On the rank of a finite group of odd order with an involutory automorphism

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Abstract. Let G be a finite group of odd order admitting an involutory automorphism ϕ, and let $G_{-\phi}$ be the set of elements of G transformed by ϕ into their inverses. Note that $[G, \phi]$ is precisely the subgroup generated by $G_{-\phi}$. Suppose that each subgroup generated by a subset of $G_{-\phi}$ can be generated by at most r elements. We show that the rank of $[G, \phi]$ is r-bounded.

1. Introduction

Let G be a finite group of odd order admitting an involutory automorphism ϕ. Here the term "involutory automorphism" means an automorphism ϕ such that $\phi^2 = 1$. We let $G_{-\phi}$ stand for the set $\{g \in G \mid g^\phi = g^{-1}\}$ and G_ϕ for the centralizer of ϕ, that is, the subgroup of fixed points of ϕ. As usual we denote by $[G, \phi]$ the subgroup generated by all elements of G that can be written as $g^{-1}g^\phi$ for a suitable $g \in G$. It is well known that $[G, \phi]$ is normal in G and ϕ induces the trivial automorphism of $G/[G, \phi]$. Observe that $[G, \phi]$ is precisely the subgroup generated by $G_{-\phi}$. This is because an automorphism of order at most two of a group of odd order is nontrivial if and only if $G_{-\phi} \neq \{1\}$ (cf Lemma 2.1(i) in the next section).The following theorem was proved in [**10**, Theorem B].

Theorem 1.1. *Let G a finite group of odd order admitting an involutory automorphism ϕ such that the rank of G_ϕ is at most r. Then the rank of $[G, \phi]'$ is r-bounded.*

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Recall that a rank of a finite group G is the least number r such that each subgroup of G can be generated by at most r elements. Throughout this manuscript we use the term "$(a, b, c \dots)$-bounded" to mean "bounded from above by some function depending only on the parameters $a, b, c \dots$".

Since in a finite group of odd order with an involutory automorphism ϕ there is a kind of (very vague) duality between G_ϕ and $G_{-\phi}$, in this paper we address the question whether a rank condition imposed on the set $G_{-\phi}$ has an impact on the structure of G. We emphasize that $G_{-\phi}$ in general is not a subgroup of G and therefore the usual concept of rank does not apply to $G_{-\phi}$. Instead we impose the condition that each subgroup of G generated by a subset of $G_{-\phi}$ can be generated by at most r elements. Our main result is as follows.

THEOREM 1.2. *Let G be a group of odd order admitting an involutory automorphism ϕ and suppose that any subgroup generated by a subset of $G_{-\phi}$ can be generated by r elements. Then $[G, \phi]$ has r-bounded rank.*

It is noteworthy that in the literature there are several papers dealing with finite groups admitting a (not necessarily involutory) automorphism whose fixed-point subgroup is of rank r (see for example [**6, 5**]). In particular, [**5**] contains a result similar to the above Theorem 1.1. Thus, it seems plausible that some analogues of Theorem 1.2 are valid for the case where the order of ϕ is bigger than two.

2. Nilpotent groups with involutory automorphisms

We start with a collection of well-known facts about involutory automorphisms of groups of odd order (see for example [**3**, Lemma 4.1, Chap. 10]).

LEMMA 2.1. *Let G be a finite group of odd order admitting an involutory automorphism ϕ. The following conditions hold:*

(i) $G = G_\phi G_{-\phi} = G_{-\phi} G_\phi$ *and* $|G_{-\phi}| = [G : G_\phi]$;
(ii) *The subgroup generated by $G_{-\phi}$ is exactly $[G, \phi]$;*
(iii) *If N is any ϕ-invariant normal subgroup of G we have $(G/N)_\phi = G_\phi N/N$, and $(G/N)_{-\phi} = \{gN \mid g \in G_{-\phi}\}$;*
(iv) *If N is any ϕ-invariant normal subgroup of G such that $N = N_{-\phi}$ or $N = N_\phi$, then $[G, \phi]$ centralizes N;*
(v) *The subgroup G_ϕ normalizes $G_{-\phi}$.*

It is well known that a maximal abelian normal subgroup of a nilpotent group coincides with its centralizer. We will require the following related result.

