Regularity and long-time behavior for a thermodynamically consistent model for complex fluids in two space dimensions*

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Abstract

We consider a thermodynamically consistent model for the evolution of thermally conducting two-phase incompressible fluids. Complementing previous results, we prove additional regularity properties of solutions in the case when the evolution takes place in the two-dimensional flat torus with periodic boundary conditions. Thanks to improved regularity, we can also prove uniqueness and characterize the long-time behavior of trajectories showing existence of the global attractor in a suitable phase-space.

Keywords: Cahn-Hilliard, Navier-Stokes, incompressible non-isothermal binary fluid, thermodynamically consistent model, regularity of solutions, long-time behavior.

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1 Introduction

We consider here a mathematical model for two-phase flows of non-isothermal incompressible fluids in a bounded container $\Omega \subset \mathbb{R}^2$. The model consists in a PDE system describing the evolution of the unknown variables $\boldsymbol{u}$ (macroscopic velocity), φ (order parameter), μ (chemical potential), ϑ (absolute temperature) and taking the form

$$\operatorname{div} \boldsymbol{u} = 0, \tag{1.1}$$

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \Delta \boldsymbol{u} - \operatorname{div}(\nabla \varphi \otimes \nabla \varphi), \tag{1.2}$$

$$\varphi_t + \boldsymbol{u} \cdot \nabla \varphi = \Delta \mu, \tag{1.3}$$

$$\mu = -\Delta \varphi + F'(\varphi) - \vartheta, \tag{1.4}$$

$$\vartheta_t + \boldsymbol{u} \cdot \nabla \vartheta + \vartheta\big(\varphi_t + \boldsymbol{u} \cdot \nabla \varphi\big) - \operatorname{div}(\kappa(\vartheta)\nabla \vartheta) = |\nabla \boldsymbol{u}|^2 + |\nabla \mu|^2. \tag{1.5}$$

Relation (1.2), with the incompressibility constraint (1.1), represents a variant of the Navier-Stokes equations; (1.3)-(1.4) correspond to a form of the Cahn-Hilliard system [5] for phase separation, while (1.5) is the internal energy equation describing the evolution of temperature. As usual, the variable p in the Navier-Stokes system (1.2) represents the (unknown) pressure. The function F whose derivative appears in (1.4) is a possibly non-convex potential whose minima represent the least

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energy configurations of the phase variable. Here we will assume that F is smooth and has a power-like growth at infinity. Finally, the function $\kappa(\vartheta)$ in (1.5) denotes the heat conductivity coefficient, assumed to grow at infinity like a sufficiently high power of ϑ (see (K1) below).

The system is highly nonlinear and contains complicated coupling terms; however these features arise naturally and are directly related to the thermodynamical consistency of the model. In particular, the quadratic terms on the right hand side of (1.5), which constitute the main difficulty in the mathematical analysis, describe the heat production coming from dissipation of kinetic and chemical energy, respectively. We may also note that transport effects are admitted for all variables in view of the occurrence of material derivatives in (1.2), (1.3) and (1.5). In order to avoid complications related to interactions with the boundary, we will assume here $\Omega = [0,1] \times [0,1]$ to be the two-dimensional flat torus. Correspondingly, we will take periodic boundary conditions for all unknowns.

System (1.1)-(1.5) has been first introduced in [8] and can be considered as a coupling between the Navier-Stokes equations and the thermodynamically consistent model for phase transitions proposed by M. Frémond in [3] and extensively studied in recent years (see, for instance, [6, 15, 16, 17, 23] and the references therein, we also quote [7] for applications to elastoplasticity with hysteresis and [10] for liquid crystals). Other nonisothermal models for phase-changing fluids can be obtained by linearization around the critical temperature, which simplifies the mathematical analysis but gives rise at least to a partial loss of thermodynamical consistency.

A mathematical study of (1.1)-(1.5) has been first attempted in [8, 9], which refer to the three- and two-dimensional setting, respectively. Referring to these articles for a physical justification of the equations and a more comprehensive survey of the related mathematical literature, here we just recall that, in [8], existence of solutions was shown (under slightly different assumptions on coefficients with respect to those considered here) for a *very weak* formulation, where the heat equation (1.5) was replaced by a differential equality accounting for the balance of total energy, complemented with an "entropy production" differential inequality. This approach follows an idea originally devised in [4] for heat conducting fluids and later used in other contexts (see, e.g., [10, 11] and [19] for applications to nematic liquid crystals and damaging models, respectively). Indeed, in view of the upper regularity threshold for the 3D Navier-Stokes system, the quadratic terms on the right hand side of (1.5) (and particularly the one depending on $\nabla \boldsymbol{u}$) may be only estimated in L^1, which gives rise to defect measures when taking the limit in an approximation scheme. Actually, the method of [4] permits to overcome this difficulty, but at the price of dealing with a weaker concept of solution.

Later, the two-dimensional case was analyzed in [9] where stronger results were obtained. Actually, in 2D one can deduce additional estimates and, in particular, the quadratic terms in (1.5) can be controlled in L^2 (though the procedure to get such a bound is not trivial). Using that information the authors of [9] could obtain existence of "strong solutions" satisfying the equations (1.1)-(1.5) with the initial and boundary conditions in the usual (distributional) sense (hence avoiding the occurrence of differential inequalities). On the other hand, the results of [9] leave several open questions which we would like to answer here. In particular, taking essentially the same assumptions on coefficients and data considered there (here we only need to specify in a more precise way the hypotheses on the initial temperature), we will extend and complement the results of [9] in the following three directions:

- We will improve the results on regularity, defining a class of slightly smoother solutions (called *"stable solutions"* in the sequel);
- We will prove that for stable solutions uniqueness holds (hence we have well-posedness in this class);
- We will characterize the long-time behavior of stable solutions showing that they constitute a strongly continuous dynamical process which admits the global attractor.

We refer the reader to the next Section 2 for a rigorous motivation and a detailed explanation of the aspects under which our result improve and complement those given in [9]. In particular, the terminology "stable" solutions will be clarified there. Here, we conclude the introduction with the plan of the rest of the paper: Section 3 is devoted to presenting the precise statements of our results, whose proofs are split into several parts. Namely, in Section 4 further regularity on finite time intervals is discussed; in Section 5 uniqueness is proved for "stable solutions"; finally, in Section 6 the long-time behavior of trajectories is characterized and the existence of non-empty ω-limit sets is shown. Moreover, the global attractor in the sense of infinite-dimensional dynamical systems is proved to exist.

2 Assumptions and motivation

We start by introducing some notation. Recalling that $\Omega = [0,1]\times[0,1]$, we denote as $H := L^2_{\rm per}(\Omega)$ the space of functions in $L^2(\mathbb{R}^2)$ which are Ω-periodic (i.e., 1-periodic both in x_1 and in x_2). Analogously, we set $V := H^1_{\rm per}(\Omega)$. The spaces H and V are endowed with the norms of $L^2(\Omega)$ and $H^1(\Omega)$, respectively. For brevity, the norm in H will be simply indicated by $\|\cdot\|$. We will note by $\|\cdot\|_X$ the norm in the generic Banach space X. The symbol $\langle\cdot,\cdot\rangle$ will indicate the duality between V' and V and $(\cdot,\cdot)$ will stand for the scalar product of H. We also write $L^p(\Omega)$ in place of $L^p_{\rm per}(\Omega)$, and the same for other spaces; indeed, no confusion should arise since periodic boundary conditions are assumed to hold for all unknowns. We denote by $H^m_{\rm per}(\Omega)$ (or for brevity simply $H^m(\Omega)$) the space of functions which are $H^m_{\rm loc}(\mathbb{R}^2)$ and Ω-periodic. For $m \in \mathbb{N}$ they are introduced by means of the corresponding Fourier series and then they can be extended for general $m \in \mathbb{R}$, $m \geq 0$. In particular, for $m = 0$ we have $H^0_{\rm per}(\Omega) = L^2_{\rm per}(\Omega)$.

For any function $v \in H$, we will set

$$v_\Omega := \frac{1}{|\Omega|}\int_\Omega v = \int_\Omega v, \tag{2.1}$$

to indicate the spatial mean of v. If the integral is replaced with a duality, the above can be extended to $v \in V'$. The symbols V_0, H_0 and V_0' denote the subspaces of V, H and, respectively, V' containing the function(al)s having zero spatial mean. Then, the distributional operator $(-\Delta)$ is invertible if seen as a mapping from V_0 to V_0' and its inverse will be indicated by $\mathcal{N}$.

Still for brevity, we use the same notation for indicating vector-valued (or tensor-valued) function spaces and related norms. For instance, writing $\boldsymbol{u} \in H$, we will in fact mean $\boldsymbol{u} \in L^2_{\rm per}(\Omega)^2$. Also the incompressibility constraint (1.1) will not be emphasized in the notation for functional spaces, unless on occurrence: in that case we set

$$\mathbb{V} := \{\mathbf{v} \in V_0 : \operatorname{div}\mathbf{v} = 0\}, \qquad \mathbb{H} := \{\mathbf{v} \in H_0 : \operatorname{div}\mathbf{v} = 0\}.$$

Otherwise, for instance the notation $\boldsymbol{u} \in H$ will also implicitly subsume that $\operatorname{div}\boldsymbol{u} = 0$ in the sense of distributions. These simplifications will allow us to shorten a bit some formulas.

Moreover, in the following we will frequently use the following 2D interpolation inequalities:

$$\|v\|_{L^4(\Omega)} \leq c\|v\|_V^{1/2}\|v\|^{1/2}, \tag{2.2}$$

$$\|v\|_{L^\infty(\Omega)} \leq c\|v\|_{H^2(\Omega)}^{1/2}\|v\|^{1/2}, \tag{2.3}$$

$$\|v\|_{L^r(\Omega)} \leq c\|v\|^{\frac{2}{r}}\|v\|_V^{1-\frac{2}{r}}, \qquad r \in [1,\infty), \tag{2.4}$$

$$\|v\|_{H^s(\Omega)} \leq c\|v\|_{H^{s_1}(\Omega)}^{1-\theta}\|v\|_{H^{s_2}(\Omega)}^{\theta}, \qquad \theta = \frac{s-s_1}{s_2-s_1}, \tag{2.5}$$

$$\|v - v_\Omega\| \leq c\|\nabla v\|, \tag{2.6}$$

holding for any sufficiently smooth function v and for suitable embedding constants, all denoted by the same symbol $c > 0$ for brevity.

We will also use the following nonlinear version of the Poincaré inequality (see [12])

$$\|v^{p/2}\|_V^2 \leq c_p\big(\|v\|_{L^1(\Omega)}^p + \|\nabla v^{p/2}\|^2\big), \tag{2.7}$$

holding for all nonnegative $v \in L^1(\Omega)$ such that $\nabla v^{p/2} \in L^2(\Omega)$, and for all $p \in [2,\infty)$. We also recall that

$$\|v\| \leq c\|\nabla v\|^{1/2}\|v\|_{V'}^{1/2} \quad \text{for all } v \in V_0, \tag{2.8}$$

as one can prove simply combining the standard interpolation inequality $\|v\| \le c\|v\|_V^{1/2}\|v\|_{V'}^{1/2}$ with the Poincaré-Wirtinger inequality (2.6).

With the above notation at disposal, we can present our main assumptions on the nonlinear terms. Basically, these assumptions will be retained for all our results. They also essentially coincide with the hypotheses considered in [9] (differences will be observed on occurrence). First of all, we ask the configuration potential F to satisfy:

$$F \in C^3(\mathbb{R};\mathbb{R}), \quad \liminf_{|r|\to\infty} \frac{F(r)}{|r|} > 0, \tag{F1}$$

$$F''(r) \ge -\lambda \quad \text{for some } \lambda \ge 0 \quad \text{and all } r \in \mathbb{R}, \tag{F2}$$

$$|F'''(r)| \le \tilde{c}_F\big(1+|r|^{p_F-1}\big) \quad \text{for some } \tilde{c}_F \ge 0,\ p_F \ge 1, \quad \text{and all } r \in \mathbb{R}. \tag{F3}$$

We remark that (F3) implies

$$|F''(r)| \le c_F\big(1+|r|^{p_F}\big) \quad \text{for some } c_F \ge 0,\ p_F \ge 0, \quad \text{and all } r \in \mathbb{R}. \tag{2.9}$$

Assumption (F1) postulates regularity and coercivity of F, (F2) is λ-convexity, and (F3) prescribes a polynomial growth at infinity. Note that (F1) implies that

$$F(s) \ge -c_0 \qquad \forall s \in \mathbb{R} \tag{2.10}$$

and some constant $c_0 > 0$. Observe also that in [9] it was just assumed that $F \in C^2$ (in place of C^3) in (F1); moreover (2.9) was taken in place of (F3). Here we are asking more regularity because we will look for smoother solutions.

Next, we assume the heat conductivity to be given (exactly as in [9]) by

$$\kappa(r) = 1 + r^q, \quad q \in [2,\infty), \quad r \ge 0. \tag{K1}$$

Correspondingly, we define

$$K(r) := \int_0^r \kappa(s)\,\mathrm{d}s = r + \frac{1}{q+1} r^{q+1}, \quad r \ge 0. \tag{2.11}$$

We then observe that, for some $k_q > 0$,

$$\int_\Omega \kappa(\vartheta)^2 |\nabla\vartheta|^2 = \|\nabla K(\vartheta)\|^2 \ge \|\nabla\vartheta\|^2 + k_q \|\nabla \vartheta^{q+1}\|^2. \tag{2.12}$$

Let us now discuss the conditions on initial data. Here we need to give some more words of explanation in view of the fact that different assumptions on data automatically generate different concepts of solutions.

To start, we observe that, in view of the periodic boundary conditions and of the absence of external forces, some physical quantities are necessarily *conserved* during the evolution. Namely, any (reasonably defined) solution to the system must satisfy

$$\boldsymbol{u}(t)_\Omega = \boldsymbol{u}(0)_\Omega, \quad \varphi(t)_\Omega = \varphi(0)_\Omega, \qquad \mathcal{E}(\boldsymbol{u}(t), \varphi(t), \vartheta(t)) = \mathcal{E}(\boldsymbol{u}(0), \varphi(0), \vartheta(0)). \tag{2.13}$$

This corresponds to conservation of momentum, of mass, and of the "total energy" $\mathcal{E}$ defined as

$$\mathcal{E}(\boldsymbol{u}, \varphi, \vartheta) = \int_\Omega \left(\frac{1}{2}|\boldsymbol{u}|^2 + \frac{1}{2}|\nabla\varphi|^2 + F(\varphi) + \vartheta \right).$$

Physically speaking, $\mathcal{E}$ is the sum of the kinetic, interfacial, configuration and thermal energies. Then, while the first principle of Thermodynamics yields conservation of $\mathcal{E}$, the second principle prescribes the production of entropy, and this fact can be checked directly from the equations simply by testing (1.5) by $-\vartheta^{-1}$. Hence, in order for the initial entropy to be finite, one needs to assume $\vartheta_0 > 0$ almost everywhere and $\log\vartheta_0 \in L^1(\Omega)$. This property, together with the finiteness of the initial energy, leads naturally to define the "energy-entropy space" of data as

$$\mathcal{H} = \big\{ z = (\boldsymbol{u}, \varphi, \vartheta) \in H \times V \times L^1(\Omega) : \ \operatorname{div} \boldsymbol{u} = 0,\ \vartheta > 0 \text{ a.e. in } \Omega,\ \log\vartheta \in L^1(\Omega) \big\}. \tag{2.14}$$

The space $\mathcal{H}$ is not a Banach space (in view of the occurrence of the nonlinear logarithm function); nevertheless, it can be endowed with a (complete) metric. Namely, for $z_i = (\boldsymbol{u}_i, \varphi_i, \vartheta_i) \in \mathcal{H}$, $i = 1, 2$, we may set

$$\mathrm{dist}_{\mathcal{H}}(z_1, z_2) := \|\boldsymbol{u}_1 - \boldsymbol{u}_2\| + \|\varphi_1 - \varphi_2\|_V + \|\vartheta_1 - \vartheta_2\|_{L^1(\Omega)} + \|\log\vartheta_1 - \log\vartheta_2\|_{L^1(\Omega)}. \tag{2.15}$$

Then, what we get from Thermodynamics is that any eventual solution being in $\mathcal{H}$ at the initial time will remain in $\mathcal{H}$ in the evolution. Indeed, this regularity setting corresponds to that of the "weak solutions" considered in [8] in the 3D case (in particular, the terminology is consistent with that commonly used for the Navier-Stokes system). On the other hand, even in 2D, we expect that for "weak solutions" defect measures would appear in (1.5) due to the fact that the right hand side is only controlled in L^1. This is the reason that led the authors of [9] to postulate additional regularity on the initial data in order to get a stronger and more satisfactory concept of solution. Indeed, in [9] the following result was proved:

Theorem 2.1. *Let us assume* (F1) *(with C^2 in place of C^3),* (F2), (2.9) *and* (K1). *Let also $T > 0$ and let $z_0 = (\boldsymbol{u}_0, \varphi_0, \vartheta_0) \in \mathcal{H}$ additionally satisfy*

$$z_0 \in V \times H^3(\Omega) \times V. \tag{2.16}$$

Moreover, assume that

$$\exists\, \underline{\vartheta} > 0 \ \text{ such that } \vartheta_0(x) \geq \underline{\vartheta} \ \text{ a.e. in } \Omega. \tag{2.17}$$

Then, there exists at least one "strong solution" *to the non-isothermal model for two-phase fluid flows, namely, one quadruple $(\boldsymbol{u}, \varphi, \mu, \vartheta)$ with*

$$\boldsymbol{u} \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)), \tag{2.18}$$

$$\varphi \in W^{1,\infty}(0,T;V') \cap H^1(0,T;V) \cap L^2(0,T;H^3(\Omega)), \tag{2.19}$$

$$\mu \in H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega)), \tag{2.20}$$

$$\vartheta \in H^1(0,T;V') \cap L^\infty(0,T;L^{q+2}(\Omega)) \cap L^2(0,T;V), \quad \vartheta > 0 \text{ a.e. in } (0,T) \times \Omega, \tag{2.21}$$

$$K(\vartheta) \in L^2(0,T;V), \tag{2.22}$$

such that the equations of the system (1.1)-(1.4) *hold in the sense of distributions as well as almost everywhere in $(0,T) \times \Omega$, while* (1.5) *holds in V' for a.e. $t \in (0,T)$. Moreover, the following initial conditions hold a.e. in Ω:*

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad \varphi|_{t=0} = \varphi_0, \quad \vartheta|_{t=0} = \vartheta_0. \tag{2.23}$$

Remark 2.2. In Theorem 2.1 we have noted the solution as a quadruple $(\boldsymbol{u}, \varphi, \mu, \vartheta)$. However, in view of the fact that μ can be regarded as an auxiliary variable, in some situations it will be convenient to "exclude" μ from the definition and interpret the solution just as a triple z, namely setting $z := (\boldsymbol{u}, \varphi, \vartheta)$. Indeed, one can easily rewrite the system (1.3)-(1.4) as a single equation where μ no longer appears. This interpretation is particularly useful when we consider the dynamical system associated with (stable) solutions. Indeed, here z is the natural variable and the set $\mathcal{V}^r$ can be seen as a *phase space* for solution trajectories. On the other hand, the situation is somehow articulated, because we will see that, for determining ω-limit sets, also the limit value of μ will play a specific role.

It is worth discussing a bit more the hypotheses on initial data considered in the above theorem. First of all, (2.17) (i.e., the fact that the initial temperature is assumed to be uniformly strictly positive) was just taken for simplicity, whereas in fact a weaker assumption suffices for the proof. On the contrary, conditions (2.16) were essential. These correspond in fact to the regularity of "strong solutions" for the Navier-Stokes system (available in 2D) and to the so-called "second energy estimate" for the Cahn-Hilliard equation. Correspondingly, an improvement of the regularity of ϑ_0 is also required.

Conditions (2.16) give rise to a new functional setting suitable for "strong solutions". Namely, one may define

$$\mathcal{V} := \big\{ z = (\boldsymbol{u}, \varphi, \vartheta) \in \mathcal{H} \cap (V \times H^3(\Omega) \times V) \big\}. \tag{2.24}$$

In this notation, existence was proved in [9] for initial data $z_0 \in \mathcal{V}$ additionally satisfying (2.17). On the other hand, one immediately sees from the statement of Theorem 2.1 that from $z_0 \in \mathcal{V}$ does not seem to follow that $z(t)$ is controlled in $\mathcal{V}$ uniformly in time. In other words, strong solutions appear to exhibit some regularity loss, or, in the terminology of dynamical systems, $\mathcal{V}$ seems not to be a good phase space. In addition to that, it is also not clarified whether property (2.17) is preserved in the evolution.

The main reason for this regularity gap is probably due to the use of limit-case two dimensional embeddings in the a-priori estimates. In particular, properties (2.18), (2.20) imply that the right hand side of (1.5) lies *exactly* in $L^2(0,T;H)$ and this information seems not sufficient in order to get any additional regularity on ϑ. In particular, an L^∞-bound is lacking (Moser iterations do not work for L^2 right hand side), which would be crucial in order to manage some coefficients of (1.5) which grow like powers of ϑ. In particular, the possibility to control ϑ in the space V uniformly in time is tied to the a-priori estimate obtained testing (1.5) by ϑ_t (or, more naturally, by $K(\vartheta)_t$). Without a previous L^∞-control of ϑ, this procedure does not seem to be available.

On the basis of these considerations, in order to avoid the regularity gap occurring in Theorem 2.1, we decided to consider a concept of solution which is *slightly* more regular with respect to "strong solutions". There are probably several ways to do this; in our approach we will "fractionally" improve the regularity asked for the initial velocity in such a way to eventually get a control of the right hand side of (1.5) in L^p for some p *strictly* greater than 2. Hence, for $r \in (0,1/2]$ assigned but otherwise arbitrary, we introduce the functional class

$$\mathcal{V}^r := \big\{ z = (\boldsymbol{u}, \varphi, \vartheta) \in \mathcal{H} \cap (H^{r+1}(\Omega) \times H^3(\Omega) \times V) : K(\vartheta) \in V,\ 1/\vartheta \in L^1(\Omega) \big\}. \tag{2.25}$$

Compared to (2.16), we have made a number of changes, which it is worth explaining in some detail. First of all, we have improved a bit the regularity asked on $\boldsymbol{u}_0$; secondly, we have made precise the positivity condition on the temperature, namely, we have replaced (2.17) with the much weaker condition $1/\vartheta \in L^1(\Omega)$. This property is also more natural in view of the fact that it is easy to show (see Lemma 4.1 below) that it is preserved in the time evolution; i.e., if it holds for the initial datum, then it keeps holding with time. Finally, in place of $\vartheta \in V$ we required $K(\vartheta) \in V$. Actually, both conditions are stated in the definition; however, it is easy to check that $K(\vartheta) \in V$ implies $\vartheta \in V$, but the converse is not true because V-functions in 2D are not necessarily bounded in the uniform norm.

In the sequel we will actually prove that if the initial datum lie in $\mathcal{V}^r$ then the evolution takes place in $\mathcal{V}^r$ for any $t > 0$ (and, also, one has a uniform in time control of the "magnitude" of the solution in $\mathcal{V}$). In other words there is no regularity loss, which justifies our choice to denote as "stable solutions" those solutions that start from initial data in $\mathcal{V}^r$. Leaving to the next section the detailed presentation of our main results, here we just recall that the set $\mathcal{V}^r$ is also a metric space with respect to the distance

$$\mathrm{dist}_{\mathcal{V}^r}(z_1, z_2) := \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{H^{r+1}(\Omega)} + \|\varphi_1 - \varphi_2\|_{H^3(\Omega)} + \|K(\vartheta_1) - K(\vartheta_2)\|_V + \|1/\vartheta_1 - 1/\vartheta_2\|_{L^1(\Omega)}. \tag{2.26}$$

This distance will be later used in order to properly settle the (dissipative) dynamical process associated with stable solutions.

