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# Profinite groups with restricted centralizers of $\pi$ -elements

Cristina Acciarri and Pavel Shumyatsky

ABSTRACT. A group  $G$  is said to have restricted centralizers if for each  $g$  in  $G$  the centralizer  $C_G(g)$  either is finite or has finite index in  $G$ . Shalev showed that a profinite group with restricted centralizers is virtually abelian. Given a set of primes  $\pi$ , we take interest in profinite groups with restricted centralizers of  $\pi$ -elements. It is shown that such a profinite group has an open subgroup of the form  $P \times Q$ , where  $P$  is an abelian pro- $\pi$  subgroup and  $Q$  is a pro- $\pi'$  subgroup. This significantly strengthens a result from our earlier paper.

## 1. Introduction

A group  $G$  is said to have restricted centralizers if for each  $g$  in  $G$  the centralizer  $C_G(g)$  either is finite or has finite index in  $G$ . This notion was introduced by Shalev in [13] where he showed that a profinite group with restricted centralizers is virtually abelian. We say that a profinite group has a property virtually if it has an open subgroup with that property. The article [3] handles profinite groups with restricted centralizers of  $w$ -values for a multilinear commutator word  $w$ . The theorem proved in [3] says that if  $w$  is a multilinear commutator word and  $G$  is a profinite group in which the centralizer of any  $w$ -value is either finite or open, then the verbal subgroup  $w(G)$  is virtually abelian. In [1] we study profinite groups in which  $p$ -elements have restricted centralizers, that is, groups in which  $C_G(x)$  is either finite or open for any  $p$ -element  $x$ . The following theorem was proved.

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**THEOREM 1.1.** *Let  $p$  be a prime and  $G$  a profinite group in which the centralizer of each  $p$ -element is either finite or open. Then  $G$  has a normal abelian pro- $p$  subgroup  $N$  such that  $G/N$  is virtually pro- $p'$ .*

The present paper grew out of our desire to determine whether this result can be extended to profinite groups in which the centralizer of each  $\pi$ -element, where  $\pi$  is a fixed set of primes, is either finite or open. As usual, we say that an element  $x$  of a profinite group  $G$  is a  $\pi$ -element if the order of the image of  $x$  in every finite continuous homomorphic image of  $G$  is divisible only by primes in  $\pi$  (see [10, Section 2.3] for a formal definition of the order of a profinite group).

It turned out that the techniques used in the proof of Theorem 1.1 were not quite adequate for handling the case of  $\pi$ -elements. The basic difficulty stems from the fact that (pro)finite groups in general do not possess Hall  $\pi$ -subgroups.

In the present paper we develop some new techniques and establish the following theorem about finite groups.

If  $\pi$  is a set of primes and  $G$  a finite group, write  $O^{\pi'}(G)$  for the unique smallest normal subgroup  $M$  of  $G$  such that  $G/M$  is a  $\pi'$ -group. The conjugacy class containing an element  $g \in G$  is denoted by  $g^G$ .

**THEOREM 1.2.** *Let  $n$  be a positive integer,  $\pi$  be a set of primes, and  $G$  a finite group such that  $|g^G| \leq n$  for each  $\pi$ -element  $g \in G$ . Let  $H = O^{\pi'}(G)$ . Then  $G$  has a normal subgroup  $N$  such that*

- (1) *The index  $[G : N]$  is  $n$ -bounded;*
- (2)  *$[H, N] = [H, H]$ ;*
- (3) *The order of  $[H, N]$  is  $n$ -bounded.*

Throughout the article we use the expression “ $(a, b, \dots)$ -bounded” to mean that a quantity is finite and bounded by a certain number depending only on the parameters  $a, b, \dots$ .

The proof of Theorem 1.2 uses some new results related to Neumann’s BFC-theorem [8]. In particular, an important role in the proof is played by a recent probabilistic result from [2]. Theorem 1.2 provides a highly effective tool for handling profinite groups with restricted centralizers of  $\pi$ -elements. Surprisingly, the obtained result is much stronger than Theorem 1.1 even in the case where  $\pi$  consists of a single prime.

