A compactness result for the Sobolev embedding via potential theory *

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Devoted to Emmanuele Di Benedetto in occasion of his 70th birthday

Abstract

In this note we give a proof of the Sobolev and Morrey embedding theorems based on the representation of functions in terms of the fundamental solution of suitable partial differential operators. We also prove the compactness of the Sobolev embedding. We first describe this method in the classical setting, where the fundamental solution of the Laplace equation is used, to recover the classical Sobolev and Morrey theorems. We next consider degenerate Kolmogorov equations. In this case, the fundamental solution is invariant with respect to a non-Euclidean translation group and the usual convolution is replaced by an operation that is defined in accordance with this geometry. We recover some known embedding results and we prove the compactness of the Sobolev embedding. We finally apply our regularity results to a kinetic equation.

Keywords: Sobolev spaces, Sobolev embedding, Morrey embedding, Compactness, Fundamental solution, Kolmogorov equation.

1 Introduction

Sobolev and Morrey embedding theorems are fundamental tools in the regularity theory for Elliptic and Parabolic second order Partial Differential Equations (*PDEs* in the sequel). In particular, they play a crucial role in the natural setting for the study of uniformly elliptic PDEs in divergence form, that is the Sobolev space $W^{1,p}$.

There are several proofs of the Sobolev and Morrey embedding theorems, all of them rely on some integral representation of a general function $u \in W^{1,p}$ in terms of its gradient. Here we focus in particular on representation formulas based on the fundamental solution of the Laplace equation.

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Consider a function $u \in C_0^\infty(\mathbb{R}^n)$. By the very definition of fundamental solution Γ, the following identity holds

$$u(x) = -\int_{\mathbb{R}^n} \Gamma(x-y)\Delta u(y)\, dy, \qquad \text{for every } x \in \mathbb{R}^n, \tag{1.1}$$

and an integration by parts immediately gives

$$u(x) = \int_{\mathbb{R}^n} \langle \nabla_y \Gamma(x-y), \nabla u(y)\rangle\, dy, \qquad \text{for every } x \in \mathbb{R}^n, \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ and ∇ denote the usual inner product in $\mathbb{R}^n$ and the gradient, respectively. We recall that the gradient of the fundamental solution of the Laplace equation writes as follows

$$\nabla \Gamma(x-y) = -\frac{1}{n\omega_n |x-y|^n}(x-y), \qquad x \neq y, \tag{1.3}$$

where ω_n is the measure of the n-dimensional unit ball. In particular, $\nabla\Gamma$ is an homogeneous function of degree $-n+1$, and there exists a positive constant c_n such that

$$|\nabla\Gamma(x-y)| \le c_n |x-y|^{1-n}, \tag{1.4}$$

thus (1.2) yields the following inequality:

$$|u(x)| \le c_n \int_{\mathbb{R}^n} |x-y|^{1-n} |\nabla u(y)| dy. \tag{1.5}$$

The Young inequality for convolution with homogeneous kernels (see, for instance, Theorem 1, p. 119 in [16]) then gives

$$\|\nabla\Gamma * \nabla u\|_{L^{p^*}(\mathbb{R}^n)} \le C_p \|\nabla u\|_{L^p(\mathbb{R}^n)}, \qquad 1 < p < n, \tag{1.6}$$

where $p^* = \frac{pn}{n-p}$ is the Sobolev conjugate of p, and C_p is a positive constant which only depends on p and on the dimension n. Here and in the sequel the dependence on n will be often omitted. As a consequence we find

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C_p \|\nabla u\|_{L^p(\mathbb{R}^n)}, \qquad \text{for every } u \in C_0^\infty(\mathbb{R}^n), \quad 1 < p < n. \tag{1.7}$$

From the above inequality we plainly obtain the following Sobolev inequality for any open set $\Omega \subseteq \mathbb{R}^n$

$$\|u\|_{L^q(\Omega)} \le C_{p,q} \|u\|_{W^{1,p}(\Omega)}, \qquad \text{for every } u \in W_0^{1,p}(\Omega), \tag{1.8}$$

with $1 < p < n$ and $p \le q \le p^*$. Here $C_{p,q}$ is a positive constant which only depends on p, q and n. By a standard argument (1.6) also gives the Sobolev embedding theorem for $W^{1,p}(\Omega)$ provided that the boundary of Ω is sufficiently smooth.

The Morrey inequality (see Theorem 2.4 below) can be obtained by the representation formula (1.2), by using the following fact: there exists a positive constant M_n, only depending on n, such that with

$$|\partial_{x_j}\Gamma(x) - \partial_{x_j}\Gamma(y)| \le M_n \frac{|x-y|}{|x|^n}, \quad \text{for } j = 1, \dots, n, \tag{1.9}$$

for every $x, y \in \mathbb{R}^n \setminus \{0\}$ such that $|x-y| \le |x|/2$. Indeed, a rather simple argument based on (1.9) provides us with the following bound: if $u \in W_0^{1,p}(\Omega)$, with $p > n$, then

$$|u(x) - u(y)| \le \widetilde{C}_p \, \|\nabla u\|_{L^p(\Omega)} |x-y|^{1-\frac{n}{p}}, \qquad \text{for every } x, y \in \Omega, \tag{1.10}$$

for some positive constant $\widetilde{C}_p$ only depending on p and n.

It is worth noting that the inequality (1.9) can be also used to prove the compactness of the Sobolev embedding (1.8) for $p < q < p^*$, if Ω is a bounded open set. As we will see in the sequel, the following estimates holds for $p < q < p^*$: there exists a positive constant $\widetilde{C}_{p,q}$ such that

$$\|u(h + \cdot) - u\|_{L^q(\Omega)} \le \widetilde{C}_{p,q} \, \|\nabla u\|_{L^p(\Omega)} \; |h|^{n\left(\frac{1}{q} - \frac{1}{p^*}\right)}, \tag{1.11}$$

for every $u \in C_0^\infty(\Omega)$ and for every $h \in \mathbb{R}^n$ sufficiently small. Note that the exponent in the right hand side of (1.11) belongs to the interval $]0,1[$ if, and only if, $p < q < p^*$, then in this case we have

$$\|u(h + \cdot) - u\|_{L^q(\Omega)} \to 0 \qquad \text{as} \qquad |h| \to 0.$$

This inequality provides us with the *integral uniform continuity*, which is needed for the compactness in the L^q spaces. We also observe that $n\Big(\dfrac{1}{q} - \dfrac{1}{p^*}\Big) \to 1$ as $q \to p$. We then retrieve a known result contained for instance in [16], Chapter V, Section 3.5.

The advantage of the method described above, with respect to other ones, is in that it only requires the existence of a *fundamental solution* and its *homogeneity* properties. In particular, it applies to the function spaces introduced by Folland [8] for the study second order linear differential operators that satisfy the Hörmander's condition (see [10]). It should be noticed that this approach has also a drawback, in that it does not provide us with the Sobolev inequality for $p = 1$. On the other hand it is unifying, as it gives the Sobolev and Morrey embedding theorems and a compactness result by using a single representation formula.

To clarify the use of this method to the study of the so-called Hörmander's operators we next focus on the degenerate Kolmogorov $\mathscr{L}_0$ on $\mathbb{R}^{2n+1}$, which is one of the simplest examples belonging to this class. Let Ω be an open subset of $\mathbb{R}^{2n+1}$ and let u be a smooth real valued function defined on Ω. We denote the variable of $\mathbb{R}^{2n+1}$ as follows $z = (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, and we set

$$\mathscr{L}_0 u := \Delta_x u + \langle x, \nabla_y u \rangle - \partial_t u, \qquad \Delta_x u := \sum_{j=1}^n \partial_{x_j}^2 u. \tag{1.12}$$

As we will see in the sequel (see equation (4.1) below) the function Γ defined as

$$\begin{cases} \Gamma(x,y,t) = \dfrac{\widetilde{c}_n}{t^{2n}} \exp\left(-\frac{|x|^2}{t} - 3\frac{\langle x, y \rangle}{t^2} - 3\frac{|y|^2}{t^3} \right), & \text{for } (x,y,t) \in \mathbb{R}^{2n} \times]0, +\infty[, \\ \Gamma(x,y,t) = 0, & \text{for } (x,y,t) \in \mathbb{R}^{2n} \times]-\infty, 0], \end{cases}$$

is the fundamental solution of $\mathscr{L}_0$. Here $\widetilde{c}_n = \frac{3^{n/2}}{(2\pi)^n}$. In particular, in analogy with the heat equation, we have that the function u defined as

$$
\begin{aligned}
u(x,y,t) = & \int_{\mathbb{R}^{2n}} \Gamma(x-\xi, y+t\xi-\eta, t-t_0)\varphi(\xi,\eta) d\xi\, d\eta - \\
& \int_{\mathbb{R}^{2n}\times]t_0,t[} \Gamma(x-\xi, y+(t-\tau)\xi-\eta, t-\tau) f(\xi,\eta,\tau) d\xi\, d\eta\, d\tau
\end{aligned}
\tag{1.13}
$$

is a solution to the following Cauchy problem

$$
\begin{cases}
\mathscr{L}_0 u = f & \text{in} \quad \mathbb{R}^{2n}\times]t_0, +\infty[, \\
u_{|t=t_0} = \varphi & \text{in} \quad \mathbb{R}^{2n}.
\end{cases}
$$

whenever f and φ are bounded continuous functions.

A remarkable fact is that a kind of convolution is hidden in the expression (1.13). More specifically, we define the operation "$\circ$" by setting

$$
(x,y,t)\circ(\xi,\eta,\tau) := (x+\xi, y+\eta+\tau x, t+\tau), \qquad (x,y,t),(\xi,\eta,\tau)\in\mathbb{R}^{2n+1}, \tag{1.14}
$$

and we note that $\big(\mathbb{R}^{2n+1},\circ\big)$ is a non commutative group. The identity of the group is $(0,0,0)$ and the inverse of (x,y,t) is $(-x,-y+xt,-t)$. With this notation, it is easy to check that the expression appearing in (1.13) can be written as follows

$$
(x-\xi, y+(t-\tau)\xi-\eta, t-\tau) = (\xi,\eta,\tau)^{-1}\circ(x,y,t).
$$

Moreover, the group $\big(\mathbb{R}^{2n+1},\circ\big)$ is homogeneous with respect to the dilation defined as $d_r(x,y,t) := \big(rx, r^3y, r^2t\big)$, in the sense that

$$
d_r\big((x,y,t)\circ(\xi,\eta,\tau)\big) = d_r(x,y,t)\circ d_r(\xi,\eta,\tau), \qquad (x,y,t),(\xi,\eta,\tau)\in\mathbb{R}^{2n+1}, r>0. \tag{1.15}
$$

This algebraic structure was introduced and studied by Lanconelli and Polidoro in [11]. In [11] it was also noticed that Γ is homogeneous of degree $-4n$ with respect to $(d_t)_{r>0}$, that is

$$
\Gamma\big(d_r(x,y,t)\big) = \frac{1}{r^{4n}}\Gamma(x,y,t), \qquad (x,y,t)\in\mathbb{R}^{2n+1}, r>0. \tag{1.16}
$$

