Effective viscoelastic properties of short-fiber reinforced composites.

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Abstract.

The paper focuses on the calculation of the effective viscoelastic properties of a short fiber reinforced composite. The orientation distribution of the fibers is described by a scatter parameter, varying from perfectly aligned fibers to randomly oriented ones. Both matrix and fibers are assumed to be isotropic. The viscoelastic behavior is described using fraction-exponential operators of Scott Blair-Rabotnov. Results are obtained in closed form.

Keywords: fiber reinforced composites, viscoelasticity, fraction-exponential operators, homogenization

1. Introduction.

The present paper is motivated mostly by needs in development of synthetic fiber reinforced concrete (see review of Yin *et al*., 2015) that attracted attention of researchers when it was shown that surface treatment of synthetic fibers may significantly increase the bond strength with the concrete matrix (Di Maida *et al*., 2015; Radi *et al*., 2015). To the best of our knowledge, no analytical model that describes the mechanical behavior of such materials has been proposed. In the text to follow, we calculate effective viscoelastic properties of a material reinforced with short fibers accounting for the orientation distribution of the fibers that may vary from perfectly aligned to randomly oriented ones. Both the matrix and the fibers are assumed to be isotropic. The viscoelasticity is described using fraction-exponential operators of Scott Blair-Rabotnov that allow explicit closed form results.

Viscoelastic properties of reinforced composites are studied from first quarter of XX century and from the very beginning it was proposed to use elastic-viscoelastic analogy to obtain the solution. Implementation of various homogenization techniques in this approach is a straightforward way. Nguyen and Pastor (1993) implemented homogenization tools to determine the mechanical viscoelastic characteristics of unidirectional continuous fiber composites. Luciano and Barbero (1995) consider periodic composite with linear viscoelastic matrix and elastic fibers. The results were obtained in the form of triple infinite series. Shibuya (1997) considered hexagonal array of elastic transversely isotropic fibers embedded in linear viscoelastic isotropic matrix and evaluated effective creep compliance of the composite. Sevostianov et al (1998) applied the homogenization technique to drying ceramics that they modelled as a composite material with viscoelastic phases. Using effective field method, they were able to model dynamics of the properties variation in the technological process. Viet and Pastor (1998) used self-consistent scheme to predict moduli of a fiber reinforced composite. Viscoelastic properties of the constituents were described using complex representation. The authors also provided comparison with experimental data.

Mercier et al (2012) discussed homogenization of linear viscoelastic and non-linear viscoplastic composite materials. The authors compared two homogenization schemes based on the Mori-Tanaka method coupled with the additive interaction law or coupled with a concentration law based on translated fields. In particular, they showed a good agreement between these two methods. Klasztorny and Wilczynski (2000) considered a viscoelastic isotropic polymer matrix and elastic monotropic fibres. Viscoelastic properties were described using Mittag-Leffler fractional exponential functions. The authors also developed and verified an iterative optimization procedure for theoretical prediction of the viscoelastic constants of the composite

Kondo and Takiguchi (2002) studied the creep behavior of continuous fiber reinforced unidirectional composites by using the finite element method. They assumed that the fibers are made of a linearly elastic and transversely isotropic material and that the matrix is isotropic, linearly elastic and nonlinearly viscoelastic. They also compared their results with experimental data. Kim and Muliana (2010) used FEM technique together with analytical micromechanical approach to predict effective properties of hybrid composites consisting of unidirectional short-fiber reinforcements and a matrix system, which is composed of solid spherical particle tillers dispersed in a homogeneous polymer constituent. A combined Schapery's viscoelastic integral model and Valanis's endochronic viscoplastic model has been used for the polymer constituent, while the particle and fiber constituents has been assumed linear elastic.

Sejnoha and Zeman (2002) and Sejnoha *et al*. (2004) considered nonlinear viscoelastic response of fibrous graphite-epoxy composite systems with random distribution of fibers within a transverse plane section of the composite aggregate. The matrix was assumed to be viscoelastic and the fibers showed elastic behavior. Aboudi (2004) derived macroscopic constitutive equations that model the response of multiphase materials undergoing finite deformations in which any phase behaves as a rubber-like thermoviscoelastic material. He modeled the thermoviscoelastic effects by a free-energy function which is given by a sum of a long-term contribution - that is based on the entropic elasticity for thermoelastic polymers - plus a non-equilibrium part which characterizes the viscoelastic (dissipative) mechanism. For calculation of the effective properties of the composite, micromechanical homogenization technique was used. Dubouloz Monnet *et al*. (2006) used homogenization methods combined with 2D image processing to predict the reinforcement effect of polymers by particulates as well as unidirectional fibers in dependence on temperature.

Abdessamad et al (2007, 2009) used the Kelvin-Voigt model of viscoelasticity with rapidly oscillating space and time dependent coefficients to describe periodic viscoelastic composites solidifying under a heating process. Amosov et al (2013) extended this approach to account for chemical processes. Dutra et al (2010) used Mori-Tanaka scheme to predict effective properties of fiber reinforced concrete and validated the model with experimental data. Andrianov et al (2011) applied asymptotic homogenization techniques to the analysis of viscoelastic-matrix fibrous composites with square-lattice reinforcement. The authors mentioned, however, that their approach requires a computationally intense solution. Andrianov et al (2012) used similar technique to calculate effective properties of viscoelastic composite materials with fibres of diamond-shaped cross-section. Pyatigorets and Mogilevskaya (2011) proposed a computational approach for calculation of the effective transverse mechanical properties of unidirectional fiber-reinforced composites with linear viscoelastic matrix and elastic fibers. The effective properties are found from the assumption that the viscoelastic stresses at the distances far away from the cluster are the same as those from a single equivalent inhomogeneity. Hoang-Duc and Bonnet (2014) recently obtained a series solution for the homogenization problem of a linear viscoelastic periodic incompressible composite. The terms of the Neumann series in their paper appear as decoupled, containing geometry dependent terms and viscoelastic properties dependent terms that are polynomial fractions whose inverse Laplace transforms are provided explicitly.

The main challenge appearing in using elastic-viscoelastic analogy is to obtain analytical formulas for inverse Laplace transform. This difficulty constitutes the main reason to use the oversimplified dashpot-spring models. Unfortunately, the simplest models are not sufficiently flexible to match experimental data for real materials. An alternative approach has been proposed by Scott Blair and Coppen (1939, 1943) (experimentally) and by Rabotnov (1948) (theoretically). They suggested to use fraction-exponential operators that, on one hand can describe experimental data of real materials with sufficient accuracy and, on the other hand, allow inverse Laplace transforms in explicit analytical form. Detailed description of the approach is given, for example, in the books of Rabotnov (1977) and Gorenflo et al. (2014). As mentioned by Brenner and Suquet (2013), a major issue in the evaluation of the overall properties of viscoelastic composites is the description of the interaction between elastic and viscous deformation mechanisms within the material. Fractional-differential operators automatically yield solution for this problem in explicit form (Di Paola *et al*, 2013).