Lemma 2.2. *Let G be a nilpotent group of odd order, ϕ an involutory automorphism of G, and A a maximal ϕ-invariant abelian normal subgroup of G. Then $A = C_G(A)$.*

Proof. Let $C = C_G(A)$ and assume that the result is false, that is, $A < C$. Then C/A is a nontrivial ϕ-invariant normal subgroup of G/A. The nilpotency of G/A implies that $C/A \cap Z(G/A) \neq 1$.

Let U be the full inverse image of $C/A \cap Z(G/A)$ in G. Since $C/A \cap Z(G/A) \neq 1$, the subgroup A is properly contained in U. From Lemma 2.1(i) we know that $U = U_\phi U_{-\phi}$. Thus, either $U_\phi \not\leq A$ or $U_{-\phi} \not\leq A$. In any case we can choose $u \in U \setminus A$ satisfying either $u^\phi = u$ or $u^\phi = u^{-1}$. Take $H = A\langle u\rangle$ and note that $A < H$. Furthermore, H is a ϕ-invariant abelian normal subgroup of G. This yields a contradiction. □

Note that the previous lemma fails if ϕ is allowed to be a coprime automorphism of arbitrary order. For example, the quaternion group of order 8 admits an automorphism α of order 3 and the maximal α-invariant abelian normal subgroup is central.

Lemma 2.3. *Let p be an odd prime and G a p-group admitting an involutory automorphism ϕ such that $G = [G, \phi]$. Let M be a ϕ-invariant normal subgroup of G and assume that $|M_{-\phi}| = p^n$ for some nonnegative integer n. Then $M \leq Z_{2n+1}(G)$.*

Proof. If $n = 0$, then the result follows from Lemma 2.1(iv), so assume that $n \geq 1$ and use induction on n.

Let $N = M \cap Z_2(G)$. If $N \not\leq Z(G)$, then Lemma 2.1(iv) implies that $N_{-\phi} \neq 1$, in which case we have $|(M/N)_{-\phi}| < |M_{-\phi}| = p^n$. By induction $M/N \leq Z_{2n-1}(G/N)$, whence $M \leq Z_{2n+1}(G)$. If $N \leq Z(G)$, then it turns out that $M \cap Z(G) = M \cap Z_i(G)$ for any $i \geq 2$ and so, obviously, $M \leq Z(G)$. This concludes the proof. □

We now fix some notation and hypotheses that will be used throughout this section.

Hypothesis 2.4. *Let p be an odd prime, r a positive integer and G a finite p-group with an involutory automorphism ϕ such that $G = [G, \phi]$. Assume that any subgroup generated by a subset of $G_{-\phi}$ can be generated by r elements.*

Lemma 2.5. *Assume Hypothesis 2.4 and suppose that G is of exponent p. There exists a number $l = l(r)$, depending on r only, such that the rank $r(G)$ of G is at most l.*

Proof. Let A be a maximal ϕ-invariant abelian normal subgroup of G. The subgroup $\langle A_{-\phi}\rangle$ is an r-generated abelian subgroup of exponent p and so $|A_{-\phi}| \leq p^r$. Lemma 2.3 implies that $A \leq Z_{2r+1}(G)$. Since

$\gamma_{2r+1}(G)$ commutes with $Z_{2r+1}(G)$, we deduce that $\gamma_{2r+1}(G)$ centralizes A. Furthermore, by Lemma 2.2, $A = C_G(A)$. Thus $\gamma_{2r+1}(G) \leq A$, that is, the quotient group G/A is nilpotent of class $2r$. We deduce that G has r-bounded nilpotency class as well. Since $G = \langle G_{-\phi} \rangle$ is r-generated by hypothesis, it follows that the rank $r(G)$ of G is r-bounded, as desired. □

The following result from [**10**, Lemma 2.2] is also useful.