3 Main results

We are now ready to present our main results. Our first theorem is devoted to proving well-posedness of system (1.1)-(1.5) in the space $\mathcal{V}^r$ on finite time intervals:

Theorem 3.1. *Let us assume* (F1)-(F3) *and* (K1). *Let also* $T > 0$. *Then, given any* $z_0 \in \mathcal{V}^r$, *there* *exists a unique* stable solution *to our problem, namely a quadruple* $(\boldsymbol{u}, \varphi, \mu, \vartheta)$ *with the regularity*

$$\boldsymbol{u} \in H^1(0,T;H^r(\Omega)) \cap L^\infty(0,T;H^{1+r}(\Omega)) \cap L^2(0,T;H^{2+r}(\Omega)), \tag{3.1}$$

$$\varphi \in W^{1,\infty}(0,T;V') \cap H^1(0,T;V) \cap L^\infty(0,T;H^3(\Omega)), \tag{3.2}$$

$$\mu \in H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega)), \tag{3.3}$$

$$\vartheta \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)), \quad \vartheta > 0 \ \text{ a.e. in } (0,T)\times\Omega, \tag{3.4}$$

$$K(\vartheta) \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)), \tag{3.5}$$

$$1/\vartheta \in L^\infty(0,T;L^1(\Omega)), \tag{3.6}$$

satisfying equations (1.1)-(1.5) *a.e. in* $\Omega \times (0,T)$ *and complying with the initial conditions*

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad \varphi|_{t=0} = \varphi_0, \quad \vartheta|_{t=0} = \vartheta_0 \tag{3.7}$$

almost everywhere in Ω.

Once we have well-posedness on finite time intervals, we can consider the dynamical process associated with stable solutions. To this aim we need to introduce a further (more regular) functional set:

$$\mathcal{W} := \big\{ z = (\boldsymbol{u}, \varphi, \vartheta) \in \mathcal{H} \cap (H^2(\Omega) \times H^4(\Omega) \times H^2(\Omega)) : \ 1/\vartheta \in L^4(\Omega), \ \nabla(1/\vartheta) \in L^1(\Omega) \big\}. \tag{3.8}$$

It is a standard matter to verify that $\mathcal{W}$ is embedded continuously and compactly into $\mathcal{V}^r$. As above, $\mathcal{W}$ is a (complete) metric space with the distance

$$\begin{aligned} \text{dist}_{\mathcal{W}}(z_1, z_2) &:= \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{H^2(\Omega)} + \|\varphi_1 - \varphi_2\|_{H^4(\Omega)} + \|\vartheta_1 - \vartheta_2\|_{H^2(\Omega)} \\ &\quad + \|1/\vartheta_1 - 1/\vartheta_2\|_{L^4(\Omega)} + \|\nabla(1/\vartheta_1) - \nabla(1/\vartheta_2)\|_{L^1(\Omega)}. \end{aligned} \tag{3.9}$$

In the sequel, in order to estimate the "magnitude" of the elements of $\mathcal{W}$ we will often write

$$\|z\|_{\mathcal{W}} := \|\boldsymbol{u}\|_{H^2(\Omega)} + \|\varphi\|_{H^4(\Omega)} + \|\vartheta\|_{H^2(\Omega)} + \|1/\vartheta\|_{L^4(\Omega)} + \|\nabla(1/\vartheta)\|_{L^1(\Omega)}. \tag{3.10}$$

This is of course somehow an abuse of notation because the above is not a norm. Mimicking (3.10), we will also write

$$\|z\|_{\mathcal{H}} := \|\boldsymbol{u}\| + \|\varphi\|_V + \|\vartheta\|_{L^1(\Omega)} + \|\log\vartheta\|_{L^1(\Omega)}, \tag{3.11}$$

as well as $\|z\|_{\mathcal{V}}$, $\|z\|_{\mathcal{V}^r}$, etc., with obvious corresponding notation.

Remark 3.2. One may wonder where does the L^4-regularity in (3.9) (and (3.10)) come out. Actually, this is somehow an arbitrary choice (taken just for simplicity), as it is also essentially arbitrary the L^1-condition $1/\vartheta$ in the definition of $\mathcal{V}^r$. Indeed, at the price of technicalities, we expect that one could prove that, in general, if $\vartheta_0 > 0$ a.e. in Ω and there exists (an arbitrarily small) $\alpha > 0$ such that $\vartheta_0^{-\alpha} \in L^1(\Omega)$, then it turns out that $\vartheta^{-1}(t) \in L^\infty(\Omega)$ for any $t > 0$. In other words, the temperature should become strictly positive instantaneously in the uniform norm. The more natural way to prove this form of the minimum principle argument would exploit Lemma 4.1 below together with a Moser iteration procedure. However, the argument may involve quite a relevant amount of technicalities (cf., e.g., [20] for a similar situation) and we omit it because it is not essential for our purposes.

Of course, in order to analyze the long-time behavior of solutions we need to consider the natural problem constraints, therefore we set

Definition 3.3. *Given* $m, M \in \mathbb{R}$ *and* $\boldsymbol{m} \in \mathbb{R}^2$, *we introduce the spaces* $\mathcal{H}_{\boldsymbol{m},m,M}$, $\mathcal{V}_{\boldsymbol{m},m,M}$, $\mathcal{V}^r_{\boldsymbol{m},m,M}$, $\mathcal{W}_{\boldsymbol{m},m,M}$ *of triplets* $z = (\boldsymbol{u}, \varphi, \vartheta)$ *respectively in* $\mathcal{H}$, $\mathcal{V}$, $\mathcal{V}^r$, $\mathcal{W}$, *subject to the constraints*

$$\boldsymbol{u}_\Omega = \boldsymbol{m}, \tag{3.12}$$

$$\varphi_\Omega = m, \tag{3.13}$$

$$\frac{1}{2}\|\boldsymbol{u}\|^2 + \frac{1}{2}\|\nabla\varphi\|^2 + \int_\Omega \big(F(\varphi) + \vartheta\big) = M. \tag{3.14}$$

and endowed with the corresponding distances introduced above.

Given any triplet $(\boldsymbol{m}, m, M)$, if $z_0 = (\boldsymbol{u}_0, \varphi_0, \vartheta_0) \in \mathcal{V}^r_{\boldsymbol{m},m,M}$, then by the conservation properties (2.13) and the regularity provided by Theorem 3.1, the evolution of the system operates as a trajectory in the space $\mathcal{V}_{\mathbf{m},m,M}$. Hence, we can introduce the solution operator

$$S(t) : \mathcal{V}^r_{\boldsymbol{m},m,M} \to \mathcal{V}^r_{\boldsymbol{m},m,M}, \qquad t \geq 0,$$
$$z_0 \mapsto S(t)z_0 = (\boldsymbol{u}(t), \varphi(t), \vartheta(t)).$$

Remark 3.4. One can easily prove that S is a semigroup on $\mathcal{V}^r_{\boldsymbol{m},m,M}$, i.e.,

$$S(0) = \mathrm{Id},$$
$$S(t+s) = S(t)S(s), \quad \forall s,t \geq 0.$$

More precisely, using interpolation it is not difficult to deduce from (3.1)-(3.6) that S is a continuous semigroup. Namely, trajectories are continuous with values in $\mathcal{V}^r$, moreover,

$$S(t) : \mathcal{V}^r_{\mathbf{m},m,M} \to \mathcal{V}^r_{\mathbf{m},m,M}$$

is a continuous mapping for every $t \geq 0$.

Section 6 will be devoted to the study of the asymptotic behaviour of our model system. In particular, we will prove existence of nonempty ω-limit sets of trajectories in the case when the spatial mean of the initial velocity is $\mathbf{0}$. In particular, this will permit us to prove our main result regarding the long-time behavior of the system, i.e., existence of the global attractor for the dynamical process associated to stable solutions.

Before stating it, we need however to introduce a further subclass of solutions. Namely, for given $R \in \mathbb{R}$, we set $\mathcal{V}^{r,R} := \{z \in \mathcal{V}^r : (-\log\vartheta)_\Omega \leq R\}$ as well as the subclasses $\mathcal{V}^{r,R}_{\boldsymbol{m},m,M}$, $\mathcal{V}^{r,R}_{\mathbf{0},m,M}$. The space $\mathcal{V}^{r,R}$ can be easily proved to be a closed (hence complete) metric subspace of $\mathcal{V}^r$. Observing that for the present model the entropy density is given by $\log\vartheta$, $\mathcal{V}^{r,R}$ turns out to contain the configurations for which the global entropy is greater or equal than $-R$. Referring to the monographs [13, 18, 22, 25] for the basic concepts and definitions from the theory of infinite-dimensional dynamical systems, we can prove existence of the global attractor for solutions taking values in the phase space $\mathcal{V}^{r,R}_{\mathbf{0},m,M}$.

Theorem 3.5. *Assume that* (F1)-(F3) *and* (K1) *hold true. Let also $m, M, R \in \mathbb{R}$. Then, the space $\mathcal{V}^{r,R}_{\mathbf{0},m,M}$ is positively invariant for the semigroup $S(t)$. Moreover, there exists the global attractor $\mathcal{A} = \mathcal{A}^R(\mathbf{0}, m, M)$. Namely, $\mathcal{A}$ is a compact subset of $\mathcal{V}^{r,R}_{\mathbf{0},m,M}$ and is completely invariant for the flow $S(t)$ (i.e., $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$). Moreover, $\mathcal{A}$ uniformly attracts the trajectory bundles starting from any bounded set $B \subset \mathcal{V}^{r,R}_{\mathbf{0},m,M}$, i.e., we have*

$$\lim_{t\nearrow\infty} \mathrm{Dist}_{\mathcal{V}^r}(S(t)B, \mathcal{A}) = 0, \tag{3.15}$$

where $\mathrm{Dist}_{\mathcal{V}^r}$ denotes the Hausdorff semidistance associated to the metric of $\mathcal{V}^r$, namely, if E, F are subsets of $\mathcal{V}^r$, we have set

$$\mathrm{Dist}_{\mathcal{V}^r}(E, F) := \sup_{e\in E} \inf_{f\in F} \mathrm{dist}_{\mathcal{V}^r}(e, f).$$

The reason for restricting ourselves to the case when the spatial mean of the initial velocity is $\mathbf{0}$ stands in the fact that, for $\boldsymbol{m} \neq \mathbf{0}$, solution trajectories asymptotically tend to rotate around the flat torus with constant speed $\boldsymbol{m}$, which makes the long-time analysis more difficult. We will give more details in Section 6 where in particular we will see that a description of the long-time behavior of trajectories in the case $\boldsymbol{m} \neq \mathbf{0}$ can be provided by means of a suitable change of variables. In the proof we will also explain the reason why we need to restrict ourselves to the subclass $\mathcal{V}^{r,R}_{\mathbf{0},m,M}$, namely, why we need to impose a lower bound to the spatial mean of the entropy.

4 Proof of Theorem 3.1 part I: existence of solutions with additional regularity

The proof will be carried out by performing a number of *formal* a-priori estimates holding for any hypothetical solution to our system. These estimates could be made rigorous by adapting them to the regularization argument given in [9]. In the sequel we shall denote by c a generic positive constant depending only on the assigned parameters of the system (but independent of the time variable). In an expression like $Q(a, b, \dots)$, Q will denote a generic (computable) positive function increasingly monotone in each of its arguments. For instance, $Q(\|z_0\|_{\mathcal{V}^r})$ represents a computable quantity depending only of the fixed parameters of the system, increasingly depending on the $\mathcal{V}^r$-"magnitude" of the initial datum, and independent of the time variable.

4.1 Preliminaries and technical results

We start by presenting a technical lemma that provides an estimate of negative powers of the temperature on time intervals of arbitrary length.

Lemma 4.1. *Let $z = (\boldsymbol{u}, \varphi, \vartheta)$ be a solution to our system defined over the generic time interval (S, T) with $0 \le S < T \le +\infty$ and satisfying, for some $\alpha > 1$,*

$$N := \|\nabla\vartheta\|_{L^2(S,T;H)} < \infty, \qquad A := \|\vartheta^{1-\alpha}(S)\|_{L^1(\Omega)} < \infty. \tag{4.1}$$

Then, we have

$$\vartheta^{1-\alpha} \in L^\infty(S,T;L^1(\Omega)), \qquad \nabla(\vartheta^{\frac{1-\alpha}{2}}) \in L^2(S,T;L^2(\Omega)),$$

with the quantitative estimate

$$\|\vartheta^{1-\alpha}\|_{L^\infty(S,T;L^1(\Omega))} + \|\nabla(\vartheta^{\frac{1-\alpha}{2}})\|_{L^2(S,T;H)} \le Q(N, A). \tag{4.2}$$

Proof. We did not specify in the statement the concept of weak solutions we are dealing with because, in fact, what we are proving is a structural property of any solution of the system (even of very weak ones, provided they comply with the energy conservation and entropy production principle at least in a regularization).

That said, we multiply (1.5) by $-\vartheta^{-\alpha}$, for some $\alpha > 1$ getting

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\vartheta^{1-\alpha}}{\alpha-1} - \int_\Omega \vartheta^{1-\alpha}\Delta\mu + \int_\Omega \Delta[K(\vartheta)]\vartheta^{-\alpha} + \int_\Omega (|\nabla\boldsymbol{u}|^2 + |\nabla\mu|^2)\vartheta^{-\alpha} = 0,$$

where the third term on the left hand side reads as

$$\int_\Omega \Delta[K(\vartheta)]\vartheta^{-\alpha} = \frac{4\alpha}{(1-\alpha)^2}\|\nabla(\vartheta^{\frac{1-\alpha}{2}})\|^2 + \frac{4\alpha}{(q+1-\alpha)^2}\|\nabla(\vartheta^{\frac{1-\alpha+q}{2}})\|^2,$$

while the second one is

$$\begin{aligned} -\int_\Omega \vartheta^{1-\alpha}\Delta\mu &= (1-\alpha)\int_\Omega \vartheta^{-\alpha}\nabla\vartheta\nabla\mu \ge -\frac{1}{2}\int_\Omega \vartheta^{-\alpha}|\nabla\mu|^2 - \frac{(1-\alpha)^2}{2}\int_\Omega \vartheta^{-\alpha}|\nabla\vartheta|^2 \\ &\ge -\frac{1}{2}\int_\Omega \vartheta^{-\alpha}|\nabla\mu|^2 - \frac{2\alpha}{(1-\alpha)^2}\int_\Omega |\nabla(\vartheta^{\frac{1-\alpha}{2}})|^2 - c\int_\Omega |\nabla\vartheta|^2 \end{aligned}$$

having observed that, due to Young's inequality,

$$\int_\Omega \vartheta^{-\alpha}|\nabla\vartheta|^2 \le \delta\int_\Omega \vartheta^{-1-\alpha}|\nabla\vartheta|^2 + c_\delta\int_\Omega |\nabla\vartheta|^2,$$

for any fixed $\delta > 0$. Collecting the above estimates, we are lead to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\vartheta^{1-\alpha}}{\alpha-1} &+ \frac{2\alpha}{(1-\alpha)^2}\|\nabla(\vartheta^{\frac{1-\alpha}{2}})\|^2 + \frac{4\alpha}{(q+1-\alpha)^2}\|\nabla(\vartheta^{\frac{1-\alpha+q}{2}})\|^2 \\ &+ \int_\Omega (|\nabla\boldsymbol{u}|^2 + \frac{1}{2}|\nabla\mu|^2)\vartheta^{-\alpha} \le c\|\nabla\vartheta\|^2. \end{aligned}$$

Integrating this inequality over (S, t), $t \in (S, T]$, we then deduce

$$\frac{1}{\alpha-1}\int_\Omega \vartheta^{1-\alpha}(t) + \frac{2\alpha}{(1-\alpha)^2}\int_S^t \|\nabla(\vartheta^{\frac{1-\alpha}{2}})\|^2\,\mathrm{d}s \le \frac{1}{\alpha-1}\int_\Omega \vartheta^{1-\alpha}(S) + c\int_S^t \|\nabla\vartheta(s)\|^2\,\mathrm{d}s. \tag{4.3}$$

Taking the essential supremum as t varies in $(S, T]$, recalling (4.1), we get the assert. ∎

4.2 Higher regularity - part I

Here we will derive some estimates holding on the assigned bounded interval $(0,T)$ of finite length, whereas in Section 6 we will look for uniform in time estimates. For this reason in this section we will allow the generic (positive) constants C to depend on T. The value of C may vary on occurrence.

Hence, to start the proof of Theorem 3.1, we take a "strong solution" $z = (\boldsymbol{u}, \varphi, \mu, \vartheta)$, as provided by Theorem 2.1, and we will show that, under our slightly stronger assumptions (particularly on the initial datum $z_0 \in \mathcal{V}^r$), z is in fact a "stable solution". To this aim, we will start proving the following regularity properties:

(i) $\boldsymbol{u} \in L^\infty(0,T;H^{r+1}(\Omega)) \cap L^2(0,T;H^{r+2}(\Omega))$,

(ii) $\vartheta \in H^1(0,T;H) \cap L^\infty(0,T;V)$ and $K(\vartheta) \in L^\infty(0,T;V)$,

(iii) $\varphi \in L^\infty(0,T;H^3(\Omega))$,

(iv) $1/\vartheta \in L^\infty(0,T;L^1(\Omega))$.

• PROOF OF (i). First of all, we observe that

$$\varphi \in L^\infty(0,T;H^2(\Omega)). \tag{4.4}$$

This simply comes from (1.4) using (2.20), (2.21) and (2.9) combined with the fact that, due to (2.19) $\varphi \in L^\infty(0,T;L^p(\Omega))$ for all $p > 1$. This estimate will be used several times later on.

To achieve (i) we need now to introduce the Stokes operator (for more details see for instance [22, Par. 3.8] or [24, Par. 2.2]) $A : \mathbb{V} \to \mathbb{V}'$, where $A\boldsymbol{u} := \mathbb{P}(-\Delta \boldsymbol{u})$ for $\boldsymbol{u} \in D(A) = \{\mathbf{v} \in \mathbb{H}, \Delta\mathbf{v} \in \mathbb{H}\}$ and $\mathbb{P}$ denotes the so called Leray (or Helmholtz) projection. It can be shown (cf. [22, Theorem 38.6]) that the operator A is an unbounded, positive, linear, selfadjoint operator on the space $\mathbb{H}$. Therefore, we can define the powers A^s, $s \in \mathbb{R}$, with domain $D(A^s)$ in $\mathbb{H}$. Setting

$$\mathbb{V}^s = D(A^{s/2}),$$

then $\mathbb{V}^s$ is a closed subspace of $H^s_0(\Omega)$ (here denoting the subspace of $H^s(\Omega)$ containing the function with zero spatial mean) and indeed

$$\mathbb{V}^s = \{\mathbf{v} \in H^s_0(\Omega) : \operatorname{div}\mathbf{v} = 0\}.$$

In particular $\mathbb{V}^2 = D(A)$, $\mathbb{V}^1 = \mathbb{V}$, $\mathbb{V}^0 = \mathbb{H}$. Moreover A is an isomorphism from $\mathbb{V}^{s+2}$ onto $\mathbb{V}^s$, from $D(A)$ onto $\mathbb{H}$, from $\mathbb{V}$ onto $\mathbb{V}'$ and so on. Finally, the norm $\|A^{s/2}\mathbf{v}\|$ on $\mathbb{V}^s$ is equivalent to the norm induced by $H^s_0(\Omega)$.

The subsequent step will be to test (1.2) by $A^{r+1}\hat{\boldsymbol{u}}$, where r is as in (2.25) and where $\hat{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_\Omega$. Working in divergence-free spaces, the key point will be to project equation (1.2) into the space $\mathbb{H}$ by applying $\mathbb{P}$ and then to test by $A^{r+1}\hat{\boldsymbol{u}}$. Let us note that $\mathbb{P}\boldsymbol{u} = \boldsymbol{u}$ due to (1.1); moreover also $\mathbb{P}\boldsymbol{u}_t = \boldsymbol{u}_t$. The pressure term is missing because $\mathbb{P}\nabla p = 0$, whence we are left with the following term:

$$\mathbb{P}(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) = \mathbb{P}(\boldsymbol{u}\cdot\nabla\hat{\boldsymbol{u}}) = \mathbb{P}(\hat{\boldsymbol{u}}\cdot\nabla\hat{\boldsymbol{u}} + \boldsymbol{u}_\Omega\cdot\nabla\hat{\boldsymbol{u}}) = \mathbb{P}(\hat{\boldsymbol{u}}\cdot\nabla\hat{\boldsymbol{u}}).$$

Next, we set

$$f := \mathbb{P}(-\operatorname{div}(\nabla\varphi\otimes\nabla\varphi)) \tag{4.5}$$

and define the trilinear form

$$b(\boldsymbol{u}, \mathbf{v}, \mathbf{w}) := \langle(\boldsymbol{u}\cdot\nabla)\mathbf{v}, \mathbf{w}\rangle.$$

As a consequence, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{\boldsymbol{u}}\|^2_{\mathbb{V}^{r+1}} + \|\hat{\boldsymbol{u}}\|^2_{\mathbb{V}^{r+2}} = -b(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}, A^{r+1}\hat{\boldsymbol{u}}) + \langle A^{\frac{r}{2}}f, A^{\frac{r}{2}+1}\hat{\boldsymbol{u}}\rangle \tag{4.6}$$

and we observe that $f_\Omega = 0$. If $r \in (0,1)$ and $f \in \mathbb{V}^r$, then [24, Part I, Sec. 2.3, formula (2.20)]

$$\|A^{\frac{r}{2}}f\| = \|f\|_{\mathbb{V}^r} \leq \|f\|^{1-r}\|f\|^r_{\mathbb{V}} \leq \|f\|^{1-r}\|\nabla f\|^r. \tag{4.7}$$

Now, using (2.2), it is not difficult to show that

$$\|f\| \le C\|\nabla^2\varphi\|_{L^4(\Omega)}\,\|\nabla\varphi\|_{L^4(\Omega)} \le C\|\varphi\|^{1/2}_{H^3(\Omega)}\|\varphi\|_{H^2(\Omega)}\|\varphi\|^{1/2}_V \overset{(4.4)}{\le} C\,\|\varphi\|^{1/2}_{H^3(\Omega)}.$$

This, together with Poincaré's inequality, entails

$$\begin{aligned}
\|\nabla f\| &= \left\|\nabla\left(\nabla\varphi\Delta\varphi + \nabla\left(\frac{|\nabla\varphi|^2}{2}\right)\right)\right\| \\
&= \left\|\nabla^2\varphi\Delta\varphi + \nabla\varphi\otimes\nabla\Delta\varphi + \nabla^2\left(\frac{|\nabla\varphi|^2}{2}\right)\right\| \\
&\le \left(\|\nabla^2\varphi\|_{L^4(\Omega)}\|\Delta\varphi\|_{L^4(\Omega)} + \|\nabla\varphi\|_{L^\infty(\Omega)}\|\nabla\Delta\varphi\| + \left\|\nabla^2\left(\frac{|\nabla\varphi|^2}{2}\right)\right\|\right) \\
&\overset{(2.2),(2.3)}{\le} C\left(\|\varphi\|_{H^2(\Omega)}\|\varphi\|_{H^3(\Omega)} + \|\nabla\varphi\|^{1/2}\|\varphi\|^{\frac{3}{2}}_{H^3(\Omega)} + \|\nabla\varphi\|_{L^\infty(\Omega)}\|\varphi\|_{H^3(\Omega)}\right) \\
&\overset{(2.3),(4.4)}{\le} C\left(\|\varphi\|^{\frac{3}{2}}_{H^3(\Omega)} + 1\right).
\end{aligned} \tag{4.8}$$

This implies, going back to (4.7),

$$\|f\|_{\mathbb{V}^r} \le C\,\|\varphi\|^{\frac{1-r}{2}}_{H^3(\Omega)}\left(\|\varphi\|^{\frac{3}{2}r}_{H^3(\Omega)} + 1\right) \le C\left(\|\varphi\|^{\frac{1+2r}{2}}_{H^3(\Omega)} + 1\right) \tag{4.9}$$

and in turn

$$\langle A^{\frac{r}{2}}f, A^{\frac{r}{2}+1}\hat{\boldsymbol{u}}\rangle \le C\,\|f\|_{\mathbb{V}^r}\|\hat{\boldsymbol{u}}\|_{\mathbb{V}^{r+2}} \le \frac{1}{4}\|\hat{\boldsymbol{u}}\|^2_{\mathbb{V}^{r+2}} + C\,\|f\|^2_{\mathbb{V}^r} \le \frac{1}{4}\|\hat{\boldsymbol{u}}\|^2_{\mathbb{V}^{r+2}} + C\left(\|\varphi\|^{2r+1}_{H^3(\Omega)} + 1\right).$$

