On groups in which Engel sinks are cyclic

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Abstract. For an element g of a group G, an Engel sink is a subset $\mathcal{E}(g)$ such that for every $x \in G$ all sufficiently long commutators $[x, g, g, \dots, g]$ belong to $\mathcal{E}(g)$. We conjecture that if G is a profinite group in which every element admits a sink that is a procyclic subgroup, then G is procyclic-by-(locally nilpotent). We prove the conjecture in two cases – when G is a finite group, or a soluble pro-p group.

1. Introduction

A group G is called an *Engel group* if for every $x, g \in G$ the equation $[x, g, g, \dots, g] = 1$ holds, where g is repeated in the commutator sufficiently many times depending on x and g. (Throughout the paper, we use the left-normed simple commutator notation $[a_1, a_2, a_3, \dots, a_r] = [...[[a_1, a_2], a_3], \dots, a_r]$.) Of course, any nilpotent group is an Engel group. For finite groups the converse is also known to be true: a finite Engel group is nilpotent by Zorn's theorem [**15**]. Given arbitrary elements x, g in a group G, here and in what follows, for any $n \geq 1$, we will denote by $[x,_n g]$ the commutator of the form $[x, \underbrace{g, \dots, g}_{n}]$.

Recently, groups that are 'almost Engel' in the sense of restrictions on so-called Engel sinks were given some attention. An Engel sink of

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an element $g \in G$ is a set $\mathcal{E}(g)$ such that for every $x \in G$ all sufficiently long commutators $[x, g, g, \ldots, g]$ belong to $\mathcal{E}(g)$, that is, for every $x \in G$ there is a positive integer $n(x, g)$ such that

$$[x,_n g] \in \mathcal{E}(g) \qquad \text{for all } n \geq n(x, g).$$

Engel groups are precisely the groups for which we can choose $\mathcal{E}(g) = \{1\}$ for all $g \in G$. In [**5**] finite, profinite, and compact groups in which every element has a finite Engel sink were considered. It was proved that compact groups with this property are finite-by-(locally nilpotent). Similar result for linear groups was established in [**9**] (see also [**12**] for a shorter proof). Recall that a group G is locally nilpotent if every finitely generated subgroup of G is nilpotent. According to an important theorem, due to Wilson and Zelmanov [**14**], a profinite group is locally nilpotent if and only if it is Engel.

In [**6**] finite groups in which there is a bound for the ranks of the subgroups generated by Engel sinks were considered. Recall that the rank of a finite group is the minimum number r such that every subgroup can be generated by r elements. It was shown that if G is a finite group such that for every $g \in G$ the Engel sink $\mathcal{E}(g)$ generates a subgroup of rank r, then the rank of $\gamma_\infty(G)$ is bounded in terms of r. Here $\gamma_\infty(G)$ stands for the intersection of all terms of the lower central series of G.

The goal of this article is to establish some substantial evidence in favor of the following conjecture.

Conjecture 1.1. *Let G be a profinite group in which every element admits an Engel sink that generates a procyclic subgroup. Then G is procyclic-by-(locally nilpotent).*

First, we consider finite groups in which all elements admit Engel sinks generating cyclic subgroups.

Theorem 1.2. *Let G be a finite group in which every element admits an Engel sink generating a cyclic subgroup. Then $\gamma_\infty(G)$ is cyclic.*

Recall that a profinite group is a topological group that is isomorphic to an inverse limit of finite groups. The reader is referred to textbooks [**7**] and [**13**] for background information on profinite groups. In the context of such groups all the usual concepts of groups theory are interpreted topologically. In particular, by a subgroup of a profinite group we always mean a closed subgroup. The next result deals with soluble pro-p groups in which every element admits an Engel sink generating a procyclic subgroup.

Theorem 1.3. *Let G be a soluble pro-p group in which every element admits an Engel sink generating a procyclic subgroup. Then G has a normal procyclic subgroup K such that G/K is locally nilpotent.*

In the next section we deal with the proof of Theorem 1.2. The proof of Theorem 1.3 is given in Section 3.