Another challenge that has never been addressed even for purely elastic materials is orientation distribution of fibers. Usually, they are assumed to be perfectly aligned or randomly oriented. However, it is usually inapplicable for short fiber reinforced materials (Figure 1).

Recently, Sevostianov and Levin (2015) introduced creep and relaxation contribution tensors that allow one to describe the effect of inhomogeneities on the overall viscoelastic properties in a unified way and thus, to extend any of known micromechanical scheme from elastic materials to viscoelastic ones. In the present work, we use their results to calculate effective viscoelastic properties of short fiber reinforced composites with preferentially oriented fibers. The orientation scatter describes, as limiting cases, strictly parallel and perfectly randomly orientations. The overall properties are calculated here in the framework of non-interaction approximation that besides being rigorous at small concentration of inhomogeneities, serves as a basic building block for various homogenization schemes (see Sevostianov and Kachanov, 2013).

2. Background material.

2.1. Property contribution tensors are used in the context of homogenization problems to describe contribution of a single inhomogeneity into the property of interest – elasticity, thermal or electrical conductivity, diffusion coefficient etc. In the context of the effective elastic properties, one can use compliance contribution tensor of an inhomogeneity $\boldsymbol{H}$ that gives the extra strain produced by introduction of the inhomogeneity into the otherwise uniform stress field or stiffness contribution tensor $\boldsymbol{N}$ that gives the extra stress due to inhomogeneity when it is placed into the otherwise uniform strain field.

Compliance contribution tensors have been first introduced in the context of pores and cracks by Horii and Nemat-Nasser (1983) (see also detailed discussion in the book of Nemat-Nasser and Hori, 1993). Components of this tensor were calculated for 2-D pores of various shape and 3-D ellipsoidal pores in isotropic material by Kachanov *et al* (1994). For general case of elastic inhomogeneities, these tensors were calculated for ellipsoidal shapes by Sevostianov and Kachanov (1999, 2002). Sevostianov et al (2005) calculated components of this tensor for a spheroidal inhomogeneity embedded in a transversely-isotropic material. For reader's convenience, we provide below a brief description of the property contribution tensors

We first consider a homogeneous *elastic* material (matrix), with the compliance and stiffness tensors $\boldsymbol{S}^0$ and $\boldsymbol{C}^0$ assumed to be isotropic. It contains an inhomogeneity, of volume $V^{(1)}$, of a different elastic material with the compliance and stiffness tensors $\boldsymbol{S}^1$ and $\boldsymbol{C}^1$. The contribution of the inhomogeneity to the overall strain, per representative volume V (the extra strain, as compared to the homogeneous matrix) is given by the fourth-rank tensor $\boldsymbol{H}$ – the compliance contribution tensor of the inhomogeneity – defined by

$$\Delta\boldsymbol{\varepsilon} = \frac{V^{(1)}}{V}\boldsymbol{H} : \boldsymbol{\sigma}^{\infty} \tag{2.1}$$

where $\boldsymbol{\sigma}^{\infty}$ is the "remotely applied" stress field, that, in absence of the inhomogeneity, would have been uniform within its site ("homogeneous boundary conditions", Hashin, 1983); a colon denotes contraction over two indices. Similarly, the stiffness contribution tensor $\boldsymbol{N}$, dual to $\boldsymbol{H}$, can be introduced:

$$\Delta\boldsymbol{\sigma} = \frac{V^{(1)}}{V}\boldsymbol{N} : \boldsymbol{\varepsilon}^{\infty} \tag{2.2}$$

where $\boldsymbol{\varepsilon}^{\infty}$ is the "remotely applied" strain.

For the ellipsoidal inhomogeneity, the fourth-order tensors $\boldsymbol{H}$ and $\boldsymbol{N}$ can be expressed in terms of elastic contrast and fourth-order Hill's tensors $\boldsymbol{P}$ and $\boldsymbol{Q}$ that describe the effects of shape of the inhomogeneity:

$$\boldsymbol{H} = \left[\left(\boldsymbol{S}^1 - \boldsymbol{S}^0 \right)^{-1} + \boldsymbol{Q} \right]^{-1}, \qquad \boldsymbol{N} = \left[\left(\boldsymbol{C}^1 - \boldsymbol{C}^0 \right)^{-1} + \boldsymbol{P} \right]^{-1} \tag{2.3}$$

(i.e. effects of elastic contrast and shape of the inhomogeneity can be separated for ellipsoidal shapes). The fourth-order Hill's (1965) tensor $\boldsymbol{P}$ is the integral over volume of the inhomogeneity from the second gradient of Green's tensor in terms of Green's tensor and tensor $\boldsymbol{Q}$ is related to $\boldsymbol{P}$ as follows (Walpole, 1966):

$$Q_{ijkl} = C^0_{ijmn}\left(J_{mnkl} - P_{mnrs} C^0_{rskl} \right) \tag{2.4}$$

Here, $J_{ijkl} = \left(\delta_{ik}\delta_{lj} + \delta_{il}\delta_{kj} \right)/2$ and the inverse of symmetric (with respect to $i \leftrightarrow j$ and $k \leftrightarrow l$) fourth-order tensor X^{-1}_{ijkl} is defined by $X^{-1}_{ijmn} X_{mnkl} = X_{ijmn} X^{-1}_{mnkl} = J_{ijkl}$.

For a spheroidal inhomogeneity with semi-axes $a_3; a_1 = a_2$ embedded in an isotropic matrix, it is convenient to use representation of these tensors in terms of standard tensor basis $\boldsymbol{T}^{(1)}, \ldots, \boldsymbol{T}^{(6)}$ (see Appendix A for detail):

$$\boldsymbol{P} = \sum_{k=1}^{6} p_k \boldsymbol{T}^{(k)}\ ; \quad \boldsymbol{Q} = \sum_{k=1}^{6} q_k \boldsymbol{T}^{(k)}\ ; \quad \boldsymbol{H} = \sum_{k=1}^{6} h_k \boldsymbol{T}^{(k)}\ ; \quad \boldsymbol{N} = \sum_{k=1}^{6} n_k \boldsymbol{T}^{(k)} \tag{2.5}$$

so that finding out these tensors reduces to calculation of factors p_k, q_k, h_k and n_k. The following relations for coefficients p_i, q_i take place (see, for example, Sevostianov and Kachanov, 2002):

