

$$\lim_{t\to 0} \sup_{x\in K} \left(\frac{T(t)\varphi(x) - \varphi(x)}{t} - \psi(x) \right) = 0;$$

(ii) $\sup_{t\in(0,1]} [t^{-1}\|T(t)\varphi - \varphi\|_{b,2}] < +\infty$.

In this case $L_{b,2}\varphi = \psi$.

The action of $L_{b,2}$ is known on a space of smooth functions.

$$\xi_A(\mathcal{K}) := \text{span}\{\text{real and imaginary parts of the functions } x \mapsto e^{i\langle x,h\rangle_{\mathcal{K}}} \mid h \in \text{Dom}(A^*)\}. \tag{1.10.5}$$

We remark that $\xi_A(\mathcal{K})$ is a subset of the space $\mathcal{FC}_b^\infty(\mathcal{K})$. Indeed for $\varphi \in \xi_A(\mathcal{K})$ there exist $m, n \in \mathbb{N}$, $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{R}$ and
$h_1, \dots, h_m, k_1, \dots, k_n \in A^*$ such that

$$\varphi(x) = \sum_{i=1}^m a_i \sin(\langle x, h_i\rangle) + \sum_{j=1}^n b_j \cos(\langle x, k_j\rangle).$$

Proposition 1.10.5. *$L_{b,2}$ is the closure in $C_{b,2}(\mathcal{K})$, endowed with the mixed topology, of the operator L_0 defined as*

$$L_0\varphi(x) := \frac{1}{2}\text{Tr}[C\nabla^2\varphi(x)] + \langle x, A^*\nabla\varphi(x)\rangle_{\mathcal{K}}, \qquad x \in \mathcal{K}, \varphi \in \xi_A(\mathcal{K}). \tag{1.10.6}$$

We have also the following result about the resolvent of $L_{b,2}$.

Proposition 1.10.6. *For any $\lambda > 0$ and $\varphi \in C_{b,2}(\mathcal{K})$, the improper Riemann integral*

$$J(\lambda)\varphi := \int_0^{+\infty} e^{-\lambda t} T(t)\varphi dt,$$

is well defined. Moreover, for every $\lambda > 0$, the operator

$$J(\lambda) : (C_{b,2}(\mathcal{K}), \tau_M) \to (C_{b,2}(\mathcal{K}), \tau_M)$$

is continuous (here τ_M denotes the mixed topology), and $J(\lambda)\varphi = R(\lambda, L_{b,2})\varphi$.

We remark that, by [62, Remark 4.3], Theorem 1.10.1 and Proposition 1.10.6, the operator $L_{b,2}$ is the weak infinitesimal generator of the semigroup $T(t)$ on $C_{b,2}(\mathcal{K})$ in the sense of [18, 19]. By this fact we can use the following approximation result.

Proposition 1.10.7 (Propositions 2.5 and 2.6 of [40])**.** *Let $\varphi \in \text{Dom}(L_{b,2}) \cap C_b^1(\mathcal{K})$. There exists a family $\{\varphi_{n_1,n_2,n_3,n_4} \mid n_1, n_2, n_3, n_4 \in \mathbb{N}\} \subseteq \xi_A(\mathcal{K})$ such that for every $x \in \mathcal{K}$*

$$\lim_{n_1\to+\infty}\lim_{n_2\to+\infty}\lim_{n_3\to+\infty}\lim_{n_4\to+\infty}\varphi_{n_1,n_2,n_3,n_4}(x) = \varphi(x);$$
$$\lim_{n_1\to+\infty}\lim_{n_2\to+\infty}\lim_{n_3\to+\infty}\lim_{n_4\to+\infty}\nabla\varphi_{n_1,n_2,n_3,n_4}(x) = \nabla\varphi(x);$$
$$\lim_{n_1\to+\infty}\lim_{n_2\to+\infty}\lim_{n_3\to+\infty}\lim_{n_4\to+\infty}L_{b,2}\varphi_{n_1,n_2,n_3,n_4}(x) = L_{b,2}\varphi(x).$$

Furthermore there exists a positive constant C_φ such that, for any $n_1, n_2, n_3, n_4 \in \mathbb{N}$ and $x \in \mathcal{K}$, it holds

$$|\varphi_{n_1,n_2,n_3,n_4}(x)| + \|\nabla\varphi_{n_1,n_2,n_3,n_4}(x)\|_{\mathcal{K}} + |L_{b,2}\varphi_{n_1,n_2,n_3,n_4}(x)| \le C_\varphi(1 + \|x\|_{\mathcal{K}}^2). \tag{1.10.7}$$

For a proof we refer to [30, Section 2.8.3] or [40, Section 2]. See also [36, Section 8].

1.10.2 The invariant measure of the Ornstein-Uhlenbeck semigroup

Until here we discussed the behavior of the Ornstein-Uhlenbeck semigroup in spaces of continuous functions. In this subsection we are interested in its behavior in L^p spaces, with $p \geq 1$. In the finite dimensional case the Ornstein-Uhlenbeck semigroup has been studied in L^p spaces with respect to both the Lebesgue measure and to its invariant measure. In the infinite dimensional case we can only consider the latter. We refer to [41, 42] for the results in this subsection.

We set in the same framework of the previous subsection, and we assume that there exist $w > 0$ and $M > 0$ such that

$$\left\|e^{tA}\right\|_{\mathcal{L}(\mathcal{K})} \leq Me^{-wt}, \quad t \geq 0.$$

Proposition 1.10.8. *The following statements are equivalent.*

(i) $\sup_{t\geq 0} \mathrm{Tr}[Q_t] < +\infty$

(ii) *$T(t)$ has unique invariant measure $\mu = \mathcal{N}(0, Q_\infty)$ where*

$$Q_\infty = \int_0^\infty e^{sA}RR^*e^{sA^*}.$$

Now we can extend the Ornstein-Uhlenbeck semigroup in $L^p(\mathcal{K}, \mu)$, for any $p \geq 1$.

Proposition 1.10.9. *$T(t)$ is uniquely extendable to a strongly continuous and contraction semigroup $T_p(t)$ in $L^p(\mathcal{K}, \mu)$, for any $p \geq 1$. Moreover the infinitesimal generator L_2 of $T_2(t)$ is the closure in $L^2(\mathcal{K}, \mu)$ of operator L_0 defined in* (1.10.6)*.*

We assume that $\mathrm{Ker}(R) = \{0\}$ and $Q_\infty^{1/2}(\mathcal{K}) \subset R(\mathcal{K})$.

Proposition 1.10.10. *$T_2(t)$ is self-adjoint in $L^2(\mathcal{K}, \mu)$ if and only if one of the following conditions is verified.*

(i) $Q_\infty e^{tA^*} = e^{tA}Q_\infty$.

(ii) $Re^{tA^*} = e^{tA}R$.

In that case $Q_\infty = -\frac{1}{2}A^{-1}C$ and

$$\int_{\mathcal{K}} \psi L_2 \varphi d\mu = -\frac{1}{2}\int_{\mathcal{K}} \left\langle C^{1/2}\nabla\varphi, C^{1/2}\nabla\psi\right\rangle_{\mathcal{K}} d\mu, \quad \varphi, \psi \in W_R^{1,2}(\mathcal{K}, \mu),$$

where $W_R^{1,2}(\mathcal{K}, \mu)$ is the Sobolev space defined in Proposition 1.6.17.

1.11 Regularizing sequence for dissipative functions

In this section we introduce a useful regularizing sequence for dissipative functions. We refer to [20, Appendix A] and [43, Appendix D] for the results in this section. Let $\mathcal{K}$ be a separable Banach space and let $F : \mathrm{Dom}(F) \subseteq \mathcal{K} \to \mathcal{K}$ be a possibly non linear function. We assume that there exists $\zeta_F \in \mathbb{R}$ such that $F - \zeta_F \mathrm{I}$ is m-dissipative. For any $\delta > 0$ and $x \in \mathcal{K}$, let $J_\delta(x) \in \mathrm{Dom}(F)$ be the unique solution of

$$y - \delta(F(y) - \zeta_F y) = x. \tag{1.11.1}$$

The existence of $J_\delta(x)$, for every $x \in \mathcal{K}$ and $\delta > 0$, is guaranteed by the m-dissipativity of F. We define $F_\delta : \mathcal{K} \to \mathcal{K}$ as

$$F_\delta(x) := F(J_\delta(x)), \qquad x \in \mathcal{K},\ \delta > 0.$$

Lemma 1.11.1. *The following statements hold.*

$$\lim_{\delta\to 0} \|J_\delta(x) - x\|_{\mathcal{K}} = 0, \quad x \in \mathrm{Dom}(F). \tag{1.11.2}$$

For any $0 < \delta < |\zeta_F|^{-1}$, the function $F_\delta - \zeta_F \mathrm{I}_{\mathcal{K}}$ is dissipative on $\mathcal{K}$. Moreover for any $\delta > 0$ it holds

$$\|J_\delta(x) - x\|_{\mathcal{K}} \leq \delta \|F(x) - \zeta_F x\|_{\mathcal{K}}, \qquad x \in \mathrm{Dom}(F); \tag{1.11.3}$$

$$\|F_\delta(x)\|_{\mathcal{K}} \leq (3 + \delta|\zeta_F|)(\|F(x)\|_{\mathcal{K}} + \|x\|_{\mathcal{K}}), \qquad x \in \mathrm{Dom}(F); \tag{1.11.4}$$

and

$$\|F_\delta(x_1) - F_\delta(x_2)\|_K \leq \left(\frac{2}{\delta} + |\zeta_F|\right) \|x_1 - x_2\|_K, \qquad x_1, x_2 \in \mathcal{K}. \tag{1.11.5}$$

Proof. We apply [41, Proposition 5.5.3] to the function $G : \mathrm{Dom}(F) \subseteq \mathcal{K} \to \mathcal{K}$

$$G(x) := F(x) - \zeta_F x, \qquad x \in \mathrm{Dom}(F). \tag{1.11.6}$$

Throughout the proof we let $G_\delta(x) := G(J_\delta(x))$ for any $x \in \mathcal{K}$ and $\delta > 0$, where $J_\delta(x)$ is defined in (1.11.1). We remark that (1.11.2) follows by [41, Proposition 5.5.3(iii)], while (1.11.3) follows by [20, Proposition A.2.2(4)]. Moreover for any $\delta > 0$, G_δ is dissipative on $\mathcal{K}$ and

$$\begin{aligned}\|F_\delta(x)\|_{\mathcal{K}} &\leq \|G(x)\|_{\mathcal{K}}, \qquad x \in \mathrm{Dom}(F);\\ \|G_\delta(x_1) - G_\delta(x_2)\|_K &\leq \frac{2}{\delta}\|x_1 - x_2\|_K, \qquad x_1, x_2 \in \mathcal{K}.\end{aligned}$$

Now we show that $F_\delta - \zeta_F \mathrm{I}_{\mathcal{K}}$ is dissipative in $\mathcal{K}$. Let $\alpha > 0$, $\delta < |\zeta_F|^{-1}$ and $x_1, x_2 \in \mathcal{K}$. By (1.11.1) and (1.11.6) we have

$$\begin{aligned}&\|x_1 - x_2 - \alpha[F_\delta(x_1) - \zeta_F x_1 - F_\delta(x_2) + \zeta_F x_2]\|_{\mathcal{K}}\\ =&\|x_1 - x_2 - \alpha[F_\delta(x_1) - \zeta_F[J_\delta(x_1) - \delta G_\delta(x_1)] - F_\delta(x_2) + \zeta_F[J_\delta(x_2) - \delta G_\delta(x_2)]]\|_{\mathcal{K}}\\ =&\|x_1 - x_2 - \alpha[G_\delta(x_1) - \zeta_F \delta G_\delta(x_1) - G_\delta(x_2) + \zeta_F \delta G_\delta(x_2)]\|_{\mathcal{K}}\\ =&\|x_1 - x_2 - \alpha(1 - \delta\zeta_F)[G_\delta(x_1) - G_\delta(x_2)]\|_{\mathcal{K}} \geq \|x_1 - x_2\|_{\mathcal{K}},\end{aligned}$$

and so $F_\delta - \zeta_F \mathrm{I}_{\mathcal{K}}$ is dissipative on $\mathcal{K}$.

Now we show (1.11.5). By (1.11.6), for any $x_1, x_2 \in K$ and $\delta > 0$

$$\begin{aligned}\|F_\delta(x_1) - F_\delta(x_2)\|_{\mathcal{K}} &\le \|G_\delta(x_1) - G_\delta(x_2)\|_{\mathcal{K}} + |\zeta_F| \|J_\delta(x_1) - J_\delta(x_2)\|_{\mathcal{K}} \\ &\le \left(\frac{2}{\delta} + |\zeta_F|\right) \|x_1 - x_2\|_{\mathcal{K}}.\end{aligned}$$

This conclude the proof of (1.11.5).

We concludes by proving (1.11.4). By (1.11.3), (1.11.6) for any $x \in \mathcal{K}$ and $\delta > 0$ it holds

$$\begin{aligned}\|F_\delta(x)\|_{\mathcal{K}} &\le \|F(x)\|_{\mathcal{K}} + \|F_\delta(x) - F(x)\|_{\mathcal{K}} \\ &\le \|F(x)\|_{\mathcal{K}} + \|G_\delta(x) - G(x)\|_{\mathcal{K}} + |\zeta_F| \|J_\delta(x) - x\|_{\mathcal{K}} \\ &\le \|F(x)\|_{\mathcal{K}} + \|G_\delta(x)\|_{\mathcal{K}} + \|G(x)\|_{\mathcal{K}} + \delta|\zeta_F| \|G(x)\|_{\mathcal{K}} \\ &\le \|F(x)\|_{\mathcal{K}} + (2 + \delta|\zeta_F|) \|G(y)\|_{\mathcal{K}} \\ &\le (3 + \delta|\zeta_F|)(\|F(x)\| + |\zeta_2| \|x\|_{\mathcal{K}}).\end{aligned}$$

So (1.11.4) holds true. □

The following corollary is an immediate consequence of the Lemma 1.11.1.

Corollary 1.11.2. *Let E be a separable Banach space continuously embedded in $\mathcal{K}$ such that $F(E \cap \mathrm{Dom}(F)) \subset E$. Then the function $F_{|E \cap \mathrm{Dom}(F)} : E \cap \mathrm{Dom}(F) \to E$ verifies all the statements of the Lemma 1.11.1 with $F = F_{|E \cap \mathrm{Dom}(F)}$, $\mathcal{K} = E$ and $\mathrm{Dom}(F) = E \cap \mathrm{Dom}(F)$.*

Now we assume that $\mathcal{K}$ is a separable Hilbert space. We introduce a further regularization, through a smoothing Ornstein-Uhlenbeck semigroup, corresponding to the choice $R = \mathrm{I}$, $A = -\frac{1}{2}Q^{-1}$ in (1.10.4), with Q positive and trace class. For every $\delta, s > 0$ and $x \in \mathcal{K}$, we define

$$F_{\delta,s}(x) := \int_{\mathcal{K}} F_\delta(y + e^{sA}x) \mathcal{N}(0, Q_s)(dy), \tag{1.11.7}$$

where $Q_s = Q(\mathrm{I} - e^{2sA})$. By (1.11.7), for any $\delta, s > 0$ and $x, z \in \mathcal{K}$, $F_{\delta,s}(x)$ is Lipschitz continuous and

$$\langle F_{\delta,s}(x) - F_{\delta,s}(z), x - z \rangle_K \le \zeta_2 \|x - z\|_K^2.$$

For any $\delta > 0$ and $x \in \mathcal{K}$ we have

$$\lim_{s \to 0} \|F_{\delta,s}(x) - F(x)\|_{\mathcal{K}} = 0. \tag{1.11.8}$$

By the same arguments using in [43, Theorem 9.26], for any $s, \delta > 0$, $F_{\delta,s}$ is Frechét differentiable.

Proposition 1.11.3. *Assume that the following hypotheses are verified.*

1. *There exists a separable Banach space $E \subset \mathrm{Dom}(F)$ continuously embedded in $\mathcal{K}$ such that $F(E) \subset E$. Moreover $F_{|E} : E \to E$ is continuous, $F - \zeta_F \mathrm{I}_E$ is m-dissipative on E, and*

there exist $M > 0$ and $m \in \mathbb{N}$ such that

$$\|F(x)\|_E \le M(1 + \|x\|_H^m), \quad x \in E. \tag{1.11.9}$$

2. *There exists a probability measure ν on $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$ such that $\nu(E) = 1$ and it has finite moments of every order with respect the norm of E.*

Then

$$\lim_{\delta\to 0}\lim_{s\to 0} \|F_{\delta,s} - F\|_{L^2(\mathcal{K},\nu)} = 0.$$

Proof. For any $\delta > 0$, F_δ is Lipschitz continuous, so there exists C_δ such that

$$\|F(x)\|_{\mathcal{K}} \le C_\delta(1 + \|x\|_{\mathcal{K}}), \quad x \in \mathcal{K},$$

hence, for any $\delta > 0$, $s \ge 0$ and $T > 0$ we have

$$F_{\delta,s} \le C_\delta \left(1 + \int_{\mathcal{K}} \|y\| \mathcal{N}(0, Q_s)(dy) + \|x\|\right), \quad x \in \mathcal{K}. \tag{1.11.10}$$

By (1.11.8), (1.11.10) and applying the dominated convergence theorem for any $\delta > 0$ we have

$$\lim_{s\to 0} \|F_{\delta,s} - F_\delta\|_{L^2(\mathcal{K},\nu)} = 0. \tag{1.11.11}$$

By Corollary 1.11.2 (see (1.11.2)), the continuity of $F_{|E} : E \to E$ and the fact that E is continuously embedded in $\mathcal{K}$ we have

$$\lim_{\delta\to 0} \|F_\delta(x) - F(x)\|_{\mathcal{K}} = 0, \quad x \in E. \tag{1.11.12}$$

By (1.11.4), (1.11.9), (1.11.12) and the dominated convergence theorem we have

$$\lim_{\delta\to 0} \|F_\delta - F\|_{L^2(\mathcal{K},\nu)} = 0. \tag{1.11.13}$$

Finally by (1.11.11) and (1.11.13) we obtain the statement. $\square$

Chapter 2

The transition semigroup

In this chapter we will study the nonlinear SPDEs (Stochastic Partial Differential Equations) and the associated transition semigroups that will be the main object of study of this thesis. In contrast with the case of linear SPDEs, that give rise to Ornstein-Uhlenbeck semigroups (see Sections 1.10), the transition semigroup has not a simple representation formula such as (1.10.4). We follow the approach of the books [20, 41, 42, 43].

Let $\mathcal{X}$ be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $\{W(t)\}_{t\geq 0}$ be a $\mathcal{X}$-cylindrical Wiener process defined on a normal filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Let $R \in \mathcal{L}(\mathcal{X})$, let $A : \mathrm{Dom}(A) \subseteq \mathcal{X} \to \mathcal{X}$ be the infinitesimal generator of a strongly continuous semigroup e^{tA} on $\mathcal{X}$. Let $F : \mathrm{Dom}(F) \subseteq \mathcal{X} \to \mathcal{X}$ (possibly non linear). We introduce the SPDE

$$\begin{cases} dX(t,x) = \big(AX(t,x) + F(X(t,x))\big)dt + RdW(t), & t > 0; \\ X(0,x) = x \in \mathcal{X}. \end{cases} \tag{2.0.1}$$

This type of SPDE is widely studied in the literature, see for example [1, 10, 11, 20, 28, 30, 31, 32, 37, 43, 41, 55, 62, 78, 85]. In this thesis we focus on the case of dissipative systems, where A and F satisfy a joint dissipativity condition (see Hypotheses 2.1.1(iv)).

If $\mathrm{Dom}(F) = \mathcal{X}$, for any $x \in \mathcal{X}$ it is possible to consider the solution of the mild form of (2.0.1), namely a process $\{X(t,x)\}_{t\geq 0}$ that satisfies

$$X(t,x) = e^{tA}x + \int_0^t e^{(t-s)A}F(X(s,x))ds + W_A(t), \quad \mathbb{P}\text{-a.s.} \tag{2.0.2}$$

where $\{W_A(t)\}_{t>0}$ is the stochastic convolution process defined by

$$W_A(t) := \int_0^t e^{(t-s)A}RdW(s).$$

However, if $\mathrm{Dom}(F) \subset \mathcal{X}$ is a proper subset of $\mathcal{X}$, (2.0.2) may not make sense for every $x \in \mathcal{X}$. Hence we need a more general notion of solution. Around the nineties S. Cerrai G. Da Prato and J. Zabczyk used the notion of generalized mild solution to avoid the problem of $\mathrm{Dom}(F)$. The idea to construct a generalized mild solution is to assume that there exists a Banach space $E \subseteq \mathrm{Dom}(F)$ densely and continuously embedded in $\mathcal{X}$ such that $F_{|E} : E \to E$ is locally Lipschitz continuous. Then, under suitable Hypotheses 2.1.1, it is possible to prove that for any $x \in E$, the SPDE (2.0.1) has a unique mild solution $\{X(t,x)\}_{t\geq 0}$ such that its trajectories take values in E. After, exploiting the density of E, one proves that for any $x \in \mathcal{X}$ there exists a process $\{X(t,x)\}_{t\geq 0}$, such that

$$\lim_{n\to\infty} \|X(\cdot,x_n) - X(\cdot,x)\|_{C([0,T],\mathcal{X})} = 0, \quad \forall\, T > 0,\ \mathbb{P}\text{-a.s.} \tag{2.0.3}$$

for any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ converging to x and $X(t,x_n)$ is the unique mild solution of (2.0.1), with initial datum x_n. We call the limit $\{X(t,x)\}_{t\geq 0}$ in (2.0.3) generalized mild solution of (2.0.1).

Under suitable assumptions, in Section (2.1) we will prove that (2.0.1) has a unique mild solution, for any $x \in E$. In Section (2.2) we will prove that (2.0.1) has a unique generalized mild solution $\{X(t,x)\}_{t\geq 0}$, for any $x \in \mathcal{X}$, and we will define the transition semigroup

$$P(t)\varphi(x) := \mathbb{E}[\varphi(X(t,x))], \quad \varphi \in B_b(\mathcal{X}),\ x \in \mathcal{X},\ t > 0.$$

In Section 2.3 we recall some standard results about the space regularity of the mild solution in the case where $F : \mathcal{X} \to \mathcal{X}$ is Frechét differentiable and Lipschitz continuous. Finally in Section (2.4) we will comment our hypotheses in view of results already known in the literature.

2.1 The mild solution for x belonging to E

Our hypotheses in this subsection are the following.

Hypotheses 2.1.1.

(i) $R \in \mathcal{L}(\mathcal{X})$.

(ii) *There exists a Banach space $E \subseteq \mathrm{Dom}(F)$ which is Borel measurable, densely and continuously embedded in $\mathcal{X}$ and invariant for F, namely $F(E) \subseteq E$.*

(iii) *A generates a strongly continuous semigroup e^{tA} on $\mathcal{X}$ and A_E (the part of A in E) generates an analytic semigroup e^{tA_E} on E.*

(iv) *There exists $\zeta \in \mathbb{R}$ such that*

  (a) $A + F - \zeta \mathrm{I}$ *is dissipative in* $\mathcal{X}$*;*

  (b) $A_E + F_{|E} - \zeta \mathrm{I}$ *is dissipative in* E.

(v) *For any $T > 0$, we have*
$$\int_0^T \mathrm{Tr}[e^{sA}RR^*e^{sA^*}]ds < +\infty.$$
Moreover the process $\{W_A(t)\}_{t\geq 0}$ is continuous.

(vi) *There exist $M > 0$ and $m \in \mathbb{N}$ such that*

$$\|F(x)\|_E \leq M(1 + \|x\|_E^m), \quad x \in E.$$

(vii) $F_{|E} : E \to E$ *is locally Lipschitz continuous on* E*, namely* $F_{|E}$ *is Lipschitz continuous on bounded sets of* E*.*

Remark 2.1.2.

1. *Hypotheses 2.1.1*(vi) *or 2.1.1*(vii) *imply that* $F_{|E}$ *maps bounded sets of* E *into bounded sets of* E*, and so, since* E *is continuously embedded in* $\mathcal{X}$*,* F *maps bounded sets of* E *into bounded sets of* $\mathcal{X}$*.*

2. *Hypothesis 2.1.1*(vii) *does not imply that* $F : \mathrm{Dom}(\mathcal{X}) \subseteq \mathcal{X} \to \mathcal{X}$ *is continuous, however it implies that* $F_{|E} : E \to \mathcal{X}$ *is continuous.*

Remark 2.1.3. *By Proposition 1.8.10 and Hypotheses 2.1.1(ii-v-vi), for any* $T > 0$ *and* $p \geq 1$ *we have*

$$\mathbb{E}[\sup_{t\in[0,T]} \|F(W_A(t))\|_E^p + \sup_{t\in[0,T]} \|W_A(t)\|_E^p] < +\infty.$$

$$\mathbb{E}[\sup_{t\in[0,T]} \|F(W_A(t))\|^p + \sup_{t\in[0,T]} \|W_A(t)\|^p] < +\infty.$$

Now we define rigorously the notion of mild solution and the Banach spaces in which we will construct it [20, Section 6.2]).

Definition 2.1.4. *For any* $x \in E$ *we call mild solution of* (2.0.1) *any* E*-valued process* $\{X(t,x)\}_{t\geq 0}$ *such that, for any* $t \geq 0$*, we have*

$$X(t,x)(\omega) = e^{tA}x + \int_0^t e^{(t-s)A}F(X(s,x)(\omega))ds + W_A(t)(\omega), \quad \mathbb{P}\text{-a.s.}, \tag{2.1.1}$$

Moreover we say that the mild solution of (2.0.1) *is unique if whenever two* E*-valued process* $\{X_1(t,x)\}_{t\geq 0}$*,* $\{X_2(t,x)\}_{t\geq 0}$ *satisfy* (2.1.1)*,* $\{X_1(t,x)\}_{t\geq 0}$ *is a version of* $\{X_2(t,x)\}_{t\geq 0}$*.*

Definition 2.1.5.

1. *Let* I *be an interval contained in* $\mathbb{R}$ *and* $p \geq 1$*. We denote by* $\mathcal{K}^p(I)$ *the space of progressive measurable* $\mathcal{K}$*-valued processes* $\{Y(t)\}_{t\in I}$ *endowed with the norm*

$$\|\{Y(t)\}_{t\in I}\|^p_{\mathcal{K}^p(I)} := \sup_{t\in \overline{I}} \mathbb{E}[\|Y(t)\|^p_{\mathcal{K}}].$$

2. *Let* I *be an interval contained in* $\mathbb{R}$ *and* $p \geq 1$*. We denote by* $C_p(I, \mathcal{K})$ *the space of* $\mathcal{K}$*-valued continuous processes* $\{Y(t)\}_{t\in I} \in \mathcal{PC}_b(I, \mathcal{K})$ *endowed with the norm*

$$\|\{Y(t)\}_{t\in I}\|^p_{C_p(I,\mathcal{K})} := \mathbb{E}[\sup_{t\in\overline{I}} \|Y(t)\|^p_{\mathcal{K}}].$$

Before starting the construction of the generalized mild solution, we should recall two inequalities that we will use frequently. The first on is

$$ab \leq \frac{(q-1)(\epsilon a)^{q/(q-1)}}{q} + \frac{(b/\epsilon)^q}{q}, \quad \forall a, b, \epsilon > 0,\ q > 1. \tag{2.1.2}$$

If $\mathcal{K}$ is a Banach space for every $h_1, h_2 \in \mathcal{K}$ and $r \geq 1$ it holds

$$\|h_1 - h_2\|_{\mathcal{K}}^r \geq 2^{1-r}\|h_1\|_{\mathcal{K}}^r - \|h_2\|_{\mathcal{K}}^r. \tag{2.1.3}$$

To prove that, for any $x \in E$, the SPDE (2.0.1) has a unique mild solution $\{X(t,x)\}_{t\geq 0}$ we exploit an approximating problem. For simplicity, from here on we still denote by A the part of A in E. For any $x \in E$ and large $n \in \mathbb{N}$, we introduce the approximating problem

$$\begin{cases} dX_n(t,x) = \big(AX_n(t,x) + F(X_n(t,x))\big)dt + RdW(t), & t > 0; \\ X_n(0,x) = nR(n,A)x. \end{cases} \tag{2.1.4}$$

Remark 2.1.6. *By Hypotheses 2.1.1*(iii), e^{tA} *verifies* (1.4.1) *with some constant* $\eta_0 \in \mathbb{R}$. *Hence* $R(n,A)$ *is defined only for* $n > \eta_0$. *Hence if* $\eta_0 \geq 1$, *then we consider* (2.1.5) *only for* $n > \eta_0$.

Now we are going to prove that, for any $x \in E$ and large $n \in \mathbb{N}$ the SPDE (2.1.4) has a unique mild solution $\{X_n(t,x)\}_{t\geq 0} \in C_p([0,T],E)$, for any $p \geq 1$ and $T > 0$ (see Definition (2.1.5)). To do this we consider the equation

$$\begin{cases} \dfrac{dY_n}{dt}(t,x) = AY_n(t,x) + F(Y_n(t,x) + W_A(t)), & t > 0; \\ Y_n(0,x) = nR(n,A)x. \end{cases} \tag{2.1.5}$$

If we show that, for any $x \in E$ and large $n \in \mathbb{N}$, equation (2.1.5) has a unique mild solution $\{Y_n(t,x)\}_{t\geq 0} \in C_p([0,T],E)$, for any $p \geq 1$ and $T > 0$, then by Remark 2.1.3, the process $\{X_n(t,x)\}_{t\geq 0}$ defined by

$$X_n(t,x) := Y_n(t,x) + W_A(t), \quad \mathbb{P}\text{-a.s.}, \tag{2.1.6}$$

is the unique mild solution of (2.1.4) in $C_p([0,T],E)$, for any $p \geq 1$ and $T > 0$.

Proposition 2.1.7. *Assume that Hypotheses 2.1.1 hold true. For any* $x \in E$ *and large* $n \in \mathbb{N}$ *problem* (2.1.5) *has a unique mild solution* $\{Y_n(t,x)\}_{t\geq 0} \in C_p([0,T],E)$, *for any* $p \geq 1$ *and* $T > 0$. *Moreover there exists a sequence of processes* $\{\{Y_{n,k}(t,x)\}_{t\geq 0}\}_{k\in\mathbb{N}}$ *such that*

$$t \to Y_{n,k}(t,x) \in C^1([0,T],E) \cap C([0,T],\mathrm{Dom}(A)), \quad \forall T > 0,\ \forall k \in \mathbb{N}, \quad \mathbb{P}\text{-}a.s.$$

$$\lim_{k\to+\infty} \|Y_{n,k}(\cdot,x) - Y_n(\cdot,x)\|_{C([0,T],E)} = 0, \ \lim_{k\to+\infty} \|o_{n,k}(x)\|_{C([0,T],E)} = 0, \ \forall T > 0 \ \mathbb{P}\text{-}a.s. \tag{2.1.7}$$

where

$$o_{n,k}(t,x) = \frac{dY_{n,k}}{dt}(t,x) - AY_{n,k}(t,x) - F(Y_{n,k}(t,x) + W_A(t)), \quad \mathbb{P}\text{-}a.s. \tag{2.1.8}$$

In addition for any $p \geq 1$ *there exist* $C_p := C_p(\zeta) > 0$ *and* $\kappa_p := \kappa_p(\zeta) \in \mathbb{R}$ *such that for any* $x \in E$, *large* $n \in \mathbb{N}$ *and* $t > 0$

$$\|Y_n(t,x)\|^p \leq C_p \left(e^{\kappa_p t}\|x\|^p + \int_0^t e^{\kappa_p(t-s)}\|F(W_A(s))\|^p ds \right), \quad \mathbb{P}\text{-}a.s. \tag{2.1.9}$$

$$\|Y_n(t,x)\|_E^p \le C_p\left(e^{\kappa_p t}\|x\|_E^p + \int_0^t e^{\kappa_p(t-s)}\|F(W_A(s))\|_E^p ds\right) \quad \mathbb{P}\text{-}a.s. \tag{2.1.10}$$

Proof. We prove the statements for a fixed large $n \in \mathbb{N}$ and $x \in E$. By Hypotheses 2.1.1(v), the trajectories of the process $\{W_A(t)\}_{t\ge 0}$ are continuous $\mathbb{P}$-a.s. In this proof we work pathwise, and we will denote by $w_A(\cdot)$ a fixed arbitrary trajectory of $\{W_A(t)\}_{t\ge0}$. We fix $T>0$ and we consider the equation

$$\begin{cases} \dfrac{dy_n}{dt}(t,x) = Ay_n(t,x) + F(y_n(t,x)+w_A(t)), \quad t\in[0,T]; \\ y_n(0,x) = nR(n,A)x, \end{cases} \tag{2.1.11}$$

and the operator V in the space $C([0,T];E)$ defined by

$$V(y)(t) := e^{tA}nR(n,A)x + \int_0^t e^{(t-s)A}F(y(s)+w_A(s))ds, \quad y\in C([0,T],E),\ t\in[0,T].$$

Let $R > M_0\|x\|_E \sup_{t\in[0,T]} e^{t\eta_0}$. By (1.4.3), (1.4.7), Remark 2.1.6 and the local lipschitzianity of F, for any $y,z \in C([0,T],E)$ such that $\|y\|_{C([0,T],E)}, \|z\|_{C([0,T],E)} \le R$, we have

$$\|V(y)\|_{C([0,T];E)} \le M_0\|x\|_E \sup_{t\in[0,T]} e^{t\eta_0} + M_0 \sup_{t\in[0,T]}\|F(y(t)+w_A(t))\|_E \sup_{t\in[0,T]}\int_0^t e^{(t-s)\eta_0}ds$$

$$\|V(y)-V(z)\|_{C([0,T],E)} \le L_R M_0\|y-z\|_{C([0,T],E)} \sup_{t\in[0,T]}\int_0^t e^{(t-s)\eta_0}ds.$$

where M_0 and η_0 are the constants in Remark 2.1.6 and $L_R>0$ is the Lipschitz constant of F on the ball $B_E(0,R)$. By Remark (2.1.2) for $T_0\in[0,T]$ small enough $V(B(0,R))\subseteq B(0,R)$ and V is a contraction in $B(0,R)$ where $B(0,R)$ is the ball in $C([0,T_0],E)$ with center 0 and radius R. Hence by the contraction mapping theorem the problem (2.1.11) has a unique mild solution $y_{n,T_0}(\cdot,x)\in B(0,R)$. To prove that there exists a global solution $y_{n,T}$ of (2.1.11) in $C([0,T],E)$ it is sufficient to prove an estimate for $\|y_{n,T_0}(\cdot,x)\|_{C([0,T_0],E)}$ independent of T_0. By [73, Proposition 4.1.8] $y_{n,T_0}(\cdot,x)$ is the strong solution of

$$\begin{cases} \dfrac{dv_n}{dt}(t,x) = Av_n(t,x) + F(y_{n,T_0}(t,x)+w_A(t)), \quad t\in[0,T_0]; \\ v_n(0,x) = nR(n,A)x, \end{cases}$$

namely there exists a sequence $\{y_{n,k,T_0}(\cdot,x)\}_{k\in\mathbb{N}} \subseteq C^1([0,T_0],E)\cap C([0,T_0],\mathrm{Dom}(A))$ such that

$$\begin{aligned} &\lim_{k\to+\infty}\|y_{n,k,T_0}(\cdot,x)-y_{n,T_0}(\cdot,x)\|_{C([0,T_0],E)} = 0, \\ &\lim_{k\to+\infty}\left\|\frac{dy_{n,k,T_0}}{dt}(\cdot,x) - Ay_{n,k,T_0}(\cdot,x) - F(y_{n,T_0}(\cdot,x)+w_A(\cdot))\right\|_{C([0,T_0],E)} = 0. \end{aligned} \tag{2.1.12}$$

For any $t\in[0,T_0]$, $x\in E$ and $n,k\in\mathbb{N}$ we set

$$o_{n,k,T_0}(t,x) = \frac{dy_{n,k,T_0}}{dt}(t,x) - Ay_{n,k,T_0}(t,x) - F(y_{n,k,T_0}(t,x)+w_A(t)),$$

hence we have

$$\begin{aligned}\|o_{n,k,T_0}(t,x)\|_E \leq& \left\| \frac{dy_{n,k,T_0}}{dt}(t,x) - Ay_{n,k,T_0}(t,x) - F(y_{n,T_0}(t,x)+w_A(t)) \right\|_E \\ &+ \|F(y_{n,T_0}(t,x)+w_A(t)) - F(y_{n,k,T_0}(t,x)+w_A(t))\|_E \\ \leq& \left\| \frac{dy_{n,k,T_0}}{dt}(t,x) - Ay_{n,k,T_0}(t,x) - F(y_{n,T_0}(t,x)+w_A(t)) \right\|_E \\ &+ L_R\|y_{n,T_0}(t,x) - y_{n,k,T_0}(t,x)\|_E,\end{aligned}$$

and so, by (2.1.12), for any large $n \in \mathbb{N}$ we obtain

$$\lim_{k\to+\infty} \|o_{n,k,T_0}(x)\|_{C([0,T_0],E)} = 0, \quad \mathbb{P}\text{-a.s.}$$

Let $x \in E$, $p \geq 1$, $k, n \in \mathbb{N}$ and $t \in [0, T_0]$. By (1.3.1)-(2.1.8) and Hypotheses 2.1.1(iv), there exists $y^* \in \partial\|y_{n,k}(t,x)\|_E$, such that

$$\begin{aligned}\frac{1}{p}\frac{d^-\|y_{n,k,T_0}(t,x)\|_E^p}{dt} \leq& \|y_{n,k,T_0}(t,x)\|_E^{p-1}{}_E\langle Ay_{n,k,T_0}(t,x), y^*\rangle_{E^*} \\ &+ \|y_{n,k,T_0}(t,x)\|_E^{p-1}{}_E\langle F(y_{n,k,T_0}(t,x)+w_A(t)), y^*\rangle_{E^*} \\ &+ \|y_{n,k,T_0}(t,x)\|_E^{p-1}{}_E\langle o_{n,k,T_0}(t,x), y^*\rangle_{E^*} \\ =& \|y_{n,k,T_0}(t,x)\|_E^{p-1}{}_E\langle Ay_{n,k,T_0}(t,x), y^*\rangle_{E^*} \\ &+ \|y_{n,k,T_0}(t,x)\|_E^{p-1}{}_E\langle F(y_{n,k,T_0}(t,x)+w_A(t)) - F(w_A(t)), y^*\rangle_{E^*} \\ &+ \|y_{n,k,T_0}(t,x)\|_E^{p-1}{}_E\langle F(w_A(t)), y^*\rangle_{E^*} \\ &+ \|y_{n,k,T_0}(t,x)\|_E^{p-1}{}_E\langle o_{n,k,T_0}(t,x), y^*\rangle_{E^*} \\ \leq& \zeta\|y_{n,k,T_0}(t,x)\|_E^p \\ &+ \|y_{n,k,T_0}(t,x)\|_E^{p-1}\left(\|F(w_A(t))\|_E + \|o_{n,k,T_0}(t,x)\|_E\right). \end{aligned} \tag{2.1.13}$$

We claim that there exists $C_1 := C_1(\zeta, p)$ such that

$$\frac{1}{p}\frac{d^-\|y_{n,k,T_0}(t,x)\|_E^p}{dt} \leq C_1\|y_{n,k,T_0}(t,x)\|_E^p + \frac{1}{p}\left(\|F(w_A(t))\|_E + \|o_{n,k,T_0}(t,x)\|_E\right)^p. \tag{2.1.14}$$

Indeed for $p = 1$, (2.1.14) is verified with $C_1 = \zeta$, instead, for $p > 1$, applying (2.1.2) in (2.1.13) with $a = \|y_{n,k,T_0}(t,x)\|_E^{p-1}$, $b = \left(\|F(w_A(t))\|_E + \|o_{n,k,T_0}(t,x)\|_E\right)$, $q = p$ and $\epsilon = 1$ we obtain

$$\begin{aligned}\frac{1}{p}\frac{d^-\|y_{n,k,T_0}(t,x)\|_E^p}{dt} \leq& (\zeta + \frac{p-1}{p})\|y_{n,k,T_0}(t,x)\|_E^p \\ &+ \frac{1}{p}\left(\|F(w_A(t))\|_E + \|o_{n,k,T_0}(t,x)\|_E\right)^p,\end{aligned}$$

and so (2.1.14) is verified with $C_1 = \zeta + \frac{p-1}{p}$. By (1.3.2), (1.4.1), Remark 2.1.6 and (2.1.14) we get

$$\|y_{n,k,T_0}(t,x)\|_E^p \leq e^{pC_1t}\|x\|_E^p + \int_0^t e^{pC_1(t-s)}(\|F(w_A(t))\|_E + \|o_{n,k,T_0}(t,x)\|_E)^p ds,$$

and letting $k \to +\infty$, by (2.1.7),

$$\|y_{n,T_0}(t,x)\|_E^p \leq e^{pC_1 t}\|x\|_E^p + \int_0^t e^{pC_1(t-s)}\|F(w_A(t))\|_E^p ds. \tag{2.1.15}$$

By Remark 2.1.3 and recalling that $T_0 \in [0,T]$, for any $t>0$ we obtain

$$\|y_{n,T_0}(t,x)\|_E^p \leq \|x\|_E^p + \frac{1}{pC_1}(e^{pC_1 t}-1)\sup_{t\in[0,T]}\|F(w_A(t))\|_E^p. \tag{2.1.16}$$

and so there exists a global solution $y_{n,T}$ of (2.1.11) in $C([0,T],E)$. The uniqueness of $y_{n,T}$ follows immediately by (2.1.16), the local lipschitzianity of F and the Gronwall inequality.

We have proved that, for any $T>0$ the equation (2.1.5), has a unique mild solution $y_{n,T} \in C([0,T],E)$. We consider the continuous function $y_n(\cdot,x) : [0,+\infty) \to E$ defined by

$$y_n(\cdot,x)_{|[0,T]} = y_{n,T}(\cdot,x), \quad \forall\, T>0.$$

Exploiting [73, Proposition 4.1.8] (as we have already done for y_{n,T_0}) for any $T>0$, there exists a sequence $\{y_{n,k,T_0}(\cdot,x)\}_{k\in\mathbb{N}} \subseteq C^1([0,T],E)\cap C([0,T],\mathrm{Dom}(A))$ such that

$$\lim_{k\to+\infty}\|y_{n,k}(\cdot,x)-y_n(\cdot,x)\|_{C([0,T],E)}=0, \quad \lim_{k\to+\infty}\|o_{n,k}(x)\|_{C([0,T],E)}=0, \quad \forall\ \mathbb{P}\text{-a.s.}$$

where

$$o_{n,k}(t,x) = \frac{dy_{n,k}}{dt}(t,x) - Ay_{n,k}(t,x) - F(y_{n,k}(t,x)+w_A(t)), \quad \mathbb{P}\text{-a.s.}$$

Moreover $y_n(\cdot,x)$ verifies (2.1.15), for any $p\geq 1$ and $t>0$. The process $\{Y_n(t,x)\}_{t\geq 0}$ whose trajectories are the functions $y(\cdot,x)$ verifies the statements of the proposition. Uniqueness follows by (2.1.15), local lipschitzianity of F and the Gronwall inequality. Estimates (2.1.9) follows in exactly the same way as (2.1.10) using the inner product of $\mathcal{X}$ instead of the duality product of E and E^*. □

Remark 2.1.8. *If e^{tA} is strongly continuous also on E, then it is possible to replace the initial datum $nR(n,A)x$ by x, in* (2.1.4).

By Remark 2.1.3, (2.1.3) (with $h_1 = X_n(t,x)$, $h_2 = W_A(t)$ and $r=p$) and Proposition 2.1.7 we obtain immediately the following result.

Proposition 2.1.9. *Assume that Hypotheses 2.1.1 hold true. For any large $n\in\mathbb{N}$ and $x\in E$ the process $\{X_n(t,x)\}_{t\geq 0}$, defined in* (2.1.6)*, is the unique mild solution of* (2.1.4) *in $C_p([0,T],E)$, for any $p\geq 1$ and $T>0$. In addition, for any $p\geq 1$, large $n\in\mathbb{N}$, $x\in E$ and $t>0$, we have*

$$\|X_n(t,x)\|^p \leq C_p'\left(e^{\kappa_p t}\|x\|^p + \int_0^t e^{\kappa_p(t-s)}\|F(W_A(s))\|^p ds + \|W_A(t)\|^p\right), \ \mathbb{P}\text{-}a.s. \tag{2.1.17}$$

$$\|X_n(t,x)\|_E^p \leq C_p'\left(e^{\kappa_p t}\|x\|_E^p + \int_0^t e^{\kappa_p(t-s)}\|F(W_A(s))\|_E^p ds + \|W_A(t)\|_E^p\right), \ \mathbb{P}\text{-}a.s. \tag{2.1.18}$$

where $C_p' := \max\left(2^{p-1}C_p, 2^{p-1}\right)$, and C_p, κ_p are the constants of Proposition 2.1.7.

Now we prove a convergence result for $\{X_n(t,x)\}_{t\geq 0}$.

