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ON WORDS THAT ARE CONCISE IN RESIDUALLY FINITE GROUPS

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ABSTRACT. A group-word w is called concise if whenever the set of w-values in a group G is finite it always follows that the verbal subgroup $w(G)$ is finite. More generally, a word w is said to be concise in a class of groups X if whenever the set of w-values is finite for a group $G \in X$, it always follows that $w(G)$ is finite. P. Hall asked whether every word is concise. Due to Ivanov the answer to this problem is known to be negative. Dan Segal asked whether every word is concise in the class of residually finite groups. In this direction we prove that if w is a multilinear commutator and q is a prime-power, then the word w^q is indeed concise in the class of residually finite groups. Further, we show that in the case where $w = \gamma_k$ the word w^q is boundedly concise in the class of residually finite groups. It remains unknown whether the word w^q is actually concise in the class of all groups.

1. INTRODUCTION

Let w be a group-word in n variables, and let G be a group. The verbal subgroup $w(G)$ of G determined by w is the subgroup generated by the set G_w consisting of all values $w(g_1, \ldots, g_n)$, where $g_1, \ldots, g_n$ are elements of G. A word w is said to be concise if whenever G_w is finite for a group G, it always follows that $w(G)$ is finite. More generally, a word w is said to be concise in a class of groups X if whenever G_w is finite for a group $G \in X$, it always follows that $w(G)$ is finite. P. Hall asked whether every word is concise, but later Ivanov proved that this problem has a negative solution in its general form [3] (see also [6, p. 439]). On the other hand, many relevant words are known to be concise. For instance, it was shown in [16] that the multilinear commutator words are concise. Such words are also known under the name of outer commutator words and are precisely the words that can be written in the form of multilinear Lie monomials. Merzlyakov showed that every word is concise in the class of linear groups [5] while Turner-Smith proved that every word is concise in the class of residually finite groups all of whose quotients are again residually finite [15]. There is an open problem,

due to Dan Segal [8, p. 15], whether every word is concise in the class of residually finite groups.

In the present paper we obtain the following result.

Theorem 1.1. *Let w be a multilinear commutator word and q a prime-power. The word w^q is concise in the class of residually finite groups.*

It remains unknown whether the word w^q is actually concise in the class of all groups. Following [1] we say that a word w is boundedly concise in a class of groups X if for every integer m there exists a number $\nu = \nu(X, w, m)$ such that whenever $|G_w| \leq m$ for a group $G \in X$ it always follows that $|w(G)| \leq \nu$. Fernández-Alcober and Morigi [2] showed that every word which is concise in the class of all groups is actually boundedly concise. Moreover they showed that whenever w is a multilinear commutator word having at most m values in a group G, one has $|w(G)| \leq (m-1)^{(m-1)}$. In view of our Theorem 1.1 it would be interesting to determine whether the word w^q is boundedly concise in the class of residually finite groups. In this direction we obtain the following theorem.

Theorem 1.2. *Let $w = \gamma_k$ be the kth lower central word and q a prime-power. The word w^q is boundedly concise in the class of residually finite groups.*

Recall that the word γ_k is defined inductively by the formulae

$$\gamma_1 = x_1, \qquad \gamma_k = [\gamma_{k-1}, x_k] = [x_1, \ldots, x_k], \quad \text{for } k \geq 2.$$

The corresponding verbal subgroup $\gamma_k(G)$ is the familiar kth term of the lower central series of G. It remains unknown whether the word γ_k^q is concise in the class of all groups.

The proofs of both Theorem 1.1 and Theorem 1.2 are based on techniques created by Zelmanov in his solution of the Restricted Burnside Problem (see [17] or [18]). Some adjustments to those techniques are described in [12]. We have been unable to decide whether the condition that q is a prime-power is necessary in our results.

Throughout the paper we use the abbreviation, say, "$(a, b, \ldots)$-bounded" for "bounded above in terms of $a, b, \ldots$ only".

2. Preliminaries

In this section we collect some results that will be useful later. Recall that a group G is said to be perfect if $G = G'$. As usual if $L, M \leq G$, we write $[L, M]$ to denote the subgroup generated by all commutators $[l, m]$ where $l \in L$ and $m \in M$.