LEMMA 2.6. *Let G be a group of prime exponent p and rank r_0. Then there exists a number $s = s(r_0)$, depending only on r_0, such that $|G| \leq p^s$.*

LEMMA 2.7. *Let G be a group satisfying Hypothesis 2.4. There exists a number $\lambda = \lambda(r)$, depending only on r, such that $\gamma_{2\lambda+1}(G)$ is powerful.*

PROOF. Let $s(r_0)$ be as in Lemma 2.6 and let $l(r)$ be as in Lemma 2.5. Take $N = \gamma_{2\lambda+1}(G)$, where $\lambda = s(l(r))$. In order to show that $N' \leq N^p$, we assume that N is of exponent p and prove that N is abelian.

Note that the subgroup $\langle N_{-\phi} \rangle$ is of exponent p. By Lemma 2.5 the rank of $\langle N_{-\phi} \rangle$ is at most $l(r)$. It follows from Lemma 2.6 that $|N_{-\phi}| \leq p^{s(l(r))} = p^{\lambda}$. Now Lemma 2.3 yields $N \leq Z_{2\lambda+1}(G)$. By using the well-known fact that $[\gamma_i(G), Z_i(G)] = 1$, for any positive integer i and any group G, we conclude that N is abelian, as required. □

LEMMA 2.8. *Assume Hypothesis 2.4. For any $i \geq 1$, there exists a number $m_i = m_i(i, r)$, depending only on i and r, such that $\gamma_i(G)$ is an m_i-generated group.*

PROOF. Let $N = \gamma_i(G)$. In view of the Burnside Basis Theorem [**9**, 5.3.2], we can pass to the quotient $G/\Phi(N)$ and assume that N is elementary abelian. Now $\langle N_{-\phi} \rangle$ is an elementary abelian r-generated group, so $|\langle N_{-\phi} \rangle| \leq p^r$. Thus, by Lemma 2.3, we have $N \leq Z_{2r+1}(G)$ and deduce that G has nilpotency class bounded only in terms of i and r. Since $G = \langle G_{-\phi} \rangle$ is r-generated, we conclude that $r(G)$ is (i, r)-bounded as well. Therefore N is m_i-generated for some (i, r)-bounded number m_i. This concludes the proof. □

PROPOSITION 2.9. *Under Hypothesis 2.4 the rank of G is r-bounded.*

PROOF. Let $s(r_0)$ be as in Lemma 2.6 and $l(r)$ as in Lemma 2.5. Take $N = \gamma_{2\lambda+1}(G)$, where $\lambda = \lambda(r) = s(l(r))$. Let d be the minimal number such that N is d-generated. Lemma 2.8 tells us that d is an r-bounded integer and N is powerful by Lemma 2.7. It follows from

[**1**, Theorem 2.9] that $r(N) \leq d$, and so the rank of N is r-bounded. Since the nilpotency class of G/N is r-bounded (recall that λ depends only on r) and $G = \langle G_{-\phi} \rangle$ is r-generated, we conclude that $r(G/N)$ is r-bounded as well. Now $r(G) \leq r(G/N) + r(N)$ and the result follows. □

3. Main results

Throughout this section the Feit-Thompson Theorem [**2**] is used without explicit references and p stands for a fixed odd prime. Given a finite soluble group G, we denote by $r_p(G)$ and $l_p(G)$ the rank of a Sylow p-subgroup and the p-length of G, respectively. Recall that $l_p(G)$ is by definition the number of p-factors (that is, factors that are p-groups) of the lower p-series of G given by:

$$1 \leq O_{p'}(G) \leq O_{p',p}(G) \leq O_{p',p,p'}(G) \leq \cdots .$$

We aim to establish the following generalisation of Proposition 2.9.

THEOREM 3.1. *Let G be a group of odd order admitting an involutory automorphism ϕ such that $G = [G, \phi]$. Let r be a positive integer and assume that any subgroup generated by a subset of $G_{-\phi}$ can be generated by r elements, then $r_p(G)$ is r-bounded.*

We start with an extension of Lemma 2.3.