**THEOREM 1.3.** *Let  $\pi$  be a set of primes and  $G$  a profinite group in which the centralizer of each  $\pi$ -element is either finite or open. Then  $G$  has an open subgroup of the form  $P \times Q$ , where  $P$  is an abelian pro- $\pi$  subgroup and  $Q$  is a pro- $\pi'$  subgroup.*

Thus, the improvement over Theorem 1.1 is twofold – the result now covers the case of  $\pi$ -elements and provides additional details clarifying the structure of groups in question. Furthermore, it is easy to see that Theorem 1.3 extends Shalev’s result [13] which can be recovered by considering the case where  $\pi = \pi(G)$  is the set of all prime divisors of the order of  $G$ .

We now have several results showing that if the elements of a certain subset  $X$  of a profinite group  $G$  have restricted centralizers, then the structure of  $G$  is very special. This suggests the general line of research whose aim would be to determine which subsets of  $G$  have the above property. At present we are not able to provide any insight on the problem. Perhaps one might start with the following question:

*Let  $n$  be a positive integer. What can be said about a profinite group  $G$  such that if  $x \in G$  then  $C_G(x^n)$  is either finite or open?*

Proofs of Theorems 1.2 and 1.3 will be given in Sections 2 and 3, respectively.

## 2. Proof of Theorem 1.2

The following lemma is taken from [1]. If  $X \subseteq G$  is a subset of a group  $G$ , we write  $\langle X \rangle$  for the subgroup generated by  $X$  and  $\langle X^G \rangle$  for the minimal normal subgroup of  $G$  containing  $X$ .

LEMMA 2.1. *Let  $i, j$  be positive integers and  $G$  a group having a subgroup  $K$  such that  $|x^G| \leq i$  for each  $x \in K$ . Suppose that  $|K| \leq j$ . Then  $\langle K^G \rangle$  has finite  $(i, j)$ -bounded order.*

If  $K$  is a subgroup of a finite group  $G$ , we denote by

$$Pr(K, G) = \frac{|\{(x, y) \in K \times G : [x, y] = 1\}|}{|K||G|}$$

the relative commutativity degree of  $K$  in  $G$ , that is, the probability that a random element of  $G$  commutes with a random element of  $K$ . Note that

$$Pr(K, G) = \frac{\sum_{x \in K} |C_G(x)|}{|K||G|}.$$

It follows that if  $|x^G| \leq n$  for each  $x \in K$ , then  $Pr(K, G) \geq \frac{1}{n}$ .

The next result was obtained in [2, Proposition 1.2]. In the case where  $K = G$  this is a well known theorem, due to P. M. Neumann [9].

PROPOSITION 2.2. *Let  $\epsilon > 0$ , and let  $G$  be a finite group having a subgroup  $K$  such that  $Pr(K, G) \geq \epsilon$ . Then there is a normal subgroup  $T \leq G$  and a subgroup  $B \leq K$  such that the indexes  $[G : T]$  and  $[K : B]$ , and the order of the commutator subgroup  $[T, B]$  are  $\epsilon$ -bounded.*

We will now embark on the proof of Theorem 1.2.

Assume the hypothesis of Theorem 1.2. Let  $X$  be the set of all  $\pi$ -elements of  $G$ . Clearly,  $H = \langle X \rangle$ . Given an element  $g \in H$ , we write  $l(g)$  for the minimal number  $l$  with the property that  $g$  can be written as a product of  $l$  elements of  $X$ . The following result is straightforward from [4, Lemma 2.1].

**LEMMA 2.3.** *Let  $K \leq H$  be a subgroup of index  $m$  in  $H$ , and let  $b \in H$ . Then the coset  $Kb$  contains an element  $g$  such that  $l(g) \leq m-1$ .*

Let  $m$  be the maximum of indices of  $C_H(x)$  in  $H$  where  $x \in X$ . Obviously, we have  $m \leq n$ .