$$\begin{aligned}
& p_1 = \frac{1}{2\mu_0}\left[1 - \kappa_0 \left(f_0 - f_1 \right) \right], \quad p_2 = \frac{1}{2\mu_0}\left[2 - \kappa_0 \left(f_0 - f_1 \right) \right], \quad p_3 = p_4 = -\frac{\kappa_0}{\mu_0} f_1 \\
& p_5 = \frac{1}{\mu_0}\left[1 - f_0 - 4\kappa_0 f_1 \right], \quad p_6 = \frac{1}{\mu_0}\left[\left(1 - 2f_0 \right) - 2\kappa_0 \left(f_0 - f_1 \right) \right] \qquad (2.6) \\
& q_1 = \mu_0 \left[-1 + 2f_0 + 2\kappa_0 \left(2 - 3f_0 - f_1 \right) \right], \quad q_2 = 2\mu_0 \left[1 - 2f_0 + \kappa_0 \left(f_0 - f_1 \right) \right] \\
& q_3 = q_4 = 2\mu_0 \left[-f_0 + 2\kappa_0 \left(f_0 + f_1 \right) \right], \quad q_5 = 4\mu_0 \left[f_0 + 4\kappa_0 f_1 \right], \quad q_6 = 8\mu_0 \kappa_0 \left(f_0 - f_1 \right)
\end{aligned}$$

where

$$\kappa_0 = \frac{1}{2\left(1 - \nu_0 \right)} = \frac{3K_0 + \mu_0}{3K_0 + 4\mu_0}, \quad f_0 = \frac{\gamma^2 \left(1 - g \right)}{2\left(\gamma^2 - 1 \right)}, \quad f_1 = \frac{\gamma^2}{4\left(\gamma^2 - 1 \right)^2}\left[\left(2\gamma^2 + 1 \right) g - 3 \right] \tag{2.7}$$

and shape factor g is expressed in terms of the spheroid's aspect ratio γ as follows

$$g(\gamma)=\begin{cases}\dfrac{1}{\gamma\sqrt{1-\gamma^2}}\arctan\dfrac{\sqrt{1-\gamma^2}}{\gamma}, & \text{oblate shape } (\gamma<1)\\ \dfrac{1}{2\gamma\sqrt{\gamma^2-1}}\ln\dfrac{\gamma+\sqrt{\gamma^2-1}}{\gamma-\sqrt{\gamma^2-1}}, & \text{prolate shape } (\gamma>1)\end{cases} \tag{2.8}$$

In the case of a strongly prolate spheroid ($a_3 >> a_1 = a_2$), the n_i factors of the $\boldsymbol{N}$ - tensor are:

$$n_1=\frac{c_1\mu_0}{\mu_0+(1-\kappa_0)c_1};\quad n_2=\frac{4c_2\mu_0}{4\mu_0+(2-\kappa_0)c_2};\quad n_3=n_4=\frac{c_3\mu_0}{\mu_0+(1-\kappa_0)c_1}$$

$$n_5=\frac{8c_5\mu_0}{8\mu_0+c_5};\quad n_6=\frac{2c_6\mu_0+(1-\kappa_0)(c_1c_6-c_3c_4)}{2[\mu_0+(1-\kappa_0)c_1]} \tag{2.9}$$

where

$$c_1=K_1-K_0+\frac{1}{3}(\mu_1-\mu_0),\ c_2=2(\mu_1-\mu_0),\ c_3=K_1-K_0-\frac{2}{3}(\mu_1-\mu_0)$$

$$c_5=4(\mu_1-\mu_0),\ c_6=K_1-K_0+\frac{4}{3}(\mu_1-\mu_0). \tag{2.10}$$

We will use these relations to calculate relaxation contribution tensors and effective viscoelastic properties of a composite reinforced with short fibers.

2.2. Elastic-viscoelastic analogy in terms of fraction-exponential operators.

To describe viscoelastic behavior, we use the most general form of the governing equation in the form of Stieltjes convolution

$$\sigma_{ij}(x,t)=C_{ijkl}\varepsilon_{kl}(x,t)+\int_0^t\Phi_{ijkl}(t-\tau)\varepsilon_{kl}(x,\tau)d\tau \tag{2.11}$$

where ε_{ij} and σ_{kl} are the strain and the stress tensors, respectively, C_{ijkl} is a fourth rank tensor of instantaneous elastic stiffness and $\Phi_{ijkl}(t)$ is time dependent forth rank tensor (the creep kernel) satisfying the fading memory principle $\Phi_{ijkl}(t)\to 0$ as $t\to\infty$. We consider isotropic materials and take into account that the volume change during the deformation is a purely elastic process, whereas viscoelastic effects are reflected in the deviatoric operator. Then expression (3.1) takes the following form

$$\sigma_{ij}(x,t) = 3K\left(\frac{1}{3}\delta_{ij}\right)\varepsilon_{kk}(x,t) + \left(2\mu^*\right)\left(\varepsilon_{ij}(x,t) - \frac{1}{3}\delta_{ij}\varepsilon_{kk}(x,t)\right) \tag{2.12}$$

where K is the elastic bulk modulus of the material and

$$\mu^*[f(x,t)] = \mu^0 f(x,t) + \int_0^t \mu(t-\tau) f(x,\tau) d\tau \,. \tag{2.13}$$

The most widely used approach to solve similar problems for linear viscoelastic materials consists in using Laplace (or other integral) transform (see, for example, Christensen, 1982):

$$\bar{f}(p) = \int_0^\infty f(t) \mathrm{e}^{-pt}\, dt \,. \tag{2.14}$$

Then, relation (2.11) may be rewritten as

$$\bar{\varepsilon}_{ij}(p) = \bar{S}_{ijkl}(p)\bar{\sigma}_{kl}(p) \tag{2.15}$$

and thus solution for viscoelastic problem can be obtained from the corresponding elastic solution by using inverse Laplace transform. The main challenge of this approach is that only simplest kernels in (2.11) (for example, exponential ones) allow explicit analytical inversion. In other cases, solutions for viscoelastic problems can be obtained only numerically. For most materials, however, the simplest exponential kernels do not fit experimental data properly.

Scott Blair and Coppen (1939, 1943) and Rabotnov (1948) independently proposed to use fraction-exponential functions

$$\mathrm{Э}_\alpha(\beta, t-\tau) = (t-\tau)^\alpha \sum_{n=0}^{\infty} \frac{\beta^n (t-\tau)^{n(1+\alpha)}}{\Gamma[(n+1)(1+\alpha)]}, \qquad (-1 < \alpha < 0) \tag{2.16}$$

for kernels in viscoelastic operators that allow explicit analytical solution using Laplace transform and, at the same time, are sufficiently general to provide a good agreement with the experimental data. To satisfy the fading memory principle, the following restrictions on the parameters entering (2.16) have to be satisfied:

$$\beta < 0; \qquad -1 < \alpha \le 0 \tag{2.17}$$

Operator with such a kernel acts onto constant function c as follows:

$$\mathrm{Э}_\alpha(\beta) \cdot c = \frac{c}{\beta}\left[\mathrm{M}_{1+\alpha}\left(\beta t^{1+\alpha}\right) - 1\right] \tag{2.18}$$

where $\mathrm{M}_\lambda(z)$ is the Mittag-Lefler’s function (Gorenflo *et al*, 2014):

$$\mathrm{M}_{\lambda}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(n\lambda+1)} \tag{2.19}$$

which decreases monotonically from 1 to 0 so that

$$\lim_{t\to\infty}\left[\ni_{\alpha}^{*}(\beta,t)\cdot 1\right]=\frac{-1}{\beta} \tag{2.20}$$