Theorem 2.1.10. *Assume that Hypotheses 2.1.1 hold true. For any $x \in E$, there exists $\{X(t,x)\}_{t\geq 0} \in C_p((0,T],E) \cap C_p([0,T],\mathcal{X})$, for any $p \geq 1$ and $T > 0$, such that*

$$\lim_{n\to+\infty} \|X_n(\cdot,x) - X(\cdot,x)\|_{C([0,T],\mathcal{X})} = 0, \quad \forall\, T > 0,\ \mathbb{P}\text{-a.s.}, \tag{2.1.19}$$

$$\lim_{n\to\infty} \|X_n(\cdot,x) - X(\cdot,x)\|_{C([\epsilon,T],E)} = 0, \quad \forall\, 0 < \epsilon \leq T,\ \mathbb{P}\text{-a.s.} \tag{2.1.20}$$

For any $p \geq 1$, let C'_p be the constant of Proposition 2.1.9 and let κ_p be the constants of Proposition 2.1.7. For any $p \geq 1$, $x \in E$ and $t > 0$, we have

$$\|X(t,x)\|^p \leq C'_p \left(e^{\kappa_p t}\|x\|^p + \int_0^t e^{\kappa_p(t-s)}\|F(W_A(s))\|^p ds + \|W_A(t)\|^p \right),\ \mathbb{P}\text{-a.s.} \tag{2.1.21}$$

$$\|X(t,x)\|_E^p \leq C'_p \left(e^{\kappa_p t}\|x\|_E^p + \int_0^t e^{\kappa_p(t-s)}\|F(W_A(s))\|_E^p ds + \|W_A(t)\|_E^p \right),\ \mathbb{P}\text{-a.s.} \tag{2.1.22}$$

Moreover there exists a constant $\eta \in \mathbb{R}$ such that, for any $x, y \in E$ and $t > 0$, we have

$$\|X(t,x) - X(t,y)\| \leq e^{\eta t}\|x - y\|, \quad \mathbb{P}\text{-a.s.}, \tag{2.1.23}$$

$$\|X(t,x) - X(t,y)\|_E \leq e^{\eta t}\|x - y\|_E, \quad \mathbb{P}\text{-a.s.} \tag{2.1.24}$$

Proof. As in the proof of Proposition 2.1.7 we work pathwise, so we denote by $y_{n,k}(\cdot,x)$, $y_n(\cdot,x)$ and $w_A(\cdot)$ fixed trajectories of the processes $\{Y_{n,k}(t,x)\}_{t\geq 0}$, $\{Y_n(t,x)\}_{t\geq 0}$ and $\{W_A(t)\}_{t\in[0,T]}$ respectively.

We begin to prove (2.1.19) for a fixed $T > 0$. Let $x \in E$, $k, n \in \mathbb{N}$, $t \in [0,T]$. We define

$$z_{n,k}(t,x) := y_{n,k}(t,x) + w_A(t), \quad n, k \in \mathbb{N}.$$

We stress that $z_{n,k}(t,x) - z_{m,k}(t,x) = y_{n,k}(t,x) - y_{m,k}(t,x)$, for any $n, m \in \mathbb{N}$. For any $n, m \in \mathbb{N}$, by (2.1.8), we have

$$\begin{aligned}\frac{1}{2}\frac{d\|z_{n,k}(t,x) - z_{m,k}(t,x)\|^2}{dt} &\leq \langle A(z_{n,k}(t,x) - z_{m,k}(t,x)), z_{n,k}(t,x) - z_{m,k}(t,x)\rangle \\ &+ \langle F(z_{n,k}(t,x)) - F(z_{m,k}(t,x)), z_{n,k}(t,x) - z_{m,k}(t,x)\rangle \\ &+ \langle o_{n,k}(t,x) - o_{m,k}(t,x), z_{n,k}(t,x) - z_{m,k}(t,x)\rangle.\end{aligned}$$

By Hypotheses 2.1.1(iv) we have

$$\begin{aligned}\frac{1}{2}\frac{d\|z_{n,k}(t,x) - z_{m,k}(t,x)\|^2}{dt} &\leq \zeta\|z_{n,k}(t,x) - z_{m,k}(t,x)\|^2 \\ &+ \|o_{n,k}(t,x) - o_{m,k}(t,x)\|\|z_{n,k}(t,x) - z_{m,k}(t,x)\|.\end{aligned}$$

By (2.1.2) (with $\epsilon = 1$ and $q = 2$) we have

$$\frac{1}{2}\frac{d\|z_{n,k}(t,x) - z_{m,k}(t,x)\|^2}{dt} \leq (\zeta + \frac{1}{2})\|z_{n,k}(t,x) - z_{m,k}(t,x)\|^2$$

$$+\frac{1}{2}(\|o_{n,k}(t,x)\| + \|o_{m,k}(t,x)\|)^2.$$

We set $H_1 = \zeta + \frac{1}{2}$. By (1.3.2) we obtain

$$\|z_{n,k}(t,x) - z_{m,k}(t,x)\|^2 \leq e^{2H_1 t}\|(nR(n,A) - mR(m,A))x\| + \int_0^t e^{-2H_1(t-s)}(\|o_{n,k}(t,x)\| + \|o_{m,k}(t,x)\|)^2 ds.$$

Letting $k \to +\infty$, by (2.1.7) and Remark 2.1.2 we have

$$\|z_n(t,x) - z_m(t,x)\|^2 \leq e^{2H_1 t}\|(nR(n,A) - mR(m,A))x\|.$$

where

$$z_n(t,x) = X_n(t,x)(w) = y_n(t,x) + w_A(t).$$

By (1.4.6), we obtain that, for any $T > 0$ and $x \in E$, the sequence $\{z_n(\cdot,x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T],\mathcal{X})$ and we denote by $z_T(\cdot,x) \in C_b([0,T],\mathcal{X})$ its limit. A continuous function $z(\cdot,x) : [0,+\infty) \to \mathcal{X}$ such that

$$z(\cdot,x)_{|[0,T]} := z_T(\cdot,x), \quad \forall\, T > 0, \tag{2.1.25}$$

is well defined. So the process $\{X(t,x)\}_{t\geq 0}$, whose trajectories are the functions $z(\cdot,x)$, verifies (2.1.19). (2.1.17) and (2.1.19) yields (2.1.21) and, by Remark 2.1.3 and (2.1.21), we have $\{X(t,x)\}_{t\geq 0} \in C_p([0,T],\mathcal{X})$ for any $p \geq 1$ and $T > 0$.

Now we prove (2.1.20) for fixed $\epsilon, T > 0$. By (2.1.18), for any $x \in E$, there exists $R := R(x,T) > 0$ such that for any large $n \in \mathbb{N}$ and $t \in [\epsilon, T]$ we have

$$\|z_n(t,x)\|_E \leq R.$$

Let $L := L(x,T) > 0$ be the Lipschitz constant of F on $B_E(0,R)$. So, for any $x \in E$, large $n, m \in \mathbb{N}$ and $t \in [\epsilon, T]$, by (1.4.4) Remark 2.1.6 we have

$$\|z_n(t,x) - z_m(t,x)\|_E \leq \left\|(nR(n,A) - mR(m,A))e^{tA}x\right\|_E + M_0 L\int_0^t e^{t\eta_0}\|z_n(s,x) - z_m(s,x)\|_E ds.$$

Hence, by the Gronwall inequality, there exists $K_2 := K_2(x,T) > 0$ such that

$$\|z_n(t,x) - z_m(t,x)\|_E \leq K_2\left\|(nR(n,A) - mR(m,A))e^{tA}x\right\|_E. \tag{2.1.26}$$

Letting $m, n \to +\infty$ in (2.1.26), by (1.4.6) we obtain that, for any $T > 0$, $\epsilon > 0$ and $x \in E$, the sequence $\{z_n(\cdot,x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([\epsilon,T],E)$ and, since E is continuously embedded in $\mathcal{X}$, its limit is the same in $C([0,T],\mathcal{X})$. So the function defined in (2.1.25) is continuous from $(0,+\infty)$ to E and the process $\{X(t,x)\}_{t\geq 0}$, which verifies (2.1.19), verifies also (2.1.20). (2.1.18) and (2.1.20) yield (2.1.22), and by Remark 2.1.3 and (2.1.22), we have

$\{X(t,x)\}_{t\geq 0} \in C_p((0,T],E)$.

Now we prove (2.1.23). Let $T>0$ and $x,y\in E$. For any, $t\in[0,T]$, $k,n\in\mathbb{N}$, by (2.1.8) we have

$$\begin{aligned}\frac{1}{2}\frac{d\|z_{k,n}(t,x)-z_{k,n}(t,y)\|^2}{dt} &\leq \langle A(z_{k,n}(t,x)-z_{k,n}(t,y)), z_{k,n}(t,x)-z_{k,n}(t,y)\rangle\\ &+\langle F(z_{k,n}(t,x)+w_A(t))-F(z_{k,n}(t,y)+w_A(t)), z_{k,n}(t,x)-y_{k,n}(t,y)\rangle\\ &+\langle o_{k,n}(t,x)-o_{k,n}(t,y), z_{k,n}(t,x)-z_{k,n}(t,y)\rangle\end{aligned}$$

and by Hypotheses 2.1.1(iv) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d\|z_{k,n}(t,x)-z_{k,n}(t,y)\|^2}{dt} \leq&\zeta\|z_{k,n}(t,x)-z_{k,n}(t,y)\|^2\\ &+\|o_{k,n}(t,x)-o_{k,n}(t,y)\|\|z_{k,n}(t,x)-z_{k,n}(t,y)\|.\end{aligned}$$

By (2.1.2)(with $\epsilon=1$ and $q=2$) we have

$$\frac{d\|z_{k,n}(t,x)-z_{k,n}(t,y)\|^2}{dt}\leq 2\eta\|z_{k,n}(t,x)-z_{k,n}(t,y)\|^2+\frac{1}{2}\|o_{k,n}(t,x)-o_{k,n}(t,y)\|^2, \quad (2.1.27)$$

where $\eta=\zeta+\frac{1}{2}$. By 1.3.2 and letting $k\to+\infty$ we obtain

$$\|z_n(t,x)-z_n(t,y)\|^2\leq e^{2\eta t}\|x-z\|^2.$$

Taking the square root and letting $n\to+\infty$, by (2.1.19) we obtain

$$\|z(t,x)-z(t,y)\|\leq e^{\eta t}\|x-y\|,\quad t\in[0,T],\ x,y\in E$$

for any $T>0$ and for $\mathbb{P}$-a.a trajectory of $\{X(t,x)\}_{t\geq 0}$, so (2.1.23) is verified. Finally (2.1.24) follows from (2.1.20) using similar arguments. □

We make some remarks about possible variations of Theorem 2.1.10.

Corollary 2.1.11. *If the constant ζ in Hypotheses 2.1.1*(iv) *is negative, then the constants κ_p and η are negative.*

Proof. Applying (2.1.2) with $\epsilon=\zeta$ if $\zeta\in(0,1]$, or with $\epsilon=1/\zeta$ if $\zeta>1$, we obtain that the constants C_1 of (2.1.14) and η of (2.1.27) are negative. □

Remark 2.1.12. *If in addition the semigroup generated by the part of A in E is strongly continuous we can take $\epsilon=0$ in* (2.1.20)*, and $\{X(t,x)\}_{t\geq0}\in C_p([0,T],E)$, for any $p\geq1$ and $T>0$.*

Let $x\in E$ and let $\{X(t,x)\}_{t\geq0}$ be the process defined in Theorem 2.1.10. Now we prove that it is the unique mild solution of(2.0.1).

Theorem 2.1.13. *Assume that Hypotheses 2.1.1 hold true. For any $x\in E$, the process $\{X(t,x)\}_{t\geq0}$ is the unique mild solution of the SPDE* (2.0.1) *in $C_p([0,T],\mathcal{X})\cap C_p((0,T],E)$, for any $p\geq1$ and $T>0$.*

Proof. We begin to prove uniqueness. Let $x \in E$ and let $\{X_1(t,x)\}_{t\geq 0}, \{X_2(t,x)\}_{t\geq 0} \in C_p((0,T],E)$, for any $p \geq 1$ and $T > 0$, be two mild solution of (2.0.1). For any $0 < t \leq T$, by Remark 2.1.6, we have

$$\|X_1(t,x) - X_2(t,x)\|_E \leq M_0 \int_0^t e^{(t-s)\eta_0} \|F(X_1(t,x)) - F(X_2(t,x))\|_E ds, \quad \mathbb{P}\text{-a.s.}$$

Since $\{X_1(t,x)\}_{t\geq 0}, \{X_2(t,x)\}_{t\geq 0} \in C_p((0,T],E)$, with $p \geq 1$, then

$$\sup_{t\in[0,T]} \|X_1(t,x)\|_E, \sup_{t\in[0,T]} \|X_2(t,x)\|_E < +\infty, \quad \mathbb{P}\text{-a.s.}$$

so by the local lipschitzianity of F, there exists $L := L(x,T) > 0$ such that

$$\|X_1(t,x) - X_2(t,x)\|_E \leq M_0 L \int_0^t e^{(t-s)\eta_0} \|X_1(t,x) - X_2(t,x)\|_E ds, \quad \mathbb{P}\text{-a.s.}$$

and by the Gronwall inequality we obtain

$$X_1(t) = X_2(t), \quad \mathbb{P}\text{-a.s.}$$

for any $t \in [0,T]$ and $T > 0$, and so we have the uniqueness.

Now we prove that, for any $x \in E$, the process $\{X(t,x)\}_{t\geq 0}$ is the mild solution of (2.0.1). Let $T > 0$ and large $n \in \mathbb{N}$. We recall that, for any $t \in [0,T]$, we have

$$X_n(t,x) := Y_n(t,x) + W_A(t), \quad \mathbb{P}\text{-a.s.}$$

hence, by Proposition 2.1.7

$$X_n(t,x) = e^{tA} nR(n,A)x + \int_0^t e^{(t-s)A} F(X_n(s,x))ds + W_A(t), \quad \mathbb{P}\text{-a.s.} \tag{2.1.28}$$

By (1.4.3), Remarks 2.1.2-2.1.2, (2.1.22), (2.1.20) and the dominated convergence theorem, we have

$$\lim_{n\to+\infty} \left\| \int_0^t e^{(t-s)A} \left(F(X_n(s,x)) - F(X(s,x))\right) ds \right\|_E = 0, \quad \mathbb{P}\text{-a.s.},$$

so, letting $n \to +\infty$ in (2.1.28), by (1.4.6) we have

$$X(t,x) = e^{tA}x + \int_0^t e^{(t-s)A} F(X(s,x))ds + W_A(t) \quad \mathbb{P}\text{-a.s.}$$

for any $t \in [0,T]$ and $T > 0$. □

2.2 Generalized mild solution and transition semigroup

Now we exploit the density of E in $\mathcal{X}$ to define a process $\{X(t,x)\}_{t\geq 0}$ for any $x \in \mathcal{X}$.

Proposition 2.2.1. *Assume that Hypotheses 2.1.1 hold true. For any $x \in \mathcal{X}$ there exists a unique process $\{X(t,x)\}_{t\geq 0} \in C_p([0,T],E)$, for any $p \geq 1$ and $T > 0$, such that*

$$\lim_{n\to+\infty} \|X(\cdot,x_n) - X(\cdot,x)\|_{C([0,T],\mathcal{X})} = 0, \quad \forall\, T > 0,\ \mathbb{P}\text{-}a.s., \tag{2.2.1}$$

where $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ converges to x and $\{X(t,x_n)\}$ is the unique mild solution of (2.0.1) *with initial datum x_n. In addition, for any $p \geq 1$, $x, y \in \mathcal{X}$ and $t > 0$, we have*

$$\|X(t,x)\|^p \leq C'_p \left(e^{\kappa_p t}\|x\|^p + \int_0^t e^{\kappa_p(t-s)} \left(\|F(W_A(s))\|^p + \|W_A(s)\|^p\right) ds + \|W_A(t)\|^p \right), \tag{2.2.2}$$

$$\|X(t,x) - X(t,y)\| \leq e^{\eta t}\|x-y\|, \quad \mathbb{P}\text{-}a.s., \tag{2.2.3}$$

where κ_p is the constant of Proposition 2.1.7, C'_p is the constant of Proposition 2.1.9 and η is the constant of Theorem 2.1.10. Moreover, for any $x \in \mathcal{X}$, $p \geq 1$ and $T > 0$ we have

$$\lim_{n\to+\infty} \|\{X(t,x_n)\}_{t\geq 0} - \{X(t,x)\}_{t\geq 0}\|^p_{C_p([0,T],\mathcal{X})} = 0. \tag{2.2.4}$$

Proof. Since E is dense in $\mathcal{X}$, for any $x \in \mathcal{X}$ there exists a sequence $\{x_m\}_{m\in\mathbb{N}} \subseteq E$ such that

$$\lim_{m\to+\infty} \|x_m - x\| = 0.$$

We consider the sequence of mild solutions $\{\{X(t,x_m)\}_{t\in[0,T]}\}_{m\in\mathbb{N}} \subseteq C_p([0,T],\mathcal{X})$, for any $p \geq 1$ and $T > 0$, given by Theorem 2.1.13. We have

$$\{\{X(t,x_n)\}_{t\geq 0}\}_{n\in\mathbb{N}} \subseteq C([0,T],\mathcal{X}), \quad \mathbb{P}\text{-a.s.}$$

Moreover by (2.1.23), for any $T > 0$ and $n_1, n_2 \in \mathbb{N}$, we have

$$\lim_{n_1,n_2\to+\infty} \|X(\cdot,x_{n_1}) - X(\cdot,x_{n_2})\|_{C([0,T],\mathcal{X})} = 0, \quad \mathbb{P}\text{-a.s.}$$

So there exists a unique process $\{X(t,x)\}_{t\geq 0} \in \mathcal{PC}([0,T],\mathcal{X})$ (see Definition 2.1.5) that verifies (2.2.1). By (2.2.1) the process $\{X(t,x)\}_{t\geq 0}$ verifies (2.2.2), (2.2.3) and, by Remark 2.1.3, $\{X(t,x)\}_{t\geq 0} \in C_p([0,T],\mathcal{X})$, for any $p \geq 1$ and $T > 0$. Finally (2.2.3) yields (2.2.4). □

Definition 2.2.2. *For any $x \in \mathcal{X}$ we call generalized mild solution of* (2.0.1) *the limit $\{X(t,x)\}_{t\geq 0}$ of Corollary 2.2.1.*

Until now we have shown that

1. for any $x \in E$ the SPDE (2.0.1) has a unique mild solution $\{X(t,x)\}_{t\geq 0} \in C_p((0,T],E) \cap C_p([0,T],\mathcal{X})$, for any $p \geq 1$ and for any $T > 0$, in the sense of Definition 2.1.4;

2. for any $x \in \mathcal{X}$ the SPDE (2.0.1) has a unique generalized mild solution $\{X(t,x)\}_{t\geq 0} \in C_p([0,T],\mathcal{X})$, for any $p \geq 1$ and for any $T > 0$, in the sense of Definition 2.2.2. In particular if $x \in E$ then the generalized mild solution of (2.0.1) is the mild solution of (2.0.1).

So we define the following families of operators.

Definition 2.2.3. *For every $t > 0$ we set*

$$P(t)\varphi(x) := \mathbb{E}[\varphi(X(t,x))] = \int_\Omega \varphi(X(t,x)(\omega))\mathbb{P}(d\omega) \quad \varphi \in B_b(\mathcal{X}), x \in \mathcal{X},$$

where $\{X(t,x)\}_{t\geq 0}$ is the unique generalized mild solution of (2.0.1)*. Similarly we set*

$$P^E(t)\varphi(x) := \mathbb{E}[\varphi(X(t,x))] = \int_\Omega \varphi(X(t,x)(\omega))\mathbb{P}(d\omega) \quad \varphi \in B_b(E), x \in E,$$

where $\{X(t,x)\}_{t\geq 0}$ is the unique mild solution of (2.0.1)*.*

By the same arguments of [43][Proposition 9.14 and Corollary 9.15] and taking into account (2.1.24) and (2.2.3), we get the following result.

Proposition 2.2.4. *$\{P(t)\}_{t\geq 0}$ and $\{P^E(t)\}_{t\geq 0}$ are two contraction positive and Feller semigroups on $B_b(X)$ and $B_b(E)$ respectively.*

Proof. We prove the statements for $P(t)$, the proof for $\{P^E(t)\}_{t\geq 0}$ is the same. By uniqueness of the generalized mild solution $\{X(t,x)\}_{t\geq 0}$, for any $t, s \geq 0$ we have

$$X(t+s,x) = X(t, X(s,x)), \quad \mathbb{P}\text{-a.s.}$$

So, for any $\varphi \in B_b(X)$ and $x \in \mathcal{X}$, we have

$$P(t+s)\varphi(x) = \mathbb{E}\left[\varphi(X(t+s,x))\right] = \mathbb{E}\left[\mathbb{E}\left[\varphi(X(t,X(s,x))|X(s,x)\right]\right], \tag{2.2.5}$$

where we denote by $\mathbb{E}\left[\varphi(X(t,X(s,x))|X(s,x)\right]$ the conditional expectation of the random variable $\varphi(X(t,X(s,x))$ with respect to random variable $X(s,x)$. We denote by $\sigma\left(X(s,x)\right)$ the σ-algebra generated by $X(s,x)$. If we prove that, for any $\varphi \in B_b(\mathcal{X})$ and $\xi \in L^1((\Omega,\mathbb{P}),\mathcal{X})$ measurable with respect to $\sigma\left(X(s,x)\right)$, we have

$$\mathbb{E}\left[\varphi(X(t,\xi))|\xi\right] = P(t)\varphi(\xi), \mathbb{P}\text{-a.s.}, \tag{2.2.6}$$

then by (2.2.5) we obtain that $P(t)$ is a semigroup, namely

$$P(t+s)\varphi(x) = \mathbb{E}\left[\mathbb{E}\left[\varphi(X(t,X(s,x))|X(s,x)\right]\right] = \mathbb{E}\left[P(t)\varphi(X(s,x))\right] = P(t)P(s)\varphi(x).$$

We begin to prove (2.2.6) for a simple function ξ, namely

$$\xi(\omega) = \sum_{i=1}^n x_i \mathbb{I}_{\Gamma_i}(\omega), \quad \omega \in \Omega,$$

where $n \in \mathbb{N}$, $x_1, ..x_n \in \mathcal{X}$ and $\{\Gamma_1, ..., \Gamma_n\} \subseteq \sigma(X(s,x))$ is a partition of Ω. In this case (2.2.6) becomes

$$\mathbb{E}\left[\varphi(X(t,\xi))|\xi\right](\omega) = \sum_{i=1}^n \mathbb{E}\left[\varphi(X(t,x_i))\mathbb{I}_{\Gamma_i}|\xi = x_i\right](\omega).$$

$X(t,x_i)$ is independent of $\sigma(X(s,x))$ and Γ_i is measurable with respect to $\sigma(X(s,x))$, for any $i=1,...,n$. Then by [68, Theorem 23.5-23.6] we obtain

$$\begin{aligned}\mathbb{E}\left[\varphi(X(t,\xi))|\xi\right](\omega) &= \sum_{i=1}^{n}\mathbb{E}\left[\varphi(X(t,x_i))\mathbb{I}_{\Gamma_i}|\xi=x_i\right](\omega)\\ &= \sum_{i=1}^{n}P(t)\varphi(x_i)\mathbb{I}_{\Gamma_i}(\omega) = P(t)\varphi(\xi(\omega)).\end{aligned}$$

Since ξ belongs to $L^1((\Omega,\mathbb{P}),\mathcal{X})$ and it is measurable with respect to $\sigma(X(s,x))$, then there exists a sequence $\{\xi_n\}_{n\in\mathbb{N}}$ of simple functions such that

$$\lim_{n\to+\infty}\mathbb{E}[\|\xi_n-\xi\|] = \lim_{n\to+\infty}\int_\Omega\|\xi_n(\omega)-\xi(\omega)\|\mathbb{P}(d\omega)=0.$$

Hence there exists a subsequence of $\{\xi_n\}_{n\in\mathbb{N}}$ converging to ξ in $\mathcal{X}$ $\mathbb{P}$-a.s.. So, since (2.2.6) is verified for simple functions, then it is verified for a general $\xi\in L^1(\Omega,\mathbb{P})$. Hence (2.2.5) is verified and $P(t)$ is a semigroup. By the integral expression of $P(t)$ we see that $P(t)$ is positive and contractive and, by (2.2.3) the semigroup is also Feller. □

Since E is continuously embedded in $\mathcal{X}$, for any $f\in C_b(\mathcal{X})$ the restriction f_E of f to E belongs to $C_b(E)$. So by Theorem 1.9.3 and Proposition 2.2.4, we have the following result.

Corollary 2.2.5. *$\{P(t)\}_{t\geq 0}$ and $\{P^E(t)\}_{t\geq 0}$ are the transition semigroups associated to the generalized mild solution and to the mild solution of* (2.0.1) *in the sense of Definition 1.9.4. Moreover for any $t\geq 0$, $\varphi\in\mathcal{B}_b(\mathcal{X})$ and $x\in E$ we have*

$$P(t)\varphi(x)=P^E(t)\varphi(x).$$

2.3 The Gateaux derivative of the mild solution

In this section we assume that the following additional hypotheses hold.

Hypotheses 2.3.1. *Assume that Hypotheses 2.1.1 hold true with* $\mathrm{Dom}(F)=\mathcal{X}=E$ *and that $F:\mathcal{X}\to\mathcal{X}$ is Fréchet differentiable and Lipschitz continuous.*

Now recall a standard result about the Gateaux differentiability of the mild solution of (2.0.1).

Theorem 2.3.2 (Theorem 9.8 of [43])**.** *Assume that Hypotheses 2.3.1 hold true. The map $x\mapsto X(\cdot,x)$ is Gateaux differentiable as a function from $\mathcal{X}$ to $\mathcal{X}^p([0,T])$. For every $x,h\in\mathcal{X}$, the process $\{\mathcal{D}^G X(t,x)h\}_{t\geq 0}$ is the unique mild solution of*

$$\begin{cases}\frac{d}{dt}S_x(t,h)=\big(A+\mathcal{D}F(X(t,x))\big)S_x(t,h), & t>0;\\ S_x(0,h)=h.\end{cases}\tag{2.3.1}$$

By Hypotheses 2.1.1(iv-b) and an easy calculation, for any $t>0$ and $x,h\in\mathcal{X}$, we have

$$\langle A+\mathcal{D}F(x)h,h\rangle\leq\zeta_2\|h\|^2.\tag{2.3.2}$$

Proposition 2.3.3. *Assume that Hypotheses 2.3.1 hold true. For any $t > 0$ and $x, h \in \mathcal{X}$, it holds*

$$\|\mathcal{D}^G X(t,x)h\| \le e^{\zeta t}\|h\|,$$

where ζ is the constant appearing in Hypotheses 2.1.1(iv).

Proof. We assume that $\{\mathcal{D}^G X(t,x)h\}_{t\ge 0}$ is the strict solution of (2.3.1), otherwise we can approximate as in Proposition 2.1.7. We scalarly multiply both members of (2.3.1) by $\mathcal{D}^G X(t,x)h$. In the left hand side we obtain

$$\left\langle \frac{d}{dt}\mathcal{D}^G X(t,x)h, \mathcal{D}^G X(t,x)h \right\rangle = \frac{1}{2}\frac{d}{dt}\|\mathcal{D}^G X(t,x)h\|^2.$$

In the right hand side, by (2.3.2), we get

$$\big\langle (A + \mathcal{D}F(X(t,x)))\mathcal{D}^G X(t,x)h, \mathcal{D}^G X(t,x)h \big\rangle \le \zeta \big\|\mathcal{D}^G X(t,x)h\big\|^2.$$

Hence we obtain $\frac{d}{dt}\|\mathcal{D}^G X(t,x)h\|^2 \le 2\zeta\big\|\mathcal{D}^G X(t,x)h\big\|^2$, and so by the Gronwall inequality $\big\|\mathcal{D}^G X(t,x)h\big\|^2 \le e^{2\zeta t}\|h\|^2$. □

2.4 Remarks and examples

The results presented in this chapter are contained in the paper [9]. We make some remarks about Hypotheses 2.1.1. In [43, Sections 7.2] the authors prove existence and uniqueness of the generalized mild solution of (2.0.1) in many settings that include our own. However, they do not provide estimates like those in Theorem 2.1.10. Instead in [30, Chapters 4] and [20, Chapters 6-7] the authors prove some estimates like those in Theorem 2.1.10, but in a specific context (in the same context, see [21, 22] for the case of multiplicative noise and [24] for the nonautonomous case). Hypotheses 2.1.1 cover the case presented in [20, Chapters 6] (see [20, Section 6.1] for the definition of F, A and C). Let $\mathcal{O}$ be an open set of R^n, with $n \le 3$ and let λ be the Lebesgue measure. In the case where $\mathcal{X} = L^2(\mathcal{O}, \lambda)$ and $E = C(\overline{\mathcal{O}})$, in [20, Section 6.1] and [43, Section 5.5] the authors provide an overview of the operators A and R that satisfy Hypotheses 2.1.1. Concerning the perturbation F, in [30, Chapters 4] and [20, Chapters 6-7] the authors assume that F is a Nemytskii operator. In the next subsection we will present an example of F that it is not of this type but it verifies Hypotheses 2.1.1.

2.4.1 Infinite dimensional polynomial

We recall the notion of infinite dimensional polynomial (see [25, 47, 82]). For every $n \in \mathbb{N}$, we say that a map $V : \mathcal{X}^n \to \mathcal{X}$ is n-multilinear if it is linear in each variable separately. A n-multilinear map V is said to be symmetric if

$$V(x_1, \dots, x_n) = V(x_{\sigma(1)}, \dots, x_{\sigma(n)}), \tag{2.4.1}$$

for any permutation σ of the set $\{1,\ldots,n\}$. We say that a function $P_n : \mathcal{X} \to \mathcal{X}$ is a homogeneous polynomial of degree $n \in \mathbb{N}$ if there exists a n-multilinear symmetric map B such that for every $x \in \mathcal{X}$

$$P(x) = V(x,\ldots,x). \tag{2.4.2}$$

We consider the function $F : \mathcal{X} \to \mathcal{X}$ defined by

$$F(x) := P_n(x) + \zeta_2 x,$$

where $x \in \mathcal{X}$, $\zeta_2 \in \mathbb{R}$ and P_n is a homogeneous polynomial of degree n such that,

$$\langle V(h,x,\ldots,x), h\rangle \leq 0, \tag{2.4.3}$$

where V is the n-multilinear map defined by (2.4.2). By [25, Theorem 3.4], there exists $d > 0$ such that

$$\|F(x)\| \leq d(1 + \|x\|^n), \quad x \in \mathcal{X}. \tag{2.4.4}$$

Moreover, for any $x, h \in \mathcal{X}$, we have

$$\mathcal{D}P_n(x)h = nV(h,x,\ldots,x),$$

and so, by (2.4.3), for any $x, y \in \mathcal{X}$, we obtain

$$\langle F(x) - F(y), x - y\rangle \leq \zeta_2 \|x - y\|^2. \tag{2.4.5}$$

Let us consider a particular case. Let $E = \mathcal{X} = L^2([0,1],\lambda)$. Let $K \in L^2([0,1]^4)$ and assume that K is symmetric ((2.4.1)). Let

$$[P_3(f)](\xi) := \int_0^1\int_0^1\int_0^1 K(\xi_1,\xi_2,\xi_3,\xi) f(\xi_1)f(\xi_2)f(\xi_3)d\xi_1 d\xi_2 d\xi_3 \tag{2.4.6}$$

for $f \in L^2([0,1])$. P is a homogeneous polynomial of degree three on $L^2([0,1])$ (see [47, Exercise 1.73]). (2.4.3) holds whenever K has negative values. Indeed observe that, for $f_1, f_2, f_3 \in L^2([0,1])$,

$$B(f_1,f_2,f_3) = \int_0^1\int_0^1\int_0^1 K(\xi_1,\xi_2,\xi_3,\xi) f_1(\xi_1)f_2(\xi_2)f_3(\xi_3)d\xi_1 d\xi_2 d\xi_3,$$

and for $f, h \in L^2([0,1])$

$$\langle B(h,f,f), h\rangle = \int_0^1\int_0^1\int_0^1\int_0^1 K(\xi_1,\xi_2,\xi_3,\xi) f(\xi_1)f(\xi_2)h(\xi_3)h(\xi)d\xi_1 d\xi_2 d\xi_3 d\xi.$$

A standard argument allows to deduce that $\langle B(h,f,f),h\rangle = 0$ if, and only if, $f = 0$ a.e. or $h = 0$ a.e. So by the continuity of $\langle B(h,f,f),h\rangle$ with respect to h (for a fixed f) and the fact that

$$\langle B(-h,f,f),-h\rangle = \langle B(h,f,f),h\rangle,$$

the claim follows. Similarly it is possible to consider a general infinite dimensional polynomial of odd degree $n \in \mathbb{N}$.

Chapter 3

Regularization results

Let $\mathcal{X}$ be a separable Hilbert space.

Let $T(t)$ be a semigroup on $B_b(\mathcal{X})$. We say that $T(t)$ is a strong Feller semigroup if for any $t > 0$ we have
$$T(t)\left(B_b(\mathcal{X})\right) \subseteq C_b(\mathcal{X}).$$
Let A be the infinitesimal generator of a strongly continuous semigroup e^{tA} on $\mathcal{X}$ and let $R \in \mathcal{L}(\mathcal{X})$ satisfy (1.10.1). If $T(t)$ is the Ornstein-Uhlenbeck semigroup defined by the Mehler formula
$$T(t)f(x) := \int_0^t f(y)\mathcal{N}(e^{tA}x, Q_t), \quad Q_t := \int_0^t e^{sA}RR^*e^{sA^*}, \quad t > 0,\ f \in \mathcal{B}_b(\mathcal{X}),$$
In [43, Section 9.4] the authors show that $T(t)$ is a strong Feller semigroup if and only if
$$e^{tA}(\mathcal{X}) \subseteq Q_t^{1/2}(\mathcal{X}), \quad t > 0. \tag{3.0.1}$$
Let $G : \mathcal{X} \to \mathcal{X}$ be a Lipschitz continuous function and let $\{W(t)\}_{t\geq 0}$ be a $\mathcal{X}$-cylindrical Wiener process defined on a normal filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. In [14, 61] the authors consider the transition semigroup defined by
$$P(t)f(x) := \mathbb{E}[f(X(t,x))], \quad t > 0,\ f \in \mathcal{B}_b(\mathcal{X}), \tag{3.0.2}$$
where $\{X(t,x)\}_{t\geq 0}$ is the unique mild solution of the SPDE
$$\begin{cases} dX(t,x) = \big(AX(t,x) + RG(X(t,x))\big)dt + RdW(t), & t > 0; \\ X(0,x) = x \in \mathcal{X}. \end{cases} \tag{3.0.3}$$
Using the Girsanov theorem, they prove that, if 3.0.1 holds then $P(t)$ is a strong Feller semigroup.

If we take that $A = -\frac{1}{2}\mathrm{I}_{\mathcal{X}}$ and $R = Q^{1/2}$ where Q is a positive and trace class operator, then condition (3.0.1) is not verified. In that case it is known that $T(t)$ is not strong Feller but $T(t)$ regularizes only along $Q^{1/2}(\mathcal{X})$. In particular for any $T > 0$ there exists $K_T > 0$ such that, for any $t \in [0,T]$, $x \in \mathcal{X}$, $h \in Q^{1/2}(\mathcal{X})$ and $\varphi \in B_b(\mathcal{X})$, we have

$$|T(t)\varphi(x+h) - T(t)\varphi(x)| \leq \frac{K_T}{\sqrt{t}} \left\| Q^{-1/2}h \right\| \|\varphi\|_\infty$$

In this chapter we study the regularity properties of the semigroup $P(t)$ defined in (3.0.2) not assuming that (3.0.1) holds, in particular we will show that, under suitable conditions, $P(t)$ regularizes along $R(\mathcal{X})$.

Now we specify the hypotheses under which we will work.

Hypotheses 3.0.1. *We assume that the following conditions hold true.*

(i) $R \in \mathcal{L}(\mathcal{X})$ *is non-negative.*

(ii) A *is the infinitesimal generator of a strongly continuous semigroup* e^{tA} *on* $\mathcal{X}$ *and there exists* $w_{\mathcal{X}} \in \mathbb{R}$ *such that*

$$\langle Ax, x \rangle \leq -w_{\mathcal{X}} \|x\|^2, \quad x \in \mathrm{Dom}(A).$$

(iii) $G : \mathcal{X} \to \mathcal{X}$ *is a Lipschitz continuous function, with Lipschitz constant* L_G.

(iv) *There exists* $\eta \in (0,1)$ *such that, for any* $T > 0$, *we have*

$$\int_0^T \frac{1}{t^\eta} \mathrm{Tr}[e^{tA} R^2 e^{tA^*}] dt < +\infty.$$

By Hypotheses 3.0.1(ii-iii), for any $x, y \in \mathcal{X}$, we have

$$\langle Ax - Ay + RG(x) - RG(y), x - y \rangle \leq \zeta_{\mathcal{X}} \|x - y\|^2, \quad \zeta_{\mathcal{X}} = w_{\mathcal{X}} + \|R\|_{\mathcal{L}(\mathcal{X})} L_G.$$

Moreover by 3.0.1(iv) and Proposition 1.8.11, Hypotheses 2.1.1(v) are verified, so Hypotheses 3.0.1 implies Hypotheses 2.1.1 (with $E = \mathcal{X}$). We can apply Theorem 2.1.13 (with $E = \mathcal{X}$) and obtain that, for any $x \in \mathcal{X}$, the SPDE (3.0.3) has a unique mild solution $\{X(t,x)\}_{t \geq 0} \in C_p([0,T], \mathcal{X})$, for any $T > 0$ and $p \geq 1$. Hence the semigroup $P(t)$ given by (3.0.2) is well defined.

We now present the hypothesis that will replace the condition (3.0.1). By the results of Subsection 1.2.5 the following Hilbert space is well defined.

Definition 3.0.2. *We denote by* $(H_R, \langle \cdot, \cdot \rangle_R)$ *the separable Hilbert space defined by*

$$H_R := R(\mathcal{X}), \quad \|x\|_R := \left\| R^{-1}x \right\|, \quad \langle x, y \rangle_R := \left\langle R^{-1}x, R^{-1}y \right\rangle, \quad x, y \in H_R,$$

where R^{-1} *is the pseudo inverse of* R.

Hypotheses 3.0.3. *Assume that Hypotheses 3.0.1 hold true, that* A_R *(the part of* A *in* H_R*) generates a strongly continuous semigroup in* H_R, *and that there exists* $w_R \in \mathbb{R}$ *such that, for any* $h \in \mathrm{Dom}(A_R)$, *we have*

$$\langle Ah, h \rangle_R \leq w_R \|h\|_R^2.$$

Under Hypotheses 3.0.3 we are going to prove that for any for any $t > 0$, $x \in \mathcal{X}$, $h \in H_R$ and $\varphi \in B_b(\mathcal{X})$, we have

$$|P(t)\varphi(x+h) - P(t)\varphi(x)| \leq K(t)\|h\|_R. \tag{3.0.4}$$

where $K(\cdot) : (0, +\infty) \to (0, +\infty)$ is continuous and for small t

$$K(t) \approx \frac{k}{\sqrt{t}},$$

for some $k > 0$. To prove (3.0.4) we will not use the Girsanov theorem, but we will exploit a technique similar to the one presented in [43, Section 9.4.2] or [85]. A fundamental step in this technique is to prove a Bismut–Elworthy–Li formula for the semigroup $P(t)$. To do that, we need to study some properties of the mild solution in the space H_R.

3.1 The H_R-differentiability

First of all we show some properties of the Hilbert space H_R.

Proposition 3.1.1. *H_R is Borel measurable and continuously embedded in $\mathcal{X}$. Moreover, for any $h \in H_R$, we have*

$$\|h\| \leq \|R\|_{\mathcal{L}(\mathcal{X})} \|h\|_R, \tag{3.1.1}$$

$$\|R\|_{\mathcal{L}(H_R)} = \sup_{\{h \in H_R \,:\, \|h\|_R = 1\}} \|Rh\|_R \leq \|R\|_{\mathcal{L}(\mathcal{X})}. \tag{3.1.2}$$

Proof. (3.1.1) and (3.1.2) follow immediately by the assumption $R \in \mathcal{L}(\mathcal{X})$. To prove that H_R is Borel measurable it is enough to observe that if $\{g_k\}_{k \in \mathbb{N}}$ is an orthonormal basis of $\operatorname{Ker}(R)^\perp$ then

$$\begin{aligned} H_R &= \left\{ x \in \mathcal{X} \,\middle|\, \sum_{k=1}^{+\infty} \langle x, Rg_k \rangle_R < +\infty \right\} \\ &= \bigcup_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \left\{ x \in \mathcal{X} \,\middle|\, \sum_{k=1}^{n} \langle x, Rg_k \rangle_R \leq m \right\}. \end{aligned}$$

For every $m, n \in \mathbb{N}$ the set $\{x \in \mathcal{X} \,|\, \sum_{k=1}^n \langle x, g_k \rangle_R \leq m\}$ is closed, since the maps $x \mapsto \langle x, Rg_k \rangle_R$ are continuous for every $k \in \mathbb{N}$. □

Now we specify the regularity properties along H_R that are of interest for us.

Definition 3.1.2.

(i) *Let $\Phi : \mathcal{X} \to \mathbb{R}$ be a measurable function. We say that Φ is H_R-Lipschitz if there exists $C > 0$ such that for every $x \in \mathcal{X}$ and $h \in H_R$*

$$\|\Phi(x+h) - \Phi(x)\|_R \leq C\|h\|_R. \tag{3.1.3}$$

We denote by $Lip_{H_R}(\mathcal{X})$ the sets of Borel measurable, H_R-Lipschitz functions, and by $Lip_{b,H_R}(\mathcal{X})$ the subset of $Lip_{H_R}(\mathcal{X})$ consisting of bounded functions. We call H_R-Lipschitz constant of Φ the infimum of all the constants $C > 0$ verifying (3.1.3).

(ii) *Let $\Phi : \mathcal{X} \to \mathcal{X}$ be a Borel measurable function. We say that Φ is H_R-differentiable at $x \in \mathcal{X}$,* *if the function $\varphi_x : H_R \to \mathcal{X}$ defined as*

$$\varphi_x(h) := \Phi(x+h) - \Phi(x)$$

is H_R-valued and there exists $L \in \mathcal{L}(H_R)$ such that for every $h \in H_R$

$$\lim_{\|h\|_R \to 0} \left\| \frac{1}{\|h\|_R} \varphi_x(h) - Lh \right\|_R = 0.$$

When it exists, the operator L is unique and we set $\nabla_R \Phi(x) := L$.

(iii) *Let $\Phi : \mathcal{X} \to \mathbb{R}$ be a Borel measurable function. We say that Φ is H_R-differentiable at $x \in \mathcal{X}$, if there exists $L \in H_R^*$ such that*

$$\lim_{\|h\|_R \to 0} \frac{|\Phi(x+h) - \Phi(x) - Lh|}{\|h\|_R} = 0.$$

When it exists, the operator L is unique and we set $\mathcal{D}_R \Phi(x) := L$. Since $L \in H_R^*$, then there exists $k \in H_R$ such that $Lh = \langle h, k \rangle_R$ for any $h \in H_R$. We set $\nabla_R \Phi(x) := k$ and we call it H_R-gradient of Φ at $x \in \mathcal{X}$. For any $k \geq 2$, in the same way we define the k-times H_R-differentiable functions and we denote by $\nabla_R^k \Phi(x)$ their k-derivative. We denote by $C_{H_R}^k(\mathcal{X})$ the space of k-times H_R-differentiable functions from $\mathcal{X}$ to $\mathbb{R}$ and by $C_{b,H_R}^k(\mathcal{X})$ the space of bounded and k-times H_R-differentiable functions from $\mathcal{X}$ to $\mathbb{R}$.

(iv) *Let $\Phi : \mathcal{X} \to \mathcal{X}$ be a Borel measurable function. We say that a function Φ is H_R-Gateaux differentiable at $x \in \mathcal{X}$ if for every $h \in H_R$ there exists $\varepsilon_{x,h} > 0$ such that the function $\varphi_{x,h} : (-\varepsilon_{x,h}, \varepsilon_{x,h}) \to \mathcal{X}$ defined as*

$$\varphi_{x,h}(r) := \Phi(x + rh) - \Phi(x)$$

is H_R-valued and there exists $L \in \mathcal{L}(H_R)$ such that for every $h \in H_R$

$$\lim_{r \to 0} \left\| \frac{1}{r} \varphi_{x,h}(r) - Lh \right\|_R = 0. \tag{3.1.4}$$

We observe a relationship between Fréchet differentiability and H_R-differentiability.

Proposition 3.1.3. *Let $\Phi : \mathcal{X} \to \mathbb{R}$. If Φ is Fréchet differentiable at $x \in \mathcal{X}$, then it is H_R-differentiable at x and*

$$\nabla_R \varphi(x) = R^2 \nabla \varphi(x).$$

Proof. By the Fréchet differentiability of Φ at x we have

$$\lim_{\|h\| \to 0} \frac{|\Phi(x+h) - \Phi(x) - \mathcal{D}\Phi(x)h|}{\|h\|} = 0.$$

By (3.1.1) we have $\|h\| \to 0$ whenever $\|h\|_R \to 0$, and

$$\begin{aligned}
0 \leq &\lim_{\|h\|_R \to 0} \frac{|\Phi(x+h) - \Phi(x) - \mathcal{D}\Phi(x)h|}{\|h\|_R} \\
= &\lim_{\|h\|_R \to 0} \frac{|\Phi(x+h) - \Phi(x) - \mathcal{D}\Phi(x)h|}{\|h\|} \frac{\|h\|}{\|h\|_R} \\
\leq &\|R\|_{\mathcal{L}(\mathcal{X})} \lim_{\|h\|_R \to 0} \frac{|\Phi(x+h) - \Phi(x) - \mathcal{D}\Phi(x)h|}{\|h\|_R} = 0
\end{aligned}$$

Moreover, for every $x \in \mathcal{X}$ and $h \in H_R$, we have

$$\begin{aligned}
\langle \nabla_R \varphi(x), h \rangle_R &= \mathcal{D}_R \varphi(x)h = \mathcal{D}\varphi(x)h = \langle \nabla\varphi(x), h \rangle \\
&= \langle R\nabla\varphi(x), Rh \rangle_R = \langle R^2 \nabla\varphi(x), h \rangle_R,
\end{aligned}$$

hence $\nabla_R \varphi(x) = R^2 \nabla\varphi(x)$. □

We assume that Hypotheses 3.0.3 hold true and that G is Fréchet differentiable. Let $\{X(t,x)\}_{t \geq 0}$ be the mild solution of (3.0.3). By Theorem 2.3.2, the map $x \mapsto X(\cdot, x)$ is Gateaux differentiable as a function from $\mathcal{X}$ to $\mathcal{X}^p([0,T])$, for any $T > 0$. Moreover for every $x, h \in \mathcal{X}$, the process $\{\mathcal{D}^G X(t,x)h\}_{t \geq 0}$ is the unique mild solution of

$$\begin{cases} \frac{d}{dt} S_x(t,h) = \big(A + R\mathcal{D}G(X(t,x))\big) S_x(t,h), & t > 0; \\ S_x(0,h) = h. \end{cases} \tag{3.1.5}$$

We are going to show that the mild solution is H_R-differentiable.