2. Proof of Theorem 1.2

We start with a collection of well-known facts about coprime automorphisms that we will use throughout the article. Given a group G acted on by a group A we write $C_G(A)$ for the subgroup of fixed points of A in G and $[G, A]$ for the subgroup generated by all elements of the form $x^{-1}x^a$, where $x \in G$ and $a \in A$.

Lemma 2.1. *Let A be a group of automorphisms of a finite group G such that $(|G|, |A|) = 1$. Then*

(i) $G = C_G(A)[G, A]$;
(ii) $[G, A, A] = [G, A]$;
(iii) $C_{G/N}(A) = C_G(A)N/N$ *for any A-invariant normal subgroup N of G;*
(iv) *If G is cyclic of prime-power order, then A is cyclic;*
(v) *If G is cyclic of 2-power order, then $A = 1$.*

The assumption of coprimeness is unnecessary in the following lemma.

Lemma 2.2. *Let G be a cyclic group. The group of automorphisms of G is abelian.*

Recall that a normal subgroup N of a finite group G is a normal p-complement (for a prime p) if $N = O_{p'}(G)$ and G/N is a p-group. The well-known theorem of Frobenius states that G possesses a normal p-complement if and only if $N_G(H)/C_G(H)$ is a p-group for every nontrivial p-subgroup H of G (see [**3**, Theorem 7.4.5]).

Obviously, in a finite group G every element has the smallest Engel sink, so throughout this section, we use the term Engel sink for the minimal Engel sink, denoted by $\mathcal{E}(g)$, of an element $g \in G$.

Lemma 2.3. *Let G be a finite group in which for each $g \in G$ the Engel sink $\mathcal{E}(g)$ generates a cyclic subgroup. Then G has a normal 2-complement.*

Proof. Suppose that this is false. Then G has an element x of odd order and a 2-subgroup H such that x normalizes but not centralizes H. Let $E = H \cap \mathcal{E}(x)$. Observe that x normalizes $\langle E \rangle$. In view of Lemma 2.1(v), we deduce that x centralizes $\langle E \rangle$. Therefore for every $h \in H$ we have $[h, x, x, \dots, x] = 1$ if x is repeated in the commutator sufficiently many times. In other words, x is Engel in the group $H\langle x \rangle$ and we deduce that $[H, x] = 1$. This yields a contradiction. $\square$

In view of the Feit-Thompson Theorem on solubility of groups of odd order [**2**] the following corollary is straightforward.

Corollary 2.4. *Let G be a finite group in which the Engel sink $\mathcal{E}(g)$ generates a cyclic subgroup for each $g \in G$. Then G is soluble.*

Recall that a group G is metanilpotent if it has a normal subgroup N such that both N and G/N are nilpotent. It is easy to see that a finite group G is metanilpotent if and only if $\gamma_\infty(G)$ is nilpotent. The next result is well known (see for example [**1**, Lemma 2.4] for the proof).

Lemma 2.5. *Let G be a finite metanilpotent group. Assume that P is a Sylow p-subgroup of $\gamma_\infty(G)$ and H is a Hall p'-subgroup of G. Then $P = [P, H]$.*

We will now prove Theorem 1.2 under the additional assumption that G is metanilpotent.

Lemma 2.6. *Let G be a finite metanilpotent group in which for each $g \in G$ the Engel sink $\mathcal{E}(g)$ generates a cyclic subgroup. Then $\gamma_\infty(G)$ is cyclic.*

Proof. Since $\gamma_\infty(G)$ is nilpotent, it is sufficient to show that each Sylow subgroup of $\gamma_\infty(G)$ is cyclic. Thus, let P be a Sylow subgroup of $\gamma_\infty(G)$ for some prime p. In view of Lemma 2.5 we have $P = [P, H]$, where H is a Hall p'-subgroup of G. Without loss of generality we can assume that $G = PH$. Replacing if necessary P by $P/\Phi(P)$ and H by $H/C_H(P)$, we can assume that P is an elementary abelian p-group (a vector space over the field with p elements) on which the nilpotent group H acts faithfully by linear transformations.