LEMMA 3.2. *Let G be a group of odd order admitting an involutory automorphism ϕ such that $G = [G, \phi]$. Let M be a ϕ-invariant normal subgroup of G and assume that $|M_{-\phi}| \leq p^n$, for some nonnegative integer n. Then $M \leq Z_{2n+1}(O_p(G))$.*

PROOF. The proof can be reproduced word-by-word following that of Lemma 2.5. We argue by induction on n, being Lemma 2.1(iv) the case $n = 0$. Let $n \geq 1$. If $M \not\leq Z(O_p(G))$, then by Lemma 2.1(iv) we have $N_{-\phi} \neq 1$, where $N = M \cap Z_2(O_p(G))$. This implies that $|(M/N)_{-\phi}| < |M_{-\phi}|$. Thus we can pass to the quotient G/N and use the inductive hypothesis. The result follows. □

For the sake of simplicity we fix the following hypothesis that we will use in the next arguments.

HYPOTHESIS 3.3. *Let r be a positive integer and G a group of odd order admitting an involutory automorphism ϕ such that $G = [G, \phi]$. Assume that any subgroup generated by a subset of $G_{-\phi}$ can be generated by r elements.*

As usual, we denote by $F(G)$ the Fitting subgroup of a group G. Write $F_0(G) = 1, F_1(G) = F(G)$ and let $F_{i+1}(G)$ be the inverse image of $F(G/F_i(G))$. If G is soluble, then the least number h such that $F_h(G) = G$ is called the Fitting height of G.

One key step forward to the proof of Theorem 3.1 consists in showing that there exists an r-bounded number f such that the fth term of the derived series of G is nilpotent. For our purpose we will require the following result which is an immediate corollary of Hartley-Isaacs Theorem B in [**4**].

Proposition 3.4. *Let H be a finite group of odd order admitting an involutory automorphism ϕ such that $H = [H, \phi]$. Let k be a field with characteristic different from 2 and V a simple $k\langle\phi\rangle H$-module. Suppose that $\dim V_{-\phi} = r$. There exists an r-bounded number $\delta = \delta(r)$ such that $\dim V \leq \delta$.*

In the proof of the next proposition we will use the well-known theorem of Zassenhaus (see [**11**, Satz 7] or [**8**, Theorem 3.23]) stating that for any $n \geq 1$ there exists a number $j = j(n)$, depending only on n, such that, whenever k is a field, the derived length of any soluble subgroup of $GL(n, k)$ is at most j.

Proposition 3.5. *Assume Hypothesis 3.3. There exists a number $f = f(r)$, depending only on r, such that the fth term $G^{(f)}$ of the derived series of G is nilpotent.*

Proof. Let $\delta = \delta(r)$ be as in Proposition 3.4 and $f = j(\delta)$ the number given by the Zassenhaus theorem.

Suppose that the proposition is false and let G be a group of minimal possible order for which Hypothesis 3.3 holds while $G^{(f)}$ is not nilpotent. Then G has a unique minimal ϕ-invariant normal subgroup M. Indeed, suppose that G has two minimal ϕ-invariant normal subgroups, say M_1 and M_2. Then $M_1 \cap M_2 = 1$, being both elementary abelian p-groups for some prime p. Since $|G/M_1| < |G|$, the minimality of G implies that $(G/M_1)^{(f)}$ is nilpotent. For a symmetric argument $(G/M_2)^{(f)}$ is nilpotent too. This yields a contradiction since $G^{(f)}$ can be embedded into a subgroup of $G/M_1 \times G/M_2$ which is nilpotent, being isomorphic to the direct product of $(G/M_1)^{(f)}$ and $(G/M_2)^{(f)}$.

We claim that $M = C_G(M)$. Since M is a p-subgroup, for some prime p and it is unique, the Fitting subgroup $F = F(G)$ is a p-subgroup too. If $\Phi(F)$ is nontrivial, then we immediately get a contradiction because $F(G/\Phi(F)) = F/\Phi(F)$ and, again by the minimality of G, we know that $(G/\Phi(F))^{(f)}$ is nilpotent, so in particular $G^{(f)} \leq F$.

