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# The Möbius function of $\mathrm{PSU}(3, 2^{2^n})$

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**Abstract**

Let $G$ be the simple group $\mathrm{PSU}(3, 2^{2^n})$, $n > 0$. For any subgroup $H$ of $G$, we compute the Möbius function $\mu_L(H, G)$ of $H$ in the subgroup lattice $L$ of $G$, and the Möbius function $\mu_{\bar{L}}([H], [G])$ of $[H]$ in the poset $\bar{L}$ of conjugacy classes of subgroups of $G$. For any prime $p$, we provide the Euler characteristic of the order complex of the poset of non-trivial $p$-subgroups of $G$.

*Keywords: Unitary groups, Möbius function, subgroup lattice.*

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## 1 Introduction

The Möbius function $\mu(H, G)$ on the subgroups of a finite group $G$ is defined recursively by $\mu(G, G) = 1$ and $\sum_{K \geq H} \mu(K, G) = 0$ if $H < G$. This function was used in 1936 by Hall [12] to enumerate $k$-tuples of elements of $G$ which generate $G$, for a given $k$.

The combinatorial and group-theoretic properties of the Möbius function were investigated by many authors; see the paper [14] by Hawkes, Isaacs, and Özaydin. The Möbius function is defined more generally on a locally finite poset $(\mathcal{P}, \leq)$ by the recursive definition $\mu(x, x) = 1$, $\mu(x, y) = 0$ if $x \not\leq y$, and $\sum_{x \leq z \leq y} \mu(z, y) = 0$ if $x \leq y$; for instance, the poset taken into consideration may be the subgroup lattice $L$ of a finite group $G$ ordered by inclusion. Mann [19, 20] studied $\mu(H, G)$ in the broader context of profinite groups $G$ and defined a probabilistic zeta function $P(G, s)$ associated to $G$, related to the probability of generating $G$ with $s$ elements when $G$ is positively finitely generated.

The Möbius function on a poset $\mathcal{P}$ also appears in the context of topological invariants of the order simplicial complex $\Delta(\mathcal{P})$ associated to $\mathcal{P}$, see the works of Brown [2] and

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Quillen [25]; if $\mathcal{P}$ is the subgroup lattice of a finite group $G$, then the reduced Euler characteristic of $\Delta(\mathcal{P})$ is equal to $\mu(\{1\}, G)$. This motivates the search for $\mu(\{1\}, G)$ independently of the knowledge of $\mu(H, G)$ for other subgroups $H$ of $G$, see for instance [26, 27] and the references therein; $\mu(\{1\}, G)$ is often called the *Möbius number* of $G$. Shareshian provided a formula in [26] for $\mu(\{1\}, \mathrm{Sym}(n))$, and computed $\mu(\{1\}, G)$ in [27] when $G \in \{\mathrm{PGL}(2,q), \mathrm{PSL}(2,q), \mathrm{PGL}(3,q), \mathrm{PSL}(3,q), \mathrm{PGU}(3,q), \mathrm{PSU}(3,q)\}$ with $q$ odd or $G$ is a Suzuki group $\mathrm{Sz}(2^{2h+1})$.

Consider the poset $\bar{L}$ of conjugacy classes $[H]$ of subgroups $H$ of a finite group $G$, ordered as follows: $[H] \leq [K]$ if and only if $H$ is contained in some conjugate of $K$ in $G$. After Hawkes, Isaacs, and Özaydin [14], we denote by $\lambda(H, G)$ the Möbius function $\mu([H], [G])$ in $\bar{L}$, while $\mu(H, G)$ is the Möbius function in $L$. Some attempt was done to search relations between the Möbius functions $\mu(H, G)$ and $\lambda(H, G)$; Hawkes, Isaacs, and Özaydin [14] proved that, if $G$ is solvable, then

$$\mu(\{1\}, G) = |G'| \cdot \lambda(\{1\}, G). \tag{1.1}$$

The property (1.1), which we call $(\mu, \lambda)$-property, does not hold in general for non-solvable groups; see [1]. Pahlings [23] proved that, if $G$ is solvable, then

$$\mu(H, G) = [N_{G'}(H) : H \cap G'] \cdot \lambda(H, G) \tag{1.2}$$

for any subgroup $H$ of $G$. The analysis of the generalized $(\mu, \lambda)$-property (1.2), although false in general for non-solvable groups, is of interest since it relates the Möbius functions $\mu(H, G)$ and $\lambda(H, G)$.

A lot of work was done by several authors about probabilistic functions for groups; see for instance [6, 10, 19, 20]. In particular, Mann posed in [19] a conjecture, the validity of which would imply that the sum

$$\sum_{H} \frac{\mu(H, G)}{[G : H]^s}$$

over all subgroups $H < G$ of finite index of a positively finitely generated profinite group $G$ is absolutely convergent for $s$ in some right complex half-plane and, for $s \in \mathbb{N}$ large enough, represents the probability of generating $G$ with $s$ elements. Lucchini [18] showed that this problem can be reduced so that Mann's conjecture is reformulated as follows: there exist two constants $c_1, c_2 \in \mathbb{N}$ such that, for any finite monolithic group $G$ with non-abelian socle,

1. $|\mu(H, G)| \leq [G : H]^{c_1}$ for any $H < G$ such that $G = H\,\mathrm{soc}(G)$, and
2. the number of subgroups $H < G$ of index $n$ in $G$ such that $H\,\mathrm{soc}(G) = G$ and $\mu(H, G) \neq 0$ is upper bounded by $n^{c_2}$, for any $n \in \mathbb{N}$.

It seems natural to investigate this conjecture on finite monolithic groups starting by almost simple groups. Mann's conjecture has been shown to be satisfied by the alternating and symmetric groups [3], as well as by those families of groups $G$ for which $\mu(H, G)$ has been computed for any subgroup $H$; namely, $\mathrm{PSL}(2, q)$ [8, 12], $\mathrm{PGL}(2, q)$ [8], the Suzuki groups $\mathrm{Sz}(2^{2h+1})$ [9], and the Ree groups $R(3^{2h+1})$ [24].

In this paper, we take into consideration the three dimensional projective special unitary group $G = \mathrm{PSU}(3, q)$ over the field with $q = 2^{2^n}$ elements, for any positive $n$ (note that $\mathrm{PSU}(3, q) = \mathrm{PGU}(3, q)$ as $3 \nmid (q + 1)$). In particular, the following results are obtained.

(i) We compute $\mu(H, G)$ for any subgroup $H$ of $G$, as summarized in Table 1. This shows that the groups $\mathrm{PSU}(3, 2^{2^n})$ satisfy Mann's conjecture.

(ii) We compute $\lambda(H, G)$ for any subgroup $H$ of $G$, as summarized in Table 1. This shows that the groups $\mathrm{PSU}(3, 2^{2^n})$ satisfy the $(\mu, \lambda)$-property, but do not satisfy the generalized $(\mu, \lambda)$-property.

(iii) We compute the Euler characteristic $\chi(\Delta(L_p \setminus \{1\}))$ of the order complex of the poset $L_p \setminus \{1\}$ of non-trivial $p$-subgroups of $G$, for any prime $p$, as summarized in Table 2.

For the subgroups listed in Table 1, the isomorphism type determines a unique conjugacy class in $G$.

Table 1: Subgroups $H$ of $G = \mathrm{PSU}(3, q)$, $q = 2^{2^n}$, with $\mu(H) \neq 0$ or $\lambda(H) \neq 0$.

| Isomorphism type of $H$ | $\lvert H\rvert$ | $N_G(H)$ | $\mu(H,G)$ | $\lambda(H,G)$ |
|---|---|---|---|---|
| $G$ | $q^3(q^3+1)(q^2-1)$ | $H$ | $1$ | $1$ |
| $(E_q \,.\, E_{q^2}) \rtimes C_{q^2-1}$ | $q^3(q^2-1)$ | $H$ | $-1$ | $-1$ |
| $\mathrm{PSL}(2,q) \times C_{q+1}$ | $q(q^2-1)(q+1)$ | $H$ | $-1$ | $-1$ |
| $(C_{q+1} \times C_{q+1}) \rtimes \mathrm{Sym}(3)$ | $6(q+1)^2$ | $H$ | $-1$ | $-1$ |
| $C_{q^2-q+1} \rtimes C_3$ | $3(q^2-q+1)$ | $H$ | $-1$ | $-1$ |
| $E_q \rtimes C_{q^2-1}$ | $q(q^2-1)$ | $H$ | $1$ | $1$ |
| $(C_{q+1} \times C_{q+1}) \rtimes C_2$ | $2(q+1)^2$ | $H$ | $1$ | $1$ |
| $\mathrm{Sym}(3)$ | $6$ | $\mathrm{Sym}(3) \times C_{q+1}$ | $q+1$ | $1$ |
| $C_3$ | $3$ | $C_{q^2-1} \rtimes C_2$ | $\frac{2(q^2-1)}{3}$ | $1$ |
| $C_2$ | $2$ | $(E_q \,.\, E_{q^2}) \rtimes C_{q+1}$ | $-\frac{q^3(q+1)}{2}$ | $-1$ |

Table 2: Euler characteristic of the order complex of the poset of proper $p$-subgroups of $G$.

| Prime $p$ | $p \nmid \lvert G\rvert$ | $p=2$ | $p \mid (q+1)$ | $p \mid (q-1)$ | $p \mid (q^2-q+1)$ |
|---|---|---|---|---|---|
| $\chi(\Delta(L_p \setminus \{1\}))$ | $0$ | $q^3+1$ | $-\frac{q^6-2q^5-q^4+2q^3-3q^2}{3}$ | $\frac{q^6+q^3}{2}$ | $-\frac{q^6+q^5-q^4-q^3}{3}$ |

The paper is organized as follows. Section 2 contains preliminary results on the Möbius functions $\mu(H, G)$ and $\lambda(H, G)$ and the relation between the Möbius function and the Euler characteristic of the order complex; this section contains also preliminary results on the groups $G = \mathrm{PSU}(3, 2^{2^n})$, whose elements are described geometrically in their action on the Hermitian curve associated to $G$. Sections 3 and 4 are devoted to the determination of $\mu(H, G)$ and $\lambda(H, G)$, respectively, for any subgroup $H$ of $G$. Section 5 provides the Euler characteristic of the order complex of the poset of proper $p$-subgroups of $G$, for any prime $p$.

## 2 Preliminary results

Let $(\mathcal{P}, \leq)$ be a finite poset. The Möbius function $\mu_{\mathcal{P}} \colon \mathcal{P} \times \mathcal{P} \to \mathbb{Z}$ is defined recursively as follows:

$$\mu_{\mathcal{P}}(x,y) = 0 \quad \text{if} \quad x \not\leq y; \qquad \mu_{\mathcal{P}}(x,x) = 1; \qquad \sum_{x \leq z \leq y} \mu_{\mathcal{P}}(z,y) = 0 \quad \text{if} \quad x < y.$$

If $x < y$, then $\mu_{\mathcal{P}}(x,y)$ can be equivalently defined by

$$\sum_{x \leq z \leq y} \mu_{\mathcal{P}}(x,z) = 0.$$

To the poset $\mathcal{P}$ we can associate a simplicial complex $\Delta(\mathcal{P})$ whose vertices are the elements of $\mathcal{P}$ and whose $i$-dimensional faces are the chains $a_0 < \cdots < a_i$ of length $i$ in $\mathcal{P}$; $\Delta(\mathcal{P})$ is called the *order complex* of $\mathcal{P}$. Provided that $\mathcal{P}$ has a least element $0$, the Euler characteristic of the order complex of $\mathcal{P} \setminus \{0\}$ is computed as follows (see [28, Proposition 3.8.6]):

$$\chi(\Delta(\mathcal{P} \setminus \{0\})) = -\sum_{x \in \mathcal{P} \setminus \{0\}} \mu_{\mathcal{P}}(0,x).$$

Given a finite group $G$, we will consider the following two Möbius functions associated to $G$.

(i) The Möbius function on the subgroup lattice $L$ of $G$, ordered by inclusion. We will denote $\mu_L(H,G)$ simply by $\mu(H)$.

(ii) The Möbius function on the poset $\bar{L}$ of conjugacy classes $[H]$ of subgroups $H$ of $G$, ordered as follows: $[H] \leq [K]$ if and only if $H$ is contained in the conjugate $gKg^{-1}$ for some $g \in G$. We will denote $\mu_{\bar{L}}([H],[G])$ simply by $\lambda(H)$.

Two facts will be used to compute $\mu(H)$. The first easy fact is that, if $H$ and $K$ are conjugate in $G$, then $\mu(H) = \mu(K)$. The second fact is due to Hall [12, Theorem 2.3], and is stated in the following lemma.

