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# On groups in which Engel sinks are cyclic

Cristina Acciarri and Pavel Shumyatsky

ABSTRACT. For an element  $g$  of a group  $G$ , an Engel sink is a subset  $\mathcal{E}(g)$  such that for every  $x \in G$  all sufficiently long commutators  $[x, g, g, \dots, g]$  belong to  $\mathcal{E}(g)$ . We conjecture that if  $G$  is a profinite group in which every element admits a sink that is a procyclic subgroup, then  $G$  is procyclic-by-(locally nilpotent). We prove the conjecture in two cases – when  $G$  is a finite group, or a soluble pro- $p$  group.

## 1. Introduction

A group  $G$  is called an *Engel group* if for every  $x, g \in G$  the equation  $[x, g, g, \dots, g] = 1$  holds, where  $g$  is repeated in the commutator sufficiently many times depending on  $x$  and  $g$ . (Throughout the paper, we use the left-normed simple commutator notation  $[a_1, a_2, a_3, \dots, a_r] = [\dots[[a_1, a_2], a_3], \dots, a_r]$ .) Of course, any nilpotent group is an Engel group. For finite groups the converse is also known to be true: a finite Engel group is nilpotent by Zorn's theorem [15]. Given arbitrary elements  $x, g$  in a group  $G$ , here and in what follows, for any  $n \geq 1$ , we will denote by  $[x, {}_n g]$  the commutator of the form  $[x, \underbrace{g, \dots, g}_n]$ .

Recently, groups that are ‘almost Engel’ in the sense of restrictions on so-called Engel sinks were given some attention. An Engel sink of

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an element  $g \in G$  is a set  $\mathcal{E}(g)$  such that for every  $x \in G$  all sufficiently long commutators  $[x, g, g, \dots, g]$  belong to  $\mathcal{E}(g)$ , that is, for every  $x \in G$  there is a positive integer  $n(x, g)$  such that

$$[x, {}_n g] \in \mathcal{E}(g) \quad \text{for all } n \geq n(x, g).$$

Engel groups are precisely the groups for which we can choose  $\mathcal{E}(g) = \{1\}$  for all  $g \in G$ . In [5] finite, profinite, and compact groups in which every element has a finite Engel sink were considered. It was proved that compact groups with this property are finite-by-(locally nilpotent). Similar result for linear groups was established in [9] (see also [12] for a shorter proof). Recall that a group  $G$  is locally nilpotent if every finitely generated subgroup of  $G$  is nilpotent. According to an important theorem, due to Wilson and Zelmanov [14], a profinite group is locally nilpotent if and only if it is Engel.

In [6] finite groups in which there is a bound for the ranks of the subgroups generated by Engel sinks were considered. Recall that the rank of a finite group is the minimum number  $r$  such that every subgroup can be generated by  $r$  elements. It was shown that if  $G$  is a finite group such that for every  $g \in G$  the Engel sink  $\mathcal{E}(g)$  generates a subgroup of rank  $r$ , then the rank of  $\gamma_\infty(G)$  is bounded in terms of  $r$ . Here  $\gamma_\infty(G)$  stands for the intersection of all terms of the lower central series of  $G$ .

The goal of this article is to establish some substantial evidence in favor of the following conjecture.

**Conjecture 1.1.** *Let  $G$  be a profinite group in which every element admits an Engel sink that generates a procyclic subgroup. Then  $G$  is procyclic-by-(locally nilpotent).*

First, we consider finite groups in which all elements admit Engel sinks generating cyclic subgroups.

**Theorem 1.2.** *Let  $G$  be a finite group in which every element admits an Engel sink generating a cyclic subgroup. Then  $\gamma_\infty(G)$  is cyclic.*

Recall that a profinite group is a topological group that is isomorphic to an inverse limit of finite groups. The reader is referred to textbooks [7] and [13] for background information on profinite groups. In the context of such groups all the usual concepts of groups theory are interpreted topologically. In particular, by a subgroup of a profinite group we always mean a closed subgroup. The next result deals with soluble pro- $p$  groups in which every element admits an Engel sink generating a procyclic subgroup.

**Theorem 1.3.** *Let  $G$  be a soluble pro- $p$  group in which every element admits an Engel sink generating a procyclic subgroup. Then  $G$  has a normal procyclic subgroup  $K$  such that  $G/K$  is locally nilpotent.*

In the next section we deal with the proof of Theorem 1.2. The proof of Theorem 1.3 is given in Section 3.

## 2. Proof of Theorem 1.2

We start with a collection of well-known facts about coprime automorphisms that we will use throughout the article. Given a group  $G$  acted on by a group  $A$  we write  $C_G(A)$  for the subgroup of fixed points of  $A$  in  $G$  and  $[G, A]$  for the subgroup generated by all elements of the form  $x^{-1}x^a$ , where  $x \in G$  and  $a \in A$ .