Taking into account that H is nilpotent, we note that $\mathcal{E}(h) = [P, h]$ for each nontrivial $h \in H$. Therefore, if $H = \langle g \rangle$ is cyclic, then $P = \mathcal{E}(g)$ is cyclic, too. Hence, we assume that H is noncyclic.

Suppose first that H contains a noncyclic abelian subgroup A. Choose a nontrivial element $a_1 \in A$. The cyclic subgroup $[P, a_1]$ is A-invariant and, by Lemma 2.1(iv), the quotient $A/C_A([P, a_1])$ is cyclic. In particular $C_A([P, a_1]) \neq 1$ so we choose a nontrivial element $a_2 \in C_A([P, a_1])$. Since a_2 centralizes $[P, a_1]$, it follows that $[P, a_1][P, a_2]$ is not cyclic. Moreover, it is clear that a_1 centralizes $[P, a_2]$. Hence, $[P, a_1][P, a_2] \leq [P, a_1a_2]$. This shows that $\mathcal{E}(a_1a_2)$ is not cyclic, a contradiction. Therefore all abelian subgroups of H are cyclic.

It follows (see for example [**3**, Theorem 4.10(ii), p. 199]) that H is isomorphic to $Q \times C$, where Q is a generalized quaternion group and C is a cyclic group of odd order. Let a_0 be the unique involution of H. It is clear that a_0 is contained in all maximal cyclic subgroups of H. Thus we have $[P, h] = [P, a_0]$ for any $h \in H$ and so $[P, H] = [P, a_0]$. Note that $[P, a_0]$ is an H-invariant subgroup of order p. In view of Lemma 2.1(iv), note that H induces a cyclic group of automorphisms of $[P, a_0]$.

We deduce that a_0 acts trivially on $[P, a_0]$ and hence on P. This is a final contradiction. It shows that P is cyclic, as required. $\square$

Recall that the Fitting height of a finite soluble group G is the minimum number $h = h(G)$ such that G possesses a normal series $1 = G_0 \leq G_1 \leq \cdots \leq G_h = G$ all of whose factors are nilpotent. We say that a system of subgroups $P_1, \ldots, P_k$ of G is a tower of height k if

- Each subgroup P_i has prime-power order.
- P_j is normalized by P_i whenever $1 \leq i \leq j \leq k$.
- $P_{i+1} = \gamma_\infty(P_{i+1}P_i)$ for each $i = 1, 2, \ldots, k-1$.

Every finite soluble group of Fitting height h possesses a tower of height h (see for example [**11**]).

We are now ready to prove the theorem on finite groups.

Proof of Theorem 1.2. Recall that G is a finite group in which for each $g \in G$ the Engel sink $\mathcal{E}(g)$ generates a cyclic subgroup. We need to show that $\gamma_\infty(G)$ is cyclic. By Corollary 2.4, the group G is soluble. Suppose that the theorem is false and let G be a counter-example of minimal order. Lemma 2.6 shows that $h(G) \geq 3$.

Choose three subgroups P_1, P_2, P_3 which form a tower of height 3. Since $P_3 = \gamma_\infty(P_3P_2)$, because of Lemma 2.6 we conclude that P_3 is cyclic. By Lemma 2.2 the subgroup P_2P_1 induces an abelian group of automorphisms of P_3. Since $P_2 = \gamma_\infty(P_2P_1)$, we conclude that P_2 acts on P_3 trivially. In other words, P_2 centralizes P_3. In view of the equality $P_3 = \gamma_\infty(P_3P_2)$ we have a contradiction. This completes the proof. $\square$

3. Proof of Theorem 1.3

Our purpose in this section is to prove Theorem 1.3. Given an element g of a group G, for each $n \geq 1$, we will denote by $E_n(g)$ the subgroup of G generated by all commutators of the form $[x,_n g]$, with x in G.