Moreover, if we let $z=(x,y,t), \zeta=(\xi,\eta,\tau)$, then (1.13) can be written as follows

$$
u(z) = \int_{\mathbb{R}^{2n}} \Gamma((\xi,\eta,t_0)^{-1}\circ z)\varphi(\xi,\eta) d\xi\, d\eta - \int_{\mathbb{R}^{2n}\times]t_0,t[} \Gamma(\zeta^{-1}\circ z) f(\zeta) d\zeta. \tag{1.17}
$$

In particular, if $u\in C_0^\infty(\mathbb{R}^{2n+1})$ and $\mathrm{supp}(u)\subset\big\{t>t_0\big\}$, then we have that

$$
u(z) = -\int_{\mathbb{R}^{2n+1}} \Gamma(\zeta^{-1}\circ z)\mathscr{L}_0 u(\zeta)\, d\zeta, \quad \text{for every } z\in\mathbb{R}^{2n+1}, \tag{1.18}
$$

which is analogous to (1.1). Summarizing: the operation in (1.18) is considered here as a *convolution* with respect to the non-Euclidean operation "$\circ$" defined in (1.14), with a kernel Γ that is homogeneous whit respect to the *anisotropic* dilation d_r. Based on this representation formula, we prove Sobolev and Morrey theorems for solutions to Kolmogorov equations in divergence form $\mathscr{L}u = \mathrm{div}_x F + f$, where

$$\mathscr{L}u := \mathrm{div}_x \left(A(z)\nabla_x u\right) + \langle x, \nabla_y u\rangle - \partial_t u. \tag{1.19}$$

Here A is a $n \times n$ symmetric matrix with bounded and measurable coefficients and, for every vector field $F \in C^1(\mathbb{R}^{2n+1}, \mathbb{R}^n)$ we denote $\mathrm{div}_x F(x,y,t) := \sum_{j=1}^n \partial_{x_j} F_j(x,y,t)$. In order to simplify our treatment, we suppose that $F = 0$ and $f = 0$, so that u is a solution of $\mathscr{L}u = 0$. In this case we have that $\mathscr{L}_0 u = \mathrm{div}_x (I_n - A)\nabla_x u$, where I_n denotes the $n \times n$ identity matrix. Then, an integration by parts in (1.18) gives

$$u(z) = \int_{\mathbb{R}^{2n+1}} \langle (I_n - A(\zeta))\nabla_\xi \Gamma(\zeta^{-1} \circ z), \nabla_\xi u(\zeta)\rangle \, d\zeta, \tag{1.20}$$

for every solution u to $\mathscr{L}u = 0$. It is known that the derivatives $\partial_{\xi_1}\Gamma, \dots, \partial_{\xi_n}\Gamma$ are homogeneous functions of degree $-(2n+1)$ with respect to the dilation $(d_r)_{r>0}$. Moreover, the coefficients of the matrix $I_n - A$ are bounded, then the above identity provides us with the analogous of (1.2) for the solutions u to the equation $\mathscr{L}u = 0$.

We point out that only the derivatives with respect to the first n variables of the gradient of u appear in the representation formula (1.20), then a Sobolev inequality holding for all functions cannot be obtained from (1.20), because of the lack of information on the remaining n direction. Nevertheless, this formula is used by Cinti, Pascucci and Polidoro in [14, 5] to prove a Sobolev embedding theorem for *solutions* to the Kolmogorov equation $\mathscr{L}u = 0$. Indeed, in [14, 5] the Sobolev theorem for *solutions* is combined with a Caccioppoli inequality, still for *solutions*, in order to apply the Moser's iterative method and prove an L^∞_{loc} estimate for the solutions to $\mathscr{L}u = 0$. We also recall that a Morrey result for the solutions to $\mathscr{L}u = \mathrm{div}_x F$ was proven by Manfredini and Polidoro in [13], and later by Polidoro and Ragusa in [15] by the same method used here.

In this note we are concerned with the compactness of the Sobolev embedding for the solutions to $\mathscr{L}u = 0$ for a family of degenerate Kolmogorov equations, defined on $\mathbb{R}^{N+1}$, that will be still denoted by $\mathscr{L}$. As we will see in Section 3, the operator (1.19) is the prototype of this family of degenerate operators, and in this case, $N = 2n$. In Section 3 we introduce the notation that will be used in the following part of this introduction, and we will state the conditions (H.1) and (H.2) that ensure that the *principal part* $\mathscr{L}_0$ of $\mathscr{L}$ has a smooth fundamental solution Γ, which is invariant with respect to a translation analogous to (1.14), and homogeneous of degree $-Q$, with respect to a dilation analogous to (1.15). We will refer to the positive integer $Q+2$ as *homogeneous dimension* of the space $\mathbb{R}^{N+1}$ and plays the role of n in the Euclidean setting $\mathbb{R}^n$ where the elliptic operators are studied. In the sequel p^* and p^{**} denote the positive numbers such that

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{Q+2}, \qquad \frac{1}{p^{**}} = \frac{1}{p} - \frac{2}{Q+2}. \tag{1.21}$$

Clearly, p^* and p^{**} are finite and positive whenever $1 \le p < Q+2$ and $1 \le p < \dfrac{Q+2}{2}$, respectively.

Our main result is the following Theorem. It provides us with some estimates of the convolution of a function belonging to some L^p space with the fundamental solution Γ and with its derivatives $\partial_{x_j}\Gamma$, $j = 1, \dots, m_0$, with $m_0 \le N$. These estimates, applied to the representation formula for solutions to $\mathscr{L}u = 0$ given in Theorem 4.1, yield Sobolev theorems, Morrey theorems and the compactness of the Sobolev embedding.

Theorem 1.1 *Let $\mathscr{L}$ be an operator in the form* (3.1)*, satisfying the hypotheses* (H.1) *and* (H.2) *in Section 3, and let Γ be the fundamental solution of its* principal part*. Let also $Q+2$ be the homogeneous dimension of the space $\mathbb{R}^{N+1}$, and let p be such that $1 \le p < +\infty$. For every $f, g_j \in L^p(\mathbb{R}^{N+1})$ we let u, v_j be defined as follows*

$$u(z) = \int_{\mathbb{R}^{N+1}} \Gamma(\zeta^{-1} \circ z) f(\zeta)\, \mathrm{d}\zeta, \quad v_j(z) = \int_{\mathbb{R}^{N+1}} \partial_{x_j}\Gamma(\zeta^{-1} \circ z) g_j(\zeta)\, \mathrm{d}\zeta, \quad j = 1, \dots, m_0.$$

Then, for every $j = 1, \dots, m_0$ we have:

- *(Sobolev) if $1 < p < Q+2$, then there exists a positive constant C_p such that*

$$\|v_j\|_{L^{p^*}(\mathbb{R}^{N+1})} \le C_p \|g_j\|_{L^p(\mathbb{R}^{N+1})},$$

- *(Compactness) if moreover $p < q < p^*$, then there exists a positive constant $\widetilde{C}_{p,q}$ such that*

$$\|v_j(\cdot \circ h) - v_j\|_{L^q(\mathbb{R}^{N+1})} \le \widetilde{C}_{p,q} \|g_j\|_{L^p(\mathbb{R}^{N+1})}\ \|h\|^{(Q+2)\left(\frac{1}{q} - \frac{1}{p^*}\right)},$$

  for every $h \in \mathbb{R}^{N+1}$,

- *(Morrey) if $p > Q+2$, then there exists a positive constant $\widetilde{C}_p$ such that*

$$|v_j(z) - v_j(\zeta)| \le \widetilde{C}_p \|g_j\|_{L^p(\mathbb{R}^{N+1})} \|\zeta^{-1} \circ z\|^{1 - \frac{Q+2}{p}}, \quad \text{for every } z, \zeta \in \mathbb{R}^{N+1}.$$

We also have

- *(Sobolev) if $1 < p < \dfrac{Q+2}{2}$, then there exists a positive constant C_p such that*

$$\|u\|_{L^{p^{**}}(\mathbb{R}^{N+1})} \le C_p \|f\|_{L^p(\mathbb{R}^{N+1})},$$

- *(Compactness) if $p^* < q < p^{**}$, then there exists a positive constant $\widetilde{C}_{p,q}$ such that*

$$\|u(\cdot \circ h) - u\|_{L^q(\mathbb{R}^{N+1})} \le \widetilde{C}_{p,q} \|f\|_{L^p(\mathbb{R}^{N+1})}\ \|h\|^{(Q+2)\left(\frac{1}{q} - \frac{1}{p^{**}}\right)},$$

  for every $h \in \mathbb{R}^{N+1}$,

- *(Morrey) if $\dfrac{Q+2}{2} < p < Q+2$, then there exists a positive constant $\widetilde{C}_p$ such that*

$$|u(z) - u(\zeta)| \le \widetilde{C}_p \|f\|_{L^p(\mathbb{R}^{N+1})} \|\zeta^{-1} \circ z\|^{2 - \frac{Q+2}{p}}, \qquad \text{for every } z, \zeta \in \mathbb{R}^{N+1}.$$

From the above result and a representation formula for the solution to $\mathscr{L}u = 0$ we obtain the following result.

Theorem 1.2 *Let Ω be an open set of $\mathbb{R}^{N+1}$, and let u be a weak solution to $\mathscr{L}u = 0$ in Ω. Suppose that $u, \partial_{x_1}u, \dots, \partial_{x_{m_0}}u \in L^p(\Omega)$. Then for every compact set $K \subset \Omega$, there exist a positive constant $\widetilde{\varrho}$ such that we have:*

- *(Sobolev embedding) if $1 < p < Q+2$, then there exists a positive constant C_p such that*
$$\|u\|_{L^{p^*}(K)} \le C_p \bigg(\|u\|_{L^p(\Omega)} + \sum_{j=1}^{m_0} \|\partial_{x_j}u\|_{L^p(\Omega)} \bigg),$$
- *(Compactness) if moreover $p < q < p^*$, then there exists a positive constant $\widetilde{C}_{p,q}$ such that*
$$\|u(\cdot \circ h) - u\|_{L^q(K)} \le \widetilde{C}_{p,q} \bigg(\|u\|_{L^p(\Omega)} + \sum_{j=1}^{m_0} \|\partial_{x_j}u\|_{L^p(\Omega)} \bigg) \; \|h\|^{(Q+2)\left(\frac{1}{q} - \frac{1}{p^*}\right)},$$
for every $h \in \mathbb{R}^{N+1}$ such that $\|h\| \le \widetilde{\varrho}$,
- *(Morrey embedding) if $p > Q+2$, then there exists a positive constant $\widetilde{C}_p$ such that*
$$|u(z) - u(\zeta)| \le \widetilde{C}_p \bigg(\|u\|_{L^p(\Omega)} + \sum_{j=1}^{m_0} \|\partial_{x_j}u\|_{L^p(\Omega)} \bigg) \|\zeta^{-1} \circ z\|^{1 - \frac{Q+2}{p}},$$
for every $z, \zeta \in K$ such that $\|\zeta^{-1} \circ z\| \le \widetilde{\varrho}$.

The following Theorem is related to the main result of the article [2] by Bouchut, where the regularity of the solution of the kinetic equation

$$\partial_t f + \langle v, \nabla_x f \rangle = g, \qquad (t, x, v) \in \Omega \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n, \tag{1.22}$$

is considered. Note that the differential operator appearing in the left hand side of (1.22) agrees with the first order part of $\mathscr{L}$ defined in (1.19). Actually, the notation of the following result refers to this operator, and, in particular, the homogeneous dimension of the space $\mathbb{R}^{2n+1}$ is in this case $Q + 2 = 4n + 2$.