**Lemma 2.1.** *Let  $A$  be a group of automorphisms of a finite group  $G$  such that  $(|G|, |A|) = 1$ . Then*

- (i)  $G = C_G(A)[G, A]$ ;
- (ii)  $[G, A, A] = [G, A]$ ;
- (iii)  $C_{G/N}(A) = C_G(A)N/N$  for any  $A$ -invariant normal subgroup  $N$  of  $G$ ;
- (iv) If  $G$  is cyclic of prime-power order, then  $A$  is cyclic;
- (v) If  $G$  is cyclic of 2-power order, then  $A = 1$ .

The assumption of coprimeness is unnecessary in the following lemma.

**Lemma 2.2.** *Let  $G$  be a cyclic group. The group of automorphisms of  $G$  is abelian.*

Recall that a normal subgroup  $N$  of a finite group  $G$  is a normal  $p$ -complement (for a prime  $p$ ) if  $N = O_{p'}(G)$  and  $G/N$  is a  $p$ -group. The well-known theorem of Frobenius states that  $G$  possesses a normal  $p$ -complement if and only if  $N_G(H)/C_G(H)$  is a  $p$ -group for every nontrivial  $p$ -subgroup  $H$  of  $G$  (see [3, Theorem 7.4.5]).

Obviously, in a finite group  $G$  every element has the smallest Engel sink, so throughout this section, we use the term Engel sink for the minimal Engel sink, denoted by  $\mathcal{E}(g)$ , of an element  $g \in G$ .

**Lemma 2.3.** *Let  $G$  be a finite group in which for each  $g \in G$  the Engel sink  $\mathcal{E}(g)$  generates a cyclic subgroup. Then  $G$  has a normal 2-complement.*

PROOF. Suppose that this is false. Then  $G$  has an element  $x$  of odd order and a 2-subgroup  $H$  such that  $x$  normalizes but not centralizes  $H$ . Let  $E = H \cap \mathcal{E}(x)$ . Observe that  $x$  normalizes  $\langle E \rangle$ . In view of Lemma 2.1(v), we deduce that  $x$  centralizes  $\langle E \rangle$ . Therefore for every  $h \in H$  we have  $[h, x, x, \dots, x] = 1$  if  $x$  is repeated in the commutator sufficiently many times. In other words,  $x$  is Engel in the group  $H\langle x \rangle$  and we deduce that  $[H, x] = 1$ . This yields a contradiction.  $\square$

In view of the Feit-Thompson Theorem on solubility of groups of odd order [2] the following corollary is straightforward.

**Corollary 2.4.** *Let  $G$  be a finite group in which the Engel sink  $\mathcal{E}(g)$  generates a cyclic subgroup for each  $g \in G$ . Then  $G$  is soluble.*

Recall that a group  $G$  is metanilpotent if it has a normal subgroup  $N$  such that both  $N$  and  $G/N$  are nilpotent. It is easy to see that a finite group  $G$  is metanilpotent if and only if  $\gamma_\infty(G)$  is nilpotent. The next result is well known (see for example [1, Lemma 2.4] for the proof).

**Lemma 2.5.** *Let  $G$  be a finite metanilpotent group. Assume that  $P$  is a Sylow  $p$ -subgroup of  $\gamma_\infty(G)$  and  $H$  is a Hall  $p'$ -subgroup of  $G$ . Then  $P = [P, H]$ .*

We will now prove Theorem 1.2 under the additional assumption that  $G$  is metanilpotent.

**Lemma 2.6.** *Let  $G$  be a finite metanilpotent group in which for each  $g \in G$  the Engel sink  $\mathcal{E}(g)$  generates a cyclic subgroup. Then  $\gamma_\infty(G)$  is cyclic.*

PROOF. Since  $\gamma_\infty(G)$  is nilpotent, it is sufficient to show that each Sylow subgroup of  $\gamma_\infty(G)$  is cyclic. Thus, let  $P$  be a Sylow subgroup of  $\gamma_\infty(G)$  for some prime  $p$ . In view of Lemma 2.5 we have  $P = [P, H]$ , where  $H$  is a Hall  $p'$ -subgroup of  $G$ . Without loss of generality we can assume that  $G = PH$ . Replacing if necessary  $P$  by  $P/\Phi(P)$  and  $H$  by  $H/C_H(P)$ , we can assume that  $P$  is an elementary abelian  $p$ -group (a vector space over the field with  $p$  elements) on which the nilpotent group  $H$  acts faithfully by linear transformations.