The next two results, whose proofs can be found in [**10**, Lemmas 2.1 and 2.2] respectively, state general facts about nilpotent groups and Engel elements.

Lemma 3.1. *Let $G = H\langle a\rangle$, where H is a normal nilpotent subgroup of class c and a is an n-Engel element. Then G is nilpotent with class at most cn.*

Lemma 3.2. *For any positive integers c, n there exists an integer $f = f(c, n)$ with the following property. Let $G = H\langle a\rangle$, where H is a normal nilpotent subgroup of class c. Then $\gamma_f(G) \le E_n(a)$.*

Here and throughout the article $\gamma_f(G)$ denotes the fth term of the lower central series of G.

The following lemma concerns profinite groups and Engel elements.

Lemma 3.3. *Let $G = M\langle a\rangle$ be a profinite group with an abelian normal subgroup M and an Engel element a. Then G is nilpotent.*

Proof. For any nonnegative integer i set

$$B_i = \{x \in M \mid [x,_i a] = 1\}.$$

Each set B_i is closed, and $\bigcup_{i\ge 0} B_i = M$. By Baire's Category Theorem [**4**, p. 200] at least one of these sets has non-empty interior. Therefore there exist an integer n, an element b in M and an open normal subgroup N contained in M such that $[y,_n a] = 1$ for any $y \in bN$. From this we deduce that $[x,_n a] = 1$ for any x in N. Since N is open in M, there exists a positive integer k such that $[z,_k a] \in N$ for any $z \in M$. Thus $[M,_{n+k} a] = 1$ and the result follows. $\square$

Note that, for an element g of a group G, once a sink $\mathcal{E}(g)$ is chosen, the subgroup $\langle\mathcal{E}(g)\rangle$ generated by $\mathcal{E}(g)$ is also a sink for g. In the remaining part of this article it will be convenient to use the term "sink $\mathcal{E}(g)$ of g" meaning a subgroup containing all sufficiently long commutators $[x,_i g]$ with $x \in G$.

Lemma 3.4. *Let G be a metabelian profinite group and let a be an element of G. Then, for any choice of a sink $\mathcal{E}(a)$, there exists an integer n such that $E_n(a) \leq \mathcal{E}(a)$.*

Proof. If $\mathcal{E}(a)$ is finite, then a is Engel in G. Set $K = G'\langle a\rangle$. By Lemma 3.3 the subgroup K is nilpotent and $[G',_{n-1} a] = 1$, for some integer n. Therefore $[G,_n a] = 1$ and so $E_n(a) \leq \mathcal{E}(a)$.

Assume that $\mathcal{E}(a)$ is infinite. Let E_1 be the subgroup generated by all commutators $[x, a, \dots, a] \in \mathcal{E}(a)$ such that $x \in G'$. Note that $E_1 \leq \mathcal{E}(a)$ and E_1 is a normal subgroup of G. Moreover a is Engel in G/E_1. In view of Lemma 3.3 the subgroup $G'\langle a\rangle$ is nilpotent modulo E_1 and the result follows. □

Lemma 3.5. *Let G be a metabelian profinite group and $a \in G$. For each $n \geq 1$ the subgroup $E_n(a)$ is normal in G.*

Proof. For any $i \geq 1$, any $g \in G'$ and $y \in G$ we have

$$[g,_i a]^y = [g^y,_i a] \text{ and } [g^{-1},_i a] = [g,_i a]^{-1}.$$

Moreover, for any $x, y \in G$, the equality $[x, a]^y = [xy, a][y, a]^{-1}$ holds.