**Lemma 2.1.** *If $H < G$ satisfies $\mu(H) \neq 0$, then $H$ is the intersection of maximal subgroups of $G$.*

For any prime $p$, let $L_p$ be the subposet of $L$ given by all $p$-subgroups of $G$, so that

$$\chi(\Delta(L_p \setminus \{1\})) = -\sum_{H \in L_p \setminus \{1\}} \mu_{L_p}(\{1\},H). \tag{2.1}$$

By a result of Brown [2], $\chi(\Delta(L_p \setminus \{1\}))$ is congruent to 1 modulo the order $|G|_p$ of a Sylow $p$-subgroup of $G$. In order to compute explicitly $\chi(\Delta(L_p \setminus \{1\}))$ we will use the following result of Hall [12, Equation (2.7)]:

**Lemma 2.2.** *Let $H$ be a $p$-group of order $p^r$. If $H$ is not elementary abelian, then $\mu_{L_p}(\{1\},H) = 0$. If $H$ is elementary abelian, then $\mu_{L_p}(\{1\},H) = (-1)^r p^{\binom{r}{2}}$.*

We describe now the group $G$ which will be considered in the following sections. Let $n$ be a positive integer, $q = 2^{2^n}$, $\mathbb{F}_q$ be the finite field with $q$ element, and $\bar{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$. Let $\mathcal{U}$ be a non-degenerate unitary polarity of the plane $\mathrm{PG}(2, q^2)$ over $\mathbb{F}_{q^2}$, and $\mathcal{H}_q \subset \mathrm{PG}(2, \bar{\mathbb{F}}_q)$ be the Hermitian curve defined by $\mathcal{U}$. The following homogeneous equations define models for $\mathcal{H}_q$ which are projectively equivalent over $\mathbb{F}_{q^2}$:

$$X^{q+1} + Y^{q+1} + Z^{q+1} = 0; \tag{2.2}$$

$$X^q Z + X Z^q - Y^{q+1} = 0. \tag{2.3}$$

The models (2.2) and (2.3) are called the Fermat and the Norm-Trace model of $\mathcal{H}_q$, respectively. The set of $\mathbb{F}_{q^2}$-rational points of $\mathcal{H}_q$ is denoted by $\mathcal{H}_q(\mathbb{F}_{q^2})$, and consists of the $q^3 + 1$ isotropic points of $\mathcal{U}$. The full automorphism group $\mathrm{Aut}(\mathcal{H}_q)$ of $\mathcal{H}_q$ is defined over $\mathbb{F}_{q^2}$, and coincides with the unitary subgroup $\mathrm{PGU}(3, q)$ of $\mathrm{PGL}(3, q^2)$ stabilizing $\mathcal{H}_q(\mathbb{F}_{q^2})$, of order $|\,\mathrm{PGU}(3, q)| = q^3(q^3 + 1)(q^2 - 1)$.

The combinatorial properties of $\mathcal{H}_q(\mathbb{F}_{q^2})$ can be found in [16]. In particular, any line $\ell$ of $\mathrm{PG}(2, q^2)$ has either 1 or $q+1$ common points with $\mathcal{H}_q(\mathbb{F}_{q^2})$, that is, $\ell$ is either a tangent line or a chord of $\mathcal{H}_q(\mathbb{F}_{q^2})$; in the former case $\ell$ contains its pole with respect to $\mathcal{U}$, in the latter case $\ell$ doesn't. Also, $\mathrm{PGU}(3, q)$ acts 2-transitively on $\mathcal{H}_q(\mathbb{F}_{q^2})$ and transitively on $\mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$; $\mathrm{PGU}(3, q)$ acts transitively also on the non-degenerate self-polar triangles $T = \{P_1, P_2, P_3\} \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ with respect to $\mathcal{U}$. Recall that, if $\sigma \in \mathrm{PGU}(3, q)$ stabilizes a point $P \in \mathrm{PG}(2, q^2)$, then $\sigma$ stabilizes also the polar line of $P$ with respect to $\mathcal{U}$, and vice versa.

The curve $\mathcal{H}_q$ is non-singular and $\mathbb{F}_{q^2}$-maximal of genus $g = \frac{q(q-1)}{2}$, that is, the size of $\mathcal{H}_q(\mathbb{F}_{q^2})$ attains the Hasse-Weil upper bound $q^2 + 1 + 2gq$. This implies that $\mathcal{H}_q$ is $\mathbb{F}_{q^4}$-minimal and $\mathbb{F}_{q^6}$-maximal, so that $\mathcal{H}_q(\mathbb{F}_{q^4}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2}) = \emptyset$ and $|\mathcal{H}_q(\mathbb{F}_{q^6}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2})| = q^6 + q^5 - q^4 - q^3$. Let $\Phi_{q^2}$ be the Frobenius map $(X, Y, Z) \mapsto (X^{q^2}, Y^{q^2}, Z^{q^2})$ over $\mathrm{PG}(2, \bar{\mathbb{F}}_{q^2})$; then the $\mathbb{F}_{q^6} \setminus \mathbb{F}_{q^2}$-rational points of $\mathcal{H}_q$ split into $\frac{q^6+q^5-q^4-q^3}{3}$ non-degenerate triangles $\{P, \Phi_{q^2}(P), \Phi_{q^2}^2(P)\}$. The group $\mathrm{PGU}(3, q)$ is transitive on such triangles.

Since $3 \nmid (q + 1)$, we have $\mathrm{PGU}(3, q) = \mathrm{PSU}(3, q)$; henceforth, we denote by $G$ the simple group $\mathrm{PSU}(3, q)$. The following classification of subgroups of $G$ goes back to Hartley [13]; here we use that $\log_2(q)$ has no odd divisors different from 1. The notation $S_2$ stands for a Sylow 2-subgroup of $G$, which is a non-split extension $E_q \,.\, E_{q^2}$ of its elementary abelian center of order $q$ by an elementary abelian group of order $q^2$.

**Theorem 2.3.** *Let $n > 0$, $q = 2^{2^n}$, and $G = \mathrm{PSU}(3, q)$. Up the conjugation, the maximal subgroups of G are the following.*

*(i) The stabilizer $M_1(P) \cong S_2 \rtimes C_{q^2-1}$ of a point $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$, of order $q^3(q^2 - 1)$.*

*(ii) The stabilizer $M_2(P) \cong \mathrm{PSL}(2, q) \times C_{q+1}$ of a point $P \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q(\mathbb{F}_{q^2})$, of order $q(q^2 - 1)(q + 1)$.*

*(iii) The stabilizer $M_3(T) \cong (C_{q+1} \times C_{q+1}) \rtimes \mathrm{Sym}(3)$ of a non-degenerate self-polar triangle $T = \{P, Q, R\} \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ with respect to $\mathcal{U}$, of order $6(q + 1)^2$.*

*(iv) The stabilizer $M_4(T) \cong C_{q^2-q+1} \rtimes C_3$ of a triangle $T = \{P, \Phi_{q^2}(P), \Phi_{q^2}^2(P)\} \subset \mathcal{H}_q(\mathbb{F}_{q^6}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2})$, of order $3(q^2 - q + 1)$.*

For a detailed description of the maximal subgroups of $G$, both from an algebraic and a geometric point of view, we refer to [11, 21, 22].

In our investigation it is useful to know the geometry of the elements of $\mathrm{PGU}(3,q)$ on $\mathrm{PG}(2,\bar{\mathbb{F}}_q)$, and in particular on $\mathcal{H}_q(\mathbb{F}_{q^2})$. This can be obtained as a corollary of Theorem 2.3, and is stated in Lemma 2.2 with the usual terminology of collineations of projective planes; see [16]. In particular, a linear collineation $\sigma$ of $\mathrm{PG}(2,\bar{\mathbb{F}}_q)$ is a $(P,\ell)$-*perspectivity*, if $\sigma$ preserves each line through the point $P$ (the *center* of $\sigma$), and fixes each point on the line $\ell$ (the *axis* of $\sigma$). A $(P,\ell)$-perspectivity is either an *elation* or a *homology* according to $P\in\ell$ or $P\notin\ell$. Lemma 2.4 was obtained in [21] in a more general form (i.e., for any prime power $q$).

**Lemma 2.4.** *For a nontrivial element $\sigma\in G=\mathrm{PSU}(3,q)$, $q=2^{2^n}$, one of the following cases holds.*

- *(A)* $\mathrm{ord}(\sigma)\mid(q+1)$ *and $\sigma$ is a homology, with center $P\in\mathrm{PG}(2,q^2)\setminus\mathcal{H}_q$ and axis $\ell_P$ which is a chord of $\mathcal{H}_q(\mathbb{F}_{q^2})$; $(P,\ell_P)$ is a pole-polar pair with respect to $\mathcal{U}$.*
- *(B)* $2\nmid\mathrm{ord}(\sigma)$ *and $\sigma$ fixes the vertices $P_1,P_2,P_3$ of a non-degenerate triangle $T\subset$* $\mathrm{PG}(2,q^6)$.
  - *(B1)* $\mathrm{ord}(\sigma)\mid(q+1)$, *$P_1,P_2,P_3\in\mathrm{PG}(2,q^2)\setminus\mathcal{H}_q$, and the triangle $T$ is self-polar with respect to $\mathcal{U}$.*
  - *(B2)* $\mathrm{ord}(\sigma)\mid(q^2-1)$ *and* $\mathrm{ord}(\sigma)\nmid(q+1)$*; $P_1\in\mathrm{PG}(2,q^2)\setminus\mathcal{H}_q$ and $P_2,P_3\in\mathcal{H}_q(\mathbb{F}_{q^2})$.*
  - *(B3)* $\mathrm{ord}(\sigma)\mid(q^2-q+1)$ *and $P_1,P_2,P_3\in\mathcal{H}_q(\mathbb{F}_{q^6})\setminus\mathcal{H}_q(\mathbb{F}_{q^2})$.*
- *(C)* $\mathrm{ord}(\sigma)=2$*; $\sigma$ is an elation with center $P\in\mathcal{H}_q(\mathbb{F}_{q^2})$ and axis $\ell_P$ which is tangent to $\mathcal{H}_q$ at $P$, such that $(P,\ell_P)$ is a pole-polar pair with respect to $\mathcal{U}$.*
- *(D)* $\mathrm{ord}(\sigma)=4$*; $\sigma$ fixes a point $P\in\mathcal{H}_q(\mathbb{F}_{q^2})$ and a line $\ell_P$ which is tangent to $\mathcal{H}_q$ at $P$, such that $(P,\ell_P)$ is a pole-polar pair with respect to $\mathcal{U}$.*
- *(E)* $\mathrm{ord}(\sigma)=2d$ *where $d$ is a nontrivial divisor of $q+1$; $\sigma$ fixes two points $P\in\mathcal{H}_q(\mathbb{F}_{q^2})$ and $Q\in\mathrm{PG}(2,q^2)\setminus\mathcal{H}_q$, the polar line $PQ$ of $P$, and the polar line of $Q$ which passes through $P$.*

For a detailed description of the elements and subgroups of $G$, both from an algebraic and a geometric point of view, we refer to [11, 21, 22], on which our geometric arguments are based.

Throughout the paper, a nontrivial element of $G$ is said to be of type (A), (B), (B1), (B2), (B3), (C), (D), or (E), as given in Lemma 2.4. Also, the polar line to $\mathcal{H}_q$ at $P\in\mathrm{PG}(2,q^2)$ is denoted by $\ell_P$. Note that, under our assumptions, any element of order 3 in $G$ is of type (B2). We will denote a cyclic group of order $d$ by $C_d$ and an elementary abelian group of order $d$ by $E_d$. The center $Z(S_2)$ of $S_2$ is elementary abelian of order $q$, and any element in $S_2\setminus Z(S_2)$ has order 4; see [11, Section 3].

# 3 Determination of $\mu(H)$ for any subgroup $H$ of $G$

Let $n>0$, $q=2^{2^n}$, $G=\mathrm{PSU}(3,q)$. This section is devoted to the proof of the following theorem.

**Theorem 3.1.** *Let $H$ be a proper subgroup of $G$. Then $H$ is the intersection of maximal subgroups of $G$ if and only if $H$ is one of the following groups:*

$$\begin{array}{lll} S_2 \rtimes C_{q^2-1}, & \mathrm{PSL}(2,q)\times C_{q+1}, & C_{q^2-q+1}\rtimes C_3, \\ (C_{q+1}\times C_{q+1})\rtimes \mathrm{Sym}(3), & E_q \rtimes C_{q^2-1}, & (C_{q+1}\times C_{q+1})\rtimes C_2, \\ C_{q+1}\times C_{q+1}, & C_{q^2-1}, & C_{2(q+1)}, \\ C_{q+1} = Z(M_2(P)) \text{ for some } P, & E_q, & \mathrm{Sym}(3), \\ C_3, & C_2, & \{1\}. \end{array} \tag{3.1}$$

*Given a type of groups in Equation (3.1), there is just one conjugacy class of subgroups of $G$ of that isomorphism type.*

*The normalizer $N_G(H)$ of $H$ in $G$ for the groups $H$ in Equation (3.1) are, respectively:*

$$\begin{array}{lll} H, & H, & H, \\ H, & H, & H, \\ H \rtimes \mathrm{Sym}(3), & H \rtimes C_2, & E_q \times C_{q+1}, \\ \mathrm{PSL}(2,q)\times H, & S_2 \rtimes C_{q^2-1}, & H \times C_{q+1}, \\ C_{q^2-1}\rtimes C_2, & S_2 \rtimes C_{q+1}, & G. \end{array} \tag{3.2}$$

*The values $\mu(H)$ for the groups $H$ in Equation (3.1) are, respectively:*

$$\begin{array}{ccc} -1, & -1, & -1, \\ -1, & 1, & 1, \\ 0, & 0, & 0, \\ 0, & 0, & q+1, \\ \dfrac{2(q^2-1)}{3}, & -\dfrac{q^3(q+1)}{2}, & 0. \end{array} \tag{3.3}$$

The proof of Theorem 3.1 is divided into several propositions.

**Proposition 3.2.** *The group $G$ contains exactly one conjugacy class for any group in Equation (3.1).*

*Proof.* **Case 1:** The first four groups in Equation (3.1), i.e.,

$$S_2 \rtimes C_{q^2-1},\ \mathrm{PSL}(2,q)\times C_{q+1},\ C_{q^2-q+1}\rtimes C_3,\ \text{ and }\ (C_{q+1}\times C_{q+1})\rtimes \mathrm{Sym}(3),$$

are the maximal subgroups of $G$, for which there is just one conjugacy class by Theorem 2.3.