Taking into account that  $H$  is nilpotent, we note that  $\mathcal{E}(h) = [P, h]$  for each nontrivial  $h \in H$ . Therefore, if  $H = \langle g \rangle$  is cyclic, then  $P = \mathcal{E}(g)$  is cyclic, too. Hence, we assume that  $H$  is noncyclic.

Suppose first that  $H$  contains a noncyclic abelian subgroup  $A$ . Choose a nontrivial element  $a_1 \in A$ . The cyclic subgroup  $[P, a_1]$  is  $A$ -invariant and, by Lemma 2.1(iv), the quotient  $A/C_A([P, a_1])$  is cyclic. In particular  $C_A([P, a_1]) \neq 1$  so we choose a nontrivial element  $a_2 \in C_A([P, a_1])$ . Since  $a_2$  centralizes  $[P, a_1]$ , it follows that  $[P, a_1][P, a_2]$  is not cyclic. Moreover, it is clear that  $a_1$  centralizes  $[P, a_2]$ . Hence,  $[P, a_1][P, a_2] \leq [P, a_1a_2]$ . This shows that  $\mathcal{E}(a_1a_2)$  is not cyclic, a contradiction. Therefore all abelian subgroups of  $H$  are cyclic.

It follows (see for example [3, Theorem 4.10(ii), p. 199]) that  $H$  is isomorphic to  $Q \times C$ , where  $Q$  is a generalized quaternion group and  $C$  is a cyclic group of odd order. Let  $a_0$  be the unique involution of  $H$ . It is clear that  $a_0$  is contained in all maximal cyclic subgroups of  $H$ . Thus we have  $[P, h] = [P, a_0]$  for any  $h \in H$  and so  $[P, H] = [P, a_0]$ . Note that  $[P, a_0]$  is an  $H$ -invariant subgroup of order  $p$ . In view of Lemma 2.1(iv), note that  $H$  induces a cyclic group of automorphisms of  $[P, a_0]$ .

We deduce that  $a_0$  acts trivially on  $[P, a_0]$  and hence on  $P$ . This is a final contradiction. It shows that  $P$  is cyclic, as required.  $\square$

Recall that the Fitting height of a finite soluble group  $G$  is the minimum number  $h = h(G)$  such that  $G$  possesses a normal series  $1 = G_0 \leq G_1 \leq \dots \leq G_h = G$  all of whose factors are nilpotent. We say that a system of subgroups  $P_1, \dots, P_k$  of  $G$  is a tower of height  $k$  if

- Each subgroup  $P_i$  has prime-power order.
- $P_j$  is normalized by  $P_i$  whenever  $1 \leq i \leq j \leq k$ .
- $P_{i+1} = \gamma_\infty(P_{i+1}P_i)$  for each  $i = 1, 2, \dots, k-1$ .

Every finite soluble group of Fitting height  $h$  possesses a tower of height  $h$  (see for example [11]).

We are now ready to prove the theorem on finite groups.

**PROOF OF THEOREM 1.2.** Recall that  $G$  is a finite group in which for each  $g \in G$  the Engel sink  $\mathcal{E}(g)$  generates a cyclic subgroup. We need to show that  $\gamma_\infty(G)$  is cyclic. By Corollary 2.4, the group  $G$  is soluble. Suppose that the theorem is false and let  $G$  be a counter-example of minimal order. Lemma 2.6 shows that  $h(G) \geq 3$ .

Choose three subgroups  $P_1, P_2, P_3$  which form a tower of height 3. Since  $P_3 = \gamma_\infty(P_3P_2)$ , because of Lemma 2.6 we conclude that  $P_3$  is cyclic. By Lemma 2.2 the subgroup  $P_2P_1$  induces an abelian group of automorphisms of  $P_3$ . Since  $P_2 = \gamma_\infty(P_2P_1)$ , we conclude that  $P_2$  acts on  $P_3$  trivially. In other words,  $P_2$  centralizes  $P_3$ . In view of the equality  $P_3 = \gamma_\infty(P_3P_2)$  we have a contradiction. This completes the proof.  $\square$

### 3. Proof of Theorem 1.3

Our purpose in this section is to prove Theorem 1.3. Given an element  $g$  of a group  $G$ , for each  $n \geq 1$ , we will denote by  $E_n(g)$  the subgroup of  $G$  generated by all commutators of the form  $[x, {}_n g]$ , with  $x$  in  $G$ .

The next two results, whose proofs can be found in [10, Lemmas 2.1 and 2.2] respectively, state general facts about nilpotent groups and Engel elements.