For an isotropic material, the shear (or deviatoric) operator of relaxation can be written in terms of Scott Blair –Rabotnov (SBR) kernel (2.18) as

$$(\mu^{*})[\varepsilon(x,t)]=\mu^{0}\left[\varepsilon(x,t)+\lambda\int_{0}^{t}\ni_{\alpha}(\beta,t-t')\varepsilon(x,t')dt'\right] \tag{2.21}$$

This formula, in particular, clarifies the physical meaning of parameter β - it is inverse of the relaxation time τ to the power 1 + α taken with negative sign:

$$\beta=\frac{1}{\tau^{1+\alpha}} \tag{2.22}$$

From (2.20) and (2.21) it follows that

$$\lambda=\beta\left(1-\varepsilon_{max}\right)=\frac{\mu_{0}-\mu_{\infty}}{\mu_{0}}\beta \tag{2.23}$$

where μ^{∞} is the shear modulus at $t\to\infty$, μ^{0} is the instantaneous shear modulus and $\varepsilon_{\max}$ is the maximal shear strain. Therefore, the viscoelastic shear behavior of materials is described by four parameters: μ^{0}, α, β (or τ), and λ. Since in the processes of creep and relaxation the shear modulus is a decreasing function of time ($\mu^{0}\geq\mu^{\infty}$), we have

$$\beta<\lambda<0. \tag{2.24}$$

For $\alpha=0$, the kernel of SBR operator is reduced to the ordinary exponential function. In this case it describes the properties of standard viscoelastic material (Kelvin material) representing combination of two springs with stiffnesses E_1 and E_2, and a dashpot of viscosity η:

$$\left(\frac{1}{E_1}+\frac{1}{E_2}\right)\sigma+\frac{\eta}{E_1E_2}\dot{\sigma}=\varepsilon+\frac{\eta}{E_2}\dot{\varepsilon} \tag{2.25}$$

The convenience of the introduced operators with the Rabotnov’s kernels is that the algebra of these operators is well developed (see Rabotnov, 1977). In particular, we have

$$\Im_\alpha^*(\beta_1)\cdot\Im_\alpha^*(\beta_2)=\frac{\Im_\alpha^*(\beta_1)-\Im_\alpha^*(\beta_2)}{\beta_1-\beta_2}, \qquad (\beta_1\neq\beta_2)$$
$$\Im_\alpha^*(\beta)\cdot\Im_\alpha^*(\beta)=\frac{\partial}{\partial\beta}\Im_\alpha^*(\beta) \tag{2.26}$$

Laplace transform of the Rabotnov's kernel has the following form:

$$\mathcal{L}\left[\Im_\alpha^*(\beta,t)\right]\equiv\int_0^\infty \Im_\alpha^*(\beta,t)e^{-tp}dt=\frac{1}{p^{1-\alpha}+\beta} \tag{2.27}$$

Therefore, if the elastic solution can be represented as a rational function of parameter $x=p^{1+\alpha}$, then its inverse Laplace transform can be obtained analytically in explicit form.

3. Relaxation contribution tensor of a viscoelastic short fiber.

We now discuss the case of a linear viscoelastic material containing a single viscoelastic fiber and derive expression for the tensor describing the contribution of this fiber into overall relaxation process. We assume that viscoelastic properties of the matrix material are relevant under shear loading only (i.e. its response to the hydrostatic loading is completely elastic). The viscoelastic shear operators can be written in the form (2.12) with kernel (2.16). Now, the average, over representative volume element (RVE) V stress $\sigma_{ij}(x,t)$ can be represented as a sum

$$\sigma_{ij}(x,t)=C_{ijkl}\varepsilon_{kl}(x,t)+\int_0^t \Phi_{ijkl}(t-\tau)\varepsilon_{kl}(x,\tau)d\tau+\Delta\sigma_{ij}(x,t) \tag{3.1}$$

where the first two terms represent viscoelastic deformation of the matrix material described by (2.12), $\varepsilon_{kl}(x,t)$ is remotely applied strain (in absence of inhomogeneities, it would have been uniform in V), and $\Delta\sigma_{ij}(x,t)$ is the extra (average over V) stress due to the presence of the inhomogeneity.

The materials are assumed to be *linear viscoelastic*, hence $\sigma_{ij}(x,t)$ is related to applied strain through a linear operator:

$$\Delta\sigma_{ij}(x,t)=\frac{V^{(1)}}{V}N_{ijkl}^*\varepsilon_{kl}=\frac{V^*}{V}\left[N_{ijkl}^0\varepsilon_{kl}+\int_0^t N_{ijkl}(t-\tau)\varepsilon_{kl}(x,\tau)d\tau\right] \tag{3.2}$$

where N_{ijkl}^{*} is a fourth-rank tensor *relaxation contribution operator* of the inhomogeneity and N_{ijkl}^{0} is its instantaneous value. This operator is dual to the creep contribution operator discussed by Sevostianov and Levin (2015) and by Sevostianov et al (2015) in the context of pores and cracks in viscoelastic material. In the case of multiple fibers, the extra relaxation due to their presence is given by

$$\Delta\sigma_{ij}(x,t)=\left[\frac{1}{V}\sum V^{(1)}N_{ijkl}^{*}\right]\varepsilon_{kl} \tag{3.3}$$

If the matrix material is described using SBR operators, expressions for components of N_{ijkl}^{*} in the framework of non-interacting inhomogeneities can be obtained using Laplace transform, expressions (2.26), (2.27) and results for elastic stiffness contribution tensors outlined in Section 2. Indeed, Laplace transform for operator μ^{*} with kernel $\mu(t)=\lambda\,\mu^{0}\,\ni_{\alpha}(\beta,t)$, according to (2.27) is

$$\mu(\,p\,)=\mu^{0}\left(1+\frac{\lambda}{x+\beta}\right),\qquad x\equiv p^{1-\alpha} \tag{3.4}$$

Strictly speaking, Poisson's ratio entering κ_0 in expressions (2.7) and (2.9) is also an operator. It leads to significant complication of formulas for viscoelastic properties of composites (may be, beyond the practical applicability). Rabotnov (1977) proposed to treat Poisson's ratio as a constant. Sevostianov *et al* (2015) checked this hypothesis comparing exact solutions with those obtained by assuming a constant Poisson's ratio. They considered the following two values of the Poisson's ratio:

$$\nu^{0}=\frac{3K-2\mu^{0}}{2\left(3K+\mu^{0}\right)},\qquad \nu^{\infty}=\frac{3K-2\mu^{\infty}}{2\left(3K+\mu^{\infty}\right)}, \tag{3.5}$$

corresponding to the instantaneous Poisson's ratio and its value as $t\to\infty$, respectively. Their results show that even ν_0 gives good agreement (better than 5%). Using ν_∞ reduces the disagreement to less than 2% along entire interval of time. Rabotnov (1977) suggested to use $(\nu_\infty+\nu_0)/2$ as a constant Poisson's ratio. This value, however, does not produce noticeable improvement of the approximation as compared to ν_∞. In the following formulas, we use the values of the Poisson's ratios of both matrix and inhomogeneities at $t\to\infty$.