Next, we have

$$b(\hat{\boldsymbol{u}},\hat{\boldsymbol{u}},A^{r+1}\hat{\boldsymbol{u}}) = \langle\hat{\boldsymbol{u}}\cdot\nabla\hat{\boldsymbol{u}}, A^{r+1}\hat{\boldsymbol{u}}\rangle = \langle A^{\frac{r}{2}}(\hat{\boldsymbol{u}}\cdot\nabla\hat{\boldsymbol{u}}), A^{\frac{r}{2}+1}\hat{\boldsymbol{u}}\rangle \le \|\hat{\boldsymbol{u}}\cdot\nabla\hat{\boldsymbol{u}}\|_{\mathbb{V}^r}\,\|\hat{\boldsymbol{u}}\|_{\mathbb{V}^{r+2}}. \tag{4.10}$$

Now, also on account of the Friedrichs inequality (2.6),

$$\begin{aligned}
\|\hat{\boldsymbol{u}}\cdot\nabla\hat{\boldsymbol{u}}\|_{\mathbb{V}^r} &\le \|\hat{\boldsymbol{u}}\cdot\nabla\hat{\boldsymbol{u}}\|^{1-r}\,\|\hat{\boldsymbol{u}}\cdot\nabla\hat{\boldsymbol{u}}\|^r_{\mathbb{V}} \\
&\le C\,\|\hat{\boldsymbol{u}}\|^{1-r}_{L^\infty(\Omega)}\,\|\nabla\hat{\boldsymbol{u}}\|^{1-r}\left(\|\nabla\hat{\boldsymbol{u}}\|^{2r}_{L^4(\Omega)} + \|\hat{\boldsymbol{u}}\|^r_{L^\infty(\Omega)}\,\|\nabla^2\hat{\boldsymbol{u}}\|^r\right) =: V_1 + V_2.
\end{aligned}$$

We have that

$$V_1 \overset{(2.3)}{\le} C\|\hat{\boldsymbol{u}}\|^{\frac{1-r}{2}}\|\hat{\boldsymbol{u}}\|^{\frac{1-r}{2}}_{\mathbb{V}^2}\,\|\nabla\hat{\boldsymbol{u}}\|^{1-r}\,\|\nabla\hat{\boldsymbol{u}}\|^r\,\|\nabla\hat{\boldsymbol{u}}\|^r_{\mathbb{V}} \le C\,\|\hat{\boldsymbol{u}}\|^{\frac{r+1}{2}}_{\mathbb{V}^2} \le C\,\|\hat{\boldsymbol{u}}\|_{\mathbb{V}^2},$$

where we used (2.18) and the fact that $0 < r \le 1/2$. Analogously

$$V_2 \le \|\hat{\boldsymbol{u}}\|_{L^\infty(\Omega)}\|\hat{\boldsymbol{u}}\|^{1-r}_{\mathbb{V}}\|\hat{\boldsymbol{u}}\|^r_{\mathbb{V}^2} \overset{(2.3)}{\le} \|\hat{\boldsymbol{u}}\|^{\frac{1}{2}+r}_{\mathbb{V}^2}\,\|\hat{\boldsymbol{u}}\|^{\frac{1}{2}}\,\|\hat{\boldsymbol{u}}\|^{1-r}_{\mathbb{V}} \le C\,\|\hat{\boldsymbol{u}}\|^{\frac{1+2r}{2}}_{\mathbb{V}^2} \le C\,\|\hat{\boldsymbol{u}}\|_{\mathbb{V}^2}.$$

Summing up, we deduce

$$\|\hat{\boldsymbol{u}}\cdot\nabla\hat{\boldsymbol{u}}\|_{\mathbb{V}^r} \le C\,\|\hat{\boldsymbol{u}}\|_{\mathbb{V}^2}. \tag{4.11}$$

This fact, using interpolation together with (4.10), entails that

$$\begin{aligned}
b(\hat{\boldsymbol{u}},\hat{\boldsymbol{u}},A^{r+1}\hat{\boldsymbol{u}}) &\le \|\hat{\boldsymbol{u}}\cdot\nabla\hat{\boldsymbol{u}}\|_{\mathbb{V}^r}\,\|\hat{\boldsymbol{u}}\|_{\mathbb{V}^{r+2}} \le C\,\|\hat{\boldsymbol{u}}\|_{\mathbb{V}^2}\|\hat{\boldsymbol{u}}\|_{\mathbb{V}^{r+2}} \\
&\le C\,\|\hat{\boldsymbol{u}}\|^r_{\mathbb{V}^{r+1}}\|\hat{\boldsymbol{u}}\|^{2-r}_{\mathbb{V}^{r+2}} \le \frac{1}{4}\|\hat{\boldsymbol{u}}\|^2_{\mathbb{V}^{r+2}} + C\,\|\hat{\boldsymbol{u}}\|^2_{\mathbb{V}^{r+1}}.
\end{aligned}$$

Coming back to (4.6), we finally obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{\boldsymbol{u}}\|^2_{\mathbb{V}^{r+1}} + \|\hat{\boldsymbol{u}}\|^2_{\mathbb{V}^{r+2}} \le C\left(\|\varphi\|^{1+2r}_{H^3(\Omega)} + 1\right) + C\,\|\hat{\boldsymbol{u}}\|^2_{\mathbb{V}^{r+1}} \tag{4.12}$$

and the right hand side is summable since $0 < r \le \frac{1}{2}$. Then, (i) follows from Gronwall's lemma. It is also worth remarking that (i) yields in particular $\boldsymbol{u} \in L^\infty(0,T;L^\infty(\Omega))$.

• PROOF OF (ii). We can now address (ii), passing through the intermediate step

$$\vartheta \in L^\infty(0,T;L^p(\Omega)) \qquad \text{for all } p > 1. \tag{4.13}$$

In order to show this property, we multiply (1.5) by ϑ^p and integrate over Ω. We remark that

$$\int_\Omega (\boldsymbol{u}\cdot\nabla\vartheta)\vartheta^p = 0$$

due to (1.1) and the choice of periodic boundary conditions, whence we have

$$\begin{aligned}\frac{1}{p+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \vartheta^{p+1} + \frac{4p}{(p+1)^2}\int_\Omega \left|\nabla\vartheta^{\frac{p+1}{2}}\right|^2 + \frac{4p}{(p+q+1)^2}\int_\Omega \left|\nabla\vartheta^{\frac{p+q+1}{2}}\right|^2 \\ \le \int_\Omega |\Delta\mu|\vartheta^{p+1} + \int_\Omega |\nabla\boldsymbol{u}|^2\vartheta^p + \int_\Omega |\nabla\mu|^2\vartheta^p.\end{aligned} \tag{4.14}$$

At this point, from (4.14) and (2.7), we deduce

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \vartheta^{p+1} + \frac{4p}{c_p(p+1)}\left\|\vartheta^{\frac{p+1}{2}}\right\|_V^2 \\ &\le \frac{4p}{(p+1)}\|\vartheta\|_{L^1(\Omega)}^{p+1} + (p+1)\int_\Omega |\Delta\mu|\vartheta^{p+1} + (p+1)\int_\Omega |\nabla\boldsymbol{u}|^2\vartheta^p + (p+1)\int_\Omega |\nabla\mu|^2\vartheta^p \\ &\le (p+1)\int_\Omega (|\Delta\mu|+1)\vartheta^{p+1} + (p+1)\int_\Omega (|\nabla\boldsymbol{u}|^2 + |\nabla\mu|^2)\vartheta^p =: I_a + I_b,\end{aligned} \tag{4.15}$$

having observed that $\frac{4p}{p+1} \le p+1$ and where c_p is the Poincaré constant in (2.7). Next,

$$\begin{aligned}I_a &:= (p+1)\int_\Omega (|\Delta\mu|+1)\vartheta^{p+1} = (p+1)\int_\Omega \vartheta^{\frac{p+1}{2}}(|\Delta\mu|+1)\vartheta^{\frac{p+1}{2}} \\ &\le c(p)\left\|\vartheta^{\frac{p+1}{2}}\right\|_V \left\|\vartheta^{\frac{p+1}{2}}(|\Delta\mu|+1)\right\|_{V'} \le \frac{2p}{c_p(p+1)}\left\|\vartheta^{\frac{p+1}{2}}\right\|_V^2 + c(p)\left\|\vartheta^{\frac{p+1}{2}}(|\Delta\mu|+1)\right\|_{L^{6/5}(\Omega)}^2,\end{aligned}$$

where the exponent 6/5 is taken just for computational convenience and where from now on $c(p)$ denotes a positive constant depending on p, possibly varying from line to line. Now, applying Hölder's inequality with exponents $\frac{5}{3}$ and $\frac{5}{2}$, we obtain

$$\left\|\vartheta^{\frac{p+1}{2}}(|\Delta\mu|+1)\right\|_{L^{6/5}(\Omega)}^2 = \left(\int_\Omega \left[\vartheta^{\frac{p+1}{2}}(|\Delta\mu|+1)\right]^{\frac{6}{5}}\right)^{\frac{5}{3}} \le \||\Delta\mu|+1\|_{L^3(\Omega)}^2 \left(\int_\Omega \vartheta^{p+1}\right).$$

Thus we end up with

$$I_a := (p+1)\int_\Omega (|\Delta\mu|+1)\vartheta^{p+1} \le \frac{2p}{c_p(p+1)}\left\|\vartheta^{\frac{p+1}{2}}\right\|_V^2 + c(p)\||\Delta\mu|+1\|_{L^3(\Omega)}^2 \left(\int_\Omega \vartheta^{p+1}\right). \tag{4.16}$$

On the other hand, by Hölder's inequality with exponents $p+1$ and $\frac{p+1}{p}$ we deduce

$$\begin{aligned}I_b &:= (p+1)\int_\Omega (|\nabla\boldsymbol{u}|^2 + |\nabla\mu|^2)\vartheta^p \le c(p)\left(\|\nabla\boldsymbol{u}\|_{L^{2(p+1)}(\Omega)}^2 + \|\nabla\mu\|_{L^{2(p+1)}(\Omega)}^2\right)\left(\int_\Omega \vartheta^{p+1} + 1\right) \\ &\le c(p)\left(\|\boldsymbol{u}\|_{H^2(\Omega)}^2 + \|\mu\|_{H^2(\Omega)}^2\right)\left(\int_\Omega \vartheta^{p+1} + 1\right).\end{aligned}$$

Summing up, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \vartheta^{p+1} \le c(p)\left(\||\Delta\mu|+1\|_{L^3(\Omega)}^2 + \|\boldsymbol{u}\|_{H^2(\Omega)}^2 + \|\mu\|_{H^2(\Omega)}^2\right)\left(\int_\Omega \vartheta^{p+1} + 1\right).$$

Then, the conclusion comes from the Gronwall inequality (see for instance [18, Lemma 2.8]) by recalling (2.20), (2.18) and (2.17).

We can now address the proof of $\vartheta \in L^\infty(0,T;V)$. To this aim, we formally multiply (1.5) by $\partial_t K(\vartheta) = \kappa(\vartheta)\vartheta_t$. We deduce

$$\begin{aligned}
&\int_\Omega |\sqrt{\kappa(\vartheta)}\vartheta_t|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla K(\vartheta)|^2 \\
&\quad = -\int_\Omega \boldsymbol{u}\cdot\nabla\vartheta\,\kappa(\vartheta)\vartheta_t - \int_\Omega \vartheta\Delta\mu\kappa(\vartheta)\vartheta_t + \int_\Omega |\nabla\boldsymbol{u}|^2\kappa(\vartheta)\vartheta_t + \int_\Omega |\nabla\mu|^2\kappa(\vartheta)\vartheta_t \\
&\quad =: I + II + III + IV.
\end{aligned}$$

First of all, we have

$$\begin{aligned}
I &:= -\int_\Omega \boldsymbol{u}\cdot\nabla\vartheta\,\kappa(\vartheta)\vartheta_t \le \int_\Omega |\boldsymbol{u}||\nabla K(\vartheta)||\vartheta_t| \le C\|\boldsymbol{u}\|_{L^\infty(\Omega)}\|\nabla K(\vartheta)\|\|\vartheta_t\| \\
&\le \frac{1}{4}\|\sqrt{\kappa(\vartheta)}\vartheta_t\|^2 + C\|\boldsymbol{u}\|^2_{L^\infty(\Omega)}\|\nabla K(\vartheta)\|^2 \le \frac{1}{4}\|\sqrt{\kappa(\vartheta)}\vartheta_t\|^2 + C\|\nabla K(\vartheta)\|^2
\end{aligned}$$

where we used the fact that $|\vartheta_t| \le |\sqrt{\kappa(\vartheta)}\vartheta_t|$ because $\kappa(\vartheta) \ge 1$ and the fact that $\boldsymbol{u}$ is bounded in the uniform norm as a consequence of (i) and of the continuous embedding $H^{1+r}(\Omega) \subset L^\infty(\Omega)$.

Let us now deal with the term II. We have

$$\begin{aligned}
II &:= -\int_\Omega \vartheta\Delta\mu\kappa(\vartheta)\vartheta_t \le \int_\Omega \vartheta\sqrt{\kappa(\vartheta)}|\Delta\mu|\sqrt{\kappa(\vartheta)}|\vartheta_t| \\
&\le \frac{1}{4}\|\sqrt{\kappa(\vartheta)}\vartheta_t\|^2 + C\|\Delta\mu\|^2_{L^4(\Omega)}\|\sqrt{\kappa(\vartheta)}\vartheta\|^2_{L^4(\Omega)} \\
&\overset{(2.2),(4.13),(2.20)}{\le} \frac{1}{4}\|\sqrt{\kappa(\vartheta)}\vartheta_t\|^2 + C\|\Delta\mu\|^2_V.
\end{aligned}$$

The most difficult term to be estimated is III: indeed, as remarked above, we are lacking the information $\vartheta \in L^\infty((0,T)\times\Omega)$ (we only have (4.13) at this level) and for this reason we need to use the improved regularity on $\boldsymbol{u}$ in the following way:

$$\begin{aligned}
III &:= \int_\Omega |\nabla\boldsymbol{u}|^2\sqrt{\kappa(\vartheta)}\sqrt{\kappa(\vartheta)}\vartheta_t \\
&\le C\|\nabla\boldsymbol{u}\|^2_{L^{2s}(\Omega)}\|\sqrt{\kappa(\vartheta)}\|_{L^{s'}(\Omega)}\|\sqrt{\kappa(\vartheta)}\vartheta_t\| \\
&\le \frac{1}{4}\|\sqrt{\kappa(\vartheta)}\vartheta_t\|^2 + C\|\nabla\boldsymbol{u}\|^4_{L^{2s}(\Omega)},
\end{aligned}$$

for some exponents $s, s' > 2$ such that $\frac{1}{s} + \frac{1}{s'} = \frac{1}{2}$; we will specify these exponents below. Here we used (4.13). At this point we look for a suitable exponent s such that

$$\int_0^T \|\nabla\boldsymbol{u}(t)\|^4_{L^{2s}(\Omega)}\,\mathrm{d}t \le C.$$

First of all we apply (2.4) with the choices $r := 2s$ and $v := \nabla\boldsymbol{u}$. We have

$$\|\nabla\boldsymbol{u}\|^4_{L^{2s}(\Omega)} \le C\|\nabla\boldsymbol{u}\|^{\frac{4}{s}}\|\nabla\boldsymbol{u}\|^{4\left(1-\frac{1}{s}\right)}_{H^1(\Omega)}. \tag{4.17}$$

At this point we use (2.5) with the choices $s = 1$, $s_1 = 0$, $s_2 = 1 + r$. We obtain

$$\|\nabla\boldsymbol{u}\|_{H^1(\Omega)} \le C\|\nabla\boldsymbol{u}\|^{\frac{r}{r+1}}\|\nabla\boldsymbol{u}\|^{\frac{1}{r+1}}_{H^{r+1}(\Omega)}.$$

Hence, combining the latter estimate with (4.17), we deduce

$$
\begin{aligned}
\int_0^T \|\nabla \boldsymbol{u}\|_{L^{2s}(\Omega)}^4 &\leq C \int_0^T \|\nabla \boldsymbol{u}\|^{\frac{4}{s}} \|\nabla \boldsymbol{u}\|_{H^1(\Omega)}^{4\left(1-\frac{1}{s}\right)} \\
&\leq C \int_0^T \|\nabla \boldsymbol{u}\|^{\frac{4}{s}} \left[\|\nabla \boldsymbol{u}\|^{\frac{r}{r+1}} \|\nabla \boldsymbol{u}\|_{H^{r+1}(\Omega)}^{\frac{1}{r+1}} \right]^{4\left(1-\frac{1}{s}\right)} \\
&\leq C \int_0^T \|\nabla \boldsymbol{u}\|^{\frac{4}{s}+\frac{4r(s-1)}{(r+1)s}} \|\nabla \boldsymbol{u}\|_{H^{r+1}(\Omega)}^{\frac{4(s-1)}{s(r+1)}}.
\end{aligned}
$$

The first term in the last product is controlled uniformly in time since $\nabla \boldsymbol{u} \in L^\infty(0,T;L^2(\Omega))$. Hence the integral is bounded provided (see (i))

$$
\frac{4(s-1)}{s(r+1)} \leq 2 \Leftrightarrow s \leq \frac{2}{1-r}.
$$

We recall that $0 < r \leq 1/2$. So for instance if $r = 1/2$ then we can choose $s = 4$.

Finally, the estimate of the term IV can be done in the same way as we did for the term III (note that $\nabla\mu$ has even more regularity than $\nabla \boldsymbol{u}$). We then conclude by the Gronwall inequality that

$$
\vartheta \in H^1(0,T;L^2(\Omega)) \qquad K(\vartheta) \in L^\infty(0,T;V), \tag{4.18}
$$

so that (ii) holds true. Note that at this point it is crucial to assume $K(\vartheta_0) \in V$ in place of the sole property $\vartheta_0 \in V$ considered in [9].

• PROOF OF (iii). Now, $\varphi \in L^\infty(0,T;H^3(\Omega))$, that is (iii), follows easily: indeed, reading (1.4) as the elliptic equation $-\Delta\varphi + F'(\varphi) = \vartheta + \mu \in L^\infty(0,T;V)$, we readily obtain the desired conclusion.

• PROOF OF (iv). This is directly achieved from Lemma 4.1 applied over the interval $(0,T)$ with $\alpha = 2$. Note that we use here (2.21) and the assumption $(1/\vartheta_0) \in L^1(\Omega)$ resulting from the choice of $z_0 \in \mathcal{V}^r$.

4.3 Higher regularity - part II

We will now show that, under the assumptions of Theorem 3.1, the following additional regularity properties hold:

(v) $\boldsymbol{u} \in H^1(0,T;H^r(\Omega))$,

(vi) $\vartheta \in L^2(0,T;H^2(\Omega))$, $K(\vartheta) \in L^2(0,T;H^2(\Omega))$.

Actually, this completes the proof that z is a stable solution.

• PROOF OF (v). To prove the additional regularity for $\boldsymbol{u}$, we proceed as in the proof of (i) in section 4.2: we first project equation (1.2) into the space $\mathbb{H}$ by applying the operator $\mathbb{P}$, then we test by $A^r \boldsymbol{u}_t$. Setting $\hat{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_\Omega$ (so that $\hat{\boldsymbol{u}}_t = \boldsymbol{u}_t$) and recalling (4.5), we obtain

$$
\|\hat{\boldsymbol{u}}_t\|_{\mathbb{V}^r}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{\boldsymbol{u}}\|_{\mathbb{V}^{r+1}}^2 = -b(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}, A^r \hat{\boldsymbol{u}}_t) + \langle A^{r/2} f, A^{r/2} \hat{\boldsymbol{u}}_t \rangle.
$$

We can estimate the trilinear term working as in (4.10), namely we have

$$
-b(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}, A^r \hat{\boldsymbol{u}}_t) \leq C\|\hat{\boldsymbol{u}} \cdot \nabla \hat{\boldsymbol{u}}\|_{\mathbb{V}^r} \|\hat{\boldsymbol{u}}_t\|_{\mathbb{V}^r} \overset{(4.11)}{\leq} C\|\hat{\boldsymbol{u}}\|_{\mathbb{V}^2} \|\hat{\boldsymbol{u}}_t\|_{\mathbb{V}^r} \leq \frac{1}{4}\|\hat{\boldsymbol{u}}_t\|_{\mathbb{V}^r}^2 + C\|\hat{\boldsymbol{u}}\|_{\mathbb{V}^2}^2. \tag{4.19}
$$

On the other hand,

$$
\begin{aligned}
\langle A^{r/2} f, A^{r/2} \hat{\boldsymbol{u}}_t \rangle &\leq \|f\|_{\mathbb{V}^r} \|\hat{\boldsymbol{u}}_t\|_{\mathbb{V}^r} \\
&\overset{(4.9)}{\leq} C \left(\|\varphi\|_{H^3(\Omega)}^{\frac{1+2r}{2}} + 1 \right) \|\hat{\boldsymbol{u}}_t\|_{\mathbb{V}^r} \leq \frac{1}{4}\|\hat{\boldsymbol{u}}_t\|_{\mathbb{V}^r}^2 + C \left(\|\varphi\|_{H^3(\Omega)}^{1+2r} + 1 \right).
\end{aligned} \tag{4.20}
$$

Summing up, we finally obtain (we use here the fact that $r \leq 1/2$)

$$\|\hat{\boldsymbol{u}}_t\|^2_{\mathbb{V}^r} + \frac{\mathrm{d}}{\mathrm{d}t}\|\hat{\boldsymbol{u}}\|^2_{\mathbb{V}^{r+1}} \leq C\left(\|\varphi\|^2_{H^3(\Omega)} + 1\right) + C\|\hat{\boldsymbol{u}}\|^2_{\mathbb{V}^2},$$

and the conclusion comes by (2.18) and (2.19).

• PROOF OF (vi). Comparing terms in equation (1.5) and using in particular (3.4) together with the already observed fact that $\boldsymbol{u}$ is bounded in the uniform norm, we easily deduce

$$\Delta K(\vartheta) \in L^2(0,T;L^2(\Omega)), \tag{4.21}$$

whence the second of (vi). Now we would like to conclude that $\vartheta \in L^2(0,T;H^2(\Omega))$. Actually, using the second (3.4) and interpolation, we first notice that

$$K(\vartheta) \in L^4(0,T;W^{1,4}(\Omega)),$$

which, in view of (2.11)), is equivalent to

$$\vartheta + \vartheta^{q+1} \in L^4(0,T;W^{1,4}(\Omega)). \tag{4.22}$$

We now observe that

$$-\Delta K(\vartheta) = -\kappa(\vartheta)\Delta\vartheta - \kappa'(\vartheta)|\nabla\vartheta|^2.$$

Therefore,

$$\int_0^T\int_\Omega(-\kappa(\vartheta)\Delta\vartheta - \kappa'(\vartheta)|\nabla\vartheta|^2)^2 = \|-\Delta K(\vartheta)\|^2_{L^2(0,T;L^2(\Omega))} \leq C.$$

Let us deal with the integral in the left hand side. We have

$$\int_0^T\int_\Omega(-\kappa(\vartheta)\Delta\vartheta - \kappa'(\vartheta)|\nabla\vartheta|^2)^2 =: \int_0^T\int_\Omega \kappa^2(\vartheta)|\Delta\vartheta|^2 + \int_0^T\int_\Omega(\kappa'(\vartheta))^2|\nabla\vartheta|^4 + J,$$

where

$$\begin{aligned} J &:= 2\int_0^T\int_\Omega \kappa(\vartheta)\kappa'(\vartheta)\Delta\vartheta|\nabla\vartheta|^2 \\ &\leq \frac{1}{2}\int_0^T\int_\Omega \kappa^2(\vartheta)|\Delta\vartheta|^2 + C\int_0^T\int_\Omega(\kappa'(\vartheta))^2|\nabla\vartheta|^4 \\ &= \frac{1}{2}\int_0^T\int_\Omega \kappa^2(\vartheta)|\Delta\vartheta|^2 + C\int_0^T\int_\Omega \vartheta^{2q-2}|\nabla\vartheta|^4, \end{aligned}$$

and the last term in the right hand side can be rewritten as

$$C\int_0^T\int_\Omega |\nabla\vartheta^{\frac{q+1}{2}}|^4,$$

which is controlled by (4.22) (in fact this implies that we can control all the intermediate powers of ϑ between 1 and $q+1$). Hence, using the fact that $\kappa(\vartheta) \geq 1$ and applying elliptic regularity, we finally come to the goal

$$\vartheta \in L^2(0,T;H^2(\Omega)). \tag{4.23}$$

5 Proof of Theorem 3.1 part II: uniqueness

We now prove the uniqueness part of Theorem 3.1. Let then $z_0 \in \mathcal{V}^r$, and let $(\boldsymbol{u}_i, \varphi_i, \mu_i, \vartheta_i)$, $i = 1,2$, be a couple of stable solutions both emanating from z_0 over the interval $(0,T)$. Setting $(\boldsymbol{u}, \varphi, \mu, \vartheta) :=$

$(\boldsymbol{u}_1-\boldsymbol{u}_2,\varphi_1-\varphi_2,\mu_1-\mu_2,\vartheta_1-\vartheta_2)$, it is readily seen that, then,

$$\operatorname{div}\boldsymbol{u}=0 \tag{5.1}$$

$$\boldsymbol{u}_t+\boldsymbol{u}_1\cdot\nabla\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}_2=\Delta\boldsymbol{u}-\operatorname{div}(\nabla\varphi_1\otimes\nabla\varphi)-\operatorname{div}(\nabla\varphi\otimes\nabla\varphi_2) \tag{5.2}$$

$$\varphi_t+\boldsymbol{u}_1\cdot\nabla\varphi+\boldsymbol{u}\cdot\nabla\varphi_2=\Delta\mu \tag{5.3}$$

$$\mu=-\Delta\varphi+F'(\varphi_1)-F'(\varphi_2)-\vartheta \tag{5.4}$$

$$\begin{aligned}&\vartheta_t+\boldsymbol{u}_1\cdot\nabla\vartheta+\boldsymbol{u}\cdot\nabla\vartheta_2+\vartheta_1\Delta\mu+\vartheta\Delta\mu_2-\Delta[K(\vartheta_1)-K(\vartheta_2)]\\&\qquad=(\nabla\boldsymbol{u}_1+\nabla\boldsymbol{u}_2)\cdot\nabla\boldsymbol{u}+(\nabla\mu_1+\nabla\mu_2)\cdot\nabla\mu\end{aligned} \tag{5.5}$$

supplemented with null initial data. This guarantees for instance that $\varphi_\Omega(t)=0$ and $\boldsymbol{u}_\Omega(t)=\mathbf{0}$ for all $t\geq 0$.

