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Journal of Algebra

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# On finite groups in which coprime commutators are covered by few cyclic subgroups

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## ARTICLE INFO

*Article history:*

Received 27 February 2014

Available online 12 April 2014

Communicated by E.I. Khukhro

*MSC:*

primary 20F14

secondary 20D25

*Keywords:*

Finite groups

Cyclic subgroups

Commutators

## ABSTRACT

The coprime commutators  $\gamma_j^*$  and  $\delta_j^*$  were recently introduced as a tool to study properties of finite groups that can be expressed in terms of commutators of elements of coprime orders. Every element of a finite group  $G$  is both a  $\gamma_1^*$ -commutator and a  $\delta_0^*$ -commutator. Now let  $j \geq 2$  and let  $X$  be the set of all elements of  $G$  that are powers of  $\gamma_{j-1}^*$ -commutators. An element  $g$  is a  $\gamma_j^*$ -commutator if there exist  $a \in X$  and  $b \in G$  such that  $g = [a, b]$  and  $(|a|, |b|) = 1$ . For  $j \geq 1$  let  $Y$  be the set of all elements of  $G$  that are powers of  $\delta_{j-1}^*$ -commutators. An element  $g$  is a  $\delta_j^*$ -commutator if there exist  $a, b \in Y$  such that  $g = [a, b]$  and  $(|a|, |b|) = 1$ . The subgroups of  $G$  generated by all  $\gamma_j^*$ -commutators and all  $\delta_j^*$ -commutators are denoted by  $\gamma_j^*(G)$  and  $\delta_j^*(G)$ , respectively. For every  $j \geq 2$  the subgroup  $\gamma_j^*(G)$  is precisely the last term  $\gamma_\infty(G)$  of the lower central series of  $G$ , while for every  $j \geq 1$  the subgroup  $\delta_j^*(G)$  is precisely the last term of the lower central series of  $\delta_{j-1}^*(G)$ , that is,  $\delta_j^*(G) = \gamma_\infty(\delta_{j-1}^*(G))$ .

In the present paper we prove that if  $G$  possesses  $m$  cyclic subgroups whose union contains all  $\gamma_j^*$ -commutators of  $G$ , then  $\gamma_j^*(G)$  contains a subgroup  $\Delta$ , of  $m$ -bounded order, which is normal in  $G$  and has the property that  $\gamma_j^*(G)/\Delta$  is cyclic.

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<sup>1</sup> This research was supported by CNPq.

If  $j \geq 2$  and  $G$  possesses  $m$  cyclic subgroups whose union contains all  $\delta_j^*$ -commutators of  $G$ , then the order of  $\delta_j^*(G)$  is  $m$ -bounded.

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## 1. Introduction

A covering of a group  $G$  is a family  $\{S_i\}_{i \in I}$  of subsets of  $G$  such that  $G = \bigcup_{i \in I} S_i$ . If  $\{H_i\}_{i \in I}$  is a covering of  $G$  by subgroups, it is natural to ask what information about  $G$  can be deduced from properties of the subgroups  $H_i$ . In the case where the covering is finite actually quite a lot about the structure of  $G$  can be said. In particular, as was first pointed out by Baer (see [10, p. 105]), a group covered by finitely many cyclic subgroups is either cyclic or finite. More recently Fernández-Alcober and Shumyatsky proved that if  $G$  is a group in which the set of all commutators is covered by finitely many cyclic subgroups, then  $G'$  is either finite or cyclic [4]. This suggests the question about the structure of a group in which the set of all  $\gamma_j$ -commutators (or of all  $\delta_j$ -commutators) is covered by finitely many cyclic subgroups. Here the words  $\gamma_j$  and  $\delta_j$  are defined by the positions  $\gamma_1 = \delta_0 = x_1$ ,  $\gamma_{j+1} = [\gamma_j, x_{j+1}]$  and  $\delta_{j+1} = [\delta_j, \delta_j]$ .

In [3] Cutolo and Nicotera showed that if  $G$  is a group in which the set of all  $\gamma_j$ -commutators is covered by finitely many cyclic subgroups, then  $\gamma_j(G)$  is finite-by-cyclic. They also showed that  $\gamma_j(G)$  can be neither cyclic nor finite. It is still unknown whether a similar result holds for the derived words  $\delta_j$ .