Proposition 3.1.4. *Assume that Hypotheses 3.0.3 hold true and that G is Fréchet differentiable. Then for any $x \in \mathcal{X}$, $h \in H_R$ and $t > 0$, we have*

$$\|\mathcal{D}^G X(t,x)h\|_R \leq e^{\zeta_R t} \|h\|_R, \quad \mathbb{P}\text{-a.s.}$$

where $\zeta_R := w_R + \|R\|_{\mathcal{L}(\mathcal{X})} L_G$. Moreover for every $t > 0$, the map $x \mapsto X(t,x)$ is $\mathbb{P}$-a.s.. H_R-Gateaux differentiable and for any $x \in \mathcal{X}$ and $h \in H_R$ its H_R-Gateaux derivative along $h \in H_R$ is $\mathcal{D}^G X(t,x)h$.

Proof. All estimates in this proof should be understood to be true $\mathbb{P}$-a.s. By Hypotheses 3.0.3 and Definition 3.0.2, for any $x \in \mathcal{X}$ and $h \in H_R$ we have

$$\langle [A + R\nabla G(x)]h, h \rangle_R \leq \zeta_R \|h\|_R^2, \tag{3.1.6}$$

where $\zeta_R := w_R + \|R\|_{\mathcal{L}(\mathcal{X})} L_G$. Since the map $t \in [0, +\infty) \to X(t,x) \in \mathcal{X}$ is continuous $\mathbb{P}$-a.s. then $t \in [0, +\infty) \to R\mathcal{D}G(X(t,x)) \in H_R$ is continuous $\mathbb{P}$-a.s., so we can study (3.1.5) in H_R. For any $x \in \mathcal{X}$, $h \in H_R$ and $t > 0$, we scalarly multiply both members of (3.1.5) by $\mathcal{D}^G X(t,x)h$. In the left hand side we obtain

$$\left\langle \frac{d}{dt} \mathcal{D}^G X(t,x)h, \mathcal{D}^G X(t,x)h \right\rangle_R = \frac{1}{2} \frac{d}{dt} \|\mathcal{D}^G X(t,x)h\|_R^2.$$

In the right hand side, by (2.3.2) we get

$$\big\langle (A + \mathcal{D}F(X(t,x)))\mathcal{D}^G X(t,x)h, \mathcal{D}^G X(t,x)h \big\rangle_R \le \zeta_R \big\| \mathcal{D}^G X(t,x)h \big\|_R^2.$$

Hence we obtain $\frac{d}{dt}\|\mathcal{D}^G X(t,x)h\|_R^2 \le 2\zeta_R \big\|\mathcal{D}^G X(t,x)h\big\|_R^2$, and so by the Gronwall inequality $\big\|\mathcal{D}^G X(t,x)h\big\|_R^2 \le e^{2\zeta_R t}\|h\|_R^2$.

By Hypotheses 3.0.3, for any $x \in \mathcal{X}$, $h \in H_R$ and $r, t > 0$ we get

$$\begin{aligned}
&\left\| \frac{X(t,x+rh) - X(t,x)}{r} - \mathcal{D}^G X(t,x)h \right\|_R \\
&\le \int_0^t \left\| \frac{G(X(s,x+rh)) - G(X(s,x))}{r} - \mathcal{D}G(X(s,x))\mathcal{D}^G X(s,x)h \right\| ds,
\end{aligned}$$

hence, since $x \to X(t,x)$ is Gateaux differentiable and G is Lipschitz continuous and Frechét differentiable, by the dominated convergence theorem we have

$$\lim_{r\to 0} \left\| \frac{X(t,x+rh) - X(t,x)}{r} - \mathcal{D}^G X(t,x)h \right\|_R = 0.$$

□

Remark 3.1.5. *It is possible to assume* (3.1.6) *as Hypothesis in order to avoid ζ_R depending on the Lipschitz constant of G.*

Corollary 3.1.6. *Assume that Hypotheses 3.0.3 hold true and G is Frechét differentiable. If $g : \mathcal{X} \to \mathbb{R}$ is a function belonging to $C_b^1(\mathcal{X})$ and $h \in H_R$, then for any $x \in X$ and $t \ge 0$*

$$((\mathcal{D}_R^G(g \circ X))(t,x))h = \big\langle (\nabla_R g)\,(X(t,x)), \mathcal{D}^G X(t,x)h \big\rangle_R.$$

Proof. Since $g \in C_b^1(\mathcal{X})$, by Proposition 3.1.3, g is also H_R differentiable, then for every $x \in \mathcal{X}$ and $h \in H_R$

$$g(x + \epsilon h) = g(x) + \varepsilon \langle \nabla_R g(x), h\rangle_R + o(\varepsilon) \qquad \varepsilon \to 0.$$

We define for $x \in \mathcal{X}$, $h \in H_R$, $t \ge 0$ and $\varepsilon > 0$

$$K_\varepsilon(t,x,h) := X(t,x+\varepsilon h) - X(t,x) - \varepsilon \mathcal{D}^G X(t,x)h = X(t,x+h) - X(t,x) - \varepsilon \mathcal{D}^G X(t,x)h.$$

By Proposition 3.1.4, for any $T > 0$ we have $\sup_{t\in[0,T]} \mathbb{E}[\|K_\varepsilon(\cdot,x,h)\|_R^2] = o(\varepsilon)$, when ε goes to zero. Hence for $\varepsilon \to 0$

$$\begin{aligned}
g\big(X(t,x+\varepsilon h)\big) &= g\big(X(t,x) + \varepsilon \mathcal{D}^G X(t,x)h + K_\varepsilon(t,x,h)\big) \\
&= g\big(X(t,x) + \varepsilon(\mathcal{D}^G X(t,x)h + \varepsilon^{-1} K_\varepsilon(t,x,h)\big) \\
&= g(X(t,x)) + \big\langle (\nabla_R g)(X(t,x)), \varepsilon \mathcal{D}^G X(t,x)h + K_\varepsilon(t,x,h) \big\rangle_R + o(\varepsilon) \\
&= g(X(t,x)) + \varepsilon \big\langle (\nabla_R g)(X(t,x)), \mathcal{D}^G X(t,x)h \big\rangle_R \\
&+ \langle (\nabla_R g)(X(t,x)), K_\varepsilon(t,x,h)\rangle_R + o(\varepsilon).
\end{aligned}$$

So for $\varepsilon \to 0$ we get

$$\begin{aligned}
0 &\le \mathbb{E}\Big[\big|g\big(X(t,x+\varepsilon h)\big) - g(X(t,x)) - \varepsilon\big\langle(\nabla_R g)(X(t,x)), \mathcal{D}^G X(t,x)h\big\rangle_R\big|^2\Big] \\
&\le \sup_{t\in[0,T]} \mathbb{E}\Big[\big|g\big(X(t,x+\varepsilon h)\big) - g(X(t,x)) - \varepsilon\big\langle(\nabla_R g)(X(t,x)), \mathcal{D}^G X(t,x)h\big\rangle_R\big|^2\Big] \\
&= \sup_{t\in[0,T]} \mathbb{E}\Big[|\langle(\nabla_R g)(X(t,x)), K_\varepsilon(t,x,h)\rangle_R|^2\Big] + o(\varepsilon) \\
&\le \left(\sup_{x\in\mathcal{X}} \|\mathcal{D}_R g(x)\|_{\mathcal{L}(R)}\right)\left(\sup_{t\in[0,T]} \mathbb{E}\Big[\|K_\varepsilon(t,x,h)\|_R^2\Big]\right) + o(\varepsilon) \\
&= \left(1 + \sup_{x\in\mathcal{X}} \|\mathcal{D}_R g(x)\|_{\mathcal{L}(H_R)}\right) o(\varepsilon)
\end{aligned}$$

This implies that $((\mathcal{D}_R^G(g\circ X))(t,x))h = \big\langle(\nabla_R g)\,(X(t,x)), \mathcal{D}^G X(t,x)h\big\rangle_R$ $\mathbb{P}$-a.s., and the proof is concluded. □

3.2 The Bismut–Elworthy–Li formula

Now we prove a variant of the Bismut–Elworthy–Li formula.

Proposition 3.2.1. *Assume that Hypotheses 3.0.3 hold true and that $G \in C_b^2(\mathcal{X},\mathcal{X})$. Let $\varphi \in C_b^2(\mathcal{X})$. For every $x \in \mathcal{X}$, $h \in H_R$ and $t \ge 0$*

$$\langle \nabla_R P(t)\varphi(x), h\rangle_R = \frac{1}{t}\mathbb{E}\left[\varphi(X(t,x))\int_0^t \big\langle \mathcal{D}^G X(s,x)h, RdW(s)\big\rangle_R\right]. \tag{3.2.1}$$

Furthermore

$$|\langle \nabla_R P(t)\varphi(x), h\rangle_R|^2 \le \frac{1}{t^2}\|\varphi\|_\infty^2 \mathbb{E}\left[\int_0^t \big\|\mathcal{D}_R^G X(s,x)h\big\|_R^2 ds\right]. \tag{3.2.2}$$

Proof. (3.2.2) is a standard consequence of (3.2.1) and the Itô isometry (see [84, Lemma 3.1.5]) so we will only show (3.2.1). By [85, Lemma 2.3], for any $\varphi \in C_b^2(\mathcal{X})$, $t > 0$ and $x \in \mathcal{X}$ we have

$$\varphi(X(t,x)) = P(t)\varphi(x) + \int_0^t \langle \nabla P(t-s)\varphi(X(s,x)), RdW(s)\rangle. \tag{3.2.3}$$

By Proposition 3.1.3, (3.2.3) becomes

$$\varphi(X(t,x)) = P(t)\varphi(x) + \int_0^t \langle \nabla_R P(t-s)\varphi(X(s,x)), RdW(s)\rangle_R. \tag{3.2.4}$$

For any $h \in H_R$, we consider the process

$$\left\{\int_0^t \big\langle \mathcal{D}^G X(s,x)h, RdW(s)\big\rangle_R\right\}_{t\ge 0}. \tag{3.2.5}$$

Multiplying both sides of (3.2.4) by (3.2.5) and taking the expectations we get

$$\begin{aligned}
&\mathbb{E}\left[\varphi(X(t,x))\int_0^t \left\langle \mathcal{D}^G X(s,x)h, RdW(s)\right\rangle_R\right]\\
=&\mathbb{E}\left[P(t)\varphi(x)\int_0^t \left\langle \mathcal{D}^G X(s,x)h, RdW(s)\right\rangle_R\right]\\
+&\mathbb{E}\left[\int_0^t \left\langle \nabla_R P(t-s)\varphi(X(s,x)), RdW(s)\right\rangle_R \int_0^t \left\langle \mathcal{D}^G X(s,x)h, RdW(s)\right\rangle_R\right].
\end{aligned}$$

Since $R:\mathcal{X}\to H_R$ is continuous, then $\{RW(t)\}_{t\geq 0}$ is a H_R-cylindrical Wiener process (see [43, Remark 5.1]). By Proposition 3.1.4 for every $t\in[0,T]$ and $h\in H_R$

$$\int_0^t \mathbb{E}\big[\|\mathcal{D}^G X(s,x)h\|_R^2\big]ds<+\infty,$$

and so by [52, Remark 2], the process $\{\int_0^t \left\langle \mathcal{D}^G X(s,x)h, RdW(s)\right\rangle_R\}_{t\geq 0}$ is a martingale. Hence for every $t\in[0,T]$, $x\in\mathcal{X}$ and $h\in H_R$

$$\mathbb{E}\left[P(t)\varphi(x)\int_0^t \left\langle \mathcal{D}^G X(s,x)h, RdW(s)\right\rangle_R\right]=0.$$

We recall that since $L^2(\Omega,\mathcal{F},\mathbb{P})$ is a Hilbert space, then for every $\xi_1,\xi_2\in L^2(\Omega,\mathcal{F},\mathbb{P})$ we have

$$\mathbb{E}[\xi_1\xi_2]=\frac{1}{4}\mathbb{E}\big[|\xi_1+\xi_2|^2\big]-\frac{1}{4}\mathbb{E}\big[|\xi_1-\xi_2|^2\big]. \tag{3.2.6}$$

Let $\Phi(s):=\nabla_R P(t-s)\varphi(X(s,x))$ and $\Gamma(s):=\mathcal{D}^G X(s,x)h$. We apply (3.2.6) with $\xi_1=\int_0^t\langle\Phi(s),RdW(s)\rangle_R$ and $\xi_2=\int_0^t\langle\Gamma(s),RdW(s)\rangle_R$ and using the Itô isometry we get

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t\langle\Phi,RdW\rangle_R\int_0^t\langle\Gamma,RdW\rangle_R\right]\\
=&\frac{1}{4}\mathbb{E}\left[\left(\int_0^t\langle\Phi+\Gamma,RdW\rangle_R\right)^2\right]-\frac{1}{4}\mathbb{E}\left[\left(\int_0^t\langle\Phi-\Gamma,RdW\rangle_R\right)^2\right]\\
=&\frac{1}{4}\mathbb{E}\left[\int_0^t\|\Phi+\Gamma\|_R^2ds\right]-\frac{1}{4}\mathbb{E}\left[\int_0^t\|\Phi-\Gamma\|_R^2ds\right]=\mathbb{E}\left[\int_0^t\langle\Phi,\Gamma\rangle_R ds\right].
\end{aligned}$$

Hence by (3.1.6), with $g=P(t-s)\varphi$ we obtain

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t\langle(\nabla_R P(t-s)\varphi)(X(s,x)),RdW(s)\rangle_\alpha\int_0^t\left\langle\mathcal{D}^G X(s,x)h,RdW(s)\right\rangle_R\right]\\
=&\mathbb{E}\left[\int_0^t\left\langle(\nabla_R P(t-s)\varphi)(X(s,x)),\mathcal{D}^G X(s,x)h\right\rangle_R ds\right]\\
=&\mathbb{E}\left[\int_0^t \mathcal{D}^G_R(((P(t-s)\varphi)\circ X)(s,x))hds\right]\\
=&\int_0^t\big(\mathcal{D}^G_R\mathbb{E}\big[(P(t-s)\varphi\circ X)(s,x)\big]\big)hds.
\end{aligned}$$

By the very definition of $P(t)$ we know that $\mathbb{E}[(P(t-s)\varphi\circ X)(s,x)] = (P(s)P(t-s)\varphi)(x) = P(t)\varphi(x)$. Recalling that $P(t)\varphi$ belongs to $C_b^2(\mathcal{X})$, so that it is also is H_R-differentiable, it holds $\mathcal{D}_R^G P(t)\varphi(x) = \mathcal{D}_R P(t)\varphi(x)$. So we conclude

$$\begin{aligned}\mathbb{E}\left[\varphi(X(t,x))\int_0^t \left\langle \mathcal{D}^G X(s,x)h, RdW(s)\right\rangle_R\right] &= \int_0^t \langle \nabla_R P(t)\varphi(x), h\rangle_R ds \\ &= t\langle \nabla_R P(t)\varphi(x), h\rangle_R.\end{aligned}$$

□

3.3 H_R regularity for the transition semigroup

The last step before proving the main result of this Chapter is the following Lemma.

Lemma 3.3.1. *Assume that Hypotheses 3.0.3 hold true and that $G\in C_b^2(\mathcal{X};\mathcal{X})$. For every $t>0$, $x\in\mathcal{X}$, $h\in H_R$ and $\varphi\in C_b^2(\mathcal{X})$*

$$|P(t)\varphi(x+h) - P(t)\varphi(x)| \le K(t)\|\varphi\|_\infty \|h\|_R, \tag{3.3.1}$$

where

$$K(t) := \begin{cases} \frac{\sqrt{\zeta_R^{-1}(e^{\zeta_R t}-1)}}{t}, & \zeta_R\neq 0 \\ \frac{1}{\sqrt{t}}. & \zeta_R = 0. \end{cases} \tag{3.3.2}$$

Proof. Taking into account Proposition 3.1.4 and (3.2.2) we obtain the gradient estimate

$$\|\nabla_R P(t)\varphi(x)\|_R \le K(t)\|\varphi\|_\infty, \qquad t\in(0,T],\ x\in\mathcal{X}. \tag{3.3.3}$$

Let $x\in\mathcal{X}$ and $h\in H_R$. By the mean value theorem there exists $c_h\in(0,1)$ such that

$$P(t)\varphi(x+h) - P(t)\varphi(x) = \langle \nabla_R P(t)\varphi(x+c_h h), h\rangle.$$

So, by (3.3.3), the thesis follows. □

Now we prove the main result of this chapter.

Theorem 3.3.2. *Assume that Hypotheses 3.0.3 hold true. For every $t>0$, $x\in\mathcal{X}$, $h\in H_R$ and $\varphi\in B_b(\mathcal{X})$*

$$|P(t)\varphi(x+h) - P(t)\varphi(x)| \le K(t)\|\varphi\|_\infty \|h\|_R,$$

where $K(t)$ is defined in (3.3.2).

Proof. As a first step, we prove that (3.3.1) is verified for $\varphi\in C_b^2(\mathcal{X})$ if F satisfies Hypotheses 3.0.3.

Since G is Lipschitz continuous, it is possible to construct a sequence $\{G^{(n)}\}_{n\in\mathbb{N}}\subseteq C_b^2(\mathcal{X};\mathcal{X})$ (see [85, Lemma 2.5]) such that the functions $G^{(n)}$ are Lipschitz continuous with Lipschitz constants less or equal than L_G, and

$$\lim_{n\to+\infty}\|G^{(n)}(h)-G(h)\|=0, \qquad h\in\mathcal{X}.$$

We consider the transitions semigroup

$$P^{(n)}(t)\varphi(x):=\mathbb{E}\big[\varphi(X^{(n)}(t,x))\big], \qquad \varphi\in C_b(\mathcal{X}),$$

where $X^{(n)}(t,x)$ is the mild solution of

$$\begin{cases} dX_n(t,x)=\big(AX(t,x)+RG^{(n)}(X(t,x))\big)dt+RdW(t), & t>0;\\ X(0,x)=x. \end{cases}$$

Fix $\varphi\in C_b^2(\mathcal{X})$. Then by (3.3.1) for every $x\in\mathcal{X}$, $h\in H_R$, $T>0$ and $t\in(0,T]$, we get

$$|P^{(n)}(t)\varphi(x+h)-P^{(n)}(t)\varphi(x)|\leq K(t)\|\varphi\|_\infty\|h\|_R.$$

By [85, Theorem A.1] there exists a subsequence $\{X^{(n_k)}(t,x)\}_{k\in\mathbb{N}}$ such that, as k goes to infinity

$$X^{(n_k)}(t,x)\to X(t,x),$$

where the convergence is almost sure with respect to $\mathbb{P}$. Since φ is bounded and continuous, then

$$P^{(n_k)}(t)\varphi(x)=\mathbb{E}\Big[\varphi(X^{(n_k)}(t,x))\Big]\to\mathbb{E}\big[\varphi(X(t,x))\big]=P(t)\varphi(x).$$

So for every $x\in\mathcal{X}$, $h\in H_R$, $T>0$ and $t\in(0,T]$, we get

$$|P^{(n)}(t)\varphi(x+h)-P^{(n)}(t)\varphi(x)|\leq K(t)\|\varphi\|_\infty\|h\|_R.$$

As second step we show that since (3.3.1) is verified for $\varphi\in C_b^2(\mathcal{X})$ then it also holds for $\varphi\in B_b(\mathcal{X})$. We recall that by [103, Theorem 5.4], if $\varphi\in C_b(\mathcal{X})$ then there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}}\subseteq C_b^2(\mathcal{X})$ such that, for every $x\in\mathcal{X}$,

$$\lim_{n\to+\infty}\varphi_n(x)=\varphi(x), \qquad \|\varphi_n\|_\infty\leq\|\varphi\|_\infty, \quad \forall n\in\mathbb{N}.$$

So for every $x\in\mathcal{X}$, $h\in H_R$, $T>0$ and $t\in(0,T]$, we get

$$|P^{(n)}(t)\varphi(x+h)-P^{(n)}(t)\varphi(x)|\leq K(t)\|\varphi\|_\infty\|h\|_R.$$

Observe that by the dominated convergence theorem $P(t)\varphi_n(x+h)$ and $P(t)\varphi_n(x)$ converge to $P(t)\varphi(x+h)$ and $P(t)\varphi(x)$, respectively. Therefore (3.3.1) is verified also for $\varphi\in C_b(\mathcal{X})$.

By the Riesz representation theorem and (3.3.1), for every $x\in\mathcal{X}$, $h\in H_R$ and $t\in(0,T]$, we have the following estimate for the total variation of the finite measure $\mathscr{L}(X(x+h,t))-\mathscr{L}(X(x,t))$

$$\begin{aligned}
\mathrm{Var}\big(\mathscr{L}(X(t,x+h))-\mathscr{L}(X(t,x))\big) &:= \sup_{\substack{\varphi\in C_b(\mathcal{X})\\ \|\varphi\|_\infty\le 1}} \left|\int_{\mathcal{X}}\varphi d\Big(\mathscr{L}\big(X(t,x+h)\big)-\mathscr{L}\big(X(t,x)\big)\Big)\right| \\
&= \sup_{\substack{\varphi\in C_b(\mathcal{X})\\ \|\varphi\|_\infty\le 1}} \left|\int_{\mathcal{X}}\varphi d\mathscr{L}\big(X(t,x+h)\big)-\int_{\mathcal{X}}\varphi d\mathscr{L}\big(X(t,x)\big)\right| \\
&= \sup_{\substack{\varphi\in C_b(\mathcal{X})\\ \|\varphi\|_\infty\le 1}} \Big|\mathbb{E}\big[\varphi(X(t,x+h))\big]-\mathbb{E}\big[\varphi(X(t,x))\big]\Big| \\
&= \sup_{\substack{\varphi\in C_b(\mathcal{X})\\ \|\varphi\|_\infty\le 1}} |P(t)\varphi(x+h)-P(t)\varphi(x)| \le K(t)\|h\|_R.
\end{aligned}$$

Let $\varphi\in B_b(\mathcal{X})$. Then for $t\in(0,T]$, $x\in\mathcal{X}$ and $h\in H_R$

$$\begin{aligned}
|P(t)\varphi(x+h)-P(t)\varphi(x)| &= \left|\int_{\mathcal{X}}\varphi d\Big(\mathscr{L}\big(X(t,x+h)\big)-\mathscr{L}\big(X(t,x)\big)\Big)\right| \\
&\le \|\varphi\|_\infty K(t)\|h\|_R.
\end{aligned}$$

□

Remark 3.3.3. *We stress that, for small $t>0$*

$$K(t)\approx\frac{c}{\sqrt{t}},$$

where $c>0$ and $K(t)$ is defined in (3.3.2).

3.3.1 Strong-Feller case

Let $P(t)$ the transition semigroup associated to stochastic equation

$$\begin{cases} dX(t,x)=\big(AX(t,x)+F(X(t,x))\big)dt+RdW(t), & t>0; \\ X(0,x)=x\in\mathcal{X}, \end{cases} \tag{3.3.4}$$

where A and R verify Hypotheses 3.0.1 and $F:\mathcal{X}\to\mathcal{X}$ is a Lipschitz continuous function with Lipschitz constant L_F.

In this subsection we replace Hypotheses 3.0.3 by the following condition.

Hypotheses 3.3.4. *There exist $\epsilon\in\mathbb{R}$, $K_\epsilon>0$ and $\gamma\in(0,1/2)$ such that*

$$e^{tA}(\mathcal{X})\subseteq R(\mathcal{X}),\qquad \|R^{-1}e^{tA}\|_{\mathcal{L}(\mathcal{X})}\le K_\epsilon e^{\epsilon t}t^{-\gamma},\quad t>0.$$

We remark that Hypotheses 3.3.4 implies (3.0.1) (see [43, Corollary 9.30]). By Hypotheses 3.3.4 it is possible to prove that for any $t\ge 0$ we have

$$P(t)\left(B_b(\mathcal{X})\right)\subseteq Lip_b(\mathcal{X}). \tag{3.3.5}$$

As in Section 3.2 we assume that $F \in C_b^2(\mathcal{X})$. By Theorem 2.3.2 and Proposition 2.3.3, for any $T > 0$, the map $x \mapsto \{X(t,x)\}_{t\in[0,T]}$ is Gateaux differentiable as a map from $\mathcal{X}$ to $\mathcal{X}^2([0,T])$ and, for any $x, y \in \mathcal{X}$, it holds

$$\|\mathcal{D}_G X(t,x)y\| \leq e^{\zeta_{\mathcal{X}} t}\|y\|, \qquad t \geq 0,$$

where $\zeta_{\mathcal{X}} = w_{\mathcal{X}} + L_F$. Let $T > 0$, $t \in [0,T]$, $x, h \in \mathcal{X}$, we have

$$\begin{aligned}\left\|R^{-1}\mathcal{D}_G X(t,x)h\right\| &\leq \|R^{-1}e^{tA}h\| + \int_0^t \|R^{-1}e^{(t-s)A}\mathcal{D}F(X(s,x))\mathcal{D}_G X(s,x)h\|ds \\ &\leq K_\epsilon e^{\epsilon t}t^{-\gamma}\|h\| + K_\epsilon L_F\|R\|_{\mathcal{L}(\mathcal{X})}\|h\|_R \int_0^t \frac{e^{(\zeta_{\mathcal{X}}+\epsilon)s}}{(t-s)^\gamma}ds \\ &= K_\epsilon e^{\epsilon t}t^{-\gamma}\|h\| + K_\epsilon L_F\|R\|_{\mathcal{L}(\mathcal{X})}\|h\|_R \int_0^t \frac{e^{(\zeta_{\mathcal{X}}+\epsilon)(t-s)}}{s^\gamma}ds. \end{aligned} \tag{3.3.6}$$

Let $0 < t_0 < \min(1,t)$, by (3.3.6), we have

$$\begin{aligned}\|R^{-1}\mathcal{D}_G X(t,x)h\|_R &\leq K_\epsilon e^{\epsilon t}t^{-\gamma}\|h\| + K_\epsilon L_F\|R\|_{\mathcal{L}(\mathcal{X})}\|h\|_R e^{(\zeta_{\mathcal{X}}+\epsilon)t}\left(\int_0^{t_0}\frac{1}{s^\gamma}ds + \int_{t_0}^t e^{-(\zeta_{\mathcal{X}}+\epsilon)s}ds\right) \\ &\leq \left(K_\epsilon e^{\epsilon t}t^{-\gamma} + K_\epsilon L_F\|R\|_{\mathcal{L}(\mathcal{X})}\left(\frac{e^{(\zeta_{\mathcal{X}}+\epsilon)t}}{1-\gamma} + \frac{e^{(\zeta_{\mathcal{X}}+\epsilon)(t-t_0)}-1}{\zeta_{\mathcal{X}}+\epsilon}\right)\right)\|h\|,\end{aligned}$$

hence, for any $T > 0$ there exists $K : (0,T] \to (0,+\infty)$ such that, for any $t \in (0,T]$ and $x, h \in \mathcal{X}$

$$\|R^{-1}\mathcal{D}_G X(t,x)h\|_R \leq K(t)\|h\|. \tag{3.3.7}$$

Moreover for small $t > 0$

$$K(t) \approx ct^{-\gamma},$$

for some $c > 0$.

Due to (3.3.7), in the proof of Proposition 3.2.1 we can multiply (3.2.3) by

$$\left\{\int_0^t \left\langle R^{-1}\mathcal{D}^G X(s,x)h, dW(s)\right\rangle\right\}_{t\geq 0}$$

instead of (3.2.5), and so we obtain that for every $\varphi \in C_b^2(\mathcal{X})$, $x, h \in \mathcal{X}$ and $t \geq 0$

$$\langle \nabla P(t)\varphi(x), h\rangle = \frac{1}{t}\mathbb{E}\left[\varphi(X(t,x))\int_0^t \left\langle R^{-1}\mathcal{D}^G X(s,x)h, dW(s)\right\rangle\right]. \tag{3.3.8}$$

Hence using (3.3.7), (3.3.8) and the same arguments used in [43, Section 9.4.2], we conclude that (3.3.5) holds.

3.4 Remarks and examples

This chapter is a reworked version of [10]. In such a paper we have considered a special case in which A and R were powers of the same trace class and positive operator, and the perturbation

RG was replaced by a more general perturbation F. In particular we assumed that F was just H_R-Lipschitz and not Lipschitz continuous.

Moreover we suggest that it is possible to generalize Theorem 3.3.2 even in another way. We consider the SPDE

$$\begin{cases} dX(t,x) = \big(AX(t,x) + F(X(t,x))\big)dt + RdW(t), & t > 0; \\ X(0,x) = x \in \mathcal{X}. \end{cases}$$

where A and R verify Hypotheses 3.0.3. Instead F verifies Hypotheses 2.1.1, $F(H_R) \subset H_R \subset E$ (where E is the Banach space of Hypotheses 2.1.1) and there exists ζ_F such that

$$\langle F(x) - F(y), x - y\rangle_R \leq \zeta_F \|x - y\|_R^2, \quad x, y \in H_R.$$

By approximating F with the approximating sequence defined in Subsection 1.11 and proving that [85, Lemma 2.3] is also verified for these approximations we think that it is possible to obtain a result analogous to Theorem 3.3.2.

In [20, Chapters 6-7] the author studies the strong Feller properties in the case where the perturbation F is not defined on the whole space $\mathcal{X}$. However in this work more restrictive hypotheses on A and R (see [20, Hypotheses 6.1]) than those in [14] (see (3.0.1)) are assumed. In particular the author assumes that A is a non positive operator, R is a positive operator and that there exists $\epsilon < 1$ such that

$$\mathrm{Dom}(A^{\epsilon}) \subseteq R(\mathcal{X}).$$

In [57, 58, 77, 78] the authors work in a separable Banach space with a Schauder basis, and they study a problem similar to ours. They define the following differential operator.

Definition 3.4.1. *Let $f : \mathcal{X} \to \mathbb{R}$ be a continuous function, the R-directional derivative $\nabla^R f(x; y)$ at a point $x \in \mathcal{X}$ in the direction $y \in \mathcal{X}$ is defined as:*

$$\nabla^R f(x; y) := \lim_{s \to 0} \frac{f(x + sRy) - f(x)}{s},$$

provided that the limit exists and the map $y \mapsto \nabla^R f(x; y)$ belongs to $\mathcal{X}^*$.

The authors of [57, 58, 77, 78], using the Girsanov theorem, prove that, for every $\varphi \in B_b(\mathcal{X})$, the function $P(t)\varphi$ admits R-directional derivatives in along every $y \in \mathcal{X}$.In this chapter we obtain a H_R Lipschitzianity result, instead, in [57, 58, 77, 78], the authors cannot achieve a similar result, with Definition 3.4.1. As we already said in the introduction Hypotheses 3.0.3 cover the case where $A = -\frac{1}{2}\mathrm{I}_{\mathcal{X}}$ and $R = Q^{1/2}$ where Q is a positive and trace class operator. In Subsection 3.4.1 we will show an interesting example that is included the case just mentioned. Instead in Subsection 3.4.2 we will present a suitable choice of A and R which simplifies the checking of the Hypotheses 3.0.3.

3.4.1 A classical example

Consider the space $\mathcal{X} = L^2([0,1], d\xi)$ where $d\xi$ denotes the Lebesgue measure on $[0,1]$ and let $Q : L^2([0,1], d\xi) \to L^2([0,1], d\xi)$ be covariance operator of the Wiener measure on $L^2([0,1], d\xi)$, namely the positive and self-adjoint operator defined as

$$Qf(\xi)=\int_0^1 \max\{\xi,\eta\}f(\eta)d\eta.$$

If $R=Q^{1/2}$, it is known that H_R is the space $W_0^{1,2}([0,1],d\xi)$. Moreover the norm $\|\cdot\|_R$ is equivalent to the norm

$$\|f\|_{W_0^{1,2}([0,1],d\xi)}:=\|f'\|_{L^2([0,1],d\xi)}.$$

For all these results see [12, Remark 2.3.13 and Lemma 2.3.14]. Let $G:L^2([0,1],d\xi)\to L^2([0,1],d\xi)$ defined by choosing $x_1,\dots,x_n\in L^2([0,1],d\xi)$ and a function $f:[0,1]\times\mathbb{R}^n\to\mathbb{R}$, $(\xi,y_1,\dots,y_n)\mapsto f(\xi,y_1,\dots,y_n)$ and setting

$$(G(g))(\xi):=f\Big(\xi,\int_0^1 g(\eta)x_1(\eta)d\eta,\dots,\int_0^1 g(\eta)x_n(\eta)d\eta\Big).$$

Assume that $x_1,\dots,x_n$ are orthonormal and for every $i=1,\dots,n$

$$\begin{aligned}&f,\frac{\partial f}{\partial\xi},\frac{\partial f}{\partial y_i}\ \text{are bounded and continuous on } [0,1]\times\mathbb{R}^n;\\&f(0,y_1,\dots,y_n)=0,\ \text{for every } y_1,\dots,y_n\in\mathbb{R}.\end{aligned}$$

Then $G(L^2([0,1],d\xi))\subseteq W_0^{1,2}([0,1],d\xi)$, since

$$(G(g))(0)=f\Big(0,\int_0^1 g(\eta)x_1(\eta)d\eta,\dots,\int_0^1 g(\eta)x_n(\eta)d\eta\Big)=0$$

and

$$(G(g))'(\xi)=\frac{\partial f}{\partial\xi}\Big(\xi,\int_0^1 g(\eta)x_1(\eta)d\eta,\dots,\int_0^1 g(\eta)x_n(\eta)d\eta\Big)\le\Big\|\frac{\partial f}{\partial\xi}\Big\|_\infty.$$

Moreover for $g_1,g_2\in L^2([0,1],d\xi)$

$$\begin{aligned}&\|G(g_1)-G(g_2)\|^2_{L^2([0,1],d\xi)}\\&=\int_0^1\Big|f\Big(\xi,\int_0^1 g_1x_1d\eta,\dots,\int_0^1 g_1x_nd\eta\Big)-f\Big(\xi,\int_0^1 g_2x_1d\eta,\dots,\int_0^1 g_2x_nd\eta\Big)\Big|\\&\le n^2\sup_{i=1,\dots,n}\Big\{\Big\|\frac{\partial f}{\partial\xi}\Big\|^2_\infty,\Big\|\frac{\partial f}{\partial y_i}\Big\|^2_\infty\Big\}\sum_{i=1}^n\Big|\int_0^1(g_1-g_2)x_id\eta\Big|^2\\&\le n^2\sup_{i=1,\dots,n}\Big\{\Big\|\frac{\partial f}{\partial\xi}\Big\|^2_\infty,\Big\|\frac{\partial f}{\partial y_i}\Big\|^2_\infty\Big\}\|g_1-g_2\|^2_{L^2([0,1],d\xi)}.\end{aligned}$$

Let $P(t)$ be the transition semigroup associated to stochastic differential equation

$$\begin{cases}dX(t,x)=\big(-\frac{1}{2}X(t,x)+F(X(t,x))\big)dt+Q^{1/2}dW(t), & t\in(0,T];\\ X(0,x)=x\in\mathcal{X},\end{cases}$$

by Theorem 3.3.2, for every $t > 0$, $x \in L^2([0,1], d\xi)$, $h \in W_0^{1,2}([0,1], d\xi)$ and $\varphi \in B_b(\mathcal{X})$ we have

$$|P(t)\varphi(x+h) - P(t)\varphi(x)| \leq K(t)\|\varphi\|_\infty \|h\|_{W_0^{1,2}([0,1],d\xi)},$$

where $K(t)$ is defined in (3.3.2).

We emphasize that we have assumed as Q the covariance operator of the Wiener measure on $L^2([0,1], d\xi)$, but we could consider any Q such that $Q(L^2([0,1], d\xi)) \subseteq W_0^{1,2}([0,1], d\xi)$, where $W_0^{1,2}([0,1], d\xi)$ is the set of the real-valued functions f defined on $[0,1]$ such that f is absolutely continuous, $f' \in L^2([0,1], d\xi)$ and $f(0) = 0$.

3.4.2 A suitable choice of A and R

Let Q be a positive and compact operator. For $\alpha, \beta \geq 0$ we set $A = -(1/2)Q^{-\beta} : Q^\beta(\mathcal{X}) \subseteq \mathcal{X} \to \mathcal{X}$ and $R = Q^\alpha$. Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{X}$ consisting of eigenvectors of Q, and let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the eigenvalues associated with $\{e_k\}_{k\in\mathbb{N}}$. Since Q is a compact and positive operator, there exists $k_0 \in \mathbb{N}$ such that $0 < \lambda_k \leq \lambda_{k_0}$, for any $k \in \mathbb{N}$. Without loss of generality we assume $k_0 = 1$. Hence, for any $x \in Q^\beta(\mathcal{X})$, we have

$$\langle Ax, x\rangle = \sum_{k=1}^{+\infty} -\frac{1}{2}\lambda_k^{-\beta}\langle x, e_k\rangle^2 \leq -\frac{1}{2}\lambda_1^{-\beta}\|x\|^2. \tag{3.4.1}$$

Moreover, since Q is a compact and positive operator, $\mathrm{Dom}(A) = Q^\beta(\mathcal{X})$ is dense in $\mathcal{X}$, so A generates a strongly continuous contraction semigroup in $\mathcal{X}$. Let A_α be the part of A in $H_\alpha := H_{Q^\alpha}$, we recall that

$$\mathrm{Dom}(A_\alpha) := \{x \in Q^\alpha(\mathcal{X}) \cap Q^\beta(\mathcal{X}) \ : \ Ax \in Q^\alpha(\mathcal{X})\}.$$

By (3.4.1), for any $x \in \mathrm{Dom}(A_\alpha)$, we have

$$\langle Ax, x\rangle_\alpha = \langle Q^{-\alpha}Ax, Q^{-\alpha}x\rangle = \langle AQ^{-\alpha}x, Q^{-\alpha}x\rangle \leq -\lambda_1^{-\beta}\|Q^{-\alpha}x\|^2 = -\frac{1}{2}\lambda_1^{-\beta}\|x\|_\alpha^2. \tag{3.4.2}$$

Since $Q^{\alpha+\beta}(\mathcal{X})$ is dense in $\mathcal{X}$ and $Q^{-\alpha}$ is a closed operator in $\mathcal{X}$, then $Q^{\alpha+\beta}(\mathcal{X})$ is dense in H_α, moreover $Q^{\alpha+\beta}(\mathcal{X}) = \mathrm{Dom}(A_\alpha)$. Hence A generates a strongly continuous and contraction semigroup in H_α. We refer to [43, Section 5.4-5.5] for a study of Hypothesis 3.0.1(iv).

In this setting, we stress that by [73, Proposition 2.1.1], the Hypotheses 3.3.4 are verified when $\beta > 2\alpha$.

Chapter 4

Logarithmic Harnack inequalities for transition semigroups

The first formulation of the Harnack inequality dates back to 1887 and can be found in his seminal paper [66], and concerns positive harmonic functions. After some partial extensions, the most important contribution is due to J. Moser [79] which proved the Harnack inequality for positive (weak) solutions of uniformly elliptic linear equations with bounded coefficients in variational form. Moser also stresses the usefulness of such kind of estimates to deduce regularity results, such as the local hölderianity of the solutions. The further passage towards non-linear elliptic equations was made first by J. Serrin [94] and then by N.S. Trudinger [97] a few years later, and is based on Moser's approach.

The first parabolic version of the Harnack inequality is proved separately from J. Hadamard [65] and B. Pini [87] for positive solutions of the heat equation. Many years later this kind of estimates have been extended to positive solutions of more general linear parabolic equations by Moser himself [80]. Hence the extension to almost linear parabolic equations was due to D. G. Aronson and J. Serrin [7] and N.S. Trudinger [97]. Differently from the elliptic case, however, the case of operators with non-linear coefficients turned out to be more difficult and remained unresolved for a long time. In this direction we refer to [45, 46] where an *intrinsic* Harnack type inequality was proved for solutions of a large class of nonlinear equations and for operators with nonlinear coefficients. The techniques used in these latter results were inspired by the method of E. De Giorgi and J. Nash (see [44, 83]) to show boundedness and regularity for certain classes of functions (the so-called De Giorgi classes), which contain in particular the solutions of some elliptic equations.

We refer to [69] and the reference therein for a more in-depth analysis of the Harnack inequality. In all the quoted results, the formulation of the Harnack inequality allows to compare the values of a positive solution of some elliptic or parabolic differential equation, at two different points. All these Harnack inequalities are dimension-dependent and thus they cannot pass to infinite dimension. A possibility to get the Harnack-type inequality in an infinite dimensional setting consists in replacing the classical formulation to the dimension-free logarithmic Harnack Inequality (LHI) firtst introduced by F.-Y. Wang in [99] for the study of diffusion semigroups on a Riemannian manifold M. It reads as

$$(P(t)f)^{\alpha}(x) \leq (P(t)f^{\alpha})(x)e^{c(t)\rho(x,y)}, \qquad t>0,\ x,y\in M \tag{4.0.1}$$

which holds true for any positive and Borel bounded function f, any $\alpha > 1$ and some continuous function $c(t)$. Here ρ is a Riemannian metric on M. Also in the infinite dimensional setting, this kind of inequality has been used to obtain a lot of results, like some regularizing effects of the semigroup (see, for example, [38, Proposition 4.1], [92, Corollary 1.2] and [101, Corollary 7.3.14]) as well as some hyperboundedness properties for the semigroup $\{P(t)\}_{t\geq 0}$ (see, for example, [92] and [100]). We refer to [101] and the reference therein for a discussion of this inequality and its consequences.

Let $\mathcal{X}$ be a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let $\{W(t)\}_{t\geq 0}$ be a $\mathcal{X}$-cylindrical Wiener process defined on a normal filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Let $R\in\mathcal{L}(\mathcal{X})$, let $A:\mathrm{Dom}(A)\subseteq\mathcal{X}\to\mathcal{X}$ be the infinitesimal generator of a strongly continuous semigroup e^{tA} on $\mathcal{X}$. Let $F:\mathrm{Dom}(F)\subseteq\mathcal{X}\to\mathcal{X}$ (possibly non linear). We introduce the SPDE

$$\begin{cases} dX(t,x) = \big(AX(t,x)+F(X(t,x))\big)dt + RdW(t), & t>0; \\ X(0,x) = x\in\mathcal{X}. \end{cases} \tag{4.0.2}$$

Under suitable hypotheses (4.0.2) has unique generalize mild solution, we denote by $P(t)$ the transition semigroup associated to 4.0.2 (see Chapter 2). The aim of this chapter consists in proving a (LHI) similar to (4.0.1), for $P(t)$. To be more precise, in the same spirit as Chapter 3, we will prove a (LHI) type inequality along H_R (see Definition), namely

$$|P(t)\varphi(x+h)|^p \leq P(t)|\varphi(x)|^p e^{c(t)\|R^{-1}h\|^2}, \qquad t>0,\ x\in\mathcal{X},\ h\in R(\mathcal{X}); \tag{4.0.3}$$

for any bounded and Borel measurable function $\varphi:\mathcal{X}\to\mathbb{R}$, any $p>1$ and some continuous function $c:(0,+\infty)\to\mathbb{R}$. In Section 4.1 we will prove (4.0.3) in the case where the perturbation F of SPDE (2.0.1) is Lipschitz continuous. In Section 4.2 we will prove (4.0.3) in the case where $F:\mathrm{Dom}(F)\subseteq\mathcal{X}\to\mathcal{X}$ satisfies a dissipativity hypothesis in H_R. The key tool we use to prove the (LHI) (4.1.16) and (4.2.9) in both cases is an approximation method. In the Lipschitz continuous case the approximants which allow us to get our estimate are suitable finite dimensional semigroups which satisfy suitable gradient estimates (see Subsection 4.1.6). On the other hand, the dissipative case is solved by using a double approximation procedure which consider a finite dimensional approximation of the Yosida approximants (see Subsection 4.2.1).

Finally in Section 4.3 we will comment the results of this chapter in view of results already known in the literature. Moreover we will present some consequences of (4.0.3) and some examples of semigroups which satisfy (4.0.3).

4.1 Logarithmic Harnack inequality: the Lipschitz continuous case

In this section we assume the following hypotheses.

Hypotheses 4.1.1.

(i) $R \in \mathcal{L}(\mathcal{X})$ *is non-negative.*

(ii) $A : \mathrm{Dom}(A) \subseteq \mathcal{X} \to \mathcal{X}$ *generates a strongly continuous semigroup* e^{tA} *on* $\mathcal{X}$ *and there exists* $w_{\mathcal{X}} \in \mathbb{R}$ *such that* $A - w_{\mathcal{X}} \mathrm{I}_{\mathcal{X}}$ *is dissipative.*

(iii) *There exists* $\eta \in (0,1)$*, such that for any* $t > 0$ *we have*

$$\int_0^t \frac{1}{s^\eta} \mathrm{Tr}[e^{2sA}R^2]ds < +\infty. \tag{4.1.1}$$

(iv) $F : \mathcal{X} \to \mathcal{X}$ *is a Fréchet differentiable and Lipschitz continuous function with Lipschitz constant* L_F*.*

By Hypotheses 4.1.1(ii-iv), for any for any $x, h \in \mathcal{X}$

$$\langle [A + \mathcal{D}F(x)]h, h\rangle \leq \zeta_{\mathcal{X}} \|h\|^2, \quad \zeta_{\mathcal{X}} := w_{\mathcal{X}} + L_F. \tag{4.1.2}$$

By Hypotheses 4.1.1, Theorem 2.1.13 (with $\mathcal{X} = E$) and Proposition 2.2.4, the stochastic partial differential equation (4.0.2) admits a unique mild solution $\{X(t,x)\}_{t\geq 0}$ and the family of operators $\{P(t)\}_{t\geq 0}$ defined as

$$P(t)\varphi(x) := \mathbb{E}[\varphi(X(t,x))], \qquad x \in \mathcal{X},\ t \geq 0,\ \varphi \in B_b(\mathcal{X}),$$

is a semigroup. Moreover by Theorem 2.3.2 and Proposition 2.3.3, for any $T > 0$, the map $x \mapsto \{X(t,x)\}_{t\in[0,T]}$ is Gateaux differentiable as a map from $\mathcal{X}$ to $\mathcal{X}^2([0,T])$ and, for any $x, y \in \mathcal{X}$, its Gateaux derivative is the unique mild solution of

$$\begin{cases} dY_x(t,y) = [A + \mathcal{D}F(X(t,x))]Y_x(t,y)dt, & t > 0; \\ Y_x(0,y) = y, \end{cases} \tag{4.1.3}$$

namely, for every $x, y \in \mathcal{X}$ the process $\{\mathcal{D}_G X(t,x)y\}_{t\geq 0}$ satisfies the mild form of (4.1.3)

$$\mathcal{D}_G X(t,x)y = e^{tA}y + \int_0^t e^{(t-s)A}\mathcal{D}F(X(s,x))\mathcal{D}_G X(s,x)yds, \qquad t \geq 0. \tag{4.1.4}$$

Furthermore, by Proposition 2.3.3, for every $x, y \in \mathcal{X}$ it holds

$$\|\mathcal{D}_G X(t,x)y\| \leq e^{\zeta_{\mathcal{X}} t}\|y\|, \qquad t \geq 0. \tag{4.1.5}$$

4.1.1 H_R-regularity

To study the H_R-regularity of the mild solution, we need some additional hypotheses.