Taking advantage of the regularity properties (3.1)-(3.5), we then have (actually even something more is true)

$$\begin{cases}\|\boldsymbol{u}_i(t)\|_V+\|\varphi_i(t)\|_{H^3(\Omega)}+\|\mu_i(t)\|_V+\|\vartheta_i(t)\|_V\leq c, \quad t\in(0,T)\\ \|\boldsymbol{u}_i\|_{L^2(0,T;H^2(\Omega))}+\|\mu_i\|_{L^2(0,T;H^3(\Omega))}+\|\vartheta_i\|_{L^2(0,T;H^2(\Omega))}\leq c\end{cases} \tag{5.6}$$

for some positive constant c depending on T and on the initial data. We will use the above properties repeatedly in the sequel. The proof is actually based on the combination of several estimates, presented in separate subsections for more clarity.

5.1 Preliminary estimates

Having in mind to test (5.3) by $\mathcal{N}\varphi_t$, we first estimate $\nabla\mu$ multiplying (5.3) by $\mu-\mu_\Omega$. This gives

$$\begin{aligned}\|\nabla\mu\|^2&=-\langle\varphi_t,\mu-\mu_\Omega\rangle-\langle\boldsymbol{u}_1\cdot\nabla\varphi,\mu-\mu_\Omega\rangle-\langle\boldsymbol{u}\cdot\nabla\varphi_2,\mu-\mu_\Omega\rangle\\&\leq\|\varphi_t\|_{V'}\|\mu-\mu_\Omega\|_V+\|\boldsymbol{u}_1\|_{L^3(\Omega)}\|\nabla\varphi\|\|\mu-\mu_\Omega\|_{L^6(\Omega)}+\|\boldsymbol{u}\|\|\nabla\varphi_2\|_{L^4(\Omega)}\|\mu-\mu_\Omega\|_{L^4(\Omega)}.\end{aligned}$$

Then Friedrich's inequality and (5.6) yield

$$\|\nabla\mu\|^2\leq c(\|\varphi_t\|^2_{V'}+\|\nabla\varphi\|^2+\|\boldsymbol{u}\|^2). \tag{5.7}$$

Next, we control $\Delta\varphi$ by taking the product of (5.4) with $-\Delta\varphi$:

$$\begin{aligned}\|\Delta\varphi\|^2&=-\langle\mu-\mu_\Omega,\Delta\varphi\rangle-\langle\vartheta-\vartheta_\Omega,\Delta\varphi\rangle+\langle F'(\varphi_1)-F'(\varphi_2),\Delta\varphi\rangle\\&\leq c\|\Delta\varphi\|(\|\nabla\mu\|+\|\vartheta-\vartheta_\Omega\|+\|F'(\varphi_1)-F'(\varphi_2)\|)\end{aligned}$$

Recalling (F3) and exploiting Hölder's inequality with exponents 3 and 3/2 together with (5.6), we deduce

$$\|F'(\varphi_1)-F'(\varphi_2)\|^2\leq c\langle(1+|\varphi_1|^{2p_F}+|\varphi_2|^{2p_F}),\varphi^2\rangle\leq c\|\varphi\|^2_{L^3(\Omega)}\leq c\|\varphi\|^2_V. \tag{5.8}$$

Thus, also on account of (5.7) we conclude that

$$\|\Delta\varphi\|^2\leq c(\|\nabla\mu\|^2+\|\vartheta-\vartheta_\Omega\|^2+\|\varphi\|^2_V)\leq c(\|\boldsymbol{u}\|^2+\|\varphi_t\|^2_{V'}+\|\vartheta-\vartheta_\Omega\|^2+\|\varphi\|^2_V). \tag{5.9}$$

5.2 Difference of fluid velocities

The product of (5.2) by $\boldsymbol{u}$, on account of (5.1), leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|^2+\|\nabla\boldsymbol{u}\|^2=-\langle\boldsymbol{u}\cdot\nabla\boldsymbol{u}_2,\boldsymbol{u}\rangle+\langle\nabla\varphi_1\otimes\nabla\varphi,\nabla\boldsymbol{u}\rangle+\langle\nabla\varphi\otimes\nabla\varphi_2,\nabla\boldsymbol{u}\rangle, \tag{5.10}$$

where the right hand side is easily controlled thanks to (5.6). Indeed, by (2.2) and (2.6),

$$\begin{aligned}\text{RHS}&\leq\|\nabla\boldsymbol{u}_2\|\|\boldsymbol{u}\|^2_{L^4(\Omega)}+(\|\nabla\varphi_1\|_{L^\infty(\Omega)}+\|\nabla\varphi_2\|_{L^\infty(\Omega)})\|\nabla\varphi\|\|\nabla\boldsymbol{u}\|\\&\leq\frac{1}{2}\|\nabla\boldsymbol{u}\|^2+c\big[\|\boldsymbol{u}\|^2+(\|\varphi_1\|_{H^3(\Omega)}+\|\varphi_2\|_{H^3(\Omega)})\|\nabla\varphi\|^2\big].\end{aligned}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|^2+\|\nabla\boldsymbol{u}\|^2\overset{(5.6)}{\leq}c(\|\boldsymbol{u}\|^2+\|\nabla\varphi\|^2). \tag{5.11}$$

5.3 Difference of temperatures' means

Integrating (5.5) over Ω we obtain

$$|\Omega|(\vartheta_\Omega)_t = \langle \nabla\vartheta_1, \nabla\mu\rangle - \langle \vartheta - \vartheta_\Omega, \Delta\mu_2\rangle + \langle(\nabla\boldsymbol{u}_1 + \nabla\boldsymbol{u}_2), \nabla\boldsymbol{u}\rangle + \langle(\nabla\mu_1 + \nabla\mu_2), \nabla\mu\rangle, \tag{5.12}$$

which, by (5.6), entails

$$|(\vartheta_\Omega)_t| \le c(\|\nabla\mu\| + \|\mu_2\|_{H^3(\Omega)}\|\vartheta - \vartheta_\Omega\|_{V'} + \|\nabla\boldsymbol{u}\|), \tag{5.13}$$

Moreover, the product of (5.12) by ϑ_Ω gives, for (small) $\alpha, \beta > 0$ to be chosen later and corresponding (large) $c > 0$,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}{\vartheta_\Omega}^2 &\le c|\vartheta_\Omega|(\|\nabla\mu\| + \|\mu_2\|_{H^3(\Omega)}\|\vartheta - \vartheta_\Omega\|_{V'} + \|\nabla\boldsymbol{u}\|) \\ &\le \beta\|\nabla\boldsymbol{u}\|^2 + \alpha\varepsilon\|\nabla\mu\|^2 + c({\vartheta_\Omega}^2 + \|\mu_2\|^2_{H^3(\Omega)}\|\vartheta - \vartheta_\Omega\|^2_{V'}).\end{aligned} \tag{5.14}$$

5.4 Difference of order parameters

Multiplying (5.3) by $\mathcal{N}\varphi_t$, we obtain

$$\|\varphi_t\|^2_{V'} + \langle\mu, \varphi_t\rangle = -\langle \boldsymbol{u}_1 \cdot \nabla\varphi, \mathcal{N}\varphi_t\rangle - \langle \boldsymbol{u} \cdot \nabla\varphi_2, \mathcal{N}\varphi_t\rangle,$$

where, taking the product of (5.4) by φ_t, the second term reads

$$\langle\mu, \varphi_t\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla\varphi\|^2 - 2\langle\vartheta - \vartheta_\Omega, \varphi\rangle\big) + \langle F'(\varphi_1) - F'(\varphi_2), \varphi_t\rangle + \langle\vartheta_t, \varphi\rangle.$$

Combining the above relations, we then deduce

$$\begin{aligned}&\|\varphi_t\|^2_{V'} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla\varphi\|^2 - 2\langle\vartheta - \vartheta_\Omega, \varphi\rangle) \\ &\quad= -\langle F'(\varphi_1) - F'(\varphi_2), \varphi_t\rangle - \langle \boldsymbol{u}_1 \cdot \nabla\varphi, \mathcal{N}\varphi_t\rangle - \langle \boldsymbol{u} \cdot \nabla\varphi_2, \mathcal{N}\varphi_t\rangle - \langle\vartheta_t, \varphi\rangle.\end{aligned} \tag{5.15}$$

In order to estimate the fourth term on the right hand side, we multiply (5.5) by φ:

$$\begin{aligned}\langle\vartheta_t, \varphi\rangle &= \langle K(\vartheta_1) - K(\vartheta_2), \Delta\varphi\rangle + \langle \boldsymbol{u}_1(\vartheta - \vartheta_\Omega), \nabla\varphi\rangle - \langle \boldsymbol{u} \cdot \nabla\vartheta_2, \varphi\rangle + \langle\nabla\mu, \nabla\vartheta_1\,\varphi\rangle \\ &\quad+ \langle\nabla\mu, \vartheta_1\nabla\varphi\rangle - \langle\vartheta\Delta\mu_2, \varphi\rangle + \langle(\nabla\boldsymbol{u}_1 + \nabla\boldsymbol{u}_2) \cdot \nabla\boldsymbol{u} + (\nabla\mu_1 + \nabla\mu_2) \cdot \nabla\mu, \varphi\rangle.\end{aligned}$$

Using (5.4), the first term on the right hand side gives

$$\begin{aligned}&\langle K(\vartheta_1) - K(\vartheta_2), \Delta\varphi\rangle \\ &\quad= \langle K(\vartheta_1) - K(\vartheta_2), -\mu + \mu_\Omega + F'(\varphi_1) - F'(\varphi_2) - \vartheta + \vartheta_\Omega\rangle - (\mu_\Omega + \vartheta_\Omega)\int_\Omega [K(\vartheta_1) - K(\vartheta_2)] \\ &\quad= \langle K(\vartheta_1) - K(\vartheta_2), -\mu + \mu_\Omega + F'(\varphi_1) - F'(\varphi_2) - (F'(\varphi_1) - F'(\varphi_2))_\Omega - \vartheta + \vartheta_\Omega\rangle \\ &\quad= \langle K(\vartheta_1) - K(\vartheta_2), -\mu + \mu_\Omega + F'(\varphi_1) - F'(\varphi_2) - (F'(\varphi_1) - F'(\varphi_2))_\Omega\rangle \\ &\qquad\quad - \langle K(\vartheta_1) - K(\vartheta_2), \vartheta - \vartheta_\Omega\rangle.\end{aligned}$$

Hence,

$$\begin{aligned}\langle\vartheta_t, \varphi\rangle &= -\langle K(\vartheta_1) - K(\vartheta_2), \mu - \mu_\Omega - F'(\varphi_1) + F'(\varphi_2) + (F'(\varphi_1) - F'(\varphi_2))_\Omega\rangle \\ &\quad - \langle K(\vartheta_1) - K(\vartheta_2), \vartheta - \vartheta_\Omega\rangle + \langle \boldsymbol{u}_1(\vartheta - \vartheta_\Omega), \nabla\varphi\rangle - \langle \boldsymbol{u} \cdot \nabla\vartheta_2, \varphi\rangle + \langle\nabla\mu, \nabla\vartheta_1\,\varphi\rangle \\ &\quad + \langle\nabla\mu, \vartheta_1\nabla\varphi\rangle - \langle\vartheta\Delta\mu_2, \varphi\rangle + \langle(\nabla\boldsymbol{u}_1 + \nabla\boldsymbol{u}_2) \cdot \nabla\boldsymbol{u} + (\nabla\mu_1 + \nabla\mu_2) \cdot \nabla\mu, \varphi\rangle.\end{aligned}$$

Combining the above relation with (5.15), we then deduce

$$\begin{aligned}&\|\varphi_t\|^2_{V'} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla\varphi\|^2 - 2\langle\vartheta - \vartheta_\Omega, \varphi\rangle) - \langle K(\vartheta_1) - K(\vartheta_2), \vartheta - \vartheta_\Omega\rangle \\ &\quad= -\langle F'(\varphi_1) - F'(\varphi_2), \varphi_t\rangle + \langle \boldsymbol{u}_1\varphi + \boldsymbol{u}\varphi_2, \nabla\mathcal{N}\varphi_t\rangle \\ &\qquad + \langle K(\vartheta_1) - K(\vartheta_2), \mu - \mu_\Omega - F'(\varphi_1) + F'(\varphi_2) + (F'(\varphi_1) - F'(\varphi_2))_\Omega\rangle \\ &\qquad - \langle \boldsymbol{u}_1(\vartheta - \vartheta_\Omega), \nabla\varphi\rangle + \langle \boldsymbol{u} \cdot \nabla\vartheta_2, \varphi\rangle \\ &\qquad - \langle\nabla\mu, \nabla\vartheta_1\,\varphi\rangle - \langle\nabla\mu, \vartheta_1\nabla\varphi\rangle + \langle\vartheta\Delta\mu_2, \varphi\rangle \\ &\qquad - \langle(\nabla\boldsymbol{u}_1 + \nabla\boldsymbol{u}_2) \cdot \nabla\boldsymbol{u} + (\nabla\mu_1 + \nabla\mu_2) \cdot \nabla\mu, \varphi\rangle =: \mathcal{I}_1.\end{aligned} \tag{5.16}$$

Owing to (F3), we have

$$\|\nabla(F'(\varphi_1) - F'(\varphi_2))\| \le c\| |\nabla\varphi|(1+|\varphi_1|^{p_F}) + |\varphi|(1+|\varphi_1|^{p_F-1} + |\varphi_2|^{p_F-1})|\nabla\varphi_2| \| \le c\|\varphi\|_V. \quad (5.17)$$

Thus, recalling that $\langle\varphi_t, 1\rangle = 0$, we obtain

$$-\langle F'(\varphi_1) - F'(\varphi_2), \varphi_t\rangle \le C\|\nabla(F'(\varphi_1) - F'(\varphi_2))\|\|\varphi_t\|_{V'} \le c\|\varphi\|_V\|\varphi_t\|_{V'}.$$

Next, owing to (2.2) and (2.6),

$$\begin{aligned}\langle \boldsymbol{u}_1\varphi + \boldsymbol{u}\varphi_2, \nabla\mathcal{N}\varphi_t\rangle &\le c\|\varphi_t\|_{V'}(\|\boldsymbol{u}_1\|_{L^4(\Omega)}\|\varphi\|_{L^4(\Omega)} + \|\boldsymbol{u}\|\|\varphi_2\|_{L^\infty(\Omega)})\\ &\le c\|\varphi_t\|_{V'}(\|\nabla\varphi\| + \|\boldsymbol{u}\|).\end{aligned}$$

Having observed that

$$\|K(\vartheta_1) - K(\vartheta_2)\|_{3/2} \le c(\|\vartheta - \vartheta_\Omega\| + |\vartheta_\Omega|)(1 + \|\vartheta_1\|_V^q + \|\vartheta_2\|_V^q) \le c(\|\vartheta-\vartheta_\Omega\| + |\vartheta_\Omega|),$$

thanks to (5.17), it is straightforward to obtain

$$\begin{aligned}&\langle K(\vartheta_1) - K(\vartheta_2), \mu - \mu_\Omega - F'(\varphi_1) + F'(\varphi_2) + (F'(\varphi_1) - F'(\varphi_2))_\Omega\rangle\\ &\le \|K(\vartheta_1) - K(\vartheta_2)\|_{L^{3/2}(\Omega)}\big(\|\mu - \mu_\Omega\|_{L^3(\Omega)} + \|F'(\varphi_1) - F'(\varphi_2) - (F'(\varphi_1) - F'(\varphi_2))_\Omega\|_{L^3(\Omega)}\big)\\ &\le c(\|\vartheta - \vartheta_\Omega\| + |\vartheta_\Omega|)(\|\nabla\mu\| + \|\varphi\|_V).\end{aligned}$$

Next, also on account of Agmon's inequality, we can estimate the remaining summands in $\mathcal{I}_1$ as follows:

$$\begin{aligned}&-\langle \boldsymbol{u}_1(\vartheta - \vartheta_\Omega), \nabla\varphi\rangle + \langle \boldsymbol{u}\cdot\nabla\vartheta_2, \varphi\rangle\\ &\quad\le \|\boldsymbol{u}_1\|_{L^\infty(\Omega)}\|\vartheta - \vartheta_\Omega\|\|\nabla\varphi\| + \|\boldsymbol{u}\|\|\nabla\vartheta_2\|_{L^4(\Omega)}\|\varphi\|_{L^4(\Omega)}\\ &\quad\le c\left(\|\boldsymbol{u}_1\|_{H^2(\Omega)}^{1/2}\|\vartheta - \vartheta_\Omega\| + \|\boldsymbol{u}\|\|\vartheta_2\|_{H^2(\Omega)}\right)\|\nabla\varphi\|;\end{aligned}$$

By (2.2), (2.3) and (2.6)

$$\begin{aligned}-\langle\nabla\mu, \nabla\vartheta_1\varphi + \vartheta_1\nabla\varphi\rangle &\le c\|\nabla\mu\|\left(\|\nabla\vartheta_1\|\|\varphi\|_{L^\infty(\Omega)} + \|\nabla\varphi\|_{L^4(\Omega)}\|\vartheta_1\|_{L^4(\Omega)}\right)\\ &\le c\|\nabla\mu\|\left(\|\varphi\|^{1/2}\|\varphi\|_{H^2(\Omega)}^{1/2} + \|\nabla\varphi\|^{1/2}\|\varphi\|_{H^2(\Omega)}^{1/2}\right)\\ &\le c\|\nabla\mu\|\|\nabla\varphi\|^{1/2}\|\varphi\|_{H^2(\Omega)}^{1/2};\end{aligned}$$

Exploiting (2.6) and the injection $V \subset L^p(\Omega)$, for $p \ge 1$,

$$\begin{aligned}\langle\vartheta\Delta\mu_2, \varphi\rangle &= \vartheta_\Omega\langle\Delta\mu_2, \varphi\rangle + \langle(\vartheta - \vartheta_\Omega)\Delta\mu_2, \varphi\rangle\\ &\le c|\vartheta_\Omega|\|\nabla\mu_2\|\|\nabla\varphi\| + c\|\vartheta - \vartheta_\Omega\|\|\Delta\mu_2\|_{L^4(\Omega)}\|\varphi\|_{L^4(\Omega)}\\ &\le c|\vartheta_\Omega|\|\nabla\varphi\| + c\|\vartheta - \vartheta_\Omega\|\|\mu_2\|_{H^3(\Omega)}\|\nabla\varphi\|;\end{aligned}$$

Finally, by the same token and (2.2),

$$\begin{aligned}&-\langle(\nabla\boldsymbol{u}_1 + \nabla\boldsymbol{u}_2)\cdot\nabla\boldsymbol{u} + (\nabla\mu_1 + \nabla\mu_2)\cdot\nabla\mu, \varphi\rangle\\ &\quad\le (\|\nabla\boldsymbol{u}_1 + \nabla\boldsymbol{u}_2\|_{L^4(\Omega)}\|\nabla\boldsymbol{u}\| + \|\nabla\mu_1 + \nabla\mu_2\|_{L^4(\Omega)}\|\nabla\mu\|)\|\varphi\|_{L^4(\Omega)}\\ &\le \big[(\|\boldsymbol{u}_1\|_{H^2(\Omega)}^{1/2} + \|\boldsymbol{u}_2\|_{H^2(\Omega)}^{1/2})\|\nabla\boldsymbol{u}\| + (\|\mu_1\|_{H^2(\Omega)}^{1/2} + \|\mu_2\|_{H^2(\Omega)}^{1/2})\|\nabla\mu\|\big]\|\nabla\varphi\|.\end{aligned}$$

Collecting the above computations, we finally arrive at

$$\begin{aligned}\mathcal{I}_1 &\le c(1 + \|\boldsymbol{u}_1\|_{H^2(\Omega)}^{1/2} + \|\boldsymbol{u}_2\|_{H^2(\Omega)}^{1/2})\|\nabla\boldsymbol{u}\|\|\nabla\varphi\| + c(1 + \|\mu_1\|_{H^2(\Omega)}^{1/2} + \|\mu_2\|_{H^2(\Omega)}^{1/2})\|\nabla\mu\|\|\nabla\varphi\|\\ &\quad + c\|\nabla\mu\|\|\nabla\varphi\|^{1/2}\|\varphi\|_{H^2(\Omega)}^{1/2} + c(\|\vartheta - \vartheta_\Omega\| + |\vartheta_\Omega|)\|\nabla\mu\|\\ &\quad + c(\|\vartheta - \vartheta_\Omega\| + |\vartheta_\Omega|)(1 + \|\mu_2\|_{H^3(\Omega)})\|\varphi\|_V + c\|\varphi_t\|_{V'}(\|\varphi\|_V + \|\boldsymbol{u}\|)\\ &\quad + c\|\vartheta - \vartheta_\Omega\|\|\boldsymbol{u}_1\|_{H^2(\Omega)}^{1/2}\|\nabla\varphi\| + c\|\vartheta_2\|_{H^2(\Omega)}\|\boldsymbol{u}\|\|\nabla\varphi\|\\ &\le \frac{1}{2}\|\varphi_t\|_{V'}^2 + \frac{\alpha}{2}\|\nabla\mu\|^2 + \frac{c}{\alpha}\|\vartheta - \vartheta_\Omega\|^2 + \frac{\beta}{2}\|\nabla\boldsymbol{u}\|^2 + \frac{\alpha}{2}\|\Delta\varphi\|^2 + g(t)(\|\varphi\|_V^2 + \|\boldsymbol{u}\|^2 + \vartheta_\Omega{}^2)\\ &\le \frac{1}{2}\|\varphi_t\|_{V'}^2 + \left(\frac{\alpha}{2} + \alpha c\right)\|\nabla\mu\|^2 + \left(\frac{c}{\alpha} + \alpha c\right)\|\vartheta - \vartheta_\Omega\|^2 + \frac{\beta}{2}\|\nabla\boldsymbol{u}\|^2 + g(t)(\|\varphi\|_V^2 + \|\boldsymbol{u}\|^2 + \vartheta_\Omega{}^2).\end{aligned}$$