Lemma 2.1. *Let G be a perfect group and A a normal abelian subgroup of G. Then $[A, G] = [A, G, G]$.*

Proof. Without loss of generality assume that $[A, G, G] = 1$. Note that $[G, A, G] = [A, G, G] = 1$. In view of the Three Subgroups Lemma [7, 5.1.10],

we have $[G, G, A] = 1$. Since G is perfect, we conclude that $[A, G] = 1$, as desired. $\qquad\square$

Let w be a group-word and G a group. A subgroup $N \leq G$ will be called w-subgroup if N is generated by cyclic subgroups contained in G_w. By weight of a multilinear commutator we mean the number of variables involved in the word. It is clear that any multilinear commutator w of weight $k \geq 2$ can be written in the form $w = [w_1, w_2]$ where w_1 and w_2 are multilinear commutators of smaller weights.

Lemma 2.2. *Let G be a perfect group and A a normal abelian subgroup of G. Then $[A, G]$ is a w-subgroup for any multilinear commutator w.*

Proof. Choose a multilinear commutator word w and let k be the weight of w. If $k = 1$, then $w = x$ and $[A, G]$ is a w-subgroup. Thus, assume that $k \geq 2$ and show that $[A, G]$ is a w-subgroup using induction on k. Let $w = [w_1, w_2]$, where w_1 and w_2 are two multilinear commutators of smaller weights, say k_1 and k_2, such that $k = k_1 + k_2$. By the inductive hypothesis $[A, G]$ is a w_1-subgroup in G. Moreover since w is a multilinear commutator and G is perfect, we have $G = w_2(G)$, and so G is generated by w_2-values. By Lemma 2.1 we know that $[A, G] = [A, G, G]$. Hence $[A, G]$ is generated by elements of the form $[x, y]$, where $\langle x \rangle \subseteq G_{w_1}$ and $y \in G_{w_2}$. Since $[A, G]$ is abelian, we obtain that

$$[x, y]^j = [x^j, y] \in [G_{w_1}, G_{w_2}] \subseteq G_w$$

for any j. Thus we conclude that $[A, G]$ is a w-subgroup, as desired. $\qquad\square$

Another result of similar nature is the following lemma.

Lemma 2.3. *Let K be a normal subgroup of a group G and $w = [w_1, w_2]$ a multilinear commutator word. Suppose that K is nilpotent of class two. If $K/Z(K)$ is a w_1-subgroup in $G/Z(K)$ and if it is generated by w_2-values in $G/Z(K)$, then K' is a w-subgroup in G.*

Proof. By the hypothesis K has a generating set X such that $x^i \in G_{w_1}Z(K)$, for every $x \in X$ and every integer i. Similarly K has a generating set Y such that $y \in G_{w_2}Z(K)$ for every $y \in Y$.

Since K is nilpotent of class two, it follows that K' is generated by elements of the form $[x, y]$, where $x \in X$ and $y \in Y$. Now we have

$$[x, y]^i = [x^i, y] \in [G_{w_1}Z(K), G_{w_2}Z(K)] \subseteq G_w$$

for any i. Thus K' is a w-subgroup in G, as desired. $\qquad\square$

The next lemma ensures us that in any soluble-by-finite group G we can find a term of the derived series of G which is a perfect group.

Lemma 2.4. *Let G be a group having a normal soluble subgroup with finite index n and derived length d. Then $G^{(i)} = G^{(i+1)}$ for some integer $i \leq d+n$.*

Proof. It is clear that any soluble quotient of G must have derived length at most $d + n$. Therefore we have $G^{(i)} = G^{(i+1)}$ for some $i \leq d + n$. $\square$

To any finite ordered set of subgroups of a group G, say $\{N_1, \ldots, N_s\}$, we associate the *triple* $(s, s-i, s-j)$, where s is the number of subgroups in the set, i is the maximal index such that N_i is central in G and j is the maximal index such that N_j is abelian. If there are no central subgroups, or no abelian subgroups in the set, then we put $i = 0$ or $j = 0$, respectively. The set of all such possible triples is naturally endowed with the lexicographical order. In the sequel we say that one triple is smaller than another meaning that it is smaller with respect to the lexicographical order. Our goal in the remaining part of this section is to prove that the verbal subgroup $w(G)$ of a soluble-by-finite group G possesses a normal series with some very specific properties. Similar series were considered in [1] and [2]. We start with the case where G is perfect.