Assume now that $\Phi(F) = 1$ and so F is elementary abelian. If $M = F$, then $M = C_G(M)$, since the Fitting subgroup of a soluble group contains its own centralizer (see, for example, [**3**, Theorem 1.3, Chap. 6]). Thus we can assume that $M < F$. By hypotheses, on one hand, we know that $G^{(f)} \leq F_2(G)$ (to clarify, for the minimality of G the quotient $(G/F)^{(f)}$ is nilpotent, so it is contained in $F(G/F)$) and, on the other hand, that $(G/M)^{(f)}$ is nilpotent (again by the minimality of G). Now let T be a ϕ-invariant Hall p'-subgroup of $G^{(f)}$. It follows that both FT and MT are ϕ-invariant normal subgroups of G. Indeed, FT/F is normal in G/F, since $(G/F)^{(f)}$ is nilpotent and, similarly, MT/M is normal in G/M, being $(G/M)^{(f)}$ nilpotent as well.

Suppose first that $C_F(T) \neq 1$. Note that $C_F(T) = Z(FT)$, since F is abelian. Thus $C_F(T)$ is a ϕ-invariant normal subgroup of G, because FT is normal and ϕ-invariant. Hence $M \leq C_F(T)$. This implies that T centralizes M and so $MT = T \times M$. Recall that $T \leq F_2(G)$ and $T \cap F = 1$. It follows that T is nilpotent. Then $T \times M$ is normal nilpotent and $T \leq F$, a contradiction.

Thus, $C_F(T) = 1$. On the other hand, we see that $[F, T] \leq M$, since the nilpotent p'-subgroup MT/M and the p-subgroup F/M are both contained in $F(G/M)$ and commute, being $F(G/M)$ nilpotent. Now we have $M < F$ and $F = [F, T] \times C_F(T)$, so it should be $C_F(T) \neq 1$, a contradiction. Thus $M = C_G(M)$, as claimed above.

Then G/M acts faithfully and irreducibly on M. Moreover $\langle M_{-\phi} \rangle$ is r-generated and elementary abelian, so $|\langle M_{-\phi} \rangle| \leq p^r$. Now we can view M as a $G/M\langle\phi\rangle$-module over the field with p elements. By Proposition 3.4 we have $\dim(M) \leq \delta(r)$. Applying the theorem of Zassenhaus the derived length of G/M is at most $f = f(\delta(r))$. Then $G^{(f)} \leq F$, which concludes the proof. □

As a by-product of the previous result we obtain a bound for the p-length of G.

Corollary 3.6. *Assume Hypothesis 3.3. Then $l_p(G)$ is r-bounded, for any $p \in \pi(G)$.*

Proof. By Proposition 3.5 we know that $G^{(f)}$ is nilpotent for some r-bounded number f. This implies that the Fitting height $h(G) \leq f$. The result easily follows since it can be shown, by induction on the Fitting height $h(K)$, that $l_p(K) \leq h(K)$ for any finite soluble group K and for any prime $p \in \pi(K)$. □

The next result will be useful for a reduction argument inside the proof of Theorem 3.1.

LEMMA 3.7. *Let G be a group of odd order admitting an involutory automorphism ϕ. Assume that $G = PB$, where P is a ϕ-invariant normal elementary abelian p-subgroup and B is a cyclic subgroup such that $B = B_{-\phi}$. If $r(P_{-\phi}) = r$, then the rank of $[P, B]$ is at most $2r$.*

PROOF. Let $B = \langle b \rangle$, where b is a generator of B. Let $C = P_\phi$ and $C_0 = C \cap C^b$. Then it follows from Lemma 2.1(i) that

$$[P : C_0] \leq [P : C][P : C^b] \leq p^{2r},$$

since $r(P_{-\phi}) = r$. We claim that $C_0 \leq C_G(b)$. Indeed, choose $x \in C$ such that $x^b \in C$. Then, we have $x^b = (x^b)^\phi = x^{b^{-1}}$ and so x commutes with b^2. Since b has odd order, it follows that $C_0 \leq C_G(b)$, as claimed. Thus $C_0 \leq Z(G)$. Choose now $a_1, \ldots, a_{2r}$ elements that generate P modulo C_0. By using linearity in P and the fact that C_0 is central in G, we deduce that $[P, b]$ is generated by $[a_1, b], \ldots, [a_{2r}, b]$. Hence the result. □

We are ready to embark on the proof of Theorem 3.1.