Hypotheses 4.1.2. *Assume that Hypotheses 4.1.1 hold true. Moreover* A_R *(the part of* A *in* H_R*) generates a strongly continuous semigroup* e^{tA_R} *in* H_R *and one of the following conditions hold true:*

1. *There exist $\epsilon \in \mathbb{R}$, $K_\epsilon > 0$ and $\gamma \in (0,1)$ such that*

$$e^{tA}(\mathcal{X}) \subseteq R(\mathcal{X}), \qquad \|R^{-1}e^{tA}\|_{\mathcal{L}(\mathcal{X})} \leq K_\epsilon e^{\epsilon t} t^{-\gamma}.$$

2. *There exists $w_R \in \mathbb{R}$ such that $A_R - w_R \mathrm{I}_{H_R}$ is dissipative in H_R, moreover $F = RG$, where $G : \mathcal{X} \to \mathbb{R}$ is a Fréchet differentiable and Lipschitz continuous function with Lipschitz constant L_G.*

Since A_R generates a strongly continuous semigroup e^{tA_R} in H_R, then there exists $w_0 \in \mathbb{R}$ and $K_0 > 0$ such that

$$\|e^{tA_R}\|_{\mathcal{L}(H_R)} \leq K_0 e^{w_0 t}. \tag{4.1.6}$$

The following proposition gives us information about the H_R differentiability of the mild solution.

Proposition 4.1.3. *Assume that Hypotheses 4.1.2 hold true. For any $T > 0$ there exists a constant $C_T := C_T(A, R, F) > 0$ such that for every $t \in [0, T]$, $x \in \mathcal{X}$ and $h \in H_R$*

$$\|\mathcal{D}_G X(t, x)h\|_R \leq C_T \|h\|_R, \quad \mathbb{P}\text{-}a.s. \tag{4.1.7}$$

Moreover for every $t > 0$, the map $x \mapsto X(t, x)$ is $\mathbb{P}$-a.s.. H_R-Gateaux differentiable and for any $x \in \mathcal{X}$ and $h \in H_R$ its H_R-Gateaux derivative along $h \in H_R$ is $\mathcal{D}^G X(t, x)h$.

Proof. We already know that the statements hold under Hypotheses 4.1.2(1), by Proposition 3.1.4. So it is sufficient consider the case where Hypotheses 4.1.2(2) hold. All the calculations in this proof will hold $\mathbb{P}$-a.e. Let $T > 0$, $t \in [0, T]$, $x \in \mathcal{X}$ and $h \in H_R$. By (4.1.4) we have

$$\|\mathcal{D}_G X(t, x)h\|_R \leq \|e^{tA_R} h\|_R + \int_0^t \|R^{-1} e^{(t-s)A} \mathcal{D}F(X(s, x))\mathcal{D}_G X(s, x)h\| ds$$

By the Lipschitz continuity of F, Hypotheses 4.1.2(1), (4.1.5) and (4.1.6) we have

$$\begin{aligned}\|\mathcal{D}_G X(t, x)h\|_R &\leq K_0 e^{w_0 t}\|h\|_R + K_\epsilon L_F \|R\|_{\mathcal{L}(\mathcal{X})}\|h\|_R \int_0^t \frac{e^{(\zeta_{\mathcal{X}}+\epsilon)s}}{(t-s)^\gamma} ds \\ &= K_0 e^{w_0 t}\|h\|_R + K_\epsilon L_F \|R\|_{\mathcal{L}(\mathcal{X})}\|h\|_R \int_0^t \frac{e^{(\zeta_{\mathcal{X}}+\epsilon)(t-s)}}{s^\gamma} ds.\end{aligned} \tag{4.1.8}$$

Let $0 < t_0 < \min(1, t)$, by (4.1.8), we have

$$\begin{aligned}\|\mathcal{D}_G X(t, x)h\|_R &\leq K_0 e^{w_0 t}\|h\|_R + K_\epsilon L_F \|R\|_{\mathcal{L}(\mathcal{X})}\|h\|_R e^{(\zeta_{\mathcal{X}}+\epsilon)t}\left(\int_0^{t_0} \frac{1}{s^\gamma} ds + \int_{t_0}^t e^{-(\zeta_{\mathcal{X}}+\epsilon)s} ds\right) \\ &\leq \left[K_0 e^{w_0 t} + K_\epsilon L_F \|R\|_{\mathcal{L}(\mathcal{X})}\left(\frac{e^{(\zeta_{\mathcal{X}}+\epsilon)t}}{1-\gamma} + \frac{e^{(\zeta_{\mathcal{X}}+\epsilon)(t-t_0)} - 1}{\zeta_{\mathcal{X}}+\epsilon}\right)\right]\|h\|_R\end{aligned}$$

So (4.1.7) is verified with

$$C_T := \sup_{t\in[0,T]} \left\{ K_0 e^{w_R t} + K_\epsilon L_F \|R\|_{\mathcal{L}(\mathcal{X})} \left(\frac{t_0 e^{(\zeta_{\mathcal{X}}+\epsilon)t}}{1-\gamma} + \frac{e^{(\zeta_{\mathcal{X}}+\epsilon)(t-t_0)}-1}{\zeta_{\mathcal{X}}+\epsilon} \right) \right\}.$$

Now we prove the moreover part of the proposition. First of all we prove that for any fixed $t > 0$, $x \in \mathcal{X}$ and $h \in H_C$, the function $\varphi_{x,h} : \mathbb{R} \to \mathcal{X}$ defined as

$$\varphi_{x,h}(r) := X(t, x + rh) - X(t, x),$$

is H_C-valued. Some standard calculations give

$$\varphi_{x,h}(r) = re^{tA}h + \int_0^t e^{(t-s)A}\big[F(X(s, x + rh)) - F(X(s, x))\big]ds, \qquad r > 0.$$

for every $r \in [0, t]$, it holds

$$\begin{aligned}
\|\varphi_{x,h}(r)\|_R &= \left\| re^{tA_R}h + \int_0^t e^{(t-s)A}\big[F(X(s, x + rh)) - F(X(s, x))\big]ds \right\|_R \\
&\le \|re^{tA_R}h\|_R + \left\| \int_0^t e^{(t-s)A}\big[F(X(s, x + rh)) - F(X(s, x))\big]ds \right\|_R \\
&\le rK_0 e^{w_0 t}\|h\|_R + \int_0^t \left\|e^{(t-s)A}\big[F(X(s, x + rh)) - F(X(s, x))\big]\right\|_R ds \\
&\le rK_0 e^{w_0 t}\|h\|_R + \int_0^t \left\|R^{-1}e^{(t-s)A}\big[F(X(s, x + rh)) - F(X(s, x))\big]\right\| ds \\
&\le rK_0 e^{w_0 t}\|h\|_R + \int_0^t \|R^{-1}e^{(t-s)A}\|_{\mathcal{L}(\mathcal{X})}\|F(X(s, x + rh)) - F(X(s, x))\| ds,
\end{aligned}$$

By (2.1.23) and Hypotheses 4.1.2(ii-1) we have

$$\begin{aligned}
\|\varphi_{x,h}(r)\|_R &\le rK_0 e^{w_0 t}\|h\|_R + r\|h\| \int_0^t K_\epsilon e^{\epsilon(t-s)}(t-s)^{-\gamma} ds \\
&\le rK_0 e^{w_0 t}\|h\|_R + rK_\epsilon\|h\| \left(\frac{t^{1-\gamma}}{1-\gamma} \right) \sup_{s\in[0,t]} (e^{s\epsilon}).
\end{aligned}$$

So $\varphi_{x,h}$ is $R(\mathcal{X})$-valued.

Since $\mathcal{D}X(t, x) \in \mathcal{L}(H_R)$ for every $x \in \mathcal{X}$ and $t > 0$, to conclude the proof we just need to check that (3.1.4) holds. Let $x \in \mathcal{X}$, $h \in R(\mathcal{X})$ and $t, r > 0$. Then

$$\begin{aligned}
&\mathbb{E}\left[\left\| \frac{1}{r}\varphi_{x,h}(r) - \mathcal{D}_G X(t, x)h \right\|_R \right] \\
&= \mathbb{E}\left[\left\| \int_0^t R^{-1}e^{(t-s)A} \left(\frac{F(X(s, x + rh)) - F(X(s, x))}{r} - \mathcal{D}F(X(s, x))\mathcal{D}_G X(s, x)h \right) ds \right\| \right].
\end{aligned}$$

By Hypotheses 4.1.2(1) we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left\|\frac{1}{r}\varphi_{x,h}(r)-\mathcal{D}_G X(t,x)h\right\|_R\right]\\
&\leq K_\epsilon\int_0^t \frac{e^{\epsilon(t-s)}}{(t-s)^\gamma}\mathbb{E}\left[\left\|\frac{F(X(s,x+rh))-F(X(s,x))}{r}-\mathcal{D}F(X(s,x))\mathcal{D}_G X(s,x)h\right\|\right]ds.
\end{aligned}$$

Recalling that F is Fréchet differentiable and Lipschitz continuous, recalling (2.1.23) and $\gamma\in(0,1)$, we apply the Dominated Convergence theorem and we get

$$\lim_{r\to 0}\mathbb{E}\left[\left\|\frac{1}{r}\varphi_{x,h}(r)-\mathcal{D}_G X(t,x)h\right\|_R\right]=0,$$

which concludes the proof. □

Remark 4.1.4. *If we assume that there exists $\zeta<0$ verifying* (4.1.2) *then the computations of Proposition 4.1.3 yield that there exists $C>0$ such that $C_T\leq C$, for any $T>0$.*

4.1.2 Finite dimensional approximating

A key tool to prove the (LHI) stated in Theorem 4.1.9 is a finite-dimensional approximation procedure which allows us to approximate the transition semigroup $\{P(t)\}_{t\geq 0}$ by means of a sequence of transition semigroups $\{P^n(t)\}_{t\geq 0}$ associated to suitable finite-dimensional stochastic differential equations. The idea of such approximation comes from [38]. Here, for the sake of completeness and to point out the minimal assumptions needed for such kind of procedure, we recall it and we provide a proof of the main approximation result (Proposition 4.1.6).

We need one more assumption.

Hypotheses 4.1.5. *Assume that H_R is dense in $\mathcal{X}$ and that there exists a sequence of A-invariant and R-invariant finite dimensional subspaces $\mathcal{X}_n\subseteq \mathrm{Dom}(A_R)$ such that $\bigcup_{n=1}^\infty \mathcal{X}_n$ is dense in H_R.*

Hypotheses 4.1.5 hold true for instance, if A is a self-adjoint positive operator and R admits a continuous inverse $R^{-1}\in\mathcal{L}(X)$ or if A and R are simultaneously diagonalizable.

In view of Hypothesis 4.1.5 we can consider $\{e_k\}_{k\in\mathbb{N}}\in\mathrm{Dom}(A_R)$ such that for any $n\in\mathbb{N}$

$$\mathcal{X}_n=\mathrm{span}\{e_1,\dots,e_n\},$$

and the family $\{e_k\,|\,k\in\mathbb{N}\}$ is an orthonormal basis of H_R. Further, let $\pi_n:\mathcal{X}\to\mathcal{X}_n$ be the orthogonal projection with respect to $(\mathcal{X},\langle\cdot,\cdot\rangle)$, for any $n\in\mathbb{N}$ we define $A_n:\mathcal{X}\to\mathcal{X}_n$, $R_n:\mathcal{X}\to\mathcal{X}_n$ and $F_n:\mathcal{X}\to\mathcal{X}_n$ by

$$A_n:=\pi_nA\pi_n(=A\pi_n),\qquad R_n:=\pi_nR\pi_n(=R\pi_n)\qquad\text{and}\qquad F_n:=\pi_nF\pi_n$$

Now, fix $n\in\mathbb{N}$ and consider

$$\begin{cases}dX_n(t)=[A_nX_n(t)+F_n(X_n(t))]dt+R_ndW_n(t), & t>0,\\ X_n(0)=x\in\mathcal{X}_n.\end{cases}\tag{4.1.9}$$

Here $W_n(t) := \pi_n W(t) = \sum_{k=1}^n \langle W(t), e_k\rangle e_k$.

It is straightforward to see that A_n, R_n and F_n satisfy Hypotheses 4.1.1. Moreover, being R an injective operator it follows that R_n is bijective hence $R_n\mathcal{X}_n = \mathcal{X}_n$ for any $n \in \mathbb{N}$. Therefore, fixed $x \in \mathcal{X}_n$, by Theorem 2.1.13 we can deduce existence and uniqueness of a mild solution $\{X_n(t,x)\}_{t\geq 0}$ of (4.1.9) and consequently well-posedness for the associated transition semigroup defined for $f \in B_b(\mathcal{X}_n)$ as

$$P_n(t)f(x) := \mathbb{E}[f(X_n(t,x))], \qquad t > 0,\ x \in \mathcal{X}_n. \tag{4.1.10}$$

We recall that for any $n \in \mathbb{N}$ and $\varphi \in C_b^2(\mathcal{X}_n)$ we have

$$\lim_{t\to 0} \frac{P(t)\varphi(x) - \varphi(x)}{t} = \mathcal{N}_n\varphi(x), \quad x \in \mathcal{X}_n,$$

where

$$\mathcal{N}_n\varphi(x) = \frac{1}{2}\mathrm{Tr}[R_n^2 D^2\varphi(x)] + \langle A_n x + F_n(x), D\varphi(x)\rangle. \tag{4.1.11}$$

Now we are able to state the main finite-dimensional approximation result.

Proposition 4.1.6. *Assume that Hypotheses 4.1.2 and 4.1.5 hold true. For any $f \in C_b(\mathcal{X})$, $t \geq 0$ and $x \in \mathcal{X}_{n_0}$, for some $n_0 \in \mathbb{N}$, it holds*

$$\lim_{n\to+\infty} P_n(t)f(x) = P(t)f(x).$$

Proof. Let $\{X(t,x)\}_{t\geq 0}$ be the unique mild solution of (4.0.2). For any $t \geq 0$, we set $Z(t) := X(t) - W_A(t)$, where $\{W_A(t)\}_{t\geq 0}$ is the stochastic convolution process. For any fixed $x \in \mathcal{X}_{n_0}$ the process $\{Z(t)\}_{t\geq 0}$ is the unique mild solution of

$$\begin{cases} dZ(t) = (AZ(t) + F(Z(t) + W_A(t)))dt, & t > 0; \\ Z(0) = x, \end{cases}$$

and it satisfies

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|Z(t)\|^2\right] < +\infty$$

Since

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|W_A(t)\|^2\right] < +\infty,$$

by the dominated convergence theorem it is easy to see that $\pi_n W_A(t)$ converges to $W_A(t)$ in $L^2(\Omega, \mathbb{P})$, as n tends to infinity.

Setting $W_n(t) := \pi_n W(t)$ and

$$W_{A_n}(t) := \int_0^t e^{(t-s)A_n} R_n dW_n(s), \qquad t \geq 0,$$

then the process $Z_n(t) := X_n(t,x) - W_{A_n}(t)$ $(n \geq n_0)$ satisfies

$$\begin{cases} dZ_n(t) = (A_n Z_n(t) + F_n(Z_n(t) + W_{A_n}(t)))dt, & t > 0; \\ Z_n(0) = x. \end{cases}$$

Now we split the proof in two steps. In the first one we show that $W_{A_n}(t) - W_A(t)$ converges to 0 in $L^2((\Omega, \mathcal{F}, \mathbb{P}); (\mathcal{X}, \mathcal{B}(\mathcal{X})))$ and that, consequently $\|Z_n(t)\|$ is uniformly bounded with respect to n. In the second one we complete the proof.
*S*tep 1. Using that $e^{tA}_{|\mathcal{X}_n} = e^{tA_n}$ for every $t \geq 0$ we write

$$W_A(s) - W_{A_n}(s) = \int_0^s e^{(s-r)A}(R - \pi_n R \pi_n) dW(r), \qquad s \in [0,T].$$

By the Itô formula we obtain

$$\mathbb{E}(\|W_A(s) - W_{A_n}(s)\|^2) = \int_0^s \left\| e^{(s-r)A}(R - \pi_n R \pi_n) \right\|^2_{\mathcal{L}(\mathcal{X})} dr$$

and since the integrand converges to zero as $n \to \infty$ uniformly with respect to $r \in (0,s)$, by the dominated convergence theorem we get the claim. Now, scalarly multiplying $dZ_n(t) = (A_n Z_n(t) + F_n(Z_n(t) + W_{A_n}(t)))dt$ by $Z_n(t)$ and using the Hypotheses 4.1.1 with A and F replaced by A_n and F_n we deduce

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|Z_n(t)\|^2 =& \langle A_n Z_n(t) + F_n(Z_n(t) + W_{A_n}(t)), Z_n(t) \rangle \\ =& \langle A_n Z_n(t) + F_n(Z_n(t) + W_{A_n}(t)) \pm F_n(W_{A_n}(t)), Z_n(t) \rangle \\ \leq& \zeta_{\mathcal{X}} \|Z_n(t)\|^2 + \langle F_n(W_{A_n}(t)), Z_n(t) \rangle \\ \leq& \left(\zeta_{\mathcal{X}} + \frac{1}{2}\right) \|Z_n(t)\|^2 + \frac{1}{2}\|F_n(W_{A_n}(t))\|^2 \\ \leq& \left(\zeta_{\mathcal{X}} + \frac{1}{2}\right) \|Z_n(t)\|^2 + M 2^{-1} \left(1 + \|W_{A_n}(t)\|^2\right), \qquad t > 0. \end{aligned}$$

The Gronwall lemma and the uniform boundedness of $\|W_{A_n}(t)\|$ with respect to n allows to deduce that $\|Z_n(t)\|$ is uniformly bounded with respect to n.
*S*tep 2. To conclude the proof we show that $Z_n(t) - Z(t)$ converges to 0 in $L^2((\Omega, \mathcal{F}, \mathbb{P}); (\mathcal{X}, \mathcal{B}(\mathcal{X})))$ as $n \to \infty$. This will imply that $X_n(t,x)$ converges to $X(t,x)$ in $L^2((\Omega, \mathcal{F}, \mathbb{P}); (\mathcal{X}, \mathcal{B}(\mathcal{X})))$ as $n \to \infty$, by (4.1.10) we will conclude. By Hypothesis 4.1.5 we have

$$\begin{aligned} d(Z(t) - Z_n(t)) &= (AZ(t) - A_n Z_n(t) + F(X(t,x)) - F_n(X_n(t,x)))dt \\ &= (A(Z(t) - Z_n(t)) + F(X(t,x)) - F_n(X_n(t,x)))dt, \qquad t > 0; \end{aligned}$$

so, scalarly multiplying by $Z(t) - Z_n(t)$ we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|Z(t) - Z_n(t)\|^2 =& \langle (A(Z(t) - Z_n(t)) + F(X(t,x)) - F_n(X_n(t,x))), Z(t) - Z_n(t) \rangle \\ =& \langle (A(Z(t) - Z_n(t)) + F(Z(t) + W_A(t)) \end{aligned}$$

$$- F_n(Z_n(t) + W_{A_n}(t))), Z(t) - Z_n(t)\rangle.$$

Adding and subtracting $F(Z_n(t) + W_A(t))$ and $F(Z_n(t) + W_{A_n}(t))$ and using Hypotheses 4.1.1 we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|Z(t) - Z_n(t)\|^2 \leq{}& \zeta_{\mathcal{X}}\|Z(t) - Z_n(t)\|^2 \\
&+ \langle F(Z_n(t) + W_A(t)) - F(Z_n(t) + W_{A_n}(t)), Z(t) - Z_n(t)\rangle \\
&+ \langle F(Z_n(t) + W_{A_n}(t)) - \pi_n F(Z_n(t) + W_{A_n}(t)), Z(t) - Z_n(t)\rangle \\
\leq{}& \zeta_{\mathcal{X}}\|Z(t) - Z_n(t)\|^2 \\
&+ \|F(Z_n(t) + W_A(t)) - F(Z_n(t) + W_n(t))\|\|Z(t) - Z_n(t)\| \\
&+ \|(\mathrm{Id}_{\mathcal{X}} - \pi_n)F(Z_n(t) + W_{A_n}(t))\|\|Z(t) - Z_n(t)\| \\
\leq{}& \zeta_{\mathcal{X}}\|Z(t) - Z_n(t)\|^2 + L_F\|W_A(t) - W_{A_n}(t)\|\|Z(t) - Z_n(t)\| \\
&+ \frac{1}{2}\|(\mathrm{Id}_{\mathcal{X}} - \pi_n)F(Z_n(t) + W_{A_n}(t))\|^2 + \frac{1}{2}\|Z(t) - Z_n(t)\|^2 \\
\leq{}& (\zeta_{\mathcal{X}} + 1)\|Z(t) - Z_n(t)\|^2 + \frac{1}{2}L_F^2\|W_A(t) - W_{A_n}(t)\|^2 \\
&+ \frac{1}{2}\|(\mathrm{Id}_{\mathcal{X}} - \pi_n)F(Z_n(t) + W_{A_n}(t))\|^2.
\end{aligned}$$

where in the last two lines we have used the Young inequality. Integrating over $[0, t]$, we obtain

$$\begin{aligned}
&\frac{1}{2}\|Z(t) - Z_n(t)\|^2 \\
&\leq \int_0^t \left(\zeta_{\mathcal{X}} + \frac{1}{2}\right)\|Z(s) - Z_n(s)\|^2 + \|F(X(s)) - F(X_n(s))\|^2 + \|(1 - \pi_n)F(X_n(s))\|^2 ds \\
&\leq \int_0^t \left(\zeta_{\mathcal{X}} + \frac{1}{2} + 2L_F^2\right)\|Z(s) - Z_n(s)\|^2 + 2L_F^2\|W_A(s) - W_n(s)\|^2 + \|(1 - \pi_n)F(X_n(s))\|^2 ds.
\end{aligned}$$

Applying again the Gronwall lemma we obtain

$$\begin{aligned}
\|Z(t) - Z_n(t)\|^2 \leq{}& L_F^2 e^{2(\zeta_{\mathcal{X}}+1)t}\int_0^t \|W_A(s) - W_{A_n}(s)\|^2 ds \\
&+ e^{2(\zeta_{\mathcal{X}}+1)t}\int_0^t \|(\mathrm{Id}_{\mathcal{X}} - \pi_n)F(Z_n(s) + W_{A_n}(s))\|^2 ds
\end{aligned} \tag{4.1.12}$$

and using the results in Step 1 we infer that the right hand side of (4.1.12) vanishes as $n \to \infty$, concluding the proof. □

We point out that the assumption of R-invariance of $\mathcal{X}_n$ can be dropped in order to prove Proposition 4.1.6. However it is essential to apply the results in the previous sections to the finite-dimensional approximating operators. Indeed, assuming Hypotheses 4.1.5, it is easy to see that if A, R and F satisfy Hypotheses 4.1.2(1) then A_n, R_n and F_n satisfy them as well with the same constants. So all the results of the previous sections hold true even for the mild solution of (4.1.9) and for the semigroup in (4.1.10). In particular for every $h \in H_{R_n}$

$$\|\mathcal{D}_G X_n(t, x)h\|_{R_n} \leq C_T\|h\|_{R_n}, \qquad \text{for } x \in \mathcal{X}_n \text{ and } \mathbb{P}\text{-a.e.}, \tag{4.1.13}$$

where C_T is the constant given by Proposition 4.1.3.

4.1.3 Proof of the LHI

The next result is a gradient estimate which is interesting in its own, and it is a fundamental tool to prove Theorem 4.1.9.

Theorem 4.1.7. *Assume that Hypotheses 4.1.2 and 4.1.5 hold true. Then for every $\varphi \in C_b^1(\mathcal{X})$ and $T > 0$ it holds that $P(t)\varphi$ is Fréchet differentiable and we have*

$$\|\mathcal{D}_R P(t)\varphi(x)\|_R \le C_T(P(t)\|\mathcal{D}_R\varphi\|_R)(x), \quad x \in \mathcal{X},\ t \in [0,T], \tag{4.1.14}$$

and

$$\|\mathcal{D}_{R_n} P_n(t)\varphi(x)\|_{R_n} \le C_T(P(t)\|\mathcal{D}_{R_n}\varphi\|_{R_n})(x) \quad x \in \mathcal{X},\ t \in [0,T], \tag{4.1.15}$$

where C_T is the constant given by Proposition 4.1.3.

Proof. We only prove estimate (4.1.14) since (4.1.15) can be obtained similarly using (4.1.13) instead of (4.1.7). If φ belongs to $C_b^1(\mathcal{X})$, then, by (4.1.5), Corollary 3.1.6 and [86, Fact 1.13(b), p. 8], $P(t)\varphi$ is also Fréchet differentiable. Using Proposition 3.1.3 we get that $P(t)\varphi$ is H_R-Fréchet differentiable. By Proposition 4.1.3 and Corollary 3.1.6 for every $T > 0$ $t \in [0,T]$, $x \in \mathcal{X}$ and $h \in H_R$ we have

$$\begin{aligned}
\langle \mathcal{D}_R P(t)\varphi(x), h\rangle_R &= \langle \mathcal{D}_R \mathbb{E}[\varphi(X(t,x)), h\rangle_R \\
&= \mathbb{E}\big[\langle \mathcal{D}_R\varphi(X(t,x)), \mathcal{D}_G X(t,x)h\rangle_R\big] \\
&\le \mathbb{E}\big[\|\mathcal{D}_R\varphi(X(t,x))\|_R \|\mathcal{D}_G X(t,x)h\|_R\big] \\
&\le C_T\|h\|_R(P(t)\|\mathcal{D}_R\varphi\|_R)(x).
\end{aligned}$$

Now (4.1.14) follows by a standard argument. $\square$

We recall a version of the monotone class theorem that we will use in the next proof.

Theorem 4.1.8 (Theorem 6.3 of [89])**.** *Let $\mathcal{M}$ be a class of bounded functions from a set Ω to $\mathbb{R}$ and let $\mathcal{H}$ be a vector space of functions such that $\mathcal{M} \subseteq \mathcal{H}$. If*

(i) *$\mathcal{M}$ is closed under multiplication, i.e. if $f, g \in \mathcal{M}$, then $fg \in \mathcal{M}$;*

(ii) *$\mathcal{H}$ contains the constant functions;*

(iii) *for any sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H}$ such that if*

$$0 \le f_1 \le f_2 \le f_3 \le \cdots$$

and the pointwise limit f of $\{f_n\}_{n\in\mathbb{N}}$ is bounded, then f belongs to $\mathcal{H}$;

then $\mathcal{H}$ contains all bounded $\sigma(\mathcal{M})$-measurable functions from Ω to $\mathbb{R}$, where

$$\sigma(\mathcal{M}) := \{f^{-1}(B) \,|\, B \in \mathcal{B}(\mathcal{X}),\ f \in \mathcal{M}\}.$$

We have all the results we need to prove Theorem 4.1.9.

Theorem 4.1.9. *Assume that Hypotheses 4.1.2 and 4.1.5 hold true, then for any $T > 0$, $\varphi \in B_b(\mathcal{X})$, $t \in [0,T]$, $x \in \mathcal{X}$, $h \in H_R$ and $p > 1$ it holds*

$$|(P(t)\varphi)(x+h)|^p \leq (P(t)|\varphi|^p)(x)\exp\bigg(\frac{pC_T^2}{t(p-1)}\|h\|_R^2\bigg), \tag{4.1.16}$$

where C_T is the constant given by Proposition 4.1.3.

Proof. Fix $n, n_0 \in \mathbb{N}$ with $n > n_0$ and $x, h \in \mathcal{X}_{n_0}$. We claim that estimate (4.1.16) holds true with $P(t)$, C, $\mathcal{X}$ and φ replaced respectively by $P_n(t)$, C_n, $\mathcal{X}_n$ and $f \in \mathcal{F}C_b^2(\mathcal{X})$ with positive infimum. To this aim, fix $\varepsilon > 0$ and $f \in \mathcal{F}C_b^2(\mathcal{X})$ be such that $\inf_{x\in\mathcal{X}} f(x) > \varepsilon$. We set

$$g_n(r,x) := P_n(r)f(x), \qquad x \in \mathcal{X}_{n_0},\ r \geq 0,$$

We note that the function $g_n : [0,T]\times\mathcal{X}_{n_0} \to \mathbb{R}$ belongs to $C^{1,2}([0,T]\times\mathcal{X}_{n_0})$ and solves

$$\begin{cases} D_r g_n(r,x) = \mathcal{N}_n g_n(r,x), & r > 0; \\ g_n(0,x) = f(x), \end{cases}$$

where $\mathcal{N}_n$ is the operator defined by (4.1.11). Moreover $g_n(r,x) \geq \varepsilon$ for any $r \geq 0$ and $x \in \mathcal{X}_{n_0}$ (see [72, Theorem 1.2.5]). Now fix $t > 0$ and consider the function

$$G_n(r) := (P_n(t-r)g_n^p(r,\cdot))\big(x + rt^{-1}h\big), \qquad r \in [0,t],\ x,h \in \mathcal{X}_{n_0}.$$

For the sake of simplicity we let $\psi_h(r) := x + rt^{-1}h$. We differentiate the map $r \mapsto \ln(G(r))$.

$$\begin{aligned}\frac{d}{dr}\ln G_n(r) &= (G_n(r))^{-1}P_n(t-r)\Big(-\mathcal{N}_n g_n^p(r,\cdot) + D_r g_n^p(r,\cdot)\Big)(\psi_h(r)) \\ &\quad + (tG_n(r))^{-1}\langle \mathcal{D}P_n(t-r)g_n^p(\psi_h(r)), h\rangle.\end{aligned}$$

where we used that the semigroup and its generator commute on smooth functions. A straightforward computation yields that

$$-\mathcal{N}_n g_n^p + D_r g_n^p = -p(p-1)g_n^{p-2}\|C_n^{1/2}\mathcal{D}g_n\|_{\mathcal{X}_n}^2$$

whereas by estimate (4.1.15) we infer

$$\begin{aligned}\langle \mathcal{D}P_n(t-r)g_{n,n_0}^p(r,\psi_h(r)), h\rangle &= \langle R_n\mathcal{D}P_n(t-r)g_{n,n_0}^p(r,\psi_h(r)), R_n h\rangle \\ &\leq C_T\|R_n h\|_{\mathcal{X}_n}P_n(t-r)\big[pg_{n,n_0}^{p-1}(r,\cdot)\|R_n\mathcal{D}g_{n,n_0}(r,\cdot)\|_{\mathcal{X}_n}\big](\psi_h(r)).\end{aligned}$$

Hence we obtain

$$\frac{d}{dr}\ln G(r) \leq \frac{1}{G(r)}P_n(t-r)[-p(p-1)g_{n,n_0}^p(r,\cdot)(-v^2+\beta v)](\psi_h(r))$$

where

$$v := g_n^{-1}(r,\psi_h(r))\|R_n\mathcal{D}g(r,\psi_h(r))\|_{\mathcal{X}_n}, \qquad \beta := \frac{C_T}{t(p-1)}\|R_nh\|_{\mathcal{X}_n}.$$

The elementary inequality $a^2 - ab + 4^{-1}b^2 \geq 0$ which holds true for any $a, b \in \mathbb{R}$ allows us to estimate $-v^2 + \beta v \leq \beta^2$ and, consequently, by the positivity of $P_n(t)$, we get

$$\frac{d}{dr}\ln G(r) \leq \frac{pC_T^2\|R_nh\|_{\mathcal{X}_n}^2}{t^2(p-1)G(r)}P_n(t-r)[g_{n,n_0}^p(r,\cdot)](\psi_h(r)) = \frac{pC_T^2}{t^2(p-1)}\|R_nh\|_{\mathcal{X}_n}^2.$$

whence, integrating from 0 to t with respect to r we obtain

$$|(P_n(t)f)(x+h)|^p \leq (P_n(t)|f|^p)(x)\exp\left(\frac{pC_T^2}{t(p-1)}\|h\|_{R_n}^2\right), \qquad t > 0,$$

for any $f \in \mathcal{F}C_b^2(\mathcal{X})$ with positive infimum, $x, h \in \mathcal{X}_n$. The estimate above continues to hold replacing f by $|f|$ for a general $f \in \mathcal{F}C_b^2(\mathcal{X})$. This can be obtained approximating pointwise the function $|f|$ by the sequence $f_n = (f^2 + n^{-1})^{1/2}$ and using the dominated convergence theorem. Further, the Jensen inequality yields that

$$|(P_n(t)f)(x+h)|^p \leq (P_n(t)|f|^p)(x)\exp\left(\frac{pC_T^2}{t(p-1)}\|h\|_{R_n}^2\right), \qquad t > 0,$$

for any $f \in \mathcal{F}C_b^2(\mathcal{X})$, $x \in \mathcal{X}_n$. Proposition 4.1.6 and the fact that $\|h\|_{R_n}$ converges to $\|h\|_R$ for any $h \in H_R$ as $n \to \infty$ imply that

$$|(P(t)f)(x+h)|^p \leq (P(t)|f|^p)(x)\exp\left(\frac{pC_T^2}{t(p-1)}\|h\|_R^2\right), \qquad t > 0, \tag{4.1.17}$$

for any $f \in \mathcal{F}C_b^2(\mathcal{X})$ and $x, h \in \bigcup_{i\in\mathbb{N}}\mathcal{X}_i$ (recall that $R_n^{1/2}\mathcal{X}_n = \mathcal{X}_n$ for any $n \in \mathbb{N}$). Since $\mathcal{F}C_b^2(\mathcal{X})$ is dense in $C_b(\mathcal{X})$ with respect to the mixed topology (see [62, Lemma 2.6 and Theorem 4.1(b)]), then (4.1.17) is verified for any $f \in C_b(\mathcal{X})$.

Now we claim that estimate (4.1.17) can be extended to any $x \in \mathcal{X}$ and $h \in H_R$. To this aim, thanks to Hypothesis 4.1.5 we can consider two sequences $(x_n)_{n\in\mathbb{N}}$ and $(h_m)_{m\in\mathbb{N}}$ belonging to $\bigcup_{i\in\mathbb{N}}\mathcal{X}_i$ converging respectively to x in $\mathcal{X}$ and h in H_R as $n, m \to \infty$ (recall that, by Hypotheses 4.1.5, H_R is dense in $\mathcal{X}$). Writing (4.1.17) with x and h being replaced by x_n and h_m we deduce

$$|(P(t)f)(x_n+h_m)|^p \leq (P(t)|f|^p)(x_n)\exp\left(\frac{pC_T^2}{t(p-1)}\|h_m\|_R^2\right), \qquad t > 0,\ f \in C_b(\mathcal{X}).$$

By the continuity of the map $x \mapsto P(t)f(x)$, we get the claim.

Finally, we let $\mathcal{M} = C_b(\mathcal{X})$ and let $\mathcal{H}$ be the biggest closed vector space in $(B_b(\mathcal{X}), \|\cdot\|_\infty)$ whose elements satisfy estimate (4.1.17) for any $x \in \mathcal{X}$ and $h \in H_C$. Observe that the hypotheses of Theorem 4.1.8 are satified ((iii) follows by the monotone convergence theorem) and so we conclude that (4.1.17) holds true for any $f \in B_b(\mathcal{X})$ as well. $\square$

4.2 Logarithmic Harnack inequality: the dissipative case

The aim of this section is proving some (LHI) when no hypotheses of global Lipschitzianity for F is done but only some m-dissipativity along H_R. The main tool is again an approximation procedure this time using the Yosida approximants.

Hypotheses 4.2.1. *Assume that Hypotheses 2.1.1 hold true, R is non-negative and there exists $\zeta_F \in \mathbb{R}$ such that $F - \zeta_F \mathrm{I}_{\mathcal{X}}$ is m-dissipative.*

Remark 4.2.2. *Since $F - \zeta_F \mathrm{I}_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$ is dissipative then by Hypotheses 2.1.1*(iv) *there exists $\zeta_A \in \mathbb{R}$ such that $A - \zeta_A \mathrm{I}_{\mathcal{X}}$ is dissipative.*

Under Hypotheses 2.1.1, by Propositions 2.2.1 2.2.4 and Corollary 2.2.5, for any $x \in \mathcal{X}$, the SPDE (4.0.2) has unique generalized mild solution $\{X(t,x)\}_{t\geq 0}$, we denote by $P(t)$ the transition semigroup associated to (4.0.2).

4.2.1 The Yosida approximating

Let $\{F_\delta\}_{\delta>0}$ be the Yosida approximations of F defined in Proposition 1.11.1. For every $\delta > 0$ and $x \in \mathcal{X}$ the we consider the SPDE

$$\begin{cases} dX_\delta(t,x) = \big[AX_\delta(t,x) + F_\delta(X_\delta(t,x))\big]dt + RdW(t), \quad t > 0; \\ X_\delta(0,x) = x. \end{cases} \tag{4.2.1}$$

by Theorem 2.1.13 (with $\mathcal{X} = E$) and Lemma 1.11.1, (4.2.1) has a unique mild solution $\{X_\delta(t,x)\}_{t\geq 0}$. Moreover by (2.1.22) and (1.11.4), for any $p \geq 1$ there exists $C_p, \kappa_p > 0$ such that, for any $\delta > 0$, $x \in E$ and $t > 0$ we have $\mathbb{P}$-a.e.

$$\|X_\delta(t,x)\|_E^p \leq C_p\bigg(e^{-\kappa_p t}\|x\|_E^p + \|W_A(t)\|_E^p + \int_0^t e^{-\kappa_p(t-s)}\left(\|F(W_A(s))\|_E^p + \|W_A(s)\|_E^p\right)ds\bigg). \tag{4.2.2}$$

We denote by $P_\delta(t)$ the transition semigroup associated to (4.2.1).

The following proposition is the approsimation result we need to prove Theorem 4.2.9.

Proposition 4.2.3. *If Hypotheses 2.1.1 hold true, then for any $T > 0$, $\varphi \in C_b(\mathcal{X})$ and $x \in E$*

$$\lim_{\delta\to 0}\sup_{t\in[0,T]}\|X_\delta(t,x) - X(t,x)\|_E = 0, \qquad \mathbb{P}\text{-}a.s.; \tag{4.2.3}$$

$$\lim_{\delta\to 0}|P_\delta(t)\varphi(x) - P(t)\varphi(x)| = 0, \qquad t > 0. \tag{4.2.4}$$

Proof. First of all let us observe that (4.2.4) immediately follows by (4.2.3). Therefore we just prove (4.2.3). To this aim we start pointing out that (4.2.2) implies that the function $(t,x) \mapsto \|X_\delta(t,x)\|_E$, as a map from $[0,T] \times E$ into $\mathbb{R}$, is bounded by a positive constant $C = C(T,x)$ independent of δ. This fact, together with estimates (4.2.2) and (1.11.3) implies that for any

$\delta > 0$, $x \in E$ and $T > 0$

$$K(T,x) := \sup_{t\in[0,T]} \left(\|J_\delta(X_\delta(t,x))\|_E + \|X_\delta(t,x)\|_E + \|X(t,x)\|_E\right) < +\infty \tag{4.2.5}$$

and $K(T,x)$ is independent of δ.

Now if $L := L(x,T) > 0$ denotes the Lipschitz constant of the restriction of F to the ball $B(0,K(T,x))$ (see Hypotheses 2.1.1(iii)), then we have

$$\begin{aligned}\|F_\delta(X_\delta(t,x)) - F(X(t,x))\|_E &= \|F(J_\delta(X_\delta(t,x))) - F(X(t,x))\|_E\\ &\le L\|J_\delta(X_\delta(t,x)) - X(t,x)\|_E\\ &\le L\|J_\delta(X_\delta(t,x)) - X_\delta(t,x)\|_E + L\|X_\delta(t,x) - X(t,x)\|_E. \end{aligned}\tag{4.2.6}$$

By (1.11.3), (4.2.5) and (4.2.6) we can conclude that

$$\|F_\delta(X_\delta(t,x)) - F(X(t,x))\|_E \le \delta M' + L\|X_\delta(t,x) - X(t,x)\|_E. \tag{4.2.7}$$

for some positive $M' = M'(K,M,L,m,\zeta_F)$. Thus, by using the definition of mild solution and estimate (4.2.7) we obtain

$$\|X_\delta(t,x) - X(t,x)\|_E \le \delta M_0 M' \int_0^t e^{(t-s)\eta_0} ds + M_0 L \int_0^t e^{(t-s)\eta_0} \|X_\delta(t,x) - X(t,x)\|_E ds.$$

Applying the Gronwall lemma we complete the proof. □

Let $H_R := R(\mathcal{X})$ be the Hilbert space defined in Section 3.1, as announced we need an additional assumption of dissipativity on F.

Hypotheses 4.2.4. *Assume that Hypotheses 4.2.1 hold true, that $F(\mathrm{Dom}(F) \cap H_R) \subseteq H_R$ and $F_{|_{H_R}} - \zeta_F \mathrm{Id}_{H_R}$ is m-dissipative, where $F_{|_{H_R}} : \mathrm{Dom}(F_{|_{H_R}}) = \mathrm{Dom}(F) \cap H_R \subseteq H_R \to H_R$.*

By Corollary 1.11.2 we have the following result

Corollary 4.2.5. *Assume that Hypotheses 4.2.4 hold true. For any $0 < \delta < |\zeta_F|^{-1}$, $F_\delta - \zeta_F \mathrm{I}_{\mathcal{X}}$ is dissipative as a function from H_R to H_R.*

4.2.2 Proof of the LHI

We want to prove a result similar to Theorem 4.1.9, to do so we need an approximating sequence as the one in Proposition 4.1.6 and an estimate similar to (4.1.15). However to prove this estimate we have to assume a stronger hypothesis than Hypotheses 4.1.5.

Hypotheses 4.2.6. *H_R is dense in $\mathcal{X}$. There exists a basis $\{e_k\}_k$ of H_R consisting of eigenvectors of A, moreover R is diagonalizable with respect to this basis.*

Remark 4.2.7. *Hypotheses 4.2.6 imply Hypotheses 4.1.5, indeed we can set*

$$\mathcal{X}_n := span\{e_1, ..., e_n\}.$$

We assume that Hypotheses 4.2.4 and 4.2.6 hold. For any $0 < \delta < |\zeta_F|^{-1}$ and $n \in \mathbb{N}$, let $P_{\delta,n}$ be the transition semigroup of equation

$$\begin{cases} dX_{\delta,n}(t) = \big(A_n X_{\delta,n}(t) + F_{\delta,n}(X_{\delta,n}(t))\big)dt + R_n dW_n(t), t > 0, \\ X_{\delta,n}(0) = z. \end{cases} \tag{4.2.8}$$

See Section 4.1.2 for the definitions of A_n, R_n and F_n.

Lemma 4.2.8. *Assume Hypotheses 4.2.4 and 4.2.6 hold true. For any $0 < \delta < |\zeta_F|^{-1}$, $n \in \mathbb{N}$, $t > 0$, $f \in \mathcal{F}C^1_{n,b}(\mathcal{X})$ (see Remark 1.6.16) and $x \in \mathcal{X}_n$ we have*

$$\|\mathcal{D}_{R_n} P_{\delta,n}(t) f(x)\|_{R_n} \leq e^{t\zeta_R} P_{\delta,n}(t)(\|\mathcal{D}_{R_n} f\|_{R_n})(x),$$

where $\zeta_R = \zeta_A + \zeta_F$ (see Hypotheses 4.2.1 and Remark 4.2.2).

Proof. We fix $n \in \mathbb{N}$ and $0 < \delta < |\zeta_F|^{-1}$. Let $\{X_n(t,x)\}_{t\geq 0}$ and $\{Y_n(t,y)\}_{t\geq 0}$ be the mild solutions of (4.2.8) with initial datum x and y respectively. We assume that $\{X(t,x)\}_{t\geq 0}$ and $\{Y(t,y)\}_{t\geq 0}$ are strict solutions of (4.2.8), otherwise we proceed as in Proposition 2.1.7 approximating them by means of a sequence of more regular processes. For any $t \geq 0$ we have

$$\begin{aligned} d\|X(t,x) - Y(t,y)\|^2_{R_n} &= \langle A\left(X(t,x) - Y(t,y)\right), X(t,x) - Y(t,y)\rangle_{R_n} \\ &\quad + \langle F_n(X(t,x)) - F_n(Y(t,y)), X(t,x) - Y(t,y)\rangle_{R_n}. \end{aligned}$$

$\mathcal{X}_n$ is finite dimensional space and the operators A and R are diagonalizable with respect to the same basis, so they commute. Hence by Hypotheses 4.2.1 and Remark 4.2.2 for any $t \geq 0$ we have

$$d\|X(t,x) - Y(t,y)\|^2_{R_n} \leq \zeta_R \|X(t,x) - Y(t,y)\|^2_{R_n},$$

where $\zeta_R = \zeta_A + \zeta_F$. By the Gronwall inequality, for any $t \geq 0$ and $x, y \in \mathcal{X}_n$ we obtain

$$\|X(t,x) - Y(t,y)\|_{R_n} \leq e^{t\zeta_R}\|x - y\|^2_{R_n}.$$

Therefore, for any $f \in \mathcal{F}C^1_{n,b}(\mathcal{X})$, $t \geq 0$ and $x \in \mathcal{X}_n$ we have

$$\begin{aligned} \|\mathcal{D}_{R_n} P_{\delta,n}(t) f(x)\|_{R_n} &= \limsup_{y\to x} \frac{|P_{\delta,n}(t)f(x) - P_{\delta,n}(t)f(y)|}{\|x-y\|_{R_n}} \\ &= \limsup_{y\to x} \left(\frac{|P_{\delta,n}(t)f(x) - P_{\delta,n}(t)f(y)|}{\|X(t,x) - Y(t,y)\|_{R_n}}\right)\left(\frac{\|X(t,x) - Y(t,y)\|_{R_n}}{\|x-y\|_{R_n}}\right) \\ &\leq e^{t\zeta_R} P_{\delta,n}(t)\|\mathcal{D}_{R_n} f(x)\|_{R_n}. \end{aligned}$$

□

We now have all the results we need to prove Theorem 4.2.9.