Proposition 2.5. *There exist (d, n)-bounded integers m and k with the following property: Let G be a perfect group having a normal soluble subgroup of finite index n and with derived length d. Then G has a characteristic series*

$$1 = T_1 \leq T_2 \leq \cdots \leq T_m \leq G$$

such that every quotient T_i/T_{i-1} is an abelian w-subgroup in G/T_{i-1} for every multilinear commutator word w and the index $[G : T_m]$ is at most k.

Proof. Since $G = G'$, we have $G = w(G)$ for any w. Let S be the soluble radical of G and let d_1 be the derived length of S. It is clear that $d_1 \leq d+n$. If $d_1 = 0$, then G is finite of order n and so there is nothing to prove.

Thus, suppose that $d_1 \geq 1$ and use induction. Let A be the last nontrivial term of the derived series of S. We consider the quotient group $\overline{G} = G/A$ and, as usual, if $X \leq G$, we denote by $\overline{X}$ the image of X in $\overline{G}$. By the inductive hypothesis $\overline{G}$ has a characteristic series

$$\overline{1} = \overline{K_1} \leq \overline{K_2} \leq \cdots \leq \overline{K_s} \qquad (*)$$

such that every quotient $\overline{K_i}/\overline{K}_{i-1}$ is an abelian w-subgroup in $\overline{G}/\overline{K}_{i-1}$ for every w while s and the index $[\overline{G} : \overline{K_s}]$ are both (d, n)-bounded. We choose the series $(*)$ in such a way that the triple $(s, s - i, s - j)$ associated with the set $\{K_1 = A, K_2, \ldots, K_s\}$ is as small as possible. We now argue by induction on the triple.

If K_s is central in G, then the index $[G : Z(G)]$ is (d, n)-bounded. Thus, by Schur's theorem [7, 10.1.4] $|G'|$ is finite and (d, n)-bounded so there is nothing to prove. Suppose that K_s is not central in G. By the definition of the triple, K_{i+1} is the first term which is not central in G. We define a subgroup D in the following way:

  (1) if K_{i+1} is abelian, then $D = [K_{i+1}, G]$;
  (2) if K_{i+1} is not abelian, then $D = [K_{i+1}, K_{i+1}]$.

In the case (1) Lemma 2.2 tells us that D is a w-subgroup in G for every multilinear commutator word w. In the case (2) the subgroup K_{i+1} is nilpotent of class two and we are in the position to apply Lemma 2.3. Indeed, note that $K_{i+1}/Z(K_{i+1})$ is isomorphic to a quotient group of K_{i+1}/K_i, which is a w-subgroup in G/K_i for every multilinear commutator word w and moreover $K_i \leq Z(K_{i+1})$. Thus, also in case (2) we conclude that D is a w-subgroup in G for all w. It is clear that D is abelian.

Next, we consider the quotient group $\widetilde{G} = G/D$. Observe that in the case (1) the subgroup $\widetilde{K_{i+1}}$ is central in $\widetilde{G}$ and, in the case (2), the subgroup $\widetilde{K_{i+1}}$ is abelian. Therefore in both cases the triple associated to the set $\{\widetilde{K_1}, \widetilde{K_2}, \ldots, \widetilde{K_s}\}$ is smaller. By induction on the triple we deduce that $\widetilde{G}$ has a characteristic series

$$\widetilde{1} = \widetilde{D_1} \leq \widetilde{D_2} \leq \cdots \leq \widetilde{D_l}$$

such that every quotient $\widetilde{D_i}/\widetilde{D_{i-1}}$ is an abelian w-subgroup in $\widetilde{G}/\widetilde{D_{i-1}}$ for all w, and the index $[\widetilde{G} : \widetilde{D_l}]$ and l are (d,n)-bounded. Put $T_1 = 1, T_2 = D, \ldots, T_{l+1} = D_l$. The characteristic series

$$1 = T_1 \leq T_2 \leq \cdots \leq T_{l+1}$$

has all the required properties. This completes the proof. $\square$

Now we analyze the general case where G is a soluble-by-finite group.

Proposition 2.6. *Let w be a multilinear commutator. There exist a (d, n, w)-bounded integer s and a (d, n)-bounded integer h with the following property: Let G be a group having a normal soluble subgroup of finite index n and derived length d. Then G has a series of normal subgroups*

$$1 = T_1 \leq T_2 \leq \cdots \leq T_s = w(G)$$

such that every quotient T_i/T_{i-1} is an abelian w-subgroup in G/T_{i-1}, except possibly one quotient whose order is at most h.