**Lemma 3.1.** *Let  $G = H\langle a \rangle$ , where  $H$  is a normal nilpotent subgroup of class  $c$  and  $a$  is an  $n$ -Engel element. Then  $G$  is nilpotent with class at most  $cn$ .*

**Lemma 3.2.** *For any positive integers  $c, n$  there exists an integer  $f = f(c, n)$  with the following property. Let  $G = H\langle a \rangle$ , where  $H$  is a normal nilpotent subgroup of class  $c$ . Then  $\gamma_f(G) \leq E_n(a)$ .*

Here and throughout the article  $\gamma_f(G)$  denotes the  $f$ th term of the lower central series of  $G$ .

The following lemma concerns profinite groups and Engel elements.

**Lemma 3.3.** *Let  $G = M\langle a \rangle$  be a profinite group with an abelian normal subgroup  $M$  and an Engel element  $a$ . Then  $G$  is nilpotent.*

PROOF. For any nonnegative integer  $i$  set

$$B_i = \{x \in M \mid [x, {}_i a] = 1\}.$$

Each set  $B_i$  is closed, and  $\bigcup_{i \geq 0} B_i = M$ . By Baire's Category Theorem [4, p. 200] at least one of these sets has non-empty interior. Therefore there exist an integer  $n$ , an element  $b$  in  $M$  and an open normal subgroup  $N$  contained in  $M$  such that  $[y, {}_n a] = 1$  for any  $y \in bN$ . From this we deduce that  $[x, {}_n a] = 1$  for any  $x$  in  $N$ . Since  $N$  is open in  $M$ , there exists a positive integer  $k$  such that  $[z, {}_k a] \in N$  for any  $z \in M$ . Thus  $[M, {}_{n+k} a] = 1$  and the result follows.  $\square$

Note that, for an element  $g$  of a group  $G$ , once a sink  $\mathcal{E}(g)$  is chosen, the subgroup  $\langle \mathcal{E}(g) \rangle$  generated by  $\mathcal{E}(g)$  is also a sink for  $g$ . In the remaining part of this article it will be convenient to use the term "sink  $\mathcal{E}(g)$  of  $g$ " meaning a subgroup containing all sufficiently long commutators  $[x, {}_i g]$  with  $x \in G$ .



**Lemma 3.4.** *Let  $G$  be a metabelian profinite group and let  $a$  be an element of  $G$ . Then, for any choice of a sink  $\mathcal{E}(a)$ , there exists an integer  $n$  such that  $E_n(a) \leq \mathcal{E}(a)$ .*

PROOF. If  $\mathcal{E}(a)$  is finite, then  $a$  is Engel in  $G$ . Set  $K = G'\langle a \rangle$ . By Lemma 3.3 the subgroup  $K$  is nilpotent and  $[G',_{n-1} a] = 1$ , for some integer  $n$ . Therefore  $[G,_{n-1} a] = 1$  and so  $E_n(a) \leq \mathcal{E}(a)$ .

Assume that  $\mathcal{E}(a)$  is infinite. Let  $E_1$  be the subgroup generated by all commutators  $[x, a, \dots, a] \in \mathcal{E}(a)$  such that  $x \in G'$ . Note that  $E_1 \leq \mathcal{E}(a)$  and  $E_1$  is a normal subgroup of  $G$ . Moreover  $a$  is Engel in  $G/E_1$ . In view of Lemma 3.3 the subgroup  $G'\langle a \rangle$  is nilpotent modulo  $E_1$  and the result follows.  $\square$

**Lemma 3.5.** *Let  $G$  be a metabelian profinite group and  $a \in G$ . For each  $n \geq 1$  the subgroup  $E_n(a)$  is normal in  $G$ .*

PROOF. For any  $i \geq 1$ , any  $g \in G'$  and  $y \in G$  we have

$$[g,_{i-1} a]^y = [g^y,_{i-1} a] \text{ and } [g^{-1},_{i-1} a] = [g,_{i-1} a]^{-1}.$$

Moreover, for any  $x, y \in G$ , the equality  $[x, a]^y = [xy, a][y, a]^{-1}$  holds.

We only need to prove the lemma with  $n \geq 2$  since for  $n = 1$  the result is well known even without the assumption that  $G$  is metabelian. For arbitrary elements  $x, y \in G$ , by using the standard commutator laws, write

$$[x,_{n-1} a]^y = [[x, a]^y,_{n-1} a] = [[xy, a][y, a]^{-1},_{n-1} a] = [xy,_{n-1} a][y,_{n-1} a]^{-1}.$$

The formula above shows that  $[x,_{n-1} a]^y \in E_n(a)$  and the lemma follows.  $\square$

We write  $C_n$  to denote the cyclic group of order  $n$  and  $\mathbb{Z}_p$  the additive group of  $p$ -adic integers. Recall that the group of automorphisms of  $\mathbb{Z}_p$  is isomorphic to  $\mathbb{Z}_p \oplus C_{p-1}$  if  $p \geq 3$  and  $\mathbb{Z}_2 \oplus C_2$  if  $p = 2$  (see for example [7, Theorem 4.4.7]). Note that all nontrivial subgroups of  $\mathbb{Z}_p$  have finite index in  $\mathbb{Z}_p$  (see for example [8, Proposition and Corollary 1 at p. 23]).