We only need to prove the lemma with $n \geq 2$ since for $n = 1$ the result is well known even without the assumption that G is metabelian. For arbitrary elements $x, y \in G$, by using the standard commutator laws, write

$$[x,_n a]^y = [[x, a]^y,_{n-1} a] = [[xy, a][y, a]^{-1},_{n-1} a] = [xy,_n a][y,_n a]^{-1}.$$

The formula above shows that $[x,_n a]^y \in E_n(a)$ and the lemma follows. □

We write C_n to denote the cyclic group of order n and $\mathbb{Z}_p$ the additive group of p-adic integers. Recall that the group of automorphisms of $\mathbb{Z}_p$ is isomorphic to $\mathbb{Z}_p \oplus C_{p-1}$ if $p \geq 3$ and $\mathbb{Z}_2 \oplus C_2$ if $p = 2$ (see for example [**7**, Theorem 4.4.7]). Note that all nontrivial subgroups of $\mathbb{Z}_p$ have finite index in $\mathbb{Z}_p$ (see for example [**8**, Proposition and Corollary 1 at p. 23]).

Lemma 3.6. *Let G be a pro-p group and K a normal infinite procyclic subgroup of G. If $a \notin C_G(K)$, then for any $i \geq 1$, the subgroup $[K,_i a]$ has finite index in K. In particular, if G is locally nilpotent, then K is central in G.*

Proof. Let α be the automorphism of K induced by the conjugation by the element a. Write R for the ring of the p-adic integers and regard K as the additive group of R. There exists $b \in R$ such that $x^a = x \cdot b$, for each $x \in K$. Note that the subgroup $[K,_n a]$ consists of elements of the form $x \cdot (b-1)^n$, where x ranges over K. Moreover the set $\{x \cdot (b-1)^n \mid x \in K\}$ is infinite for each $n \geq 1$, since R has no zero divisors. The lemma follows. $\square$

We now can prove Theorem 1.3 in the particular case where G is metabelian. The general case will require considerably more efforts.

Proposition 3.7. *Let G be a metabelian pro-p group such that $\mathcal{E}(g)$ can be chosen procyclic for each g in G. Then G has a normal procyclic subgroup K such that G/K is locally nilpotent.*

Proof. If G is Engel, then it is locally nilpotent and there is nothing to prove. Assume that G is not Engel and let X be the set of all non-Engel elements in G. In view of Lemmas 3.4 and 3.5 we can assume that all $\mathcal{E}(g)$ are chosen procyclic and normal in G. Indeed, by Lemmas 3.4 and 3.5, for each g in G there exists an integer n such that $E_n(g)$ is a normal subgroup in $\mathcal{E}(g)$, so we can take such $E_n(g)$ as the sink $\mathcal{E}(g)$ of g. In view of Lemma 3.6 each subgroup $[\mathcal{E}(x),_i x]$ has finite index in $\mathcal{E}(x)$ whenever $x \in X$.

Given $a, b \in X$, suppose first that $\mathcal{E}(a) \cap \mathcal{E}(b) = 1$. On the one hand, a acts on $\mathcal{E}(a)$ in such a way that, for any $i \geq 1$, the subgroup $[\mathcal{E}(a),_i a]$ has finite index in $\mathcal{E}(a)$. On the other hand, a centralizes $\mathcal{E}(b)$, since the intersection of the two sinks is trivial. A similar remarks applies to b. Note that ab acts on $\mathcal{E}(a) \oplus \mathcal{E}(b)$ in the following way: it acts as the element a on $\mathcal{E}(a)$ and as b on $\mathcal{E}(b)$. This implies that, for any n, the subgroup $E_n(ab)$ contains a subgroup, which is the direct sum of a finite index subgroup in $\mathcal{E}(a)$ and a finite index subgroup in $\mathcal{E}(b)$, isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$. Thus $\mathcal{E}(ab)$ is not procyclic, a contradiction.