Theorem 1.3 *Let Ω be an open set of $\mathbb{R}^{2n+1}$, and let $f \in L^2_{\mathrm{loc}}(\Omega)$ be a weak solution to (1.22). Suppose that $g, f, \partial_{v_1}f, \dots, \partial_{v_n}f \in L^p(\Omega)$. Then for every compact set $K \subset \Omega$, there exist a positive constant $\widetilde{\varrho}$ such that we have:*

- *(Sobolev embedding) if $1 < p < 4n+2$, then there exists a positive constant C_p such that*
$$\|f\|_{L^{p^*}(K)} \le C_p \bigg(\|g\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \sum_{j=1}^{n} \|\partial_{v_j}f\|_{L^p(\Omega)} \bigg),$$

- *(Compactness) if moreover $p < q < p^*$, then there exists a positive constant $\widetilde{C}_{p,q}$ such that*

$$\|f(\cdot \circ h) - f\|_{L^q(K)} \le \widetilde{C}_{p,q}\left(\|g\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \sum_{j=1}^n \|\partial_{v_j} f\|_{L^p(\Omega)} \right) \|h\|^{(4n+2)\left(\frac{1}{q}-\frac{1}{p^*}\right)},$$

for every $h \in \mathbb{R}^{2n+1}$ such that $\|h\| \le \widetilde{\varrho}$,

- *(Morrey embedding) if $p > 4n+2$, then there exists a positive constant $\widetilde{C}_p$ such that*

$$|f(z) - f(\zeta)| \le \widetilde{C}_p\left(\|g\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \sum_{j=1}^n \|\partial_{v_j} f\|_{L^p(\Omega)} \right) \|\zeta^{-1} \circ z\|^{1-\frac{4n+2}{p}},$$

for every $z, \zeta \in K$ such that $\|\zeta^{-1} \circ z\| \le \widetilde{\varrho}$.

The proof of Theorems 1.2 and 1.3 is given in Section 4.

We next give some comments to our main results. We still refer here to the notation relevant to the operator $\mathscr{L}$ defined in (1.19), and to the representation formula (1.20). As we said above, it holds for solutions to $\mathscr{L}u = 0$ then, for this reason, it seems to be weaker than the usual Sobolev inequality. On the other hand, due to the strong degeneracy of the operator $\mathscr{L}$, its natural Sobolev space $W^{1,p}_{\mathscr{L}}$ is the space of the functions $u \in L^p$ with weak derivatives $\partial_{x_1} u, \dots, \partial_{x_n} u \in L^p$. In particular, it is impossible to prove a Sobolev inequality unless some information is given on u with respect to the remaining variables $y_1, \dots, y_n$ and t. We obtain this missing information from the fact that u is a solution to $\mathscr{L}u = 0$ (or, in a more general case, to $\mathscr{L}u = \mathrm{div}_x F + f$). We also note that the regularity property of the operator $\mathscr{L}$ is quite unstable. Indeed, let us fix any $x_0 \in \mathbb{R}^n$ and consider the operator $\widetilde{\mathscr{L}_0}$, acting on $(x, y, t) \in \mathbb{R}^{2n+1}$ as follows

$$\widetilde{\mathscr{L}_0} u := \Delta_x u + \langle x_0, \nabla_y u\rangle - \partial_t u.$$

Its natural Sobolev spaces agrees with that of $\mathscr{L}$, however it is known that a fundamental solution for $\widetilde{\mathscr{L}_0}$ does not exists and our method for the proof of the Sobolev inequality fails in this case. Actually, it is not difficult to check that the Sobolev inequality does not hold for the solutions to $\widetilde{\mathscr{L}_0} u = 0$.

We conclude this discussion with a simple remark. Also when we consider the more familiar uniformly parabolic equations, we find that the natural Sobolev space only contains the *spatial* derivatives, and it is not possible to find a simple natural space for the time derivative. As a matter of facts, several regularity results for parabolic equations depend on some *fractional Sobolev spaces.* The situation becomes more complicated when we consider second order PDEs with non-negative characteristic form analogous to $\mathscr{L}$. An alternative approach to our method, that only relies on a representation formula in terms of the fundamental solution, is the use of *fractional Sobolev spaces* (we refer to the articles by Bochut [2], see also Golse, Imbert, Mouhot and Vasseur [9]) to recover the missing information with respect to the variables $y_1, \dots, y_n$ and t.

This article is organized as follows. In Section 2 we give a comprehensive proof of the Sobolev embedding, of its compactness, and the Morrey embedding, following the method above outlined. In Section 3 we recall the tools of the Real Analysis on Lie groups we need to prove Theorem 1.1, and we give its proof. In Section 4 we discuss some applications of Theorem 1.1 to the solutions of $\mathscr{L} u = 0$. Section 5 contains some comments about the possible extension of Theorem 1.1 to a family of more general operators considered by Cinti and Polidoro in [6].

2 Continuous and compact embeddings: the Euclidean case

In this Section we give a comprehensive proof of the Sobolev embedding (1.8), the Morrey embedding (1.10), and of the inequality (1.11) from which the compactness of the Sobolev embedding follows. As said in the Introduction, all these results rely on the representation formula (1.2), which gives the bound (1.5) that we recall below

$$|u(x)| \leq c_n \int_{\mathbb{R}^n} |x-y|^{1-n} |\nabla u(y)| \, dy, \qquad \text{for every } u \in C_0^\infty(\mathbb{R}^n).$$

With this aim, we first recall the weak Young inequality that gives the Sobolev embedding, then we prove (1.9) and we deduce from this and (1.5) the Morrey embedding (1.10), and that stated in the inequality (1.11).

2.1 Some preliminary results

For a given positive α we denote by K_α any continuous homogeneous function of degree $-\alpha$, that is a function satisfying

$$K_\alpha(rx) = r^{-\alpha} K_\alpha(x), \quad \text{for every } x \in \mathbb{R}^n \setminus \{0\}, \text{ and } r > 0.$$

We easily see that

$$|K_\alpha(x)| \leq \frac{c_\alpha}{|x|^\alpha}, \quad \text{for every } x \in \mathbb{R}^n \setminus \{0\}, \tag{2.1}$$

where $c_\alpha := \max_{|x|=1} |K_\alpha(x)|$. In particular, K_α belongs to the space $L^q_{\text{weak}}(\mathbb{R}^n)$, for $q = \dfrac{n}{\alpha}$, that is

$$\text{meas}\big\{x \in \mathbb{R}^n \mid |K_\alpha(x)| \geq \lambda\big\} \leq \left(\frac{C}{\lambda}\right)^q, \quad \text{for every} \quad \lambda > 0, \tag{2.2}$$

for some non-negative constant C. Here meas E denotes the Lebesgue measure of the set E. Moreover we define the *seminorm* of K_α as follows

$$\|K_\alpha\|_{L^q_{\text{weak}}(\mathbb{R}^n)} := \inf\big\{C \geq 0 \mid (2.2) \text{ holds}\big\}.$$

From (2.1) it plainly follows that $C \leq c_\alpha \omega_n^{\alpha/n}$. We next recall two elementary inequalities that will be useful in the sequel. For every $R > 0$ we have that:

- $K_\alpha \in L^q\big(\{x \in \mathbb{R}^n \mid |x| \le R\}\big)$ if, and only if, $q < \dfrac{n}{\alpha}$. Moreover, there exists a positive constant $c_{\alpha,q}$, only depending on K_α, n and q, such that

$$\|K_\alpha\|_{L^q(\{x\in\mathbb{R}^n \mid |x|\le R\})} \le c_{\alpha,q} R^{\frac{n}{q}-\alpha}, \tag{2.3}$$

- $K_\alpha \in L^q\big(\{x \in \mathbb{R}^n \mid |x| \ge R\}\big)$ if, and only if, $q > \dfrac{n}{\alpha}$. Moreover, there exists a positive constant $c_{\alpha,q}$, only depending on K_α, n and q, such that

$$\|K_\alpha\|_{L^q(\{x\in\mathbb{R}^n \mid |x|\ge R\})} \le c_{\alpha,q} R^{\frac{n}{q}-\alpha}. \tag{2.4}$$

The following weak Young inequality holds (see Theorem 1, p. 119 in [16], where this result is referred to as Hardy-Littlewood-Sobolev theorem for fractional integration).

Theorem 2.1 *Let K_α be a continuous homogeneous function of degree $-\alpha$, with $0 < \alpha < n$. Let p, q be such that $1 \le p < q < +\infty$ and that $1 + \dfrac{1}{q} = \dfrac{1}{p} + \dfrac{\alpha}{n}$. Then, for every $f \in L^p(\mathbb{R}^n)$ the integral $K_\alpha * f(x)$ is convergent for almost every $x \in \mathbb{R}^n$. Moreover,*
- if $p > 1$, then there exists $C_{\alpha,p} > 0$ such that $\|K_\alpha * f\|_{L^q(\mathbb{R}^n)} \le C_{\alpha,p}\|f\|_{L^p(\mathbb{R}^n)}$,
- if $p = 1$, then there exists $C_{\alpha,1} > 0$ such that $\|K_\alpha * f\|_{L^q_{\text{weak}}(\mathbb{R}^n)} \le C_{\alpha,1}\|f\|_{L^1(\mathbb{R}^n)}$.

In order to prove the Morrey embedding (1.10) and the compactness estimate (1.11), we state and prove the following lemma.

Lemma 2.2 *Let $K_\alpha \in C^1(\mathbb{R}^n\backslash\{0\})$ be any homogeneous function of degree $-\alpha$, with $0 < \alpha < n$. Then there exists a positive constant M_α such that*

$$|K_\alpha(x) - K_\alpha(y)| \le M_\alpha \frac{|x-y|}{|x|^{\alpha+1}}, \quad \textit{for every} \quad x, y \in \mathbb{R}^n\backslash\{0\} \textit{ such that } |x-y| \le \frac{|x|}{2}.$$

Proof. We first prove the result for x such that $|x| = 1$. In this case $|x-y| \le \dfrac{1}{2}$ and by the Mean Value Theorem there exists $\theta \in (0,1)$ such that

$$|K_\alpha(x) - K_\alpha(y)| = |\langle (x-y), \nabla K_\alpha(\theta x + (1-\theta y))\rangle| \le M_\alpha |x-y|,$$

where

$$M_\alpha = \max_{\frac{1}{2}\le|z|\le\frac{3}{2}} |\nabla K_\alpha(z)|. \tag{2.5}$$