**Lemma 3.6.** *Let  $G$  be a pro- $p$  group and  $K$  a normal infinite procyclic subgroup of  $G$ . If  $a \notin C_G(K)$ , then for any  $i \geq 1$ , the subgroup  $[K, {}_i a]$  has finite index in  $K$ . In particular, if  $G$  is locally nilpotent, then  $K$  is central in  $G$ .*

PROOF. Let  $\alpha$  be the automorphism of  $K$  induced by the conjugation by the element  $a$ . Write  $R$  for the ring of the  $p$ -adic integers and regard  $K$  as the additive group of  $R$ . There exists  $b \in R$  such that  $x^a = x \cdot b$ , for each  $x \in K$ . Note that the subgroup  $[K, {}_n a]$  consists of elements of the form  $x \cdot (b - 1)^n$ , where  $x$  ranges over  $K$ . Moreover the set  $\{x \cdot (b - 1)^n \mid x \in K\}$  is infinite for each  $n \geq 1$ , since  $R$  has no zero divisors. The lemma follows.  $\square$

We now can prove Theorem 1.3 in the particular case where  $G$  is metabelian. The general case will require considerably more efforts.

**Proposition 3.7.** *Let  $G$  be a metabelian pro- $p$  group such that  $\mathcal{E}(g)$  can be chosen procyclic for each  $g$  in  $G$ . Then  $G$  has a normal procyclic subgroup  $K$  such that  $G/K$  is locally nilpotent.*

PROOF. If  $G$  is Engel, then it is locally nilpotent and there is nothing to prove. Assume that  $G$  is not Engel and let  $X$  be the set of all non-Engel elements in  $G$ . In view of Lemmas 3.4 and 3.5 we can assume that all  $\mathcal{E}(g)$  are chosen procyclic and normal in  $G$ . Indeed, by Lemmas 3.4 and 3.5, for each  $g$  in  $G$  there exists an integer  $n$  such that  $E_n(g)$  is a normal subgroup in  $\mathcal{E}(g)$ , so we can take such  $E_n(g)$  as the sink  $\mathcal{E}(g)$  of  $g$ . In view of Lemma 3.6 each subgroup  $[\mathcal{E}(x), {}_i x]$  has finite index in  $\mathcal{E}(x)$  whenever  $x \in X$ .

Given  $a, b \in X$ , suppose first that  $\mathcal{E}(a) \cap \mathcal{E}(b) = 1$ . On the one hand,  $a$  acts on  $\mathcal{E}(a)$  in such a way that, for any  $i \geq 1$ , the subgroup  $[\mathcal{E}(a), {}_i a]$  has finite index in  $\mathcal{E}(a)$ . On the other hand,  $a$  centralizes  $\mathcal{E}(b)$ , since the intersection of the two sinks is trivial. A similar remarks applies to  $b$ . Note that  $ab$  acts on  $\mathcal{E}(a) \oplus \mathcal{E}(b)$  in the following way: it acts as the element  $a$  on  $\mathcal{E}(a)$  and as  $b$  on  $\mathcal{E}(b)$ . This implies that, for any  $n$ , the subgroup  $E_n(ab)$  contains a subgroup, which is the direct sum

of a finite index subgroup in  $\mathcal{E}(a)$  and a finite index subgroup in  $\mathcal{E}(b)$ , isomorphic to  $\mathbb{Z}_p \oplus \mathbb{Z}_p$ . Thus  $\mathcal{E}(ab)$  is not procyclic, a contradiction.

Hence,  $\mathcal{E}(a) \cap \mathcal{E}(b) \neq 1$ , for any  $a, b \in X$ . Let  $K = \mathcal{E}(a)$  for some  $a \in X$ . We see that, for any  $g \in G$ , the image in  $G/K$  of the sink  $\mathcal{E}(g)$  is finite. Indeed, if  $g$  is Engel in  $G$ , then the claim is obvious. Otherwise  $g \in X$  and the image of  $\mathcal{E}(g)$  in  $G/K$  is isomorphic to  $\mathcal{E}(g)/(\mathcal{E}(a) \cap \mathcal{E}(g))$  which is finite. It follows that  $G/K$  is Engel, hence locally nilpotent by the Wilson-Zelmanov theorem. The proof is complete.  $\square$

Next, we consider another particular case of Theorem 1.3.