Following the idea of elastic-viscoelastic analogy, we treat the expressions (2.9) as ones in Laplace space and take their inverse Laplace transforms. We start by replacing the elastic constants μ_0 and μ in (2.9) and (2.10) by the Laplace transforms of the corresponding operators. Then, we can find Laplace transform of relaxation contribution operator

$$N_{ijkl}^{*}(p)=\sum_{k=1}^{6} n_i^{*}(p)\boldsymbol{T}^{(k)} .$$

As an example, we illustrate the entire procedure for deriving the expression for the operator n_1^{*}

$$n_1^{*}(p)=\frac{c_1(p)\mu_0(p)}{\mu_0(p)+(1-\kappa_0)c_1(p)}, \tag{3.6}$$

$$c_1(p)=K_1-K_0+\frac{1}{3}\left(\mu_1(p)-\mu_0(p)\right) \tag{3.7}$$

Using SBR operators and introducing the notation $x=p^{1-\alpha}$, one can re-write (3.7) as

$$\begin{aligned} c_1(p)&=K_1-K_0+\frac{1}{3}\left[\mu_1^0\left(1+\frac{\lambda_1}{x+\beta_1}\right)-\mu_0^0\left(1+\frac{\lambda_0}{x+\beta_0}\right)\right] \\ &=\frac{1}{3(x+\beta_1)(x+\beta_0)}\left[\left(3(K_1-K_0)+\mu_1^0-\mu_0^0\right)(x+\beta_1)(x+\beta_0)\right. \\ &\quad\left.+\mu_1^0\lambda_1(x+\beta_0)-\mu_0^0\lambda_0(x+\beta_1)\right], \end{aligned} \tag{3.8}$$

where μ_0^0 and μ_1^0 are the instantaneous shear moduli of the matrix and the fibers, respectively. It yields the following expression for the denominator of (3.6)

$$\begin{aligned} &\mu_0(p)+(1-\kappa_0)\left(K_1-K_0+\frac{1}{3}\left(\mu_1(p)-\mu_0(p)\right)\right) \\ &=(1-\kappa_0)(K_1-K_0)+\frac{1}{3}(2+\kappa_0)\mu_0(p)+(1-\kappa_0)\mu_1(p) \\ &=\frac{1}{3(x+\beta_1)(x+\beta_0)}\left\{\left[3(1-\kappa_0)(K_1-K_0)+(2+\kappa_0)\mu_0^0+(1-\kappa_0)\mu_1^0\right](x+\beta_1)(x+\beta_0)+\right. \\ &\left.+(2+\kappa_0)\mu_0^0\lambda_0(x+\beta_1)+(1-\kappa_0)\mu_1^0\lambda_1(x+\beta_0)\right\}. \end{aligned} \tag{3.9}$$

Thus

$$\frac{c_1(p)}{\mu_0(p)+(1-\kappa_0)c_1(p)}=\frac{P_1(x)}{\alpha_1\,F_1(x)}, \tag{3.10}$$

where

$$P_1(x) = a_1(x+\beta_0)(x+\beta_1) + b_1(x+\beta_0) + d_1(x+\beta_1),$$

$$a_1 = 3(K_1 - K_0) + \mu_1^0 - \mu_0^0, \quad b_1 = \mu_1^0\lambda_1, \quad d_1 = -\mu_0^0\lambda_0.$$

$$F_1(x) = (x+\beta_0)(x+\beta_1) + \gamma_1(x+\beta_0) + \zeta_1(x+\beta_1),$$

$$\alpha_1 = 3(1-\kappa_0)(K_1 - K_0) + (2+\kappa_0)\mu_0^0 + (1-\kappa_0)\mu_1^0$$

$$\gamma_1 = \frac{\mu_1^0}{\alpha_1}(1-\kappa_0)\lambda_1, \quad \zeta_1 = \frac{\mu_0^0}{\alpha_1}(2+\kappa_0)\lambda_0. \tag{3.11}$$

The right hand side of (3.10) can now be represented as

$$\frac{P_1(x)}{\alpha_1 F_1(x)} = \frac{P_1(x)}{\alpha_1(x+x_1^{(1)})(x+x_2^{(1)})}, \tag{3.12}$$

Where $x_1^{(1)}$ and $x_2^{(1)}$ are the opposites of the roots of the equation $F_1(x) = 0$. Adding $\mu_0(p)$ to this expression yields.

$$\frac{P_1(x)(x+\gamma_0)}{\alpha_1(x+x_1^{(1)})(x+x_2^{(1)})(x+x_3^{(1)})}, \quad \gamma_0 = \beta_0 + \lambda_0, \quad x_3^{(1)} = \beta_0 \tag{3.13}$$

Operation of partial fractions then gives

$$n_1^*(p) = \frac{\mu_0^0}{\alpha_1}\left(a_1 + \sum_{k=1}^{3}\frac{m_k^{(1)}}{x+x_k^{(1)}}\right), \quad m_k^{(1)} = \frac{P_1(-x_k^{(1)})(\gamma_0 - x_k^{(1)})}{\prod_{\substack{j=1 \\ (j\neq k)}}^{3}(x_k^{(1)} - x_j^{(1)})} \tag{3.14}$$

Taking inverse Laplace transform, we obtain

$$n_1^*(t) = \frac{\mu_0^0}{\alpha_1}\left(a_i + \sum_{k=1}^{3} m_k^{(1)}\mathrm{Э}_\alpha^*\left(x_k^{(1)}, t\right)\right) \tag{3.15}$$

In the same manner all other coefficients are derived. Details of the derivation are given in the Appendix B.

$$n_i^*(t) = \frac{\mu_0^0}{\alpha_i}\left(a_i + \sum_{k=1}^{3} m_k^{(i)}\mathrm{Э}_\alpha^*\left(x_k^{(i)}, t\right)\right), \; i = 1,2,3,4,5$$

$$n_6^*(t) = \frac{\mu_0^0 a_6}{\alpha_1} + \frac{\mu_1^0 - \mu_0^0}{2\alpha_1} + \sum_{k=1}^{2} l_k \mathrm{Э}_\alpha^*\left(x_k^{(1)}, t\right) + \frac{\mu_0^0 m_3^{(6)}}{\alpha_1}\mathrm{Э}_\alpha^*\left(x_3^{(1)}, t\right) \tag{3.16}$$

Expressions (3.15)-(3.16) give the explicit representation of the components of relaxation contribution operator N_{ijkl}^* of a spheroidal short fiber in terms of tensor basis (A.1). These formulas serve as the basic building block for calculation of the effective viscoelastic properties of fiber reinforced composites.