**Case 2:** Let $\alpha_1,\alpha_2 \in G$ have order 3, so that they are of type (B2) and $\alpha_i$ fixes two distinct points $P_i, Q_i \in \mathcal{H}_q(\mathbb{F}_{q^2})$. The group $G$ is 2-transitive on $\mathcal{H}_q(\mathbb{F}_{q^2})$, and the pointwise stabilizer of $\{P_i, Q_i\}$ is cyclic of order $q^2-1$. Hence, $\langle\alpha_1\rangle$ and $\langle\alpha_2\rangle$ are conjugated in $G$.

**Case 3:** Let $\alpha_1,\alpha_2 \in G$ have order 2, so that they are of type (C) and $\alpha_i$ fixes exactly one point $P_i$ on $\mathcal{H}_q(\mathbb{F}_{q^2})$. Up to conjugation $P_1 = P_2$, as $G$ is transitive on $\mathcal{H}_q(\mathbb{F}_{q^2})$. The involutions fixing $P_1$ in $G$, together with the identity, form an elementary abelian group $E_q$, which is normalized by a cyclic group $C_{q-1}$; no nontrivial element of $C_{q-1}$ commutes with any nontrivial element of $E_q$ (see [11, Section 4]). Hence, $\alpha_1$ and $\alpha_2$ are conjugated under an element of $C_{q-1}$.

**Case 4:** Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in G$ satisfy $o(\alpha_i) = 3$, $o(\beta_i) = 2$, and $H_i := \langle \alpha_i, \beta_i \rangle \cong \mathrm{Sym}(3)$. As shown in the previous point, we can assume $\alpha_1 = \alpha_2$ up to conjugation. Let $P, Q \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $R \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ be the fixed points of $\alpha_1$. Since $\beta_i \alpha_1 \beta_i^{-1} = \alpha_1^{-1}$, we have that $\beta_i$ fixes $R$ and interchanges $P$ and $Q$; $\beta$ is then uniquely determined from the $\mathbb{F}_{q^2}$-rational point of $PQ$ fixed by $\beta$ (namely, the intersection between $PQ$ and the axis of $\beta$). Since the pointwise stabilizer $C_{q^2-1}$ of $\{P, Q\}$ acts transitively on $PQ(\mathbb{F}_{q^2}) \setminus \mathcal{H}_q$, $\beta_1$ and $\beta_2$ are conjugated, and the same holds for $H_1$ and $H_2$.

**Case 5:** Any two groups isomorphic to $C_{q^2-1}$ are conjugated in $G$, because they are generated by elements of type (B2) and $G$ is 2-transitive on $\mathcal{H}_q(\mathbb{F}_{q^2})$.

**Case 6:** Any two groups isomorphic to $E_q$ are conjugated in $G$, because any such group fixes exactly one point $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$, $G$ is transitive on $\mathcal{H}_q(\mathbb{F}_{q^2})$, and the stabilizer $G_P = M_1(P)$ contains just one subgroup $E_q$.

**Case 7:** Any two groups $H_1, H_2 \cong E_q \rtimes C_{q^2-1}$ are conjugated in $G$. In fact, their Sylow 2-subgroups $E_q$ coincide up to conjugation, as shown in the previous point. The normalizer $N_G(E_q)$ fixes the fixed point $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ of $E_q$, and hence $N_G(E_q) = M_1(P) = S_2 \rtimes C_{q^2-1}$. The complements $C_{q^2-1}$ are conjugated by Schur-Zassenhaus Theorem; hence, $H_1$ and $H_2$ are conjugated.

**Case 8:** Any two groups isomorphic to $C_{2(q+1)}$ are conjugated in $G$, because they are generated by elements of type (E) and two elements $\alpha_1, \alpha_2$ of type (E) of the same order are conjugated in $G$. In fact, $\alpha_i$ is uniquely determined by its fixed points $P_i \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $Q_i \in \ell_{P_i}(\mathbb{F}_{q^2}) \setminus \mathcal{H}_q$; here, $\ell_{P_i}$ is the polar line of $P_i$. Up to conjugation $P_1 = P_2$, from the transitivity of $G$ on $\mathcal{H}_q(\mathbb{F}_{q^2})$. Also, $S_2$ has order $q^3$ and acts on the $q^2$ points of $\ell_{P_i}(\mathbb{F}_{q^2}) \setminus \mathcal{H}_q$ with kernel $E_q$, hence transitively. We can then assume $Q_1 = Q_2$.

**Case 9:** Let $Z_{P_i}$ be the center of $M_2(P_i)$, $i = 1, 2$. As shown in [5, Section 4], $Z_{P_i} \cong C_{q+1}$ and $Z_{P_i}$ is made by the homologies with center $P_i$, together with the identity. Since $G$ is transitive on $\mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$, we have up to conjugation that $M_2(P_1) = M_2(P_2)$ and $Z_{P_1} = Z_{P_2}$.

**Case 10:** Any two groups $H_1, H_2 \cong C_{q+1} \times C_{q+1}$ are conjugated in $G$. In fact, $H_i$ is the pointwise stabilizer of a self-polar triangle $T_i = \{P_i, Q_i, R_i\} \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ (see [5, Section 3]), and the stabilizers of $T_1$ and $T_2$ are conjugated by Theorem 2.3.

**Case 11:** Any two groups $H_1, H_2 \cong (C_{q+1} \times C_{q+1}) \rtimes C_2$ are conjugated in $G$. In fact, their subgroups $C_{q+1} \times C_{q+1}$ coincide up to conjugation as shown above, and fix pointwise a self-polar triangle $T = \{P, Q, R\} \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$. Let $\beta_i \in H_i$ have order 2, $i = 1, 2$. Then $\beta_i$ fixes one vertex of $T$ and interchanges the other two vertexes. Up to conjugation in $M_3(T)$ we have $\beta_1(P) = \beta_2(P) = P$. Then $H_1 = H_2$, as they coincide with the stabilizer of $P$ in $M_3(T)$. □

**Proposition 3.3.** *The normalizers $N_G(H)$ of the groups $H$ in Equation (3.1) are given in Equation (3.2).*

*Proof.* **Case 1:** Clearly $N_G(H) = H$ for any $H$ from the first four groups of Equation (3.1) as $H$ is maximal in $G$.

**Case 2:** Let $H = E_q \rtimes C_{q^2-1}$. Then $H \leq M_2(P)$, where $P$ is the unique fixed point of $C_{q^2-1}$ in $\mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$. The group $H$ has a unique cyclic subgroup $C_{q+1}$ of order $q+1$; namely, $C_{q+1}$ is the center of $M_2(P)$ and is made by the homologies with center $P$; since $q$ is even, $H$ is a split extension $C_{q+1} \times (E_q \rtimes C_{q-1})$. Hence, $N_G(H) \leq N_G(C_{q+1}) = M_2(P)$. The group $H/C_{q+1} \cong E_q \rtimes C_{q-1}$ is maximal and hence self-normalizing in $M_2(P)/C_{q+1} = \mathrm{PSL}(2, q)$; thus, $N_G(E_q \rtimes C_{q-1}) = H$ and $N_G(H) = H$.

**Case 3:** Let $H = C_{q+1} \times C_{q+1}$. Then $N_G(H) \leq M_3(T)$, where $T$ is the self-polar triangle fixed pointwise by $H$. Since $H$ is the kernel of $M_3(T)$ in its action on $T$, we have $N_G(H) = M_3(T)$ and $|N_G(H)| = 6|H|$.

**Case 4:** Let $H = (C_{q+1} \times C_{q+1}) \rtimes C_2$. Then $C_{q+1} \times C_{q+1}$ is normal in $N_G(H)$, being the unique subgroup of index 2 in $H$. Hence $N_G(H) \leq M_3(T)$, where $T$ is the self-polar triangle fixed pointwise by $H$. Also, $N_G(H)$ fixes the vertex $P$ of $T$ fixed by $H$, so that $N_G(H) \neq M_3(T)$. This implies $N_G(H) = H$.

**Case 5:** Let $H = C_{q^2-1}$. Then $H$ is generated by an element $\alpha$ of type (B2) with fixed points $P, Q \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $R \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$. Let $\beta$ be an involution satisfying $\beta(R) = R$, $\beta(P) = Q$, and $\beta(Q) = P$; then $\beta \in N_G(H)$, because $H$ coincides with the pointwise stabilizer of $\{P, Q\}$ in $G$. An explicit description is the following: given $\mathcal{H}_q$ with equation (2.3), we can assume up to conjugation that $\alpha = \mathrm{diag}(a^{q+1}, a, 1)$ where $a$ is a generator if $\mathbb{F}_{q^2}^*$ (see [11]); then take

$$\beta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.4}$$

Since $N_G(H)$ acts on $\{P, Q\}$ and $\beta \in N_G(H)$, the pointwise stabilizer $H$ of $\{P, Q\}$ has index 2 in $N_G(H)$. This implies $N_G(H) = C_{q^2-1} \rtimes C_2$ and $|N_G(H)| = 2|H|$.

**Case 6:** Let $H = C_{2(q+1)}$, so that $H$ is generated by an element $\alpha$ of type (E) fixing exactly two points $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $Q \in \ell_P(\mathbb{F}_{q^2}) \setminus \mathcal{H}_q$. Then $N_G(H)$ fixes $P$ and $Q$. The subgroup $E_q$ of $M_1(P)$ commutes with $H$ elementwise, while any 2-element in $M_1(P) \setminus E_q$ has order 4 and does not fix $Q$; hence, the Sylow 2-subgroup of $N_G(H)$ is $E_q$. Also, $N_G(H) = E_q \rtimes C_d$, where $C_d$ is a subgroup of $C_{q^2-1}$ containing the subgroup $C_{q+1}$ of $H$. Let $C_2$ be the subgroup of $H$ of order 2; the quotient group $(C_2 \rtimes C_d)/C_{q+1} \cong C_2 \rtimes C_{\frac{d}{q+1}}$ acts faithfully as a subgroup of $\mathrm{PGL}(2, q)$ on the $q+1$ points of $\ell_Q \cap \mathcal{H}_q$. By the classification of subgroups of $\mathrm{PGL}(2, q)$ ([7]; see [17, Hauptsatz 8.27]), this implies $d = 1$; that is, $N_G(H) = E_q \rtimes C_{q+1}$ and $|N_G(H)| = \frac{q}{2}|H|$.

**Case 7:** Let $H = C_{q+1} = Z(M_2(P))$. Since $H$ is the center of $M_2(P)$, $M_2(P) \leq N_G(H)$. Conversely, $H$ is made by homologies with center $P$, and hence $N_G(H)$ fixes $P$. Thus, $N_G(H) = M_2(P)$ and $|N_G(H)| = q(q^2-1)|H|$.

**Case 8:** Let $H = E_q$. Since $E_q$ has a unique fixed point $P$ on $\mathcal{H}_q(\mathbb{F}_{q^2})$ and $E_q = Z(M_1(P))$, we have $N_G(H) \leq M_1(P)$ and $M_1(P) \leq N_G(H)$, so that $N_G(H) = M_1(P)$ and $|N_G(H)| = q^2(q^2-1)|H|$.

**Case 9:** Let $H = \mathrm{Sym}(3) = \langle \alpha, \beta \rangle$, with $o(\alpha) = 3$ and $o(\beta) = 2$. Let $P, Q \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $R \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ be the fixed points of $\alpha$; $\beta$ fixes $R$, interchanges $P$ and $Q$, and fixes another point $A_\beta$ on $\ell_R \cap \mathcal{H}_q$. The group $N_G(H)$ acts on $\{P, Q\}$ and on $\{A_\beta, A_{\alpha\beta}, A_{\alpha^2\beta}\}$.

The pointwise stabilizer $C_{q^2-1}$ has a subgroup $C_{q+1}$ which is the center of $M_2(P)$ and fixes $PQ$ pointwise, while any element in $C_{q^2-1} \setminus C_{q+1}$ acts semiregularly on $PQ \setminus \{P, Q\}$; hence, $C_{q^2-1} \cap N_G(H) = C_{3(q+1)}$. If an element $\gamma \in N_G(H)$ fixes $\{P, Q\}$ pointwise, then $\gamma$ fixes a point in $\{A_\beta, A_{\alpha\beta}, A_{\alpha^2\beta}\}$, and hence $\gamma \in \{\beta, \alpha\beta, \alpha^2\beta\}$. Therefore, $N_G(H) = C_{3(q+1)} \rtimes C_2 = H \times C_{q+1}$ and $|N_G(H)| = (q+1)|H|$.

**Case 10:** Let $H = C_3$ and $\alpha$ be a generator of $H$, with fixed points $P, Q \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $R \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$. The normalizer $N_G(H)$ fixes $R$ and acts on $\{P, Q\}$. There exists an involution $\beta \in G$ normalizing $H$ and interchanging $P$ and $Q$ (see Equation (3.4)). Then the pointwise stabilizer of $\{P, Q\}$ has index 2 in $N_G(H)$. Also, the pointwise stabilizer of $\{P, Q\}$ in $G$ is cyclic of order $q^2 - 1$. Then $N_G(H) = C_{q^2-1} \rtimes C_2$ and $|N_G(H)| = \frac{2(q^2-1)}{3}|H|$.