**Lemma 3.8.** *Let  $G$  be a soluble pro- $p$  group such that  $\mathcal{E}(g)$  can be chosen procyclic for each  $g$  in  $G$ . Assume that  $G$  has a normal nilpotent subgroup  $M$  and  $a \in G$  such that  $G = M\langle a \rangle$ . Then there exists  $n$  such that  $E_n(a)$  is procyclic and normal in  $G$ . Moreover there exists  $i$  such that  $\gamma_i(G)$  has finite index in  $E_n(a)$ .*

**PROOF.** We argue by induction on the nilpotency class of  $M$ . If  $M$  is abelian, then in view of Lemma 3.5 there exists  $n$  such that  $E_n(a)$  is procyclic and normal in  $G$ . Since  $G/E_n(a)$  is nilpotent, the result holds. Suppose that  $M$  is nonabelian and set  $Z = Z(M)$ . By induction assume that there is  $n$  such that  $L = ZE_n(a)$  is normal in  $G$  and  $L/Z$  is procyclic. Since  $L/Z$  is procyclic, the subgroup  $L$  is abelian. Now looking at the action of  $\langle a \rangle$  on  $L$  and using the fact that  $L$  is abelian, Lemma 3.4 shows that if  $j$  is big enough, then the subgroup  $E_{n+j}(a)$  is procyclic. By Lemma 3.2 there exists  $f$  such that  $\gamma_f(G) \leq E_{n+j}(a)$ . Thus,  $\gamma_f(G)$  is a normal procyclic subgroup and so all subgroups of  $\gamma_f(G)$  are normal in  $G$ . In particular,  $E_f(a)$  is normal and procyclic. Since by Lemma 3.1 the factor-group  $G/E_f(a)$  is nilpotent, there exists  $i$  such that  $\gamma_i(G) \leq E_f(a)$ . This completes the proof.  $\square$

In the sequel we will use, without mentioning explicitly, the following fact: let  $H$  be a subgroup of a profinite group  $G$  and let  $x$  be an element of  $G$  such that  $H^x \leq H$ . Then  $H^x = H$ . This is because if

$H^x < H$ , then the inequality would also hold in some finite image of  $G$ , which yields a contradiction.

The next result is a key observation that will be applied many times throughout the proof of the main result.

**Lemma 3.9.** *Let  $G$  be a profinite group and  $K$  a procyclic pro- $p$  subgroup of  $G$  such that  $K \cap K^x \neq 1$  for each  $x \in G$ . Then  $K$  contains a nontrivial subgroup  $L$  (of finite index) which is normal in  $G$ .*

PROOF. If  $K$  is finite, the result is obvious, so we assume that  $K$  is infinite. Recall that  $K \cap K^x$  has finite index in  $K$  for each  $x \in G$ . For each  $i$  set

$$S_i = \{x \in G \mid K \cap K^x \text{ has index at most } p^i \text{ in } K\}.$$

The sets  $S_i$  are closed. By Baire Category Theorem at least one of these sets has non-empty interior. Therefore there is an open normal subgroup  $N$ , an element  $d \in G$ , and a fixed  $p$ -power  $p^i$  such that  $K \cap K^x$  has index at most  $p^i$  in  $K$  for every  $x \in dN$ . Let  $K_0 = K^{p^i}$  be the subgroup of index  $p^i$  in  $K$ . We see that  $N$  normalizes  $K_0$ . Since  $N$  is open, it follows that  $K_0$  has only finitely many conjugates in  $G$ . Let  $L$  be their intersection. Obviously,  $L$  is normal in  $G$ . Since  $K_0 \cap K_0^x$  has finite index in  $K_0$  for each  $x \in G$ , the subgroup  $L$  is nontrivial.  $\square$

Now we are ready to deal with the proof of Theorem 1.3. We want to establish that if  $G$  is a soluble pro- $p$  group such that  $\mathcal{E}(g)$  can be chosen procyclic for each  $g$  in  $G$ , then  $G$  has a normal procyclic subgroup  $K$  such that  $G/K$  is locally nilpotent.

PROOF OF THEOREM 1.3. The argument will be by induction on the derived length of  $G$ . Set  $H = G'$ . By induction,  $H$  has a normal procyclic subgroup  $K$  such that  $H/K$  is locally nilpotent.

CLAIM 1.  $H$  is locally nilpotent.

If  $K$  is finite, the claim holds. So we assume that  $K$  is infinite. It is sufficient to show that  $H$  has a procyclic subgroup  $K_0$ , which is normal

in  $G$ , such that  $H/K_0$  is locally nilpotent. Indeed, once the existence of such subgroup  $K_0$  is established, observe that  $K_0 \leq Z(H)$  because  $G/C_G(K_0)$  embeds into  $\text{Aut}(\mathbb{Z}_p)$  which is abelian. Hence  $H$  is locally nilpotent. Thus, assume that  $K$  is not normal in  $G$ .