4. Calculation of the effective properties of short fiber reinforced composites accounting for fibers orientation.

We now consider a set of spheroidal fibers that tend to be aligned with x_3-axis with certain orientation scatter. For multiple fibers, their combined effect (3.3) is described by

$$\Delta\boldsymbol{\sigma}(x,t)=\left[\frac{1}{V}\sum_i V_i \boldsymbol{N}_i^*(t)\right]:\boldsymbol{\varepsilon}\,. \tag{4.1}$$

Note that summation in the latter equation can be replaced by integration over orientations. Following Sevostianov and Kachanov (2000), we describe the orientation distribution by the following function, containing the scatter parameter ζ :

$$P_\zeta(\varphi)=\frac{1}{2\pi}[(\zeta^2+1)\,e^{-\zeta\varphi}+\zeta\,e^{-\zeta\pi/2}] \tag{4.2}$$

where φ is the angle between fiber axis and x_3-axis. Parameter ζ characterizes the sharpness of the peak at $\varphi=\pi/2$ and the extent of the scatter; the extreme cases of the fully random and perfectly parallel fibers correspond to $\zeta=0$ and $\zeta=\infty$, respectively. The effective elastic moduli are relatively insensitive to the exact form of a function that has the above-mentioned features. The particular form (4.2) is chosen to keep the calculations, related to averaging over orientations, simple.

Figure 2a shows dependence of P_ζ on φ for several values of ζ and the orientation patterns that correspond to these values. For readers convention, Figure 2b shows dependence of P_ζ on ζ for several values of φ (i.e. change in probability to get fibers of specific orientation as ζ increases). Now, the following two tensors have to be averaged over the orientation of fibers:

$$A_{ij}=\frac{1}{V}\sum V^{(p)}\left(m_i m_j\right)^{(p)} \qquad B_{ijkl}=\frac{1}{V}\sum V^{(p)}\left(m_i m_j m_k m_l\right)^{(p)} \tag{4.3}$$

where $\boldsymbol{m}^{(p)}$ is a unit vector along the p-th fiber that has the following components in spherical coordinate system: $\boldsymbol{m}(\varphi,\theta)=\cos\theta\sin\varphi\,\boldsymbol{e}_1+\sin\theta\sin\varphi\,\boldsymbol{e}_2+\cos\varphi\,\boldsymbol{e}_3$. This operation is equivalent to averaging of basic tensors $\boldsymbol{T}^{(i)}$ given by (A.1) over orientation of vectors $\boldsymbol{m}^{(p)}$:

$$\langle \boldsymbol{mm}\rangle = g_1(\zeta)\boldsymbol{\theta} + g_2(\zeta)\,\boldsymbol{mm}$$

$$\langle \boldsymbol{mmmm}\rangle = g_3(\zeta)\left(\boldsymbol{T}^{(1)}+\boldsymbol{T}^{(2)}\right)+g_4(\zeta)\left(\boldsymbol{T}^{(3)}+\boldsymbol{T}^{(4)}+\boldsymbol{T}^{(5)}\right)+g_5(\zeta)\boldsymbol{T}^{(6)} \tag{4.4}$$

where

$$g_1(\zeta)=\frac{18-\zeta\left(\zeta^2+3\right)e^{-\zeta\pi/2}}{6\left(\zeta^2+9\right)};\qquad g_2(\zeta)=\frac{3\left(\zeta^2+3\right)+\zeta\left(\zeta^2+3\right)e^{-\zeta\pi/2}}{3\left(\zeta^2+9\right)} \tag{4.5}$$

$$g_3(\zeta)=\frac{30}{\left(\zeta^2+9\right)\left(\zeta^2+25\right)}-\zeta\, e^{-\zeta\pi/2}\left[\frac{1\left(\zeta^4+30\zeta^2+149\right)}{4\left(\zeta^2+9\right)\left(\zeta^2+25\right)}-\frac{2}{15}\right]$$

$$g_4(\zeta)=\frac{3\zeta^2+15}{\left(\zeta^2+9\right)\left(\zeta^2+25\right)}+\zeta\, e^{-\zeta\pi/2}\left[\frac{\zeta^4+31\zeta^2+187}{3\left(\zeta^2+9\right)\left(\zeta^2+25\right)}\right]$$

$$g_5(\zeta)=\frac{24+\left(\zeta^2+1\right)\left(\zeta^2+21\right)}{\left(\zeta^2+9\right)\left(\zeta^2+25\right)}+\zeta\, e^{-\zeta\pi/2}\left[\frac{\zeta^4+34\zeta^2-105}{5\left(\zeta^2+9\right)\left(\zeta^2+25\right)}\right] \tag{4.6}$$

and components of tensor basis $\boldsymbol{T}^{(i)}$ are given in Appendix A by (A.1). Functions $g_i(\zeta)$ are shown in Fig 3.

Finally, one can write the following formulas for the effective viscoelastic properties of a short fiber reinforced composite (with arbitrary orientation of the fibers) in the framework of non-interaction approximation:

$$c_1^*(t)=K_0+\frac{1}{3}\mu_0^*(t)+n_1^*(t)+g_1(\zeta)\left[n_3^*(t)+n_4^*(t)+\frac{1}{2}n_5^*(t)\right]+g_3(\zeta)\,n_6^*(t);$$

$$c_2^*(t)=2\mu_0^*(t)+n_2^*(t)+g_1(\zeta)\,n_5^*(t)+g_3(\zeta)\,n_6^*(t);$$

$$c_3^*(t)=K_0-\frac{2}{3}\mu_0^*(t)+g_2(\zeta)\left[n_3^*(t)+n_4^*(t)\right]+g_4(\zeta)\,n_6^*(t);$$

$$c_4^*(t)=c_3^*(t)$$

$$c_5^*(t)=4\mu_0^*(t)+g_2(\zeta)\,n_5^*(t)+g_4(\zeta)\,n_6^*(t);$$

$$c_6^*(t) = K_0 + \frac{4}{3}\mu_0^*(t) + g_5(\zeta)\, n_6^*(t). \tag{4.7}$$

where operators $n_i^*(t)$ are given by (3.15) and (3.16). Equations (4.7) represent effective elastic relaxation operators for a viscoelastic material reinforced with viscoelastic short fibers having orientation distribution (4.2).