In the last passage we have used (5.4) to control the term depending on $\Delta\varphi$, namely we noted that

$$\Delta\varphi = \Delta\varphi - (\Delta\varphi)_\Omega = F'(\varphi_1) - F'(\varphi_2) - (F'(\varphi_1) - F'(\varphi_2))_\Omega - (\vartheta - \vartheta_\Omega) - (\mu - \mu_\Omega); \tag{5.18}$$

moreover, we have set

$$g(t) := c[1 + \|\vartheta_2\|^2_{H^2(\Omega)} + \|\boldsymbol{u}_1\|^2_{H^2(\Omega)} + \|\boldsymbol{u}_2\|^2_{H^2(\Omega)} + \|\mu_1\|^2_{H^3(\Omega)} + \|\mu_2\|^2_{H^3(\Omega)}], \tag{5.19}$$

where the (large) constant $c > 0$ also depends on the choice of the (small) constants $\alpha, \beta > 0$ (the letters α and β were already used before: we may take the smaller of the two choices for α, β). Collecting the above estimates, (5.16) finally gives

$$\begin{aligned} &\|\varphi_t\|^2_{V'} + \frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla\varphi\|^2 - 2\langle \vartheta - \vartheta_\Omega, \varphi\rangle) - 2\langle K(\vartheta_1) - K(\vartheta_2), \vartheta - \vartheta_\Omega\rangle \\ &\le \alpha(1 + 2c)\|\nabla\mu\|^2 + c\left(\frac{1}{\alpha} + \alpha\right)\|\vartheta - \vartheta_\Omega\|^2 + \beta\|\nabla\boldsymbol{u}\|^2 + g(t)(\|\varphi\|^2_V + \|\boldsymbol{u}\|^2 + {\vartheta_\Omega}^2). \end{aligned} \tag{5.20}$$

5.5 Difference of temperatures

Multiplying (5.5) by $\mathcal{N}(\vartheta - \vartheta_\Omega)$ and integrating by parts, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vartheta - \vartheta_\Omega\|^2_{V'} + \langle K(\vartheta_1) - K(\vartheta_2), \vartheta - \vartheta_\Omega\rangle \\ &\quad = \langle \boldsymbol{u}_1(\vartheta - \vartheta_\Omega), \nabla\mathcal{N}(\vartheta - \vartheta_\Omega)\rangle - \langle \boldsymbol{u}\cdot\nabla\vartheta_2, \mathcal{N}(\vartheta - \vartheta_\Omega)\rangle \\ &\qquad + \langle \nabla\vartheta_1\cdot\nabla\mu, \mathcal{N}(\vartheta - \vartheta_\Omega)\rangle + \langle \vartheta_1\nabla\mu, \nabla\mathcal{N}(\vartheta - \vartheta_\Omega)\rangle \\ &\qquad - \langle (\vartheta - \vartheta_\Omega)\Delta\mu_2, \mathcal{N}(\vartheta - \vartheta_\Omega)\rangle + \vartheta_\Omega\langle \mu_2 - (\mu_2)_\Omega, \vartheta - \vartheta_\Omega\rangle \\ &\qquad + \langle (\nabla\boldsymbol{u}_1 + \nabla\boldsymbol{u}_2)\cdot\nabla\boldsymbol{u} + (\nabla\mu_1 + \nabla\mu_2)\cdot\nabla\mu, \mathcal{N}(\vartheta - \vartheta_\Omega)\rangle =: I_1 + I_2 + I_3, \end{aligned}$$

where the terms I_i, $i = 1, 2, 3$, will be specified below. First of all, for any $\sigma > 0$, by interpolation we have

$$\begin{aligned} I_1 &:= \langle \boldsymbol{u}_1(\vartheta - \vartheta_\Omega) + \vartheta_1\nabla\mu, \nabla\mathcal{N}(\vartheta - \vartheta_\Omega)\rangle \\ &\le c\|\boldsymbol{u}_1\|_\infty\|\vartheta - \vartheta_\Omega\|\|\vartheta - \vartheta_\Omega\|_{V'} + c\|\vartheta_1\|_{L^{(4+2\sigma)/\sigma}(\Omega)}\|\nabla\mu\|\|\nabla\mathcal{N}(\vartheta - \vartheta_\Omega)\|_{L^{2+\sigma}(\Omega)} \\ &\overset{(2.4)}{\le} c\|\boldsymbol{u}_1\|^{1/2}_{H^2(\Omega)}\|\vartheta - \vartheta_\Omega\|\|\vartheta - \vartheta_\Omega\|_{V'} + c\|\nabla\mu\|\|\nabla\mathcal{N}(\vartheta - \vartheta_\Omega)\|^{2/(2+\sigma)}\|\nabla\mathcal{N}(\vartheta - \vartheta_\Omega)\|^{\sigma/(2+\sigma)}_V \\ &\le c\|\boldsymbol{u}_1\|^{1/2}_{H^2(\Omega)}\|\vartheta - \vartheta_\Omega\|\|\vartheta - \vartheta_\Omega\|_{V'} + c\|\nabla\mu\|\|\vartheta - \vartheta_\Omega\|^{2/(2+\sigma)}_{V'}\|\vartheta - \vartheta_\Omega\|^{\sigma/(2+\sigma)}. \end{aligned}$$

Next, using (2.3), we notice that

$$\|\mathcal{N}(\vartheta - \vartheta_\Omega)\|_{L^\infty(\Omega)} \le c\|\vartheta - \vartheta_\Omega\|^{1/2}_{V'}\|\vartheta - \vartheta_\Omega\|^{1/2}.$$

Therefore,

$$\begin{aligned} I_2 &:= -\langle \boldsymbol{u}\cdot\nabla\vartheta_2 + \nabla\vartheta_1\cdot\nabla\mu - (\vartheta - \vartheta_\Omega)\Delta\mu_2, \mathcal{N}(\vartheta - \vartheta_\Omega)\rangle \\ &\quad + \langle (\nabla\boldsymbol{u}_1 + \nabla\boldsymbol{u}_2)\cdot\nabla\boldsymbol{u} + (\nabla\mu_1 + \nabla\mu_2)\cdot\nabla\mu, \mathcal{N}(\vartheta - \vartheta_\Omega)\rangle \\ &\le \|\mathcal{N}(\vartheta - \vartheta_\Omega)\|_{L^\infty(\Omega)}\big[\|\vartheta_2\|_V\|\boldsymbol{u}\| + \|\vartheta_1\|_V\|\nabla\mu\| + \|\mu_2\|_{H^2(\Omega)}\|\vartheta - \vartheta_\Omega\| \\ &\quad + (\|\nabla\boldsymbol{u}_1\| + \|\nabla\boldsymbol{u}_2\|)\|\nabla\boldsymbol{u}\| + (\|\nabla\mu_2\| + \|\nabla\mu_2\|)\|\nabla\mu\|\big] \\ &\le c\|\vartheta - \vartheta_\Omega\|^{1/2}_{V'}\|\vartheta - \vartheta_\Omega\|^{1/2}(\|\boldsymbol{u}\| + \|\nabla\boldsymbol{u}\| + \|\nabla\mu\|) + c\|\mu_2\|_{H^2(\Omega)}\|\vartheta - \vartheta_\Omega\|^{1/2}_{V'}\|\vartheta - \vartheta_\Omega\|^{3/2}. \end{aligned}$$

Finally,

$$I_3 := \vartheta_\Omega\langle \mu_2 - (\mu_2)_\Omega, \vartheta - \vartheta_\Omega\rangle \le c|\vartheta_\Omega|\|\mu_2 - (\mu_2)_\Omega\|_V\|\vartheta - \vartheta_\Omega\|_{V'} \le c|\vartheta_\Omega|\|\vartheta - \vartheta_\Omega\|_{V'}.$$

Collecting the last three estimates and using Young's inequality, we eventually infer

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vartheta - \vartheta_\Omega\|^2_{V'} + \langle K(\vartheta_1) - K(\vartheta_2), \vartheta - \vartheta_\Omega\rangle \le \delta\varepsilon\|\vartheta - \vartheta_\Omega\|^2 \\ &\quad + \beta\|\nabla\boldsymbol{u}\|^2 + \alpha\varepsilon\|\nabla\mu\|^2 + C_*(1 + \|\boldsymbol{u}_1\|_{H^2(\Omega)} + \|\mu_2\|^2_{H^3(\Omega)})\|\vartheta - \vartheta_\Omega\|^2_{V'} + c({\vartheta_\Omega}^2 + \|\boldsymbol{u}\|^2), \end{aligned} \tag{5.21}$$

where the "large" constant C_* may depend on the "small" constants α, β, δ, ε whose value will be specified at the end.

5.6 Conclusion

In order to accomplish our purpose, we introduce the functional

$$\mathcal{Z}(t) := 2\|\boldsymbol{u}(t)\|^2 + \varepsilon\|\nabla\varphi(t)\|^2 - 2\varepsilon\langle\vartheta(t) - \vartheta_\Omega(t), \varphi(t)\rangle + \|\vartheta(t) - \vartheta_\Omega(t)\|_{V'}^2 + \vartheta_\Omega(t)^2,$$

noticing that, provided $\varepsilon > 0$ is small enough,

$$\mathcal{Z}(t) \geq c_\varepsilon(\|\boldsymbol{u}(t)\|^2 + \|\nabla\varphi(t)\|^2 + \|\vartheta(t) - \vartheta_\Omega(t)\|_{V'}^2 + \vartheta_\Omega(t)^2).$$

Now, adding together (5.21) with (5.11), (5.14) and $\frac{\varepsilon}{2}$ times (5.20), we see that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Z} + (1-\varepsilon)\langle K(\vartheta_1) - K(\vartheta_2), \vartheta - \vartheta_\Omega\rangle + \|\nabla\boldsymbol{u}\|^2 + \frac{\varepsilon}{2}\|\varphi_t\|_{V'}^2\\ &\quad\leq \varepsilon\left(c + c\alpha^2 + \delta\right)\|\vartheta - \vartheta_\Omega\|^2 + 3\alpha\varepsilon\|\nabla\mu\|^2 + 3\beta\|\nabla\boldsymbol{u}\|^2 + g(t)\mathcal{Z},\end{aligned}$$

where g was defined in (5.19).

We now develop the second term in the left hand side. Owing to (2.11), we actually have

$$\langle K(\vartheta_1) - K(\vartheta_2), \vartheta - \vartheta_\Omega\rangle = \|\vartheta - \vartheta_\Omega\|^2 + \frac{1}{q+1}\langle \ell(\vartheta_1) - \ell(\vartheta_2), \vartheta - \vartheta_\Omega\rangle,$$

having set $\ell(\vartheta_i) = \vartheta_i^{q+1}$, $i = 1, 2$. Now

$$\ell(\vartheta_1) - \ell(\vartheta_2) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\ell(s\vartheta_1 + (1-s)\vartheta_2)\,\mathrm{d}s = \int_0^1 \ell'(s\vartheta_1 + (1-s)\vartheta_2)(\vartheta_1 - \vartheta_2)\,\mathrm{d}s = \omega(\vartheta_1, \vartheta_2)\vartheta,$$

where

$$\omega(\vartheta_1, \vartheta_2) := \int_0^1 \ell'(s\vartheta_1 + (1-s)\vartheta_2)\,\mathrm{d}s$$

and we notice that $\omega(\vartheta_1, \vartheta_2) \geq 0$ almost everywhere. We also observe that, due to (4.13),

$$|\omega(\vartheta_1, \vartheta_2)| \leq c(1 + |\vartheta_1|^q + |\vartheta_2|^q). \tag{5.22}$$

Hence, it is not difficult to deduce

$$\begin{aligned}\langle K(\vartheta_1) - K(\vartheta_2), \vartheta - \vartheta_\Omega\rangle &= \|\vartheta - \vartheta_\Omega\|^2 + \frac{1}{q+1}\langle \ell(\vartheta_1) - \ell(\vartheta_2), \vartheta - \vartheta_\Omega\rangle\\ &\geq \|\vartheta - \vartheta_\Omega\|^2 + \frac{1}{q+1}\int_\Omega \omega(\vartheta_1, \vartheta_2)\vartheta(\vartheta - \vartheta_\Omega)\\ &= \|\vartheta - \vartheta_\Omega\|^2 + \frac{1}{q+1}\int_\Omega \omega(\vartheta_1, \vartheta_2)|\vartheta - \vartheta_\Omega|^2 + \frac{1}{q+1}\int_\Omega \omega(\vartheta_1, \vartheta_2)\vartheta_\Omega(\vartheta - \vartheta_\Omega)\\ &\geq \|\vartheta - \vartheta_\Omega\|^2 + \frac{1}{q+1}\int_\Omega \omega(\vartheta_1, \vartheta_2)\vartheta_\Omega(\vartheta - \vartheta_\Omega).\end{aligned}$$

Therefore, using (4.13) and (5.22), we arrive at

$$\left|\frac{1}{q+1}\int_\Omega \omega(\vartheta_1, \vartheta_2)\vartheta_\Omega(\vartheta - \vartheta_\Omega)\right| \leq c\|\omega(\vartheta_1, \vartheta_2)\||\vartheta_\Omega|\|\vartheta - \vartheta_\Omega\| \leq \frac{1}{2}\|\vartheta - \vartheta_\Omega\|^2 + c{\vartheta_\Omega}^2,$$

and, in turn,

$$\langle K(\vartheta_1) - K(\vartheta_2), \vartheta - \vartheta_\Omega\rangle \geq \frac{1}{2}\|\vartheta - \vartheta_\Omega\|^2 - c{\vartheta_\Omega}^2.$$

Summing up, we finally see that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Z} + \frac{1-\varepsilon}{2}\|\vartheta - \vartheta_\Omega\|^2 + \|\nabla\boldsymbol{u}\|^2 + \frac{\varepsilon}{2}\|\varphi_t\|_{V'}^2 \qquad (5.23)\\ &\quad\leq \varepsilon\left(c + c\alpha^2 + \delta\right)\|\vartheta - \vartheta_\Omega\|^2 + 3\alpha\varepsilon\|\nabla\mu\|^2 + 3\beta\|\nabla\boldsymbol{u}\|^2 + g(t)\mathcal{Z}\\ &\quad\overset{(5.7)}{\leq} \varepsilon\left(c + c\alpha^2 + \delta\right)\|\vartheta - \vartheta_\Omega\|^2 + 3\alpha\varepsilon c\|\varphi_t\|_{V'}^2 + 3\beta\|\nabla\boldsymbol{u}\|^2 + g(t)\mathcal{Z}\end{aligned}$$

where in deducing the second inequality we used (5.18), the Friedrichs inequality (2.6) together with (5.17), namely,

$$\|\nabla(F'(\varphi_1) - F'(\varphi_2))\| \le c\|\nabla\varphi\|,$$

as one can verify directly by using assumption (F3), the previous estimates (cf. (5.6)), and the fact that $\varphi_\Omega = 0$. Moreover, the last inequality in (5.23) follows easily by comparing terms in (5.3) and using once more (5.6). In particular, we can observe that

$$\|\nabla\mu\| \le c\big(\|\varphi_t\|_{V'} + \|\nabla\varphi\| + \|\boldsymbol{u}\|\big).$$

In particular, we may notice that the constant c on the right hand side of (5.23) is *independent* of the parameters $\alpha, \beta, \delta, \varepsilon$ (in fact it depends only on the regularity properties of solutions collected in (5.6)). As a consequence, we can first choose $\alpha, \beta, \delta > 0$ small enough in order to absorb the last two terms on the right hand side of (5.23) with the corresponding quantities appearing in the left hand side. In a second stage, we also take ε sufficiently small (possibly depending on the other parameters), in such a way that also the first term on the right hand side is absorbed. Consequently, (5.23) eventually reduces to the simpler form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Z} + \kappa_0\left(\|\vartheta - \vartheta_\Omega\|^2 + \|\nabla\boldsymbol{u}\|^2 + \|\varphi_t\|_{V'}^2\right) \le g(t)\mathcal{Z},$$

where $\kappa_0 > 0$ and g has the same expression as in (5.19) and therefore, owing to (5.6), it is summable over the interval $(0, T)$. Being $\mathcal{Z}(0) = 0$, then using Gronwall's lemma we deduce that $\mathcal{Z}$ is identically 0 over $(0, T)$, whence the assert.

6 Proof of Theorem 3.5

The proof is divided into various steps that are discussed in separate subsections and in some occasion presented as single Lemmas. As before, Q will denote a computable positive function, increasingly monotone in each of its arguments, whose expression is independent of time unless otherwise specified. Hypotheses (F1)-(F3) and (K1) on the nonlinear terms will be always implicitly assumed in the sequel.

6.1 Uniform estimates

Theorem 3.1 provides existence and uniqueness of "stable solutions" on fixed time intervals of arbitrary length. Our first step consists in showing that the $\mathcal{V}^r$-magnitude of any such solution remains bounded in a way only depending on the initial data also as time goes to infinity.

Lemma 6.1. *Any global solution $z(t)$ originating from an initial datum $z_0 \in \mathcal{V}^r$ satisfies*

$$\|z(t)\|_{\mathcal{H}} + |\mu_\Omega(t)| \le Q(\|z_0\|_{\mathcal{H}}) \quad \forall\, t \ge 0. \tag{6.1}$$

Moreover, we have the dissipation integrals

$$\int_0^\infty \int_\Omega \left(|\nabla\boldsymbol{u}(s)|^2 + |\nabla\vartheta(s)|^2 + |\nabla\mu(s)|^2\right) \mathrm{d}s + \sup_{t\ge0}\int_t^{t+1}\int_\Omega |\nabla\Delta\varphi(s)|^2 \,\mathrm{d}s \le Q(\|z_0\|_{\mathcal{H}}). \tag{6.2}$$

Proof. Arguing as in [9], we get the energy estimate, corresponding to the energy conservation principle, testing (1.2) by $\boldsymbol{u}$, (1.3) by μ, (1.4) by φ_t and (1.5) by 1, integrating over Ω, and summing all the obtained relation together. Namely, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\boldsymbol{u}, \varphi, \vartheta) = 0,$$

where

$$\mathcal{E}(\boldsymbol{u}, \varphi, \vartheta) = \int_\Omega \left(\frac{1}{2}|\boldsymbol{u}|^2 + \frac{1}{2}|\nabla\varphi|^2 + F(\varphi) + \vartheta\right).$$

This entails in particular that

$$\mathcal{E}(z(t)) = \mathcal{E}(z_0) \qquad \forall\, t \ge 0.$$

Therefore, using (2.10), we obtain

$$\frac{1}{2}\|\boldsymbol{u}(t)\|^2 + \frac{1}{2}\|\nabla\varphi(t)\|^2 + \int_\Omega \vartheta(t) - c_0 \le \mathcal{E}(z(t)) = \mathcal{E}(z_0) \le Q(\|z_0\|_{\mathcal{H}}). \tag{6.3}$$

Next, arguing once more as in [9], we obtain the entropy estimate, corresponding to the entropy production principle. It is obtained by testing (1.5) by $-\vartheta^{-1}$ and integrating over Ω, which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (-\log\vartheta) + \int_\Omega \frac{1}{\vartheta}\left(|\nabla\boldsymbol{u}|^2 + |\nabla\mu|^2\right) + \int_\Omega \frac{\kappa(\vartheta)}{\vartheta^2}|\nabla\vartheta|^2 = 0. \tag{6.4}$$

Then, integrating in time over $(0,t)$ and recalling that, due to (2.13),

$$\varphi(t)_\Omega = (\varphi_0)_\Omega, \tag{6.5}$$

we deduce first of all that

$$-\int_\Omega \log\vartheta(t) \le -\int_\Omega \log\vartheta_0. \tag{6.6}$$

Summing the above to (6.3) and using that $|\log r| \le r - \log r$ for all $r > 0$, we then obtain

$$\|\log\vartheta(t)\|_{L^1(\Omega)} \le Q(\|z_0\|_{\mathcal{H}}), \tag{6.7}$$

for every $t \ge 0$. Combining (6.3) and (6.7), we then deduce

$$\|z(t)\|_{\mathcal{H}} = \|\boldsymbol{u}(t)\| + \|\varphi(t)\|_V + \|\vartheta(t)\|_{L^1(\Omega)} + \|\log\vartheta(t)\|_{L^1(\Omega)} \le Q(\|z_0\|_{\mathcal{H}}), \tag{6.8}$$

where the full V-norm of $\varphi(t)$ is controlled also in view of (6.5). Finally, by (1.4), we have

$$\mu(t)_\Omega = F'(\varphi(t))_\Omega - \vartheta(t)_\Omega.$$

Hence, recalling also (F3),

$$|\mu(t)_\Omega| \le Q(\|\varphi(t)\|_V, \vartheta(t)_\Omega) \le Q(\|z_0\|_{\mathcal{H}}) \quad \forall\, t \ge 0,$$

and this result together with (6.8) brings (6.1).

Let us now deal with (6.2). Integrating the entropy estimate (6.4) over $(0,t)$ for a generic $t > 0$, we deduce

$$\begin{aligned} 0 &\le \int_0^t \int_\Omega \frac{1}{\vartheta}\left(|\nabla\boldsymbol{u}|^2 + |\nabla\mu|^2\right) + \int_0^t \int_\Omega \frac{\kappa(\vartheta)}{\vartheta^2}|\nabla\vartheta|^2 \\ &= \int_\Omega \log\vartheta(t) - \int_\Omega \log\vartheta_0 \overset{(6.1)}{\le} Q(\|z_0\|_{\mathcal{H}}). \end{aligned} \tag{6.9}$$

Arguing as in [9], we then deduce

$$\begin{aligned} \int_0^t \int_\Omega |\nabla\vartheta|^2 &\le c\int_0^t \int_\Omega \left(\frac{1}{\vartheta^2} + k_q\vartheta^{q-2}\right)|\nabla\vartheta|^2 \\ &\le c\int_0^t \int_\Omega \frac{\kappa(\vartheta)}{\vartheta^2}|\nabla\vartheta|^2 \overset{(6.9)}{\le} Q(\|z_0\|_{\mathcal{H}}) \quad \forall\, t \ge 0, \end{aligned} \tag{6.10}$$

where we remark that the quantity on the right hand side is independent of t. Then, integrating (1.5) over Ω and using the periodic boundary conditions, we infer

$$\|\nabla\boldsymbol{u}\|^2 + \|\nabla\mu\|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \vartheta + \int_\Omega \vartheta\Delta\mu \le \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \vartheta + \frac{1}{2}\|\nabla\vartheta\|^2 + \frac{1}{2}\|\nabla\mu\|^2.$$

Hence, integrating in time over $(0,t)$, absorbing the last term on the right hand side with the corresponding one on the left hand side, and recalling (6.10) and (6.8), we readily arrive at

$$\int_0^t \left(\|\nabla\boldsymbol{u}\|^2 + \|\nabla\mu\|^2\right) \le Q(\|z_0\|_{\mathcal{H}}), \quad \forall\, t \ge 0. \tag{6.11}$$

Collecting (6.10) and (6.11), observing once more that the quantities on the right hand side are independent of t, and letting $t \nearrow \infty$, we then deduce the first bound in (6.2). Next, testing (1.4) by $\Delta^2\varphi$ and performing standard manipulations (see [9, (3.20)] for details), we deduce

$$\|\nabla\Delta\varphi\|^2 \le c\left(Q(\|\varphi\|_V) + \|\nabla\vartheta\|^2 + \|\nabla\mu\|^2\right) \overset{(6.8)}{\le} c\left(Q(\|z_0\|_{\mathcal{H}}) + \|\nabla\vartheta\|^2 + \|\nabla\mu\|^2\right).$$

Then, integrating the above relation over a generic interval $(t, t+1)$ and using (6.10) and (6.11), we eventually obtain the second bound in (6.2). ∎

It is worth noting that the above lemma is stated for "stable solutions" (i.e., for initial data $z_0 \in \mathcal{V}^r$); however, only the $\mathcal{H}$-regularity of the initial datum is used in the proof, and consequently the assert remains valid for more general classes of solutions. The same holds for the next result, which is an immediate consequence of the lemma.