**LEMMA 2.4.** *For any  $x \in X$  the subgroup  $[H, x]$  has  $m$ -bounded order.*

**PROOF.** Take  $x \in X$ . Since the index of  $C_H(x)$  in  $H$  is at most  $m$ , by Lemma 2.3, we can choose elements  $y_1, \dots, y_m$  in  $H$  such that  $l(y_i) \leq m-1$  and the subgroup  $[H, x]$  is generated by the commutators  $[y_i, x]$ , for  $i = 1, \dots, m$ . For any such  $i$  write  $y_i = y_{i1} \dots y_{i(m-1)}$ , with  $y_{ij} \in X$ . Using standard commutator identities we can rewrite  $[y_i, x]$  as a product of conjugates in  $H$  of the commutators  $[y_{ij}, x]$ . Let  $\{h_1, \dots, h_s\}$  be the conjugates in  $H$  of all elements from the set  $\{x, y_{ij} \mid 1 \leq i, j \leq m\}$ . Note that the number  $s$  here is  $m$ -bounded. This follows from the fact that  $C_H(x)$  has index at most  $m$  in  $H$  for each  $x \in X$ . Put  $T = \langle h_1, \dots, h_s \rangle$ . Since  $[H, x]$  is contained in the commutator subgroup  $T'$ , it is sufficient to show that  $T'$  has  $m$ -bounded order. Observe that the centre  $Z(T)$  has index at most  $m^s$  in  $T$ , since the index of  $C_H(h_i)$  is at most  $m$  in  $H$  for any  $i = 1, \dots, s$ . Thus, by Schur's theorem [11, 10.1.4], we conclude that the order of  $T'$  is  $m$ -bounded, as desired.  $\square$

Select  $a \in X$  such that  $|a^H| = m$ . Choose  $b_1, \dots, b_m$  in  $H$  such that  $l(b_i) \leq m-1$  and  $a^H = \{a^{b_i}; i = 1, \dots, m\}$ . The existence of the elements  $b_i$  is guaranteed by Lemma 2.3. Set  $U = C_G(\langle b_1, \dots, b_m \rangle)$ . Note that the index of  $U$  in  $G$  is  $n$ -bounded. Indeed, since  $l(b_i) \leq m-1$  we can write  $b_i = b_{i1} \dots b_{i(m-1)}$ , where  $b_{ij} \in X$  and  $i = 1, \dots, m$ . By the hypothesis the index of  $C_G(b_{ij})$  in  $G$  is at most  $n$  for any such element  $b_{ij}$ . Thus,  $[G : U] \leq n^{(m-1)m}$ .

The next result is somewhat analogous to [14, Lemma 4.5].

**LEMMA 2.5.** *If  $u \in U$  and  $ua \in X$ , then  $[H, u] \leq [H, a]$ .*

**PROOF.** Assume that  $u \in U$  and  $ua \in X$ . For each  $i = 1, \dots, m$  we have  $(ua)^{b_i} = ua^{b_i}$ , since  $u$  belongs to  $U$ . We know that  $ua \in X$  so taking into account the hypothesis on the cardinality of the conjugacy

class of  $ua$  in  $H$ , we deduce that  $(ua)^H$  consists exactly of the elements  $ua^{b_i}$ , for  $i = 1, \dots, m$ . Thus, given an arbitrary element  $h \in H$ , there exists  $b \in \{b_1, \dots, b_m\}$  such that  $(ua)^h = ua^b$  and so  $u^h a^h = ua^b$ . It follows that  $[u, h] = a^b a^{-h} \in [H, a]$ , and the result holds.  $\square$

LEMMA 2.6. *The order of the commutator subgroup of  $H$  is  $n$ -bounded.*

PROOF. Let  $U_0$  be the maximal normal subgroup of  $G$  contained in  $U$ . Recall that, by the remark made before Lemma 2.5,  $U$  has  $n$ -bounded index in  $G$ . It follows that the index  $[G : U_0]$  is  $n$ -bounded as well.

By the hypothesis  $a$  has at most  $n$  conjugates in  $G$ , say  $\{a^{g_1}, \dots, a^{g_n}\}$ . Let  $T$  be the normal closure in  $G$  of the subgroup  $[H, a]$ . Note that the subgroups  $[H, a^{g_i}]$  are normal in  $H$ , therefore  $T = [H, a^{g_1}] \dots [H, a^{g_n}]$ . By Lemma 2.4 each of the subgroups  $[H, a^{g_i}]$  has  $n$ -bounded order. We conclude that the order of  $T$  is  $n$ -bounded.