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Appendix A. Tensor basis in the space of transversely isotropic fourth rank tensors. Representation of certain transversely isotropic tensors in terms of the tensor basis.

We outline a convenient technique of analytic inversion and multiplication of 4th rank tensors. It is based on expressing tensors in "standard" tensor bases (see Kunin, 1983; Walpole, 1984; and Kanaun and Levin, 2008). In the case of the transversely isotropic elastic symmetry, the following basis is most convenient (it differs slightly from the one used by Kanaun and Levin, 2008):

$$T^{(1)}_{ijkl} = \theta_{ij}\theta_{kl}\,,\quad T^{(2)}_{ijkl} = \left(\theta_{ik}\theta_{lj} + \theta_{il}\theta_{kj} - \theta_{ij}\theta_{kl}\right)/2\,,\quad T^{(3)}_{ijkl} = \theta_{ij}m_k m_l\,,\quad T^{(4)}_{ijkl} = m_i m_j \theta_{kl}$$

$$T^{(5)}_{ijkl} = \left(\theta_{ik}m_l m_j + \theta_{il}m_k m_j + \theta_{jk}m_l m_i + \theta_{jl}m_k m_i\right)/4\,,\quad T^{(6)}_{ijkl} = m_i m_j m_k m_l \qquad \text{(A.1)}$$

where $\theta_{ij} = \delta_{ij} - m_i m_j$ and $\boldsymbol{m} = m_1\boldsymbol{e}_1 + m_2\boldsymbol{e}_2 + m_3\boldsymbol{e}_3$ is a unit vector along the axis of transverse symmetry.

These tensors form the closed algebra with respect to the operation of (non-commutative) multiplication (contraction over two indices):

$$\left(\boldsymbol{T}^{(\alpha)} : \boldsymbol{T}^{(\beta)}\right)_{ijkl} \equiv T^{(\alpha)}_{ijpq}T^{(\beta)}_{pqkl} \qquad \text{(A.2)}$$

The inverse of any fourth rank tensor $\boldsymbol{X}$, as well as the product $\boldsymbol{X}:\boldsymbol{Y}$ of two such tensors are readily found in the closed form, as soon as the representation in the basis

$$\boldsymbol{X} = \sum_{k=1}^{6} X_k \boldsymbol{T}^{(k)}\,,\ \boldsymbol{Y} = \sum_{k=1}^{6} Y_k \boldsymbol{T}^{(k)} \qquad \text{(A.3)}$$

are established. Indeed:

a) inverse tensor $\boldsymbol{X}^{-1}$ defined by $X^{-1}_{ijmn}X_{mnkl} = \left(X_{ijmn}X^{-1}_{mnkl}\right) = J_{ijkl}$ is given by

$$\boldsymbol{X}^{-1} = \frac{X_6}{2\Delta}\boldsymbol{T}^{(1)} + \frac{1}{X_2}\boldsymbol{T}^{(2)} - \frac{X_3}{\Delta}\boldsymbol{T}^{(3)} - \frac{X_4}{\Delta}\boldsymbol{T}^{(4)} + \frac{4}{X_5}\boldsymbol{T}^{(5)} + \frac{2X_1}{\Delta}\boldsymbol{T}^{(6)} \qquad \text{(A.4)}$$

where $\Delta = 2\left(X_1X_6 - X_3X_4\right)$.

b) product of two tensors $\boldsymbol{X}:\boldsymbol{Y}$ (tensor with $ijkl$ components equal to $X_{ijmn}Y_{mnkl}$) is

$$\begin{aligned}\boldsymbol{X}:\boldsymbol{Y} &= \left(2X_1Y_1 + X_3Y_4\right)\boldsymbol{T}^{(1)} + X_2Y_2\boldsymbol{T}^{(2)} + \left(2X_1Y_3 + X_3Y_6\right)\boldsymbol{T}^{(3)} \\ &+ \left(2X_4Y_1 + X_6Y_4\right)\boldsymbol{T}^{(4)} + \frac{1}{2}X_5Y_5\boldsymbol{T}^{(5)} + \left(X_6Y_6 + 2X_4Y_3\right)\boldsymbol{T}^{(6)}\end{aligned} \qquad \text{(A.5)}$$

General transversely isotropic fourth-rank tensor, being represented in this basis

$$\Psi_{ijkl} = \sum \psi_m T_{ijkl}^m$$

has the following components:

$$\psi_1 = \left(\Psi_{1111} + \Psi_{1122} \right)/2;\ \psi_2 = 2\Psi_{1212};\ \psi_3 = \Psi_{1133};\ \psi_4 = \Psi_{3311}; \tag{A.6}$$

$$\psi_5 = 4\Psi_{1313};\ \psi_6 = \Psi_{3333}$$

Utilizing (A.6) one obtains the following representations:

- Tensor of elastic compliances of the isotropic material $S_{ijkl} = \sum s_m T_{ijkl}^m$ has the following components

$$s_1 = \frac{1-\nu}{4\mu(1+\nu)};\ s_2 = \frac{1}{2\mu};\ s_3 = s_4 = \frac{-\nu}{2\mu(1+\nu)};\ s_5 = \frac{1}{\mu};\ s_6 = \frac{1}{2\mu(1+\nu)}. \tag{A.7}$$

- Tensor of elastic stiffness of the isotropic material by $C_{ijkl} = \sum c_m T_{ijkl}^m$ has components

$$c_1 = K + \mu/3;\ c_2 = 2\mu;\ c_3 = c_4 = K - 2\mu/3;\ c_5 = 4\mu;\ c_6 = K + 4\mu/3. \tag{A.8}$$

- Symmetric isomers of the unit fourth rank tensor are represented in the form

$$J_{ijkl}^{(1)} = \left(\delta_{ik}\delta_{lj} + \delta_{il}\delta_{kj}\right)/2 = \frac{1}{2}T_{ijkl}^{(1)} + T_{ijkl}^{(2)} + 2T_{ijkl}^{(5)} + T_{ijkl}^{(6)} \tag{A.9}$$

$$J_{ijkl}^{(2)} = \delta_{ij}\delta_{kl} = T_{ijkl}^{(1)} + T_{ijkl}^{(3)} + T_{ijkl}^{(4)} + T_{ijkl}^{(6)} \tag{A.10}$$

Appendix B. Derivation of the expressions for components of the relaxation contribution tensor.