**Case 11:** Let $H = C_2$ and $\alpha$ be a generator of $H$, with fixed point $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$. Then $N_G(H)$ fixes $P$, i.e. $N_G(H) \leq M_1(P) = S_2 \rtimes C_{q^2-1}$. Since any involution of $M_1(P)$ is in the center of $S_2$, the Sylow 2-subgroup of $N_G(H)$ has order $q^3$. Let $\beta \in C_{q^2-1}$. If $o(\beta) \mid (q+1)$, then $\beta$ commutes with any involution of $S_2$. If $o(\beta) \nmid (q+1)$, then $\beta$ does not commute with any element of $S_2$. This implies that $N_G(H) = S_2 \rtimes C_{q+1}$, and $|N_G(H)| = \frac{q^3(q+1)}{2}|H|$. □

**Lemma 3.4.** *Let $\alpha \in G$ be an involution, and hence an elation, with center $P$ and axis $\ell_P$. Then there exist exactly $q^3/2$ self-polar triangles $T_{i,j} = \{P_i, Q_{i,j}, R_{i,j}\}$, $i = 1, \ldots, q^2$, $j = 1, \ldots, \frac{q}{2}$, such that $\alpha$ stabilizes $T_{i,j}$. Also, $P_i \in \ell_P$ and $P \in Q_{i,j}R_{i,j}$ for any $i$ and $j$.*

*Proof.* The number of involutions in $G$ is $(q^3+1)(q-1)$, since for any of the $q^3+1$ $\mathbb{F}_{q^2}$-rational points $P$ of $\mathcal{H}_q$ the involutions fixing $P$ form a group $E_q$. The number of self-polar triangles $T \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ is $[G : M_3(T)] = \frac{(q^3+1)q^3(q^2-1)}{6(q+1)^2}$. For any self-polar triangle $T = \{A_1, A_2, A_3\} \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$, the number of involutions in $G$ stabilizing $T$ is $3(q+1)$. In fact, for any of the 3 vertexes of $T$ there are exactly $q+1$ involutions $\alpha_1, \ldots, \alpha_{q+1}$ fixing that vertex, say $A_1$, and interchanging $A_2$ and $A_3$; $\alpha_i$ is uniquely determined by its center $A_2A_3 \cap \mathcal{H}_q$. Then, by double counting the size of

$$\{(\beta, T) \mid \beta \in G,\ o(\beta) = 2,\ T \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q \text{ is a self-polar triangle},\ \beta \text{ stabilizes } T\},$$

$\alpha$ stabilizes exactly $\frac{q^3}{2}$ self-polar triangles $T$. For any such $T$, one vertex $P_i$ of $T$ lies on the axis of $\alpha$, because $\alpha$ is an elation, and the other two vertexes $\{Q_{i,j}, R_{i,j}\}$ of $T$ lie on the polar line $\ell_{P_i}$ of $P_i$. Since $M_1(P)$ is transitive on the $q^2$ points $P_1, \ldots, P_{q^2}$ of $\ell_P(\mathbb{F}_{q^2}) \setminus \{P\}$, any point $P_i$ is contained in the same number $\frac{q}{2}$ of self-polar triangles $T_{i,j}$ stabilized by $\alpha$. □

**Lemma 3.5.** *Let $\alpha \in G$ have order 3. Then there are exactly $\frac{q^2-1}{3}$ self-polar triangles*

$$T_i \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q, \quad i = 1, \ldots, \frac{q^2-1}{3},$$

*which are stabilized by $\alpha$. Also, there are exactly $\frac{2(q^2-1)}{3}$ triangles*

$$\tilde{T}_j = \{P_j, \Phi_{q^2}(P_j), \Phi_{q^2}^2(P_j)\} \subset \mathcal{H}_q(\mathbb{F}_{q^6}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2}), \quad j = 1, \ldots, \frac{2(q^2-1)}{3},$$

*which are stabilized by $\alpha$.*

*Proof.* By Proposition 3.2, any two subgroups of $G$ of order 3 are conjugated in $G$. Also, any element of order 3 is conjugated to its inverse by an involution of $G$. Hence, any two element of order 3 are conjugated in $G$.

Now the claim follows by double counting the size of

$$\{(\beta,T)\mid \beta\in G,\ o(\beta)=3,\ T\subset \mathrm{PG}(2,q^2)\setminus\mathcal{H}_q \text{ is a self-polar triangle}, \\ \beta \text{ stabilizes } T\},$$

and

$$\{(\beta,\tilde{T})\mid \beta\in G,\ o(\beta)=3,\ \tilde{T}=\{P,\Phi_{q^2}(P),\Phi_{q^2}^2(P)\} \text{ with} \\ P\in\mathcal{H}_q(\mathbb{F}_{q^6})\setminus\mathcal{H}_q(\mathbb{F}_{q^2}),\ \beta \text{ stabilizes } \tilde{T}\},$$

using the following facts. The number of elements of order 3 in $G$ is $\binom{q^3+1}{2}\cdot 2$. The number of self-polar triangles $T\subset\mathrm{PG}(2,q^2)\setminus\mathcal{H}_q$ is $[G:M_3(T)]$. The number of elements of order 3 stabilizing a fixed self-polar triangle $T$ is $2(q+1)^2$, because any element acting as a 3-cycle on the vertexes of $T$ has order 3 (see [5, Section 3]). The number of triangles $\tilde{T}=\{P,\Phi_{q^2}(P),\Phi_{q^2}^2(P)\}\subset\mathcal{H}_q(\mathbb{F}_{q^6})\setminus\mathcal{H}_q(\mathbb{F}_{q^2})$ is $[G:M_4(\tilde{T})]$. The number of elements of order 3 stabilizing a fixed triangle $\tilde{T}$ is $2(q^2-q+1)$, because any element in $M_4(\tilde{T})\setminus C_{q^2-q+1}$ has order 3 (see [4, Section 4]). □

**Lemma 3.6.** *Let $H<G$ be isomorphic to $\mathrm{Sym}(3)$, $H=\langle\alpha\rangle\rtimes\langle\beta\rangle$. Then there are exactly $q+1$ self-polar triangles*

$$T_i=\{P_i,Q_i,R_i\}\subset\mathrm{PG}(2,q^2)\setminus\mathcal{H}_q,\quad i=1,\dots,q+1,$$

*which are stabilized by $H$. Up to relabeling the vertexes, we have that $P_1,\dots,P_{q+1}$ lie on the axis of the elation $\beta$, $Q_1,\dots,Q_{q+1}$ lie on the axis of the elation $\alpha\beta$, and $R_1,\dots,R_{q+1}$ lie on the axis of the elation $\alpha^2\beta$.*

*Proof.* By Proposition 3.2, any two subgroups $K_1,K_2<G$ with $K_i\cong\mathrm{Sym}(3)$ are conjugated, and $|N_G(K_i)|=6(q+1)$; hence, the number of subgroups of $G$ isomorphic to $\mathrm{Sym}(3)$ is $[G:N_G(K_i)]=\frac{(q^3+1)q^3(q-1)}{6}$. The number of self-polar triangles $T$ is $[G:M_3(T)]=\frac{(q^2-q+1)q^3(q-1)}{6}$. Then the claim on the number of self-polar triangles follows by double counting the size of

$$\{(K,T)\mid K<G,\ K\cong\mathrm{Sym}(3),\ T\subset\mathrm{PG}(2,q^2)\setminus\mathcal{H}_q \text{ is a self-polar triangle}, \\ K \text{ stabilizes } T\},$$

once we show that, for any self-polar triangle $T=\{A,B,C\}$, there are in $G$ exactly $(q+1)^2$ subgroups isomorphic to $\mathrm{Sym}(3)$ which stabilize $T$.

Let $K<M_3(T)$, $K\cong\mathrm{Sym}(3)$, $K=\langle\alpha,\beta\rangle$ with $o(\alpha)=3$, $o(\beta)=2$. Let $P,Q,R$ be the fixed points of $\alpha$, with $P\in\mathrm{PG}(2,q^2)\setminus\mathcal{H}_q$, $Q,R\in\mathcal{H}_q(\mathbb{F}_{q^2})$. By Proposition 3.3, $N_G(K)=K\times C_{q+1}$ where $C_{q+1}$ is made by homologies with center $P$; this implies $N_G(K)\cap M_3(T)=K$. Hence, there are at least $[M_3(T):\mathrm{Sym}(3)]=(q+1)^2$ distinct groups $\mathrm{Sym}(3)$ stabilizing $T$, namely the conjugates of $K$ through elements of $M_3(T)$. On the other side, $M_3(T)$ contains exactly $(q+1)^2$ subgroups $K$ of order 3, with fixed points $P\in\mathrm{PG}(2,q^2)\setminus\mathcal{H}_q$, $Q,R\in\mathcal{H}_q(\mathbb{F}_{q^2})$. Any involution $\beta$ of $M_3(T)$ normalizing

$K$ is uniquely determined by the vertex of $T$ that $\beta$ fixes, because $\beta(P) = P$, $\beta(Q) = R$, and $\beta(R) = Q$. Thus, $K$ is contained in exactly one subgroup of $M_3(T)$ isomorphic to $\mathrm{Sym}(3)$. Therefore the number of subgroups isomorphic to $\mathrm{Sym}(3)$ which stabilize $T$ is $(q+1)^2$.

Finally, the configuration of the vertexes of $T_1, \dots, T_{q+1}$ on the axes of the involutions of $H$ follows from Lemma 2.4 and the fact that every involution fixes a different vertex of $T_i$. □

**Proposition 3.7.** *Any group $H$ in Equation (3.1) is the intersection of maximal subgroups of $G$.*

*Proof.* **Case 1:** The first four groups of Equation (3.1) are exactly the maximal subgroups of $G$.

**Case 2:** Let $H = E_q \rtimes C_{q^2-1}$. Let $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ be the unique point of $\mathcal{H}_q$ fixed by $E_q$; $E_q$ fixes $\ell_P$ pointwise. Also, the fixed points of $C_{q^2-1}$ are $P, Q \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $R \in \mathrm{PG}(2,q^2) \setminus \mathcal{H}_q$, where $R \in \ell_P$ and $PQ = \ell_R$. Then $H \le M_1(P) \cap M_2(R)$. Conversely, from $M_1(P) \cap M_2(R) \le M_1(P)$ follows $M_1(P) \cap M_2(R) = K \rtimes C_d$ with $K \le S_2$ and $C_d \le C_{q^2-1}$. From $M_1(P) \cap M_2(R) \le M_2(R)$ follows that $K$ does not contain any element of type (D), so that $K \le E_q$. Thus, $M_1(P) \cap M_2(R) \le H$, and $H = M_1(P) \cap M_2(R)$.

**Case 3:** Let $H = (C_{q+1} \times C_{q+1}) \rtimes C_2$. Let $T = \{P, Q, R\} \subset \mathrm{PG}(2,q^2) \setminus \mathcal{H}_q$ be the self-polar triangle fixed pointwise by $C_{q+1} \times C_{q+1}$, and let $P$ be the vertex of $T$ fixed by $C_2$. Then $H \le M_3(T) \cap M_2(P)$. Conversely, since $M_3(T) \cap M_2(P)$ fixes $P$ and acts on $\{Q, R\}$, the pointwise stabilizer $C_{q+1} \times C_{q+1}$ of $T$ has index at most 2 in $M_3(T) \cap M_2(P)$, so that $M_3(T) \cap M_2(P) \le H$. Thus, $H = M_3(T) \cap M_2(P)$.

**Case 4:** Let $H = C_{q+1} \times C_{q+1}$. Let $T = \{P, Q, R\} \subset \mathrm{PG}(2,q^2) \setminus \mathcal{H}_q$ be the self-polar triangle fixed pointwise by $C_{q+1} \times C_{q+1}$. Since $H$ is the whole pointwise stabilizer of $T$ in $G$, we have $H = M_2(P) \cap M_2(Q) \cap M_2(R)$.

**Case 5:** Let $H = C_{q^2-1}$ and let $\alpha$ be a generator of $H$, with fixed points $P, Q \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $R \in \mathrm{PG}(2,q^2) \setminus \mathcal{H}_q$. The pointwise stabilizer of $\{P, Q\}$ in $G$ is exactly $H$; thus, $H = M_1(P) \cap M_2(Q)$.

**Case 6:** Let $H = C_{2(q+1)}$ and let $\alpha$ be a generator of $H$, of type (E), with fixed points $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $Q \in \ell_P(\mathbb{F}_{q^2}) \setminus \mathcal{H}_q$. By Lemma 3.4 there are $\frac{q}{2}$ self-polar triangles stabilized by the involution $\alpha^{q+1}$ having one vertex in $Q$ and two vertexes on $\ell_Q$; let $T = \{Q, R_1, R_2\}$ be one of these triangles. Then $H \le M_1(P) \cap M_2(Q) \cap M_3(T)$.

Conversely, let $\sigma \in (M_1(P) \cap M_2(Q) \cap M_3(T)) \setminus \{1\}$. If $\sigma$ fixes $\{R_1, R_2\}$ pointwise, then from $\sigma \in M_1(P)$ follows that $\sigma$ is in the kernel $C_{q+1} \le H$ of the action of $M_2(Q)$ on $\ell_Q$. The quotient $(M_1(P) \cap M_2(Q) \cap M_3(T))/C_{q+1}$ acts on $\ell_Q$ as a subgroup of $\mathrm{PSL}(2,q)$ fixing $P$ and interchanging $R_1$ and $R_2$. From [17, Hauptsatz 8.27] follows $(M_1(P) \cap M_2(Q) \cap M_3(T))/C_{q+1} \cong C_2$, and hence $H = M_1(P) \cap M_2(Q) \cap M_3(T)$.