Let  $Y = Xa^{-1} \cap U$ . Note that for any  $y \in Y$  the product  $ya$  belongs to  $X$ . Therefore, by Lemma 2.5, for any  $y \in Y$ , the subgroup  $[H, y]$  is contained in  $[H, a]$ . Thus,

$$(1) \quad [H, Y] \leq T.$$

Observe that for any  $u \in U_0$  the commutator  $[u, a^{-1}] = a^u a^{-1}$  lies in  $Y$  and so

$$(2) \quad [H, [U_0, a^{-1}]] \leq [H, Y].$$

Since  $[U_0, a^{-1}] = [U_0, a]$ , we deduce from (1) and (2) that

$$(3) \quad [H, [U_0, a]] \leq T.$$

Since  $T$  has  $n$ -bounded order, it is sufficient to show that the derived group of the quotient  $H/T$  has finite  $n$ -bounded order. We pass now to the quotient  $G/T$  and for the sake of simplicity the images of  $G$ ,  $H$ ,  $U$ ,  $U_0$ ,  $X$  and  $Y$  will be denoted by the same symbols. Note that by (1) the set  $Y$  becomes central in  $H$  modulo  $T$ . Containment (3) shows that  $[U_0, a] \leq Z(H)$ . This implies that if  $b \in U_0$  is a  $\pi$ -element, then  $[b, a] \in Z(H)$  and the subgroup  $\langle a, b \rangle$  is nilpotent. Thus the product  $ba$  is a  $\pi$ -element too and so  $b \in Y$ . Hence, all  $\pi$ -elements of  $U_0$  are contained in  $Y$  and, in view of (1), we deduce that they are contained in  $Z(H)$ .

Next we consider the quotient  $G/Z(H)$ . Since the image of  $U_0$  in  $G/Z(H)$  is a  $\pi'$ -group and has  $n$ -bounded index in  $G$ , we deduce that the order of any  $\pi$ -subgroup in  $G/Z(H)$  is  $n$ -bounded. In particular, there is an  $n$ -bounded constant  $C$  such that for every  $p \in \pi$  the order of the Sylow  $p$ -subgroup of  $G/Z(H)$  is at most  $C$ . Because of Lemma

2.1 for any  $p \in \pi$  each Sylow  $p$ -subgroup of  $G/Z(H)$  is contained in a normal subgroup of  $n$ -bounded order. We deduce that all such Sylow subgroups of  $G/Z(H)$  are contained in a normal subgroup of  $n$ -bounded order. Since  $H$  is generated by  $\pi$ -elements, it follows that the order of  $H/Z(H)$  is  $n$ -bounded. Thus, in view of Schur's theorem [11, 10.1.4], we conclude that  $|H'|$  is  $n$ -bounded, as desired.  $\square$

We will now complete the proof of Theorem 1.2.

PROOF. Assume first that  $H$  is abelian. In this case the set  $X$  of  $\pi$ -elements is a subgroup, that is,  $X = H$ . By the hypothesis we have  $|x^G| \leq n$  for any element  $x \in H$  and so the relative commutativity degree  $Pr(H, G)$  of  $H$  in  $G$  is at least  $\frac{1}{n}$ . Thus, by virtue of Proposition 2.2, there is a normal subgroup  $T \leq G$  and a subgroup  $B \leq H$  such that the indexes  $[G : T]$  and  $[H : B]$ , and the order of the commutator subgroup  $[T, B]$  are  $n$ -bounded.

Since  $H$  is a normal  $\pi$ -subgroup and  $[G : H]$  is a  $\pi'$ -number, by the Schur–Zassenhaus Theorem [5, Theorem 6.2.1] the subgroup  $H$  admits a complement  $L$  in  $G$  such that  $G = HL$  and  $L$  is a  $\pi'$ -subgroup. Set  $T_0 = T \cap L$ . Observe that the index  $[L : T_0]$  is  $n$ -bounded since it is at most the index of  $T$  in  $G$ . Thus we deduce that the index of  $HT_0$  is  $n$ -bounded in  $G$ , as well.