Consider now a general choice of $x, y \in \mathbb{R}^n\backslash\{0\}$ with $|x-y| \le \dfrac{|x|}{2}$. Being K_α homogeneous of degree $-\alpha$, we obtain

$$|K_\alpha(x) - K_\alpha(y)| = \frac{1}{|x|^\alpha}\left|K_\alpha\left(\frac{x}{|x|}\right) - K_\alpha\left(\frac{y}{|x|}\right)\right| \le \frac{M_\alpha}{|x|^\alpha}\left|\frac{x}{|x|} - \frac{y}{|x|}\right| = M_\alpha\frac{|x-y|}{|x|^{\alpha+1}},$$

being M_α as in (2.5), because $\left|\dfrac{x}{|x|}\right| = 1$. $\square$

In order to prove the Morrey embedding (1.10) and the compactness of the Sobolev embedding for $p < q < p^*$ we rely on the following argument. We choose any $u \in C_0^\infty(\mathbb{R}^n), h \in \mathbb{R}^n$ and we set

$$v(x) := u(x+h) - u(x), \qquad \text{for every} \quad x \in \mathbb{R}^n, \tag{2.6}$$

then

$$\begin{aligned} v(x) &= \int_{\{y\in\mathbb{R}^n:|x+h-y|\geq 2|h|\}} \langle \nabla\Gamma(x+h-y) - \nabla\Gamma(x-y), \nabla u(y)\rangle \,\mathrm{d}y \\ &+ \int_{\{y\in\mathbb{R}^n:|x+h-y|< 2|h|\}} \langle \nabla\Gamma(x+h-y), \nabla u(y)\rangle \,\mathrm{d}y \\ &+ \int_{\{y\in\mathbb{R}^n:|x+h-y|< 2|h|\}} -\langle \nabla\Gamma(x-y), \nabla u(y)\rangle \,\mathrm{d}y =: I_A(x) + I_B(x) + I_C(x). \end{aligned} \tag{2.7}$$

We next rely on Lemma 2.2 and on (1.3) to estimate the terms I_A, I_B and I_C as follows

$$\begin{aligned} |I_A(x)| &\leq M|h| \int_{\{y\in\mathbb{R}^n:|x+h-y|\geq 2|h|\}} \frac{1}{|x+h-y|^n} |\nabla u(y)| \,\mathrm{d}y, \\ |I_B(x)| &\leq c_n \int_{\{y\in\mathbb{R}^n:|x+h-y|< 2|h|\}} \frac{1}{|x+h-y|^{n-1}} |\nabla u(y)| \,\mathrm{d}y, \\ |I_C(x)| &\leq c_n \int_{\{y\in\mathbb{R}^n:|x+h-y|< 2|h|\}} \frac{1}{|x-y|^{n-1}} |\nabla u(y)| \,\mathrm{d}y, \end{aligned} \tag{2.8}$$

where $M := \max\limits_{\frac{1}{2}\leq|z|\leq\frac{3}{2}, j,k=1,\dots,n} |\partial^2_{x_jx_k}\Gamma(z)|$.

2.2 The Sobolev and Morrey embedding theorems

As we said in the Introduction, Theorem 2.1 combined with (1.5) immediately yields the following result

Theorem 2.3 *Let $1 < p < n$. Then there exists $C_p > 0$ such that:*

$$\|u\|_{p^*} \leq C_p \|\nabla u\|_p \quad \textit{for every} \quad u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \quad \textit{such that} \quad \nabla u \in L^p(\mathbb{R}^n),$$

where $p^* = \dfrac{np}{n-p}$.

We next turn our attention on the Morrey's Theorem.

Theorem 2.4 (Morrey's Theorem) *Let $u : \mathbb{R}^n \to \mathbb{R}$ and $p > n$. If $\nabla u \in L^p(\mathbb{R}^n)$, then u is continuous and*

$$|u(x+h) - u(x)| \leq C_{n,p} \|\nabla u\|_p |h|^{1-\frac{n}{p}}, \qquad \textit{for every} \quad x, h \in \mathbb{R}^n, \tag{2.9}$$

for some positive constant $C_{n,p}$ depending only on p and n.

In particular $W^{1,p}(\mathbb{R}^n)$ is continuously embedded in the space of Hölder continuous functions $C^\beta(\mathbb{R}^n)$, with $\beta = 1 - \frac{n}{p}$.

Proof. Consider a function $u \in C_0^\infty(\mathbb{R}^n)$, let $x, h \in \mathbb{R}^n$, and $v(x)$ be the function defined in (2.7). We next estimate I_A, I_B and I_C by using the Hölder inequality.

From the first inequality in (2.8) and (2.4), with $\alpha = n$ and $q = \dfrac{p}{p-1}$, we obtain

$$
\begin{aligned}
|I_A(x)| &\le M|h| \|\nabla u\|_{L^p(\mathbb{R}^n)} \left\| \frac{1}{|z|^n} \right\|_{L^q(\{|z|\ge 2|h|\})} \\
&= M|h| \|\nabla u\|_{L^p(\mathbb{R}^n)} c_{n,q} (2|h|)^{\frac{n(p-1)}{p} - n} = M_A \|\nabla u\|_{L^p(\mathbb{R}^n)} |h|^{1-\frac{n}{p}},
\end{aligned}
$$

for some positive constant M_A only depending on M, p and n. Moreover, from the second inequality in (2.8) and (2.3), with $\alpha = n-1$ and $q = \dfrac{p}{p-1}$, we find

$$
\begin{aligned}
|I_B(x)| &\le c_n \|\nabla u\|_{L^p(\mathbb{R}^n)} \left\| \frac{1}{|z|^{n-1}} \right\|_{L^q(\{|z|\le 2|h|\})} \\
&= c_n \|\nabla u\|_{L^p(\mathbb{R}^n)} c_{n-1,q} (2|h|)^{\frac{n(p-1)}{p} - n + 1} = M_B \|\nabla u\|_{L^p(\mathbb{R}^n)} |h|^{1-\frac{n}{p}},
\end{aligned}
$$

where M_B is a positive constant only depending on p and n. Finally the last term in (2.7) can be estimated similarly to the second one, observing that $|x-y| \le |x+h-y| + |h| < 3|h|$, thus getting

$$
|I_C(x)| \le c_n \|\nabla u\|_{L^p(\mathbb{R}^n)} \left\| \frac{1}{|z|^{n-1}} \right\|_{L^q(\{|z|\le 3|h|\})} = M_C \|\nabla u\|_{L^p(\mathbb{R}^n)} |h|^{1-\frac{n}{p}},
$$

for some positive constant M_C. We note that the L^q norms of the functions $z \mapsto |z|^{-n}$ and $z \mapsto |z|^{-n+1}$ appearing in the above estimates are finite thanks to the assumption $p > n$. Then (2.9) is obtained with $C_{n,p} := M_A + M_B + M_C$. This proves our claim for $u \in C_0^\infty(\mathbb{R}^n)$. The general case follows by a density argument. $\square$

We next prove the compactness of the Sobolev embedding (1.11) for $p < q < p^*$ starting again from (2.7). Here we use the Young inequality instead of the Hölder inequality.

Theorem 2.5 *Let $p, q \ge 1$ be such that $p < q < p^* := \frac{np}{n-p}$. Then there exists a positive constant $C_{p,q}$ depending on n, p, q such that*

$$
\|u(h+\cdot) - u\|_{L^q(\mathbb{R}^n)} \le C_{p,q} \|\nabla u\|_{L^p(\mathbb{R}^n)} |h|^{n\left(\frac{1}{q} - \frac{1}{p^*}\right)}.
$$

for every $u \in C_0^\infty(\mathbb{R}^n)$ and for any $h \in \mathbb{R}^n$. In particular, as long as $q < p^*$, we have

$$
\|u(h+\cdot) - u\|_{L^q(\mathbb{R}^n)} \to 0 \qquad \text{as} \qquad |h| \to 0.
$$

Proof. Consider a function $u \in C_0^\infty(\mathbb{R}^n)$, let $x, h \in \mathbb{R}^n$, and $v(x)$ be the function defined in (2.7). We next estimate the L^q norm of I_A, I_B and I_C by using the Young inequality. To this aim we introduce the exponent r defined by the identity

$$
1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}, \tag{2.10}
$$

and we note that

$$1 < r < \frac{n}{n-1} \;\Leftrightarrow\; p < q < p^*. \tag{2.11}$$

From the first inequality in (2.8) and (2.4), with $\alpha = n$, we obtain

$$\begin{aligned}\|I_A\|_{L^q(\mathbb{R}^n)} &\le M|h|\|\nabla u\|_{L^p(\mathbb{R}^n)} \left\| \frac{1}{|z|^n} \right\|_{L^r(\{|z|\ge 2|h|\})} \\ &= M|h|\|\nabla u\|_{L^p(\mathbb{R}^n)} c_{n,r} (2|h|)^{\frac{n}{r}-n} = C_A(n,r)|h|^{n\left(\frac{1}{q}-\frac{1}{p^*}\right)},\end{aligned}$$

From the second inequality in (2.8) and (2.3), with $\alpha = n-1$, we obtain

$$\begin{aligned}\|I_B\|_{L^q(\mathbb{R}^n)} &\le c_n \|\nabla u\|_{L^p(\mathbb{R}^n)} \left\| \frac{1}{|z|^{n-1}} \right\|_{L^r(\{|z|\le 2|h|\})} \\ &= c_n \|\nabla u\|_{L^p(\mathbb{R}^n)} c_{n-1,r} (2|h|)^{\frac{n}{r}-n+1} = C_B(n,r)\|\nabla u\|_{L^p(\mathbb{R}^n)}|h|^{n\left(\frac{1}{q}-\frac{1}{p^*}\right)},\end{aligned}$$

where $C_B(n,r)$ is a constant depending on n and r (and thus on n,p,q). The same argument applies to I_C, so that, provided that we consider the norm $\left\| \frac{1}{|z|^{n-1}} \right\|_{L^r(\{|z|\le 3|h|\})}$ instead of $\left\| \frac{1}{|z|^{n-1}} \right\|_{L^r(\{|z|\le 2|h|\})}$, as in the proof of the Morrey's theorem. We then find

$$\|I_C\|_{L^q(\mathbb{R}^n)} \le C_C(n,r)\|\nabla u\|_{L^p(\mathbb{R}^n)}|h|^{n\left(\frac{1}{q}-\frac{1}{p^*}\right)},$$

We note that the L^r norms of the functions $z \mapsto |z|^{-n}$ and $z \mapsto |z|^{-n+1}$ appearing in the above estimates are finite if, and only if, the condition (2.11) is satisfied. The thesis is obtained with $C_{p,q} := C_A(n,r) + C_B(n,r) + C_C(n,r)$ for $u \in C_0^\infty(\mathbb{R}^n)$. The general case follows by a density argument. □

2.3 A more general compactness result

We note that the Theorem 2.5 only applies to a kernel that is homogeneous of degree $-n+1$. Actually, the method used in its proof also applies to any general homogeneous kernel K_α, with $0 < \alpha < n$, as those considered in Theorem 2.1. In the following statement we denote by u the convolution $K_\alpha * f$, that is

$$u(x) = \int_{\mathbb{R}^n} K_\alpha(x-y) f(y)\, \mathrm{d}y.$$

Theorem 2.6 *Let K_α be a $C^1(\mathbb{R}^n \backslash \{0\})$ homogeneous function of degree $-\alpha$, with $1 < \alpha < n$, and let $p, q \ge 1$ be such that $q > p$ and*

$$1 - \frac{\alpha+1}{n} < \frac{1}{p} - \frac{1}{q} < 1 - \frac{\alpha}{n}. \tag{2.12}$$

Then there exists a positive constant $\widetilde{C}_{p,q}$, depending on n,p,q, such that

$$\|u(h+\cdot) - u\|_{L^q(\mathbb{R}^n)} \le \widetilde{C}_{p,q}\, \|f\|_{L^p(\mathbb{R}^n)}\, |h|^{\frac{n}{r}-\alpha},$$

for every $f \in L^p(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$. Here r is the constant introduced in (2.10)*, that is* $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{p}$*. Moreover the exponent $\frac{n}{r}-\alpha$ is strictly positive.*