**Case 7:** Let $H = C_{q+1} = Z(M_2(P))$. Then $H$ is made by the homologies of $G$ with center $P$, together with the identity. Thus, $H = M_1(P_1) \cap M_1(P_2) \cap M_1(P_3)$, where $P_1, P_2, P_3$ are distinct point in $\ell_P \cap \mathcal{H}_q$.

**Case 8:** Let $H = E_q$ and let $P$ be the unique point of $\mathcal{H}_q(\mathbb{F}_{q^2})$ fixed by any element in $H$. Then $H = M_2(P_1) \cap M_2(P_2) \cap M_2(P_3)$, where $P_1, P_2, P_3$ are distinct points in $\ell_P(\mathbb{F}_{q^2}) \setminus \{P\}$.

**Case 9:** Let $H = C_2$, $\alpha$ be a generator of $H$ with fixed point $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$, and $P_1, P_2, P_3 \in \ell_P(\mathbb{F}_{q^2}) \setminus \{P\}$. Let $T = \{P_1, Q_{1,1}, R_{1,1}\}$ be a self-polar triangle stabilized by $\alpha$. Then $H \leq M_2(P_1) \cap M_2(P_2) \cap M_2(P_3) \cap M_3(T)$. Since the elation $\alpha$ is uniquely determined by the image of one point not on its axis $\ell_P$, $H \leq M_3(T)$ implies $H = M_2(P_1) \cap M_2(P_2) \cap M_2(P_3) \cap M_3(T)$.

**Case 10:** Let $H = C_3$. By Lemma 3.5, $H$ stabilizes $\frac{2(q^2-1)}{3}$ triangles $\tilde{T} \subset \mathcal{H}_q(\mathbb{F}_{q^6}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2})$; let $\tilde{T}_1$ and $\tilde{T}_2$ be two of them. Then $H \leq M_4(\tilde{T}_1) \cap M_4(\tilde{T}_2)$. If $H < M_4(\tilde{T}_1) \cap M_4(\tilde{T}_2)$, then there exist a nontrivial $\sigma \in G$ stabilizing pointwise both $\tilde{T}_1$ and $\tilde{T}_2$, a contradiction to Lemma 2.4. Thus, $H = M_4(\tilde{T}_1) \cap M_4(\tilde{T}_2)$.

**Case 11:** Let $H = \mathrm{Sym}(3)$. By Lemma 3.6, $H$ stabilizes $q+1$ self-polar triangles $T_1, \ldots, T_{q+1}$, so that $H \leq M_3(T_1) \cap \cdots \cap M_3(T_{q+1})$. Suppose by contradiction that $H \neq M_3(T_1) \cap \cdots \cap M_3(T_{q+1})$. Then $M_3(T_1) \cap \cdots \cap M_3(T_{q+1})$ contains a nontrivial element $\sigma$ fixing every triangle $T_i$ pointwise. Since the triangles $T_i$'s do not have vertexes in common, this is a contradiction to Lemma 2.4. Thus, $H = M_3(T_1) \cap \cdots \cap M_3(T_{q+1})$.

**Case 12:** Let $H = \{1\}$. Since $G$ is simple, $H$ is the Frattini subgroup of $G$. □

**Proposition 3.8.** *If $H < G$ is the intersection of maximal subgroups, then $H$ is one of the groups in Equation* (3.1).

*Proof.* We proceed as follows: we take every subgroup $K < G$ in Equation (3.1), starting from the maximal subgroups $M_i$ of $G$; we consider the intersections $H = K \cap M_i$ of $K$ with the maximal subgroups of $G$; here, we assume that $K \not\leq M_i$. We show that $H$ is again one of the groups in Equation (3.1).

**Case 1:** Let $K = S_2 \rtimes C_{q^2-1} = M_1(P)$ for some $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$.

Let $H = K \cap M_1(Q)$, $Q \neq P$. Then $H$ is the pointwise stabilizer of $\{P, Q\} \subset \mathcal{H}_q(\mathbb{F}_{q^2})$, which is cyclic of order $q^2-1$, i.e. $H = C_{q^2-1}$.

Let $H = K \cap M_2(Q)$. Suppose $Q \in \ell_P$. Then $H = E_{q^2} \rtimes C_{q^2-1}$, where $E_{q^2}$ is made by the elations with axis $PQ$ and $C_{q^2-1}$ is generated by an element of type (B2) with fixed points $Q$, $P$, and another point $R \in \ell_Q$. Now suppose $Q \notin \ell_P$. Then $H$ stabilizes $\ell_Q$ and hence also the point $R = \ell_P \cap \ell_Q$. Then $H$ stabilizes $QR$ and hence also the pole $A$ of $QR$; by reciprocity, $A \in PQ$. Thus, $H$ fixes three collinear point $A, P, Q$, and hence every point on $AP$. Then $H = C_{q+1} = Z(M_2(R))$.

Let $H = K \cap M_3(T)$, $T = \{A, B, C\}$, with $P$ on a side of $T$, say $P \in AB$. Then $H$ fixes $C$ and acts on $\{A, B\}$. Thus, $H$ is generated by an element of type (E) with fixed points $P, C$ and fixed lines $PC, AB$; hence, $H = C_{2(q+1)}$.

Let $H = K \cap M_3(T)$, $T = \{A, B, C\}$, with $P$ out of the sides of $T$. By reciprocity, no vertex of $T$ lies on $\ell_P$. This implies that no elation acts on $T$, so that $2 \nmid |H|$; this also implies that no homology in $M_3(T)$ fixes $P$, so that $H$ has no nontrivial elements fixing $T$ pointwise. Thus $H \leq C_3$.

Let $H = K \cap M_4(T)$. By Lagrange's theorem, $H \leq C_3$.

**Case 2:** Let $K = \mathrm{PSL}(2, q) \times C_{q+1} = M_2(P)$ for some $P \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$.

Let $H = K \cap M_2(Q)$, $Q \neq P$, and $R$ be the pole of $PQ$. If $R \in PQ$, then $H$ is the pointwise stabilizer of $PQ$ and is made by the elations with center $R$; thus, $H = E_q$. If $R \notin PQ$, then $H$ is the pointwise stabilizer of $T = \{P, Q, R\}$; thus, $H = C_{q+1} \times C_{q+1}$.

Let $H = K \cap M_3(T)$ with $T = \{A, B, C\}$. If $P$ is a vertex of $T$, then $H = (C_{q+1} \times C_{q+1}) \rtimes C_2$. If $P$ is on a side of $T$ but is not a vertex, say $P \in AB$, then $H$ fixes the pole $D \in AB$ of $C$. Then $H$ fixes pointwise $T' = \{P, C, D\}$ and acts on $\{A, B\}$. This implies that $H$ fixes $AB$ pointwise and $H = C_{q+1} = Z(M_2(C))$. If $P$ is out of the sides of $T$, then no nontrivial element of $H$ fixes $T$ pointwise; thus, $H \leq \mathrm{Sym}(3)$.

Let $H = K \cap M_4(T)$. By Lagrange's theorem, $H \leq C_3$.

**Case 3:** Let $K = (C_{q+1} \times C_{q+1}) \rtimes \mathrm{Sym}(3) = M_3(T)$ for some self-polar triangle $T = \{A, B, C\}$.

Let $H = K \cap M_3(T')$ with $T' = \{A', B', C'\} \neq T$. If $T$ and $T'$ have one vertex $A = A'$ in common, then $H = C_{2(q+1)}$ is generated by an element of type (E) fixing $A$ and a point $D \in BC = B'C'$. If $A' \in AC \setminus \{A, C\}$, then $H$ stabilizes $B'C'$, because $B'C'$ is the only line containing 4 points of $\{A, B, C, A', B', C'\}$. Then $H$ fixes $A'$, $A$, and $C$; hence also $B$. Since $H$ acts on $\{B', C'\}$, $H$ cannot be made by nontrivial homologies of center $B$; thus, $H = \{1\}$.

Let $H = K \cap M_4(T')$. By Lagrange's theorem, $H \leq C_3$.

**Case 4:** Let $K = C_{q^2-q+1} \rtimes C_3 = M_4(T)$ for some $T \subset \mathcal{H}_q(\mathbb{F}_{q^6})$. Let $H = K \cap M_4(T')$ with $T' \neq T$. Since 3 does not divide the order of the pointwise stabilized $C_{q^2-q+1}$ of $T$, $H$ contains no nontrivial elements fixing $T$ or $T'$ pointwise. Thus, $H \leq C_3$.

**Case 5:** Let $K = E_q \rtimes C_{q^2-1}$ and $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$, $Q \in \ell_P \setminus \{P\}$ be the fixed points of $K$.

Let $H = K \cap M_1(R)$ with $R \neq P$. If $R \in \ell_Q$, then $H = C_{q^2-1}$. If $R \notin \ell_Q$, then $H$ fixes the pole $S$ of $PR$; by reciprocity $S \in PQ$, so that $H$ fixes $PQ$ pointwise and also $R \notin PQ$. Thus, $H = \{1\}$.

Let $H = K \cap M_2(R)$ with $R \neq Q$. If $R \in \ell_P$, then $H$ is the pointwise stabilizer $E_q$ of $PQ$. If $R \notin \ell_P$, then $H$ fixes pointwise the self-polar triangle $\{Q, R, S\}$ where $S$ is the pole of $QR$. Hence, either $H = C_{q+1} = Z(M_2(Q))$ or $H = \{1\}$ according to $P \in RS$ or $P \notin RS$, respectively.

Let $H = K \cap M_3(T)$ with $T = \{A, B, C\}$. If $P$ is on a side of $T$, say $P \in BC$, then either $H = \{1\}$ or $H = C_{q+1} = Z(M_2(A))$. If $P$ is out of the sides of $T$, then no nontrivial element of $H$ can fix $T$ pointwise; thus, $H \leq \mathrm{Sym}(3)$.

Let $H = K \cap M_4(T)$. By Lagrange's theorem, $H \leq C_3$.

**Case 6:** Let $K = (C_{q+1} \times C_{q+1}) \rtimes C_2 = M_3(T) \cap M_2(A)$, where $T = \{A, B, C\}$.

Let $H = K \cap M_1(P)$. If $P \in BC$, then $H = C_{2(q+1)}$ is generated by an element of type (E). If $P \notin BC$, then $H = \{1\}$.

Let $H = K \cap M_2(P)$, $P \neq A$. If $P \in \{B, C\}$, then $H$ is the pointwise stabilizer $C_{q+1} \times C_{q+1}$ of $T$. If $P \in AB \setminus \{A, B\}$ or $P \in AC \setminus \{A, C\}$, then $H = C_{q+1} = Z(M_2(C))$ or $H = C_{q+1} = Z(M_2(B))$, respectively. If $P \in BC \setminus \{B, C\}$, then $H$ fixes $A$, $P$, the pole of $AP$, and acts on $\{B, C\}$; thus, $H = C_{q+1} = Z(M_2(A))$. If $P$ is not on the sides of $T$, then no nontrivial element of $H$ can fix $T$ pointwise; thus, $H \leq C_2$.

Let $H = K \cap M_3(T')$ with $T' = \{A', B', C'\} \neq T$. Since $3 \nmid |H|$, $H$ fixes a vertex of $T'$, say $A'$. If $A' = A$, then $H = C_{2(q+1)}$. If $A' \in \{B, C\}$, then $H$ fixes $T$ pointwise and acts on $\{B', C'\}$; thus, $H = C_{q+1} = Z(M_2(A'))$. If $A' \in (AB \cup AC) \setminus \{A, B, C\}$, then $H$ fixes $AB$ or $AC$ pointwise and acts on $\{B', C'\}$; thus, $H = \{1\}$. If $A' \in BC$, then $H$

fixes $A$, $A'$, and the pole $D$ of $AA'$; as $H$ acts on $\{B, C\}$, this implies $H = \{1\}$. If $A'$ is not on the sides of $T$, then no nontrivial element of $H$ fixes $T$ pointwise and $H \leq C_2$.

Let $H = K \cap M_4(T')$. By Lagrange's theorem, $H \leq C_3$.

**Case 7:** Let $K = C_{q+1} \times C_{q+1} = M_3(T) \cap M_2(A) \cap M_2(B) \cap M_2(C)$ with $T = \{A, B, C\}$.

Let $H = K \cap M_1(P)$ or $H = K \cap M_2(P)$. If $P$ is not on the sides of $T$, then $H = \{1\}$; if $P$ is on a side of $T$, say $P \in BC$, then $H = C_{q+1} = Z(M_2(A))$.

Let $H = K \cap M_3(T')$ with $T' = \{A', B', C'\}$. Since $K$ is not divisible by 2 or 3, $H \neq \{1\}$ only if $H$ fixes $T'$ pointwise. Up to relabeling, this implies $A' = A$, $B', C' \in BC$, and $H = C_{q+1} = Z(M_2(A))$.

Let $H = K \cap M_4(T')$. By Lagrange's theorem, $H = \{1\}$.

**Case 8:** Let $K = C_{q^2-1} = \langle \alpha \rangle$, with $\alpha$ of type (B2) fixing the points $P \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ and $Q, R \in \mathcal{H}_q(\mathbb{F}_{q^2})$.

Let $H = K \cap M_1(A)$ or $H = K \cap M_2(A)$. Since the nontrivial elements of $H$ are either of type (B2) or of type (A) with axis $QR$, we have $H = \{1\}$ unless $A \in QR$; in this case, $H = C_{q+1} = Z(M_2(P))$.