Theorem 4.2.9. *Assume that Hypotheses 4.2.4 and 4.2.6 hold true. Then for any* $\varphi \in B_b(\mathcal{X})$, $t > 0$, $x \in \mathcal{X}$ *and* $p > 1$ *we have*

$$|(P(t)\varphi)(x+h)|^p \leq (P(t)|\varphi|^p) \exp\left(\frac{pe^{2t\zeta_R}}{t(p-1)}\|h\|_R^2\right), \qquad h \in H_R \cap E. \tag{4.2.9}$$

Proof. Using Lemma 4.2.8 by the same proof of Theorem 4.1.9 we obtain

$$|(P_\delta(t)\varphi)(x+h)|^p \leq (P_\delta(t)|\varphi|^p)(x) \exp\left(\frac{pe^{2t\zeta_R}}{t(p-1)}\|h\|_R^2\right). \tag{4.2.10}$$

for any $0 < \delta < |\zeta_F|^{-1}$, $p > 1$, $t > 0$, $\varphi \in C_b(\mathcal{X})$, $x \in X$ and $h \in H_R$. Now, using formula (4.2.4) and letting $\delta \to 0$ in (4.2.10) we get

$$|(P(t)\varphi)(x+h)|^p \leq (P(t)|\varphi|^p)(x) \exp\left(\frac{pe^{2t\zeta_R}}{t(p-1)}\|h\|_R^2\right). \tag{4.2.11}$$

for any $p > 1$, $t > 0$, $\varphi \in C_b(\mathcal{X})$, $x \in E$ and $h \in H_R \cap E$. Using the fact that E is densely embedded in $\mathcal{X}$ and the continuity of $P(t)\varphi$ we can extend estimate (4.2.11) to any $x \in \mathcal{X}$. Finally, using the monotone class theorem as in the proof of Theorem 4.1.9 we complete the proof. $\square$

Remark 4.2.10. *We point out that if* $H_R \cap E$ *is dense in* H_R*, then the (LHI) in* (4.2.9) *holds true for any* $h \in H_R$.

4.3 Remarks and examples

This chapter is a reworked version of [4]. Estimates like (4.0.1) for the transition semigroups of equations similar to (4.0.2) can be found for instance in [38, 56, 67, 76, 93, 102]. In all the quoted papers two different sets of assumptions for A, F and R are made in order to get inequalities like (4.0.1).

In [38, 102, 67, 76] it is required that R admits continuous and bounded linear inverse. In [67, 76] it is assumed that RR^* is invertible, while in [56] the authors restrict themselves to consider as R the identity operator.

In [93] it is not assumed that R^{-1} is linear and bounded. However the function F is Lipschitz continuous and it satisfies the following dissipativity type condition:

$$\langle F(x) - F(y), R^{-2}(x-y)\rangle \leq \zeta \|R^{-1}(x-y)\|, \qquad x - y \in R(\mathcal{X}).$$

for some $\zeta \in \mathbb{R}$.

In this chapter it is never assumed that R^{-1} is linear and bounded. Moreover in Subsection 4.2.2 we proved the same result in [93] without assuming that F is Lipschitz continuous, so, in this sense, this chapter generalizes the results contained in [38, 56, 67, 76, 93, 102].

Now we will see some consequences of the logaritmic Harnack inequalities and some examples of A, R and F that verify our hypotheses.

4.3.1 Consequences of the logaritmic Harnack inequalities

We start by stating and proving some classical consequences of the (LHI) for which we refer to [93, Corollary 1.2] and [101, Section 1.3.1]. In this subsection we assume that $P(t)$ has a invariant probability measure μ. Sufficient conditions that guarantee the existence of such a measure can be found in the next chapter (Theorem 5.1.3) or in [20, Chapter 8], [41, Chapter 6] and [43, Chapter 11]. In this case, for any $p \geq 1$, $P(t)$ is uniquely extendable to a strongly continuous and contraction semigroup in $L^p(\mathcal{X}, \mu) =: L^p_\mu$ (see Section 5.1), we still denote it by $P(t)$. For simplicity we write $\mu(f)$ to denote $\int_{\mathcal{X}} f d\mu$.

Corollary 4.3.1. *Assume that Hypotheses 4.2.4 and 4.2.6 hold true.*

(i) *For any positive $f \in B_b(\mathcal{X})$, $t > 0$, $x \in \mathcal{X}$ and $h \in H_R \cap E$*

$$[P(t)(\ln f)](x+h) \leq \ln P(t)f(x) + \frac{e^{2t\zeta_R}}{t}\|h\|_R^2. \tag{4.3.1}$$

(ii) *For every $f \in B_b(\mathcal{X})$ and $x \in \mathcal{X}$ it holds*

$$\lim_{\substack{\|h\|_R \to 0 \\ h \in H_R \cap E}} P(t)f(x+h) = P(t)f(x). \tag{4.3.2}$$

(iii) *The following entropy-cost inequality holds true*

$$\mu((P^*(t)f)\ln(P^*(t)f)) \leq \frac{e^{2t\zeta_R}}{t} W(f\mu, \mu)^2,$$

for any positive function $f \in B_b(\mathcal{X})$ such that $\mu(f) = 1$. Here $\{P^*(t)\}_{t \geq 0}$ is the adjoint semigroup of $\{P(t)\}_{t \geq 0}$ in L^2_μ and W denotes the L^2-Wasserstein distance with respect to the cost function $(x, y) \mapsto \|x - y\|_R$, namely for any two probability measure μ_1, μ_2 on $\mathcal{X}$

$$W(\mu_1, \mu_2)^2 := \inf \left\{ \int_{\mathcal{X}\times\mathcal{X}} \|x-y\|_C^2 \pi(dx, dy) \,\middle|\, \pi \in \mathscr{C}(\mu_1, \mu_2) \right\},$$

where $\mathscr{C}(\mu_1, \mu_2)$ is the set of all the couplings of μ_1 and μ_2 and we let $\|x - y\|_C = +\infty$, if $x - y$ does not belong to $H_R \cap E$.

Proof. A proof of (i) can be found in [101, Section 1.3.1]. We now prove (ii). It suffices to prove (4.3.2) for a non-negative function $f \in B_b(\mathcal{X})$. Indeed the general case can be obtained writing $f = f^+ - f^-$, being f^+ and f^- the positive and the negative part of f. So, let us fix a non-negative function f and for any $\varepsilon > 0$ we set $f_\varepsilon := 1 + \varepsilon f$. Recalling that $r \leq \ln(1 + r) + r^2$ for any $r \geq 0$ we get for every $x \in \mathcal{X}$

$$\ln f_\varepsilon(x) = \ln(1 + \varepsilon f(x)) \geq \varepsilon f(x) - \varepsilon^2 f^2(x) \geq \varepsilon f(x) - \varepsilon^2 \|f\|_\infty^2. \tag{4.3.3}$$

Now applying (4.3.1) to f_ε, using (4.3.3) and dividing by ε we get for every $x \in \mathcal{X}$ and $h \in H_R \cap E$

$$P(t)f(x+h) - \varepsilon\|f\|_\infty^2 \leq \frac{1}{\varepsilon} \ln P(t)(1 + \varepsilon f(x)) + \frac{e^{2t\zeta_R}}{\varepsilon t}\|h\|_R^2. \tag{4.3.4}$$

Taking the supremum limit as $\|h\|_R \to 0$ with $h \in H_R \cap E$ and then letting $\varepsilon \to 0$ we get

$$\limsup_{\substack{\|h\|_R \to 0\\ h\in H_R\cap E}} P(t)f(x+h) \leq P(t)f(x).$$

Recalling that $\ln(1+r) \leq r$ for any $r > -1$ and arguing as above we get that for any $\varepsilon > 0$, $x \in \mathfrak{X}$ and $h \in H_R \cap E$

$$P(t)\left(\frac{1+\varepsilon f(x)}{\varepsilon}\right) - \frac{e^{2t\zeta_R}}{\varepsilon t}\|h\|_R^2 \leq \frac{1}{\varepsilon}\ln P(t)(1+\varepsilon f(x-h)) \leq P(t)f(x-h).$$

Taking the infimum limit as $\|h\|_R \to 0$ with $h \in H_R \cap E$ and then letting $\varepsilon \to 0$ we get

$$P(t)f(x) \leq \liminf_{\substack{\|h\|_R \to 0\\ h\in H_R\cap E}} P(t)f(x-h). \tag{4.3.5}$$

Since $H_R \cap E$ is a linear space then applying (4.3.5) to $-h$ we get

$$P(t)f(x) \leq \liminf_{\substack{\|h\|_R \to 0\\ h\in H_R\cap E}} P(t)f(x+h). \tag{4.3.6}$$

By (4.3.4) and (4.3.6) we get (4.3.2).

Now a standard argument allows us to prove (iii) for a bounded Borel and positive function f with $\mu(f) = 1$. Writing (4.3.1) with $P^*(t)f$ in place of f, we get

$$[(P(t)f)(\ln P^*(t)f)](x) \leq \ln(P(t)P^*(t)f(y)) + \frac{e^{2t\zeta_R}}{t}\|x-y\|_R^2 \tag{4.3.7}$$

for any $t > 0$, $x, y \in \mathfrak{X}$ such that $x - y \in H_R \cap E$. Integrating both sides of (4.3.7) with respect to $\pi \in \mathscr{C}(f\mu, \mu)$ we get

$$\mu((P^*(t)f)(\ln P^*(t)f)) \leq \mu(\ln P(t)P^*(t)f) + \frac{e^{2t\zeta_R}}{t}\int_{\mathfrak{X}\times\mathfrak{X}} \|x-y\|_R^2 \pi(dx, dy).$$

To conclude it is sufficient to observe that the Jensen inequality yields that

$$\mu(\ln P(t)P^*(t)f) \leq \ln \mu(P(t)P^*(t)f) = \ln \mu(f) = 0,$$

whence the claim. □

Remark 4.3.2. *We stress that Corollary 4.3.1 remains true (with the constant $e^{2t\zeta_R}$ replaced by C_T^2) if we assume that the hypotheses of Theorem 4.1.9 hold true.*

Another classical consequence of (LHI) is a hypercontractivity type estimate for the semigroup $P(t)$ in L^p_μ. Such estimate relies on the Hölder inequality and some integrability conditions with respect to μ of some exponential functions.

Corollary 4.3.3. *Assume that the hypotheses of Theorem 4.1.9 hold true. If, in addition there* *exists $\varepsilon > 0$ such that*

$$\int_{\mathcal{X}}\int_{\mathcal{X}} e^{\varepsilon\|x-y\|_R^2}\mu(dx)\mu(dy) < +\infty, \tag{4.3.8}$$

then, for any $p \geq 2$, there exists $t_0 > 0$ and a positive constant C such that

$$\|P(t)f\|_{L^p_\mu} \leq C\|f\|_{L^2_\mu} \tag{4.3.9}$$

for any $t \geq t_0$ and any $f \in L^2_\mu$.

Proof. Let us consider $f \in L^2_\mu$ and $\vartheta \in (1,2)$. By (4.1.16) we deduce that for any $t > 0$

$$\begin{aligned}
\int_{\mathcal{X}} |P(t)f|^{2\vartheta}(x)\mu(dx) &= \int_{\mathcal{X}}\int_{\mathcal{X}} |P(t)f|^{2\vartheta}(x)\mu(dx)\mu(dy)\\
&= \int_{\mathcal{X}}\int_{\mathcal{X}} |P(t)f(x)|^{\vartheta}(|P(t)f(x)|^2)^{\vartheta/2}\mu(dx)\mu(dy)\\
&\leq \int_{\mathcal{X}}\int_{\mathcal{X}} |P(t)f(x)|^{\vartheta}(P(t)f^2(y))^{\vartheta/2}e^{\frac{\vartheta C_T^2}{t}\|x-y\|_R^2}\mu(dx)\mu(dy)\\
&=: \int_{\mathcal{X}}\int_{\mathcal{X}} h(x,y)g(x,y)\mu(dx)\mu(dy),
\end{aligned}$$

where $h(x,y) := |P(t)f(x)|^{\vartheta}(P(t)f^2(y))^{\vartheta/2}$ and $g(x,y) := e^{\frac{\vartheta C_T^2}{t}\|x-y\|_R^2}$. Applying the Hölder inequality with respect to the measure $\mu\otimes\mu$ we get

$$\int_{\mathcal{X}}\int_{\mathcal{X}} h(x,y)g(x,y)\mu(dx)\mu(dy) \leq \|h\|_{L^{2/\vartheta}_{\mu\otimes\mu}}\|g\|_{L^{2/(2-\vartheta)}_{\mu\otimes\mu}}.$$

Now, the invariance of μ and the contractivity of $P(t)$ in L^2_μ allow us to estimate

$$\|h\|_{L^{2/\vartheta}_{\mu\otimes\mu}} = \|P(t)f\|^{\vartheta}_{L^2_\mu}\|f\|^{\vartheta}_{L^2_\mu} \leq \|f\|^{2\vartheta}_{L^2_\mu}.$$

Moreover, being

$$\|g\|^{2/(2-\vartheta)}_{L^{2/(2-\vartheta)}_{\mu\otimes\mu}} = \int_{\mathcal{X}}\int_{\mathcal{X}} e^{\frac{2\vartheta}{2-\vartheta}\frac{C_T^2}{t}\|x-y\|_R^2}\mu(dx)\mu(dy) =: C(\vartheta,t)$$

condition (4.3.8) ensures that there exists $\bar{t} > 0$ such that $C(\vartheta,t) < +\infty$ for any $t \geq \bar{t}$ and any $\vartheta \in (1,2)$. Consequently

$$\|P(t)f\|_{L^{2\vartheta}_\mu} \leq (C(\vartheta,t))^{\frac{2-\vartheta}{4\vartheta}}\|f\|_{L^2_\mu}.$$

i.e., $P(t)$ maps L^2_μ into $L^{2\vartheta}_\mu$ for $t \geq \bar{t}$. Since $\vartheta > 1$, $P(t)$ actually improves summability of the initial datum when $t \geq \bar{t}$. To go further we use the semigroup law. Indeed, if $f \in L^2_\mu$, then $P(\bar{t})f \in L^{2\vartheta}_\mu$, i.e. $|P(\bar{t})f|^{\vartheta} \in L^2_\mu$. Using again the first part of the proof, we deduce that $P(t)|P(\bar{t})f|^{\vartheta} \in L^{2\vartheta}_\mu$ for $t \geq \bar{t}$. Since, by the Jensen inequality and the positivity of $P(t)$ we can estimate

$$+\infty > \|P(t)|P(\bar{t})f|^{\vartheta}\|^{2\vartheta}_{L^{2\vartheta}_\mu} = \int_{\mathcal{X}} |P(t)|P(\bar{t})f|^{\vartheta}|^{2\vartheta}d\mu$$

$$
\begin{aligned}
&\geq \int_{\mathcal{X}} |P(t)|P(\bar{t})f||^{2\vartheta^2} d\mu \\
&= \int_{\mathcal{X}} (P(t)|P(\bar{t})f|)^{2\vartheta^2} d\mu \\
&\geq \int_{\mathcal{X}} |P(t)P(\bar{t})f|^{2\vartheta^2} d\mu \\
&= \int_{\mathcal{X}} |P(t+\bar{t})f|^{2\vartheta^2} d\mu, \qquad t \geq \bar{t}
\end{aligned}
$$

we infer that $P(t)$ maps L^2_μ into $L^{2\vartheta^2}_\mu$ for any $t \geq 2\bar{t}$. Iterating this procedure we can prove that for any $p > 2$ there exists $t_0 = t_0(p) > 0$ such that $P(t)$ maps L^2_μ into L^p_μ for any $t \geq t_0$ and estimate (4.3.9) holds true. □

Remark 4.3.4. *Note that the result in Corollary 4.3.3 continue to hold true if we assume that the hypotheses of Theorem 4.2.9 (with constant $\zeta_R < 0$) hold and that* (4.3.8) *is satisfied with $\varepsilon = 1$.*

4.3.2 An example in $L^2([0,1],\lambda)$

Hypotheses 4.1.2(2) and 4.1.5 are verified in the case considered in Subsection 3.4.2. Now we present some examples that satisfy the other sets of Hypotheses.

Let $\mathcal{X} = L^2([0,1],\lambda)$ where λ is the Lebesgue measure. Let $-Q^{-1}$ be the realization of the second order derivative in $L^2([0,1],\lambda)$ with Dirichlet boundary condition. Hence Q is a positive and trace class operator. Let $A = -(1/2)Q^{-\beta}$ and $R = Q^\alpha$, with $\alpha, \beta \geq 0$ such that (4.1.1) is verified. The constant λ_1 in (3.4.1) and (3.4.2) is equal to π^{-2} and A generates a strongly continuous analytic semigroup e^{tA} such that

$$
\left\| e^{tA} \right\|_{\mathcal{L}(\mathcal{X})} \leq e^{-\frac{1}{2}\pi^2 t}, \quad t \geq 0.
$$

see [30, Chapter 4]. Moreover by Proposition [73, Proposition 2.1.1], if $\alpha < \beta$ then (4.1.2)(1) is verified with $\gamma = \frac{\alpha}{\beta}$ and $\epsilon = 0$. Since Q is a trace class operator then Hypotheses 4.1.5 are verified.

4.3.3 Infinite dimensional polynomial

Let $E = \mathcal{X} = L^2([0,1],\lambda)$ and A and R as in Subsection 4.3.2 with $\alpha = \frac{1}{2}$ and $\beta > \frac{1}{2}$. Since $\alpha = \frac{1}{2}$ and $-Q^{-1}$ is be the realization in $L^2([0,1],\lambda)$ of the second order derivative with Dirichlet boundary condition, then $H_R = W^{1,2}_0([0,1],\lambda)$ where λ is the Lebesgue measure and (4.1.1) is verified. We take F as in Subsection 2.4.1. In addition we assume that K has weak derivative with respect to the fourth variable, such that

$$
\frac{\partial K}{\partial \xi} \in L^2([0,1]^4,\lambda).
$$

Let $f \in W^{1,2}([0,1],\lambda)$ we have $F(f) = P_3(f) + \zeta_2 f \in W^{1,2}([0,1],\lambda)$(see (2.4.6)) and its weak derivative is

$$(F(f))' = \int_0^1 \int_0^1 \int_0^1 \frac{\partial K}{\partial \xi}(\xi_1, \xi_2, \xi_3, \xi) f(\xi_1) f(\xi_2) f(\xi_3) d\xi_1 d\xi_2 d\xi_3 + \zeta_2 f'. \tag{4.3.10}$$

If we assume that $(\partial K/\partial \xi) \in L^2([0,1]^4, \lambda)$ is symmetric (see (2.4.1)) and it has negative value, then by (4.3.10), (2.4.4) and (2.4.5) are verified in $W^{1,2}([0,1],\lambda)$. Hence Hypotheses 4.2.4 is verified. Clearly, by the choice of A and R, Hypotheses 4.2.6 is verified, so Theorem 4.2.9 can be applied.

4.3.4 A reaction-diffusion system

Assume that $\mathcal{X} = L^2([0,1],\lambda)$ (where λ is the Lebesgue measure), $E = C([0,1])$, A is the realization in $L^2([0,1],\lambda)$ of the second order derivative operator with Dirichlet boundary condition and $R = \mathrm{I}_{\mathcal{X}}$. In order to define the function F we consider a decreasing function $\varphi \in C^1(\mathbb{R})$ such that

$$|\varphi'(\xi)| \le d_1(1 + |\xi|^m), \qquad \xi \in \mathbb{R},$$

for some constants $d_1 > 0$ and $m \in \mathbb{N}$. Let $\zeta_F > 0$. We set

$$[F(f)](\xi) = \begin{cases} \varphi(f(\xi)) - \frac{\zeta_F}{2} f(\xi)^2, & f \in C([0,1]),\ \xi \in [0,1]; \\ 0, & \text{otherwise.} \end{cases}$$

By [20, Section 6.1, Lemma 6.1.2 and Lemma 8.2.1] and [43, Example D.7] it follows that Hypotheses 4.2.4 are verified. Finally, taking into account that $-A^{-1}$ is a positive and trace class operator and $R = \mathrm{I}_{\mathcal{X}}$ we can conclude that Hypotheses 4.1.5 are verified too and so Theorem 4.2.9 can be applied.

Chapter 5

Behavior in L^p type spaces

In this chapter, under Hypotheses 2.1.1, we study the transition semigroup $P(t)$ (see Definition 2.2.3) in L^p type spaces, with $p \geq 1$.

As in the Ornstein–Uhlenbeck case, the best setting are the spaces $L^p(\mathcal{X}, \nu)$ where ν is an invariant measure of $P(t)$. Indeed, it is easy to see that if $P(t)$ has an invariant measure ν, then it is extendable to a strongly continuous contraction semigroup $P_p(t)$ in $L^p(\mathcal{X}, \nu)$, for every $p \geq 1$. Particular attention will be paid to the case $p = 2$. Denoting by N_2 the infinitesimal generator of $P_2(t)$ we will find out a core of regular functions (the space $\xi_A(\mathcal{X})$ defined in (1.10.5)) on which N_2 has an explicit expression as a perturbation of the operator defined in (1.10.6)

$$N_0\varphi(x) := L_0\varphi(x) + \langle F_0(x), \nabla\varphi(x)\rangle, \quad \varphi \in \xi_A(\mathcal{X}),\ x \in \mathcal{X}, \tag{5.0.1}$$

where

$$F_0(x) = \begin{cases} F(x) & x \in E, \\ 0 & x \in \mathcal{X}\backslash E. \end{cases}$$

Preliminarily, we prove that $\nu(E) = 1$ and ν has finite moments of every order.

5.1 Existence and Uniqueness of the invariant measure

In this Section we are going to prove that the semigroup $P(t)$ has a unique invariant measure ν verifying some useful properties. To do this we need an additional hypothesis.

Hypotheses 5.1.1. *Assume that Hypotheses 2.1.1 hold true. Moreover we assume that the constant ζ in Hypothesis 2.1.1*(iv) *is negative and that*

$$\sup_{t\geq 0} \mathbb{E}[\|W_A(t)\|_E^p] < +\infty, \quad \forall\, p \geq 1. \tag{5.1.1}$$

By Hypotheses 2.1.1(vi) and (5.1.1) we have

$$\Sigma_{p,E} := \sup_{t\geq 0} \mathbb{E}[\|F(W_A(t))\|_E^p + \|W_A(t)\|_E^p] < +\infty, \quad \forall\, p \geq 1,$$

and, since E is continuously embedded in $\mathcal{X}$, we have

$$\Sigma_{p,\mathcal{X}} := \sup_{t\geq 0} \mathbb{E}[\|F(W_A(t))\|^p + \|W_A(t)\|^p] < +\infty, \quad \forall\, p \geq 1.$$

For any $p \geq 1$ we set

$$\Sigma_p := \max\{\Sigma_{p,\mathcal{X}}, \Sigma_{p,E}\}. \tag{5.1.2}$$

By (2.1.22), (2.2.2), (5.1.2) and Corollary 2.1.11 we obtain the following result.

Proposition 5.1.2. *Assume that Hypotheses 5.1.1 hold true and let $\{X(t,x)\}_{t\geq 0}$ be the generalized mild solution of (2.0.1). If $x \in \mathcal{X}$ then $\{X(t,x)\}_{t\geq 0} \in \mathcal{X}^p([0,\infty))$, for any $p \geq 1$, if $x \in E$ then $\{X(t,x)\}_{t\geq 0} \in E^p([0,\infty))$, for any $p \geq 1$; (see Definition 2.1.5). In particular, for any $p \geq 1$, there exists $K_p := K_p(\Sigma_p, C'_p)$ (where C'_p is the constant of Theorem 2.1.9), such that*

$$\begin{aligned} &\mathbb{E}[\|X(t,x)\|^p] \leq K_p(1 + e^{\kappa_p t}\|x\|^p), \quad \forall\, t > 0, \forall\, x \in \mathcal{X}, \\ &\mathbb{E}[\|X(t,x)\|_E^p] \leq K_p(1 + e^{\kappa_p t}\|x\|_E^p), \quad \forall\, t > 0, \forall\, x \in E. \end{aligned} \tag{5.1.3}$$

where $\kappa_p < 0$ is the constant of Proposition 2.1.7.

The semigroups $P(t)$ and $P^E(t)$ are two transition semigroups. In particular the transition probabilities of $P^E(t)$ are determined by the unique mild solution of (2.0.1) in the following way,

$$p_t^E(x,\cdot) = \mathscr{L}(X(t,x))(\cdot), \quad t \geq 0, x \in E.$$

The transition probabilities $p_t(x,\cdot)$ of $P(t)$ are determined in the same way by the unique generalized mild solution of (2.0.1). Moreover, by the definition of generalized mild solution, for any $x \in E$ and $t \geq 0$ we have $p_t(x,\cdot) = p_t^E(x,\cdot)$, so we denote both by $p_t(x,\cdot)$.

Theorem 5.1.3. *Assume that Hypotheses 5.1.1 hold true. There exists $\nu \in \mathscr{P}(\mathcal{X})$ such that it is the unique invariant measure of both semigroups $P^E(t)$ and $P(t)$. Moreover $\nu(E) = 1$ and it satisfies*

$$\int_{\mathcal{X}} \|x\|^p \nu(dx) < +\infty, \quad \forall\, p \geq 1, \tag{5.1.4}$$

$$\int_E \|x\|_E^p \nu(dx) < +\infty, \quad \forall\, p \geq 1. \tag{5.1.5}$$

Moreover we have

$$\lim_{t\to+\infty} P(t)\varphi(x) = \int_{\mathcal{X}} \varphi(y)\nu(dy), \quad \varphi \in C_b(\mathcal{X}),\ x \in \mathcal{X}, \tag{5.1.6}$$

$$\lim_{t\to+\infty} P(t)\varphi(x) = \int_E \varphi(y)\nu(dy), \quad \varphi \in C_b(E),\ x \in E. \tag{5.1.7}$$

Proof. To prove existence and uniqueness of the invariant measures ν and ν^E of $P(t)$ and $P^E(t)$

respectively we exploit similar arguments to [43, Theorems 11.33-11.34].

We begin to prove that $P^E(t)$ has a unique invariant measure. By [43, Proposition 11.1], for any $x \in E$ and $t \in [0,T]$, we have

$$U(t)\delta_x := \int_{\mathcal{X}} p_t(x,\cdot)\delta_x(dx) = \mathscr{L}(X(t,x)).$$

Since $P^E(t)$ is Feller, by Proposition 1.9.10, if there exists $\nu^E \in \mathscr{P}(E)$ such that, for any $x \in E$, we have

$$U(t)\delta_x := \mathscr{L}(X(t,x)) \to_* \nu^E, \quad \text{as } t \to +\infty, \tag{5.1.8}$$

then ν^E is the unique invariant measure of $P^E(t)$. To prove (5.1.8) we consider the SPDE (2.0.1) with an arbitrary $s \in \mathbb{R}$ instead of 0 as initial time. Let $\{W'(t)\}_{t\geq 0}$ be another $\mathcal{X}$-cylindrical Wiener process independent of $\{W(t,x)\}_{t\geq 0}$. For any $t \in \mathbb{R}$ we define the process

$$\widehat{W}(t) := \begin{cases} W(t) & t \geq 0 \\ W'(-t) & t < 0. \end{cases}$$

For any $s \in \mathbb{R}$ and $x \in \mathcal{X}$, we consider the SPDE

$$\begin{cases} dX(t,s,x) = \big(AX(t,s,x) + F(X(t,s,x))\big)dt + Cd\widehat{W}(t), & t \geq s, \\ X(s,s,x) = x, \end{cases} \tag{5.1.9}$$

We emphasize that the method used to prove Theorem 2.1.13 and to define the generalized mild solution (see Corollary 2.2.1) also works by replacing the initial time 0 by an arbitrary $s \in \mathbb{R}$. Hence, for any $x \in \mathcal{X}$ and $s \in \mathbb{R}$, the SPDE (5.1.9) has a unique generalized mild solution $\{X(t,s,x)\}_{t\geq 0}$. Moreover, as in Proposition 5.1.2, for any $p \geq 1$ we have the following estimates in $\mathcal{X}$

$$\begin{aligned} &\mathbb{E}[\|X(t,s,x)\|_E^p] \leq K_p(1 + e^{\kappa_p(t-s)}\|x\|_E^p), \quad t \geq s, x \in \mathcal{X} \\ &\mathbb{E}[\|X(t,s,x) - X(t,s,z)\|] \leq e^{\eta(t-s)}\|x-z\|, \quad t \geq s, x, z \in \mathcal{X}, \end{aligned}$$

and the following in E

$$\mathbb{E}[\|X(t,s,x)\|_E^p] \leq K_p(1 + e^{\kappa_p(t-s)}\|x\|_E^p), \quad t \geq s, x \in E \tag{5.1.10}$$

$$\mathbb{E}[\|X(t,s,x) - X(t,s,z)\|_E] \leq e^{\eta(t-s)}\|x-z\|, \quad t \geq s, x, z \in E \tag{5.1.11}$$

where κ_p is the constant of Proposition 2.1.7, η is the constant of Proposition 2.1.10 and K_p is the constant of Proposition 5.1.2. By Corollary 2.1.11 the constants η and κ are negative.

Now we prove that there exists a random variable $\xi \in L^2((\Omega,\mathbb{P}),E)$, such that, for any $x \in E$, we have

$$\lim_{s\to+\infty} \mathbb{E}\left[\|X(0,-s,x) - \xi\|_E^2\right] = 0, \tag{5.1.12}$$

and after we will prove that the law of ξ is the measure ν^E that verifies (5.1.8).

We can assume that $\{X(t,s,x)\}_{t\geq s}$ is a strict solution of (5.1.9), otherwise we approximate it as in Proposition 2.1.7. For $\mathbb{P}$-a.a. $\omega \in \Omega$, for any $x \in E$, $s \in \mathbb{R}$, $t \geq s$ and $h \in [s,t]$, taking into account (1.3.1) and by Hypotheses 2.1.1(iv), there exist $z^* \in \partial(\|X(t,s,x)(\omega) - X(t,h,x)(\omega)\|_E)$ such that

$$\begin{aligned}\frac{1}{2}\frac{d^-\|X(t,s,x)-X(t,h,x)\|_E}{dt} &= {}_E\langle A(X(t,s,x)-X(t,h,x)), z^*\rangle_{E^*}\\ &+ {}_E\langle F(X(t,s,x)) - F(X(t,h,x)), z^*\rangle_{E^*}\\ &\leq -\zeta\|X(t,s,x)(\omega) - X(t,h,x)(\omega)\|_E.\end{aligned}$$

By (1.3.2), taking the expectation we obtain

$$\begin{aligned}\mathbb{E}\left[\|X(t,s,x)-X(t,h,x)\|_E^2\right] &\leq e^{-4\zeta(t-h)}\mathbb{E}\left[\|X(h,s,x)-x\|_E^2\right]\\ &\leq 2e^{-4\zeta(t-h)}(\mathbb{E}\left[\|X(h,s,x)\|_E^2\right] + \|x\|_E^2),\end{aligned}$$

and so

$$\mathbb{E}\left[\|X(t,s,x)-X(t,h,x)\|_E^2\right] \leq e^{-4\zeta(t-h)}C_x, \tag{5.1.13}$$

where $C_x := 2\sup_{r\geq s}\left(\mathbb{E}\left[\|X(r,s,x)\|_E^2\right]\right) + 2\|x\|_E^2$ is finite by (5.1.10). For any $x \in \mathcal{X}$, by (5.1.13), when t goes to $+\infty$ the family $\{X(0,-t,x)\}_{t\geq 0}$ is Cauchy in $L^2((\Omega,\mathcal{F},\mathbb{P}),(E,\mathcal{B}(E)))$, namely

$$\lim_{s,t\to+\infty}\mathbb{E}\left[\|X(0,-t,x)-X(0,-s,x)\|_E^2\right] = 0$$

Since $L^2((\Omega,\mathbb{P}),E)$ is complete, then $\{X(0,-t,x)\}_{t\geq 0}$ converges in $L^2((\Omega,\mathbb{P}),E)$ and by (5.1.11) its limit does not depend on x, so (5.1.12) is verified. Let $\nu^E = \mathscr{L}(\xi)$, where ξ is the random variable that verifies (5.1.12). We prove that it verifies (5.1.8). Since $\{W'(t)\}_{t\geq 0}$ and $\{W(t)\}_{t\geq 0}$ are $\mathcal{X}$-cylindrical Wiener processes; they have the same law, and so, for any $x \in E$ and $t \geq 0$, we have

$$\mathscr{L}(X(t,x)) = \mathscr{L}(X(0,-t,x)).$$

Let $\varphi \in C_b(E)$. For any $x \in \mathcal{X}$, $t \geq 0$, we have

$$\begin{aligned}\int_{\mathcal{X}}\varphi(y)p_t(x,dy) = \int_{\mathcal{X}}\varphi(y)\mathscr{L}(X(t,x))(dy) &= \int_{\mathcal{X}}\varphi(y)\mathscr{L}(X(0,-t,x))(dy)\\ = \int_{\Omega}\varphi(X(0,-t,x)(\omega))\mathbb{P}(d\omega).\end{aligned}$$

Since $\varphi \in C_b(E)$, by (5.1.12) and the dominated convergence theorem we have

$$\begin{aligned}\lim_{t\to+\infty}\int_{\mathcal{X}}\varphi(y)p_t(x,dy) &= \lim_{t\to+\infty}\int_{\Omega}\varphi(X(0,-t,x)(\omega))\mathbb{P}(d\omega)\\ &= \int_{\Omega}\varphi(\xi(\omega))\mathbb{P}(d\omega) = \int_{\mathcal{X}}\varphi(y)\nu^E(dy),\end{aligned} \tag{5.1.14}$$

hence (5.1.8) is verified and so, by Proposition 1.9.10, the measure ν^E is the unique invariant measure of the transition semigroup $P^E(t)$. (5.1.7) follows immediately by the definition of transition semigroup $P^E(t)$ and (5.1.14). Now we prove (5.1.5). For $p \geq 1$ and $b > 0$ we have

$$\int_E \frac{\|y\|_E^p}{1 + b\|y\|_E^p} p_t(x, dy) \leq \int_E \|y\|_E^p p_t(x, dy) = \mathbb{E}[\|X(t, x)\|_E^p].$$

Then, by (5.1.3), (5.1.7) and the monotone convergence theorem, we conclude

$$\int_{\mathcal{X}} \|y\|_E^p \nu^E(dy) = \lim_{b \to 0} \lim_{t \to +\infty} \int_E \frac{\|y\|_E^p}{1 + b\|y\|_E^p} p_t(x, dy) < +\infty.$$

In the same way, we can prove that the semigroup $P(t)$ has a unique invariant measure ν that verifies (5.1.4) and (5.1.6).

[13, Lemma 2.1.1] claim that

$$\mathcal{B}(E) = \{E \cap B \ : \ B \in \mathcal{B}(\mathcal{X})\},$$

so the measure ν' defined by

$$\nu'(\Gamma) = \nu^E(\Gamma \cap E), \quad \Gamma \in \mathcal{B}(\mathcal{X}),$$

is a Borel measure. Let $f \in C_b(\mathcal{X})$. Taking into account Corollary 2.2.5 and that ν^E is the invariant measure of $P^E(t)$ we have

$$\int_{\mathcal{X}} P(t)f(x)\nu'(dx) = \int_E (P(t)f)(x)\nu^E(dx) = \int_E f(x)\nu^E(dx) = \int_{\mathcal{X}} f(x)\nu'(dx),$$

hence ν' is invariant for the semigroup $P(t)$. By the uniqueness we conclude $\nu = \nu'$. □

Remark 5.1.4. *Theorem 5.1.3 yields that $\nu(H) = 1$, for any $H \subseteq \mathcal{X}$ satisfying Hypotheses 5.1.1 (see the example of subsection 5.4).*

Remark 5.1.5. *In some specific settings it is possible to prove Theorem 5.1.3 replacing the condition $\zeta < 0$ by other hypotheses on F (e.g. [20, Chapter 8]).*

Now, thanks to the invariance of ν, we can prove that the transition semigroup $P(t)$ is uniquely extendable to a contraction strongly continuous semigroup $P_p(t)$ in $L^p(\mathcal{X}, \nu)$, for any $p \geq 1$.

We recall a couple of inequalities that follow immediately from the Hölder and Jensen inequalities. For every $\varphi, \psi \in B_b(\mathcal{X})$, $t > 0$, $p, q \in [1, \infty]$, such that $1/p + 1/q = 1$ (with the usual convention that if $p = 1$, then $q = \infty$, and viceversa) and for any convex function $f : \mathbb{R} \to \mathbb{R}$ we have

$$|P(t)\varphi\psi| \leq (P(t)|\varphi|^q)^{1/q}(P(t)|\psi|^p)^{1/p}; \tag{5.1.15}$$

$$(f \circ P(t))\varphi \leq P(t)(f \circ \varphi); \tag{5.1.16}$$

Proposition 5.1.6. *The transition semigroup $P(t)$ is uniquely extendable to a contraction strongly continuous semigroup $P_p(t)$ in $L^p(\mathcal{X}, \nu)$, for any $p \geq 1$.*

Proof. Let $p \geq 1$. By (5.1.16) and the invariance of ν, for any $\varphi \in C_b(\mathcal{X})$ we have

$$\|P(t)\varphi\|^p_{L^p(\mathcal{X},\nu)} = \int_{\mathcal{X}} |P(t)\varphi|^p d\nu \leq \int_{\mathcal{X}} P(t)(|\varphi|^p) d\nu \leq \int_{\mathcal{X}} |\varphi|^p d\nu = \|\varphi\|^p_{L^p(\mathcal{X},\nu)}. \tag{5.1.17}$$

We know that, for any $\varphi \in C_b(\mathcal{X})$ and $x \in \mathcal{X}$

$$\lim_{t\to 0^+} P(t)\varphi(x) = \varphi(x),$$

so by the dominated convergence theorem we have

$$\lim_{t\to 0^+} \|P(t)\varphi - \varphi\|_{L^p(\mathcal{X},\nu)} = 0 \tag{5.1.18}$$

We recall that $C_b(\mathcal{X})$ is dense in $L^p(\mathcal{X},\nu)$. Observe that if $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq C_b(\mathcal{X})$ converges to φ in $L^p(\mathcal{X},\nu)$, then for any $t \geq 0$ the sequence $\{P(t)\varphi_n\}_{n\in\mathbb{N}}$ is Cauchy in $L^p(\mathcal{X},\nu)$. Indeed, by (5.1.17), we have

$$\|P(t)\varphi_n - P(t)\varphi_m\|_{L^p(\mathcal{X},\nu)} \leq \|\varphi_n - \varphi_m\|_{L^p(\mathcal{X},\nu)}.$$

Hence the transition semigroup $P(t)$ is uniquely extendable to a semigroup $P_p(t)$ in $L^p(\mathcal{X},\nu)$. By (5.1.17) and (5.1.18), $P_p(t)$ is contractive and strongly continuous. $\square$

Definition 5.1.7. *We denote by N_2 the infinitesimal generator of $P_2(t)$.*

Remark 5.1.8. *In a similar way, it is possible to prove that the semigroup $P^E(t)$ is uniquely extendable to a strongly continuous semigroup $P_2^E(t)$ in $L^2(E,\nu)$. In the rest of this paper we will not study $P_2^E(t)$ but only $P_2(t)$. However it is possible to prove a result analogous to Theorem 5.3.3 for $P_2^E(t)$ (see [23]).*

5.2 Behavior on $\xi_A(\mathcal{X})$

In this section we study the behavior of N_2 on the space $\xi_A(\mathcal{X})$ defined in (1.10.5). In particular we will prove that N_2 coincides with N_0 (defined in (5.0.1)) on $\xi_A(\mathcal{X})$.

Remark 5.2.1. *In this section we will study the behaviour of N_0 in $L^2(\mathcal{X},\nu)$, hence is not significant how we define F in $\mathcal{X}\backslash E$, since by Theorem 5.1.3, $\nu(E) = 1$.*

For every $\varphi \in \xi_A(\mathcal{X})$, there exist $m, n \in \mathbb{N}$, $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{R}$ and $h_1, \ldots, h_m, k_1, \ldots, k_n \in A^*$ such that

$$\varphi(x) = \sum_{i=1}^m a_i \sin(\langle x, h_i\rangle) + \sum_{j=1}^n b_j \cos(\langle x, k_j\rangle).$$

Easy computations give for $x \in \mathcal{X}$

$$N_0\varphi(x) = \sum_{i=1}^m a_i \left(\langle x, Ah_i\rangle + \langle F_0(x), h_i\rangle - \frac{1}{2}\|Rh_i\| \right) \sin(\langle x, h_i\rangle)$$

$$+\sum_{j=1}^{n} b_j\left(\langle x, Ak_j\rangle + \langle F_0(x), k_j\rangle - \frac{1}{2}\|Rk_j\|\right)\cos(\langle x, k_j\rangle),$$

moreover by Hypothesis 2.1.1(vi) and Theorem 5.1.3 we have

$$\int_{\mathcal{X}} \|F_0(x)\|^p d\nu(x) < +\infty, \quad \forall\, p \geq 1, \tag{5.2.1}$$

and so $N_0\varphi$ belongs to $L^2(\mathcal{X}, \nu)$.

Proposition 5.2.2. *Assume that Hypotheses 5.1.1 hold true. N_0 is closable in $L^2(\mathcal{X}, \nu)$ and its closure $\overline{N}_0$ is dissipative in $L^2(\mathcal{X}, \nu)$. Moreover N_2 is an extension of $\overline{N}_0$, namely* $\mathrm{Dom}(\overline{N}_0) \subseteq \mathrm{Dom}(N_2)$ *and*

$$\overline{N}_0\varphi = N_2\varphi, \quad \varphi \in \mathrm{Dom}(\overline{N}_0). \tag{5.2.2}$$

Proof. By Theorem 2.1.13, for any $x \in E$, the trajectories of $\{X(t, x)\}_{t\geq 0}$ take values in E. So by [30, Proof of Theorem 3.19], for any $\varphi \in \xi_A(\mathcal{X})$ and $x \in E$, we have

$$\begin{aligned} P_2(t)\varphi(x) = \mathbb{E}[\varphi(X(t, x))] &= \varphi(x) + \mathbb{E}\left[\int_0^t N_0\varphi(X(s, x))ds\right] \\ &= \varphi(x) + \int_0^t P(s)N_0\varphi(X(s, x))ds, \end{aligned} \tag{5.2.3}$$

and so

$$\lim_{t\to 0} \frac{P_2(t)\varphi(x) - \varphi(x)}{t} = N_0\varphi(x). \tag{5.2.4}$$

To obtain (5.2.2) we need to prove that

$$\lim_{t\to 0} \int_{\mathcal{X}} \left\|\frac{P_2(t)\varphi(x) - \varphi(x)}{t} - N_0\varphi(x)\right\|^2 \nu(dx) = 0, \quad \forall\, \varphi \in \xi_A(\mathcal{X}). \tag{5.2.5}$$

We recall the Vitali convergence theorem (see [60, Theorem 2.24]): (5.2.5) is verified if and only if the following three conditions are verified.

1. There exists $B \in \mathcal{B}(\mathcal{X})$ such that $\nu(B) = 1$ and $\{\frac{P(t)\varphi(x)-\varphi(x)}{t}\}_{t\geq 0}$ converges for any $x \in B$.
2. For any $\varepsilon > 0$ there exists $\Gamma \in \mathcal{B}(\mathcal{X})$ such that $\nu(\Gamma) < +\infty$ and
$$\frac{1}{t^2}\int_{(\mathcal{X}-\Gamma)} |P(t)\varphi(x) - \varphi(x))|^2\nu(dx) \leq \varepsilon \quad \forall\, t > 0.$$
3. For any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\Gamma \in \mathcal{B}(\mathcal{X})$ with $\nu(\Gamma) < \delta$ we have
$$\frac{1}{t^2}\int_{\Gamma} |P(t)\varphi(x) - \varphi(x))|^2\nu(dx) \leq \varepsilon \quad \forall\, t > 0.$$

By (5.2.4) and $\nu(E) = 1$, (1) is verified. Since ν is a probability measure then (3) implies (2). We prove (3). We fix $\varepsilon > 0$. Since $N_0\varphi \in L^2(\mathcal{X}, \nu)$, there exists $\delta > 0$ such that whenever $\Gamma \in \mathcal{B}(\mathcal{X})$ with $\nu(\Gamma) < \delta$, then

$$\int_{\Gamma} |N_0\varphi(x)|^2\nu(dx) < \varepsilon.$$

Recalling that $\nu(E) = 1$, by the Hölder inequality, the invariance of $P(t)$ with respect to ν and (5.2.3) we have

$$\begin{aligned}
\frac{1}{t^2}\int_\Gamma |P(t)\varphi(x) - \varphi(x))|^2\nu(dx) &= \frac{1}{t^2}\int_{\Gamma\cap E} |P(t)\varphi(x) - \varphi(x))|^2\nu(dx)\\
&= \int_{\Gamma\cap E}\left|\int_0^t P(s)N_0\varphi(x)\frac{ds}{t}\right|^2\nu(dx)\\
&\le \frac{1}{t}\int_0^t\left(\int_{\Gamma\cap E}|P(s)(N_0\varphi)(x)|^2\nu(dx)\right)ds\\
&\le \frac{1}{t}\int_0^t\left(\int_{\Gamma\cap E}P(s)(|N_0\varphi|^2)(x)\nu(dx)\right)ds\\
&= \frac{1}{t}\int_0^t\left(\int_{\Gamma\cap E}|N_0\varphi(x)|^2\nu(dx)\right)ds = \frac{1}{t}\int_0^t \varepsilon ds = \varepsilon.
\end{aligned}$$

Hence, by the Vitali convergence theorem, we obtain (5.2.5) and so (5.2.2). In particular, since ν is the invariant measure of $P_2(t)$, for any $\varphi \in \xi_A(\mathcal{X})$, we have

$$\int_{\mathcal{X}} N_0\varphi d\nu = \int_E N_2\varphi d\nu = 0. \tag{5.2.6}$$

Moreover by standard calculations we obtain

$$N_0\varphi^2(x) = 2\varphi(x)N_0\varphi(x) + \|R\nabla\varphi(x)\|^2.$$

Hence integrating with respect to ν and exploiting (5.2.6) we get

$$\int_{\mathcal{X}}(N_0\varphi(x))\varphi(x)\nu(dx) = -\frac{1}{2}\int_{\mathcal{X}}\|R\nabla\varphi(x)\|^2\nu(x), \quad \forall\, \varphi \in \xi_A(\mathcal{X}),$$

so, since $\xi_A(\mathcal{X})$ is dense in $L^2(\mathcal{X},\nu)$, N_0 is closable in $L^2(\mathcal{X},\nu)$ and its closure $\overline{N}_0$ is dissipative in $L^2(\mathcal{X},\nu)$. □

We conclude this subsection with a useful criterium to check ff a function $\varphi : \mathcal{X} \to \mathbb{R}$ belongs to $\mathrm{Dom}(\overline{N}_0)$.