Hence, $\mathcal{E}(a) \cap \mathcal{E}(b) \neq 1$, for any $a, b \in X$. Let $K = \mathcal{E}(a)$ for some $a \in X$. We see that, for any $g \in G$, the image in G/K of the sink $\mathcal{E}(g)$ is finite. Indeed, if g is Engel in G, then the claim is obvious. Otherwise $g \in X$ and the image of $\mathcal{E}(g)$ in G/K is isomorphic to $\mathcal{E}(g)/(\mathcal{E}(a) \cap \mathcal{E}(g))$ which is finite. It follows that G/K is Engel, hence locally nilpotent by the Wilson-Zelmanov theorem. The proof is complete. □

Next, we consider another particular case of Theorem 1.3.

Lemma 3.8. *Let G be a soluble pro-p group such that $\mathcal{E}(g)$ can be chosen procyclic for each g in G. Assume that G has a normal nilpotent subgroup M and $a \in G$ such that $G = M\langle a \rangle$. Then there exists n such that $E_n(a)$ is procyclic and normal in G. Moreover there exists i such that $\gamma_i(G)$ has finite index in $E_n(a)$.*

Proof. We argue by induction on the nilpotency class of M. If M is abelian, then in view of Lemma 3.5 there exists n such that $E_n(a)$ is procyclic and normal in G. Since $G/E_n(a)$ is nilpotent, the result holds. Suppose that M is nonabelian and set $Z = Z(M)$. By induction assume that there is n such that $L = ZE_n(a)$ is normal in G and L/Z is procyclic. Since L/Z is procyclic, the subgroup L is abelian. Now looking at the action of $\langle a \rangle$ on L and using the fact that L is abelian, Lemma 3.4 shows that if j is big enough, then the subgroup $E_{n+j}(a)$ is procyclic. By Lemma 3.2 there exists f such that $\gamma_f(G) \leq E_{n+j}(a)$. Thus, $\gamma_f(G)$ is a normal procyclic subgroup and so all subgroups of $\gamma_f(G)$ are normal in G. In particular, $E_f(a)$ is normal and procyclic. Since by Lemma 3.1 the factor-group $G/E_f(a)$ is nilpotent, there exists i such that $\gamma_i(G) \leq E_f(a)$. This completes the proof. □

In the sequel we will use, without mentioning explicitly, the following fact: let H be a subgroup of a profinite group G and let x be an element of G such that $H^x \leq H$. Then $H^x = H$. This is because if

$H^x < H$, then the inequality would also hold in some finite image of G, which yields a contradiction.

The next result is a key observation that will be applied many times throughout the proof of the main result.

Lemma 3.9. *Let G be a profinite group and K a procyclic pro-p subgroup of G such that $K \cap K^x \neq 1$ for each $x \in G$. Then K contains a nontrivial subgroup L (of finite index) which is normal in G.*

Proof. If K is finite, the result is obvious, so we assume that K is infinite. Recall that $K \cap K^x$ has finite index in K for each $x \in G$. For each i set

$$S_i = \{x \in G \mid K \cap K^x \text{ has index at most } p^i \text{ in } K\}.$$

The sets S_i are closed. By Baire Category Theorem at least one of these sets has non-empty interior. Therefore there is an open normal subgroup N, an element $d \in G$, and a fixed p-power p^i such that $K \cap K^x$ has index at most p^i in K for every $x \in dN$. Let $K_0 = K^{p^i}$ be the subgroup of index p^i in K. We see that N normalizes K_0. Since N is open, it follows that K_0 has only finitely many conjugates in G. Let L be their intersection. Obviously, L is normal in G. Since $K_0 \cap K_0^x$ has finite index in K_0 for each $x \in G$, the subgroup L is nontrivial. □

Now we are ready to deal with the proof of Theorem 1.3. We want to establish that if G is a soluble pro-p group such that $\mathcal{E}(g)$ can be chosen procyclic for each g in G, then G has a normal procyclic subgroup K such that G/K is locally nilpotent.

Proof of Theorem 1.3. The argument will be by induction on the derived length of G. Set $H = G'$. By induction, H has a normal procyclic subgroup K such that H/K is locally nilpotent.

Claim 1. H is locally nilpotent.