Proof. We choose $x, h \in \mathbb{R}^n$ and let v be defined as in (2.6): $v(x) = u(x+h) - u(x)$, and we consider three integrals $v(x) = I_A(x) + I_B(x) + I_C(x)$ as in (2.7). We proceed as we did in the proof of Theorem 2.4. We find

$$\begin{aligned}
|I_A(x)| &\le M_\alpha |h| \int_{\{y\in\mathbb{R}^n:|x+h-y|\ge 2|h|\}} \frac{1}{|x+h-y|^{\alpha+1}}|f(y)|\,\mathrm{d}y,\\
|I_B(x)| &\le c_\alpha \int_{\{y\in\mathbb{R}^n:|x+h-y|< 2|h|\}} \frac{1}{|x+h-y|^{\alpha}}|f(y)|\,\mathrm{d}y,\\
|I_C(x)| &\le c_\alpha \int_{\{y\in\mathbb{R}^n:|x+h-y|< 2|h|\}} \frac{1}{|x-y|^{\alpha}}|f(y)|\,\mathrm{d}y,
\end{aligned}$$

being $M_\alpha := \max_{\frac{1}{2}\le|z|\le\frac{3}{2}} |\nabla K_\alpha(z)|$. In order to use the Young inequality, we recall that

$$\left\| \frac{1}{|z|^{\alpha+1}} \right\|_{L^r(\{y\in\mathbb{R}^n:|z|\ge 2|h|\})} = c_{\alpha+1,r}(2|h|)^{\frac{n}{r}-(\alpha+1)},$$

where r is the exponent introduced in (2.10) and we note that the above integral converges if, and only if, $(\alpha+1)r > n$. Moreover,

$$\left\| \frac{1}{|z|^{\alpha}} \right\|_{L^r(\{z\in\mathbb{R}^n:|z|\le 2|h|\})} = c_{\alpha,r}(2|h|)^{\frac{n}{r}-\alpha},$$

and the above integral converges if, and only if, $\alpha r < n$. Summarizing, the two above integrals are finite if, and only if,

$$\frac{n}{\alpha+1} < r < \frac{n}{\alpha}, \tag{2.13}$$

which is equivalent to (2.12). We also note that the exponent $\frac{n}{r}-\alpha$ is strictly positive.

We next proceed as in the proof of Theorem 2.4. By the Young inequality we deduce

$$\begin{aligned}
\|I_A\|_{L^q(\mathbb{R}^n)} &\le \widetilde{C}_A(n,\alpha,r)|h|^{\left(\frac{n}{r}-\alpha\right)} \|f\|_{L^p(\mathbb{R}^n)},\\
\|I_B\|_{L^q(\mathbb{R}^n)} &\le \widetilde{C}_B(n,\alpha,r)|h|^{\left(\frac{n}{r}-\alpha\right)} \|f\|_{L^p(\mathbb{R}^n)},\\
\|I_C\|_{L^q(\mathbb{R}^n)} &\le \widetilde{C}_C(n,\alpha,r)|h|^{\left(\frac{n}{r}-\alpha\right)} \|f\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

where $\widetilde{C}_A(n,\alpha,r), \widetilde{C}_B(n,\alpha,r)$, and $\widetilde{C}_C(n,\alpha,r)$ are positive constants only depending on n, p, q, on c_α and on M_α. The thesis is obtained with $\widetilde{C}_{p,q} := \widetilde{C}_A(n,\alpha,r) + \widetilde{C}_B(n,\alpha,r) + \widetilde{C}_C(n,\alpha,r)$. □

Remark 2.7 *The statement of Theorem 2.6 is more involved with respect to that of Theorem 2.5 as we don't have a natural counterpart of the Sobolev exponent p^* for any $\alpha \in]0, n[$. We list here the explicit conditions on q for the validity of* (2.12)*. We discuss specifically the case $1 < p < n$ as we are interested in the compactness of the Sobolev embedding.*

(i) *If* $\boxed{0 < \alpha < n-2}$ *then*

(i.1) *If* $1 < p < \dfrac{n}{n-\alpha}$ *then*

$$\frac{np}{n-p(n-1-\alpha)} < q < \frac{np}{n-p(n-\alpha)}$$

(i.2) *If* $\dfrac{n}{n-\alpha} < p < \dfrac{n}{n-1-\alpha}$ *then*

$$q > \max\left\{1, \frac{np}{n-p(n-1-\alpha)}\right\} = \frac{np}{n-p(n-\alpha)}$$

(i.3) *If* $\dfrac{n}{n-1-\alpha} < p < n$ *then no values of* q *are available*

(ii) *If* $\boxed{\alpha = n-2}$ *then*

(ii.1) *If* $1 < p < \dfrac{n}{2}$ *then*

$$p^* = \frac{np}{n-p} < q < \frac{np}{n-2p}$$

(ii.2) *If* $\dfrac{n}{2} < p < n$ *then*

$$q > \max\left\{1, \frac{np}{n-p}\right\} = \frac{np}{n-p} = p^*$$

(iii) *If* $\boxed{n-2 < \alpha < n-1}$ *then*

(iii.1) *If* $1 < p < \dfrac{n}{n-\alpha}$ *then*

$$\frac{np}{n-p(n-1-\alpha)} < q < \frac{np}{n-p(n-\alpha)}$$

(iii.2) *If* $\dfrac{n}{n-\alpha} < p < n$ *then*

$$q > \max\left\{1, \frac{np}{n-p(n-\alpha)}\right\} = \frac{np}{n-p(n-1-\alpha)}$$

(iv) *If* $\boxed{\alpha = n-1}$ *then for any* $1 < p < n$

$$p < q < p^* = \frac{np}{n-p}$$

(v) *If* $\boxed{n-1 < \alpha < n}$ *then for any* $1 < p < n$

$$\frac{np}{n-p(n-1-\alpha)} < q < \frac{np}{n-p(n-\alpha)}.$$

3 Continuous and compact embeddings for degenerate Kolmogorov equations

The aim of this section is to prove compactness estimates for weak solutions to a family of degenerate Kolmogorov operators that includes the one in (1.19) as the simplest prototype. Specifically, we consider operators in this form

$$\mathscr{L}u(x,t) := \sum_{i,j=1}^{m_0} \partial_{x_i}(a_{i,j}(x,t)\partial_{x_j}u(x,t)) + \sum_{i,j=1}^{N} b_{i,j}x_i\partial_{x_j}u(x,t) - \partial_t u(x,t), \tag{3.1}$$

with $(x,t) = (x_1, ..., x_N, t) \in \mathbb{R}^{N+1}$. Here $m_0 \in \mathbb{N}$ is such that $1 \le m_0 \le N$. In the sequel we will also use the following notation $z := (x,t)$, and we always assume the following hypotheses:

(H.1) The matrix $A = (a_{i,j}(z))_{i,j=1,...,m_0}$ is symmetric, with measurable coefficients and there exists a positive constant μ such that

$$\mu^{-1}|\xi|^2 \le \sum_{i,j=1}^{m_0} a_{i,j}(z)\xi_i\xi_j \le \mu|\xi|^2$$

for all $z \in \mathbb{R}^{N+1}$ and $\xi \in \mathbb{R}^{m_0}$.

(H.2) The matrix B has constant coefficients. Moreover there exists a basis of $\mathbb{R}^N$ such that the matrix B can be written in a canonical form:

$$B = \begin{pmatrix} 0 & B_1 & 0 & ... & 0 \\ 0 & 0 & B_2 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & B_r \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}$$

where B_k is a matrix $m_{k-1} \times m_k$ with rank m_k, $k = 1, 2, ..., r$ with

$$m_0 \ge m_1 \ge ... \ge m_r \ge 1, \quad \text{e} \quad \sum_{k=0}^{r} m_k = N.$$

We prove in detail Theorem 1.1 along the same techniques outlined in the Introduction for the operator $\mathscr{L}$ in (1.19). In particular, we will rely on a representation formula analogous to (1.2) in terms of the fundamental solution to the operator

$$\mathscr{L}_0 u := \sum_{i=1}^{m_0} \partial_{x_i}^2 u + \sum_{i,j=1}^{N} b_{i,j}x_i\partial_{x_j}u - \partial_t u. \tag{3.2}$$

Remark 1. It is know that Assumption (H.2) is equivalent to the assumption of hypoellipticity of $\mathscr{L}_0$ (see [11] and its bibliography). This means that any function u which is a distributional solution to $\mathscr{L}_0 u = f$ in some open subset Ω of $\mathbb{R}^{N+1}$ is a C^∞ function whenever f is C^∞.

In order to simplify our statements, in the sequel we adopt the following compact notation. If I_{m_0} is the identity matrix $m_0 \times m_0$, we set

$$\Delta_{m_0} = \sum_{i=1}^{m_0} \partial_{x_i}^2, \qquad Y = \sum_{i,j=1}^{N} b_{i,j} x_i \partial_{x_j} - \partial_t, \qquad A_0 = \begin{pmatrix} I_{m_0} & 0 \\ 0 & 0 \end{pmatrix}.$$

In particular we will write

$$\mathscr{L}_0 = \Delta_{m_0} + Y = \mathrm{div}(A_0 \nabla) + Y.$$

We next introduce the non-Euclidean geometric setting suitable for the study of $\mathscr{L}$, the fundamental solution of $\mathscr{L}_0$ and the definition of the convolution with homogeneous kernels.

3.1 Dilation and translation groups associated to $\mathscr{L}$

We recall here some invariance properties of the operator $\mathscr{L}_0$. We refer to [11] where the definition of translation group and dilation group for Kolmogorov operators have been given for the first time. Let

$$(x,t) \circ (\xi, \tau) = (\xi + E(\tau)x, t + \tau), \quad E(t) = \exp(-tB^T), \ (x,t), (\xi, \tau) \in \mathbb{R}^{N+1} \tag{3.3}$$

$$D(\lambda) = \mathrm{diag}(\lambda I_{m_0}, \lambda^3 I_{m_1}, ..., \lambda^{2r+1} I_{m_r}, \lambda^2), \quad \lambda > 0 \tag{3.4}$$

where I_{m_j} denotes the identity matrix $m_j \times m_j$. It is known that $(\mathbb{R}^{N+1}, \circ)$ is a *non commutative* group, and $\mathscr{L}_0$ is invariant with respect to the *left translations* of $(\mathbb{R}^{N+1}, \circ)$, in the following sense: if we choose any $\zeta \in \mathbb{R}^{N+1}$ and we set $v(z) := u(\zeta \circ z)$ and $g(z) := f(\zeta \circ z)$, then the have

$$\mathscr{L}_0 u(z) = f(z) \quad \Leftrightarrow \quad \mathscr{L}_0 v(z) = g(z).$$

Moreover, $\mathscr{L}_0$ is invariant with respect to $(D(\lambda))_{\lambda>0}$, with the following meaning:

$$\mathscr{L}_0 u(z) = f(z) \quad \Leftrightarrow \quad \mathscr{L}_0 w(z) = \lambda^2 h(z).$$

Now $w(z) := u(D(\lambda)z)$, $h(z) := f(D(\lambda)z)$, and λ is any positive constant. The zero of the group is $(0,0)$ and the inverse of (ξ, τ) is $(\xi, \tau)^{-1} = (-E(\tau)\xi, -\tau)$. Moreover the following *distributive property* holds:

$$D(\lambda)\,(z \circ \zeta) = (D(\lambda)z) \circ (D(\lambda)\zeta)$$

We summarize the above properties by saying that $\mathscr{L}_0$ is invariant with respect to the *homogeneous group* $\left(\mathbb{R}^{N+1}, \circ, (D(\lambda))_{\lambda>0}\right)$.