Proof. By Lemma 2.4 there exists an integer $j \leq d + n$ such that $G^{(j)} = G^{(j+1)}$. Since $G^{(j)}$ is perfect, we have $G^{(j)} = w(G^{(j)})$, and so $G^{(j)} \leq w(G)$. By Proposition 2.5 $G^{(j)}$ has characteristic subgroups

$$1 = S_1 \leq S_2 \leq \cdots \leq S_m$$

such that each quotient S_i/S_{i-1} is an abelian w-subgroup in $G^{(j)}/S_{i-1}$ while the index $[G^{(j)} : S_m]$ and m are (d,n)-bounded.

The quotient $\overline{G} = G/G^{(j)}$ is soluble with derived length at most j. It follows from [2, Theorem B] that $\overline{G}$ has a normal series

$$\overline{1} = \overline{P_1} \leq \overline{P_2} \leq \cdots \leq \overline{P_t} = w(\overline{G})$$

such that every quotient $\overline{P_i}/\overline{P_{i-1}}$ is an abelian w-subgroup in $\overline{G}/\overline{P_{i-1}}$ and t is (d, n, w)-bounded. Put $T_1 = S_1, \ldots, T_m = S_m, T_{m+1} = G^{(j)}, T_{m+2} = P_2, \ldots, T_{m+t} = P_t$. By the construction the index $[T_{m+1} : T_m]$ is (d, n)-bounded. Moreover every quotient T_i/T_{i-1} with $i \neq m + 1$ is an abelian

w-subgroup in G/T_{i-1}, and $m + t$ is (d, n, w)-bounded. Therefore $1 = T_1 \leq T_2 \leq \cdots \leq T_{m+t} = w(G)$ is the series with the desired properties. This completes the proof. $\qquad\square$

We conclude this section with the following corollary of Proposition 2.6.

Corollary 2.7. *Let G be a soluble-by-finite group, w a multilinear commutator word, and $m \geq 1$ an integer. Suppose that G has only finitely many w^m-values. Then $w(G)$ has finite exponent. If G has only t values of w^m and the soluble radical of G has index n and derived length d, then the exponent of $w(G)$ is (d, m, n, t, w)-bounded.*

Proof. It follows from Theorem 2.6 that G has a series,

$$1 = T_1 \leq T_2 \leq \cdots \leq T_s = w(G)$$

such that s is a (d, n, w)-bounded number and every quotient T_i/T_{i-1} is an abelian w-subgroup in G/T_{i-1}, except possibly one quotient whose order is (d, n)-bounded. By the hypothesis G has only t values of w^m. Therefore every quotient T_i/T_{i-1} must have finite exponent bounded in terms of m and t, except possibly one finite quotient whose order is (d, n)-bounded. Thus we conclude that $w(G)$ has finite exponent bounded in terms of d, m, n, t and w, as claimed. $\qquad\square$

3. Proof of the main results

The aim of this section is to prove Theorem 1.1 and Theorem 1.2. For the reader's convenience we recall some useful facts.

Lemma 3.1. *Let w be any word and G a group such that G_w is finite. Then $w(G)$ is finite if and only if it is periodic. Moreover if G has precisely m w-values and $w(G)$ has exponent e, then the order of $w(G)$ is (e, m)-bounded.*

Proof. Assume that G_w is a finite set of m elements. Let $C = C_G(G_w)$. Since the set G_w is normal, it follows that the index of C in G is at most $m!$. Since $w(G)$ is generated by at most m w-values, $C \cap w(G)$ is generated by an m-bounded number of elements. Moreover $C \cap w(G)$ is an abelian subgroup with m-bounded index in $w(G)$. Therefore $w(G)$ is finite if and only if $C \cap w(G)$ is periodic. It is clear that if $C \cap w(G)$ has exponent e, then $w(G)$ has (e, m)-bounded order. $\qquad\square$

We need the following result that can be found in [11]. The proof is based on Lie methods in the spirit of Zelmanov's solution of the Restricted Burnside Problem.