If K is finite, the claim holds. So we assume that K is infinite. It is sufficient to show that H has a procyclic subgroup K_0, which is normal in G, such that H/K_0 is locally nilpotent. Indeed, once the existence of such subgroup K_0 is established, observe that $K_0 \leq Z(H)$ because $G/C_G(K_0)$ embeds into $\mathrm{Aut}(\mathbb{Z}_p)$ which is abelian. Hence H is locally nilpotent. Thus, assume that K is not normal in G.

For any $x \in G$ the quotient H/K^x is locally nilpotent. If there exists x such that $K^x \cap K = 1$, then H, being isomorphic to a subgroup $H/K \times H/K^x$, must be locally nilpotent, as desired. Therefore we will assume that $K^x \cap K \neq 1$ for any $x \in G$.

In view of Lemma 3.9 K contains a nontrivial subgroup L which is normal in G. Since H/K is locally nilpotent and L has finite index in K, it follows that H/L is locally nilpotent too. Moreover, since L is normal in G, it follows that L is in the center of H and so H is locally nilpotent. This establishes Claim 1.

Claim 2. Assume that G has a normal nilpotent subgroup M such that G/M is nilpotent and finitely generated. Then G has a normal procyclic subgroup M_0 such that G/M_0 is nilpotent.

Indeed, choose $a_1, \ldots, a_s$ in G such that $G = \langle M, a_1, \ldots, a_s \rangle$. We argue by induction on the nilpotency class of G/M and also use induction on s.

Assume first that G/M is abelian. The case $s = 1$ follows from Lemma 3.8 so suppose that $s \geq 2$. Let $V_j = M\langle a_j \rangle$, for $1 \leq j \leq s$. Observe that for each j the subgroup V_j is normal in G and, in view of Lemma 3.8, there exists $i(j)$ such that $\gamma_{i(j)}(V_j)$ is procyclic. If for any j the subgroup $\gamma_{i(j)}(V_j)$ is finite (or trivial), then each V_j is nilpotent and so G is nilpotent too. Thus we can assume that some $\gamma_{i(j)}(V_j)$ are procyclic infinite. Moreover, if for some j and k the subgroups $\gamma_{i(j)}(V_j)$ and $\gamma_{i(k)}(V_k)$ are infinite and satisfy $\gamma_{i(j)}(V_j) \cap \gamma_{i(k)}(V_k) = 1$, then we get a contradiction. Indeed, set $N = \gamma_{i(j)}(V_j) \oplus \gamma_{i(k)}(V_k)$ and consider the action of $a_j a_k$ on N. Arguing as in the proof of Proposition 3.7, we see that $\mathcal{E}(a_j a_k)$ is not procyclic, since for any n the subgroup $E_n(a_j a_k)$ contains a subgroup isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$. We therefore assume that all infinite subgroups $\gamma_{i(j)}(V_j)$ intersect pairwise nontrivially. In particular their intersection V is a nontrivial normal procyclic subgroup such that G/V is nilpotent. This concludes the argument in the case where G/M is abelian.

Next, suppose that G/M has nilpotency class at least two, so in particular s is bigger than one. Let $W_j = \langle a_j \rangle HM$, for $1 \le j \le s$. Note that any subgroup W_j modulo M is a finitely generated subgroup, since it is a subgroup of a finitely generated nilpotent group. Furthermore W_j modulo M has nilpotency class smaller than the nilpotency class of G/M, since it is generated by the image of H and a_j. Thus, by induction, any W_j has a normal procyclic subgroup B_j such that W_j/B_j is nilpotent. So, there exists $l(j)$ such that $\gamma_{l(j)}(W_j) \le B_j$. As in the previous paragraph, if all B_j are finite (or trivial), then G is nilpotent. If the infinite B_j intersect nontrivially, then the claim follows since their intersection B is a nontrivial normal procyclic subgroup such that G/B is nilpotent. Suppose that for some i, j the subgroups B_i and B_j are infinite and $B_i \cap B_j = 1$. Note that Claim 1 implies that both B_i and B_j are centralized by H. Set $N = B_i \oplus B_j$ and look at the action of $a_i a_j$ on N. We see that for any n the subgroup $E_n(a_i a_j)$ contains a subgroup isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$. Thus $\mathcal{E}(a_i a_j)$ is not procyclic, a contradiction. This concludes the proof of Claim 2.