In the sequel we will use the following notation

$$D(\lambda) = \mathrm{diag}(\lambda^{\alpha_1}, \lambda^{\alpha_2}, \ldots, \lambda^{\alpha_N}, \lambda^2),$$

where, in accordance with (3.4) $\alpha_1 = \alpha_2 = \cdots = \alpha_{m_0} = 1$, $\alpha_{m_0+1} = \cdots = \alpha_{m_0+m_1} = 3, \ldots, \alpha_{m_0+m_1+\cdots+m_{r-1}+1} = \cdots = \alpha_N = 2r+1$.

We define now a norm of $\mathbb{R}^{N+1}$ homogeneous of degree one with respect to the dilation introduced before.

Definition 3.1 *For all $z \in \mathbb{R}^{N+1} \setminus \{0\}$, we define the norm $\|z\| = \rho$, as the unique positive solution to:*

$$\frac{x_1^2}{\rho^{2\alpha_1}} + \frac{x_2^2}{\rho^{2\alpha_2}} + ... + \frac{x_N^2}{\rho^{2\alpha_N}} + \frac{t^2}{\rho^4} = 1$$

and $\|0\| = 0$.

This norm is homogeneous with respect to the dilation group $(D(\lambda))_{\lambda>0}$ as long as the following property holds:

$$\|D(\lambda)z\| = \lambda\|z\| \quad \forall z \in \mathbb{R}^{N+1} \setminus \{0\} \text{ and } \lambda > 0.$$

Moreover the following quasi-triangular inequality holds. There exists a constant $c_T \geq 1$ such that

$$\|z \circ \zeta\| \leq c_T(\|z\| + \|\zeta\|), \qquad \|z^{-1}\| \leq c_T\|z\|, \tag{3.5}$$

for every $z, \zeta \in \mathbb{R}^{N+1}$. We denote by $d(\zeta, z) := \|z^{-1} \circ \zeta\|$ the *quasi-distance* of z and ζ. We denote by $B_\varrho(z)$ the open ball of radius ϱ and center z with respect to the above quasi-distance

$$B_\varrho(z) := \big\{\zeta \in \mathbb{R}^{N+1} \mid \|z^{-1} \circ \zeta\| < \varrho\big\}. \tag{3.6}$$

Note that the topology induced on $\mathbb{R}^{N+1}$ from the norm introduced in Definition 3.1 is equivalent to the Euclidean one.

Remark 3.2 The Lebesgue measure is invariant with respect to the group $(\mathbb{R}^{N+1}, \circ)$. Moreover, as long as $\det D(\lambda) = \lambda^{Q+2}$, where

$$Q = m_0 + 3m_1 + ... + (2r+1)m_r$$

we have that the following identity holds:

$$\text{meas}(B_r(0)) = r^{Q+2}\text{meas}(B_1(0)),$$

where $\text{meas}(B)$ indicates the Lebesgue measure of the set B. For this reason, we will refer to $(Q+2)$ as the *homogeneous dimension* of $\mathbb{R}^{N+1}$ with respect to the dilation (3.4).

We also note that, in view of the structure of the matrix B and of the definition of $E(\tau)$, we have that $\det E(\tau) = 1$ for every τ. In particular, the Jacobian determinant of the left translation $(x,t) \mapsto (\xi,\tau) \circ (x,t)$ agrees with 1 for every $(\xi,\tau) \in \mathbb{R}^{N+1}$. The same is true for the Jacobian determinant of $(\xi,\tau) \mapsto (\xi,\tau)^{-1}$. As a consequence we have that

$$\begin{aligned}
&\int_A f(\zeta \circ z)dz = \int_{\zeta\circ A} f(w)dw, && \zeta \circ A := \big\{w = \zeta \circ z \mid z \in A\big\},\\
&\int_A f\big(z^{-1}\big)dz = \int_{A^{-1}} f(w)dw, && A^{-1} := \big\{w = z^{-1} \mid z \in A\big\},\\
&\int_A f(z^{-1} \circ \zeta)dz = \int_{A^{-1}\circ\zeta} f(w)dw, && A^{-1} \circ \zeta := \big\{w = z^{-1} \circ \zeta \mid z \in A\big\}.
\end{aligned} \tag{3.7}$$

The following results is the analogous of (2.3) and (2.4) in the setting of the homogeneous Lie group $\big(\mathbb{R}^{N+1}, \circ, (D(\lambda))_{\lambda>0}\big)$.

Lemma 3.3 *Let K_α denote any continuous function which is homogeneous of degree $-\alpha$ with respect to $(D(\lambda))_{\lambda>0}$, for some α such that $0 < \alpha < Q+2$. For every $R > 0$ we have that:*

- $K_\alpha \in L^q\big(\{z \in \mathbb{R}^{N+1} \mid \|z\| \le R\}\big)$ *if, and only if,* $q > \dfrac{Q+2}{\alpha}$. *Moreover, there exists a positive constant $\widetilde{c}_{\alpha,q}$, only depending on K_α, $(D(\lambda))_{\lambda>0}$ and q, such that*

$$\|K_\alpha\|_{L^q(\{z\in\mathbb{R}^{N+1} | \|z\|\le R\})} \le \widetilde{c}_{\alpha,q} R^{\frac{Q+2}{q}-\alpha}, \tag{3.8}$$

- $K_\alpha \in L^q\big(\{z \in \mathbb{R}^{N+1} \mid \|z\| \ge R\}\big)$ *if, and only if,* $q < \dfrac{Q+2}{\alpha}$. *Moreover, there exists a positive constant $\widetilde{c}_{\alpha,q}$, only depending on K_α, $(D(\lambda))_{\lambda>0}$ and q, such that*

$$\|K_\alpha\|_{L^q(\{z\in\mathbb{R}^{N+1} | \|z\|\ge R\})} \le \widetilde{c}_{\alpha,q} R^{\frac{Q+2}{q}-\alpha}. \tag{3.9}$$

Proof. We compute the integrals by using the "polar coordinates"

$$\begin{cases} x_1 = \rho^{\alpha_1} \cos\psi_1 \ldots \cos\psi_{N-1} \cos\psi_N \\ x_2 = \rho^{\alpha_2} \cos\psi_1 \ldots \cos\psi_{N-1} \sin\psi_N \\ \vdots \\ x_N = \rho^{\alpha_N} \cos\psi_1 \sin\psi_2 \\ t = \rho^2 \sin\psi_1. \end{cases}$$

Note that, in accordance with the Definition 3.1, the Jacobian determinant of the above change of coordinate is homogeneous of degree $Q+1$ with respect to the variable ρ, that is $J(\rho,\psi) = \rho^{Q+1} J(1,\psi)$. The claim then follows by proceedings as in the Euclidean case. □

3.2 Preliminary results on convolutions in homogeneous Lie groups

We recall some facts concerning the convolution of functions in homogeneous Lie groups. We refer to the work of Folland [10], and to its bibliography, for a comprehensive treatment of this subject. The first result is a Young inequality in the non-Euclidean setting

Theorem 3.4 *Let $p, q, r \in [1, +\infty]$ be such that:*

$$1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}. \tag{3.10}$$

If $f \in L^p(\mathbb{R}^{N+1})$ and $g \in L^r(\mathbb{R}^{N+1})$, then the function $f * g$ defined as:

$$f * g(z) := \int_{\mathbb{R}^{N+1}} f(\zeta^{-1} \circ z) g(\zeta)\, \mathrm{d}\zeta$$

belongs to $L^q(\mathbb{R}^{N+1})$ and it holds:

$$\|f * g\|_{L^q(\mathbb{R}^{N+1})} \le \|f\|_{L^p(\mathbb{R}^{N+1})} \|g\|_{L^r(\mathbb{R}^{N+1})}.$$

The following two theorems are the counterpart of Theorem 2.1 and Lemma 2.2 in Section 2, respectively.

Theorem 3.5 (Proposition (1.11) in [10]) *Let K_α be a continuous function, homogeneous of degree $-\alpha$ with $0 < \alpha < Q+2$, with respect to the dilation (3.4). Then, for every $p \in]1, +\infty[$, the convolution u of K_α with a function $f \in L^p(\mathbb{R}^{N+1})$*

$$u(z) = \int_{\mathbb{R}^{N+1}} K_\alpha(\zeta^{-1} \circ z) f(\zeta)\, \mathrm{d}\zeta, \tag{3.11}$$

is defined for almost every $z \in \mathbb{R}^{N+1}$ and is a measurable function. Moreover there exists a constant $\widehat{C}_p = \widehat{C}_p(p, Q)$ such that

$$\|u\|_{L^q(\mathbb{R}^{N+1})} \leq \widehat{C}_p \max_{\|z\|=1} |K_\alpha(z)| \|f\|_{L^p(\mathbb{R}^{N+1})},$$

for every $f \in L^p(\mathbb{R}^{N+1})$, where q is defined by

$$1 + \frac{1}{q} = \frac{1}{p} + \frac{\alpha}{Q+2}.$$

For the proof of the next result we refer to Proposition (1.11) in [10] or Lemma 5.1 in [13].

Theorem 3.6 *Let $K_\alpha \in C^1(\mathbb{R}^{N+1} \backslash \{0\})$ be a homogeneous function of degree $-\alpha$ with respect to the group $(D(\lambda))_{\lambda>0}$. Then there exist two constants $\kappa > 1$ and $M_\alpha > 0$ such that:*

$$|K_\alpha(\zeta) - K_\alpha(z)| \leq M_\alpha \frac{\|z^{-1} \circ \zeta\|}{\|z\|^{\alpha+1}}$$

for all z, ζ such that $\|z\| \geq \kappa \|z^{-1} \circ \zeta\|$.

3.3 Compactness estimates for convolutions with homogeneous kernels

Theorem 3.7 *Let K_α be a $C^1(\mathbb{R}^{N+1} \setminus \{0\})$ homogeneous function of degree $-\alpha$ with $0 < \alpha < Q+2$, with respect to the dilation (3.4). Then for every $p, q \geq 1$ such that $q > p$ and*

$$1 - \frac{\alpha+1}{Q+2} < \frac{1}{p} - \frac{1}{q} < 1 - \frac{\alpha}{Q+2}, \tag{3.12}$$

there exists a positive constant $\widetilde{C}_{p,q}$ depending on α, p, q and on the dilation group $(D(\lambda))_{\lambda>0}$ such that

$$\|u(\cdot \circ h) - u\|_{L^q(\mathbb{R}^{N+1})} \leq \widetilde{C}_{p,q} \|h\|^{\frac{Q+2}{r} - \alpha} \|f\|_{L^p(\mathbb{R}^{N+1})},$$

for every $f \in L^p(\mathbb{R}^{N+1})$, and $h \in \mathbb{R}^{N+1}$. Here r is the constant defined by (3.10), that is

$$1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p},$$

and the exponent $\frac{Q+2}{r} - \alpha$ is strictly positive, because of (3.12).