PROOF OF THEOREM 3.1. Recall that G is a group satisfying Hypothesis 3.3 and we want to show that $r_p(G)$ is r-bounded for any fixed prime $p \in \pi(G)$.

First, we show that G is generated by r-boundedly many elements from $G_{-\phi}$. If G is a p-group, then the claim follows from the Burnside Basis Theorem since $G = \langle G_{-\phi} \rangle$ is r-generated. In the case where G is nilpotent, we have $[G, \phi] = [P_1.\phi] \times \cdots \times [P_s, \phi]$, where $\{P_1, \ldots, P_s\}$ are the Sylow subgroups of G, so the result easily follows from the case of p-groups. Assume now that G is not nilpotent. Let $h = h(G) \geq 2$. Since we know from the proof of Corollary 3.6 that h is r-bounded, it is sufficient to show that G is generated by (h, r)-boundedly many elements from $G_{-\phi}$. We argue by induction on h. Let $F = F(G)$. By induction there are boundedly many elements $a_1, \ldots, a_d \in G_{-\phi}$ such that $G = F\langle a_1, \ldots, a_d \rangle$. Let $D = \langle F_{-\phi}, a_1, \ldots, a_d \rangle$. Note that D has an r-bounded number of generators from $G_{-\phi}$. Let N be the normal closure of $\langle F_{-\phi} \rangle$ in G. Then N is precisely $\langle F_{-\phi} \rangle^D$ because F normalizes $\langle F_{-\phi} \rangle$ by Lemma 2.1(v). Thus $N \leq D$. Recall that by Lemma 2.1(i) we have $F = F_\phi F_{-\phi}$. Hence the image of F in G/N is contained in $(G/N)_\phi$ and, therefore, it is central by Lemma 2.1(iv). Since $G = FD$, it follows that D/N becomes normal in G/N and, therefore, D is normal in G (because $N \leq D$). Now ϕ acts trivially on the quotient G/D, that is $[G, \phi] \leq D$. Since $G = [G, \phi]$, we have $G = D$. This concludes the proof that G can be generated by r-boundedly many elements from $G_{-\phi}$.

If G is a p-group, then the theorem follows immediately from Proposition 2.9. Assume that G is not a p-group and use induction on $l = l_p(G)$ that is r-bounded by Corollary 3.6. So it is sufficient to show that $r_p(G)$ is (l, r)-bounded. By induction assume that there exists r_1, depending only on l and r, such that $r_p(K) \leq r_1$ for any ϕ-invariant quotient K of G having $l_p(K)$ at most $l - 1$.

Since $l = l_p(G/O_{p'}(G))$, we can assume that $O_{p'}(G) = 1$. Take $P = O_p(G)$. Note that

$$r_p(G) \leq r(P) + r_p(G/P).$$

Since $l_p(G/[P, G]) \leq l - 1$, by induction the rank $r_p(G/[P, G]) \leq r_1$. Then it is sufficient to bound the rank of P.

Let us show first that P has an r-bounded number of generators. Passing to the quotient $G/\Phi(P)$, we can assume that P is elementary abelian. As showed above, we know that G can be generated by $t = t(r)$ elements from $G_{-\phi}$, say $d_1, \dots, d_t$. Note that $[P, G] = [P, d_1][P, d_2] \dots [P, d_t]$. In view of Lemma 3.7 each $[P, d_i]$ has rank at most $2r$. Therefore the rank of the image of $[P, G]$ in $G/\Phi(P)$ is at most $2rt$ and by induction on l, $r_p(G/[P, G])$ is r-bounded, so P has an r-bounded number of generators, as claimed.