Corollary 6.2. *Under the same assumptions as in* Lemma 6.1, *for any* $z_0 \in \mathcal{V}^r$ *and any* $t \ge 0$, *there exists* $s \in (t, t+1)$ *such that*

$$\|z(s)\|_{\mathcal{V}} + \|\mu(s)\|_V = \|\boldsymbol{u}(s)\|_V + \|\varphi(s)\|_{H^3(\Omega)} + \|\vartheta(s)\|_V + \|\mu(s)\|_V \le Q(\|z_0\|_{\mathcal{H}}). \tag{6.12}$$

Actually, a similar estimate holds also for the "$\mathcal{V}^r$-magnitude" of the solution (of course now assuming the initial datum to lie in $\mathcal{V}^r$ is essential). This is the object of the next

Lemma 6.3. *Given* $z_0 \in \mathcal{V}^r$, *for any* $t \ge 0$ *there exists* $s \in [t, t+1)$ *such that*

$$\begin{aligned}\|z(s)\|_{\mathcal{V}^r} + \|\mu(s)\|_V &= \|\boldsymbol{u}(s)\|_{H^{r+1}(\Omega)} + \|\varphi(s)\|_{H^3(\Omega)} + \|\vartheta(s)\|_V + \|(1/\vartheta(s))\|_{L^1(\Omega)} + \|\mu(s)\|_V \\ &\le Q(\|z_0\|_{\mathcal{V}^r}).\end{aligned} \tag{6.13}$$

Proof. Being $z_0 \in \mathcal{V}^r$, we have in particular $(1/\vartheta_0) \in L^1(\Omega)$. Hence we may apply Lemma 4.1 with $\alpha = 2$ over the time interval $(0, +\infty)$. Note, indeed, that the quantity noted as N in (4.1) is globally controlled due to (6.2). Thus, we deduce

$$\|\vartheta^{-1}(t)\|_{L^1(\Omega)} \le Q(\|z_0\|_{\mathcal{V}}), \quad \forall\, t \ge 0. \tag{6.14}$$

Hence, it only remains to improve the regularity estimate on $\boldsymbol{u}$. To this aim, let us take any $t \ge 0$ and observe that, thanks to (6.2), there exists $\tau \in (t, t+1/2)$ such that $\|\boldsymbol{u}(\tau)\|_V \le Q(\|z_0\|_{\mathcal{H}})$. Then, test (1.2) by $-\Delta\boldsymbol{u}$, and integrate over $(\tau, \tau+1/2) \times \Omega$. Performing standard manipulations and using the regularity of $\boldsymbol{u}(\tau)$ and the uniform bounds (6.1) and (6.2), it is not difficult to deduce

$$\int_\tau^{\tau+1/2} \|\Delta\boldsymbol{u}\|^2 \le Q(\|z_0\|_{\mathcal{H}}).$$

As a consequence, there exists $s \in (\tau, \tau+1/2)$ (so that in particular $s \in (t, t+1)$) such that $\|\boldsymbol{u}(s)\|_{H^2(\Omega)} \le Q(\|z_0\|_{\mathcal{H}})$. Hence we have in particular (6.13). ∎

The next property plays a key role in the asymptotic analysis of stable solutions. Namely, we can prove that if the initial datum z_0 lies in $\mathcal{V}^r$ then the $\mathcal{V}^r$-magnitude of $z(t)$ is controlled uniformly in time:

Lemma 6.4. *Given* $z_0 \in \mathcal{V}^r$, *we have the bound*

$$\|z(t)\|_{\mathcal{V}^r} \le Q(\|z_0\|_{\mathcal{V}^r}), \quad \text{for a.a. } t \in (0, \infty). \tag{6.15}$$

Moreover, for any $s \ge 0$, *there holds the following additional bound:*

$$\begin{aligned}&\|\mu\|_{L^2(s,s+2;H^3(\Omega))} + \|\boldsymbol{u}_t\|_{L^2(s,s+2;H^r(\Omega))} + \|\boldsymbol{u}\|_{L^2(s,s+2;H^{2+r}(\Omega))} + \|\varphi_t\|_{L^2(s,s+2;V)} \\ &\quad + \|\vartheta\|_{L^2(s,s+2;H^2(\Omega))} + \|\vartheta_t\|_{L^2(s,s+2;H)} \le Q(\|z_0\|_{\mathcal{V}^r}),\end{aligned} \tag{6.16}$$

with Q *independent of the choice of* s.

Proof. Given any $s \geq 0$ such that $z(s) \in \mathcal{V}^r$, we may interpret $z(s)$ as an "initial" datum and apply Theorem 3.1 over the time interval $(s, s+2)$ (in place of $(0,T)$). Then, the solution satisfies the regularity properties (3.1)-(3.6) over $(s, s+2)$. More precisely, one has the quantitative estimates

$$\|\boldsymbol{u}\|_{L^\infty(s,s+2;H^{1+r}(\Omega))} + \|\varphi\|_{L^\infty(s,s+2;H^3(\Omega))} + \|\mu\|_{L^\infty(s,s+2;V)} + \|\vartheta\|_{L^\infty(s,s+2;V)} + \|K(\vartheta)\|_{L^\infty(s,s+2;V)} + \|1/\vartheta\|_{L^\infty(s,s+2;L^1(\Omega))} \leq Q(\|z(s)\|_{\mathcal{V}^r}) \tag{6.17}$$

and

$$\|\mu\|_{L^2(s,s+2;H^3(\Omega))} + \|\boldsymbol{u}_t\|_{L^2(s,s+2;H^r(\Omega))} + \|\boldsymbol{u}\|_{L^2(s,s+2;H^{2+r}(\Omega))} + \|\varphi_t\|_{L^2(s,s+2;V)} + \|\vartheta\|_{L^2(s,s+2;H^2(\Omega))} + \|\vartheta_t\|_{L^2(s,s+2;H)} \leq Q(\|z(s)\|_{\mathcal{V}^r}), \tag{6.18}$$

where in this case Q may depend on the length of the considered time span, which is however fixed (and equal to 2). Hence, taking first $s = 0$, which is possible because $z_0 \in \mathcal{V}^r$ by assumption and then with the choices of s provided by Lemma 6.3, we readily get the assert (clearly (6.18) reduces to (6.16) and (6.17), written concisely, becomes (6.15)). Indeed, we may notice that the L^1-norm of $1/\vartheta$ (which is a summand in the quantity $\|z\|_{\mathcal{V}^r}$) has already been controlled uniformly over $(0,\infty)$ by virtue of (6.14). ∎

6.2 Asymptotic compactness

We are now ready to show asymptotic compactness of trajectories associated to stable solutions. Namely, if the initial datum z_0 lies in $\mathcal{V}^r$, then for any $t > 0$ the corresponding solutions $z(t)$ belongs to $\mathcal{W}$. Moreover, the $\mathcal{W}$-magnitude of $z(t)$ is uniformly bounded for large t. This is stated more precisely in the

Lemma 6.5. *Let $z_0 \in \mathcal{V}^r$. Then, for any $\tau > 0$ there holds*

$$\|z\|_{L^\infty(\tau,+\infty;\mathcal{W})} \leq Q(\|z_0\|_{\mathcal{V}^r}, \tau^{-1}). \tag{6.19}$$

Proof. We need to prove the following regularity properties (and control the corresponding norms by a quantity $Q(\|z_0\|_{\mathcal{V}^r}, \tau^{-1})$):

(vii) $1/\vartheta \in L^\infty(\tau, +\infty; L^4(\Omega))$ and $\nabla(1/\vartheta) \in L^\infty(\tau, +\infty; L^1(\Omega))$;

(viii) $\boldsymbol{u} \in L^\infty(\tau, +\infty; H^2(\Omega))$ and $\boldsymbol{u} \in L^2(t, t+1, H^3(\Omega))$ for all $t > \tau$;

(ix) $\vartheta \in L^\infty(\tau, +\infty; H^2(\Omega))$;

(x) $\varphi \in L^\infty(\tau, +\infty; H^4(\Omega))$.

• proof of (vii). Let us go back to formula (4.3) in Lemma 4.1 with $\alpha = 2$. This contains the additional estimate

$$\|\nabla \vartheta^{-1/2}\|_{L^2(0,+\infty;H)} \leq Q(\|z_0\|_{\mathcal{V}^r}). \tag{6.20}$$

Combining this property with the information on $1/\vartheta$ provided by (6.15) and using the continuity of the embedding $V \subset L^p(\Omega)$ for any $p \in [1,\infty)$, we deduce

$$\sup_{t \geq 0} \|\vartheta^{-1}\|_{L^1(t,t+1;L^p(\Omega))} \leq Q(\|z_0\|_{\mathcal{V}^r}, p). \tag{6.21}$$

Let now apply once more Lemma 4.1 for a generic $\alpha \geq 2$. Then, the differential version of (4.3) reads

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \vartheta^{1-\alpha} + c_\alpha \|\nabla \vartheta^{(1-\alpha)/2}\|^2 \leq c'_\alpha \|\nabla \vartheta\|^2, \tag{6.22}$$

for some $c_\alpha, c'_\alpha > 0$. For our purposes it is enough to take $\alpha = 5$. Then, from (6.21) there follows that, for any $t \geq 0$ and $\tau > 0$, there exists $s \in [t, t+\tau]$ such that

$$\|\vartheta^{-1}(s)\|_{L^4(\Omega)} \leq \frac{1}{\tau} \int_t^{t+\tau} \|\vartheta^{-1}(r)\|_{L^4(\Omega)} \, \mathrm{d}r \leq \frac{1}{\tau} Q(\|z_0\|_{\mathcal{V}^r}) = Q(\|z_0\|_{\mathcal{V}^r}, \tau^{-1}). \tag{6.23}$$

Hence, choosing $t = 0$ and $s \in (0, \tau)$ such that (6.23) holds, integrating (6.22) (with $\alpha = 5$) over $(s, +\infty)$, and recalling (6.2), we readily obtain the first of (vii). Then, recalling (6.17), for any $t \geq \tau > 0$, we have

$$\|\nabla\vartheta^{-1}(t)\|_{L^1(\Omega)} \leq \|\nabla\vartheta(t)\| \, \|\vartheta^{-1}(t)\|^2_{L^4(\Omega)} \leq Q(\|z_0\|_{\mathcal{V}^r}, \tau^{-1}),$$

whence follows the second of (vii).

Remark 6.6. An easy refinement of the argument above yields more precisely

$$\|\vartheta^{-1}\|_{L^\infty(\tau,\infty;L^p(\Omega))} \leq Q(\|z_0\|_{\mathcal{V}^r}, \tau^{-1}, p)$$

for all $\tau > 0$ and all $p \in [1, \infty)$. Actually, at the price of some additional work, in the spirit of Moser's iterations one may also prove that

$$\|\vartheta^{-1}\|_{L^\infty(\tau,\infty;L^\infty(\Omega))} \leq Q(\|z_0\|_{\mathcal{V}^r}, \tau^{-1}). \tag{6.24}$$

We omit details because we do not want to overburden the reader with technical arguments.

• PROOF OF (viii). In the sequel we shall denote by C a generic positive constant of the form $C = Q(\|z_0\|_{\mathcal{V}^r}, \tau^{-1})$. Namely, C may first of all depend on the quantities uniformly estimated by (6.15). Moreover, it may also depend on time in the sense that it may explode (in a controlled and computable way) as $\tau \searrow 0$. This is a natural behavior as we are looking for parabolic smoothing estimates.

That said, to prove the additional regularity for $\boldsymbol{u}$ we proceed as in the proof of (i) in Subsection 4.2: we first project equation (1.2) into the space $\mathbb{H}$ by applying the operator $\mathbb{P}$, then we test by $A^2\hat{\boldsymbol{u}}$, where $\hat{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_\Omega$. Recalling (4.5), we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A\hat{\boldsymbol{u}}\|^2 + \|A^{3/2}\hat{\boldsymbol{u}}\|^2 &= -b(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}, A^2\hat{\boldsymbol{u}}) + \langle f, A^2\hat{\boldsymbol{u}}\rangle \\ &\leq -b(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}, A^2\hat{\boldsymbol{u}}) + \frac{1}{4}\|A^{3/2}\hat{\boldsymbol{u}}\|^2 + C\left(\|\varphi\|^3_{H^3(\Omega)} + 1\right), \end{aligned} \tag{6.25}$$

where we also used (4.8). On the other hand,

$$\begin{aligned} |\langle(\hat{\boldsymbol{u}} \cdot \nabla)\hat{\boldsymbol{u}}, A^2\hat{\boldsymbol{u}}\rangle| &= |\langle A^{1/2}(\hat{\boldsymbol{u}} \cdot \nabla\hat{\boldsymbol{u}}), A^{3/2}\hat{\boldsymbol{u}}\rangle| \\ &\leq \int_\Omega (|\nabla\hat{\boldsymbol{u}}|^2 + |\hat{\boldsymbol{u}}||\nabla^2\hat{\boldsymbol{u}}|)\,|\nabla^3\hat{\boldsymbol{u}}| \\ &\leq \|\nabla\hat{\boldsymbol{u}}\|^2_{L^4(\Omega)}\|\nabla^3\hat{\boldsymbol{u}}\| + \|\hat{\boldsymbol{u}}\|_{L^\infty(\Omega)}\|\nabla^2\hat{\boldsymbol{u}}\|\,\|\nabla^3\hat{\boldsymbol{u}}\| \\ &\overset{(2.2),(2.3)}{\leq} C\|\nabla\hat{\boldsymbol{u}}\|\,\|\nabla^2\hat{\boldsymbol{u}}\|\,\|\nabla^3\hat{\boldsymbol{u}}\| + C\|\nabla^2\hat{\boldsymbol{u}}\|^{3/2}\,\|\hat{\boldsymbol{u}}\|^{1/2}\,\|\nabla^3\hat{\boldsymbol{u}}\| \\ &\overset{(2.5)}{\leq} \frac{1}{8}\|\nabla^3\hat{\boldsymbol{u}}\|^2 + C\|\nabla^2\hat{\boldsymbol{u}}\|^2 + C\|\nabla^3\hat{\boldsymbol{u}}\|^{7/4} \\ &\leq \frac{1}{4}\|\nabla^3\hat{\boldsymbol{u}}\|^2 + C\big(\|\nabla^2\hat{\boldsymbol{u}}\|^2 + 1\big). \end{aligned}$$

Coming back to (6.25) we then obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A\hat{\boldsymbol{u}}\|^2 + \frac{1}{2}\|A^{3/2}\hat{\boldsymbol{u}}\|^2 \leq C\left(\|\varphi\|^3_{H^3(\Omega)} + 1\right) + C\|\nabla^2\hat{\boldsymbol{u}}\|^2.$$

Then, the thesis is obtained by Gronwall's lemma by also exploiting (6.17). To be more precise, we integrate the above relation over time intervals of fixed length (for instance equal to 2) taking as starting point suitable times s such that $\|\boldsymbol{u}(s)\|_{H^2(\Omega)}$ is controlled by the $\mathcal{V}^r$-magnitude of the initial datum (these times s are characterized in the proof of Lemma 6.3). This yields (viii) away from 0. Then, in order to control the boundary layer at $t \sim 0$, we multiply the above inequality by $t \in (0, 1)$ obtaining

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{2}\|A\hat{\boldsymbol{u}}\|^2\right) + \frac{t}{2}\|A^{3/2}\hat{\boldsymbol{u}}\|^2 \leq \frac{1}{2}\|A\hat{\boldsymbol{u}}\|^2 + C\left(\|\varphi\|^3_{H^3(\Omega)} + 1\right) + C\|\nabla^2\hat{\boldsymbol{u}}\|^2.$$

Integrating over $(0,1)$ and using (6.16), we get (viii) for $t \sim 0$, as desired.

- PROOF OF (ix). We test (1.5) by $\Delta^2\vartheta$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\vartheta\|^2 + \langle \nabla\Delta K(\vartheta), \nabla\Delta\vartheta\rangle + \langle \boldsymbol{u}\cdot\nabla\vartheta, \Delta^2\vartheta\rangle + \langle \vartheta\Delta\mu, \Delta^2\vartheta\rangle = \langle |\nabla\boldsymbol{u}|^2 + |\nabla\mu|^2, \Delta^2\vartheta\rangle. \tag{6.26}$$

At this point, we notice that for a generic function ϕ

$$\|\phi\|^2 = \|\phi - \phi_\Omega + \phi_\Omega\|^2 = \|\phi - \phi_\Omega\|^2 + |\Omega|\phi_\Omega^2 \overset{(2.6)}{\leq} c_\Omega\|\nabla\phi\|^2 + |\Omega|\phi_\Omega^2.$$

On the other hand,

$$\|\nabla\phi\|^2 = \langle \phi - \phi_\Omega, -\Delta\phi\rangle \leq \|\phi - \phi_\Omega\|\,\|\Delta\phi\| \leq \frac{1}{2c_\Omega}\|\phi - \phi_\Omega\|^2 + c\|\Delta\phi\|^2 \overset{(2.6)}{\leq} \frac{1}{2}\|\nabla\phi\|^2 + c\,\|\Delta\phi\|^2.$$

Moreover, we will use repeatedly in the sequel the following facts:

$$\|\Delta\phi\|^2 \leq \|\phi\|^2_{H^2(\Omega)} \leq |\Omega|\phi_\Omega^2 + c\|\Delta\phi\|^2. \tag{6.27}$$

We analyze all terms in (6.26). We have

$$\Delta K(\vartheta) = \mathrm{div}(\kappa(\vartheta)\nabla\vartheta) = \kappa'(\vartheta)|\nabla\vartheta|^2 + \kappa(\vartheta)\Delta\vartheta.$$

Therefore,

$$\nabla\Delta K(\vartheta) = \kappa''(\vartheta)|\nabla\vartheta|^2\nabla\vartheta + 2\kappa'(\vartheta)\nabla\vartheta\cdot\nabla^2\vartheta + \kappa'(\vartheta)\nabla\vartheta\Delta\vartheta + \kappa(\vartheta)\nabla\Delta\vartheta.$$

This permits us to deal with the second term in (6.26):

$$\begin{aligned}\langle \nabla\Delta K(\vartheta), \nabla\Delta\vartheta\rangle &= \int_\Omega \kappa(\vartheta)|\nabla\Delta\vartheta|^2 + \langle \kappa''(\vartheta)|\nabla\vartheta|^2\nabla\vartheta + 2\kappa'(\vartheta)\nabla\vartheta\cdot\nabla^2\vartheta + \kappa'(\vartheta)\nabla\vartheta\Delta\vartheta, \nabla\Delta\vartheta\rangle \\ &=: \int_\Omega \big(\sqrt{\kappa(\vartheta)}|\nabla\Delta\vartheta|\big)^2 + \Theta_1 + \Theta_2 + \Theta_3.\end{aligned}$$

To estimate the right hand side let us assume, for simplicity, that $q \geq 4$ (the opposite situation being in fact easier). Then we first have

$$\begin{aligned}|\Theta_1| &\leq q(q-1)\int_\Omega \vartheta^{q-2}|\nabla\Delta\vartheta||\nabla\vartheta|^3 = q(q-1)\int_\Omega \vartheta^{q/2}|\nabla\Delta\vartheta|(\vartheta^{\frac{q}{2}-2}|\nabla\vartheta|^{\frac{1}{2}})|\nabla\vartheta|^{\frac{5}{2}} \\ &\leq C\|\sqrt{\kappa(\vartheta)}|\nabla\Delta\vartheta|\|\,\||\nabla K(\vartheta)|^{\frac{1}{2}}\|_{L^4(\Omega)}\,\||\nabla\vartheta|^{\frac{5}{2}}\|_{L^4(\Omega)} \\ &\leq C\|\sqrt{\kappa(\vartheta)}|\nabla\Delta\vartheta|\|\,\|\nabla K(\vartheta)\|^{\frac{1}{2}}\,\|\nabla\vartheta\|^{\frac{5}{2}}_{L^{10}(\Omega)} \overset{(6.17),(2.4)}{\leq} C\|\sqrt{\kappa(\vartheta)}|\nabla\Delta\vartheta|\|\left(\|\nabla\vartheta\|^{\frac{1}{5}}\,\|\vartheta\|^{\frac{4}{5}}_{H^2(\Omega)}\right)^{\frac{5}{2}} \\ &\leq C\|\sqrt{\kappa(\vartheta)}|\nabla\Delta\vartheta|\|\,\|\vartheta\|^2_{H^2(\Omega)} \leq \varepsilon\|\sqrt{\kappa(\vartheta)}|\nabla\Delta\vartheta|\|^2 + C_\varepsilon\|\vartheta\|^4_{H^2(\Omega)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}|\Theta_2| &\leq 2\int_\Omega q\vartheta^{q-1}|\nabla\vartheta|\,|\nabla^2\vartheta|\,|\nabla\Delta\vartheta| \\ &\leq C\|\vartheta^{q/2}|\nabla\Delta\vartheta|\|\,\|\vartheta^{(q-2)/2}|\nabla\vartheta|^{1/2}\|_{L^4(\Omega)}\,\||\nabla\vartheta|^{1/2}\|_{L^8(\Omega)}\,\|\nabla^2\vartheta\|_{L^8(\Omega)} \\ &\overset{(2.3),(2.2)}{\leq} C\|\sqrt{\kappa(\vartheta)}|\nabla\Delta\vartheta|\|\,\|\nabla K(\vartheta)\|^{\frac{1}{2}}\|\nabla\vartheta\|^{\frac{1}{4}}\,\|\vartheta\|^{\frac{1}{2}}_{H^2(\Omega)}\,\|\vartheta\|^{\frac{3}{4}}_{H^3(\Omega)} \\ &\overset{(3.5)}{\leq} C\|\sqrt{\kappa(\vartheta)}|\nabla\Delta\vartheta|\|^{\frac{7}{4}}\|\vartheta\|^{\frac{1}{2}}_{H^2(\Omega)} \leq \varepsilon\|\sqrt{\kappa(\vartheta)}|\nabla\Delta\vartheta|\|^2 + C_\varepsilon\|\vartheta\|^4_{H^2(\Omega)}\end{aligned}$$

and it is apparent that the term Θ_3 can be controlled analogously.

We now deal with the third term in (6.26). Due to the fact that

$$\nabla(\boldsymbol{u}\cdot\nabla\vartheta) = \nabla\vartheta\cdot\nabla\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\nabla\vartheta, \tag{6.28}$$

we then have

$$
\begin{aligned}
\langle \boldsymbol{u}\cdot\nabla\vartheta, \Delta^2\vartheta\rangle &= -\langle \nabla(\boldsymbol{u}\cdot\nabla\vartheta), \nabla\Delta\vartheta\rangle \overset{(6.28)}{=} -\langle \nabla\vartheta\cdot\nabla\boldsymbol{u}, \nabla\Delta\vartheta\rangle + \langle(\boldsymbol{u}\cdot\nabla)\nabla\vartheta, \nabla\Delta\vartheta\rangle \\
&\le 2\varepsilon\|\nabla\Delta\vartheta\|^2 + C_\varepsilon\|\nabla\vartheta\|^2_{L^4(\Omega)}\|\nabla\boldsymbol{u}\|^2_{L^4(\Omega)} + C_\varepsilon\int_\Omega |\boldsymbol{u}|^2\,|\nabla^2\vartheta|^2 \\
&\le 2\varepsilon\|\nabla\Delta\vartheta\|^2 + C_\varepsilon\|\vartheta\|_V\|\vartheta\|_{H^2(\Omega)}\|\boldsymbol{u}\|_V\|\boldsymbol{u}\|_{H^2(\Omega)} \overset{(2.4)}{+} C_\varepsilon\|\boldsymbol{u}\|^2_{L^6(\Omega)}\|\nabla^2\vartheta\|^{4/3}\|\nabla^3\vartheta\|^{2/3} \\
&\le 3\varepsilon\|\sqrt{\kappa(\vartheta)}\nabla\Delta\vartheta\|^2 \overset{(2.4)}{+} C_\varepsilon\|\nabla^2\vartheta\|^2 + C_\varepsilon,
\end{aligned}
$$

where we used (viii) of Lemma 6.5 together with (i) and (ii) of Subsection 4.2. Notice in particular that at this level the constants C (and C_ε) are allowed to depend also on the H^2-norm of $\boldsymbol{u}$, which has been estimated in (viii) far from 0. Hence, here (and below) C is of the form $C = C(t) = Q(\|z_0\|_{\mathcal{V}^r}, t^{-1})$.