Proof. We proceed as in the proof of Theorem 2.6. We choose $z, h \in \mathbb{R}^{N+1}$ and we let

$$v(z) := u(z \circ h) - u(z). \tag{3.13}$$

By the formula (3.11) we have $v(z) = \widetilde{I}_A(z) + \widetilde{I}_B(z) + \widetilde{I}_C(z)$, where

$$\begin{aligned}
\widetilde{I}_A(z) &= \int_{\{\zeta\in\mathbb{R}^{N+1}:\|\zeta^{-1}\circ z\circ h\|\geq\kappa\|h\|\}} \big(K_\alpha(\zeta^{-1}\circ z\circ h) - K_\alpha(\zeta^{-1}\circ z)\big) f(\zeta)\,\mathrm{d}\zeta,\\
\widetilde{I}_B(z) &= \int_{\{\zeta\in\mathbb{R}^{N+1}:\|\zeta^{-1}\circ z\circ h\|<\kappa\|h\|\}} K_\alpha(\zeta^{-1}\circ z\circ h) f(\zeta)\,\mathrm{d}\zeta,\\
\widetilde{I}_C(z) &= \int_{\{\zeta\in\mathbb{R}^{N+1}:\|\zeta^{-1}\circ z\circ h\|<\kappa\|h\|\}} -K_\alpha(\zeta^{-1}\circ z) f(\zeta)\,\mathrm{d}\zeta.
\end{aligned} \tag{3.14}$$

Then, as in the proof of Theorem 2.6, we find

$$\begin{aligned}
&|\widetilde{I}_A(z)| \leq c_T M_\alpha \|h\| \int_{\{\zeta\in\mathbb{R}^{N+1}:\|\zeta^{-1}\circ z\circ h\|\geq\kappa\|h\|\}} \frac{1}{\|\zeta^{-1}\circ z\circ h\|^{\alpha+1}} |f(\zeta)|\,\mathrm{d}\zeta,\\
&|\widetilde{I}_B(z)| \leq c_\alpha \int_{\{\zeta\in\mathbb{R}^{N+1}:\|\zeta^{-1}\circ z\circ h\|<\kappa\|h\|\}} \frac{1}{\|\zeta^{-1}\circ z\circ h\|^{\alpha}} |f(\zeta)|\,\mathrm{d}\zeta,\\
&|\widetilde{I}_C(z)| \leq c_\alpha \int_{\{\zeta\in\mathbb{R}^{N+1}:\|\zeta^{-1}\circ z\circ h\|<\kappa\|h\|\}} \frac{1}{\|\zeta^{-1}\circ z\|^{\alpha}} |f(\zeta)|\,\mathrm{d}\zeta.
\end{aligned}$$

The first estimate follows from Theorem 3.6 and the constant c_T is the one appearing in (3.5), while $c_\alpha := \max_{\|w\|=1} K_\alpha(w)$.

We next compute the L^r norm of the homogeneous functions appearing above. In view of Remark 3.2 and Lemma 3.3, we have that

$$\left\| \frac{1}{\|\zeta^{-1}\circ z\circ h\|^{\alpha+1}} \right\|_{L^r(\{\zeta\in\mathbb{R}^{N+1}:\|\zeta^{-1}\circ z\circ h\|\geq\kappa\|h\|\})} = \widetilde{C}_A(r,Q)\|h\|^{\frac{Q+2}{r}-(\alpha+1)},$$

where r is the exponent introduced in (2.10) and $\widetilde{C}_A(r,Q)$ is a constant depending on Q and r (hence on Q, p, q), and on the dilation group $(D(\lambda))_{\lambda>0}$. Using again Remark 3.2 and Lemma 3.3, and we also find

$$\left\| \frac{1}{\|\zeta^{-1}\circ z\circ h\|^{\alpha}} \right\|_{L^r(\{\zeta\in\mathbb{R}^{N+1}:\|\zeta^{-1}\circ z\circ h\|\leq\kappa\|h\|\})} = \widetilde{C}_B(r,Q)\|h\|^{\frac{Q+2}{r}-\alpha},$$

The same argument applies to $\widetilde{I}_C$, by using the quasi-triangular inequality (3.5), so that

$$\left\| \frac{1}{\|\zeta^{-1}\circ z\circ h\|^{\alpha}} \right\|_{L^r(\{\zeta\in\mathbb{R}^{N+1}:\|\zeta^{-1}\circ z\circ h\|\leq\kappa\|h\|\})} = \widetilde{C}_C(r,Q)\|h\|^{\frac{Q+2}{r}-\alpha},$$

Note that the three above integrals converge if, and only if,

$$\frac{Q+2}{\alpha+1} < r < \frac{Q+2}{\alpha}. \tag{3.15}$$

We note that (3.12) is equivalent to (3.15) and that the second inequality in (3.15) says that the exponent $\frac{Q+2}{r} - \alpha$ appearing in the statement of this Theorem is strictly positive. By the Young inequality (Theorem 3.4) we conclude that

$$\begin{aligned}
\left\| \widetilde{I}_A \right\|_{L^q(\mathbb{R}^{N+1})} &\le c_T M_\alpha \widetilde{C}_A(r,Q) \|h\|^{\frac{Q+2}{r}-\alpha} \|f\|_{L^p(\mathbb{R}^{N+1})}, \\
\left\| \widetilde{I}_B \right\|_{L^q(\mathbb{R}^{N+1})} &\le c_\alpha \widetilde{C}_B(r,Q) \|h\|^{\frac{Q+2}{r}-\alpha} \|f\|_{L^p(\mathbb{R}^{N+1})}, \\
\left\| \widetilde{I}_B \right\|_{L^q(\mathbb{R}^{N+1})} &\le c_\alpha \widetilde{C}_C(r,Q) \|h\|^{\frac{Q+2}{r}-\alpha} \|f\|_{L^p(\mathbb{R}^{N+1})}.
\end{aligned}$$

The thesis is obtained with $\widetilde{C}_{p,q} := c_T M_\alpha \widetilde{C}_A(r,Q) + c_\alpha \widetilde{C}_B(r,Q) + c_\alpha \widetilde{C}_C(r,Q)$. $\square$

Remark 3.8 *Similarly as we did in Remark 2.7, we can state the conditions on q for the validity of* (3.12). *They can can be obtained by substituting the* dimension n *with the* homogeneous dimension $Q+2$. *We explicitly write here the condition for $\alpha = Q$ and $\alpha = Q+1$, which occur in the representation formulas for the solutions to $\mathscr{L}u = f$. Moreover, when $\alpha = Q+1$, we only consider the case $1 < p < Q+2$, as we apply Theorem 3.7 to prove the compactness of the embedding of Theorem 3.5, which holds only for $p < Q+2$. For the same reason, when $\alpha = Q$, we only consider the case $1 < p < \dfrac{Q+2}{2}$.*

(i) If $\alpha = Q$ and $1 < p < \dfrac{Q+2}{2}$, we have that $p^* < q < p^{**}$,

(ii) If $\alpha = Q+1$ and $1 < p < Q+2$ we have that $p < q < p^*$.

3.4 Proof of our main result

Proof of Theorem 1.1. The proof of the Sobolev inequality is a direct consequence of Theorem 3.5, with $\alpha = Q+1$ when considering $v_1, \dots, v_{m_0}$, and $\alpha = Q$ as we consider u.

The compactness of the embedding is a direct consequence of Theorem 3.7. As noticed in Remark 3.8, it applies to the derivatives $\partial_{\xi_j}\Gamma$, that are homogeneous functions of degree $-(Q+1)$, only when $p < q < p^*$. Moreover, a direct computation based on (3.10) shows that

$$\frac{Q+2}{r} - (Q+1) = (Q+2)\left(\frac{1}{q} - \frac{1}{p^*}\right). \tag{3.16}$$

Analogously, as Γ is homogeneous of degree $-Q$, we need to consider $p^* < q < p^{**}$. In this case, by using again (3.10) we find

$$\frac{Q+2}{r} - Q = (Q+2)\left(\frac{1}{q} - \frac{1}{p^{**}}\right). \tag{3.17}$$

The proof of the Morrey embedding is obtained by the same argument used in the proof of Theorem 2.4. We consider the function $v(z) = u(z \circ h) - u(z)$ introduced in (3.13) and, as in the proof of Theorem 3.7, we write $v(z) = \widetilde{I}_A(z) + \widetilde{I}_B(z) + \widetilde{I}_C(z)$, where the functions $\widetilde{I}_A, \widetilde{I}_B, \widetilde{I}_C$ are defined in (3.14). The conclusion of the proof is obtained by using the Young inequality stated in Theorem 3.4. $\square$

4 Representation formulas

4.1 Fundamental solution to $\mathscr{L}_0$ and representation formula

In this Section we focus on the representation formulas for the equation $\mathscr{L}u = 0$, for $\mathscr{L}$ satisfying the assumptions (H.1) and (H.2), then we prove Theorem 1.2.

We first recall the definition of *weak solution* to $\mathscr{L}u = 0$, then we recall that, under these assumptions, the fundamental solution to $\mathscr{L}_0$ has been derived by Hörmander in [10]. We say that u is a weak solution to $\mathscr{L}u = 0$ in an open set $\Omega \subset \mathbb{R}^{N+1}$ if $u, \partial_{x_1} u, \dots, \partial_{x_{m_0}} u, Yu \in L^2_{\text{loc}}(\Omega)$ and

$$\int_\Omega -\langle A(z)\nabla u(z), \nabla\psi(z)\rangle + \psi(z) Yu(z)\, \mathrm{d}z = 0,$$

for all $\psi \in C_0^\infty(\Omega)$.

With the notations introduced in Section 3, we let

$$C(t) = \int_0^t E(s) A_0 E^T(s)\, \mathrm{d}s$$

The assumptions (H.1) and (H.2) guarantee that the matrix $C(t)$ is strictly positive for all $t > 0$. In this case its inverse $C^{-1}(t)$ is well defined and the fundamental solution to $\mathscr{L}_0$ with singularity at the origin of $\mathbb{R}^{N+1}$, is given by

$$\Gamma((x,t),(0,0)) = \begin{cases} \dfrac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det C(t)}} \exp\left(-\frac{1}{4}\left\langle C^{-1}(t)x, x\right\rangle\right), & \text{if } t > 0, \\ 0, & \text{if } t \le 0. \end{cases} \tag{4.1}$$

To simplify the notation, in the sequel we will write $\Gamma(x,t,\xi,\tau)$ instead of $\Gamma((x,t),(\xi,\tau))$, and $\Gamma(x,t)$ instead of $\Gamma(x,t,0,0)$. The fundamental solution $\Gamma(x,t,\xi,\tau)$ of $\mathscr{L}_0$ with pole at (ξ,τ) is the "left translation" of $\Gamma(\cdot,0,0)$ with respect to the group $(\mathbb{R}^{N+1}, \circ)$:

$$\Gamma(x,t,\xi,\tau) = \Gamma((\xi,\tau)^{-1} \circ (x,t), 0, 0).$$

Let us explicitly remark that $\Gamma(\cdot,0,0)$ is homogeneous of degree $-Q$ with respect to the group $(D(\lambda))_{\lambda>0}$ and $\partial_{x_j}\Gamma(\cdot,0,0)$ is homogeneous of degree $-Q-1$, for $j = 1, ..., m_0$. Moreover, also $\partial_{\xi_j}\Gamma(0,0,\cdot)$ is homogeneous of degree $-Q-1$, for $j = 1, ..., m_0$.