Let $H = K \cap M_3(T)$ or $H = K \cap M_4(T)$. By Lagrange's theorem, $H \leq C_3$.

**Case 9:** Let $K = C_{2(q+1)} = \langle \alpha \rangle$ with $\alpha$ of type (E) fixing the points $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $Q \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$.

Let $H = K \cap M_1(R)$ or $H = K \cap M_2(R)$. If $R \in \ell_Q$, then $H = C_{q+1} = Z(M_2(Q))$. If $R \notin \ell_Q$, then $H = \{1\}$.

Let $H = K \cap M_3(T)$; recall that $H < K$. If $Q$ is a vertex of $T$, then $H = C_{q+1} = Z(M_2(Q))$. If $Q$ is not a vertex of $T$, then no homology in $K$ acts on $T$; hence, $H \leq C_2$.

Let $H = K \cap M_4(T)$. By Lagrange's theorem, $H = \{1\}$.

**Case 10:** Let $K = C_{q+1} = Z(M_2(P))$ for some $P \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ and $\sigma \in K \setminus \{1\}$. Then $\sigma$ fixes no points out of $\{P\} \cup \ell_P$; also, the triangles fixed by $\sigma$ have one vertex in $P$ and two vertexes on $\ell_P$. Thus, $K \cap M_i = \{1\}$ for any maximal subgroup $M_i$ of $G$ not containing $K$.

**Case 11:** Let $K = E_q$ and $\sigma \in E_q \setminus \{1\}$. Recall that $K$ fixes one point $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and the line $\ell_P$ pointwise. Also, $\sigma$ fixes no points out of $\ell_P$. If $\sigma$ fixes a triangle $T = \{A, B, C\}$, then one vertex of $T$ lies on $\ell_P(\mathbb{F}_{q^2})$, say $A$, and $\sigma$ is uniquely determined by $\sigma(B) = C$. Thus, $K \cap M_1(Q) = K \cap M_2(Q) = K \cap M_4(T') = \{1\}$ and $K \cap M_3(T) \leq C_2$.

**Case 12:** Let $K \in \{\mathrm{Sym}(3), C_3, C_2, \{1\}\}$. Then every subgroup of $K$ is in Equation (3.1). □

**Proposition 3.9.** *The values* $\mu(H)$ *for the groups in Equation* (3.1) *are given in Equation* (3.3).

*Proof.* Let $H$ be one of the groups in Equation (3.1). By Lemma 2.1 and Proposition 3.8, $\mu(H)$ only depends on the subgroups $K$ of $G$ such that $H < K$ and $K$ is in Equation (3.1).

**Case 1:** If $H$ is one of the first four groups in Equation (3.1), then $H$ is maximal in $G$, and hence $\mu(H) = -1$.

**Case 2:** Let $H = E_q \rtimes C_{q^2-1}$. Let $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $Q \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ be the fixed points of $H$. Then $H = M_1(P) \cap M_2(Q)$ and $H$ is not contained in any other maximal subgroup of $G$. Thus, $\mu(H) = -\{\mu(G) + \mu(M_1(P)) + \mu(M_2(Q))\} = 1$.

**Case 3:** Let $H = (C_{q+1} \times C_{q+1}) \rtimes C_2$. Let $T = \{P, Q, R\}$ be the self-polar triangle stabilized by $H$, with $H(P) = P$. No point different from $P$ is fixed by $H$. Also, if a triangle $T' = \{P', Q'\} \neq T$ is fixed by $H$, then $P$ is a vertex of $T'$, say $P = P'$, and $\{Q', R'\} \subset QR$; but $C_{q+1} \times C_{q+1}$ has orbits of length $q + 1 > |\{Q', R'\}|$, so that $H$ cannot fix $T'$. Then $H = M_2(P) \cap M_3(T)$ and $H$ is not contained in any other maximal subgroup of $G$. Thus, $\mu(H) = 1$.

**Case 4:** Let $H = C_{q+1} \times C_{q+1}$ and $T = \{P, Q, R\}$ be the self-polar triangle fixed pointwise by $H$. The vertexes of $T$ are the unique fixed points of the elements of type (B1) in $H$. Also, any triangle $T' \neq T$ fixed by an element of type (A) in $H$ has two vertexes on a side $\ell$ of $T$; but $H$ has orbits of length $q + 1 > 2$ on $\ell$, so that $H$ does not fix $T'$. Then $H = M_3(T) \cap M_2(P) \cap M_2(Q) \cap M_2(R)$ and $H$ is not contained in any other maximal subgroup of $G$.

If $K$ is one of the groups $M_3(T) \cap M_2(P)$, $M_3(T) \cap M_2(P)$, $M_3(T) \cap M_2(P)$, then $K$ contains $H$ properly, and $\mu(K) = 1$ as shown in the previous point. The intersection of three groups between $M_3(T)$, $M_2(P)$, $M_2(Q)$, and $M_2(R)$ is equal to $H$. Thus, by direct computation, $\mu(H) = 0$.

**Case 5:** Let $H = C_{q^2-1}$ with fixed points $P \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ and $Q, R \in \mathcal{H}_q(\mathbb{F}_{q^2})$. Then $H = M_1(Q) \cap M_1(R) = M_1(Q) \cap M_1(R) \cap M_2(P)$. We already know $\mu(M_1(Q) \cap M_2(P)) = \mu(M_1(R) \cap M_2(P)) = 1$. Moreover, $C_{q^2-1}$ has no fixed triangles, by Lagrange's theorem, and no other fixed points. Thus, by direct computation, $\mu(H) = 0$.

**Case 6:** Let $H = C_{2(q+1)} = \langle \alpha \rangle$; $\alpha$ is of type (E), fixes the points $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $Q \in PG(2, q^2) \setminus \mathcal{H}_q$, and fixes the lines $\ell_P$ and $\ell_Q$. Since $\alpha^2$ is a homology with center $Q$, the orbits on $\ell_Q$ of $H$ coincide with the orbits on $\ell_Q$ of the elation $\alpha^{q+1}$. By Lemma 3.4, the self-polar triangles $T_i$ stabilized by $H$ have a vertex in $Q$ and two vertexes on $\ell_Q$; there are exactly $\frac{q}{2}$ such triangles $T_1, \ldots, T_{\frac{q}{2}}$. No other triangle and no other point different from $P$ and $Q$ is fixed by $H$, so that $H = M_1(P) \cap M_2(Q) \cap M_3(T_1) \cap \cdots \cap M_3(T_{\frac{q}{2}})$ and $H$ is not contained in any other maximal subgroup of $G$.

If $K$ is the intersection of $M_2(Q)$ with one of the groups $M_1(P), M_3(T_1), \ldots, M_3(T_{\frac{q}{2}})$, then $K = E_q \rtimes C_{q^2-1}$ or $K = (C_{q+1} \times C_{q+1}) \rtimes C_2$; hence, $K$ contains $H$ properly and $\mu(K) = 1$ as shown above. The intersection of $K$ with a third maximal subgroup of $G$ containing $H$ coincides with $H$. Finally, the intersection of any two groups in $\{M_1(P), M_3(T_1), \ldots, M_3(T_{\frac{q}{2}})\}$ coincides with $H$. Thus, by direct computation, $\mu(H) = 0$.

**Case 7:** Let $H = C_{q+1} = Z(M_2(P))$. Denote $\ell_P \cap \mathcal{H}_q = \{P_1, \ldots, P_{q+1}\}$ and $\ell(\mathbb{F}_{q^2}) \setminus \mathcal{H}_q = \{Q_1, \ldots, Q_{q^2-q}\}$ such that, for $i = 1, \ldots, \frac{q^2-q}{2}$, $T_i = \{P, Q_i, Q_{i+\frac{q^2-q}{2}}\}$ are the self-polar triangles with a vertex in $P$. Then

$$H = \bigcap_{i=1}^{q+1} M_1(P_i) \cap M_2(P) \cap \bigcap_{i=1}^{q^2-q} M_2(Q_i) \cap \bigcap_{i=1}^{(q^2-q)/2} M_3(T_i)$$

and $H$ is not contained in any other maximal subgroup of $G$. By direct inspection, the intersections $K$ of some (at least two) maximal subgroups of $G$ such that $H < K < G$ are exactly the following.

(i) $K = M_1(P_i) \cap M_1(P_j)$ for some $i \neq j$; in this case, $K = C_{q^2-1}$ and $\mu(K) = 0$.

(ii) $K = M_1(P_i) \cap M_2(P)$ with $i \in \{1, \ldots, q+1\}$; in this case, $K = E_q \rtimes C_{q^2-1}$ and $\mu(K) = 1$. These $q+1$ groups are pairwise distinct.

(iii) $K = M_1(P_i) \cap M_3(T_j)$ for some $i, j$; in this case, $K = C_{2(q+1)}$ and $\mu(K) = 0$.

(iv) $K = M_2(P) \cap M_2(Q_i)$ for some $i$; in this case, $K = C_{q+1} \times C_{q+1}$ and $\mu(K) = 0$.

(v) $K = M_2(P) \cap M_3(T_i)$ with $i \in \{1, \ldots, \frac{q^2-q}{2}\}$; in this case, $K = (C_{q+1} \times C_{q+1}) \rtimes C_2$ and $\mu(K) = 1$. These $\frac{q^2-q}{2}$ groups are pairwise distinct.

(vi) $K = M_2(Q_i) \cap M_3(T_i)$ or $K = M_2(Q_{i+\frac{q^2-q}{2}}) \cap M_3(T_i)$, with $i \in \{1, \ldots, \frac{q^2-q}{2}\}$; in this case, $K = (C_{q+1} \times C_{q+1}) \rtimes C_2$ and $\mu(K) = 0$. These $q^2 - q$ groups are pairwise distinct.

To sum up, the only subgroups $K$ with $H < K < G$ and $\mu(K) \neq 0$ are the maximal subgroups, $q+1$ distinct groups of type $E_q \rtimes C_{q^2-1}$, and $\frac{3(q^2-q)}{2}$ distinct groups of type $(C_{q+1} \times C_{q+1}) \rtimes C_2$. Thus, $\mu(H) = 0$.

**Case 8:** Let $H = E_q$. Let $P$ be the point of $\mathcal{H}_q(\mathbb{F}_{q^2})$ fixed by $H$; $H$ fixes $\ell_P$ pointwise. We have $H = M_1(P) \cap M_2(Q_1) \cap \cdots \cap M_2(Q_{q^2})$, where $Q_1, \ldots, Q_{q^2}$ are the $\mathbb{F}_{q^2}$-rational points of $\ell_P \setminus \{P\}$; $H$ is not contained in any other maximal subgroup of $G$. The intersections $K$ of at least two maximal subgroups of $G$ such that $H < K < G$ are exactly the $q^2$ groups $M_1(P) \cap M_2(Q_i) = E_q \rtimes C_{q^2-1}$, with $\mu(K) = 1$. Thus, by direct computation, $\mu(H) = 0$.

**Case 9:** Let $H = \mathrm{Sym}(3) = \langle \alpha, \beta \rangle$ with $o(\alpha) = 3$ and $o(\beta) = 2$. Let $P \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ and $Q, R \in \mathcal{H}_q$ be the fixed points of $\alpha$, and $A \in QR$ be the fixed point of $\beta$ on $\mathcal{H}_q$, so that $\beta$ fixes $\ell_A = AP$. By Lemma 3.6 and its proof, $H = M_2(P) \cap M_3(T_1) \cap \cdots \cap M_3(T_{q+1})$, where $T_i$ has one vertex on $\ell_A \setminus \{P, A\}$ and the other two vertexes are collinear with $A$; $H$ is not contained in any other maximal subgroup of $G$.

For any $i, j \in \{1, \ldots, q+1\}$ with $i \neq j$, no vertex of $T_j$ is on a side of $T_i$; hence, no nontrivial element of $M_3(T_i) \cap M_3(T_j)$ fixes $T_i$ pointwise. This implies $M_3(T_i) \cap M_3(T_j) = H$. Analogously, no nontrivial element in $M_3(T_i) \cap M_2(P)$ fixes $T_i$ pointwise, and this implies $M_3(T_i) \cap M_2(P) = H$. Thus, by direct computation, $\mu(H) = q + 1$.

**Case 10:** Let $H = C_3 = \langle \alpha \rangle$ with fixed points $P \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ and $Q, R \in \mathcal{H}_q$. By Lemma 3.5,

$$H = M_1(Q) \cap M_1(R) \cap M_2(P) \cap \bigcap_{i=1}^{(q^2-1)/3} M_3(T_i) \cap \bigcap_{i=1}^{2(q^2-1)/3} M_4(\tilde{T}_i)$$

and $H$ is not contained in any other maximal subgroup of $G$. By direct inspection, the intersections $K$ of at least two maximal subgroups of $G$ such that $H < K < G$ are exactly the following.

(i) $K = M_1(Q) \cap M_2(P)$ or $K = M_1(R) \cap M_2(P)$; in this case, $K = E_q \rtimes C_{q^2-1}$ and $\mu(K) = 1$.

(ii) $K = M_1(Q) \cap M_1(R)$; in this case, $K = C_{q^2-1}$ and $\mu(K) = 0$.