Let R be the last nontrivial term of the derived series of G. By induction on the derived length of G assume that for G/R the theorem holds. Thus G has a normal subgroup S, containing R, such that S/R is procyclic and G/S is locally nilpotent. Obviously, we can choose S in such way that $S \le H$. Let $a \in S$ such that $S = R\langle a \rangle$. In view of Claim 1, a is an Engel element. Thus applying Lemma 3.3 we deduce that S is nilpotent.

Claim 3. Let $a_1, \dots, a_s \in G$ and set $J = R\langle a_1, \dots, a_s \rangle$. Then J has a normal procyclic subgroup J_0 such that J/J_0 is nilpotent.

If J/R is nilpotent, then the claim follows from Claim 2. Assume that J/R is not nilpotent. Set $J_1 = \langle J, a \rangle$, where a is as above. Note that $S \leq J_1$ and J_1/S is nilpotent since G/S is locally nilpotent. Hence, again by Claim 2 there exists a normal procyclic subgroup N_0 in J_1 such that J_1/N_0 is nilpotent. In particular JN_0/N_0 is nilpotent too, so we can take $J_0 = J \cap N_0$. This concludes the proof of Claim 3.

We now embark on the final part of the proof of the theorem. Assume that the group G is not locally nilpotent. Choose elements $a_1, \ldots, a_s \in G$ such that $T = \langle a_1, \ldots, a_s \rangle$ is not nilpotent. Recall that S is a nilpotent normal subgroup of G such that G/S is locally nilpotent. By Claim 2 the group ST has a normal procyclic subgroup K_0 such that ST/K_0 is nilpotent. Without loss of generality we assume that there is a positive integer i_0 such that $K_0 = \gamma_{i_0}(ST)$. Note that K_0 here must be infinite since T is not nilpotent. Moreover we can replace K_0 by $S \cap K_0$ and simply assume that $K_0 \leq S$. Indeed, since ST/K_0 and ST/S are both nilpotent, we have $\gamma_i(ST) \leq S \cap K_0$, for some positive integer i.

Given any finite subset Y of G, we write T_Y for the subgroup $\langle Y, T \rangle$. By Claim 2 the group ST_Y has a normal procyclic subgroup K_Y such that ST_Y/K_Y is nilpotent. Again there is a positive integer i_Y such that $K_Y = \gamma_{i_Y}(ST_Y)$. Note that all subgroups K_Y are infinite and have infinite intersection with K_0. Indeed, any subgroup ST_Y contains ST, the subgroup ST is nilpotent modulo the intersection of K_Y with K_0, so if this intersection were trivial, then ST_Y would be nilpotent, a contradiction. As before, since G/S is locally nilpotent, we choose all K_Y inside S.

Now choose an arbitrary element $x \in G$ and set

$$Y(x) = \{{a_1}^x, \ldots, {a_s}^x, a_1, \ldots, a_s\}.$$

We see that $K_{Y(x)}$ has infinite intersection with each of the subgroups K_0 and K_0^x. Hence $K_0 \cap K_0^x$ is nontrivial and this holds for any choice of $x \in G$. Thus, by Lemma 3.9, K_0 contains a nontrivial subgroup L_0 which is normal in G.

Note that for any choice of a finite subset Y of G, the subgroup L_0 intersects K_Y by a finite index subgroup, since K_0 intersects K_Y

nontrivially and L_0 has finite index in K_0. Therefore every subgroup T_Y is nilpotent modulo L_0, since K_Y becomes finite modulo L_0. Hence G is locally nilpotent modulo L_0 and this concludes the proof. □

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