The other components $n_i(p)$ can be obtained by the same way as it was done in the Section 3 for $n_1(p)$. The results can be written as

$$n_i(p) = \frac{\mu_0^0}{\alpha_1}\left(a_i + \sum_{k=1}^{3}\frac{m_k^{(i)}}{x + x_k^{(i)}}\right), \quad m_k^{(i)} = \frac{P_i(-x_k^{(i)})(\gamma_0 - x_k^{(i)})}{\prod_{\substack{j=1 \\ (j\neq k)}}^{3}\left(x_k^{(i)} - x_j^{(i)}\right)}, \quad (i = 2,3,4,5) \tag{B.1}$$

Here

$$P_i(x) = a_i(x+\beta_1)(x+\beta_0) + b_i(x+\beta_0) + d_i(x+\beta_1), \tag{B.2}$$

and the following notations are used

$$\begin{aligned} &a_2 = 4(\mu_1^0 - \mu_0^0), \quad b_2 = 4\mu_1^0\lambda_1, \quad d_2 = -4\mu_0^0\lambda_0, \\ &a_3 = a_4 = 3(K_1 - K_0) - 2(\mu_1^0 - \mu_0^0), \quad b_3 = b_4 = -2\mu_1^0\lambda_1, \quad d_3 = d_4 = 2\mu_0^0\lambda_0, \\ &a_5 = 2a_2, \quad b_5 = 2b_2, \quad d_5 = 2d_2. \end{aligned} \tag{B.3}$$

The quantities $x_k^{(i)}$ ($k = 1,2$) are the roots of the quadratic equations

$$F_i(x) = (x+\beta_1)(x+\beta_0) + \gamma_i(x+\beta_0) + \zeta_i(x+\beta_1) = 0, \tag{B.4}$$

where

$$\begin{aligned} &\alpha_2 = \kappa_0\mu_0^0 + (2-\kappa_0)\mu_1^0, \quad \gamma_2 = \frac{\mu_1^0}{\alpha_2}(2-\kappa_0)\lambda_1, \quad \zeta_2 = \frac{\mu_0^0}{\alpha_2}\kappa_0\lambda_0, \\ &\alpha_3 = \alpha_4 = \alpha_1, \quad \gamma_3 = \gamma_4 = \gamma_1, \quad \zeta_3 = \zeta_4 = \zeta_1, \\ &\alpha_5 = \mu_1^0 + \mu_0^0, \quad \gamma_5 = \frac{\mu_1^0\lambda_1}{\alpha_5}, \quad \zeta_5 = \frac{\mu_0^0\lambda_0}{\alpha_5} \end{aligned} \tag{B.5}$$

Inverse Laplace transform applied to (B.1) yields

$$n_i^*(t) = \frac{\mu_0^0}{\alpha_i}\left(a_i + \sum_{k=1}^{3} m_k^{(i)}\,Э_\alpha^*\left(x_k^{(i)}, t\right)\right) \tag{B.6}$$

Somewhat lengthier derivation is required for $n_6^*(t)$. First, according to (2.9), we can represent $n_6(p)$ as a sum

$$n_6(p) = n_6^{(1)}(p) + n_6^{(2)}(p), \tag{B.7}$$

with

$$n_6^{(1)}(p) = \frac{c_6(p)\mu_0(p)}{\mu_0(p) + (1-\kappa_0)c_1(p)}, \quad c_6(p) = K_1 - K_0 + \frac{4}{3}(\mu_1(p) - \mu_0(p)),$$

$$n_6^{(2)}(p) = \frac{9(1-\kappa_0)(K_1 - K_0)(\mu_1(p) - \mu_0(p))}{2\alpha_1 F_1(x)} = \frac{9(1-\kappa_0)(K_1 - K_0)P_7(x)}{2\alpha_1 F_1(x)} \qquad \text{(B.8)}$$

where

$$P_7(x) = P_5(x)/8 \qquad \text{(B.9)}$$

First of the expressions (B.17) allows the following representation;

$$\frac{c_6(p)}{\mu_0(p) + (1-\kappa_0)c_1(p)} = \frac{P_6(x)}{\alpha_1 F_1(p)}, \qquad \text{(B.10)}$$

where

$$P_6 = a_6(x+\beta_1)(x+\beta_0) + b_6(x+\beta_0) + d_6(x+\beta_1),$$
$$a_6 = 3(K_1 - K_0) + 4(\mu_1^0 - \mu_0^0), \quad b_6 = 4\mu_1^0\lambda_1, \quad d_6 = -4\mu_0^0\lambda_0.$$

(B.11)

Finally,

$$n_6^{(1)}(p) = \frac{\mu_0^0}{\alpha_1}\left(a_6 + \sum_{k=1}^{3}\frac{m_k^{(6)}}{x + x_k^{(6)}}\right), \quad m_k^{(6)} = \frac{P_6(-x_k^{(6)})(\gamma_0 - x_k^{(6)})}{\prod_{\substack{j=1 \\ (j\neq k)}}^{3}(x_k^{(6)} - x_j^{(6)})}, \quad x_k^{(6)} = x_k^{(1)},$$

$$n_6^{(2)} = \frac{1}{2\alpha_1}\left(\mu_1^0 - \mu_0^0 + \frac{9(1-\kappa_0)(K_1 - K_0)}{(x_1^{(1)} - x_1^{(1)})}\sum_{k=1}^{2}(-1)^k \frac{P_7(-x_k^{(1)})}{x + x_k^{(1)}}\right) \qquad \text{(B.12)}$$

After some algebra, we can rewrite (B.16) as

$$n_6(p) = \frac{\mu_0^0 a_6}{\alpha_1} + \frac{\mu_1^0 - \mu_0^0}{2\alpha_1} + \sum_{k=1}^{2}\frac{l_k}{x + x_k^{(1)}} + \frac{\mu_0^0 m_3^{(6)}}{\alpha_1(x + x_3^{(1)})},$$
$$l_k = \frac{\mu_0^0 m_k^{(6)}}{\alpha_1} + \frac{9(1-\kappa_0)(K_1 - K_0)}{2\alpha_1(x_2^{(1)} - x_1^{(1)})}(-1)^k P_7(-x_k^{(1)}). \qquad \text{(B.13)}$$

Inverse Laplace transform yields

$$n_6^*(t) = \frac{\mu_0^0 a_6}{\alpha_1} + \frac{\mu_1^0 - \mu_0^0}{2\alpha_1} + \sum_{k=1}^{2} l_k \mathrm{Э}_\alpha^*\left(x_k^{(1)}, t\right) + \frac{\mu_0^0 m_3^{(6)}}{\alpha_1}\mathrm{Э}_\alpha^*\left(x_3^{(1)}, t\right) \qquad \text{(B.14)}$$

Figure captions

Figure 1. Optical micrographs showing orientation distribution of fibers: (a) δ-Al_2O_3 fibers reinforced aluminum alloy (from Kang et al, 2002); (b) polypyrrole-coated amorphous silica short fibers reinforced polyvinylidene fluoride matrix (from Arenhart et al, 2015).

Figure 2. (a) Dependence of the orientation distribution function P_λ on angle φ at several values of λ and the corresponding fiber orientation patterns; (b) dependence of P_λ on scatter parameter λ for several values of the angle φ.

Figure 3. Dependence of functions $g_i(\zeta)$ on the scatter parameter.