We claim that the order of  $[H, T_0]$  is  $n$ -bounded. Indeed, the  $\pi'$ -subgroup  $T_0$  acts coprimely on the the abelian  $\pi$ -subgroup  $B_1 = B[B, T_0]$ , and so we have  $B_1 = C_{B_1}(T_0) \times [B_1, T_0]$  ([7, Corollary 1.6.5]). Note that  $[B_1, T_0] = [B, T_0]$ . Since the order of  $[B, T_0]$  is  $n$ -bounded (being at most the order of  $[T, B]$ ), we deduce that the index  $[B_1 : C_{B_1}(T_0)]$  is  $n$ -bounded. In combination with the fact that  $[H : B]$  is  $n$ -bounded, we obtain that the index  $[H : C_{B_1}(T_0)]$  is  $n$ -bounded and so in particular  $[H : C_H(T_0)]$  is  $n$ -bounded. Since  $T_0$  acts coprimely on the abelian normal  $\pi$ -subgroup  $H$ , we have  $H = C_H(T_0) \times [H, T_0]$ . Thus we obtain that the order of the commutator subgroup  $[H, T_0]$  is  $n$ -bounded, as claimed. Let  $T_1 = C_{T_0}([H, T_0])$  and remark that the index  $[T_0 : T_1]$  of  $T_1$  in  $T_0$  is  $n$ -bounded too. Set  $N = HT_1$ . From the fact that the indexes  $[T_0 : T_1]$  and  $[G : HT_0]$  are both  $n$ -bounded, we deduce that the index of  $N$  in  $G$  is  $n$ -bounded, as well.

Note that  $N$  is normal in  $G$  since the image of  $N$  in  $G/H \cong L$  is isomorphic to  $T_1$  which is normal in  $L$ . Furthermore, we have  $[H, T_1, T_1] = 1$ , since  $T_1 = C_{T_0}([H, T_0])$ . Hence by the standard properties of coprime actions we have  $[H, T_1] = 1$  ([7, Corollary 1.6.4]). Therefore  $[H, N] = 1$ . This proves the theorem in the particular case where  $H$  is abelian.

In the general case, in view of Lemma 2.6, the commutator subgroup  $[H, H]$  is of  $n$ -bounded order. We pass to the quotient  $\overline{G} = G/[H, H]$ . The above argument shows that  $\overline{G}$  has a normal subgroup  $\overline{N}$  of  $n$ -bounded index such that  $\overline{H} \leq Z(\overline{N})$ . Here  $Z(\overline{N})$  stands for the centre of  $\overline{N}$ . Let  $N$  be the inverse image of  $\overline{N}$ . We have  $[H, N] = [H, H]$  and so  $N$  has the required properties. The proof is now complete.  $\square$

### 3. Proof of Theorem 1.3

We will require the following result taken from [1, Lemma 4.1].

LEMMA 3.1. *Let  $G$  be a locally nilpotent group containing an element with finite centralizer. Suppose that  $G$  is residually finite. Then  $G$  is finite.*

Profinite groups have Sylow  $p$ -subgroups and satisfy analogues of the Sylow theorems. Prosoluble groups satisfy analogues of the theorems on Hall  $\pi$ -subgroups. We refer the reader to the corresponding chapters in [10, Ch. 2] and [15, Ch. 2].

Recall that an automorphism  $\phi$  of a group  $G$  is called fixed-point-free if  $C_G(\phi) = 1$ , that is, the fixed-point subgroup is trivial. It is a well-known corollary of the classification of finite simple groups that if  $G$  is a finite group admitting a fixed-point-free automorphism, then  $G$  is soluble (see for example [12] for a short proof). A continuous automorphism  $\phi$  of a profinite group  $G$  is coprime if for any open  $\phi$ -invariant normal subgroup  $N$  of  $G$  the order of the automorphism of  $G/N$  induced by  $\phi$  is coprime to the order of  $G/N$ . It follows that if a profinite group  $G$  admits a coprime fixed-point-free automorphism, then  $G$  is prosoluble. This will be used in the proof of Theorem 1.3.

PROOF OF THEOREM 1.3. Recall that  $\pi$  is a set of primes and  $G$  is a profinite group in which the centralizer of every  $\pi$ -element is either finite or open. We wish to show that  $G$  has an open subgroup of the form  $P \times Q$ , where  $P$  is an abelian pro- $\pi$  subgroup and  $Q$  is a pro- $\pi'$  subgroup.

Let  $X$  be the set of  $\pi$ -elements in  $G$ . Consider first the case where the conjugacy class  $x^G$  is finite for any  $x \in X$ . For each integer  $i \geq 1$  set

$$S_i = \{x \in X; |x^G| \leq i\}.$$

The sets  $S_i$  are closed. Thus, we have countably many sets which cover the closed set  $X$ . By the Baire Category Theorem [6, Theorem 34] at least one of these sets has non-empty interior. It follows that there is a positive integer  $k$ , an open normal subgroup  $M$ , and an element  $a \in X$  such that all elements in  $X \cap aM$  are contained in  $S_k$ .