Next, concerning the fourth term in (6.26) we have

$$
|\langle \vartheta\Delta\mu, \Delta^2\vartheta\rangle| \le \int_\Omega |\Delta\mu|\,|\nabla\vartheta|\,|\nabla\Delta\vartheta| + \int_\Omega |\vartheta|\,|\nabla\Delta\mu|\,|\nabla\Delta\vartheta| =: \Theta_4 + \Theta_5,
$$

where

$$
\begin{aligned}
\Theta_4 := \int_\Omega |\Delta\mu|\,|\nabla\vartheta|\,|\nabla\Delta\vartheta| &\le \|\nabla\Delta\vartheta\|\,\|\Delta\mu\|_{L^6(\Omega)}\,\|\nabla\vartheta\|_{L^3(\Omega)} \\
&\le C\|\nabla\Delta\vartheta\|\,\|\Delta\mu\|^{\frac13}\,\|\nabla\Delta\mu\|^{\frac23}\,\|\nabla\vartheta\|^{\frac23}\,\|\vartheta\|^{\frac13}_{H^2(\Omega)} \\
&\overset{(2.5)}{\le} C\|\nabla\Delta\vartheta\|\,\|\nabla\mu\|^{\frac16}\,\|\nabla\Delta\mu\|^{\frac23+\frac16}\|\nabla\vartheta\|^{\frac23}\,\|\vartheta\|^{\frac13}_{H^2(\Omega)} \\
&\le C\|\nabla\Delta\vartheta\|\,\|\nabla\Delta\mu\|^{\frac56}\,\|\vartheta\|^{\frac13}_{H^2(\Omega)} \\
&\le \varepsilon\|\sqrt{\kappa(\vartheta)}\nabla\Delta\vartheta\|^2 + C_\varepsilon\|\nabla\Delta\mu\|^2 + C_\varepsilon\|\vartheta\|^4_{H^2(\Omega)},
\end{aligned}
$$

where we also used the uniform control (6.17) of μ. On the other hand,

$$
\begin{aligned}
\Theta_5 := \int_\Omega |\vartheta|\,|\nabla\Delta\mu|\,|\nabla\Delta\vartheta| &\\
&\le \|\sqrt{\kappa(\vartheta)}|\nabla\Delta\vartheta|\|\,\|\nabla\Delta\mu\| \\
&\le \varepsilon\|\sqrt{\kappa(\vartheta)}|\nabla\Delta\vartheta|\|^2 + C_\varepsilon\|\nabla\Delta\mu\|^2.
\end{aligned}
$$

Coming to the very last terms in (6.26), we eventually have

$$
\begin{aligned}
|\langle|\nabla\boldsymbol{u}|^2, \Delta^2\vartheta\rangle| &\le C\int_\Omega |\nabla^2\boldsymbol{u}\,\nabla\boldsymbol{u}\,\nabla\Delta\vartheta| \le C\|\nabla^2\boldsymbol{u}\|_{L^4(\Omega)}\,\|\nabla\boldsymbol{u}\|_{L^4(\Omega)}\,\|\nabla\Delta\vartheta\| \\
&\le C\|\nabla^2\boldsymbol{u}\|_V^{1/2}\,\|\nabla^2\boldsymbol{u}\|\,\|\nabla\boldsymbol{u}\|^{1/2}\,\|\nabla\Delta\vartheta\| \le \varepsilon\|\sqrt{\kappa(\vartheta)}\nabla\Delta\vartheta\|^2 + C_\varepsilon\|\nabla^3\boldsymbol{u}\|^2,
\end{aligned}
$$

where we used (6.15) and the previous estimate (viii). Similarly, we also get

$$
\begin{aligned}
|\langle|\nabla\mu|^2, \Delta^2\vartheta\rangle| &\le C\int_\Omega |\nabla^2\mu\,\nabla\mu\,\nabla\Delta\vartheta| \le C\|\nabla^2\mu\|\,\|\nabla\mu\|_{L^\infty(\Omega)}\,\|\nabla\Delta\vartheta\| \\
&\le C\|\mu\|_{H^2}\|\mu\|^{1/2}_{H^3}\|\mu\|^{1/2}_{H^1}\,\|\nabla\Delta\vartheta\| \le C\|\mu\|_{H^3}\|\mu\|_{H^1}\,\|\nabla\Delta\vartheta\| \\
&\le \varepsilon\|\sqrt{\kappa(\vartheta)}\nabla\Delta\vartheta\|^2 + C_\varepsilon\|\mu\|^2_{H^3}.
\end{aligned}
$$

Collecting the above estimates, and taking ε small enough, we finally arrive at

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\vartheta\|^2 + \|\sqrt{\kappa(\vartheta)}|\nabla\Delta\vartheta|\|^2 \le C\|\vartheta\|^2_{H^2}\,\|\Delta\vartheta\|^2 + C(1 + \|\mu\|^2_{H^3} + \|\boldsymbol{u}\|^2_{H^3} + \|\vartheta\|^2_{H^2}), \tag{6.29}
$$

where, as said, $C = C(t) = Q(\|z_0\|_{\mathcal{V}^r}, t^{-1})$. Then, similarly as before, we apply Gronwall's lemma on time intervals of the form $(s, s+2)$ starting from suitable s such that $\|\Delta\vartheta(s)\|^2$ is controlled. Indeed, from (6.16) one can easily deduce that for any $t \geq 0$ and any $\tau > 0$ there exists $s \in [t, t+\tau]$ such that $\|\vartheta(s)\|_{H^2(\Omega)} \leq Q(\|z_0\|_{\mathcal{V}^r}, \tau^{-1})$. Then, noting that the right hand side of (6.29) is controlled by means of (6.16) and (viii), we deduce (ix).

- PROOF OF (x). To derive the additional regularity of φ, we first prove that

$$\|\varphi_t\|_{L^\infty(\tau,+\infty;H)} \leq Q(\|z_0\|_{\mathcal{V}^r}, \tau^{-1}). \tag{6.30}$$

To this aim, we differentiate in time (1.3)-(1.4), obtaining

$$\varphi_{tt} + \boldsymbol{u}_t \cdot \nabla\varphi + \boldsymbol{u} \cdot \nabla\varphi_t = \Delta\mu_t, \tag{6.31}$$

$$\mu_t = -\Delta\varphi_t + F''(\varphi)\varphi_t - \vartheta_t. \tag{6.32}$$

Then we test (6.31) by φ_t and (6.32) by $\Delta\varphi_t$ and sum the resulting relations in order to erase the term $\langle \Delta\mu_t, \varphi_t \rangle$. Moreover, noticing that

$$\langle \boldsymbol{u} \cdot \nabla\varphi_t, \varphi_t \rangle = 0$$

due to (1.1), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_t\|^2 + \|\Delta\varphi_t\|^2 = -\langle \boldsymbol{u}_t \cdot \nabla\varphi, \varphi_t \rangle - \langle \vartheta_t, \Delta\varphi_t \rangle + \langle F''(\varphi)\varphi_t, \Delta\varphi_t \rangle.$$

Let us now notice that

$$-\int_\Omega \boldsymbol{u}_t \cdot \nabla\varphi\, \varphi_t \leq \|\boldsymbol{u}_t\|\, \|\nabla\varphi\|\, \|\varphi_t\|_{L^\infty(\Omega)} \overset{(2.3)}{\leq} C\|\boldsymbol{u}_t\|\, \|\nabla\varphi\|\, \|\varphi_t\|^{1/2}\, \|\Delta\varphi_t\|^{1/2}$$
$$\leq \frac{1}{6}\|\Delta\varphi_t\|^2 + C\,\|\varphi_t\|^2 + C\,\|\boldsymbol{u}_t\|^2,$$

thanks also to (6.15). On the other hand,

$$-\int_\Omega \vartheta_t \Delta\varphi_t \leq C\|\vartheta_t\|^2 + \frac{1}{6}\|\Delta\varphi_t\|^2.$$

Finally, recalling (2.9),

$$\int_\Omega F''(\varphi)\varphi_t\, \Delta\varphi_t \leq C(1 + \|\varphi\|^{p_F}_{L^\infty(\Omega)})\|\varphi_t\|\, \|\Delta\varphi_t\| \leq \frac{1}{6}\|\Delta\varphi_t\|^2 + C\|\varphi_t\|^2.$$

Summarizing,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_t\|^2 + \|\Delta\varphi_t\|^2 \leq C(1 + \|\varphi_t\|^2 + \|\boldsymbol{u}_t\|^2 + \|\vartheta_t\|^2).$$

Hence, (6.30) follows from Gronwall's lemma using the information (6.16) (and estimating the boundary layer near 0 as before).

We now claim that this is enough to get the desired estimate (x): indeed, from (1.3) we immediately obtain $\Delta\mu \in L^\infty(\tau, +\infty; H)$ (with a quantitative bound on that norm) because $\boldsymbol{u}$ and $\nabla\varphi$ are bounded uniformly. By elliptic regularity, this implies $\mu \in L^\infty(\tau, \infty; H^2(\Omega))$. Then, we also interpret (1.4) as an elliptic problem, namely we have $\Delta\varphi = g$, where $g \in L^\infty(\tau, \infty; H^2(\Omega))$ thanks to (ix), (F3) and (6.15). Hence, we deduce that $\varphi \in L^\infty(\tau, +\infty; H^4(\Omega))$ and, more precisely, the quantitative bound $\|\varphi\|_{L^\infty(\tau,+\infty;H^4(\Omega))} \leq Q(\|z_0\|_{\mathcal{V}^r}, \tau^{-1})$. This concludes the proof of Lemma 6.5. ∎

6.3 ω-limits and dissipativity

On account of the previous estimates, we can now show that the dynamical process $S(t)$ associated to "stable solutions" is *asymptotically compact*. Namely, we have the

Theorem 6.7. *Let $\{z_{0,n}\}$ be a bounded sequence in $\mathcal{V}^r$ and let $z_n(t) = S(t)z_{0,n}$ be the unique stable solution emanating from $z_{0,n}$. Then, for any sequence $t_n \nearrow +\infty$, there exist an element $\zeta \in \mathcal{V}^r$ and a (nonrelabelled) subsequence of t_n such that*

$$S(t_n)z_{0,n} \to \zeta \quad \text{in } \mathcal{V}^r.$$

Proof. Thanks to Lemma 6.5, the sequence $\{S(t_n)z_{0,n}\}$ is bounded in $\mathcal{W}$. Hence, recalling (3.10) it is clear that, for some $\zeta = (\boldsymbol{u}, \varphi, \vartheta) \in \mathcal{W}$, there holds (all the following relations are intended to hold up to the extraction of nonrelabelled subsequences)

$$S(t_n)z_{0,n} \to \zeta \quad \text{weakly in } H^2(\Omega) \times H^4(\Omega) \times H^2(\Omega), \tag{6.33}$$

whence, by Rellich's theorem,

$$S(t_n)z_{0,n} \to \zeta \quad \text{strongly in } H^{1+r}(\Omega) \times H^3(\Omega) \times V, \quad \text{and uniformly in } \Omega.$$

It is then easy to check that we also have $K(\vartheta_n) \to K(\vartheta)$ in V. Moreover, $\{1/\vartheta_n\}$ is bounded in $W^{1,1}(\Omega) \cap L^4(\Omega)$. Hence, using also pointwise convergence, we obtain that $1/\vartheta_n \to 1/\vartheta$ strongly in $L^1(\Omega)$ (actually, something more is true), which concludes the proof. Note that, in fact, our argument shows that the metric space embedding $\mathcal{W} \subset \mathcal{V}^r$ is compact. ∎

Remark 6.8. In the sequel, with some abuse of notation, when a sequence $\{\zeta_n\} = \{(\boldsymbol{u}_n, \varphi_n, \vartheta_n)\} \subset \mathcal{W}$ satisfies

$$\zeta_n \to \zeta \quad \text{weakly in } H^2(\Omega) \times H^4(\Omega) \times H^2(\Omega), \qquad 1/\vartheta_n \to 1/\vartheta \quad \text{weakly in } L^4(\Omega), \tag{6.34}$$

for some $\zeta = (\boldsymbol{u}, \varphi, \vartheta) \in H^2(\Omega) \times H^4(\Omega) \times H^2(\Omega)$ with $1/\vartheta \in L^4(\Omega)$, we will speak of "weak convergence in $\mathcal{W}$". This is in fact the type of information we obtain from asymptotic compactness. As before, by Rellich's theorem, (6.34) implies strong convergence of the components in weaker norms.

In order to show existence of the global attractor in the case when the spatial mean of the velocity is $\mathbf{0}$, we will combine the above property with the *point dissipativity* of the dynamical process generated by stable solutions. Namely, we will prove that there exists a bounded set $\mathcal{B} \subset \mathcal{V}^r$ such that for any $z_0 \in \mathcal{V}^r$ there exists $T = T(z_0)$ such that $S(t)z_0 \in \mathcal{B}$ for any $t \geq T$. In other words, $\mathcal{B}$ is a *pointwise absorbing* set. For many evolutionary system, this kind of property (and often a stronger one, i.e., the existence of a *uniformly absorbing* set) can be proved directly by showing that suitable norms of the solutions satisfy a dissipative differential inequality. Here, however, due to the presence of the quadratic source terms in (1.5) and to the physical constraints corresponding to conservation of mass, momentum, and total energy, it seems difficult to derive such a type of inequality. For this reason we will work in an alternative way, proving first of all that any trajectory has a nonempty ω-limit set, which is contained in a *proper subclass* $\mathcal{S}_0$ of the family of solutions of the stationary problem associated to (1.1)-(1.5). Pointwise dissipativity will follow from a precise characterization of $\mathcal{S}_0$. In order to start with this program, we preliminarily observe that the stationary problem associated to our system has the form

$$\operatorname{div} \boldsymbol{u} = 0, \tag{6.35}$$
$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \Delta \boldsymbol{u} - \operatorname{div}(\nabla\varphi \otimes \nabla\varphi), \tag{6.36}$$
$$\boldsymbol{u} \cdot \nabla\varphi = \Delta\mu, \tag{6.37}$$
$$\mu = -\Delta\varphi + F'(\varphi) - \vartheta, \tag{6.38}$$
$$\boldsymbol{u} \cdot \nabla\vartheta + \vartheta\Delta\mu - \operatorname{div}(\kappa(\vartheta)\nabla\vartheta) = |\nabla\boldsymbol{u}|^2 + |\nabla\mu|^2, \tag{6.39}$$

naturally complemented with periodic boundary conditions. Letting $\mathcal{S}$ be the set of solutions of (6.35)-(6.39), we will now show that any element of the ω-limit set of a stable solution of the evolution system not only belongs to $\mathcal{S}$, but it also satisfies additional structure properties in such a way that it solves in fact a much simpler system. This is the object of the following

Lemma 6.9. *Let $z_0 = (\boldsymbol{u}_0, \varphi_0, \vartheta_0) \in \mathcal{V}^r$ and let*

$$\boldsymbol{m} := (\boldsymbol{u}_0)_\Omega = \mathbf{0}, \qquad m := (\varphi_0)_\Omega, \qquad M := \frac{1}{2}\|\boldsymbol{u}_0\|^2 + \frac{1}{2}\|\nabla\varphi_0\|^2 + \int_\Omega (F(\varphi_0) + \vartheta_0) \tag{6.40}$$

be the associated conserved quantities. Let $z(t) := S(t)z_0$ and let $t_n \nearrow \infty$. Then, there exist a nonrelabelled subsequence of t_n and $z_\infty = (\mathbf{0}, \varphi_\infty, \vartheta_\infty) \in \mathcal{V}^r$ such that

$$z(t_n) = S(t_n)z_0 \to z_\infty \quad \text{in } \mathcal{V}^r \tag{6.41}$$

(and, in fact, "weakly" in $\mathcal{W}$). Moreover, there exists $\mu_\infty \in V$ such that

$$\mu(t_n) \to \mu_\infty \quad \text{strongly in } V. \tag{6.42}$$

In addition to that, if $z_\infty = (\boldsymbol{u}_\infty, \varphi_\infty, \vartheta_\infty)$ (with the auxiliary variable μ_∞) is any limit point of a sequence $S(t_n)z_0$ in the sense specified above, then we have that $\boldsymbol{u}_\infty$, μ_∞ and ϑ_∞ are constant functions with respect to the space variables with $\boldsymbol{u}_\infty \equiv \boldsymbol{m} \equiv \boldsymbol{0}$ and the stationary system reduces to the single equation

$$-\Delta\varphi_\infty + F'(\varphi_\infty) = \mu_\infty + \vartheta_\infty = \int_\Omega F'(\varphi_\infty). \tag{6.43}$$

Finally, there exists a constant $C_\infty > 0$ depending only on the conserved values $\boldsymbol{m} = \boldsymbol{0}$, m and M (and hence independent of the specific choice of z_0) such that

$$|\mu_\infty| + |\vartheta_\infty| + \|\varphi_\infty\|_{H^4(\Omega)} \le C_\infty. \tag{6.44}$$

Proof. We start pointing out that all convergence relations in the proof will be intended to hold up to the extraction of nonrelabelled subsequences of n. That said, we set $z_n(t) = S(t)z(t_n)$, for $t \in (0,1)$. Namely, we interpret $z(t_n) \in \mathcal{V}^r$ as an "initial" datum and consider the corresponding "stable" solution z_n over the time interval $(0,1)$. Then, by Lemma 6.7 (asymptotic compactness), we have

$$z(t_n) \to z_\infty = (\boldsymbol{u}_\infty, \varphi_\infty, \vartheta_\infty) \quad \text{in } \mathcal{V}^r \quad \text{(and weakly in } \mathcal{W}) \tag{6.45}$$

for some $z_\infty \in \mathcal{V}^r$. As a consequence, it is immediate to check that

$$(\boldsymbol{u}_\infty)_\Omega = \boldsymbol{m} = \boldsymbol{0}, \quad (\varphi_\infty)_\Omega = m, \quad \frac{1}{2}\|\boldsymbol{u}_\infty\|^2 + \frac{1}{2}\|\nabla\varphi_\infty\|^2 + \int_\Omega (F(\varphi_\infty) + \vartheta_\infty) = M. \tag{6.46}$$

Moreover, looking at the proof of (x) in Lemma 6.5, we can easily realize that $\mu(t_n)$ is bounded in $H^2(\Omega)$. Hence, (6.42) holds.

Let us now look at the behavior of z_n over the time interval $(0,1)$. Since $z(t_n)$ is a convergent (sub)sequence in $\mathcal{V}^r$, by weak sequential stability of solutions (cf. [9] for details), it readily follows that, correspondingly, z_n converges in a proper way to a limit $\overline{z} = (\overline{\boldsymbol{u}}, \overline{\varphi}, \overline{\vartheta})$ and $\mu_n = \mu(t_n)$ converges to $\overline{\mu}$, where $\overline{z}, \overline{\mu}$ are defined over $(0,1) \times \Omega$ and solve system (1.1)-(1.5) with the periodic boundary conditions and the initial conditions

$$\overline{\boldsymbol{u}}|_{t=0} = \boldsymbol{u}_\infty, \qquad \overline{\varphi}|_{t=0} = \varphi_\infty, \qquad \overline{\vartheta}|_{t=0} = \vartheta_\infty. \tag{6.47}$$

Let us now prove that $\overline{z}$ is independent of time. First of all, in view of the dissipation integrals (6.2), it is clear that

$$\nabla\boldsymbol{u}_n, \nabla\vartheta_n, \nabla\mu_n \to 0 \quad \text{strongly in } L^2(0,1;H). \tag{6.48}$$

Then, let us take $\xi \in H^2(\Omega) \subset L^\infty(\Omega)$ with $\|\xi\|_{H^2(\Omega)} \le 1$ and test (1.5), written for z_n on the time span $(0,1)$, by ξ. Then, performing standard manipulations and subsequently taking the supremum with respect to such test function ξ, we infer

$$\begin{aligned}\|\partial_t\vartheta_n\|_{H^2(\Omega)'} &\le \|\boldsymbol{u}_n\|\,\|\nabla\vartheta_n\| + \|\nabla\mu_n\|\,\|\nabla\vartheta_n\| + \|\nabla\mu_n\|\,\|\vartheta_n\|_{L^4(\Omega)} \\ &\quad + \|\kappa(\vartheta_n)\|_{L^4(\Omega)}\,\|\nabla\vartheta_n\| + \|\nabla\mu_n\|^2 + \|\nabla\vartheta_n\|^2 \\ &\le C\big(\|\nabla\mu_n\|^2 + \|\nabla\vartheta_n\|^2 + \|\nabla\vartheta_n\| + \|\nabla\mu_n\|\big),\end{aligned}$$

where C may also depend on the uniformly estimated norms of the other quantities (cf. (6.17), (6.16)). Squaring and integrating over $(0,1)$, we deduce

$$\|\partial_t\vartheta_n\|^2_{L^2(0,1;H^2(\Omega)')} \le C\big(1 + \|\nabla\vartheta_n\|^2_{L^\infty(0,1;H)} + \|\nabla\mu_n\|^2_{L^\infty(0,1;H)}\big)\big(\|\nabla\vartheta_n\|^2_{L^2(0,1;H)} + \|\nabla\mu_n\|^2_{L^2(0,1;H)}\big),$$

whence, by (6.17) and (6.48), $\partial_t\vartheta_n$ goes to 0 strongly in $L^2(0,1;H^2(\Omega)')$ and consequently $\overline{\vartheta} \equiv \vartheta_\infty$ over $(0,1)$. Moreover, using (6.48) again, we see that ϑ_∞ does not depend on space variables.

Next, we consider the behavior of the other variables, which is simpler to describe. First of all, notice that, by (6.48), the assumption $\boldsymbol{m} = \mathbf{0}$, and the Friedrichs inequality, we deduce that $\boldsymbol{u}_n \to \mathbf{0}$ strongly in $L^2(0,1;V)$. More precisely, since $\boldsymbol{u}_n$ converges uniformly as a consequence of (6.16), (viii) and the Aubin-Lions lemma, it turns out that $\overline{\boldsymbol{u}} \equiv \boldsymbol{u}_\infty = \boldsymbol{m} = \mathbf{0}$. Moreover, we easily deduce from (1.3) that

$$\|\partial_t \varphi_n\|_{L^2(0,1;V')} \le C\big(\|\nabla \mu_n\|_{L^2(0,1;H)} + \|\boldsymbol{u}_n\|_{L^2(0,1;V)} \|\varphi_n\|_{L^\infty(0,1;V)}\big),$$

whence, using (6.17) and (6.48), we deduce that $\partial_t \varphi_n \to 0$ strongly in $L^2(0,1;V')$. Consequently, $\overline{\varphi}$ is also independent of time and, more precisely, $\overline{\varphi} \equiv \varphi_\infty$ over $(0,1)$. Finally, concerning the auxiliary variable μ_n, from (6.48) we deduce that $\overline{\mu}$ is independent of space variables. Note that, a priori, it is not clear whether $\overline{\mu}(t)$ is also independent of $t \in (0,1)$. On the other hand, taking $n \nearrow \infty$ in (1.4) and collecting the previous information, we deduce that, a.e. in $(0,1)$ (in fact, everywhere in $[0,1]$ since one can easily realize that also μ_n converges uniformly), there holds

$$\overline{\mu} = -\Delta \overline{\varphi} + F'(\overline{\varphi}) - \overline{\vartheta} \equiv -\Delta \varphi_\infty + F'(\varphi_\infty) - \vartheta_\infty.$$

Then, in view of the fact that the right hand side is independent of time, so is also the left hand side. Hence $\overline{\mu} \equiv \mu_\infty$ and the above relation reduces to (6.43) (in particular the second equality follows by integration over Ω).