For any  $x \in G$  the quotient  $H/K^x$  is locally nilpotent. If there exists  $x$  such that  $K^x \cap K = 1$ , then  $H$ , being isomorphic to a subgroup  $H/K \times H/K^x$ , must be locally nilpotent, as desired. Therefore we will assume that  $K^x \cap K \neq 1$  for any  $x \in G$ .

In view of Lemma 3.9  $K$  contains a nontrivial subgroup  $L$  which is normal in  $G$ . Since  $H/K$  is locally nilpotent and  $L$  has finite index in  $K$ , it follows that  $H/L$  is locally nilpotent too. Moreover, since  $L$  is normal in  $G$ , it follows that  $L$  is in the center of  $H$  and so  $H$  is locally nilpotent. This establishes Claim 1.

CLAIM 2. Assume that  $G$  has a normal nilpotent subgroup  $M$  such that  $G/M$  is nilpotent and finitely generated. Then  $G$  has a normal procyclic subgroup  $M_0$  such that  $G/M_0$  is nilpotent.

Indeed, choose  $a_1, \dots, a_s$  in  $G$  such that  $G = \langle M, a_1, \dots, a_s \rangle$ . We argue by induction on the nilpotency class of  $G/M$  and also use induction on  $s$ .

Assume first that  $G/M$  is abelian. The case  $s = 1$  follows from Lemma 3.8 so suppose that  $s \geq 2$ . Let  $V_j = M\langle a_j \rangle$ , for  $1 \leq j \leq s$ . Observe that for each  $j$  the subgroup  $V_j$  is normal in  $G$  and, in view of Lemma 3.8, there exists  $i(j)$  such that  $\gamma_{i(j)}(V_j)$  is procyclic. If for any  $j$  the subgroup  $\gamma_{i(j)}(V_j)$  is finite (or trivial), then each  $V_j$  is nilpotent and so  $G$  is nilpotent too. Thus we can assume that some  $\gamma_{i(j)}(V_j)$  are procyclic infinite. Moreover, if for some  $j$  and  $k$  the subgroups  $\gamma_{i(j)}(V_j)$  and  $\gamma_{i(k)}(V_k)$  are infinite and satisfy  $\gamma_{i(j)}(V_j) \cap \gamma_{i(k)}(V_k) = 1$ , then we get a contradiction. Indeed, set  $N = \gamma_{i(j)}(V_j) \oplus \gamma_{i(k)}(V_k)$  and consider the action of  $a_j a_k$  on  $N$ . Arguing as in the proof of Proposition 3.7, we see that  $\mathcal{E}(a_j a_k)$  is not procyclic, since for any  $n$  the subgroup  $E_n(a_j a_k)$  contains a subgroup isomorphic to  $\mathbb{Z}_p \oplus \mathbb{Z}_p$ . We therefore assume that all infinite subgroups  $\gamma_{i(j)}(V_j)$  intersect pairwise nontrivially. In particular

their intersection  $V$  is a nontrivial normal procyclic subgroup such that  $G/V$  is nilpotent. This concludes the argument in the case where  $G/M$  is abelian.

Next, suppose that  $G/M$  has nilpotency class at least two, so in particular  $s$  is bigger than one. Let  $W_j = \langle a_j \rangle HM$ , for  $1 \leq j \leq s$ . Note that any subgroup  $W_j$  modulo  $M$  is a finitely generated subgroup, since it is a subgroup of a finitely generated nilpotent group. Furthermore  $W_j$  modulo  $M$  has nilpotency class smaller than the nilpotency class of  $G/M$ , since it is generated by the image of  $H$  and  $a_j$ . Thus, by induction, any  $W_j$  has a normal procyclic subgroup  $B_j$  such that  $W_j/B_j$  is nilpotent. So, there exists  $l(j)$  such that  $\gamma_{l(j)}(W_j) \leq B_j$ . As in the previous paragraph, if all  $B_j$  are finite (or trivial), then  $G$  is nilpotent. If the infinite  $B_j$  intersect nontrivially, then the claim follows since their intersection  $B$  is a nontrivial normal procyclic subgroup such that  $G/B$  is nilpotent. Suppose that for some  $i, j$  the subgroups  $B_i$  and  $B_j$  are infinite and  $B_i \cap B_j = 1$ . Note that Claim 1 implies that both  $B_i$  and  $B_j$  are centralized by  $H$ . Set  $N = B_i \oplus B_j$  and look at the action of  $a_i a_j$  on  $N$ . We see that for any  $n$  the subgroup  $E_n(a_i a_j)$  contains a subgroup isomorphic to  $\mathbb{Z}_p \oplus \mathbb{Z}_p$ . Thus  $\mathcal{E}(a_i a_j)$  is not procyclic, a contradiction. This concludes the proof of Claim 2.