Lemma 5.2.3. *Assume that Hypotheses 5.1.1 hold true. If $\varphi \in \mathrm{Dom}(L_{b,2}) \cap C_b^1(\mathcal{X})$, then $\varphi \in \mathrm{Dom}(\overline{N}_0)$ and*

$$\overline{N}_0\varphi(x) = L_{b,2}\varphi(x) + \langle F_0(x), \nabla\varphi(x)\rangle, \qquad x \in \mathcal{X};$$

where $L_{b,2}$ is the operator introduced in Theorem 1.10.1.

Proof. By Proposition 1.10.7 a family $\{\varphi_{n_1,n_2,n_3,n_4} \,|\, n_1, n_2, n_3, n_4 \in \mathbb{N}\} \subseteq \xi_A(\mathcal{X})$ exists such that, for any $x \in \mathcal{X}$,

$$\lim_{n_1\to+\infty}\lim_{n_2\to+\infty}\lim_{n_3\to+\infty}\lim_{n_4\to+\infty}\overline{N_0}\varphi_{n_1,n_2,n_3,n_4}(x) = L_{b,2}\varphi(x) + \langle F(x), \nabla\varphi(x)\rangle.$$

whenever $\varphi \in \mathrm{Dom}(L_{b,2}) \cap C_b^1(\mathcal{X})$. By (1.10.7), there exists a constant C_φ, such that for any

$x \in E$

$$|\overline{N_0}\varphi_{n_1,n_2,n_3,n_4}(x)| = |N_0\varphi_{n_1,n_2,n_3,n_4}(x)| \leq C_\varphi(1+\|x\|^{m+2})(1+\|F(x)\|^2),$$

so, since $\nu(E) = 1$, by (5.2.1), (5.1.4) and the Dominated Convergence theorem we obtain the statement. □

5.3 A core for N_2

To prove the main results of this chapter we need an additional hypothesis and a general result about closed operators. Since we were unable to find an appropriate reference in the literature we provide its proof.

Hypotheses 5.3.1. *Assume that Hypotheses 5.1.1 hold true and that there exists a constant $\zeta_2 \in \mathbb{R}$ such that $F - \zeta_2 \mathrm{I}_{\mathcal{X}} : \mathrm{Dom}(F) \subset \mathcal{X} \to \mathcal{X}$ is m-dissipative.*

Proposition 5.3.2. *Let Y be a Banach space and let $B_1 : \mathrm{Dom}(B_1) \subseteq Y \to Y$ and $B_2 : \mathrm{Dom}(B_2) \subseteq Y \to Y$ be two, possibly unbounded, operators. If*

(i) *B_1 is an extension of B_2, namely $\mathrm{Dom}(B_2) \subseteq \mathrm{Dom}(B_1)$ and, for any $x \in \mathrm{Dom}(B_2)$, it holds $B_2 x = B_1 x$;*

(ii) *there exists a dense subset D of Y such that, for some $\lambda > 0$, $R(\lambda, B_1)$ and $R(\lambda, B_2)$ are well defined, and $R(\lambda, B_1)(D) \subseteq \mathrm{Dom}(B_2)$;*

then $\mathrm{Dom}(B_1) = \mathrm{Dom}(B_2)$ and $B_1 = B_2$.

Proof. For any $x \in D$

$$x = (\mathrm{I}_Y\lambda - B_1)R(\lambda, B_1)x = \lambda R(\lambda, B_1)x - B_1 R(\lambda, B_1)x.$$

By the fact that $R(\lambda, B_1)(D) \subseteq \mathrm{Dom}(B_2)$ and that B_1 is an extension of B_2, it follows

$$x = \lambda R(\lambda, B_1)x - B_2 R(\lambda, B_1)x = (\mathrm{I}_Y\lambda - B_2)R(\lambda, B_1)x,$$

hence, for any $x \in D$, we have $R(\lambda, B_2)x = R(\lambda, B_1)x$. So by the density of D in Y, for any $x \in Y$, we have shown that $R(\lambda, B_2)x = R(\lambda, B_1)x$. Recalling that the domain of an operator coincides with the range of its resolvent, we get the thesis. □

Now we prove the main result of this chapter.

Theorem 5.3.3. *Assume that Hypotheses 5.3.1 hold true. N_2 is the closure in $L^2(\mathcal{X}, \nu)$ of the operator N_0, defined in (5.0.1). In particular $\xi_A(\mathcal{X})$ is a core for N_2.*

Proof. By Proposition 5.3.2, to prove Theorem 5.3.3 it is sufficient to show that there exists a dense subset D of $L^2(\mathcal{X}, \nu)$ such that

$$R(\lambda, N_2)(D) \subseteq \mathrm{Dom}(\overline{N_0}).$$

We split the proof in two steps. In the first step we assume that F is Gateaux differentiable and Lipschitz continuous, and we show that we can take $C_b^1(\mathcal{X})$ as the set D. In the second step we show that, in the general case, the set $(\lambda \mathrm{I}_{\mathcal{X}} - N_2)(\mathrm{Dom}(\overline{N}_0))$ is dense in $L^2(\mathcal{X},\nu)$ and it can be chosen as the set D. Throughout the proof we let $X(t,x)$ be the mild solution of (2.0.1).

Step 1. Assume that F is Gateaux differentiable and Lipschitz continuous. For $f \in C_b^1(\mathcal{X})$ and $\lambda > 0$, consider the function φ defined as

$$\varphi(x) := R(\lambda, N_2)f(x) = \int_0^{+\infty} e^{-\lambda s}P(s)f(x)ds, \qquad x \in \mathcal{X}.$$

We want to show that φ is Gateaux differentiable. We start by proving that for any $h \in \mathcal{X}$ the following limit exists

$$\lim_{\delta\to 0}\frac{\varphi(x+\delta h)-\varphi(x)}{\delta} = \lim_{\delta\to 0}\frac{\int_0^{+\infty} e^{-\lambda s}\left(\int_\Omega \left(f(X(s,x+\delta h)) - f(X(s,x))\right)\mathbb{P}(d\omega)\right)ds}{\delta}. \tag{5.3.1}$$

Since $f \in C_b^2(\mathcal{X})$ and, for any $t \geq 0$ $X(t,\cdot) : \mathcal{X} \to \mathcal{X}$ is Gateaux differentiable $\mathbb{P}$-a.s., then it is sufficient to prove that the dominated convergence theorem is applicable in (5.3.1). By Proposition 2.3.3 and Corollary 3.1.6 we have

$$\lim_{\delta\to 0}\frac{1}{\delta}[f(X(t,x+\delta h)) - f(X(t,x))] = \left\langle \nabla f(X(t,x)), \mathcal{D}^G X(t,x)h\right\rangle.$$

Furthermore for $\delta \in \mathbb{R}$ it holds

$$\begin{aligned}\frac{1}{|\delta|}|f(X(t,x+\delta h)) - f(X(t,x))| &= \frac{1}{|\delta|}\left|\int_0^\delta \left\langle \nabla f(X(t,x+sh)), \mathcal{D}^G X(t,x+sh)h\right\rangle ds\right| \\ &\leq e^{-\zeta t}\|\nabla f\|_\infty \|h\|.\end{aligned}$$

For any $x,h \in \mathcal{X}$ we set

$$Lh := \lim_{\delta\to 0}\frac{1}{\delta}\varphi(x+\delta h) - \varphi(x)$$

, we have

$$\begin{aligned}|Lh| &= \lim_{\delta\to 0}\frac{1}{|\delta|}\left|\int_0^{+\infty} e^{-\lambda s}(P(s)f(x+\delta h) - P(s)f(x))ds\right| \\ &\leq \lim_{\delta\to 0}\frac{1}{|\delta|}\int_0^{+\infty} e^{-\lambda s}\mathbb{E}[|f(X(s,x+\delta h)) - f(X(s,x))|]ds \\ &= \lim_{\delta\to 0}\frac{1}{|\delta|}\int_0^{+\infty} e^{-\lambda s}\mathbb{E}\left[\left|\int_0^\delta \left\langle \nabla f(X(s,x+rh)), \mathcal{D}^G X(s,x+rh)h\right\rangle dr\right|\right]ds \\ &\leq \|\nabla f\|_\infty\|h\|\int_0^{+\infty} e^{-(\lambda+\zeta)s}ds = \frac{1}{\lambda+\zeta}\|\nabla f\|_\infty\|h\|,\end{aligned}$$

so φ is Gateaux differentiable. Using Proposition 1.1.1 it is also possible to prove that φ is Fréchet differentiable, and so

$$\|\nabla\varphi\|_\infty \leq \frac{1}{\lambda+\zeta}\|\nabla f\|_\infty. \tag{5.3.2}$$

We are going to check the conditions of Proposition 1.10.4 to obtain that φ belongs to $\mathrm{Dom}(L_{b,2})$. We begin to check (i) of Proposition 1.10.4. Let $Z(t,x)$ be the mild solution of (2.0.1) with $F=0$, we have

$$Z(t,x) = X(t,x) - \int_0^t e^{(t-s)A} F(X(s,x))ds.$$

Then, for every $x \in \mathcal{X}$, we have

$$\begin{aligned}\frac{T(t)\varphi(x)-\varphi(x)}{t} &= \frac{\mathbb{E}[\varphi(Z(t,x))-\varphi(x)]}{t}\\ &= \frac{1}{t}\mathbb{E}\Big[\varphi\Big(X(t,x)-\int_0^t e^{(t-s)A}F(X(s,x))ds\Big)-\varphi(x)\Big].\end{aligned} \tag{5.3.3}$$

By the Taylor formula we have

$$\begin{aligned}\varphi\Big(X(t,x)-\int_0^t e^{(t-s)A}F(X(s,x))ds\Big) &= \varphi(X(t,x))\\ &\quad -\Big\langle \nabla\varphi(X(t,x)), \int_0^t e^{(t-s)A}F(X(s,x))ds\Big\rangle\\ &\quad + o\Big(\mathbb{E}\Big[\Big\|\int_0^t e^{(t-s)A}F(X(s,x))ds\Big\|\Big]\Big)\end{aligned}$$

so in the right hand side of (5.3.3), we obtain

$$\begin{aligned}\frac{T(t)\varphi(x)-\varphi(x)}{t} =& \frac{P(t)\varphi(x)-\varphi(x)}{t} - \frac{1}{t}\mathbb{E}\left[\Big\langle \nabla\varphi(X(t,x)), \int_0^t e^{(t-s)A}F(X(s,x))ds\Big\rangle\right]\\ &+ \frac{1}{t}o\Big(\mathbb{E}\Big[\Big\|\int_0^t e^{(t-s)A}F(X(s,x))ds\Big\|\Big]\Big),\end{aligned}$$

hence for any $x \in \mathcal{X}$, we have

$$\lim_{t\to 0}\frac{1}{t}(T(t)\varphi(x)-\varphi(x)) = N_2\varphi(x) - \langle \nabla\varphi(x), F(x)\rangle. \tag{5.3.4}$$

Now let K be a compact subset of $\mathcal{X}$. Since, in this step, we have assume that F is Lipschitz continuous, by (5.1.3), we get

$$\lim_{t\to 0}\sup_{x\in K}\frac{1}{t}o\Big(\mathbb{E}\Big[\Big\|\int_0^t e^{(t-s)A}F(X(s,x))ds\Big\|\Big]\Big) = 0.$$

We set for $t>0$ and $x \in \mathcal{X}$

$$\Delta_t(x) := \frac{P(t)\varphi(x)-\varphi(x)}{t}, \quad R_t(x) := \frac{1}{t}\mathbb{E}\left[\Big\langle \nabla\varphi(X(t,x)), \int_0^t e^{(t-s)A}F(X(s,x))ds\Big\rangle\right].$$

We recall that for every $t \geq 0$

$$P(t)\varphi = P(t)\int_0^{+\infty} e^{-\lambda s}P(s)f ds = e^{\lambda t}\int_t^{+\infty} e^{-\lambda s}P(s)f ds.$$

Let $x_0 \in K$. Since $f \in C_b^1(\mathcal{X})$ we know that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x_0)| \leq \varepsilon$, whenever $\|x - x_0\| \leq \delta$. Now let $\|x - x_0\| \leq \delta$

$$\begin{aligned}
|\Delta_t(x) - \Delta_t(x_0)| &= \frac{1}{t}|P(t)\varphi(x) - \varphi(x) - P(t)\varphi(x_0) + \varphi(x_0)| \\
&= \frac{1}{t}\left|e^{-\lambda t}\int_t^{+\infty} e^{-\lambda s}P(s)(f(x) - f(x_0))ds + \int_0^{+\infty} e^{-\lambda s}P(s)(f(x_0) - f(x))ds\right| \\
&= \frac{1}{t}\left|(e^{-\lambda t} - 1)\int_t^{+\infty} e^{-\lambda s}P(s)(f(x) - f(x_0))ds\right. \\
&\qquad\qquad\qquad\left. + \int_0^{t} e^{-\lambda s}P(s)(f(x_0) - f(x))ds\right| \\
&\leq \frac{e^{-\lambda t} - 1}{t}\int_t^{+\infty} e^{-\lambda s}P(s)\varepsilon ds + \frac{1}{t}\int_0^t e^{-\lambda s}P(s)\varepsilon ds \\
&\leq \varepsilon\left(\frac{e^{-\lambda t} - 1}{t}\int_t^{+\infty} e^{-\lambda s}ds + \frac{1}{t}\int_0^t e^{-\lambda s}ds\right) \\
&= \varepsilon e^{-\lambda t}\frac{1 - e^{-\lambda t}}{\lambda t} \leq \varepsilon,
\end{aligned}$$

and so

$$|\Delta_t(x) - \Delta_t(x_0)| \leq \varepsilon. \tag{5.3.5}$$

We observe that by the Lipschitz continuity of F there exists $C > 0$ such that for every $x \in \mathcal{X}$, it holds $\|F(x)\| \leq C(1 + \|x\|)$. Furthermore by Corollary 2.1.11, (2.2.3) and (5.3.2), for every $t > 0$, the functions $x \mapsto \nabla\varphi(X(t,x))$ and $x \mapsto F(X(t,x))$ are continuous uniformly with respect to $t \in [0,T]$. So for every $t \in [0,T]$, $x_0 \in K$ and $\varepsilon > 0$ there exists $\delta := \delta(\varepsilon, x_0) > 0$ such that whenever $\|x - x_0\| \leq \delta$ it holds

$$\max\{\|\nabla\varphi(X(t,x)) - \nabla\varphi(X(t,x_0))\|, \|F(X(t,x)) - F(X(t,x_0))\|\} \leq \varepsilon.$$

By the Jensen inequality and (5.1.3) we can write

$$\begin{aligned}
|R_t(x) - R_t(x_0)| &= \frac{1}{t}\left|\mathbb{E}\left[\left\langle\nabla\varphi(X(t,x)), \int_0^t e^{(t-s)A}F(X(s,x))ds\right\rangle\right]\right. \\
&\qquad\qquad\left. - \mathbb{E}\left[\left\langle\nabla\varphi(X(t,x_0)), \int_0^t e^{(t-s)A}F(X(s,x_0))ds\right\rangle\right]\right| \\
&= \frac{1}{t}\left|\mathbb{E}\left[\left\langle\nabla\varphi(X(t,x)) - \nabla\varphi(X(t,x_0)), \int_0^t e^{(t-s)A}F(X(s,x))ds\right\rangle\right]\right. \\
&\qquad\left. + \mathbb{E}\left[\left\langle\nabla\varphi(X(t,x_0)), \int_0^t e^{(t-s)A}(F(X(s,x)) - F(X(s,x_0)))ds\right\rangle\right]\right| \\
&\leq \frac{1}{t}\mathbb{E}\left[\|\nabla\varphi(X(t,x)) - \nabla\varphi(X(t,x_0))\|\left\|\int_0^t e^{(t-s)A}F(X(s,x))ds\right\|\right] \\
&\qquad + \frac{1}{t}\mathbb{E}\left[\|\nabla\varphi(X(t,x_0))\|\left\|\int_0^t e^{(t-s)A}(F(X(s,x)) - F(X(s,x_0)))ds\right\|\right]
\end{aligned}$$

$$\leq \frac{C\varepsilon}{t}\int_0^t \mathbb{E}[1+\|X(s,x)\|]ds + \frac{\|\nabla\varphi\|_\infty}{t}\mathbb{E}\left[\int_0^t \|F(X(s,x)) - F(X(s,x_0))\|ds\right],$$

so that by (5.1.3) there exists a constant $C_1 > 0$ such that

$$|R_t(x) - R_t(x_0)| \leq \varepsilon C_1(1+\|x\|+\|\nabla\varphi\|_\infty). \tag{5.3.6}$$

Hence by Proposition 1.10.2, (5.3.4), (5.3.6) and (5.3.5) we have

$$\lim_{t\to 0}\sup_{x\in K}\left|\frac{1}{t}(T(t)\varphi(x)-\varphi(x)) - N_2\varphi(x) - \langle\nabla\varphi(x), F(x)\rangle\right| = 0,$$

and so we have checked (i) of Proposition 1.10.4. Using similar arguments also condition (ii) of Proposition 1.10.4 is verified since $\varphi \in C_b^1(\mathcal{X})$ and we have assumed F to be Lipschitz continuous in this first step. So by Proposition 1.10.4, $\varphi \in \mathrm{Dom}(L_{b,2})$, in particular

$$L_{b,2}\varphi = N_2\varphi - \langle\nabla\varphi, F\rangle,$$

and

$$\lambda\varphi - L_{b,2}\varphi - \langle\nabla\varphi, F\rangle = f.$$

So $\varphi \in \mathrm{Dom}(L_{b,2}) \cap C_b^1(\mathcal{X})$ and, by Lemma 5.2.3, we conclude that $\varphi \in \mathrm{Dom}(\overline{N}_0)$ and

$$\overline{N}_0\varphi = L_{b,2}\varphi + \langle\nabla\varphi, F\rangle.$$

Step 2. Let $\{F_{\delta,s}\,|\,\delta, s > 0\}$ be the regularizing family of F defined in Section 1.11. Let $f \in C_b^1(\mathcal{X})$. For any $\delta, s > 0$, we set

$$\varphi_{\delta,s}(x) := \int_0^{+\infty} e^{-\lambda t}P_{\delta,s}(t)f(x)dt, \quad x \in \mathcal{X},$$

where $P_{\delta,s}(t)$ is the transition semigroup of the equation

$$\begin{cases} dX(t,x) = \big(AX(t,x) + F_{\delta,s}(X(t,x))\big)dt + RdW(t), & t > 0; \\ X(0,x) = x. \end{cases}$$

In Section 1.11 we have seen that, for any $\delta, s > 0$, the function $F_{\delta,s}$ is Lipschitz continuous and $F_{\delta,s} - \zeta_2\mathrm{I}$ is dissipative. Hence by Step 1, for any $\delta, s > 0$, we have $\varphi_{\delta,s} \in \mathrm{Dom}(\overline{N}_0)$ and

$$\lambda\varphi_{\delta,s} - L_{b,2}\varphi_{\delta,s} - \langle\nabla\varphi_{\delta,s}, F_{\delta,s}\rangle = f.$$

So

$$\lambda\varphi_{\delta,s} - \overline{N}_0\varphi_{\delta,s} = f + \langle\nabla\varphi_{\delta,s}, F_{\delta,s} - F\rangle,$$

and recalling that N_2 is an extension of $\overline{N}_0$ in $L^2(X,\nu)$

$$\lambda\varphi_{\delta,s} - N_2\varphi_{\delta,s} = f + \langle\nabla\varphi_{\delta,s}, F_{\delta,s} - F\rangle,$$

where the equality holds in $L^2(\mathcal{X},\nu)$. Hence noticing that estimate (5.3.2) does not depend on δ and on s and by Proposition 1.11.3 yields that

$$\lim_{\delta\to 0}\lim_{s\to 0}(\lambda \mathrm{I}_{\mathcal{X}} - N_2)\varphi_{\delta,s} = f, \qquad \text{in } L^2(X,\nu).$$

Since $\varphi_{\delta,s} \in \mathrm{Dom}(\overline{N}_0)$, by the density of $C_b^1(\mathcal{X})$ in $L^2(\mathcal{X},\nu)$ we get the density of $(\lambda \mathrm{I}_{\mathcal{X}} - N_2)(\mathrm{Dom}(\overline{N}_0))$ in $L^2(\mathcal{X},\nu)$. □

5.4 Remarks and examples

The results presented in this chapter are contained in the paper [9]. The results in this chapter are known in the literature just in some specific cases. We have proved them under general assumptions. Theorem 5.3.3 extends the results contained in [11, Section 3], [31], [30, Sections 3.5 and 4.6] and [42, Section 11.2.2]. For a study of an analogous problem in $L^2(E,\nu)$ in the case of a multiplicative noise we refer to [23]. In Subsection (5.4.1) we will present a reaction-diffusion system that verifies the hypotheses of Theorem 5.3.3. In Subsection 5.4.2 we will show an interesting application of the Theorem 5.1.3.

5.4.1 A reaction-diffusion system

Let $\mathcal{X} = L^2([0,1],\lambda)$ where λ is the Lebesgue measure and let $E = C([0,1])$. Let A be the realization in $L^2([0,1])$ of the second order derivative with Dirichlet boundary condition and set $R = \mathrm{I}_{\mathcal{X}}$.

We define the function F. Let $\varphi \in C^1(\mathbb{R})$ be a decreasing function, such that there exist $d_1 > 0$ and $m \in \mathbb{N}$ satisfying

$$|\varphi'(y)| \le d_1(1+|y|^m), \quad y \in \mathbb{R}.$$

Let $\zeta_2 > 0$. We set

$$F(f(y)) = \varphi(f(y)) - \zeta_2 f(y), \quad f \in C([0,1]), y \in [0,1].$$

By [20, Section 6.1] Hypotheses 2.1.1(iii), 2.1.1(v) are verified, in particular Hypothesis 2.1.1(iv) is verified with constant $\zeta = -\zeta_2$. By [20, Lemma 8.2.1] condition (5.1.1) of Hypotheses 5.1.1 is verified. By the definition of F, Hypotheses 2.1.1(ii) are verified. By (7.4.2) Hypotheses 2.1.1(vi) and 2.1.1(vii) are verified. By [43, Example D.7] and standard calculations Hypotheses 2.1.1(iv) are verified. So all the hypotheses of Theorem 5.1.3 are verified, so $\nu(C([0,1])) = 1$, where ν is the invariant measure of the transition semigroup $P(t)$ associated to the generalized mild solution of (2.0.1).

Moreover also the Hypotheses of Theorem 5.3.3 are verified, so the infinitesimal generator of $P(t)$ in $L^2(\mathcal{X},\nu)$ is the closure in $L^2(\mathcal{X},\nu)$ of the operator

$$N_0\psi(f) = \frac{1}{2}\mathrm{Tr}[\nabla^2\psi(f)] + \left\langle f'' + \varphi(f) - \zeta_2 f, \nabla\psi(f)\right\rangle, \quad \psi \in \xi_A(\mathcal{X}),$$

where $f \in C^2([0,1])$ and $f(0) = f(1) = 0$.

5.4.2 An application of Theorem 5.1.3

Now present a particular case of the example in Subsection 2.4.6. Let $\mathcal{X} = L^2([0,1],\lambda)$ and $E = W^{1,2}([0,1],\lambda)$. We assume that $A = -\frac{1}{2}\mathrm{I}_{\mathcal{X}}$. Let B the realization of the second order derivative in $\mathcal{X}$ with Dirichlet boundary conditions. We recall that B is negative operator, $\mathrm{Dom}((-B)^{\frac{1}{2}}) = W_0^{1,2}([0,1],\lambda)$ and $(-B)^{-\gamma}$ is a trace class operator, for any $\gamma > \frac{1}{2}$ (see [30, Section 4.1]). Let $\beta > 2$ and set $R = (-B)^{-\beta}$. Then

$$\begin{aligned}\|W_A(t)\|^2_{W^{1,2}([0,1],\lambda)} &= \left\|(-B)^{1/2}\int_0^t e^{-\frac{1}{2}(t-s)}B^{-\beta/2}dW(s)\right\|_{L^2([0,1],\lambda)}\\ &= \left\|\int_0^t e^{-\frac{1}{2}(t-s)}(-B)^{(1-\beta)/2}dW(s)\right\|_{L^2([0,1],\lambda)},\end{aligned}$$

and so by [43, Theorems 4.36 and 5.11], Hypotheses 2.1.1(v) and condition (5.1.1) of Hypotheses 5.1.1 are verified. Let F be as in Section 4.3.3, so for any $f \in W^{1,2}([0,1],\lambda)$ we have

$$F(f) = \int_0^1\int_0^1\int_0^1 K(\xi_1,\xi_2,\xi_3,\xi)f(\xi_1)f(\xi_2)f(\xi_3)d\xi_1 d\xi_2 d\xi_3 + \zeta_2 f,$$

$$(F(f))' = \int_0^1\int_0^1\int_0^1 \frac{\partial K}{\partial \xi}(\xi_1,\xi_2,\xi_3,\xi)f(\xi_1)f(\xi_2)f(\xi_3)d\xi_1 d\xi_2 d\xi_3 + \zeta_2 f',$$

with $\zeta_2 < -1/2$. Hence, by the same arguments of Subsection 4.3.3, the hypotheses of Theorem 5.1.3 are verified and so $\nu(W^{1,2}([0,1],\lambda)) = 1$.

Moreover also the Hypotheses of Theorem 5.3.3 are verified, so the infinitesimal generator of the transition semigroup $P(t)$ of (2.0.1) in $L^2(\mathcal{X},\nu)$ is the closure in $L^2(\mathcal{X},\nu)$ of the operator

$$\begin{aligned}N_0\psi(f) &= \frac{1}{2}\mathrm{Tr}[\nabla^2\psi(f)]\\ &\quad + \left\langle\left(\zeta_2 - \frac{1}{2}\right)f + \int_0^1\int_0^1\int_0^1 K(\xi_1,\xi_2,\xi_3,\xi)f(\xi_1)f(\xi_2)f(\xi_3)d\xi_1 d\xi_2 d\xi_3, \nabla\psi(f)\right\rangle,\end{aligned}$$

where $\psi \in \xi_A(\mathcal{X})$ and $f \in L^2([0,1],\lambda)$.

Chapter 6

Sobolev spaces

In the previous chapter we have shown that, under suitable hypotheses, the transition semigroup $P(t)$ is uniquely extendable to a strongly continuous and contraction semigroup $P_2(t)$ in $L^2(\mathcal{X},\nu)$, where ν is the unique invariant measure of $P(t)$. The main goal of this chapter is to define a suitable Sobolev space which contain the domain of infinitesimal generator N_2 of $P_2(t)$. In chapters 3 and 4 we have studied regularity property of the transition semigroup along the directions given by the diffusion operator R. Again, we will see that it is the diffusion operator that will determine the Sobolev space that we are going to define.

We work in the same framework of chapter 3. Let $\mathcal{X}$ be a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let $\{W(t)\}_{t\geq 0}$ be a $\mathcal{X}$-cylindrical Wiener process defined on a normal filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$. We consider the SPDE

$$\begin{cases} dX(t,x)=\big(AX(t,x)+RG(X(t,x))\big)dt+RdW(t), & t>0;\\ X(0,x)=x, \end{cases} \tag{6.0.1}$$

and the following hypotheses.

Hypotheses 6.0.1. *Assume that Hypotheses 3.0.3 hold true and that the following conditions are verified.*

(i) $G\in C^1(\mathcal{X},\mathcal{X})$ *and there exists* $\zeta>0$ *such that*

$$\langle (A+R\mathcal{D}G(x))h,h\rangle \leq -\zeta\|h\|^2,\quad x,h\in\mathcal{X}, \tag{6.0.2}$$

$$\langle (A+R\mathcal{D}G(x))h,h\rangle_R \leq -\zeta\|h\|_R^2,\quad x\in\mathcal{X},\ h\in H_R. \tag{6.0.3}$$

(ii) *For any* $p\geq 1$ *we have*

$$\sup_{t\geq 0}\mathbb{E}[\|W_A(t)\|^p]<+\infty.$$

Remark 6.0.2. (6.0.2) *and* (6.0.3) *are verified if the constants* $w_{\mathcal{X}}$ *and* w_R *in Hypotheses 3.0.1 and 3.0.3 are negative and* $\min(|w_{\mathcal{X}}|,|w_R|)>\|R\|_{\mathcal{L}(\mathcal{X})}L_G$, *where* L_G *is the Lipschitz constant of* G.

For any $x \in \mathcal{X}$, by Theorem 2.1.13 (with $\mathcal{X} = E$) the SPDE (3.0.3) has unique mild solution $\{X(t,x)\}_{t\geq 0} \in C_p([0,T],\mathcal{X})$, for any $T > 0$ and $p \geq 1$. Let $P(t)$ be the transition semigroup associated to (6.0.1).

By Theorem 5.1.3 (with $\mathcal{X} = E$) the semigroup $P(t)$ has unique invariant measure ν with finite moments of every order. As we have seen in Section 5.1, $P(t)$ is uniquely extendable to a strongly continuous semigroup $P_2(t)$ on $L^2(\mathcal{X},\nu)$, whose infinitesimal generator is denoted by N_2. By Theorem 5.3.3 N_2 is the closure in $L^2(\mathcal{X},\nu)$ of the second order Kolmogorov operator defined by

$$N_0\varphi(x) := L_0\varphi(x) + \langle F(x), \nabla\varphi(x)\rangle, \quad \varphi \in \xi_A(\mathcal{X}),\ x \in \mathcal{X},$$

where the operator L_0 and the space $\xi_A(\mathcal{X})$ are defined in (1.10.6) and (1.10.5) respectively.

First of all in Section 6.1 we define the Sobolev space $W^{1,2}_R(\mathcal{X},\nu)$ and $W^{2,2}_R(\mathcal{X},\nu)$ related to a "natural" derivative operator associated to R. In Section 6.2 we prove that $\mathrm{Dom}(N_2) \subseteq W^{1,2}_R(\mathcal{X},\nu)$ and, under some additional hypotheses, $\mathrm{Dom}(N_2) \subseteq W^{2,2}_R(\mathcal{X},\nu)$. In Section 6.3 we will prove that the logarithmic Sobolev inequality and the Poincaré inequality hold in the space $W^{1,2}_R(\mathcal{X},\nu)$, and we will see some of their consequences.

6.1 Closability of ∇_R

In this section we introduce the Sobolev spaces we will use throughout the rest of the chapter. In order to do so we need some preliminary results. Let $(H_R, \langle,\rangle_R)$ be the Hilbert space of Definition 3.0.2.

If Hypotheses 6.0.1 hold true then, by Proposition 3.1.4, for every $t > 0$ the map $x \mapsto X(t,x)$ is $\mathbb{P}$-a.e. H_R-Gateaux differentiable and for any $x \in \mathcal{X}$ and $h \in H_R$ its H_R-Gateaux derivative along $h \in H_R$ is $\mathcal{D}^G X(t,x)h$, where $\{\mathcal{D}^G X(t,x)h\}_{t\geq 0}$ is the unique mild solution of (3.1.5).

Using (6.0.3) as in the proof of Proposition 2.3.3, for any $t > 0$, $x \in \mathcal{X}$ and $h \in H_R$ we obtain

$$\left\|\mathcal{D}^G X(t,x)h\right\|_R \leq e^{-\zeta t}\|h\|_R.$$

So the following result follows in the same way of Theorem 4.1.7.

Lemma 6.1.1. *Assume that Hypotheses 6.0.1 hold true. For every $\varphi \in C^1_b(\mathcal{X})$ it holds*

$$\|\nabla_R P_2(t)\varphi(x)\|^2_R \leq e^{-2\zeta t} P_2(t)\|\nabla_R\varphi(x)\|^2_R, \quad t > 0,\ x \in \mathcal{X}. \tag{6.1.1}$$

Lemma 6.1.2. *Assume that Hypotheses 6.0.1 hold true. Let $\varphi, \psi \in \xi_A(\mathcal{X})$. Then the product $\varphi\psi$ belongs to $\xi_A(\mathcal{X})$ and*

$$N_2(\varphi\psi) = \varphi N_2\psi + \psi N_2\varphi + \langle R\nabla\varphi, R\nabla\psi\rangle = \varphi N_2\psi + \psi N_2\varphi + \langle \nabla_R\varphi, \nabla_R\psi\rangle_R. \tag{6.1.2}$$

Furthermore whenever $\varphi \in \mathrm{Dom}(N_2)$ and $g \in C^2_b(\mathbb{R})$, we have

$$\int_{\mathcal{X}} (g' \circ \varphi) N_2\varphi d\nu = -\int_{\mathcal{X}} (g'' \circ \varphi)\|R\nabla\varphi\|^2 d\nu. \tag{6.1.3}$$

Proof. The fact that $\varphi\psi$ belongs to $\xi_A(\mathcal{X})$ and (6.1.2) follows by direct calculations. We recall that $N_2 u = N_0 u$ whenever $u \in \xi_A(\mathcal{X})$ (Theorem 5.3.3). Now we prove (6.1.3). We start by showing that if ψ belongs to $\mathrm{Dom}(N_2)$ then

$$\int_{\mathcal{X}} \psi N_2 \psi d\nu = -\frac{1}{2}\int_{\mathcal{X}} \|\nabla_R \psi\|_R^2 d\nu. \tag{6.1.4}$$

To prove (6.1.4) it is enough to recall that ν is invariant. Indeed by (6.1.2) we have for $\varphi \in \xi_A(\mathcal{X})$

$$0 = \int_{\mathcal{X}} N_2 \varphi^2 d\nu = \int_{\mathcal{X}} \big(2\varphi N_2 \varphi + \|\nabla_R \varphi\|_R^2\big) d\nu.$$

Since $\xi_A(\mathcal{X})$ is a core for N_2, by (6.1.4) and the Young inequality, it follows that

$$\nabla_R : \xi_A(\mathcal{X}) \subseteq \mathrm{Dom}(N_2) \to L^2(\mathcal{X}, \nu; H_R), \qquad \varphi \mapsto \nabla_R \varphi,$$

is continuous and, consequently, it can be continuously extended to all $\mathrm{Dom}(N_2)$ (endowed with the graph norm). We shall still denote by ∇_R its extension. So (6.1.4) follows by a standard density argument. (6.1.3) follows by the dominated convergence theorem to get . □

The next result will be useful to prove the closability of the gradient operator (Proposition 6.1.4) and the Poincaré inequality (Proposition 6.3.5).

Lemma 6.1.3. *Assume that Hypotheses 6.0.1 hold true. Let $\varphi \in \xi_A(\mathcal{X})$. It holds*

$$\int_{\mathcal{X}} |P_2(t)\varphi|^2 d\nu + \int_0^t \int_{\mathcal{X}} \|\nabla_R P_2(s)\varphi\|_R^2 d\nu ds = \int_{\mathcal{X}} |\varphi|^2 d\nu. \tag{6.1.5}$$

Proof. For every $\varphi \in \xi_A(\mathcal{X})$ and $x \in \mathcal{X}$ we have

$$\frac{d}{ds}(P_2(s)\varphi)(x) = N_2(P_2(s)\varphi)(x), \qquad s > 0. \tag{6.1.6}$$

Multiplying both sides of (6.1.6) by $P_2(s)\varphi$, integrating on $\mathcal{X}$ with respect to ν, and taking into account (6.1.4), we find

$$\int_{\mathcal{X}} \frac{d}{ds}|P_2(s)\varphi|^2 d\nu = -\int_{\mathcal{X}} \|\nabla_R P_2(s)\varphi\|_R^2 d\nu. \tag{6.1.7}$$

Now the thesis follows integrating (6.1.7) with respect to s from 0 to t. □

We now can prove the closability of the derivative operators ∇_R and ∇_R^2 (see Definition 3.1.2) that we will use to define the Sobolev spaces $W_R^{1,2}(\mathcal{X}, \nu)$ and $W_R^{2,2}(\mathcal{X}, \nu)$.

Proposition 6.1.4. *Assume that Hypotheses 6.0.1 hold true and let $\mathcal{H}_R$ be the space of Hilbert–Schmidt operators on H_R. The operators $\nabla_R : \xi_A(\mathcal{X}) \subseteq L^2(\mathcal{X}, \nu) \to L^2(\mathcal{X}, \nu; H_R)$ and $(\nabla_R, \nabla_R^2) : \xi_A(\mathcal{X}) \subseteq L^2(\mathcal{X}, \nu) \to L^2(\mathcal{X}, \nu; H_R) \times L^2(\mathcal{X}, \nu; \mathcal{H}_R)$ are closable, where $\mathcal{H}_R$ is the space of the Hilbert–Schmidt operators on H_R.*

Proof. We assume that $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq \xi_A(\mathcal{X})$ is a sequence such that

$$\begin{aligned} L^2(\mathcal{X},\nu)\text{-}\lim_{n\to+\infty} \varphi_n = 0; \\ L^2(\mathcal{X},\nu;H_R)\text{-}\lim_{n\to+\infty} \nabla_R\varphi_n = \Psi, \end{aligned} \tag{6.1.8}$$

for some $\Psi \in L^2(\mathcal{X},\nu,H_R)$. By (6.1.5), the strong continuity of $P_2(t)$ and (6.1.8), we have

$$\lim_{n\to+\infty}\int_0^t\int_{\mathcal{X}} \|\nabla_R P_2(s)\varphi_n\|_R^2 d\nu ds = \lim_{n\to+\infty}\left(\int_{\mathcal{X}}|\varphi_n|^2 d\nu - \int_{\mathcal{X}}|P_2(t)\varphi_n|^2 d\nu\right) = 0 \tag{6.1.9}$$

We also claim that

$$\lim_{n\to+\infty}\int_0^t\int_{\mathcal{X}} \|\nabla_R P_2(s)\varphi_n\|_R^2 d\nu ds = \int_0^t\int_{\mathcal{X}} \left\|\mathbb{E}\big[(\mathcal{D}^G X(t,x))^*\Psi(X(t,x))\big]\right\|_R^2 \nu(dx)ds. \tag{6.1.10}$$

Indeed by Corollary 3.1.6 we have

$$\nabla_R P_2(t)\varphi_n(x) = \mathbb{E}\big[(\mathcal{D}^G X(t,x))^*\nabla_R\varphi_n(X(t,x))\big].$$

Observe that

$$\begin{aligned} &\int_0^t\int_{\mathcal{X}} \left\|\mathbb{E}\big[(\mathcal{D}^G X(s,x))^*\nabla_R\varphi_n(X(s,x))\big] - \mathbb{E}\big[(\mathcal{D}^G X(t,x))^*\Psi(X(s,x))\big]\right\|_R^2 \nu(dx)ds \\ \le &\int_0^t\int_{\mathcal{X}} e^{-2\zeta s}\|\mathbb{E}[\nabla_R\varphi_n(X(s,x)) - \Psi(X(s,x))]\|_R^2 \nu(dx)ds \\ \le &\int_0^t\int_{\mathcal{X}} e^{-2\zeta s}(P_2(s)\|\nabla_R\varphi_n - \Psi\|_R^2)(x)\nu(dx)ds. \end{aligned}$$

Recalling that ν is invariant for $P_2(t)$ we have

$$\begin{aligned} 0 \le &\limsup_{n\to+\infty}\int_0^t\int_{\mathcal{X}} \left\|\mathbb{E}\big[(\mathcal{D}^G X(s,x))^*\nabla_R\varphi_n(X(s,x))\big] - \mathbb{E}\big[(\mathcal{D}^G X(s,x))^*\Psi(X(s,x))\big]\right\|_R^2 \nu(dx)ds \\ \le &\limsup_{n\to+\infty}\int_0^t\int_{\mathcal{X}} e^{-2\zeta s}\|\nabla_R\varphi_n(x) - \Psi(x)\|_R^2 \nu(dx)ds = 0. \end{aligned}$$

This proves (6.1.10). Combining (6.1.9) and (6.1.10) we get

$$\int_0^t\int_{\mathcal{X}} \left\|\mathbb{E}\big[(\mathcal{D}^G X(s,x))^*\Psi(X(s,x))\big]\right\|_R^2 \nu(dx)ds = 0.$$

So for a.e. $s \in (0,t)$ (with respect to the Lebesgue measure) it holds

$$\int_{\mathcal{X}} \left\|\mathbb{E}\big[(\mathcal{D}^G X(s,x))^*\Psi(X(s,x))\big]\right\|_R^2 \nu(dx) = 0. \tag{6.1.11}$$

To be more precise we denote by A the subset with measure zero of $(0,t)$, such that in $(0,t)\setminus A$

(6.1.11) does not hold. For $s \in A$, by the monotone convergence theorem, we have

$$\begin{aligned}
0 =& \int_{\mathcal{X}} \left\| \mathbb{E}\left[(\mathcal{D}^G X(s,x))^* \Psi(X(s,x)) \right] \right\|_R^2 \nu(dx) \\
=& \int_{\mathcal{X}} \sum_{i=1}^{+\infty} \left| \left\langle \mathbb{E}\left[(\mathcal{D}^G X(s,x))^* \Psi(X(s,x)) \right], h_i \right\rangle_R \right|^2 \nu(dx) \\
=& \sum_{i=1}^{+\infty} \int_{\mathcal{X}} \left| \mathbb{E}\left[\left\langle (\mathcal{D}^G X(s,x))^* \Psi(X(s,x)), h_i \right\rangle_R \right] \right|^2 \nu(dx) \\
=& \sum_{i=1}^{+\infty} \int_{\mathcal{X}} \left| \mathbb{E}\left[\left\langle \Psi(X(s,x)), \mathcal{D}^G X(s,x) h_i \right\rangle_R \right] \right|^2 \nu(dx).
\end{aligned}$$

So for $s \in A$ and $i \in \mathbb{N}$

$$\int_{\mathcal{X}} \left| \mathbb{E}\left[\left\langle \Psi(X(s,x)), \mathcal{D}^G X(s,x) h_i \right\rangle_R \right] \right|^2 \nu(dx) = 0.$$

Now observe that for $s \in A$ and $i \in \mathbb{N}$ we have

$$\begin{aligned}
0 \le& \| P_2(s)(\langle \Psi(\cdot), h_i \rangle_R) \|_{L^2(\mathcal{X},\nu)} = \| \mathbb{E}[\langle \Psi(X(s,\cdot)), h_i \rangle_R] \|_{L^2(\mathcal{X},\nu)} \\
=& \| \mathbb{E}[\langle \Psi(X(s,\cdot)), h_i \rangle_R] \|_{L^2(\mathcal{X},\nu)} - \left\| \mathbb{E}\left[\left\langle \Psi(X(s,\cdot)), \mathcal{D}^G X(s,\cdot) h_i \right\rangle_R \right] \right\|_{L^2(\mathcal{X},\nu)} \\
\le& \left\| \mathbb{E}[\langle \Psi(X(s,\cdot)), h_i \rangle_R] - \mathbb{E}\left[\left\langle \Psi(X(s,\cdot)), \mathcal{D}^G X(s,\cdot) h_i \right\rangle_R \right] \right\|_{L^2(\mathcal{X},\nu)} \\
=& \left\| \mathbb{E}\left[\left\langle \Psi(X(s,\cdot)), h_i - \mathcal{D}^G X(s,\cdot) h_i \right\rangle_R \right] \right\|_{L^2(\mathcal{X},\nu)}.
\end{aligned}$$

By the continuity of $s \mapsto \mathcal{D}^G X(s,\cdot)$ and the dominated convergence theorem we get that for every $i \in \mathbb{N}$,

$$\| \langle \Psi(\cdot), h_i \rangle_R \|_{L^2(\mathcal{X},\nu)} = 0.$$

By a standard argument we get $\Psi(x) = 0$ for ν-a.e $x \in X$. This proves the closability of $\nabla_R : \xi_A(\mathcal{X}) \subseteq L^2(\mathcal{X},\nu) \to L^2(\mathcal{X},\nu; H_R)$.

Let $\mathcal{FC}_b^2(\mathcal{X};\mathcal{X})$ be the set given by Definition 1.6.16). By similar arguments to those used above we have that

$$\mathcal{D}_R : \mathcal{FC}_b^2(\mathcal{X};\mathcal{X}) \subseteq L^2(\mathcal{X},\nu;\mathcal{X}) \to L^2(\mathcal{X},\nu;\mathcal{H}_R)$$

is closable. The closability of $(\nabla_R, \nabla_R^2) : \xi_A(\mathcal{X}) \subseteq L^2(\mathcal{X},\nu) \to L^2(\mathcal{X},\nu;H_R) \times L^2(\mathcal{X},\nu;\mathcal{H}_R)$ follows by the fact that $\nabla_R^2 = \mathcal{D}_R \nabla_R$. □

We are now able to define the Sobolev spaces we will use throughout the rest of the chapter.

Definition 6.1.5. *We define the Sobolev spaces $W_R^{1,2}(\mathcal{X},\nu)$ and $W_R^{2,2}(\mathcal{X},\nu)$ as the domains of the closure of the operators $\nabla_R : \xi_A(\mathcal{X}) \subseteq L^2(\mathcal{X},\nu) \to L^2(\mathcal{X},\nu;H_R)$ and $(\nabla_R, \nabla_R^2) : \xi_A(\mathcal{X}) \subseteq L^2(\mathcal{X},\nu) \to L^2(\mathcal{X},\nu;H_R) \times L^2(\mathcal{X},\nu;\mathcal{H}_R)$ respectively. They are endowed with the graph norms of the closure of such operators. We still denote by ∇_R and ∇_R^2 the closures of operators ∇_R and ∇_R^2.*

Remark 6.1.6. *If $G \equiv 0$ in* (6.0.1) *then the Sobolev space $W_R^{1,2}(\mathcal{X},\nu)$ coincides with the Sobolev space defined in Proposition 1.6.17.*

6.2 Maximal Sobolev regularity

Now we can study the Sobolev regularity of the domain of N_2. The following theorem states that $\mathrm{Dom}(N_2)$ is continuously embedded in $W^{1,2}_R(\mathcal{X},\nu)$.