(iii) There are exactly $\frac{q-1}{3}$ groups $K$ containing $H$ with $K \cong \mathrm{Sym}(3)$, and hence $\mu(K) = q+1$. In fact, any involution $\beta \in G$ satisfying $\langle H, \beta \rangle \cong \mathrm{Sym}(3)$ interchanges $Q$ and $R$ and fixes a point of $(QR \cap \mathcal{H}_q) \setminus \{P, Q\}$; conversely, any of the $q-1$ points $A_1, \dots, A_{q-1}$ of $(QR \cap \mathcal{H}_q) \setminus \{P, Q\}$ determines uniquely the involution $\beta_i \in G$ such that $\beta(A_i)$, $\beta_i(Q) = R$, $\beta_i(R) = Q$, and hence $\langle H, \beta_i \rangle \cong \mathrm{Sym}(3)$. The involutions $\beta_i$, $\alpha\beta_i$, and $\alpha^2\beta_i$, together with $H$, generate the same group; thus, there are exactly $\frac{q-1}{3}$ groups $\mathrm{Sym}(3)$ containing $H$.

Thus, by direct computation, $\mu(H) = \frac{2(q^2-1)}{3}$.

**Case 11:** Let $H = C_2 = \langle \alpha \rangle$, where $\alpha$ has center $P$. Let $\ell_P(\mathbb{F}_{q^2}) \setminus \{P\} = \{P_1, \dots, P_{q^2}\}$. By Lemma 3.4,

$$H = M_1(P) \cap \bigcap_{i=1}^{q^2} M_2(P_i) \cap \bigcap_{i=1}^{q^2} \bigcap_{j=1}^{q/2} M_3(T_{i,j}),$$

where the triangles $T_{i,j}$ are described in Lemma 3.4; $H$ is not contained in any other maximal subgroup of $G$. By direct inspection, the intersections $K$ of at least two maximal subgroups of $G$ such that $H < K < G$ are exactly the following.

(i) $K = M_1(P) \cap M_2(P_i)$ for $i = 1, \dots, q^2$; in this case, $K = E_q \rtimes C_{q^2-1}$ and $\mu(K) = 0$.

(ii) $K = M_2(P_i) \cap M_2(P_j)$ with $i \neq j$; in this case, $K = E_q$ and $\mu(K) = 0$.

(iii) $K = M_1(P) \cap M_3(T_{i,j})$; in this case, $K = E_q \rtimes C_{2(q+1)}$ and $\mu(K) = 0$.

(iv) $K = M_2(Q_i) \cap M_3(T_{i,j})$ with $i \in \{1, \dots, q^2\}$ and $j \in \{1, \dots, \frac{q}{2}\}$; these $\frac{q^3}{2}$ distinct groups are of type $(C_{q+1} \times C_{q+1}) \rtimes C_2$, so that $\mu(K) = 1$.

(v) There are exactly $N = \frac{q^3}{2}$ groups $K$ containing $H$ such that $K \cong \mathrm{Sym}(3)$, and hence $\mu(K) = q+1$. This follows by double counting the size of

$$I = \{(H, K) \mid H, K < G,\ H \cong C_2,\ K \cong \mathrm{Sym}(3),\ H < K\}.$$

Arguing as in the proof of Lemma 3.4, $|I| = (q^3+1)(q-1)N$; arguing as in the proof of Lemma 3.6, $|I| = \frac{q^3(q^3+1)(q-1)}{6} \cdot 3$. Hence, $N = \frac{q^3}{2}$.

Thus, by direct computation, $\mu(H) = -\frac{q^3(q+1)}{2}$.

**Case 12:** Let $H = \{1\}$. Then $\mu(H) = -\sum_{\{1\}<K\leq G} \mu(K, G)$. By the values $\mu(K)$ computed in the previous cases, Propositions 3.2, and Proposition 3.3, only the following groups $K$ have to be considered:

(i) 1 group $G$;

(ii) $q^3+1$ groups $S_2 \rtimes C_{q^2-1}$;

(iii) $q^2(q^2-q+1)$ groups $\mathrm{PSL}(2,q) \times C_{q+1}$;

(iv) $\frac{q^3(q-1)(q^2-q+1)}{6}$ groups $(C_{q+1} \times C_{q+1}) \rtimes \mathrm{Sym}(3)$;

(v) $\frac{q^3(q+1)^2(q-1)}{3}$ groups $C_{q^2-q+1} \rtimes C_3$;

(vi) $(q^3+1)q^2$ groups $E_q \rtimes C_{q^2-1}$;

(vii) $\frac{q^3(q-1)(q^2-q+1)}{2}$ groups $(C_{q+1} \times C_{q+1}) \rtimes C_2$;

(viii) $\frac{q^3(q^3+1)(q-1)}{6}$ groups $\mathrm{Sym}(3)$;

(ix) $\frac{q^3(q^3+1)}{2}$ groups $C_3$;

(x) $(q^3+1)(q-1)$ groups $C_2$.

Thus, by direct computation, $\mu(H) = 0$. □

## 4 Determination of $\lambda(H)$ for any subgroup $H$ of $G$

Let $n > 0$, $q = 2^{2^n}$, $G = \mathrm{PSU}(3, q)$. This section is devoted to the proof of the following theorem.

**Theorem 4.1.** *Let $H$ be a proper subgroup of $G$. Then $\lambda(H) \neq 0$ if and only $H$ is one of the following groups:*

$$\begin{array}{lll} E_q \rtimes C_{q^2-1}, & (C_{q+1} \times C_{q+1}) \rtimes C_2, & \mathrm{Sym}(3), \\ C_3, & S_2 \rtimes C_{q^2-1}, & \mathrm{PSL}(2,q) \times C_{q+1}, \\ (C_{q+1} \times C_{q+1}) \rtimes \mathrm{Sym}(3), & C_{q^2-q+1} \rtimes C_3, & C_2. \end{array} \tag{4.1}$$

*For any isomorphism type in Equation (4.1) there is just one conjugacy class of subgroups of $G$.*

*If $H$ is in the first row of Equation (4.1), then $\lambda(H) = -1$; if $H$ is in the second row of Equation (4.1), then $\lambda(H) = 1$.*

*Proof.* By Proposition 3.2, for any isomorphism type in Equation (4.1) there is just one conjugacy class of subgroups of $G$ of that type. Hence, we can use the notation $[M_1]$, $[M_2]$, $[M_3]$ and $[M_4]$ for the conjugacy classes of $M_1(P)$, $M_2(P)$, $M_3(T)$ and $M_4(T)$, respectively. If $H = G$, then $\lambda(H) = 1$; if $H$ is one of the groups in the second row of Equation (4.1) and $H \neq C_2$, then $\lambda(H) = -1$ as $H$ is maximal in $G$.

**Case 1:** Firstly, we assume that $H$ is not a subgroup of $\mathrm{Sym}(3)$, and that $H$ is not a group of homologies, i.e. $H \not\leq C_{q+1} = Z(M_2(Q))$ for any point $Q$.

(i) Let $H < M_4(T)$ for some $T$. From $H \neq C_3$ follows that some nontrivial element in $H$ fixes $T$ pointwise; hence, $H$ is not contained in any maximal subgroup of $G$ other than $M_4(T)$. Thus, inductively, $\lambda(H) = -\{\lambda(G) + \lambda(M_4(T))\} = 0$.

(ii) Let $H < M_1(P)$ for some $P$; we assume in addition that $\gcd(|H|, q-1) > 1$. Here, the assumption $H \not\leq \mathrm{Sym}(3)$ reads $H \notin \{\{1\}, C_2, C_3\}$. If $H$ contains an element of order 4, then $H$ is not contained in any maximal subgroup of $G$ other than $M_1(P)$. Thus, inductively, $\lambda(H) = 0$.

We can then assume that the 2-elements of $H$ are involutions, so that $H = E_{2^r} \rtimes C_d$ with $0 \leq r \leq 2^n$ and $d \mid (q^2-1)$ (see [15, Theorem 11.49]). This implies that $H \leq M_1(P) \cap M_2(Q)$ for some $Q \in \ell_P$; the eventual nontrivial elements in $H$ whose order divides $q+1$ are homologies with center $Q$. Then we have $[H] \leq [M_1]$, $[H] \leq [M_2]$; by Lagrange's theorem, $[H] \not\leq [M_4]$. From the assumptions $\gcd(|H|, q-1) > 1$ and $H \not\leq \mathrm{Sym}(3)$ follows $[H] \not\leq [M_3]$.

If $H = E_q \rtimes C_{q^2-1}$, then no proper subgroup of $M_1(P)$ or $M_2(Q)$ contains $H$ properly; thus, $\lambda(H) = 1$. If $H \neq E_q \rtimes C_{q^2-1}$, then $H < E_q \rtimes C_{q^2-1} = M_1(P) \cap$

$M_2(Q)$ up to conjugation. Thus, inductively, the only classes $[K]$ with $[H] \leq [K]$ and $\lambda(K) \neq 0$ are $[K] \in \{[G], [M_1], [M_2], [E_q \rtimes C_{q^2-1}]\}$. This implies $\lambda(H) = 0$.

(iii) Let $H < M_2(Q)$ for some $Q$, and assume also $H \not\leq M_1(P)$ for any $P$. As $H \not\leq C_3$, we have $[H] \not\leq [M_4]$. The group $\bar{H} := H/(H \cap Z(M_2(Q)))$ acts as a subgroup of $\mathrm{PSL}(2,q)$ on $\ell_Q \cap \mathcal{H}_q$; we assume in this point that $H$ is one of the following groups (see [17, Hauptsatz 8.27]): $\mathrm{PSL}(2, 2^{2^h})$ with $0 < h \leq n$; a dihedral group of order $2d$ where $d$ is a divisor of $q-1$ greater than 3; $\mathrm{Alt}(5)$. Then, by Lagrange's theorem, $[H] \not\leq [M_3]$. Thus, inductively, $G$ and $M_2(Q)$ are the only groups $K$ with $H < K$ and $\lambda(K) \neq 0$, so that $\lambda(H) = 0$.

Note that, since we are under the assumptions $H \not\leq M_1(P)$ for any $P$, $H \not\leq \mathrm{Sym}(3)$, and $H \not\leq C_{q+1} = Z(M_2(Q))$, we have that the only subgroups $\bar{H}$ of $\mathrm{PSL}(2,q)$ for which $\lambda(H)$ still has not been computed are the cyclic or dihedral groups of order $d$ or $2d$ (respectively), where $d$ is a nontrivial divisor of $q+1$.

(iv) Let $H < M_3(T)$ for some $T$, and assume also $H \not\leq M_1(P)$ for any $P$. As $H \not\leq C_3$, we have $[H] \not\leq [M_4]$. Here, the assumption $H \not\leq \mathrm{Sym}(3)$ means that some nontrivial element of $H$ fixes $T$ pointwise. Hence, the assumption $H \not\leq C_{q+1} = Z(M_2(Q))$ for any vertex $Q$ of $T$, together with $H \not\leq M_1(P)$, implies that $H$ contains some element of type (B1). Write $H = L \rtimes K$, with $K \leq \mathrm{Sym}(3)$ and $L < C_{q+1} \times C_{q+1}$.

If $K = C_3$ or $K = \mathrm{Sym}(3)$, then $[H] \not\leq [M_2]$; thus, inductively, $G$ and $M_3(T)$ are the only groups $K$ with $H < K$ and $\lambda(K) \neq 0$, so that $\lambda(H) = 0$.

If $K = C_2$ and $L = C_{q+1} \times C_{q+1}$, then $H \leq M_2(Q)$ for some vertex $Q$ of $T$. Since $\bar{H} := H/(H \cap Z(M_2(Q)))$ is dihedral of order $2(q+1)$, [17, Haptsatz 8.27] implies the non-existence of groups $K$ with $H < K < M_2(Q)$ (except for $q = 4$ and $\bar{K} = \mathrm{Alt}(5)$; in this case, $\lambda(K) = 0$ by the previous point). Thus, $\lambda(H) = -\{\lambda(G) + \lambda(M_2(Q)) + \lambda(M_3(T))\} = 1$.

If $K = C_2$ and $L < C_{q+1} \times C_{q+1}$, then again $H \leq M_2(Q)$ with $Q$ vertex of $T$. The group $\bar{H}$ is dihedral of order $2d$, where $d \mid (q+1)$; $d > 1$ because $L$ contains elements of type (B1). By the previous point and [17, Hauptsatz 8.27], the only groups $K$ with $H < K < M_2(Q)$ are such that $\bar{K}$ is dihedral of order dividing $q+1$. Thus, inductively, $\lambda(H) = 0$.

If $K = \{1\}$, then $H \in M_2(Q)$ for any vertex $Q$ of $T$. The group $\bar{H} < \mathrm{PSL}(2,q)$ on the line $\ell_Q \cap \mathcal{H}_q$ is cyclic of order $d \mid (q+1)$; $d > 1$ because $H$ has elements of type (B1). By [17, Hauptsatz 8.27], the groups $K$ with $H < K < M_2(Q)$ are such that either $\bar{K}$ is cyclic of order dividing $q+1$, or we have already proved that $\lambda(K) = 0$. Thus, inductively, $\lambda(K) = 0$.

(v) Let $H < M_2(Q)$ for some $Q$. Let $\bar{H} \neq \{1\}$ be the induced subgroup of $\mathrm{PSL}(2,q)$ acting on $\ell_Q \cap \mathcal{H}_q$. If $\bar{H}$ is cyclic or dihedral of order $d$ or $2d$ (respectively) with $d \mid (q+1)$, then $H \leq M_3(T)$ for some $T$. Hence, $\lambda(H) = 0$, as already computed in the previous point in the case $K = \{1\}$ if $\bar{H}$ is cyclic, or in the case $K = C_2$ if $H$ is dihedral.