Let  $R$  be the last nontrivial term of the derived series of  $G$ . By induction on the derived length of  $G$  assume that for  $G/R$  the theorem holds. Thus  $G$  has a normal subgroup  $S$ , containing  $R$ , such that  $S/R$  is procyclic and  $G/S$  is locally nilpotent. Obviously, we can choose  $S$  in such way that  $S \leq H$ . Let  $a \in S$  such that  $S = R\langle a \rangle$ . In view of Claim 1,  $a$  is an Engel element. Thus applying Lemma 3.3 we deduce that  $S$  is nilpotent.

**CLAIM 3.** Let  $a_1, \dots, a_s \in G$  and set  $J = R\langle a_1, \dots, a_s \rangle$ . Then  $J$  has a normal procyclic subgroup  $J_0$  such that  $J/J_0$  is nilpotent.

If  $J/R$  is nilpotent, then the claim follows from Claim 2. Assume that  $J/R$  is not nilpotent. Set  $J_1 = \langle J, a \rangle$ , where  $a$  is as above. Note

that  $S \leq J_1$  and  $J_1/S$  is nilpotent since  $G/S$  is locally nilpotent. Hence, again by Claim 2 there exists a normal procyclic subgroup  $N_0$  in  $J_1$  such that  $J_1/N_0$  is nilpotent. In particular  $JN_0/N_0$  is nilpotent too, so we can take  $J_0 = J \cap N_0$ . This concludes the proof of Claim 3.

We now embark on the final part of the proof of the theorem. Assume that the group  $G$  is not locally nilpotent. Choose elements  $a_1, \dots, a_s \in G$  such that  $T = \langle a_1, \dots, a_s \rangle$  is not nilpotent. Recall that  $S$  is a nilpotent normal subgroup of  $G$  such that  $G/S$  is locally nilpotent. By Claim 2 the group  $ST$  has a normal procyclic subgroup  $K_0$  such that  $ST/K_0$  is nilpotent. Without loss of generality we assume that there is a positive integer  $i_0$  such that  $K_0 = \gamma_{i_0}(ST)$ . Note that  $K_0$  here must be infinite since  $T$  is not nilpotent. Moreover we can replace  $K_0$  by  $S \cap K_0$  and simply assume that  $K_0 \leq S$ . Indeed, since  $ST/K_0$  and  $ST/S$  are both nilpotent, we have  $\gamma_i(ST) \leq S \cap K_0$ , for some positive integer  $i$ .

Given any finite subset  $Y$  of  $G$ , we write  $T_Y$  for the subgroup  $\langle Y, T \rangle$ . By Claim 2 the group  $ST_Y$  has a normal procyclic subgroup  $K_Y$  such that  $ST_Y/K_Y$  is nilpotent. Again there is a positive integer  $i_Y$  such that  $K_Y = \gamma_{i_Y}(ST_Y)$ . Note that all subgroups  $K_Y$  are infinite and have infinite intersection with  $K_0$ . Indeed, any subgroup  $ST_Y$  contains  $ST$ , the subgroup  $ST$  is nilpotent modulo the intersection of  $K_Y$  with  $K_0$ , so if this intersection were trivial, then  $ST_Y$  would be nilpotent, a contradiction. As before, since  $G/S$  is locally nilpotent, we choose all  $K_Y$  inside  $S$ .

Now choose an arbitrary element  $x \in G$  and set

$$Y(x) = \{a_1^x, \dots, a_s^x, a_1, \dots, a_s\}.$$

We see that  $K_{Y(x)}$  has infinite intersection with each of the subgroups  $K_0$  and  $K_0^x$ . Hence  $K_0 \cap K_0^x$  is nontrivial and this holds for any choice of  $x \in G$ . Thus, by Lemma 3.9,  $K_0$  contains a nontrivial subgroup  $L_0$  which is normal in  $G$ .

Note that for any choice of a finite subset  $Y$  of  $G$ , the subgroup  $L_0$  intersects  $K_Y$  by a finite index subgroup, since  $K_0$  intersects  $K_Y$

nontrivially and  $L_0$  has finite index in  $K_0$ . Therefore every subgroup  $T_Y$  is nilpotent modulo  $L_0$ , since  $K_Y$  becomes finite modulo  $L_0$ . Hence  $G$  is locally nilpotent modulo  $L_0$  and this concludes the proof.  $\square$

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