Theorem 6.2.1. *Assume that Hypotheses 6.0.1 hold true. Let $\lambda>0$ and $f\in L^2(\mathcal{X},\nu)$, we set*

$$u:=R(\lambda,N_2)f=\int_0^{+\infty}e^{-\lambda t}P(t)fdt.$$

The function u belongs to $W^{1,2}_R(\mathcal{X},\nu)$ and

$$\|u\|_{L^2(\mathcal{X},\nu)}\leq\frac{1}{\lambda}\|f\|_{L^2(\mathcal{X},\nu)};\qquad \|\nabla_R u\|_{L^2(\mathcal{X},\nu;H_R)}\leq\sqrt{\frac{2}{\lambda}}\|f\|_{L^2(\mathcal{X},\nu)}.$$

If for any $\varphi\in\mathrm{Dom}(N_2)$ and $\psi\in W^{1,2}_R(\mathcal{X},\nu)$ it holds

$$\int_{\mathcal{X}}\psi N_2\varphi d\nu=-\frac{1}{2}\int_{\mathcal{X}}\langle\nabla_R\varphi,\nabla_R\psi\rangle_R d\nu, \tag{6.2.1}$$

then for every $\lambda>0$ and $f\in L^2(\mathcal{X},\nu)$, the function u belongs to $W^{2,2}_R(\mathcal{X},\nu)$ and

$$\|\nabla^2_R u\|_{L^2(\mathcal{X},\nu;\mathcal{H}_R)}\leq 2\sqrt{2}\|f\|_{L^2(\mathcal{X},\nu)},$$

where $\mathcal{H}_R$ is the space of the Hilbert–Schmidt operators on H_R.

Proof. Since $\xi_A(\mathcal{X})$ is a core for N_2, then a sequence $\{u_n\}_{n\in\mathbb{N}}\subseteq\xi_A(\mathcal{X})$ exists such that u_n converges to a function u in $L^2(\mathcal{X},\nu)$ and

$$L^2(\mathcal{X},\nu)\text{-}\lim_{n\to+\infty}\lambda u_n-N_2u_n=f.$$

Let $f_n:=\lambda u_n-N_2u_n$. Multiplying by u_n, integrating with respect to ν and using (6.1.3) we get

$$\int_{\mathcal{X}}f_nu_nd\nu=\lambda\int_{\mathcal{X}}u_n^2d\nu-\int_{\mathcal{X}}u_nN_2u_nd\nu=\lambda\int_{\mathcal{X}}u_n^2d\nu+\frac{1}{2}\int_{\mathcal{X}}\|\nabla_Ru_n\|_R^2d\nu.$$

By the Cauchy–Schwarz inequality we get

$$\|u_n\|_{L^2(\mathcal{X},\nu)}\leq\frac{1}{\lambda}\|f_n\|_{L^2(\mathcal{X},\nu)};\qquad \|\nabla_R u_n\|_{L^2(\mathcal{X},\nu;H_R)}\leq\sqrt{\frac{2}{\lambda}}\|f_n\|_{L^2(\mathcal{X},\nu)}.$$

Since $\{u_n\}_{n\in\mathbb{N}}$ and $\{f_n\}_{n\in\mathbb{N}}$ converge to u and f, respectively, in $L^2(\mathcal{X},\nu)$ we get

$$\|u\|_{L^2(\mathcal{X},\nu)}=\lim_{n\to+\infty}\|u_n\|_{L^2(\mathcal{X},\nu)}\leq\lim_{n\to+\infty}\frac{1}{\lambda}\|f_n\|_{L^2(\mathcal{X},\nu)}=\frac{1}{\lambda}\|f\|_{L^2(\mathcal{X},\nu)}.$$

Moreover

$$\|\nabla_Ru_n-\nabla_Ru_m\|_{L^2(\mathcal{X},\nu;H_R)}\leq\sqrt{\frac{2}{\lambda}}\|f_n-f_m\|_{L^2(\mathcal{X},\nu)},$$

then $\{\nabla_Ru_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathcal{X},\nu;H_R)$. By the closability of ∇_R in $L^2(\mathcal{X},\nu)$

(Proposition 6.1.4) it follows that $u \in W_R^{1,2}(\mathcal{X}, \nu)$ and

$$L^2(\mathcal{X}, \nu; H_R)\text{-}\lim_{n\to+\infty} \nabla_R u_n = D_R u.$$

Therefore

$$\|\nabla_R u\|_{L^2(\mathcal{X},\nu;H_R)} = \lim_{n\to+\infty} \|\nabla_R u_n\|_{L^2(\mathcal{X},\nu;H_R)} \leq \lim_{n\to+\infty} \sqrt{\frac{2}{\lambda}}\|f_n\|_{L^2(\mathcal{X},\nu)} = \sqrt{\frac{2}{\lambda}}\|f\|_{L^2(\mathcal{X},\nu)}.$$

Now we prove the moreover part of the statement. Let $\{h_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of H_R. Using (5.0.1), we differentiate the equality $\lambda u_n - N_2 u_n = f_n$ along h_j direction, we multiply the result by $\langle \nabla_R u, h_j\rangle_R$, sum over j and finally integrate over $\mathcal{X}$ with respect to ν. We obtain

$$\begin{aligned}\lambda \int_{\mathcal{X}} \|\nabla_R u_n\|_R^2 d\nu - \int_{\mathcal{X}} \langle \nabla_R u_n, A\nabla_R u_n\rangle_R d\nu + \frac{1}{2}\int_{\mathcal{X}} \|\nabla_R^2 u_n\|_{\mathcal{H}_R}^2 d\nu \\ - \int_{\mathcal{X}} \langle R^2 \nabla G \nabla_R u_n, \nabla_R u_n\rangle_R d\nu = \int_{\mathcal{X}} \langle \nabla_R f_n, \nabla_R u_n\rangle_R d\nu.\end{aligned}$$

Recalling that $\langle (A + R\mathcal{D}G(x))h, h\rangle_R \leq -\zeta\|h\|_R^2$ for every $x \in \mathcal{X}$ and $h \in H_R$ we have

$$(\lambda + \zeta)\int_{\mathcal{X}} \|\nabla_R u_n\|_R^2 d\nu + \frac{1}{2}\int_{\mathcal{X}} \|\nabla_R^2 u_n\|_{\mathcal{H}_R}^2 d\nu \leq \int_{\mathcal{X}} \langle \nabla_R f_n, \nabla_R u_n\rangle_R d\nu.$$

Finally we have

$$\frac{1}{2}\int_{\mathcal{X}} \|\nabla_R^2 u_n\|_{\mathcal{H}_R}^2 d\nu \leq \int_{\mathcal{X}} \langle \nabla_R f_n, \nabla_R u_n\rangle_R d\nu. \tag{6.2.2}$$

By (6.2.1), (6.2.2), the Cauchy–Schwarz inequality and (6.1.2) we have

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{X}} \|\nabla_R^2 u_n\|_{\mathcal{H}_R}^2 d\nu &\leq \int_{\mathcal{X}} \langle \nabla_R f_n, \nabla_R u_n\rangle_R d\nu = -2\int_{\mathcal{X}} f_n N_2 u_n d\nu \\ &= -2\int_{\mathcal{X}} f_n(\lambda u_n - f_n) d\nu \leq 4\int_{\mathcal{X}} f_n^2 d\nu.\end{aligned}$$

So we get

$$\|\nabla_R^2 u_n\|_{L^2(\mathcal{X},\nu;\mathcal{H}_R)} \leq 2\sqrt{2}\|f_n\|_{L^2(\mathcal{X},\nu)}.$$

We remark that

$$\|\nabla_R^2 u_n - \nabla_R^2 u_m\|_{L^2(\mathcal{X},\nu;\mathcal{H}_R)} \leq 2\sqrt{2}\|f_n - f_m\|_{L^2(\mathcal{X},\nu)},$$

then $\{\nabla_R^2 u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathcal{X}, \nu; \mathcal{H}_R)$. By the closability of (∇_R, ∇_R^2) in $L^2(\mathcal{X}, \nu)$ it follows that $u \in W_R^{2,2}(\mathcal{X}, \nu)$ and

$$L^2(\mathcal{X}, \nu; \mathcal{H}_R)\text{-}\lim_{n\to+\infty} \nabla_R^2 u_n = \nabla_R^2 u.$$

Therefore

$$\|\nabla_R^2 u\|_{L^2(\mathcal{X},\nu;\mathcal{H}_R)} = \lim_{n\to+\infty} \|\nabla_R^2 u_n\|_{L^2(\mathcal{X},\nu;\mathcal{H}_R)} \leq \lim_{n\to+\infty} 2\sqrt{2}\|f_n\|_{L^2(\mathcal{X},\nu)} = 2\sqrt{2}\|f\|_{L^2(\mathcal{X},\nu)},$$

and $\{u_n\}_{n\in\mathbb{N}}$ converges to u in $W^{2,2}_R(\mathcal{X},\nu)$. □

Remark 6.2.2. *If $G = R\nabla U$ for some suitable function $U : \mathcal{X} \to \mathbb{R}$ then condition 6.2.1 is verified (see [39, 40]).*

6.3 Poincaré and Logarithmic Sobolev inequalities

Logarithmic Sobolev inequalities are important tools in the study of Sobolev spaces with respect to non-Lebesgue measures. This is due to the fact that they are the counterpart of the Sobolev embeddings which in general fail to hold when the Lebesgue measure is replaced by other measures, as for example the Gaussian one. In this section we also collect some consequences of the logarithmic Sobolev inequality (6.3.1). To do this we need an additional hypothesis.

Hypotheses 6.3.1. *Assume that Hypotheses 6.0.1 hold true and that there exists an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ of $\mathcal{X}$ contained in* $\mathrm{Dom}(A)$.

In this section we will use a technique that needs lowerly bounded functions $\varphi : \mathcal{X} \to \mathbb{R}$. The space $\xi_A(\mathcal{X})$ does not contain such functions, so we will work in the larger space $\mathcal{FC}^1_b(\mathcal{X})$ (see Remark 1.6.16) which contains them.

Now we are ready to prove that the measure ν satisfies a logarithmic Sobolev inequality. We apply the Deuschel and Stroock method (see [50]).

Theorem 6.3.2. *Assume that Hypotheses 6.3.1 hold true. For $p \geq 1$ and $\varphi \in \mathcal{FC}^1_b(\mathcal{X})$, the following inequality holds:*

$$\int_{\mathcal{X}} |\varphi|^p \ln|\varphi|^p d\nu \leq \left(\int_{\mathcal{X}} |\varphi|^p d\nu\right) \ln\left(\int_{\mathcal{X}} |\varphi|^p d\nu\right) + \frac{p^2}{2\zeta}\int_{\mathcal{X}} |\varphi|^{p-2}\|\nabla_R\varphi\|^2_R \chi_{\{\varphi\neq 0\}} d\nu. \tag{6.3.1}$$

Furthermore for every $\varphi \in W^{1,2}_R(\mathcal{X},\nu)$ it holds

$$\int_{\mathcal{X}} |\varphi|^2 \ln|\varphi|^2 d\nu \leq \left(\int_{\mathcal{X}} |\varphi|^2 d\nu\right) \ln\left(\int_{\mathcal{X}} |\varphi|^2 d\nu\right) + \frac{2}{\zeta}\int_{\mathcal{X}} \|\nabla_R\varphi\|^2_R \chi_{\{\varphi\neq 0\}} d\nu. \tag{6.3.2}$$

Proof. We split the proof in two parts. In the first part we prove that the claim holds when φ satisfies some additional conditions and in the second part we show (6.3.1) in its full generality.
Step 1. Here we prove (6.3.1) for functions φ in $\mathcal{FC}^1_b(\mathcal{X})$ such that

$$c \leq \varphi \leq 1,$$

for some $c > 0$. We consider the function

$$H(t) := \int_{\mathcal{X}} (P_2(t)\varphi^p)\ln(P_2(t)\varphi^p)d\nu, \qquad t \geq 0.$$

which is well defined thanks to the contractivity and the positivity preserving property of $P_2(t)$.

Our aim is to find a lower bound for the derivative of H. Observe that by the invariance of

ν and (6.1.3) we have

$$\begin{aligned}
H'(t) &= \int_{\mathcal{X}} (N_2 P_2(t)\varphi^p)\ln(P_2(t)\varphi^p)d\nu + \int_{\mathcal{X}} N_2 P_2(t)\varphi^p d\nu \\
&= -\int_{\mathcal{X}} \frac{1}{P_2(t)\varphi^p}\|\nabla_R P_2(t)\varphi^p\|_R^2 d\nu \geq -e^{-2\zeta t}\int_{\mathcal{X}} \frac{1}{P_2(t)\varphi^p} P_2(t)\|\nabla_R \varphi^p\|_R^2 d\nu \\
&\geq -e^{-2\zeta t}\int_{\mathcal{X}} \frac{1}{P_2(t)\varphi^p}(P_2(t)\|\nabla_R\varphi^p\|_R)^2 d\nu.
\end{aligned}$$

By (5.1.15) we have $P_2(t)\|\nabla_R\varphi^p\|_R \leq [P_2(t)(\|\nabla_R\varphi^p\|_R^2\varphi^{-p})]^{1/2}(P_2(t)\varphi^p)^{1/2}$. Hence we deduce

$$\begin{aligned}
H'(t) &\geq -e^{-2\zeta t}\int_{\mathcal{X}} P_2(t)\frac{\|\nabla_R\varphi^p\|_R^2}{\varphi^p}d\nu \\
&= -e^{-2\zeta t}\int_{\mathcal{X}} \frac{\|\nabla_R\varphi^p\|_R^2}{\varphi^p}d\nu = -e^{-2\zeta t}p^2\int_{\mathcal{X}} \varphi^{p-2}\|\nabla_R\varphi\|_R^2 d\nu.
\end{aligned}$$

Integrating from 0 to $+\infty$ and using (5.1.6) we get

$$\int_{\mathcal{X}} \varphi^p \ln\varphi^p d\nu \leq \left(\int_{\mathcal{X}} \varphi^p d\nu\right)\ln\left(\int_{\mathcal{X}} \varphi^p d\nu\right) + \frac{p^2}{2\zeta}\int_{\mathcal{X}} \varphi^{p-2}\|\nabla_R\varphi\|_R^2 d\nu.$$

Step 2. Now, for any $\varphi \in \mathcal{FC}_b^1(\mathcal{X})$, consider the sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{FC}_b^1(\mathcal{X})$ defined by $\varphi_n = (1+\|\varphi\|_\infty)^{-1}\sqrt{\varphi^2+n^{-1}}$. Step 1 yields

$$\int_{\mathcal{X}} \varphi_n^p \ln(\varphi_n^p)d\nu \leq \left(\int_{\mathcal{X}} \varphi_n^p d\nu\right)\ln\left(\int_{\mathcal{X}} \varphi_n^p d\nu\right) + \frac{p^2}{2\zeta}\int_{\mathcal{X}} \varphi_n^{p-2}\|\nabla_R\varphi_n\|_R^2 d\nu. \tag{6.3.3}$$

Observing that there exists a positive constant $c_{n,p}$ such that $c_{n,p} \leq \varphi_n^p \leq 1$ for any $n \in \mathbb{N}$ and using the fact that the function $x \mapsto x|\ln x|$ is bounded in $(0,1]$, by the dominated convergence theorem the left hand side of (6.3.3) converges to

$$(1+\|\varphi\|_\infty)^{-p}\int_{\mathcal{X}} |\varphi|^p \ln\left[(1+\|\varphi\|_\infty)^{-p}|\varphi|^p\right]d\nu,$$

and the first term in the right hand side of (6.3.3) converges to

$$\left((1+\|\varphi\|_\infty)^{-p}\int_{\mathcal{X}} |\varphi|^p d\nu\right)\ln\left(\frac{\int_{\mathcal{X}} |\varphi|^p d\nu}{(1+\|\varphi\|_\infty)^p}\right).$$

Since $\|\nabla_R\varphi_n\|_R \leq (1+\|\varphi\|_\infty)^{-1}\|\nabla_R\varphi\|_R$ for every $n \in \mathbb{N}$, by the monotone convergence theorem if $p \in [1,2)$, and by the dominated convergence theorem otherwise, we obtain

$$\lim_{n\to+\infty}\int_{\mathcal{X}} {\varphi_n}^{p-2}\|\nabla_R\varphi_n\|_R^2 d\nu = (1+\|\varphi\|_\infty)^{-p}\int_{\mathcal{X}} |\varphi|^{p-2}\|\nabla_R\varphi\|_R^2\chi_{\{\varphi\neq 0\}}d\nu.$$

So the statement follows letting n to infinity in (6.3.3).

Let $\varphi \in W_R^{1,2}(\mathcal{X},\nu)$ then there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{FC}_b^1(\mathcal{X})$ such that

$$\lim_{n\to+\infty}\|\varphi_n-\varphi\|_{W_R^{1,2}(\mathcal{X},\nu)}.$$

By standard arguments there exist a sub-sequence $\{\varphi_{n_k}\}_{k\in\mathbb{N}}$ such that for a.a. $x\in\mathcal{X}$

$$\lim_{k\to+\infty}\varphi_{n_k}(x)=\varphi(x),\qquad \lim_{k\to+\infty}\nabla_R\varphi_{n_k}(x)=\nabla_R\varphi(x).$$

By the Fatou lemma we obtain

$$\begin{aligned}\int_{\mathcal{X}}|\varphi|^2\ln|\varphi|^2d\nu &\leq \liminf_{k\to+\infty}\int_{\mathcal{X}}|\varphi_{n_k}|^2\ln|\varphi_{n_k}|^2d\nu\\ &\leq \liminf_{k\to+\infty}\left[\left(\int_{\mathcal{X}}|\varphi_{n_k}|^2d\nu\right)\ln\left(\int_{\mathcal{X}}|\varphi_{n_k}|^2d\nu\right)+\frac{2}{\zeta}\int_{\mathcal{X}}\|\nabla_R\varphi_{n_k}\|_R^2\chi_{\{\varphi_{n_k}\neq 0\}}d\nu\right]\\ &\leq \left(\int_{\mathcal{X}}|\varphi|^2d\nu\right)\ln\left(\int_{\mathcal{X}}|\varphi|^2d\nu\right)+\frac{2}{\zeta}\int_{\mathcal{X}}\|\nabla_R\varphi\|_R^2\chi_{\{\varphi\neq 0\}}d\nu,\end{aligned}$$

and so we get (6.3.2). □

By [63, 64], the logarithmic Sobolev inequality is equivalent to a hypercontractivity type estimate.

Proposition 6.3.3. *Assume that Hypotheses 6.3.1 hold true. Let $t>0$ and $q,r\in(1,+\infty)$ be such that $r\leq(q-1)e^{2\zeta t}+1$. The operator $P_q(t)$ maps $L^q(\mathcal{X},\nu)$ in $L^r(\mathcal{X},\nu)$ and for every $t>0$ and $\varphi\in L^q(\mathcal{X},\nu)$ we have*

$$\|P_q(t)\varphi\|_{L^r(\mathcal{X},\nu)}\leq\|\varphi\|_{L^q(\mathcal{X},\nu)}. \tag{6.3.4}$$

Proof of Proposition 6.3.3. Let $\varphi\in\mathcal{FC}_b^1(\mathcal{X})$, have positive infimum, and let $r(t):=(q-1)e^{2\zeta t}+1$. We recall that $P_q(t)\varphi=P_2(t)\varphi$, for any $\varphi\in\mathcal{FC}_b^1(\mathcal{X})$ and $t>0$. For $s\geq 0$ we set

$$G(s):=\left(\int_{\mathcal{X}}(P_2(s)\varphi)^{r(s)}d\nu\right)^{1/r(s)}=:(R(s))^{1/r(s)}$$

and we prove that $G(s)$ is a non-increasing function. Before proceeding we want to recall that $P_2(s)$ maps $\mathcal{F}C_b^1(\mathcal{X})$ into $W_R^{1,2}(\mathcal{X},\nu)\cap L^\infty(\mathcal{X},\nu)$. This guarantees that all the integrals we are going to write in the following calculations are well defined and finite. By (6.1.3) we obtain

$$R'(s)=r'(s)\int_{\mathcal{X}}(P_2(s)\varphi)^{r(s)}\ln(P_2(s)\varphi)d\nu-r(s)(r(s)-1)\int_{\mathcal{X}}(P_2(s)\varphi)^{r(s)-2}\|\nabla_RP_2(s)\varphi\|_R^2d\nu. \tag{6.3.5}$$

Taking into account (6.3.5), if we set $u(s):=P_2(s)\varphi$ and we differentiate G, we get

$$\begin{aligned}G'(s)=&\frac{r'(s)}{r(s)\int_{\mathcal{X}}(u(s))^{r(s)}d\nu}\int_{\mathcal{X}}(u(s))^{r(s)}\ln(u(s))d\nu\\ &-\frac{r(s)-1}{\int_{\mathcal{X}}(u(s))^{r(s)}d\nu}\int_{\mathcal{X}}(u(s))^{r(s)-2}\|\nabla_Ru(s)\|_R^2d\nu-\frac{r'(s)}{r^2(s)}\ln\left(\int_{\mathcal{X}}(u(s))^{r(s)}d\nu\right)\end{aligned}$$

Since $r'(s)\geq 0$ we can apply (6.3.1) to get

$$G'(s)\leq(G(s))^{1-r(s)}\left(\frac{r'(s)}{2\zeta}-r(s)+1\right)\int_{\mathcal{X}}(P_2(s)\varphi)^{r(s)-2}\|\nabla_RP_2(s)\varphi\|_R^2d\nu=0.$$

This proves that G is a decreasing function, namely $G(0) \geq G(t)$ for every $t > 0$. So we have for every $r \leq r(t)$ and $\varphi \in \mathcal{FC}_b^1(\mathcal{X})$

$$\|P_q(t)\varphi\|_{L^r(\mathcal{X},\nu)} \leq \|P_2(t)\varphi\|_{L^{r(t)}(\mathcal{X},\nu)} \leq \|\varphi\|_{L^q(\mathcal{X},\nu)}.$$

By the same density argument used in the last part of Proof of Theorem 6.3.2, we obtain (6.3.4) for a general $\varphi \in L^q(\mathcal{X},\nu)$. □

An interesting consequence of Proposition 6.3.3 is an improvement of positivity property for the semigroup $P_2(t)$.

Corollary 6.3.4. *Assume that Hypotheses 6.3.1 hold true. For any $t > 0$ the semigroup $P_2(t)$ is positivity improving, meaning that it maps a ν-a.e. non-negative function in a ν-a.e. positive function.*

Proof. The proof is classical and we just sketch it. By [81, Theorem 1.7] and the classical reverse Hölder and Minkowski inequalities for $p < 1$ it is possible to prove a reverse hypercontratcivity estimate such as that of [15, Section 2]. This is enough to obtain positivity improving, see [15, Theorem 2.1]. □

Another classical inequality that follows from (6.1.1) is the Poincaré inequality.

Theorem 6.3.5. *Assume that Hypotheses 6.3.1 hold true. If $\varphi \in W^{1,2}_R(\mathcal{X},\nu)$, then*

$$\int_{\mathcal{X}} \left|\varphi - \int_{\mathcal{X}} \varphi d\nu\right|^2 d\nu \leq \frac{1}{2\zeta}\int_{\mathcal{X}} \|\nabla_R\varphi\|_R^2 d\nu. \tag{6.3.6}$$

Proof. We just show the theorem for $\varphi \in \xi_A(\mathcal{X})$, the general case follows by a standard approximation argument. Letting t go to infinity in (6.1.5), using (6.1.1) and the invariance of ν we get

$$\begin{aligned}
\int_{\mathcal{X}} |\varphi|^2 d\nu - \left|\int_{\mathcal{X}} \varphi d\nu\right|^2 &= \int_0^{+\infty}\int_{\mathcal{X}} \|\nabla_R P_2(s)\varphi\|_R^2 d\nu ds\\
&\leq \int_0^{+\infty} e^{-2\zeta s}\int_{\mathcal{X}} P_2(s)\|\nabla_R\varphi\|_R^2 d\nu ds\\
&= \left(\int_0^{+\infty} e^{-2\zeta s} ds\right)\left(\int_{\mathcal{X}} \|\nabla_R\varphi\|_R^2 d\nu\right)\\
&= \frac{1}{2\zeta}\int_{\mathcal{X}} \|\nabla_R\varphi\|_R^2 d\nu.
\end{aligned}$$

Recalling that $\int_{\mathcal{X}} \left|\varphi - \int_{\mathcal{X}} \varphi d\nu\right|^2 d\nu = \int_{\mathcal{X}} |\varphi|^2 d\nu - \left|\int_{\mathcal{X}} \varphi d\nu\right|^2$ we get the thesis. □

The Poincaré inequality has many interesting consequences. Here we just state two of them which are relevant to the study of the semigroup $P_2(t)$ and of its generator N_2 in $L^2(\mathcal{X},\nu)$. The next result gives us the convergence rate of $P_2(t)\varphi$ to $\int_{\mathcal{X}} \varphi d\nu$ in $L^2(\mathcal{X},\nu)$ when t goes to infinity.

Corollary 6.3.6. *Assume that Hypotheses 6.3.1 hold true. If $\varphi \in L^2(\mathcal{X},\nu)$, then*

$$\left\| P_2(t)\varphi - \int_{\mathcal{X}} \varphi d\nu \right\|_{L^2(\mathcal{X},\nu)} \le e^{-\zeta t}\|\varphi\|_{L^2(\mathcal{X},\nu)}.$$

Proof. Let $f \in \mathrm{Dom}(N_2)$, we set $G(s) := \int_{\mathcal{X}} \left|P_2(s)\varphi - \int_{\mathcal{X}} \varphi d\nu\right|^2 d\nu$. Using both (6.3.6) and (6.1.3) we get

$$\begin{aligned} G'(s) &= \frac{d}{ds}\int_{\mathcal{X}} \left|P_2(s)\varphi - \int_{\mathcal{X}} \varphi d\nu\right|^2 d\nu = 2\int_{\mathcal{X}} (P_2(s)\varphi)(N_2P_2(s)\varphi)d\nu \\ &= -\int_{\mathcal{X}} \|\nabla_R P_2(s)\varphi\|_R^2 d\nu \le -2\zeta \int_{\mathcal{X}} \left|P_2(s)\varphi - \int_X P_2(s)\varphi d\nu\right|^2 d\nu \\ &= -2\zeta \int_{\mathcal{X}} \left|P_2(s)\varphi - \int_{\mathcal{X}} \varphi d\nu\right|^2 d\nu = -2\zeta G(s). \end{aligned}$$

Thus $G(t) \le e^{-2\zeta t}G(0)$, which means

$$\begin{aligned} \int_{\mathcal{X}} \left|P_2(t)\varphi - \int_{\mathcal{X}} \varphi d\nu\right|^2 d\nu \le & e^{-2\zeta t}\int_{\mathcal{X}} \left|\varphi - \int_{\mathcal{X}} \varphi d\nu\right|^2 d\nu \\ =& e^{-2\zeta t}\int_{\mathcal{X}} |\varphi|^2 d\nu - \left|\int_{\mathcal{X}} \varphi d\nu\right|^2 d\nu \\ \le & e^{-2\zeta t}\int_{\mathcal{X}} |\varphi|^2 d\nu. \end{aligned}$$

Finally, since $\mathrm{Dom}(N_2)$ is dense in $L^2(\mathcal{X},\nu)$, we obtain the statement. □

The next proposition gives us a spectral gap for the operator N_2. We refer to [42, Proposition 10.5.1] for the proof.

Proposition 6.3.7. *If Hypotheses 6.3.1 hold true, then $\sigma(N_2) \setminus \{0\} \subseteq \{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda \le -\zeta\}$.*

6.4 Remarks and examples

The results presented in this chapter are contained in the paper [10]. Results similar to the ones of Theorems 6.3.2 and 6.3.5 are present in the literature only in specific settings. In [5, 16, 17] and in [42, Section 12] the authors assume that $F = -\mathcal{D}U$ where $U : \mathcal{X} \to \mathbb{R}$ is a convex function with Lipschitz continuous derivative and $R = \mathrm{I}_{\mathcal{X}}$ or $R = Q^{1/2}$, where Q is a positive, self-adjoint and trace class operator. In [30, Section 3.6], [32] and [42, Section 11] the authors consider a Lipschitz continuous function F and $R = \mathrm{I}_{\mathcal{X}}$. In [70] the authors assume hypotheses similar to the ones of this chapter, but they work in a finite dimensional setting.

The search for Sobolev regularity of domain of N_2 and of logarithmic Sobolev and Poincaré inequalities have already been done by various authors in the infinite dimensional setting. In [30, Section 3.6.1] and [32] the authors assume that R has a continuous inverse and work with the Sobolev space $W^{1,2}(\mathcal{X},\nu)$ defined as the domain of the closure in $L^2(\mathcal{X},\nu)$ of the standard Fréchet gradient operator $\nabla : \xi_A(\mathcal{X}) \subseteq L^2(\mathcal{X},\nu) \to L^2(\mathcal{X},\nu;\mathcal{X})$. We emphasize that the case when R has a continuous inverse presents no significant differences in defining and studying Sobolev spaces compared to the case when $R = \mathrm{I}_{\mathcal{X}}$. In [16] and [59] the authors assume that $R = Q^{1/2}$, where Q is a positive trace class operator, and $G = -Q^{1/2}\nabla U$ where $U : \mathcal{X} \to \mathbb{R}$ is a Fréchet differentiable and convex function, such that ∇U is Lipschitz continuous. They consider the Sobolev space $W^{1,2}_Q(\mathcal{X}, \nu)$ defined as the closure in $L^2(\mathcal{X}, \nu)$ of the operator $Q^{1/2}\nabla : \xi_A(\mathcal{X}) \subseteq L^2(\mathcal{X}, \nu) \to L^2(\mathcal{X}, \nu; \mathcal{X})$. We underline that, if $G = -Q^{1/2}\nabla U$, then the invariant measure ν is the weighted Gaussian measure

$$\nu(dx) = \frac{e^{-U}}{\int_{\mathcal{X}} e^{-U}\mu(dx)}\mu(dx), \quad \mu \sim \mathcal{N}(0, Q),$$

and N_2 is the self-adjoint operator associated to the quadratic form

$$G(\varphi, \psi) = \int_{\mathcal{X}} \langle Q^{1/2}\nabla\varphi, Q^{1/2}\nabla\psi\rangle d\nu, \qquad \varphi, \psi \in W^{1,2}_Q(\mathcal{X}, \nu).$$

Conversely, if G is not of that form, then N_2 is not necessarily associated to a quadratic form. In this chapter we have revised the methods of the above mentioned papers, to avoid the conditions $R^{-1} \in \mathcal{L}(\mathcal{X})$ or $G = -Q^{1/2}\nabla U$.

We conclude this section with an explicit example where Hypotheses 6.3.1 are verified.

6.4.1 An example in $L^2([0,1], \lambda)$

Let $\mathcal{X} = L^2([0,1], \lambda)$ where λ is the Lebesgue measure. Let $-Q^{-1}$ be the realization of the second order derivative with Dirichlet boundary condition in $L^2([0,1], \lambda)$. Hence Q is a positive trace class operator. Let $A = -(1/2)Q^{-\beta}$ and $R = Q^{\alpha}$, with $\alpha, \beta \geq 0$ as in Subsection 3.4.2. We refer to subsection 3.4.2 for remarks about the hypotheses on A and R. The constant λ_1 in (3.4.1) and (3.4.2) is equal to π^{-2} (see [30, Chapter 4]). Since G is Lipschitz continuous, it easy to see that $F = RG$ is Lipschitz continuous with Lipschitz constant $\|R\|L_G$. Moreover due to the choice of R we have $\|R\|L_G \leq \pi^{-2\alpha}L_G$ and so

$$\begin{aligned}
&\langle R\mathcal{D}G(x)h, h\rangle \leq \pi^{-2\alpha}L_G\|h\|^2, \quad x, h \in \mathcal{X},\\
&\langle R\mathcal{D}G(x)h, h\rangle_R \leq \pi^{-2\alpha}L_G\|h\|_R^2, \quad x \in \mathcal{X},\ h \in H_R.
\end{aligned}$$

If we assume $L_G < (1/2)\pi^{2\alpha+\beta}$ then the conditions (6.0.2) and (6.0.3) are verified and so the hypotheses of Theorems 6.2.1, 6.3.2 and 6.3.5 hold true.

Chapter 7

Dirichlet semigroup associated to a dissipative gradient systems

Assume that Hypotheses 5.3.1 hold true and let ν be the probability measure of Theorem 5.1.3. Let $\mathcal{O} \subseteq \mathcal{X}$ be an open set such that $\nu(\mathcal{O}) > 0$. We consider the Dirichlet semigroup associated to the SPDE (2.0.1), defined by

$$P^{\mathcal{O}}(t)\varphi(x) := \mathbb{E}\left[\varphi(X(t,x))\mathbb{I}_{\{\tau_x > t\}}\right] := \int_{\Omega} \varphi(X(t,x))\mathbb{I}_{\{\tau_x>t\}}d\mathbb{P}, \quad \varphi \in B_b(\mathcal{O}),$$

where $X(t,x)$ is the generalized mild solution of the SPDE (2.0.1) and τ_x is the stopping time defined by

$$\tau_x = \inf\{t > 0 \ : \ X(t,x) \in \mathcal{O}^c\}. \tag{7.0.1}$$

Now we prove that ν is subinvariant for $P^{\mathcal{O}}(t)$.

Proposition 7.0.1. *Assume that Hypotheses 5.3.1 hold true. For any $\varphi \in B_b(\mathcal{O})$, $t > 0$, we have*

$$\int_{\mathcal{O}} (P^{\mathcal{O}}(t)\varphi)^2 d\nu \leq \int_{\mathcal{O}} \varphi^2 d\nu.$$

Proof. Let

$$\widehat{\varphi}(x) = \begin{cases} \varphi(x) & x \in \mathcal{O}, \\ 0 & x \in \mathcal{O}^c, \end{cases}$$

then, by the Hölder inequality, we have

$$(P^{\mathcal{O}}(t)\varphi)^2 \leq \mathbb{E}[\varphi^2(X(t,x))\mathbb{I}_{\{\tau_x \geq t\}}] \leq \mathbb{E}[\widehat{\varphi}^2(X(t,x))\mathbb{I}_{\{\tau_x \geq t\}}] = P(t)(\widehat{\varphi}^2).$$

Since ν is invariant for $P(t)$ and $P(t)$ is non-negative (see definition at beginning of Section 1.4), then we conclude

$$\int_{\mathcal{O}} (P^{\mathcal{O}}(t)\varphi)^2 d\nu \leq \int_{\mathcal{O}} P(t)(\widehat{\varphi}^2) d\nu \leq \int_{\mathcal{X}} P(t)(\widehat{\varphi}^2) d\nu = \int_{\mathcal{X}} \widehat{\varphi}^2 d\nu = \int_{\mathcal{O}} \varphi^2 d\nu.$$

□

By Proposition 7.0.1, proceeding as in Proposition 5.1.6, the semigroup $P^{\mathcal{O}}(t)$ is uniquely extendable to a strongly continuous semigroup $P_2^{\mathcal{O}}(t)$ in $L^2(\mathcal{O},\nu)$.

Remark 7.0.2. *By the same arguments $P^{\mathcal{O}}(t)$ is uniquely extendable to a strongly continuous semigroup $L^p(\mathcal{O},\nu)$, for any $p\geq 1$.*

Definition 7.0.3. *We denote by M_2 the infinitesimal generator of $P_2^{\mathcal{O}}(t)$.*

In this chapter, under some additional hypotheses, we will characterize M_2 using a technique of [8, 33]. This technique requires that it is possible to associate a quadratic form to the operator N_2 (see Definition 5.1.7). Hence we restrict to the case where F is a gradient perturbation, namely it has a potential. In this case the invariant measure ν is a weighted Gaussian measure. In Section 7.1 we recall some known results that guarantee that it is possible in this case to define the Sobolev space $W_R^{1,2}(\mathcal{X},\nu)$ and to associate a quadratic form $\mathcal{Q}_2$ to N_2 (see Definition 5.1.7), namely

$$\int_{\mathcal{X}}(N_2\varphi)\psi d\nu=\mathcal{Q}_2(\varphi,\psi)=-\frac{1}{2}\int\langle R\nabla\varphi,R\nabla\psi\rangle d\nu,\quad \forall\varphi\in\mathrm{Dom}(N_2),\ \psi\in W_R^{1,2}(\mathcal{X},\nu).$$

see for example [2, 5, 16, 17, 29, 34, 35, 36, 39, 40, 59]. Due to this, proceeding as in [33, Section 3], we will consider a suitable space $\mathring{W}_R^{1,2}(\mathcal{X},\nu)$ of the functions $u:\mathcal{O}\to\mathbb{R}$ such that their null extension $\widehat{u}$ belongs to $W_R^{1,2}(\mathcal{X},\nu)$, and the quadratic form $\mathcal{Q}_2^{\mathcal{O}}$ on $\mathring{W}_R^{1,2}(\mathcal{O},\nu)$ defined by

$$\mathcal{Q}_2^{\mathcal{O}}(\varphi,\psi)=\mathcal{Q}_2(\widehat{\varphi},\widehat{\psi}),\quad \forall\varphi,\psi\in\mathring{W}_R^{1,2}(\mathcal{O},\nu).$$

In Section7.3 we are going to prove the main result of this chapter.

Theorem 7.0.4. *Assume that Hypotheses 7.1.2 hold true. Then the infinitesimal generator M_2 of $P_2^{\mathcal{O}}(t)$ is the operator $N_2^{\mathcal{O}}$ associated with $\mathcal{Q}_2^{\mathcal{O}}$, namely*

$$\mathrm{Dom}(N_2^{\mathcal{O}}):=\{\varphi\in\mathring{W}_R^{1,2}(\mathcal{O},\nu)\ :\ \exists\beta\in L^2(\mathcal{O},\nu)\ s.t.\ \int_{\mathcal{O}}\beta\psi d\nu=\mathcal{Q}_2^{\mathcal{O}}(\beta,\psi)\ \forall\psi\in\mathring{W}_R^{1,2}(\mathcal{X},\nu)\}$$
$$N_2^{\mathcal{O}}\varphi=\beta,\quad \varphi\in\mathrm{Dom}(N_2^{\mathcal{O}}).$$

7.1 The Sobolev spaces

We begin by stating some additional assumptions about the operators A and R in the SPDE (2.0.1).

Hypotheses 7.1.1. *Assume that Hypotheses 5.3.1 hold true and the following conditions hold true.*

(i) *R is non-negative.*

(ii) *$A:\mathrm{Dom}(A)\subset\mathcal{X}\to\mathcal{X}$ is self-adjoint and there exist $w>0$ and $M>0$ such that*

$$\left\|e^{tA}\right\|_{\mathcal{L}(\mathcal{X})}\leq Me^{-wt},\quad t\geq 0.$$

(iii) $R(\mathrm{Dom}(A)) \subseteq \mathrm{Dom}(A)$, *and* $ARx = RAx$ *for any* $x \in \mathrm{Dom}(A)$.

Under Hypotheses 7.1.1 the operator

$$Q_\infty = \int_0^\infty e^{tA} R^2 e^{tA^*} dt$$

is a positive and trace class operator. Let $\mu \sim \mathcal{N}(0, Q_\infty)$ and let $W_R^{1,2}(\mathcal{X}, \mu)$ be the Sobolev space given by Definition 6.1.5 (with $G \equiv 0$ in (6.0.1)).

Hypotheses 7.1.2. *Assume that Hypotheses 5.1.1 and 7.1.1 hold true and that there exists a lower semicontinuous function* $U : \mathcal{X} \to \mathbb{R}$ *such that*

1. $\|x\|^2 e^{-2U} \in L^1(\mathcal{X}, \mu)$ *and* $e^{-2U} \in W^{1,2}(\mathcal{X}, \mu)$*;*
2. $F = -R^2 \nabla U$.

Under Hypotheses 7.1.2 the SPDE (2.0.1) becomes

$$\begin{cases} dX(t,x) = \big(AX(t,x) - R^2 \nabla U(X(t,x))\big)dt + RdW(t), \quad t \in [0,T]; \\ X(0,x) = x \in \mathcal{X}, \end{cases}$$

the operator (5.0.1) reads as

$$N_0\varphi(x) := \frac{1}{2}\mathrm{Tr}[R^2\nabla^2\varphi(x)] + \langle x, A^*\nabla\varphi(x)\rangle - \langle R\nabla U(x), \nabla\varphi(x)\rangle, \qquad \varphi \in \xi_A(\mathcal{X}),\ x \in \mathcal{X},$$

Moreover the following results are verified.

Proposition 7.1.3. *Assume that Hypotheses 7.1.2 hold true.*

1. *The invariant measure* ν *of* $P(t)$ *has the form*

$$\nu(dx) = \frac{e^{-2U(x)}}{B}\mu(dx), \quad B := \int_{\mathcal{X}} e^{-2U(x)}\mu(dx).$$

2. *The operator*

$$R\nabla : \xi_A(\mathcal{X}) \subseteq L^2(X, \nu) \to L^2(X, \nu, \mathcal{X})$$

is closable, and we denote by $W_R^{1,2}(\mathcal{X}, \nu)$ *its domain.*

3. *For any* $\varphi \in W_R^{1,2}(X, \nu)$ *and* $\psi \in \mathrm{Dom}(N_2)$ *we have*

$$\int_{\mathcal{X}} (N_2\psi)\varphi d\nu = \mathcal{Q}_2(\varphi, \psi) := -\frac{1}{2}\int_{\mathcal{X}} \langle R\nabla\varphi, R\nabla\psi\rangle d\nu.$$

We refer to [40, Sections 3-4] or [39] for a proof. Similarly to [33, Section 2], we define the following space.

Definition 7.1.4. *We denote by* $\mathring{W}_R^{1,2}(\mathcal{O}, \nu)$ *the space of the functions* $u : \mathcal{O} \longrightarrow \mathbb{R}$ *such the extension* $\widehat{u} : \mathcal{X} \to \mathbb{R}$ *defined by*

$$\widehat{u}(x) = \begin{cases} u(x) & x \in \mathcal{O} \\ 0 & x \in \mathcal{O}^c \end{cases}$$

belongs to $W^{1,2}_R(\mathcal{O},\nu)$.

Remark 7.1.5. *If* R *is positive then* Q_∞ *is positive and* μ *is non-degenerate Gaussian measure. Hence* ν *is non-degenerate and so* $\nu(\mathcal{O}) > 0$ *for any* $\mathcal{O} \in \mathcal{B}(\mathcal{X})$.

Now we define a quadratic form on $\mathring{W}^{1,2}_R(\mathcal{O},\nu)$.

Definition 7.1.6. *We denote by* $\mathcal{Q}^{\mathcal{O}}_2$ *the quadratic form*

$$\mathcal{Q}^{\mathcal{O}}_2(\varphi,\psi) := -\frac{1}{2}\int_{\mathcal{X}} \left\langle R\nabla\widehat{\varphi}, R\nabla\widehat{\psi}\right\rangle d\nu, \qquad \varphi,\psi \in \mathring{W}^{1,2}_R(\mathcal{O},\nu).$$

Moreover, we denote by $N^{\mathcal{O}}_2$ *the self-adjoint dissipative operator associated to* $\mathcal{Q}^{\mathcal{O}}_2$*, namely*

$$\mathrm{Dom}(N^{\mathcal{O}}_2) := \{\varphi \in \mathring{W}^{1,2}_R(\mathcal{O},\nu) \ : \ \exists \beta \in L^2(\mathcal{O},\nu) \ s.t. \int_{\mathcal{O}} \beta\psi d\nu = \mathcal{Q}^{\mathcal{O}}_2(\beta,\psi) \ \ \forall \psi \in \mathring{W}^{1,2}_R(\mathcal{X},\nu)\},$$
$$N^{\mathcal{O}}_2\varphi = \beta, \quad \varphi \in \mathrm{Dom}(N^{\mathcal{O}}_2).$$

In section 7.3 we shall prove Theorem 7.0.4 using the following procedure. For $\lambda > 0$ and $f \in L^2(\mathcal{O},\nu)$, we consider the equation with unknown $\varphi \in \mathring{W}^{1,2}_R(\mathcal{O},\nu)$,

$$\lambda\int_{\mathcal{O}} \varphi v d\nu - Q^{\mathcal{O}}_2(\varphi,v) = \int_{\mathcal{O}} f v d\nu, \quad v \in \mathring{W}^{1,2}_R(\mathcal{O},\nu). \tag{7.1.1}$$

Since the quadratic form $-\mathcal{Q}^{\mathcal{O}}_2$ is continuous, nonnegative, coercive and symmetric, by the Lax–Milgram Theorem for every $\lambda > 0$ and $f \in L^2(\mathcal{O},\nu)$ there exists a unique solution $\varphi \in \mathring{W}^{1,2}_R(\mathcal{O},\nu)$ of (7.1.1). By definition of $N^{\mathcal{O}}_2$, for every $\lambda > 0$ and $f \in L^2(\mathcal{O},\nu)$, we have

$$R(\lambda, N^{\mathcal{O}}_2)f = \varphi,$$

where φ is the unique solution of (7.1.1). In Subsection 7.3, we will prove that

$$R(\lambda, M_2) = R(\lambda, N^{\mathcal{O}}_2),$$

which yields Theorem 7.0.4.

Remark 7.1.7. *We stress that in the trivial case, where* $\mathcal{O} = \mathcal{X}$*, Theorem 7.0.4 follows from Theorem 5.3.3 and Proposition 7.1.3.*

7.2 The approximating semigroups

In this section we define and study the Feynman-Kac approximating semigroups for the semigroup $P^{\mathcal{O}}_2$. For $\epsilon > 0$ we define

1. the set
$$\mathcal{O}_\epsilon := \{x \in \mathcal{O} \mid d(x,\mathcal{O}^c) > \epsilon\}; \tag{7.2.1}$$

2. the function
$$V_\epsilon(x) := \left(\frac{1}{\epsilon}d(x,\mathcal{O}_\epsilon)\right) \wedge 1, \quad x \in \mathcal{X}. \tag{7.2.2}$$

We note that $V \in C_b(\mathcal{X})$, $V \equiv 0$ on $\overline{\mathcal{O}_\epsilon}$ and $V \equiv 1$ on $\mathcal{O}^c$;

3. the semigroup

$$P^\epsilon(t)\varphi(x) = \mathbb{E}\left[\varphi(X(t,x))e^{-\frac{1}{\epsilon}\int_0^t V_\epsilon(X(s,x))ds}\right], \quad \varphi \in B_b(\mathcal{X}),\ x \in \mathcal{X},$$

where $\{X(t,x)\}_{t\geq 0}$ is the mild solution of the SPDE (2.0.1).