Next, we claim that for any $i \geq 2$ there exists a number $m_i = m_i(i, r)$, depending only on i and r, such that $V = \gamma_i(P)$ has m_i-bounded number of generators. We can pass to the quotient $G/\Phi(V)$ and assume that V is elementary abelian. Now $\langle V_{-\phi} \rangle$ is an elementary abelian r-generated group, so $|\langle V_{-\phi} \rangle| \leq p^r$. Thus, by Lemma 3.2, we have $V \leq Z_{2r+1}(P)$ and deduce that the nilpotency class of $P/\Phi(V)$ is bounded only in terms of i and r. Since P has an r-bounded number of generators, we conclude that $r(P/\Phi(V))$ is (i, r)-bounded as well. Therefore V is m_i-generated for some (i, r)-bounded number m_i, as claimed.

Let $s(r_0)$ be as in Lemma 2.6 and let $l(r)$ be as in Lemma 2.5. Take $M = \gamma_{2\lambda+1}(P)$, where $\lambda = s(l(r))$. We want to prove that M is powerful. In order to show that $M' \leq M^p$, we assume that M is of exponent p and prove that M is abelian. Note that the subgroup $\langle M_{-\phi} \rangle$ is of exponent p. By Lemma 2.5 the rank of $\langle M_{-\phi} \rangle$ is at most $l(r)$. It follows from Lemma 2.6 that $|M_{-\phi}| \leq p^{s(l(r))} = p^{\lambda}$. Now Lemma 3.2 yields that $M \leq Z_{2\lambda+1}(P)$. Since $[\gamma_i(P), Z_i(P)] = 1$, for any positive integer i, we conclude that M is abelian, as required.

Let now d_0 be the minimal number such that M is d_0-generated. It was shown above that d_0 is an r-bounded integer. Since M is powerful, it follows from [**1**, Theorem 2.9] that $r(M) \leq d_0$, and so the rank of M is r-bounded. Since the nilpotency class of P/M is r-bounded and

P has an r-bounded number of generators, we conclude that $r(P/M)$ is r-bounded as well. Now $r(P) \leq r(P/M) + r(M)$ and the result follows. □

It is now easy to give the proof of our main result, Theorem 1.2, which states that if G is a group satisfying Hypothesis 3.3, then the rank of G is r-bounded.

Proof of Theorem 1.2. Without loss of generality we can assume that $G = [G, \phi]$. By a result of Kovács [**7**] for any soluble group H we have $r(H) \leq \max\{r_p(H) \mid p \in \pi(H)\} + 1$. Therefore it is enough to check that $r_p(G)$ is bounded in terms of r only for any $p \in \pi(G)$. This is immediate from Theorem 3.1. □

References

[1] J. D. Dixon, M. P. F. du Sautoy, A. Mann and D. Segal, Analytic pro-p groups. Cambridge 1991.

[2] W. Feit and J. Thompson, Solvability of groups of odd order, *Pacific J. Math.* **13** (1963), 773–1029.

[3] D. Gorenstein, Finite Groups, Chelsea Publishing Company, New York, 1980.

[4] B. Hartley and I. M. Isaacs, On characters and fixed points of coprime operator groups. *J. Algebra* **131** (1990), 342–358.

[5] E. I. Khukhro, Groups with an automorphism of prime order that is almost regular in the sense of rank, *J. London Math. Soc.*(2) **77** (2008), 130–148.

[6] E. I. Khukhro and V. D. Mazurov, Finite groups with an automorphism of prime order whose centralizer has small rank, *J. Algebra* **301**(2006), 474–492.

[7] L. G. Kóvacs, On finite soluble groups. *Math. Z.* **103**(1967), 37–39.

[8] D. J. S. Robinson, Finiteness conditions and generalized soluble groups. part 1, Springer-Verlag, 1972.

[9] D. J. S. Robinson, A Course in the Theory of Groups, Springer-Verlag, New York (1996).

[10] P. Shumyatsky, Involutory automorphisms of finite groups and their centralizers, *Arch. Math.* **71** (1998), 425–432.

[11] H. Zassenhaus, Beweis eines Satzes über diskrete Gruppen, *Abh. Math. Sem. Univ. Hamburg* **12** (1938), 289–312.

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