Note that  $\langle a^G \rangle$  has finite commutator subgroup, which we will denote by  $T$ . Indeed, the subgroup  $\langle a^G \rangle$  is generated by finitely many elements whose centralizer is open. This implies that the centre of  $\langle a^G \rangle$  has finite index in  $\langle a^G \rangle$ , and by Schur's theorem [11, 10.1.4], we conclude that  $T$  is finite, as claimed.

Let  $x \in X \cap M$ . Note that the product  $ax$  is not necessarily in  $X$ . On the other hand,  $ax$  is a  $\pi$ -element modulo  $T$ . This is because  $\langle a^G \rangle$  becomes an abelian normal  $\pi$ -subgroup modulo  $T$  and the image of  $ax$  in the quotient  $G/\langle a^G \rangle$  is a  $\pi$ -element. In other words, there are  $y \in X \cap aM$  and  $t \in T$  such that  $ax = ty$ . Observe that  $t$  has an open centralizer in  $G$  since  $t \in T$ . In fact  $[G : C_G(t)] \leq |T|$ . From the equality  $ax = ty$  deduce that  $|x^G| \leq k^2|T|$ . This happens for any  $x \in X \cap M$ . Using a routine inverse limit argument in combination with Theorem 1.2 we obtain that  $M$  has an open normal subgroup  $N$  such that the index  $[M : N]$  and the order of  $[H, N]$  are finite. Here  $H$  stands for the subgroup generated by all  $\pi$ -elements of  $M$ . Choose an open normal subgroup  $U$  in  $G$  such that  $U \cap [H, N] = 1$ . Then  $U \cap M$  is an open normal subgroup of the form  $P \times Q$ , where  $P$  is an abelian pro- $\pi$  subgroup and  $Q$  is a pro- $\pi'$  subgroup. This proves the theorem in the case where all  $\pi$ -elements of  $G$  have open centralizers.

Assume now that  $G$  has a  $\pi$ -element, say  $b$ , of infinite order. Since the procyclic subgroup  $\langle b \rangle$  is contained in the centralizer  $C_G(b)$ , it follows that  $C_G(b)$  is open in  $G$ . This implies that all elements of  $X \cap C_G(b)$  have open centralizers (because they centralize the procyclic subgroup  $\langle b \rangle$ ). In view of the above  $C_G(b)$  has an open subgroup of the form  $P \times Q$ , where  $P$  is an abelian pro- $\pi$  subgroup and  $Q$  is a pro- $\pi'$  subgroup and we are done.

We will therefore assume that  $G$  is infinite while all  $\pi$ -elements of  $G$  have finite orders and there is at least one  $\pi$ -element, say  $d$ , such that  $C_G(d)$  is finite. The element  $d$  is a product of finitely many  $\pi$ -elements of prime power order. At least one of these elements must have finite centralizer. So without loss of generality we can assume that  $d$  is a  $p$ -element for a prime  $p \in \pi$ .

Let  $P_0$  be a Sylow  $p$ -subgroup of  $G$  containing  $d$ . Since  $P_0$  is torsion, we deduce from Zelmanov's theorem [16] that  $P_0$  is locally nilpotent. The centralizer  $C_G(d)$  is finite and so in view of Lemma 3.1 the subgroup  $P_0$  is finite. Choose an open normal pro- $p'$  subgroup  $L$  such that  $L \cap C_G(d) = 1$ . Note that any finite homomorphic image of  $L$  admits a coprime fixed-point-free automorphism (induced by the coprime action of  $d$  on  $L$ ). Hence  $L$  is prosoluble. Let  $K$  be a Hall  $\pi$ -subgroup of  $L$ . Since any element in  $K$  has restricted centralizer, Shalev's result [13] shows that  $K$  is virtually abelian. We therefore can choose an

open normal subgroup  $J$  in  $L$  such that  $J \cap K$  is abelian. If  $J \cap K$  is finite then  $G$  is virtually pro- $\pi'$  and we are done. If  $J \cap K$  is infinite, then all  $\pi$ -elements of  $J$  have infinite centralizers. This yields that all  $\pi$ -elements of  $J$  have open centralizers in  $J$  and in view of the first part of the proof,  $J$  has an open normal subgroup of the form  $P \times Q$ , where  $P$  is an abelian pro- $\pi$  subgroup and  $Q$  is a pro- $\pi'$  subgroup. This establishes the theorem.  $\square$

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