It remains to prove (6.44). First of all, we observe that, thanks to (6.46) and (2.10), we have

$$|\vartheta_\infty| + \|\varphi_\infty\|_V \le c_{M,m}$$

(again, we control the full V-norm of φ_∞ thanks to the conservation of mass). By Sobolev's embeddings, we then also deduce that $\|F'(\varphi_\infty)\|_{L^1(\Omega)} \le c_{M,m}$. Hence, from (6.43),

$$|\mu_\infty| \le |\mu_\infty + \vartheta_\infty| + |\vartheta_\infty| \le \|F'(\varphi_\infty)\|_{L^1(\Omega)} + |\vartheta_\infty| \le c_{M,m}.$$

Next, applying elliptic regularity in (6.43) (or, equivalently, testing by $(-\Delta)^3\varphi_\infty$) and exploiting (F3), we easily get the H^4-control in (6.44), which concludes the proof. ∎

To conclude this part, we would like to briefly describe the case when the initial velocity $\boldsymbol{u}_0$ has a nonzero spatial mean $\boldsymbol{m}$. In this situation, thanks to the periodic boundary conditions, we can actually perform a change of variables. Namely, we denote

$$\tilde{\zeta}(t,x) := \zeta(t, x + t\boldsymbol{m}), \quad \text{for } \zeta = \boldsymbol{u}, \varphi, \mu, \vartheta, p,$$

and we can easily check that

$$\nabla \tilde{\zeta}(t,x) = \nabla \zeta(t, x + t\boldsymbol{m}), \qquad \tilde{\zeta}_t(t,x) = \zeta_t(t, x + t\boldsymbol{m}) + \boldsymbol{m} \cdot \nabla \zeta(t, x + t\boldsymbol{m}).$$

Then, if $(\boldsymbol{u}, \varphi, \mu, \vartheta)$ is a stable solution, the Cahn-Hilliard system (1.3)-(1.4) is transformed into

$$\tilde{\varphi}_t + (\tilde{\boldsymbol{u}} - \boldsymbol{m}) \cdot \nabla \tilde{\varphi} = \Delta \tilde{\mu}, \tag{6.49}$$

$$\tilde{\mu} = -\Delta \tilde{\varphi} + F'(\tilde{\varphi}) - \tilde{\vartheta}. \tag{6.50}$$

Correspondingly, the counterparts of (1.2) and (1.5) are respectively

$$\tilde{\boldsymbol{u}}_t + (\tilde{\boldsymbol{u}} - \boldsymbol{m}) \cdot \nabla \tilde{\boldsymbol{u}} + \nabla \tilde{p} = \Delta \tilde{\boldsymbol{u}} - \operatorname{div}(\nabla \tilde{\varphi} \otimes \nabla \tilde{\varphi}), \tag{6.51}$$

$$\tilde{\vartheta}_t + (\tilde{\boldsymbol{u}} - \boldsymbol{m}) \cdot \nabla \tilde{\vartheta} + \tilde{\vartheta} \Delta \tilde{\mu} - \operatorname{div}(\kappa(\tilde{\vartheta}) \nabla \tilde{\vartheta}) = |\nabla \tilde{\boldsymbol{u}}|^2 + |\nabla \tilde{\mu}|^2 = |\nabla (\tilde{\boldsymbol{u}} - \boldsymbol{m})|^2 + |\nabla \tilde{\mu}|^2. \tag{6.52}$$

Next, we observe that (6.51) can be also rewritten as

$$(\tilde{\boldsymbol{u}} - \boldsymbol{m})_t + (\tilde{\boldsymbol{u}} - \boldsymbol{m}) \cdot \nabla (\tilde{\boldsymbol{u}} - \boldsymbol{m}) + \nabla \tilde{p} = \Delta (\tilde{\boldsymbol{u}} - \boldsymbol{m}) - \operatorname{div}(\nabla \tilde{\varphi} \otimes \nabla \tilde{\varphi})$$

and the initial conditions may be restated as

$$\tilde{\varphi}(0) = \varphi_0, \qquad \tilde{\vartheta}(0) = \vartheta_0, \qquad (\tilde{\boldsymbol{u}} - \boldsymbol{m})(0) = \boldsymbol{u}_0 - \boldsymbol{m}, \tag{6.53}$$

where, obviously,

$$(\tilde{\boldsymbol{u}} - \boldsymbol{m})_\Omega(t) \equiv (\tilde{\boldsymbol{u}}_0 - \boldsymbol{m})_\Omega = 0. \tag{6.54}$$

Hence, the variables $\tilde{z} = (\tilde{\boldsymbol{u}} - \boldsymbol{m}, \tilde{\varphi}, \tilde{\vartheta})$, with the auxiliary $\tilde{\mu}$, constitute a stable solution to (1.1)-(1.5) with the initial conditions (6.53). Hence Lemma 6.9 applies to $\tilde{z}$, which admits a nonempty ω-limit all of whose elements belong to $\mathcal{S}_0$. In particular, if $\{t_n\}$ is a diverging sequence of times, we deduce for instance that (a subsequence of) $\tilde{\vartheta}(t_n)$ converges weakly in $H^2(\Omega)$ (whence uniformly in Ω) to a positive constant ϑ_∞. On the other hand,

$$\vartheta(t_n, x) = \tilde{\vartheta}(t_n, x - t_n\boldsymbol{m}) \to \vartheta_\infty, \tag{6.55}$$

still uniformly in Ω, and the same applies to $\mu(t_n)$. On the other hand, as far as the limit of φ is concerned, the situation is more intricated. Indeed, by asymptotic compactness we can always construct a (nonrelabelled) subsequence of $n \nearrow \infty$ such that $\varphi(t_n)$ tends to a suitable limit φ_∞ and simultaneously $\tilde{\varphi}(t_n)$ tends to some limit $\tilde{\varphi}_\infty$. Moreover, convergence holds in both cases in $H^3(\Omega)$, hence uniformly in Ω. On the other hand, if we try to characterize φ_∞ as a stationary state, we have to consider that

$$\varphi(t_n, x) = \tilde{\varphi}(t_n, x - t_n\boldsymbol{m}), \tag{6.56}$$

whence, for any $\epsilon > 0$, there exists $\overline{n}(\epsilon)$ such that

$$\begin{aligned} |\varphi_\infty(x) - \tilde{\varphi}_\infty(x - t_n\boldsymbol{m})| &\le \|\varphi_\infty - \varphi(t_n, \cdot)\|_\infty + \|\tilde{\varphi}(t_n, \cdot - t_n\boldsymbol{m}) - \tilde{\varphi}_\infty(\cdot - t_n\boldsymbol{m})\|_\infty \\ &\le 2\epsilon \quad \text{for all } \ n \ge \overline{n}(\epsilon) \end{aligned} \tag{6.57}$$

and for every $x \in \Omega$. Now, $\tilde{\varphi}_\infty$ is a stationary solution (more precisely, it belongs to the subclass $\mathcal{S}_0$) in view of Lemma 6.9. Then, in order to deduce a useful information from the above relation, we may assume that the subsequence of n is chosen in such a way that, also, $t_n\boldsymbol{m} \to x_0$ for some $x_0 \in \Omega$ (of course we are using here the fact that the flat torus is a compact manifold). Hence, we eventually deduce that

$$\varphi_\infty(x) = \tilde{\varphi}_\infty(x - x_0). \tag{6.58}$$

Namely, we have proved that, if $\{t_n\}$ is a diverging sequence of times and z is a stable solution with $\boldsymbol{m} \neq \boldsymbol{0}$, then a suitable subsequence of $\varphi(t_n)$ tends to a limit φ_∞ such that

$$-\Delta\varphi_\infty(\cdot + x_0) + F'(\varphi_\infty(\cdot + x_0)) = \mu_\infty + \vartheta_\infty \tag{6.59}$$

for some $x_0 \in \Omega$, where of course x_0 may depend on the chosen subsequence of t_n, and where the constants μ_∞, ϑ_∞ are characterized as in the lemma.

6.4 End of proof of Theorem 3.5

Along this section, in view of Remark 2.2, we will often intend solutions as a quadruples instead of triples. So, with a small abuse of language, we will sometimes write $z = (\boldsymbol{u}, \varphi, \mu, \vartheta)$ in place of $z = (\boldsymbol{u}, \varphi, \vartheta)$. We also recall that here we just consider the case $\boldsymbol{m} = \boldsymbol{0}$. As in Lemma 6.9, we will note as $\mathcal{S}_0 = \mathcal{S}_0(\boldsymbol{0}, m, M)$ the set of all the quadruples $(\boldsymbol{u}_\infty, \varphi_\infty, \mu_\infty, \vartheta_\infty)$, with $\boldsymbol{u}_\infty = \boldsymbol{0} \in \mathbb{R}^2$, $\mu_\infty, \vartheta_\infty \in \mathbb{R}$, $\varphi_\infty \in H^4(\Omega)$, satisfying equation (6.43) as well as the bound (6.44) and the constraints

$$\boldsymbol{u}_\infty \equiv \boldsymbol{0}, \quad (\varphi_\infty)_\Omega = m, \quad \frac{1}{2}|\boldsymbol{u}_\infty|^2 + \frac{1}{2}\|\nabla\varphi_\infty\|^2 + \int_\Omega (F(\varphi_\infty) + \vartheta_\infty) = M. \tag{6.60}$$

We also recall that, if z_∞ is any element of the ω-limit of a stable solution, then $z_\infty \in \mathcal{S}_0$ automatically satisfies the estimate (6.44) with C_∞ as in Lemma 6.9. However, the values μ_∞ and ϑ_∞ are not univocally determined. Namely, even if $\boldsymbol{m} = \boldsymbol{0}$, m and M are assigned, different elements of the ω-limit of a solution emanating from a datum $z_0 \in \mathcal{V}^r_{\boldsymbol{0},m,M}$ may solve (6.43) for different values of μ_∞ and ϑ_∞. Physically speaking, the limit value ϑ_∞ of the ϑ-component of a trajectory tells us how much chemical and kinetic energy is converted into heat. This does not just depend on the initial value of the energy and of the mass, but also, for instance, on how far the initial value φ_0 is from the chemical equilibrium. For the same reason, we expect that, in general, we cannot obtain a *lower* bound for ϑ_∞ holding uniformly for all initial data lying in $\mathcal{S}_0 = \mathcal{S}_0(\mathbf{0}, m, M)$. In other words, it may happen that, for fixed values of the mass m and of the energy M, there exist initial data in $\mathcal{V}^r_{\mathbf{0},m,M}$ for which the "limit temperature" ϑ_∞ may be arbitrarily close to 0. This may in principle happen if the initial temperature is also close to 0 and the initial chemical configuration is so favorable that no energy (or almost no energy) is converted into heat.

So, it is to avoid this situation that we need to restrict ourselves to those configurations for which the initial entropy is greater than some assigned value $-R$, namely, for initial data lying in $\mathcal{V}^{r,R} := \{z \in \mathcal{V}^r : (-\log\vartheta)_\Omega \le R\}$. Then we can first prove that $\mathcal{V}^{r,R}$ is invariant for $S(t)$:

Lemma 6.10. *Let $z_0 \in \mathcal{V}^{r,R}$. Then, $z(t) = S(t)z_0 \in \mathcal{V}^{r,R}$ for all $t \ge 0$. Moreover, for any element $(\boldsymbol{u}_\infty, \varphi_\infty, \mu_\infty, \vartheta_\infty)$ of the ω-limit of $z(t)$ we have*

$$\vartheta_\infty \ge e^{-R}. \tag{6.61}$$

Proof. The invariance property follows from the entropy inequality (6.6). Then, by Jensen's inequality we also have

$$-\log(\vartheta(t)_\Omega) \le (-\log\vartheta(t))_\Omega \le R \quad \forall\, t \ge 0,$$

whence, letting $t_n \nearrow \infty$ in such a way that $z(t_n)$ tends to an element of the ω-limit, the second assert follows. ∎

We can now complete the proof of Theorem 3.5. Actually, thanks to the above invariance property, if $z_0 \in \mathcal{V}^{r,R}_{\mathbf{0},m,M}$, any element z_∞ of its ω-limit set not only lies in $\mathcal{S}_0$, but it also satisfies $z_\infty \in \mathcal{V}^{r,R}_{\mathbf{0},m,M}$. We can then set $\mathcal{S}_0^R := \mathcal{S}_0 \cap \mathcal{V}^{r,R}_{\mathbf{0},m,M}$. As a consequence, the limit temperature satisfies (6.61), namely, it is bounded from below by a quantity that only depends on the phase space and is actually independent of the specific choice of the initial datum.

Thanks to this fact, we may prove existence of the global attractor. As remarked before, we will combine the *asymptotic compactness* of the semiflow $S(t)$ proved in Theorem 6.7 with the following *point dissipativity* property:

Theorem 6.11. *Let $m, M, R \in \mathbb{R}$ be assigned. Then there exists a bounded subset $\mathcal{B} = \mathcal{B}(\mathbf{0}, m, M) \subset \mathcal{V}^{r,R}_{\mathbf{0},m,M}$ such that, for any $z_0 = (\boldsymbol{u}_0, \varphi_0, \vartheta_0) \in \mathcal{V}^{r,R}_{\mathbf{0},m,M}$, there exists $T_0 = T_0(z_0)$ such that $S(t)z_0 \in \mathcal{B}$ for all $t \ge T_0$.*

Proof. As usual, we proceed by contradiction. Note that, by Lemmas 6.9, 6.10, we have

$$|\mu_\infty| + |\vartheta_\infty| + \|\varphi_\infty\|_{H^4(\Omega)} \le C_\infty, \qquad \vartheta_\infty \ge e^{-R} \tag{6.62}$$

for any $z_\infty = (\boldsymbol{u}_\infty, \varphi_\infty, \mu_\infty, \vartheta_\infty)$ belonging to $\mathcal{S}_0^R$. We introduce $\mathcal{B}$ as a sort of neighbourhood of $\mathcal{S}_0^R$ in $\mathcal{V}^r_{\mathbf{0},m,M}$. Namely, we define $\mathcal{B}$ as the set of those functions $(\boldsymbol{u}, \varphi, \vartheta) \in \mathcal{V}^r$ such that $\boldsymbol{u}_\Omega = \mathbf{0}$, $\varphi_\Omega = m$, $\mathcal{E}(\boldsymbol{u}, \varphi, \vartheta) = M$ (recall that μ plays the role of an auxiliary variable), and there exists $(\boldsymbol{u}_\infty, \varphi_\infty, \mu_\infty, \vartheta_\infty) \in \mathcal{S}_0^R$ such that (6.60) holds together with

$$\|\boldsymbol{u}\|_{H^{1+r}(\Omega)} + \|\varphi - \varphi_\infty\|_{H^3(\Omega)} \le 1, \qquad \|\vartheta - \vartheta_\infty\|_{H^{3/2}(\Omega)} \le c_*, \tag{6.63}$$

where c_* is chosen in the following way: denoting as c_Ω the embedding constant of $H^{3/2}(\Omega)$ into $C(\Omega)$ we ask that $c_* c_\Omega \le \frac{e^{-R}}{4}$ in such a way that if ϑ is the last component of an element $z \in \mathcal{B}$, then for every $x \in \Omega$ there holds

$$\vartheta(x) \ge \vartheta_\infty - |\vartheta(x) - \vartheta_\infty| \ge e^{-R} - c_\Omega \|\vartheta - \vartheta_\infty\|_{H^{3/2}(\Omega)} \ge \frac{3}{4}e^{-R}. \tag{6.64}$$

Hence, ϑ is separated from 0 in the uniform norm. Let us now assume, by contradiction, that there exist $z_0 \in \mathcal{V}^{r,R}_{\mathbf{0},m,M}$ and a sequence $\{t_n\}$ with $t_n \nearrow \infty$ such that $S(t_n)z_0 \notin \mathcal{B}$ for any $n \in \mathbb{N}$. By asymptotic compactness there exists a (nonrelabelled) subsequence such that $S(t_n)z_0 \to z_\infty$ in $\mathcal{V}^r$ (and "weakly" in $\mathcal{W}$) with $z_\infty \in \mathcal{S}_0^R$ (and the last component ϑ_∞ satisfies (6.61)). Note that we actually used the fact $\vartheta(t_n) \to \vartheta_\infty$ in $H^{3/2}(\Omega)$, which is true because we have asymptotic boundedness of $\vartheta(t_n)$ in $H^2(\Omega)$. This gives a contradiction. ∎

To prove existence of the global attractor, we need to exhibit however the existence of a *uniformly absorbing* set, i.e., of a set that eventually contains bundles of trajectories starting from any bounded set in the phase space. This is a stronger property with respect of that shown above (indeed, $\mathcal{B}$ only absorbs single trajectories). This fact is proved in the following lemma that adapts the ideas of a classical argument (see, e.g., [2, 13]):

Lemma 6.12. *Let the assumptions of the previous lemma hold and let $R > 0$. Let $\mathcal{B}_0$ be defined as the set of those triples $(\boldsymbol{u}, \varphi, \vartheta) \in \mathcal{V}^{r,R}_{\mathbf{0},m,M}$ such that there exists $(\overline{\boldsymbol{u}}, \overline{\varphi}, \overline{\vartheta}) \in \mathcal{B}$ with*

$$\|\boldsymbol{u} - \overline{\boldsymbol{u}}\|_{H^{1+r}(\Omega)} + \|\varphi - \overline{\varphi}\|_{H^3(\Omega)} + \|\vartheta - \overline{\vartheta}\|_{H^{3/2}(\Omega)} < c_*. \tag{6.65}$$

Then, let also

$$\mathcal{B}_1 := \bigcup_{t \geq 0} S(t)\mathcal{B}_0. \tag{6.66}$$

Then, $\mathcal{B}_1$ is a bounded in $\mathcal{V}^r$ and is a uniformly absorbing set.

Proof. First of all, note that, by construction, if $(\boldsymbol{u}, \varphi, \vartheta) \in \mathcal{B}_0$, then $\vartheta(x) > e^{-R}/2$ for every $x \in \Omega$. Hence, in particular, $\mathcal{B}_0$ is a bounded subset of $\mathcal{V}^r$ and, consequently, $\mathcal{B}_1$ is also bounded in $\mathcal{V}^r$ by virtue of the uniform estimates (recall, e.g., (6.19)). Moreover, $\mathcal{B}_0$ is an open set in $H^{1+r}(\Omega) \times H^3(\Omega) \times H^{3/2}(\Omega)$ because it is the union over $(\overline{\boldsymbol{u}}, \overline{\varphi}, \overline{\vartheta})$ ranging in $\mathcal{B}$ of triples satisfying the strict inequality (6.65).

Let now B be a given bounded set of $\mathcal{V}^{r,R}_{\mathbf{0},m,M}$. We then claim that there exists T_B such that, for any $t \geq T_B$, $S(t)B \subset \mathcal{B}_1$. To prove this fact, let us argue once more by contradiction. Namely, let us assume there exist a sequence $\{z_n\} \subset B$ and a sequence $t_n \nearrow \infty$ such that $S(t_n)z_n \notin \mathcal{B}_1$ for any $n \in \mathbb{N}$. Then, we observe that, due to uniform smoothing of trajectories, $S(1)B$ is bounded in $\mathcal{W}$ and consequently relatively compact in $\mathcal{V}^r$. Hence, we can assume that (up to a subsequence) $S(1)z_n$ tends to some limit z^1, say, in $\mathcal{V}^r$ (of course a stronger convergence holds). We claim that $S(t)z^1 \notin \mathcal{B}_0$ for any $t \geq 1$. Indeed, let us fix $t \geq 1$. Then, at least for n large enough, we have that $S(t_n)z_n = S(t_n - t - 1)S(t)S(1)z_n$ and $S(t)S(1)z_n$ cannot lie in $\mathcal{B}_0$, otherwise, by (6.66), it would be $S(t_n)z_n \in \mathcal{B}_1$, a contradiction. Hence, $S(t)S(1)z_n$ does not lie in $\mathcal{B}_0$, at least for n large enough depending on the chosen t. This means that, for any n large enough, $S(t)S(1)z_n$ lies in the complement of $\mathcal{B}_0$, which is a closed set in $H^{1+r}(\Omega) \times H^3(\Omega) \times H^{3/2}(\Omega)$. In view of the facts that $S(1)z_n$ tends to z^1 in $H^{1+r}(\Omega) \times H^3(\Omega) \times H^{3/2}(\Omega)$ and of the continuity of the operator $S(t)$ from that space into itself, we can take the limit $n \nearrow \infty$ to deduce that $S(t)z^1$ lies in the complement of $\mathcal{B}_0$ for all $t \geq 0$. On the other hand, because $z^1 \in \mathcal{V}^{r,R}_{\mathbf{0},m,M}$, the trajectory starting from z^1 has a nonempty ω-limit set all of whose elements lie in $\mathcal{S}^R_0$, hence in $\mathcal{B}_0$, which gives a contradiction. ∎

References

[1] N.D. Alikakos, *L^p bounds of solutions of reaction-diffusion equations*, Comm. Partial Differential Equations, **4** (1979), 827–868.

[2] J.M. Ball, *Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equations*, J. Nonlinear Sci., **7** (1997), 475–502.

[3] G. Bonfanti, M. Frémond, and F. Luterotti, *Global solution to a nonlinear system for irreversible phase changes*, Adv. Math. Sci. Appl., **10** (2000), 1–24.

[4] M. Bulíček, E. Feireisl, and J. Málek, *A Navier-Stokes-Fourier system for incompressible fluids with temperature dependent material coefficients*, Nonlinear Anal. Real World Appl., **10** (2009), 992–1015.

[5] J. Cahn and J. Hilliard, *Free energy of a nonuniform system. I. Interfacial free energy*, J. Chem. Phys., **28** (1958), 258–267.

[6] P. Colli, F. Luterotti, G. Schimperna, and U. Stefanelli, *Global existence for a class of generalized systems for irreversible phase changes*, NoDEA Nonlinear Differential Equations Appl., **9** (2002), 255–276.

[7] M. Eleuteri, J. Kopfová, and P. Krejčí, *A new phase field model for material fatigue in an oscillating elastoplastic beam*, Discrete Contin. Dyn. Syst., **35** (2015), 2465–2495

[8] M. Eleuteri, E. Rocca, and G. Schimperna, *On a non-isothermal diffuse interface model for two-phase flows of incompressible fluids*, Discrete Contin. Dyn. Syst., **35** (2015), 2497–2522.

[9] M. Eleuteri, E. Rocca, and G. Schimperna, *Existence of solutions to a two-dimensional model for nonisothermal two-phase flows of incompressible fluids*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **33** (2016), 1431–1454.

[10] E. Feireisl, M. Frémond, E. Rocca, and G. Schimperna, *A new approach to non-isothermal models for nematic liquid crystals*, Arch. Ration. Mech. Anal., **205** (2012), 651–672.

[11] E. Feireisl, E. Rocca, G. Schimperna, and A. Zarnescu, *Nonisothermal nematic liquid crystal flows with the Ball-Majumdar free energy*, Ann. Mat. Pura Appl. (4), **194** (2015), 1269–1299.

[12] M. Grasselli, A. Miranville, R. Rossi, and G. Schimperna, *Analysis of the Cahn-Hilliard equation with a chemical potential dependent mobility*, Comm. Partial Differential Equations, **36** (2011), 1193–1238.

[13] J.K. Hale, Asymptotic behavior of dissipative systems. Mathematical Surveys and Monographs, 25. American Mathematical Society, Providence, RI, 1988.

[14] Ph. Laurençot, *Solutions to a Penrose-Fife model of phase-field type*, J. Math. Anal. Appl., **185** (1994), 262–274.

[15] Ph. Laurençot, G. Schimperna, and U. Stefanelli, *Global existence of a strong solution to the one-dimensional full model for irreversible phase transitions*, J. Math. Anal. Appl., **271** (2002), 426–442.

[16] F. Luterotti, G. Schimperna, and U. Stefanelli, *Existence result for a nonlinear model related to irreversible phase changes*, M3AS Math. Models Methods Appl. Sci., **11** (2001), 808–825.

[17] F. Luterotti, G. Schimperna, and U. Stefanelli, *Global solution to a phase field model with irreversible and constrained phase evolution*, Quart. Appl. Math., **60** (2002), 301–316.

[18] J.C. Robinson, Infinite-dimensional dynamical systems. An introduction to dissipative parabolic PDEs and the theory of global attractors, Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2001.

[19] E. Rocca and R. Rossi, *"Entropic" solutions to a thermodynamically consistent PDE system for phase transitions and damage*, SIAM J. Math. Anal., **47** (2015), 2519–2586.

[20] G. Schimperna, A. Segatti, and S. Zelik, *Asymptotic uniform boundedness of energy solutions to the Penrose-Fife model*, J. Evol. Equ., **12** (2012), 863–890.

[21] G. Schimperna and S. Zelik, *Existence of solutions and separation from singularities for a class of fourth order degenerate parabolic equations*, Trans. Amer. Math. Soc., **365** (2013), 3799–3829.

[22] G.R. Sell and Y. You, Dynamics of evolutionary equations, Springer, 1999.

[23] P. Sun, C. Liu, and J. Xu, *Phase field model of thermo-induced Marangoni effects in the mixtures and its numerical simulations with mixed finite element method*, Commun. Comput. Phys., **6** (2009), 1095–1117.

[24] R. Temam, Navier-Stokes equations and nonlinear functional analysis, CBMS-NSF Regional Conference Series in Applied Mathematics, 41. SIAM, Philadelphia, 1983.

[25] R. Temam, Infinite-dimensional dynamical systems in mechanics and physics. Second edition. Applied Mathematical Sciences, 68. Springer-Verlag, New York, 1997.