First we prove that $P^\epsilon(t)$ is uniquely extendable to a strongly continuous semigroup in $L^2(\mathcal{X},\nu)$.

Lemma 7.2.1. *For any $\varphi \in C_b(\mathcal{X})$ we have*

$$\|P^\epsilon(t)\varphi\|_{L^2(\mathcal{X},\nu)} \leq \|\varphi\|_{L^2(\mathcal{X},\nu)}.$$

Proof. By the Hölder inequality and the fact that V is nonnegative on $\mathcal{X}$,

$$|P^\epsilon(t)\varphi(x)|^2 \leq \mathbb{E}[\varphi^2(X(t,x))e^{-\frac{2}{\epsilon}\int_{\mathcal{X}} V_\epsilon(X(s,x))ds}] \leq P(t)(\varphi^2)(x), \quad x \in \mathcal{X}.$$

Hence, since ν is invariant for $P(t)$, we have

$$\int_{\mathcal{X}} |P^\epsilon(t)\varphi(x)|^2\nu(dx) \leq \int_{\mathcal{X}} P(t)(\varphi^2)(x)\nu(dx) \leq \int_{\mathcal{X}} \varphi^2(x)\nu(dx).$$

□

By Lemma (7.2.1) and the same procedure using in the proof of the Proposition 5.1.6, the semigroup $P^\epsilon(t)$ is uniquely extendable in $L^2(X,\nu)$ to a strongly continuous and contraction semigroup $P_2^\epsilon(t)$.

Definition 7.2.2. *We denote by N_2^ϵ the infinitesimal generator of $P_2^\epsilon(t)$.*

We recall that N_2 is both the closure of the operator N_0 in $L^2(\mathcal{X},\nu)$ and the infinitesimal generator of $P_2(t)$ (see Subsection 5.3).

Proposition 7.2.3. *Let $\lambda > 0$ and $f \in L^2(\mathcal{X},\nu)$. Then the equation*

$$\lambda\varphi_\epsilon - N_2\varphi_\epsilon + \frac{1}{\epsilon}V_\epsilon\varphi_\epsilon = f \tag{7.2.3}$$

has a unique solution $\varphi_\epsilon \in \mathrm{Dom}(N_2)$. Moreover the following estimates are verified.

$$\|\varphi_\epsilon\|^2_{L^2(\mathcal{X},\nu)} \leq \frac{1}{\lambda^2}\|f\|^2_{L^2(\mathcal{X},\nu)}. \tag{7.2.4}$$

$$\|R^2\nabla\varphi_\epsilon\|^2_{L^2(\mathcal{X},\nu,\mathcal{X})} \leq \frac{2}{\lambda}\|f\|_{L^2(\mathcal{X},\nu)}. \tag{7.2.5}$$

$$\int_{\mathcal{O}_\epsilon^c} V_\epsilon\varphi_\epsilon^2 d\nu \leq \frac{\epsilon}{\lambda}\|f\|^2_{L^2(\mathcal{X},\nu)}. \tag{7.2.6}$$

Proof. By Proposition 7.1.3, N_2 is maximal dissipative. Let $G : L^2(\mathcal{X},\nu) \to L^2(\mathcal{X},\nu)$ be the operator defined by

$$G\varphi := \frac{1}{\epsilon}V_\epsilon \cdot \varphi,$$

then $-G$ is dissipative. So the operator $K : \mathrm{Dom}(N_2) \to L^2(\mathcal{X}, \nu)$, defined by

$$K\varphi := N_2\varphi - G\varphi$$

is maximal dissipative. Therefore (7.2.3) has a unique solution $\varphi_\epsilon \in \mathrm{Dom}(N_2)$ and (7.2.4) is verified. Multiplying both sides of (7.2.3) by φ_ϵ, integrating over $\mathcal{X}$, and taking into account (7.1.3), we obtain

$$\lambda\|\varphi_\epsilon\|^2_{L^2(\mathcal{X},\nu)} + \frac{1}{2}\|R\nabla\varphi_\epsilon\|^2_{L^2(\mathcal{X},\nu)} + \frac{1}{\epsilon}\int_{\mathcal{O}^c_\epsilon} V_\epsilon\varphi^2_\epsilon d\nu = \int_{\mathcal{X}} f\varphi_\epsilon d\nu.$$

By the Hölder inequality $\int_{\mathcal{X}} |f\varphi_\epsilon| d\nu \le \|f\|_{L^2(\mathcal{X},\nu)}\|\varphi_\epsilon\|_{L^2(\mathcal{X},\nu)}$ and, by estimate (7.2.4), we obtain $\int_{\mathcal{X}} |f\varphi_\epsilon| d\nu \le \frac{1}{\lambda}\|f\|^2_{L^2(\mathcal{X},\nu)}$. Then

$$\frac{1}{2}\|R\nabla\varphi_\epsilon\|^2_{L^2(\mathcal{X},\nu)} + \frac{1}{\epsilon}\int_{\mathcal{O}^c_\epsilon} V_\epsilon\varphi^2_\epsilon d\nu \le \frac{1}{\lambda}\|f\|^2_{L^2(\mathcal{X},\nu)}.$$

which yields (7.2.5) and (7.2.6). □

Now we characterize N_2^ϵ the infinitesimal generator of $P_2^\epsilon(t)$.

Proposition 7.2.4. *For any $\epsilon > 0$, we have* $\mathrm{Dom}(N_2^\epsilon) = \mathrm{Dom}(N_2)$ *and*

$$N_2^\epsilon\varphi = N_2\varphi - \frac{1}{\epsilon}V_\epsilon\varphi, \quad \forall\varphi \in \mathrm{Dom}(N_2). \tag{7.2.7}$$

Proof. First we prove that $\mathrm{Dom}(N_2) \subset \mathrm{Dom}(N_2^\epsilon)$. We begin to show that $\xi_A(\mathcal{X}) \subset \mathrm{Dom}(N_2^\epsilon)$. Let $\varphi \in \xi_A(\mathcal{X})$. For any $x \in \mathcal{X}$ and $h > 0$, we have

$$P_2^\epsilon(h)\varphi(x) - \varphi(x) = P_2(h)\varphi(x) - \varphi(x) + \mathbb{E}[(e^{-\frac{1}{\epsilon}\int_0^h V_\epsilon(X(s,x))ds} - 1)\varphi(X(h,x))]. \tag{7.2.8}$$

Dividing both sides of (7.2.8) by $h > 0$ we obtain

$$\frac{P_2^\epsilon(h)\varphi(x) - \varphi(x)}{h} = \frac{P_2(h)\varphi(x) - \varphi(x) + \mathbb{E}[(e^{-\frac{1}{\epsilon}\int_0^h V_\epsilon(X(s,x))ds} - 1)\varphi(X(h,x))]}{h}.$$

By Theorem 5.3.3, we know that

$$\lim_{h\to 0}\frac{P_2(h)\varphi - \varphi}{h} = N_2\varphi, \quad \text{in } L^2(\mathcal{X}, \nu) \tag{7.2.9}$$

Since the generalized mild solution $X(\cdot, x)$ is continuous $\mathbb{P}$-a.s. (see Theorem 2.1.13 and Definition 2.1.5), then the functions $r \longrightarrow V_\epsilon(X(r, x))$ and $r \longrightarrow \varphi(X(r, x))$ are paths continuous, and so, recalling that $V_\epsilon \in C_b(\mathcal{X})$, for any $x \in \mathcal{X}$ we have

$$\lim_{h\to 0}\frac{\mathbb{E}[(e^{-\frac{1}{\epsilon}\int_0^h V_\epsilon(X(s,x))ds} - 1)\varphi(X(h,x))]}{h} = -\frac{1}{\epsilon}V_\epsilon(x)\varphi(x).$$

Hence, noting that

$$\frac{(e^{-\frac{1}{\epsilon}\int_0^h V_\epsilon(X(s,x))ds}-1)\varphi(X(h,x))}{h} \leq \frac{(1-e^{-\frac{1}{\epsilon}h})\|\varphi\|_\infty}{h}, \quad \mathbb{P}\text{-a.s.}$$

by the Dominated Convergence theorem it follows that

$$\lim_{h\to 0}\frac{\mathbb{E}[(e^{-\frac{1}{\epsilon}\int_0^h V_\epsilon(X(s,\cdot))ds}-1)\varphi(X(h,\cdot))]}{h} = -\frac{1}{\epsilon}V_\epsilon(\cdot)\varphi(\cdot), \quad \text{in } L^2(\mathcal{X},\nu). \tag{7.2.10}$$

So by (7.2.9) and (7.2.10) we obtain

$$N_2^\epsilon\varphi = \lim_{h\to 0}\frac{P^\epsilon(h)\varphi-\varphi}{h} = N_2\varphi - \frac{1}{\epsilon}V_\epsilon\varphi, \quad \textit{in } L^2(\mathcal{X},\nu).$$

Then, for any $\varphi\in\xi_A(\mathcal{X})$, we have $\varphi\in D(N_2^\epsilon)$ and

$$N_2^\epsilon\varphi = N_2\varphi - \frac{1}{\epsilon}V_\epsilon\varphi.$$

Let now $\varphi\in \mathrm{Dom}(N_2)$. By Theorem 5.3.3, $\xi_A(\mathcal{X})$ is a core for N_2, so we can take a sequence $(\varphi_n)_{n\in\mathbb{N}}\subset\xi_A(\mathcal{X})$ such that

$$\lim_{n\to+\infty}\varphi_n=\varphi, \quad \lim_{n\to+\infty}N_2\varphi_n = N_2\varphi, \quad \textit{in } L^2(\mathcal{X},\nu).$$

Since V_ϵ is bounded, we have

$$\lim_{n\to+\infty}\frac{1}{\epsilon}V_\epsilon\varphi_n = \frac{1}{\epsilon}V_\epsilon\varphi \quad \text{in } L^2(\mathcal{X},\nu).$$

Hence

$$\lim_{n\to+\infty}N_2^\epsilon\varphi_n = \lim_{n\to+\infty}N_2\varphi_n - \frac{1}{\epsilon}V_\epsilon\varphi_n = N_2\varphi - \frac{1}{\epsilon}V_\epsilon\varphi, \quad \text{in } L^2(\mathcal{X},\nu).$$

Then, for any $\varphi\in\mathrm{Dom}(N_2)$, we have $\varphi\in\mathrm{Dom}(N_2^\epsilon)$ and (7.2.7) holds.

Finally we prove that $\mathrm{Dom}(N_2^\epsilon)\subset\mathrm{Dom}(N_2)$. For any $\varphi\in\mathrm{Dom}(N_2^\epsilon)$, let φ_ϵ be the unique solution of (7.2.3) with $f=\lambda\varphi_\epsilon - N_2^\epsilon\varphi_\epsilon$. Then, by Proposition 7.2.3, $\varphi_\epsilon\in\mathrm{Dom}(N_2)\subset\mathrm{Dom}(N_2^\epsilon)$. Moreover $R(\lambda,N_2^\epsilon)f=\varphi_\epsilon=\varphi$ and this concludes the proof. □

Finally we prove that the semigroups $P_2^\epsilon(t)$ approximate $P_2^{\mathcal{O}}(t)$ in $L^2(\mathcal{O},\nu)$.

Proposition 7.2.5. *For any $f\in L^2(\mathcal{O},\nu)$ and $t>0$, we have*

$$\lim_{\epsilon\to 0}(P_2^\epsilon(t)\widehat{f})_{|\mathcal{O}} = P_2^{\mathcal{O}}(t)f \quad \textit{in } L^2(\mathcal{O},\nu), \tag{7.2.11}$$

and, for any $\lambda>0$,

$$\lim_{\epsilon\to 0}(R(\lambda,N_2^\epsilon)\widehat{f})_{|\mathcal{O}} = R(\lambda,M_2)f \quad \textit{in } L^2(\mathcal{O},\nu), \tag{7.2.12}$$

where $\widehat{f}$ is defined in Definition 7.1.4.

Proof. We split the proof in two steps. As a first step we prove that for any $\varphi \in C_b(\mathcal{X})$ we have

$$\lim_{\epsilon\to 0} \|P_2^\epsilon(t)\varphi - P_2^{\mathcal{O}}(t)(\varphi_{|\mathcal{O}})\|_{L^2(\mathcal{O},\nu)} = 0. \tag{7.2.13}$$

And as a second step we prove the statement of Proposition.
Step 1. Let $\varphi \in C_b(\mathcal{X})$. First of all we prove that

$$\lim_{\epsilon\to 0} P_2^\epsilon(t)\varphi(x) = P_2^{\mathcal{O}}(t)(\varphi_{|\mathcal{O}})(x) \quad x \in \mathcal{O}, t > 0. \tag{7.2.14}$$

Fixed $x \in \mathcal{O}$ we consider the stopping time τ_x defined in (7.0.1). Let $t > 0$; we define the sets

$$\Omega_1 = \{\tau_x > t\} = \{w \in \Omega \mid X(s,x)(w) \in \mathcal{O},\ \forall s \in [0,t)\},$$
$$\Omega_2 = \{\tau_x \le t\} = \{w \in \Omega \mid \exists s_0 \in (0,t] \mid X(s_0,x)(w) \in \mathcal{O}^c\}.$$

Clearly $\Omega = \Omega_1 \cup \Omega_2$ and Ω_1, Ω_2 are disjoint. We have

$$P_2^\epsilon(t)\varphi(x) = \int_{\Omega_1} \varphi(X(t,x))e^{-\frac{1}{\epsilon}\int_0^t V_\epsilon(X(s,x))ds}d\mathbb{P} + \int_{\Omega_2} \varphi(X(t,x))e^{-\frac{1}{\epsilon}\int_0^t V_\epsilon(X(s,x))ds}d\mathbb{P}. \tag{7.2.15}$$

We study separately the two integrals in the right hand side of (7.2.15). On Ω_1, $X(s,x) \in \mathcal{O}$, for any $s \in [0,t)$, and then, by definition of V_ϵ (see 7.2.2), there exist $\epsilon_0 > 0$, such that

$$V_\epsilon(X(s,x)) = 0, \quad \forall \epsilon < \epsilon_0,\ \forall s \in [0,t).$$

So

$$\lim_{\epsilon\to 0} \int_{\Omega_1} \varphi(X(t,x))e^{-\frac{1}{\epsilon}\int_0^t V_\epsilon(X(s,x))ds}d\mathbb{P} = \int_{\Omega_1} \varphi(X(t,x))d\mathbb{P}. \tag{7.2.16}$$

On Ω_2, by the fact that the generalized mild solution $X(t,x) \in \mathcal{PC}([0,+\infty),\mathcal{X})$ (see Theorem 2.1.13 and Definition 2.1.5), we know that, for $\mathbb{P}$-a.a. (almost all) $w \in \Omega_2$, there exists $s_0(w) \in (0,t]$ such that

$$X(s_0(w),x)(w) \in \partial\mathcal{O},$$

where $\partial\mathcal{O}$ is the boundary of $\mathcal{O}$. Then by definition of V_ϵ, there exists $\delta(w) > 0$ such that

$$V_\epsilon(X(s,x))(w) \ge \frac{1}{2}, \quad \forall s \in [s_0(w) - \delta(w), s_0(w)].$$

So, for the second summand of equation (7.2.15), for $\mathbb{P}$-a.a. $w \in \Omega_2$, we have

$$\lim_{\epsilon\to 0} e^{-\frac{1}{\epsilon}\int_0^t V_\epsilon(X(s,x))(w)ds} \le \lim_{\epsilon\to 0} e^{-\frac{\delta(w)}{2\epsilon}} = 0. \tag{7.2.17}$$

Therefore by (7.2.17) and the Dominated Convergence theorem, we have

$$\lim_{\epsilon\to 0} \int_{\Omega_2} \varphi(X(t,x))e^{-\frac{1}{\epsilon}\int_0^t V_\epsilon(X(s,x))ds}d\mathbb{P} = 0. \tag{7.2.18}$$

Hence, (7.2.16) and (7.2.18) yield (7.2.14). Moreover, for each $x \in \mathcal{O}$, we have
$|P_2^\epsilon(t)(\widehat{\varphi})(x)|, |P_2^{\mathcal{O}}(t)(\varphi_{|\mathcal{O}})(x)| \le \|\varphi\|_\infty$. Then, by (7.2.14) and the Dominated Convergence the-

orem, (7.2.13) is verified.

Step 2. Let $f \in L^2(\mathcal{O},\nu)$; we prove (7.2.11) . We recall that $C_b(\mathcal{X})$ is dense in $L^2(\mathcal{X},\nu)$, so there exists a sequence $(f_n) \subset C_b(\mathcal{X})$ such that, for any large $n \in \mathbb{N}$,

$$\|\widehat{f} - f_n\|_{L^2(\mathcal{X},\nu)} \leq \frac{1}{n}.$$

In particular

$$\|f - f_{n|\mathcal{O}}\|_{L^2(\mathcal{O},\nu)} = \|\widehat{f} - f_n\|_{L^2(\mathcal{X},\nu)} \leq \frac{1}{n}. \tag{7.2.19}$$

Therefore

$$\begin{aligned}\|P_2^\epsilon(t)\widehat{f} - P_2^{\mathcal{O}}(t)f\|_{L^2(\mathcal{O},\nu)} \leq \|P_2^{\mathcal{O}}(t)(f - f_{n|\mathcal{O}})\|_{L^2(\mathcal{O},\nu)} &+ \|P_2^\epsilon(t)(\widehat{f} - f_n)\|_{L^2(\mathcal{O},\nu)} \\ &+ \|P_2^\epsilon(t)f_n - P_2^{\mathcal{O}}(t)f_{n|\mathcal{O}}\|_{L^2(\mathcal{O},\nu)}.\end{aligned}$$

By Lemma 7.2.1 and Proposition 7.0.1, we have

$$\|P_2^\epsilon(t)\widehat{f} - P_2^{\mathcal{O}}(t)f\|_{L^2(\mathcal{O},\nu)} \leq \|f - f_{n|\mathcal{O}}\|_{L^2(\mathcal{O},\nu)} + \|\widehat{f} - f_n\|_{L^2(\mathcal{O},\nu)} + \|P_2^\epsilon(t)f_n - P_2^{\mathcal{O}}(t)f_{n|\mathcal{O}}\|_{L^2(\mathcal{O},\nu)}.$$

Letting $\epsilon \to 0$ and $n \to +\infty$, the first and the second summand go to zero by (7.2.19), and the third summand goes to zero by *Step* 1. We recall the following identity in $L^2(\mathcal{X},\nu)$

$$R(\lambda, N_2^\epsilon)\widehat{f} = \int_0^{+\infty} e^{-\lambda t} P_2^\epsilon(t)\widehat{f}dt,$$

taking the restriction to $\mathcal{O}$ of both sides and using (7.2.11) we obtain (7.2.12). □

7.3 Proof of Theorem 7.0.4

Finally we prove Theorem 7.0.4.

Proof of Theorem 7.0.4. First we prove that $\mathrm{Dom}(M_2) \subseteq \mathring{W}_R^{1,2}(\mathcal{O},\nu)$. For $\epsilon > 0$, $\varphi \in \mathrm{Dom}(M_2)$, $\lambda > 0$ and $f = \lambda\varphi - M_2\varphi$ we set

$$\varphi_\epsilon = R(\lambda, N_2^\epsilon)\widehat{f}.$$

By Proposition 7.2.4, φ_ϵ is the unique solution of (7.2.3), with f replaced by $\widehat{f}$. Moreover, by Proposition 7.2.3(7.2.4-7.2.5), the $W_R^{1,2}(\mathcal{X},\nu)$-norm of φ_ϵ is bounded by a constant independent of ϵ. Therefore there exists a sub-sequence (φ_{ϵ_k}) weakly convergent in $W_R^{1,2}(\mathcal{X},\nu)$ to a function ϕ. We have to prove that $\phi = \widehat{\varphi}$, namely $\phi_{|\mathcal{O}} = \varphi$ and $\phi_{|\mathcal{O}^c} = 0$. By Proposition 7.2.5(7.2.12), we know that

$$\lim_{k\to+\infty} \|\varphi - \varphi_{\epsilon_k|\mathcal{O}}\|_{L^2(\mathcal{O},\nu)} = 0,$$

so that $\phi_{|\mathcal{O}} = \varphi$. Since φ_{ϵ_k} weakly converges to ϕ in $W_R^{1,2}(\mathcal{X},\nu)$, then it weakly converges to ϕ in $L^2(\mathcal{O},\nu)$. Recalling that $V_{\epsilon_k} \equiv 1$ in $\mathcal{O}^c$ (see (7.2.2)) and using (7.2.6), we obtain

$$\|\phi\|^2_{L^2(\mathcal{O}^c,\nu)} = \int_{\mathcal{O}^c} \phi^2 d\nu = \limsup_{k\to+\infty} \int_{\mathcal{O}^c} \varphi_{\epsilon_k}\phi\, d\nu \leq \limsup_{k\to+\infty} \left(\int_{\mathcal{O}^c} \varphi^2_{\epsilon_k} V_{\epsilon_k} d\nu\right)^{\frac{1}{2}} \left(\int_{\mathcal{O}^c} \phi^2 V_{\epsilon_k} d\nu\right)^{\frac{1}{2}}$$

$$\leq \lim_{k\to+\infty}\left(\frac{\epsilon_k}{\lambda}\right)^{\frac{1}{2}}\|f\|_{L^2(H,\nu)}\left(\int_{\mathcal{O}^c}\phi^2 V_{\epsilon_k}d\nu\right)^{\frac{1}{2}}=0,$$

and so $\phi_{|\mathcal{O}^c}=0$. Therefore, $\phi=\widehat{\varphi}\in W^{1,2}_R(\mathcal{X},\nu)$, so that $\varphi\in \mathring{W}^{1,2}_R(\mathcal{O},\nu)$.

Finally we prove that φ is a solution of (7.1.1). Fixed $v\in\mathring{W}^{1,2}_R(\mathcal{O},\nu)$ and $k\in\mathbb{N}$, we multiply both members of (7.2.7) by $\widehat{v}$ and we integrate over $\mathcal{X}\backslash(\mathcal{O}\backslash\mathcal{O}_{\epsilon_k})$. Since $V_{\epsilon_k}\widehat{v}\equiv 0$ on $\mathcal{X}\backslash(\mathcal{O}\backslash\mathcal{O}_{\epsilon_k})$, we have

$$\lambda\int_{\mathcal{X}\backslash(\mathcal{O}\backslash\mathcal{O}_{\epsilon_k})}\varphi_{\epsilon_k}\widehat{v}\,d\nu+\frac{1}{2}\int_{\mathcal{X}\backslash(\mathcal{O}\backslash\mathcal{O}_{\epsilon_k})}\langle R\nabla\varphi_{\epsilon_k},R\nabla\widehat{v}\rangle\,d\nu=\int_{\mathcal{X}\backslash(\mathcal{O}\backslash\mathcal{O}_{\epsilon_k})}\widehat{f}\widehat{v}\,d\nu.$$

Recalling the definition of $\mathcal{O}_{\epsilon_k}$ (see (7.2.1)), letting $k\to+\infty$, we obtain

$$\lambda\int_{\mathcal{X}}\widehat{\varphi}\widehat{v}\,d\nu+\frac{1}{2}\int_{\mathcal{X}}\langle R\nabla\widehat{\varphi},R\nabla\widehat{v}\rangle\,d\nu=\int_{\mathcal{X}}\widehat{f}\widehat{v}\,d\nu,$$

and so we conclude that φ satisfies (7.1.1). We recall that, by the Lax-Milgram theorem, the weak solution of (7.1.1) is unique and so, for any $\lambda>0$ and $f\in L^2(\mathcal{O},\nu)$, we have

$$R(\lambda,M_2)f=R(\lambda,N_2^{\mathcal{O}})f,$$

and so Theorem 7.0.4 is proved. □

7.4 Remarks and examples

The results presented in this chapter are contained in the paper [9].

We could have used the same method to prove Theorem 7.0.4 even in the case of closed domain K as in [33] instead of an open set $\mathcal{O}$. If we had considered a closed domain K (with $\nu(K)>0$), we could have used a single potential V not dependent to ϵ. However, we should have defined differently the stopping time τ_x to take account of the value of functions at ∂K (the boundary of K).

We stress that the semigroup $P^{\mathcal{O}}(t)$ is associated to a Dirichlet problem (see [33] for the Ornstein-Uhlenbeck case). Indeed, for any $\lambda>0$ and $f\in L^2(\mathcal{O},\nu)$, we consider the Dirichlet problem

$$\begin{cases}\lambda\varphi-N_2^{\mathcal{O}}\varphi=f, & \text{on } \mathcal{O},\\ 0, & \text{on } \partial\mathcal{O}.\end{cases}\tag{7.4.1}$$

The problem (7.1.1) is a weak formulation of (7.4.1), and the spaces $\mathring{W}^{1,2}_R(\mathcal{O},\nu)$ are the natural spaces in which to set the problem. In particular, by Theorem 7.0.4, the function

$$\varphi=R(\lambda,N_2^{\mathcal{O}})f$$

is the unique weak solution of (7.4.1).

Theorem 7.0.4 generalizes the result of [33, Section 3] proved for $F=0$. For a study of an analogous problem in the case where $\mathcal{X}$ is a separable Banach space and $F=0$ we refer to [8], instead we refer to [88, 96] for other types of problems about the stopped semigroup. Now we present an interesting reaction-diffusion system that verifies the hypotheses of Theorem 7.0.4.

7.4.1 A reaction-diffusion gradient system

Let $\mathcal{X} = L^2([0,1],\lambda)$ where λ is the Lebesgue measure and let $E = C([0,1])$. Let A be the realization in $L^2([0,1])$ of the second order derivative with Dirichlet boundary condition and let $R = \mathrm{I}_{\mathcal{X}}$. By [20, Section 6.1] Hypotheses 2.1.1(iii), 2.1.1(v) and 7.1.1 are verified. By [20, Lemma 8.2.1] condition (5.1.1) of Hypotheses 5.1.1 is verified.

Now we define the function F. Let $\varphi \in C^2(\mathbb{R})$ be a function such that φ' is increasing, and there exist $d_1, d_2 > 0$ and an $m \in \mathbb{N}$ such that

$$|\varphi'(y)| \leq d_1(1 + |y|^m), \quad y \in \mathbb{R}; \tag{7.4.2}$$

$$|\varphi''(y)| \leq d_2(1 + |y|^{m-1}), \quad y \in \mathbb{R};. \tag{7.4.3}$$

Let $\zeta_2 > 0$. We consider the function $\phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(y) = \varphi(y) + \frac{\zeta_2}{2}y^2,$$

and the function $U : \mathcal{X} \to \mathbb{R}$ defined by

$$U(f) = \begin{cases} \int_0^1 \phi(f(x)), & f \in E, \\ +\infty, & f \notin E. \end{cases}$$

In this case the operator of Hypotheses 7.1.1 is $Q_\infty = A^{-1}$. Let $\mu \sim N(0, Q_\infty)$. By [34, Proposition 5.2], $U \in W_R^{1,p}(\mathcal{X}, \mu)$, for any $p \geq 1$, and

$$\nabla U(f)(x) = \phi' \circ f(x) = \varphi(f(x)) + \zeta_2 f(x), \quad \forall f \in E = C([0,1]),\ x \in [0,1].$$

We set $F = -\nabla U$, and we recall that we have taken $R = \mathrm{I}_{\mathcal{X}}$. Hence Hypotheses 2.1.1(ii) are verified. By (7.4.2) and (7.4.3) Hypotheses 2.1.1(vi) and 2.1.1(vii) are verified. By [43, Example D.7] and standard calculations Hypotheses 2.1.1(iv) are verified with $\zeta = -\zeta_2$. Therefore all the hypotheses of Theorem 5.1.3 are verified, so $\nu(C([0,1])) = 1$, where ν is the invariant measure of the transition semigroup associated to the generalize mild solution of (2.0.1). Finally, by the definition of ϕ and the Fernique theorem, the hypotheses of Theorem 7.0.4 are verified.

It is also possible to consider an operator A that verifies Hypotheses 2.1.1(iv)(a) with $\zeta_1 < 0$(see [43, Example 11.36]), in this way we can take $\zeta_2 < 0$.

Bibliography

[1] Addona, D., Bandini, E., Masiero, F. *A nonlinear Bismut-Elworthy formula for HJB equations with quadratic Hamiltonian in Banach spaces*, NODEA-Nonlinear Differential Equations AND Applications, **27** (2020)

[2] Addona, D., Cappa, G., Ferrari, S. *On the domain of elliptic operators defined in subsets of Wiener spaces*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **23** (2020).

[3] Ambrosio L., Gigli N., Savaré G. *Gradient Flows in Metric Spaces and in the Space of Probability Measures,*Lectures in Mathematics ETH Zurich, Birkhauser Basel, 205.

[4] Angiuli, L., Bignamini, D.A., Ferrari, S., *Logarithmic Harnack inequalities for transition semigroups in Hilbert spaces*, Submitted (2021). **arXiv**: 2111.13250.

[5] Angiuli, L., Ferrari, S., Pallara, D. *Gradient estimates for perturbed Ornstein–Uhlenbeck semigroups on infinite-dimensional convex domains*, J. Evol. Equ. **19** (2019), 677–715.

[6] Applebaum, D., Riedle, M. *Cylindrical Lévy processes in Banach spaces*, Proceedings of the London Mathematical Society **101** (2009), 547–552.

[7] Aronson, D. G., Serrin, J., Local behaviour of solutions of quasi-linear parabolic equations, *Arch. Rat. Mech. Anal.* **25** (1967), 81–123.

[8] Assaad, J., van Neerven, J., *L^2-theory for non-symmetric Ornstein-Uhlenbeck semigroups on domains*, J. Evol. Equ. **13**, 107-134 (2013).

[9] Bignamini, D.A., *L^2-theory for transition semigroups associated to dissipative systems*, Submitted (2021). **arXiv**: 2110.05271.

[10] Bignamini, D.A., Ferrari, S. *Regularizing Properties of (Non-Gaussian) Transition Semigroups in Hilbert Spaces*, Potential Anal (2021), **DOI**:10.1007/s11118-021-09931-2.

[11] Bignamini, D.A., Ferrari, S. *On generators of transition semigroups associated to semilinear stochastic partial differential equations*, Journal of Mathematical Analysis and Applications(2021), **DOI**: 10.1016/j.jmaa.2021.125878

[12] Bogachev, V.I. *Gaussian measures*, Mathematical Surveys and Monographs, vol. 62, American Mathematical Society, Providence, RI, 1998.

[13] Bogachev, V.I. *Measures on topological spaces*, J. of Mathematical Sciences **91** (1998), 3033–3156.

[14] Bonaccorsi, S., Fuhrman, M.: Regularity results for infinite dimensional diffusions. A Malliavin calculus approach. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 10, 35–45 (1999)

[15] Borell, C. *Positivity improving operators and hypercontractivity*, Math. Z. **180** (1982), 225–234.

[16] Cappa, G., Ferrari, S. *Maximal Sobolev regularity for solutions of elliptic equations in infinite dimensional Banach spaces endowed with a weighted Gaussian measure*, J. Differential Equations **261** (2016), 7099–7131.

[17] Cappa, G., Ferrari, S. *Maximal Sobolev regularity for solutions of elliptic equations in Banach spaces endowed with a weighted Gaussian measure: the convex subset case*, J. Math. Anal. Appl. **458** (2018), 300–331.

[18] Cerrai, S. *A Hille–Yosida theorem for weakly continuous semigroups*, Semigroup Forum **49** (1994), 349–367.

[19] Cerrai, S. *Weakly continuous semigroups in the space of functions with polynomial growth*, Dynam. Syst. Appl. **4** (1995), 351–371.

[20] Cerrai, S. *Second order PDE's in finite and infinite dimension*, Lecture Notes in Mathematics, vol. 1762, Springer-Verlag, Berlin, 2001.

[21] Cerrai, S. *Stochastic reaction-diffusion systems with multiplicative noise and non-Lipschitz reaction term*, Probability Theory and Related Fields. **125** (2003), 271–304.

[22] Cerrai, S. *Averaging Principle for Systems of Reaction-Diffusion Equations with Polynomial Nonlinearities Perturbed by Multiplicative Noise*, Dynam. Syst. Appl. **43** (2011), 2482–2518.

[23] Cerrai, S., Da Prato G. *A basic identity for Kolmogorov operators in the space of continuous functions related to RDEs with multiplicative noise*, Ann. Probab. **42** (2014), 1297–1336.

[24] Cerrai, S. Lunardi, A. *Averaging principle for nonautonomous slow-fast systems of stochastic reaction-diffusion equations: the almost periodic case*, SIAM J. Math. Anal. **49** (2017), 2843-2884.

[25] Chae, S.B. *Holomorphy and Calculus in Normed Spaces*, Chapman & Hall/CRC Pure and Applied Mathematics, vol. 92, Taylor & Francis Group, New York and Basel, 1985.

[26] Dawson D. A., Stochastic evolution equation, Math. Bios., Vol. 15, 1972

[27] Da Prato, G. *Applications croissants et équations d'évolutions dans les espaces de Banach*, Istituto Nazionale di Alta Matematica, Institutiones Mathematicae, Volume II, Academic press, London, (1976).

[28] Da Prato, G. *Bounded perturbations of Ornstein–Uhlenbeck semigroups* in "Evolution equations, semigroups and functional analysis (Milano, 2000)", Progr. Nonlinear Differential Equations Appl., vol. 50 (2002), Birkhäuser, Basel, 97–114.

[29] Da Prato, G. *Monotone gradient systems in L^2 spaces* in "Seminar on Stochastic Analysis, Random Fields and Applications, III (Ascona, 1999)", Progr. Probab., vol. 52 (2002), Birkhäuser, Basel, 73–88.

[30] Da Prato, G. *Kolmogorov equations for stochastic PDEs*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2004.

[31] Da Prato G. *Transition semigroups corresponding to Lipschitz dissipative systems*, Discrete Contin. Dyn. Syst. **10** (1–2) (2004), 177–192.

[32] Da Prato, G., Debussche, A., Goldys, B. *Invariant measures of non symmetric dissipative stochastic systems*, Probab. Th. Relat. Fields **123** (2002), 355–380.

[33] Da Prato, G., Lunardi, A. *On the Dirichlet semigroup for Ornstein–Uhlenbeck operators in subsets of Hilbert spaces*, J. Funct. Anal. **259** (2010), 2642–2672.

[34] Da Prato, G., Lunardi, A. *Sobolev regularity for a class of second order elliptic PDEs in infinite dimension*, Ann. of Prob. **47** (2014), 2113–2160.

[35] Da Prato, G., Lunardi, A. *Maximal Sobolev regularity in Neumann problems for gradient systems in infinite dimensional domains*, Ann. Inst. Henri Poincaré Probab. Stat. **51** (2015), 1102–1123.

[36] Da Prato, G., Lunardi, A., Tubaro, L. *Malliavin Calculus for non Gaussian differentiable measures and surface measures in Hilbert spaces*, Trans. Am. Math. Soc. **370** (2016), 2113–2160.

[37] Da Prato, G., Röckner, M., *Singular dissipative stochastic equations in Hilbert spaces*, Probab Theory Relat Fields **124**, 261-303, (2002).

[38] Da Prato, G., and Röckner, M., Wang, F.-Y. *Singular stochastic equations on Hilbert spaces: Harnack inequalities for their transition semigroups*, J. Funct. Anal. **257** (2009), 992–1017.

[39] Da Prato, G., Tubaro, L. *Self-adjointness of some infinite-dimensional elliptic operators and application to stochastic quantization*, Probab. Theory Related Fields **118** (2000), 131–145.

[40] Da Prato, G., Tubaro, L. *Some results about dissipativity of Kolmogorov operators*, Czechoslovak Mathematical Journal, **51** (2001), 685–699.

[41] Da Prato, G., Zabczyk, J. *Ergodicity for Infinite-Dimensional Systems*, London Mathematical Society Lecture Note Series, vol. 229, Cambridge University Press, Cambridge, 1996.

[42] Da Prato, G., Zabczyk, J. *Second Order Partial Differential Equations in Hilbert Spaces*, London Mathematical Society Lecture Note Series, vol. 293, Cambridge University Press, Cambridge, 2002.

[43] Da Prato, G., Zabczyk, J. *Stochastic equations in infinite dimensions*, Encyclopedia of Mathematics and its Applications, vol. 152, Cambridge University Press, Cambridge, 2014.

[44] De Giorgi, E. *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43.

[45] DiBenedetto, E., Gianazza, U., Vespri, V., Forward, backward and elliptic harnack inequalities for non-negative solutions to certain singular parabolic partial differential equations, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **9** (2010) 385–422.

[46] DiBenedetto, E., Gianazza, U., Vespri, V., Harnack estimates for quasi-linear degenerate parabolic differential equations, *Acta Math.* **200** (2008), 181–209.

[47] Dineen, S. *Complex Analysis on Infinite Dimensional Spaces*, Springer Monographs in Mathematics, Springer-Verlag, London, 1999.

[48] Dynkin E.B. *Markov processes (Volume 1)*,Springer-Verlag, 1965.

[49] Dynkin E.B. *Markov Processes and Related Problems of Analysis*,London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1982.

[50] Deuschel, J., Stroock, D.W. *Hypercontractivity and spectral gap of symmetric diffusions with applications to the stochastic Ising models*, J. Funct. Anal., **92** (1990), 30–48.

[51] Dunford, N., Schwartz J.T. *Linear operators. Part II*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1988.

[52] Elworthy, K.D., Li, X.-M.: Formulae for the derivatives of heat semigroups. J. Funct. Anal. 125, 252–286 (1994)

[53] Engel, K.-J., Nagel R.: One-parameter semigroups for linear evolution equations. Springer-Verlag, New York (2000)

[54] Engel, K.-J., Nagel, R. *A short course on operator semigroups*, Universitext, Springer, New York, 2006.

[55] Es-Sarhir, A., Stannat, W. *Maximal dissipativity of Kolmogorov operators with Cahn–Hilliard type drift term*, J. Differential Equations **247** (2009), 424–446.

[56] Es-Sarhir, A., von Renesse, M.-K., Scheutzow, M. *Harnack inequality for functional SDEs with bounded memory*, Electron. Commun. Probab. **14** (2009), 560–565.

[57] Federico, S., Gozzi, F., *Corrigendum to "Mild solutions of semilinear elliptic equations in Hilbert spaces" [J. Differential Equations 262 (2017) 3343–3389] [MR3584895]*, J. Differential Equations **263** (9) (2017), 6143–6144. **DOI**: 10.1016/j.jde.2017.07.026.

[58] Federico, S., Gozzi, F., *Mild solutions of semilinear elliptic equations in Hilbert spaces*, J. Differential Equations **262** (5) (2017), 3343–3389. **DOI**: 10.1016/j.jde.2016.11.031.

[59] Ferrari, S. *Sobolev spaces with respect to a weighted Gaussian measure in infinite dimensions.* Infin. Dimens. Anal. Quantum Probab. Relat. Top. **22** (2019).

[60] Fonseca, I., Leoni, G. *Modern methods in the calculus of variations: L^p spaces*, Springer monographs in Mathematics, Springer, New York, NY, 2007.

[61] Fuhrman, M.: Smoothing properties of nonlinear stochastic equations in Hilbert spaces. NoDEA Nonlinear Differential Equations Appl. 3, 445–464 (1996)

[62] Goldys, B., Kocan, M. *Diffusion Semigroups in Spaces of Continuous Functions with Mixed Topology*, J. Differential Equations **173** (2001), 17–39.

[63] Gross, L. *Logarithmic Sobolev inequalities*, Amer. J. Math. **97** (1975), 1061–1083.

[64] Gross, L. *Logarithmic Sobolev inequalities and contractivity properties of semigroups* in "Dirichlet forms (Varenna, 1992)", Lecture Notes in Math., vol. 1563 (1993), Springer, Berlin, 54–88.

[65] Hadamard, J., Extension à l'equation de la chaleur d'un thèorème de A. Harnack, *Rend. Circ. Mat. Palermo* **2** (1954), 337–346.

[66] Harnack, A. *Die Grundlagen der Theorie des logarithmischen Potentiales und der eindeutigen Potentialfunktion in der Ebene*, V. G. Teubner, Leipzig, 1887.

[67] Huang, X., Zhang, S.-Q. *Mild solutions and Harnack inequality for functional stochastic partial differential equations with Dini drift*, J. Theoret. Probab. **32** (2019), 303–329.

[68] Jacod, J., Protter, P. *Probability essentials*, Universitext, Springer-Verlag, Berlin, 2003.

[69] Kassmann, M. *Harnack inequalities: an introduction*, Bound. Value Probl. (2007), Art. ID 81415, 21.

[70] Kawabi, H. *A simple proof of log-Sobolev inequalities on a path space with Gibbs measures*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. On Reverse Hypercontractivity**9** (2006), 321–329.

[71] Liu, W., Röckner, M. *Stochastic Partial Differential Equations: An Introduction*, Universitext, Springer, Switzerland.

[72] Lorenzi, L., *Analytical Methods for Kolmogorov equations*, CRC Press, second ed., 2017.

[73] Lunardi, A. *Analytic semigroups and optimal regularity in parabolic problems*, Modern Birkhäuser Classics, Birkhäuser/Springer Basel AG, Basel, 1995.

[74] Lunardi, A., Metafune, G., Pallara, D. *The Ornstein–Uhlenbeck semigroup in finite dimension*, Philos. Trans. Roy. Soc. A **378** (2020), 20200217, 15.

[75] Lunardi, A., Pallara, D. *Ornstein–Uhlenbeck semigroups in infinite dimension*, Philos. Trans. Roy. Soc. A **378** (2020), 20190620, 19.

[76] Lv, W., Huang, X. *Harnack and shift Harnack inequalities for degenerate (functional) stochastic partial differential equations with singular drifts*, J. Theoret. Probab. **34** (2021), 827–851.

[77] Masiero F., *Semilinear Kolmogorov equations and applications to stochastic optimal control*, Appl. Math. Optim. **51** (2) (2005), 201–250. **DOI**: 10.1007/s00245-004-0810-6.

[78] Masiero, F. *Stochastic optimal control problems and parabolic equations in Banach spaces.* SIAM J. Control Optim. **47** (2008), 251–300.

[79] Moser, J. *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **14** (1961), 577–591.

[80] Moser, J., A Harnack inequality for parabolic differential equations, *Comm. Pure Appl. Math.* **17** (1964), 101–134.

[81] Mossel, E., Oleszkiewicz, K., Sen, A. *On reverse hypercontractivity*, Geom. Funct. Anal. **23** (2013), 1062–1097.

[82] Mujica, J. *Complex Analysis in Banach Spaces*, North-Holland Mathematics Studies, vol. 120, North-Holland, Amsterdam, 1985.

[83] Nash, J. *Parabolic equations*, Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 754–758.

[84] Øksendal, B. *Stochastic differential equations*, Universitext, Springer-Verlag, Berlin, 2003.

[85] Peszat, S., Zabczyk, J.: Strong Feller property and irreducibility for diffusions on Hilbert spaces. Ann. Probab. 23, 157–172, 1995

[86] Phelps, R.R. *Convex Functions, Monotone Operators and Differentiability*, Lecture Notes in Mathematics, vol. 1364, Springer-Verlag, Berlin Heidelberg, 1993.

[87] Pini, B., Sulla soluzione generalizzata di Wiener per il primo problema di valori al contorno nel caso parabolico, *Rend. Sem. Mat. Univ. Padova* **23** (1954), 422–434.

[88] Priola E.; *Dirichlet problems in a half–space of a Hilbert space*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **5** (2) (2002), 257–291.

[89] Prugovecki, E. *Quantum Mechanics in Hilbert Space*, Academic Press, New York, 1981.

[90] Reed, M., Simon, B. *Methods of modern mathematical physics. I. Functional analysis*, Academic Press, New York-London, 1972.

[91] Riedle, M. *Cylindrical Wiener Processes*, Lecture Notes in Mathematics **2006** (2008), 547–552.

[92] Röckner, M., Wang, F.-Y. *Harnack and functional inequalities for generalized Mehler semigroups*, J. Funct. Anal. **203** (2003), 237–261.

[93] Röckner, M., Wang, F.-Y. *Log-Harnack inequality for stochastic differential equations in Hilbert spaces and its consequences*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **13** (2010), 27–37.

[94] Serrin, J. *On the Harnack inequality for linear elliptic equations*, J. Analyse Math. **4** (1955/56), 292–308.

[95] Sharpe M. *General theory of markov processes*, Academic Press, Cambridge, 1982.

[96] A. Talarczyk *Dirichlet problem for parabolic equations on Hilbert spaces*, Studia Math. **141** (2000), 109–142.

[97] Trudinger, N.S., On Harnack type inequalities and their application to quasilinear elliptic equations, *Commun. Pure Appl. Math.* **20** (1967), 721–747.

[98] Walsh J. B., An introduction to stochastic partial differential equations, Ecole d'Ete de Probabilites de Saint-Flour XIV. Lecture Notes in Math. 1180, 265–439, SpringerVerlag, 1986.

[99] Wang, F.-Y. *Logarithmic Sobolev inequalities on noncompact Riemannian manifolds*, Probab. Theory Related Fields **109** (1997), 417–424.

[100] Wang, F.-Y. *Equivalence of dimension-free Harnack inequality and curvature condition*, Integral Equations Operator Theory **48** (2004), 547–552.

[101] Wang, F.-Y. *Harnack inequalities for stochastic partial differential equations*, SpringerBriefs in Mathematics, Springer, New York, 2013.

[102] Wang, F.-Y., Yuan, C. *Harnack inequalities for functional SDEs with multiplicative noise and applications*, Stochastic Process. Appl. **121** (2011), 2692–2710.

[103] Zabczyk, J.: Parabolic equations on Hilbert spaces. In Da Prato, G. (eds.) Stochastic PDE's and Kolmogorov equations in infinite dimensions (Cetraro, 1998), pp. 117–213. Springer, Berlin (1999)