(vi) Under the assumptions that $H \not\leq \mathrm{Sym}(3)$ and $H$ is not a group of homologies, the only remaining case is $H < M_1(P)$ for some $P$ with $\gcd(|H|, q-1) = 1$. In this case $H = E_{2^r} \times C_d$, where $C_d$ is cyclic of order $d \mid (q+1)$ and made by homologies, whose axis passes through $P$ and whose center $Q$ lies on $\ell_P$. We have $r > 0$, because $H \not\leq Z(M_2(Q))$.

If $r = 1$, then $H$ is cyclic of order $2d$ generated by an element of type (E). By Lemma 3.4, $H \leq M_3(T)$, where $T$ has a vertex in $Q$ and two vertexes on $\ell_Q$. Hence, $[H] \leq [M_1]$, $[H] \leq [M_2]$, $[H] \leq [M_3]$, and $[H] \not\leq [M_4]$. Let $K$ be such that $H < K \leq G$ and $K$ is not of the same type of $H$, i.e. $K$ is not cyclic of order $2d'$ with $d' \mid (q+1)$. As shown in the previous points, $\lambda(K) \neq 0$ if and only if $[K] \in \{[G], [M_1], [M_2], [M_3], [E_q \rtimes C_{q^2-1}], [(C_{q+1} \times C_{q+1}) \rtimes C_2]\}$. Thus, inductively, $\lambda(H) = 0$.

**Case 2:** Let $H \leq C_{q+1} = Z(M_2(Q))$ for some $Q$ and $K$ be a subgroup of $G$ properly containing $H$. As shown above, $\lambda(K) \neq 0$ if and only if

$$[K] \in \{[G], [M_1], [M_2], [M_3], [E_q \rtimes C_{q^2-1}], [(C_{q+1} \times C_{q+1}) \rtimes C_2]\}.$$

Thus $\lambda(Z(M_2(Q))) = 0$ and, inductively, $\lambda(H) = 0$.

**Case 3:** Let $H = \mathrm{Sym}(3) = \langle\alpha\rangle \rtimes \langle\beta\rangle$ with $o(\alpha) = 3$ and $o(\beta) = 2$. Let $P \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ and $Q, R \in \mathcal{H}_q(\mathbb{F}_{q^2})$ be the fixed point of $\alpha$, so that $\beta$ fixes $P$ and interchanges $Q$ and $R$. This implies $[H] \leq [M_2]$. By Lemma 3.6, $[H] \leq [M_3]$. From the computations above and Lagrange's theorem, no class $[K]$ with $K \leq G$ other than $[G]$, $[M_2]$ and $[M_3]$ satisfies $[H] \leq [K]$ and $\lambda(H) \neq 0$. Thus, $\lambda(H) = 1$.

**Case 4:** Let $H = C_3$. By Lagrange's theorem and Proposition 3.2, $H < K \leq G$ and $\lambda(K) \neq 0$ if and only if

$$[K] \in \{[G], [M_1], [M_2], [M_3], [M_4], [E_q \rtimes C_{q^2-1}], [\mathrm{Sym}(3)]\}.$$

Thus, $\lambda(H) = 1$.

**Case 5:** Let $H = C_2$. By Lagrange's theorem and Proposition 3.2, $H < K \leq G$ and $\lambda(K) \neq 0$ if and only if

$$[K] \in \{[G], [M_1], [M_2], [M_3], [E_q \rtimes C_{q^2-1}], [(C_{q+1} \times C_{q+1}) \rtimes C_2], [\mathrm{Sym}(3)]\}.$$

Thus, $\lambda(H) = -1$.

**Case 6:** Let $H = \{1\}$. Collecting all the classes $[K]$ with $\lambda(K) \neq 0$, we have by direct computation $\lambda(H) = 0$. □

## 5 Determination of $\chi(\Delta(L_p \setminus \{1\}))$ for any prime $p$

Let $n > 0$, $q = 2^{2^n}$, $G = \mathrm{PSU}(3, q)$. If $p$ is a prime number, we denote by $L_p$ the poset of $p$-subgroups of $G$ ordered by inclusion, by $L_p \setminus \{1\}$ its subposet of proper $p$-subgroups of $G$, and by $\Delta(L_p \setminus \{1\})$ the order complex of $L_p \setminus \{1\}$. In this section we determine the Euler characteristic $\chi(\Delta(L_p \setminus \{1\}))$ of $\Delta(L_p \setminus \{1\})$ for any prime $p$, using Equation (2.1) and Lemma 2.2. The results are stated in Theorem 5.1 and in Table 2.

**Theorem 5.1.** *For any prime number p one of the following cases holds:*

*(i)* $p \nmid |G|$ *and* $\chi(\Delta(L_p \setminus \{1\})) = 0$*;*

*(ii)* $p = 2$ *and* $\chi(\Delta(L_2 \setminus \{1\})) = q^3 + 1$*;*

*(iii)* $p \mid (q+1)$ *and* $\chi(\Delta(L_p \setminus \{1\})) = -\frac{q^6 - 2q^5 - q^4 + 2q^3 - 3q^2}{3}$*;*

*(iv)* $p \mid (q-1)$ *and* $\chi(\Delta(L_p \setminus \{1\})) = -\frac{q^6+q^3}{2}$*;*

*(v)* $p \mid (q^2-q+1)$ *and* $\chi(\Delta(L_p \setminus \{1\})) = -\frac{q^6+q^5-q^4-q^3}{3}$*.*

*Proof.* Since $|G| = q^3(q+1)^2(q-1)(q^2-q+1)$, $q$ is even, and $3 \mid (q-1)$, the cases $p \nmid |G|$, $p=2$, $p \mid (q+1)$, $p \mid (q-1)$, and $p \mid (q^2-q+1)$ are exhaustive and pairwise incompatible. We denote by $S_p$ a Sylow $p$-subgroup of $G$.

**Case 1:** Let $p \nmid |G|$. Then $\Delta(L_p \setminus \{1\}) = \emptyset$, and hence $\chi(\Delta(L_p \setminus \{1\})) = \chi(\emptyset) = 0$.

**Case 2:** Let $p=2$. The group $G$ has $q^3+1$ Sylow 2-subgroups, and any two of them intersect trivially; see [15, Theorem 11.133]. Any nontrivial element $\sigma$ of $S_2$ fixes exactly one point $P$ on $\mathcal{H}_q(\mathbb{F}_{q^2})$ which is the same for any $\sigma \in S_2$; $S_2$ is uniquely determined among the Sylow 2-subgroups of $G$ by $P$. Hence, Equation (2.1) reads

$$\chi(\Delta(L_2 \setminus \{1\})) = -(q^3+1) \sum_{H \in L_2 \setminus \{1\},\ H(P)=P} \mu_{L_2}(\{1\}, H),$$

where $P$ is a given point of $\mathcal{H}_q(\mathbb{F}_{q^2})$. By Lemma 2.2, we only consider those 2-groups in $M_1(P)$ which are elementary abelian. Then we consider all nontrivial subgroups $H$ of an elementary abelian 2-group $E_q$ of order $q$. For any such group $H = E_{2^r}$ of order $2^r$, with $1 \le r \le 2^n$, we have $\mu_{L_2}(\{1\}, H) = (-1)^r \cdot 2^{\binom{r}{2}}$ by Lemma 2.2. Thus,

$$\chi(\Delta(L_2 \setminus \{1\})) = -(q^3+1) \sum_{r=1}^{2^n} (-1)^r \, 2^{\binom{r}{2}} \binom{2^n}{r}_2$$

where the Gaussian coefficient $\binom{2^n}{r}_2$ counts the subgroups of $E_q$ of order $2^r$. Using the property

$$\binom{2^n}{r}_2 = \binom{2^n-1}{r-1}_2 + 2^r \binom{2^n-1}{r}_2$$

we obtain

$$\begin{aligned}
&\sum_{r=1}^{2^n} (-1)^r \, 2^{\binom{r}{2}} \binom{2^n}{r}_2 \\
&\quad = \sum_{r=1}^{2^n} (-1)^r \, 2^{\binom{r}{2}} \binom{2^n-1}{r-1}_2 + \sum_{r=1}^{2^n} (-1)^r \, 2^{\binom{r}{2}+r} \binom{2^n-1}{r}_2 \\
&\quad = \sum_{r=0}^{2^n-1} (-1)^{r+1} \, 2^{\binom{r+1}{2}} \binom{2^n-1}{r}_2 + \sum_{r=1}^{2^n} (-1)^r \, 2^{\binom{r+1}{2}} \binom{2^n-1}{r}_2 \\
&\quad = (-1)^0 \, 2^{\binom{1}{2}} \binom{2^n-1}{0}_2 + (-1)^{2^n} \, 2^{\binom{2^n+1}{2}} \binom{2^n-1}{2^n}_2 = -1.
\end{aligned}$$

Thus, $\chi(\Delta(L_2 \setminus \{1\})) = q^3+1$.

**Case 3:** Let $p \mid (q+1)$. Then $S_p \le C_{q+1} \times C_{q+1}$, and hence $S_p \cong C_{p^s} \times C_{p^s}$, where $p^s \mid (q+1)$ and $p^{s+1} \nmid (q+1)$. Let $H$ be a subgroup of $S_p$. By Lemma 2.2, $\mu_{L_p}(\{1\}, H) \neq 0$ only if $H$ is elementary abelian of order $p$ or $p^2$; in this cases, $\mu_{L_p}(\{1\}, C_p) = -1$ and $\mu_{L_p}(\{1\}, C_p \times C_p) = r$. Now we count the number of elementary abelian subgroups of order $p$ or $p^2$ in $G$.

(i) A subgroup $E_{p^2}$ of $G$ of type $C_p \times C_p$ is uniquely determined by the maximal subgroup $M_3(T)$ such that $E_{p^2}$ is the Sylow $p$-subgroup of $M_3(T)$. Hence, $G$ contains exactly $[G : N_G(M_3(T))] = \frac{q^3(q^2-q+1)(q-1)}{6}$ elementary abelian subgroups of order $p^2$.

(ii) A subgroup $C_p$ made by homologies is uniquely determined by its center $P \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ of homology, because the group of homologies with center $P$ is cyclic. Hence, $G$ contains exactly $|\,\mathrm{PG}(2, q^2) \setminus \mathcal{H}_q| = q^2(q^2 - q + 1)$ cyclic subgroups of order $p$ made by homologies.

(iii) A subgroup $C_p$ which is not made by homologies is made by elements of type (B1), and fixes pointwise a unique self-polar triangle $T$. The Sylow $p$-subgroup $C_p \times C_p$ of $M_3(T)$ contains exactly 3 subgroups $C_p$ made by homologies, namely the groups of homologies with center one of the vertexes of $T$. Since $C_p \times C_p$ contains $p + 1$ subgroups $C_p$ altogether, $C_p \times C_p$ contains exactly $p - 2$ subgroups $C_p$ not made by homologies. Thus, the number of subgroups $C_p$ of $G$ not made by homologies is $(p - 2) \cdot [G : N_G(M_3(T))] = \frac{q^3(q^2-q+1)(q-1)(p-2)}{6}$.

Thus, by direct computation,

$$\begin{aligned}
&\chi(\Delta(L_p \setminus \{1\})) \\
&\quad = -\Big\{\frac{q^3(q^2 - q + 1)(q - 1)(p - 2)}{6} \cdot r \\
&\qquad\quad + \Big[q^2(q^2 - q + 1) + \frac{q^3(q^2 - q + 1)(q - 1)(p - 2)}{6}\Big] \cdot (-1)\Big\} \\
&\quad = -\frac{q^6 - 2q^5 - q^4 + 2q^3 - 3q^2}{3}.
\end{aligned}$$

**Case 4:** Let $p \mid (q - 1)$. By Lemma 2.4, $S_p$ is a subgroup of the cyclic group $C_{q^2-1}$ fixing two points $P, Q$ on $\mathcal{H}_q(\mathbb{F}_{q^2})$; then a proper $p$-subgroup $H$ of $G$ satisfies $\mu_{L_p}(\{1\}) \neq 0$ if and only if $H$ has order $p$; in this case, $\mu_{L_p}(\{1\}, H) = -1$. Also, by Lemma 2.4, any two Sylow $p$-subgroups of $G$ have trivial intersection. Then the number of subgroups $C_p$ of $G$ is equal to the number $\binom{q^3+1}{1}$ of couples of points in $\mathcal{H}_q(\mathbb{F}_{q^2})$; equivalently, this number is equal to $[G : N_G(C_{q^2})]$, where $|N_G(C_{q^2-1})| = 2(q^2 - 1)$ by Proposition 3.3. Thus, $\chi(\Delta(L_p \setminus \{1\})) = -\frac{q^6+q^3}{2}$.

**Case 5:** Let $p \mid (q^2 - q + 1)$. Then $S_p \leq C_{q^2-q+1}$, and hence a proper $p$-subgroup $H$ of $G$ satisfies $\mu_{L_p}(\{1\}, H) \neq 0$ if and only if $H$ has order $p$; in this case, $\mu_{L_p}(\{1\}, H) = -1$. The number of subgroups $C_p$ of $G$ is equal to the number of subgroups $C_{q^2-q+1}$, and hence to the number $[G : N_G(M_4(\tilde{T}))] = \frac{q^3(q+1)^2(q-1)}{3}$ of maximal subgroups of type $M_4(\tilde{T})$ in $G$. Thus, $\chi(\Delta(L_p \setminus \{1\})) = -\frac{q^3(q+1)^2(q-1)}{3} = -\frac{q^6+q^5-q^4-q^3}{3}$. $\square$

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