We next represent weak solutions to $\mathscr{L}u = 0$ as convolutions with the fundamental solution Γ to $\mathscr{L}_0$ and to its derivatives $\partial_{\xi_1}\Gamma, \dots, \partial_{\xi_{m_0}}\Gamma$.

Consider any open set $\Omega \subseteq \mathbb{R}^{N+1}$ let u be a function such that $u, \partial_{x_1} u, \dots, \partial_{x_{m_0}} u, Yu \in L^p_{\text{loc}}(\Omega)$, and $\eta \in C_0^\infty(\Omega)$ is any cut-off function, then, by an elementary density argument we find

$$(\eta u)(z) = -\int_{\mathbb{R}^{N+1}} [\Gamma(z,\cdot)\mathscr{L}_0(\eta u)](\zeta)\, \mathrm{d}\zeta =$$
$$\int_{\mathbb{R}^{N+1}} [\langle A_0 \nabla_\xi \Gamma(z,\cdot), \nabla(\eta u)\rangle](\zeta)\, \mathrm{d}\zeta - \int_{\mathbb{R}^{N+1}} [\Gamma(z,\cdot) Y(\eta u)](\zeta)\, \mathrm{d}\zeta.$$

If moreover u is a weak solutions to $\mathscr{L}u = \mathrm{div}(A_0 F) + f$, then $\mathscr{L}_0 u = \mathrm{div}((A_0 - A)\nabla u + A_0 F) + f$, for some $f \in L^p_{\mathrm{loc}}(\Omega)$ and some vector valued function $F = (F_1, \dots, F_{m_0}, 0, \dots, 0)$ with $F_1, \dots, F_{m_0} \in L^p_{\mathrm{loc}}(\Omega)$, then we obtain the following representation formula introduced in Theorem 3.1 of [13], and used in the proof of Theorem 3.3 of [14].

Theorem 4.1 *If u is a weak solution to $\mathscr{L}u = \mathrm{div}(A_0 F) + f$, in some open set $\Omega \subset \mathbb{R}^{N+1}$, with $f, F_1, \dots, F_{m_0} \in L^p_{\mathrm{loc}}(\Omega)$, and η is the cut-off function defined above, then:*

$$\begin{aligned}(\eta u)(z) = &\int_{\mathbb{R}^{N+1}} [\eta \langle \nabla_\xi \Gamma(z,\cdot), (A_0 - A)\nabla u + A_0 F\rangle](\zeta)\,\mathrm{d}\zeta - \int_{\mathbb{R}^{N+1}} [\Gamma(z,\cdot)(\langle A\nabla\eta, \nabla u\rangle + \eta f)]\,\mathrm{d}\zeta \\ &+ \int_{\mathbb{R}^{N+1}} [\langle A_0 \nabla_\xi \Gamma(z,\cdot), \nabla \eta\rangle u](\zeta)\,\mathrm{d}\zeta - \int_{\mathbb{R}^{N+1}} \Gamma(z,\cdot)(Y\eta) u\,\mathrm{d}\zeta.\end{aligned} \tag{4.2}$$

In the following statement $B_\varrho(z_0)$ denotes the ball defined in (3.6), and c_T is the constant in (3.5).

Proposition 4.2 *Let Ω be an open set of $\mathbb{R}^{N+1}$, and let u be a weak solution to $\mathscr{L}u = div(A_0 F) + f$ in Ω. Suppose that $u, f, \partial_{x_1} u, \dots, \partial_{x_{m_0}} u, F_1, \dots, F_{m_0} \in L^p_{\mathrm{loc}}(\Omega)$. Then for every $z_0 \in \Omega$, and $\varrho, \sigma > 0$ such that the ball $B_\varrho(z_0)$ is contained in Ω, and $\sigma < \dfrac{\varrho}{2c_T}$, we have:*

- *(Sobolev embedding) if $1 < p < Q+2$, then there exists a positive constant C_p such that*

$$\begin{aligned}\|u\|_{L^{p^*}(B_\sigma(z_0))} \le C_p\big(&\|u\|_{L^p(B_\varrho(z_0))} + \|f\|_{L^p(B_\varrho(z_0))} \\ &+ \|A_0 \nabla u\|_{L^p(B_\varrho(z_0))} + \|A_0 F\|_{L^p(B_\varrho(z_0))}\big),\end{aligned}$$

- *(Compactness) if moreover $p < q < p^*$, then there exists a positive constant $\widetilde{C}_{p,q}$ such that*

$$\begin{aligned}\|u(\cdot \circ h) - u\|_{L^q(B_\sigma(z_0))} \le \widetilde{C}_{p,q}\big(&\|u\|_{L^p(B_\varrho(z_0))} + \|f\|_{L^p(B_\varrho(z_0))} \\ &+ \|A_0 \nabla u\|_{L^p(B_\varrho(z_0))} + \|A_0 F\|_{L^p(B_\varrho(z_0))}\big)\ \|h\|^{(Q+2)\left(\frac{1}{q} - \frac{1}{p^*}\right)},\end{aligned}$$

  for every $h \in B_\sigma(z_0)$,

- *(Morrey embedding) if $p > Q+2$, then there exists a positive constant $\widetilde{C}_p$ such that*

$$\begin{aligned}|u(z) - u(\zeta)| \le \widetilde{C}_p\big(&\|u\|_{L^p(B_\varrho(z_0))} + \|f\|_{L^p(B_\varrho(z_0))} \\ &+ \|A_0 \nabla u\|_{L^p(B_\varrho(z_0))} + \|A_0 F\|_{L^p(B_\varrho(z_0))}\big)\|\zeta^{-1} \circ z\|^{1 - \frac{Q+2}{p}},\end{aligned}$$

  for every $z, \zeta \in B_\sigma(z_0)$.

Proof. We apply Theorem 4.1 with a function η supported in the ball $B_\varrho(z_0)$ and such that $\psi(z) = 1$ for every $z \in B_{2c_T\sigma}(z_0)$. It is not difficult to check that a cut-off function with the above properties exists (see formula (3.3) in [13] for instance). Note that the integrals appearing in the equation (4.2) involving $\partial_{x_1} u, \dots, \partial_{x_{m_0}} u, F_1, \dots, F_{m_0}$ are convolutions of $\partial_{\xi_1}\Gamma, \dots, \partial_{\xi_{m_0}}\Gamma$, that are homogeneous kernels of degree $-(Q+1)$, with functions belonging to $L^p(B_\varrho(z_0))$ multiplied by bounded functions compactly supported in $B_\varrho(z_0)$. For these terms of the representation formula (4.2) the thesis then follows from a direct application of Theorem 1.1. Indeed, by our choice of η, we have $u(z) = (\eta u)(z)$ for every $z \in B_\sigma(z_0)$. Moreover, if $z, h \in B_\sigma(z_0)$ we also have that $z \circ h \in B_{2c_T\sigma}(z_0)$, by (3.5), then also $u(z \circ h) = (\eta u)(z \circ h)$.

We next consider the terms involving u and f. They are convolutions of Γ, that is a homogeneous kernel of degree $-Q$, with u and f, multiplied by bounded functions compactly supported in $B_\varrho(z_0)$. Moreover, u and f belong to $L^r(B_\varrho(z_0))$, for every r such that $1 \le r \le p$. We then choose r such that

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{Q+2}$$

and we apply again Theorem 1.1 with p replaced by r. This concludes the proof. □

Proof of Theorem 1.2. It follows from Proposition 4.2 by a simple covering argument. The constant $\widetilde{\varrho}$ can be chosen as follows. We let

$$\overline{\varrho} := \min \big\{ \varrho > 0 \mid B_\varrho(z) \subset \Omega \text{ for every } z \in K \big\},$$

then $\widetilde{\varrho} := \dfrac{\overline{\varrho}}{3c_T}$, so that we can choose $\sigma = \widetilde{\varrho}$ in every ball of the covering of K. □

Proof of Theorem 1.3. If f is a weak solution to (1.22), then it is a weak solution to

$$\partial_t f + \langle v, \nabla_x f \rangle = \Delta_v f + g - \mathrm{div}_v G, \tag{4.3}$$

where $G_j = \partial_{v_j} f, j = 1, \dots, n$. Note that the homogeneous dimension of the operator in (4.3) is $Q + 2 = n + 3n + 2$. By our assumptions $G_j \in L^p(\Omega)$ for every $j = 1, \dots, n$. Then the proof can be concluded by the same argument used in the proof of Theorem 1.2. □

5 Conclusion

The method used in this article for Kolmogorov equations can be adapted to the study of a wider family of differential operators, provided that they have a fundamental solution and that are invariant with respect to a suitable Lie group on their domain. Sobolev inequalities for operators of this kind have been proven in [6]. We recall here the assumptions on the operators.

Consider a differential operator in the form

$$\mathscr{L} u := \sum_{i,j=1}^{m} X_j \left(a_{ij}(x,t) X_i u \right) + X_0 u - \partial_t u, \tag{5.1}$$

where $(x,t) = (x_1, \dots, x_N, t)$ denotes the point in $\mathbb{R}^{N+1}$, and $1 \le m \le N$. The X_j's in (5.1) are smooth vector fields acting on $\mathbb{R}^N$, i.e.

$$X_j(x,t) = \sum_{k=1}^{N} b_k^j(x,t) \partial_{x_k}, \qquad j = 0, \dots, m,$$

and every b_k^j is a C^∞ function. In the sequel we always denote by $z = (x,t)$ the point in $\mathbb{R}^{N+1}$, and by A the $m \times m$ matrix $A = (a_{i,j})_{i,j=1,\dots,m}$. We also consider the *elliptic* analogous of $\mathscr{L}$

$$\mathscr{L}u := \sum_{i,j=1}^m X_j \left(a_{ij}(x) X_i u\right) \tag{5.2}$$

In both cases we assume that the coefficients of the matrix A are bounded measurable functions, and that A is symmetric and uniformly positive, that is, there is a positive constant μ such that

$$\sum_{i,j=1}^m a_{ij}(x,t)\xi_i, \xi_j \geq \mu |\xi|^2, \qquad \text{for every } \xi \in \mathbb{R}^m,$$

and for every $(x,t) \in \mathbb{R}^{N+1}$ (or for every $x \in \mathbb{R}^N$ as we consider the operator $\mathscr{L}$ in (5.2)).

Clearly, the Laplace operator Δ and the heat operator $\Delta - \partial_t$ write in the form (5.2) and (5.1), respectively, if we choose $X_j := \partial_{x_j}$ for $j = 1, \dots, N$, $X_0 := 0$, and the matrix A agrees with the $N \times N$ identity I_N. In the sequel we will use the following notations:

$$X = (X_1, \dots, X_m), \qquad Y = X_0 - \partial_t, \qquad \mathrm{div}_X F = \sum_{j=1}^m X_j F_j,$$

for every vector field $F = (F_1, \dots, F_m)$, so that the expression $\mathscr{L}u$ reads

$$\mathscr{L}u = \mathrm{div}_X(AXu) + Yu.$$

Finally, when A is the $m \times m$ identity matrix, we will use the notation

$$\mathscr{L}_0 := \sum_{k=1}^m X_k^2 + Y.$$

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