Theorem 3.2. *Let q be a prime-power and w a multilinear commutator word. Assume that G is a residually finite group such that any w-value in G has order dividing q. Then the verbal subgroup $w(G)$ is locally finite.*

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let G be a residually finite group in which the word $v = w^q$ has only finitely many values. In view of Lemma 3.1 it is sufficient to show that $v(G)$ is periodic. Choose a normal subgroup K in G such that the index $[G : K]$ is finite and $v(K) = 1$. All w-values in K have order dividing q. By Theorem 3.2 $w(K)$ is locally finite and so in particular it is periodic. We pass to the quotient group $G/w(K)$ and assume that $w(K) = 1$. Then K is soluble and so G is soluble-by-finite. Corollary 2.7 now tells us that $w(G)$ has finite exponent. Since every v-value in G is an element of $w(G)$, it follows that $v(G)$ is periodic too. The proof is complete. $\square$

Next we deal with the proof of Theorem 1.2. The following result will be useful.

Proposition 3.3. *Let $m, k \geq 1$ and q be a p-power for some prime p. Assume that G is a finite m-generator group such that $[x_1, \ldots, x_k]^q = 1$ for all $x_1, \ldots, x_k \in G$. Then the exponent of $\gamma_k(G)$ is (k, m, q)-bounded.*

Proof. In the case where $k = 2$ this is Lemma 3.3 of [10]. The general case is actually rather similar to the case $k = 2$. By [13, Lemma 5] $\gamma_k(G)$ is a p-group. Therefore G is metanilpotent. It follows from [14, Theorem 3.2] that there exists a number s, depending only on m, such that every element of $\gamma_k(G)$ is a product of at most s γ_k-values. Of course this also follows from Segal's theorem [9]. Let x be an arbitrary element in $\gamma_k(G)$ and write $x = x_1 \ldots x_s$, where x_i are γ_k-values. Let $H = \langle x_1, \ldots, x_s \rangle$. By Zelmanov's theory the order of H is (k, q, s)-bounded. In particular the order of the element x is (k, q, s)-bounded. Since x has been chosen in $\gamma_k(G)$ arbitrarily and since s depends only on m, the result follows. $\square$

Proposition 3.4. *Let $k, m \geq 1$ and q be a prime-power. Let $v = [x_1, \ldots, x_k]^q$ and suppose that G is a finite group having precisely m v-values. Then the order of $v(G)$ is (k, m, q)-bounded.*

Proof. Without loss of generality we can assume that G is generated by at most mk elements. Indeed, let $\{u_1, \ldots, u_m\}$ be the set of all v-values in G. Write $u_i = [g_{i1}, \ldots, g_{ik}]^q$ for $i = 1, \ldots, m$. Let K be the subgroup of G generated by $g_{11}, g_{12}, \ldots, g_{mk}$ and note that $v(G) = v(K)$. Hence, without loss of generality we can assume that G is generated by $g_{11}, g_{12}, \ldots, g_{mk}$.

Let H be the centralizer in G of all v-values. Then $[G : H] \leq m!$. Since G is generated by at most mk elements and H has m-bounded index in G, it follows that also H has a (k, m)-bounded number of generators. Consider the quotient $\overline{H} = H/Z(H)$. Since H centralizes all v-values, it follows that the law $v \equiv 1$ holds in $\overline{H}$. Therefore we are in the position to use Proposition 3.3 and deduce that the exponent of $\gamma_k(\overline{H})$ is (k, m, q)-bounded. Thus $\gamma_k(H)/Z(\gamma_k(H))$ has (k, m, q)-bounded exponent. Mann's result [4] tells us that $[\gamma_k(H), \gamma_k(H)]$ has (k, m, q)-bounded exponent. We pass to the quotient $G/[\gamma_k(H), \gamma_k(H)]$ and assume that $\gamma_k(H)$ is abelian. It follows that H is soluble with derived length at most $k + 1$. By Corollary 2.7 we

conclude that $v(G)$ has (k, m, q)-bounded exponent. Lemma 3.1 now tells us that $v(G)$ has (k, m, q)-bounded order, as desired. □

The proof of Theorem 1.2 is now immediate.

Proof of Theorem 1.2. Let $v = w^q$ and choose a residually finite group G in which the word v has at most m values. According to Proposition 3.4 there exists a (k, m, q)-bounded number ν such that the order of $v(K)$ is at most ν for every finite quotient K of G. Hence, every subgroup of finite index of G intersects $v(G)$ by a subgroup of index at most ν in $v(G)$. Since G is residually finite, we conclude that the order of $v(G)$ is at most ν. The proof is complete. □

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