In [11] the coprime commutators  $\gamma_j^*$  and  $\delta_j^*$  were introduced as a tool to study properties of finite groups that can be expressed in terms of commutators of elements of coprime orders. For the reader's convenience we recall here the definitions. Let  $G$  be a finite group. Every element of  $G$  is both a  $\gamma_1^*$ -commutator and a  $\delta_0^*$ -commutator. Now let  $j \geq 2$  and let  $X$  be the set of all elements of  $G$  that are powers of  $\gamma_{j-1}^*$ -commutators. An element  $g$  is a  $\gamma_j^*$ -commutator if there exist  $a \in X$  and  $b \in G$  such that  $g = [a, b]$  and  $(|a|, |b|) = 1$ . For  $j \geq 1$  let  $Y$  be the set of all elements of  $G$  that are powers of  $\delta_{j-1}^*$ -commutators. The element  $g$  is a  $\delta_j^*$ -commutator if there exist  $a, b \in Y$  such that  $g = [a, b]$  and  $(|a|, |b|) = 1$ . The subgroups of  $G$  generated by all  $\gamma_j^*$ -commutators and all  $\delta_j^*$ -commutators will be denoted by  $\gamma_j^*(G)$  and  $\delta_j^*(G)$ , respectively. One can easily see that if  $N$  is a normal subgroup of  $G$  and  $x$  an element whose image in  $G/N$  is a  $\gamma_j^*$ -commutator (respectively a  $\delta_j^*$ -commutator), then there exists a  $\gamma_j^*$ -commutator  $y$  in  $G$  (respectively a  $\delta_j^*$ -commutator) such that  $x \in yN$ .

It was shown in [11] that  $\gamma_j^*(G) = 1$  if and only if  $G$  is nilpotent and  $\delta_j^*(G) = 1$  if and only if the Fitting height of  $G$  is at most  $j$ . It follows that for every  $j \geq 2$  the subgroup  $\gamma_j^*(G)$  is precisely the last term of the lower central series of  $G$  (which throughout the paper will be denoted by  $\gamma_\infty(G)$ ) while for every  $j \geq 1$  the subgroup  $\delta_j^*(G)$  is precisely the last term of the lower central series of  $\delta_{j-1}^*(G)$ , that is,  $\delta_j^*(G) = \gamma_\infty(\delta_{j-1}^*(G))$ .

In the present paper we prove the following theorem.

**Theorem 1.1.** *Let  $j$  be a positive integer and  $G$  a finite group that possesses  $m$  cyclic subgroups whose union contains all  $\gamma_j^*$ -commutators of  $G$ . Then  $\gamma_j^*(G)$  contains a subgroup  $\Delta$ , of  $m$ -bounded order, which is normal in  $G$  and has the property that  $\gamma_j^*(G)/\Delta$  is cyclic.*

We note that the above result seems to be new even in the case where  $j = 1$ . Thus, one immediate corollary of [Theorem 1.1](#) is that a finite group covered by  $m$  cyclic subgroups has a normal subgroup  $\Delta$  of  $m$ -bounded order with the property that  $G/\Delta$  is cyclic. This can be easily extended to arbitrary groups.

**Corollary 1.2.** *Let  $G$  be a (possibly infinite) group covered by  $m$  cyclic subgroups. Then  $G$  has a finite normal subgroup  $\Delta$ , of  $m$ -bounded order, such that  $G/\Delta$  is cyclic.*

Indeed, let  $G$  be as in the above corollary. The classical result of B.H. Neumann [\[8\]](#) tells us that  $G$  has a cyclic subgroup of finite index. Therefore  $G$  is residually finite and all finite quotients of  $G$  satisfy the hypothesis of [Theorem 1.1](#). Hence,  $G$  has a normal subgroup  $\Delta$  of  $m$ -bounded order with the property that  $G/\Delta$  is cyclic.

We also mention that in [Theorem 1.1](#) the subgroup  $\gamma_j^*(G)$  is (of bounded order)-by-cyclic and so we observe here a phenomenon related to what was proved by Cutolo and Nicotera for the verbal subgroups  $\gamma_j(G)$ .

Having dealt with [Theorem 1.1](#), it is natural to look at finite groups in which  $\delta_j^*$ -commutators can be covered by few cyclic subgroups. Since for  $j \leq 1$  any  $\delta_j^*$ -commutator is a  $\gamma_{j+1}^*$ -commutator, the interesting cases occur when  $j \geq 2$ .

**Theorem 1.3.** *Let  $j \geq 2$  and  $G$  be a finite group that possesses  $m$  cyclic subgroups whose union contains all  $\delta_j^*$ -commutators of  $G$ . Then the order of  $\delta_j^*(G)$  is  $m$ -bounded.*

Throughout the paper we use the expression “ $(a, b, \dots)$ -bounded” to mean that the bound is a function of the parameters  $a, b, \dots$ . Henceforth all groups considered in this paper will be finite and the term “group” will mean “finite group”.

## 2. Preliminaries

We begin with some results about coprime actions of groups. Let  $H$  and  $K$  be subgroups of a group  $G$ . We denote by  $[K, H]$  the subgroup of  $G$  generated by  $\{[k, h] : k \in K, h \in H\}$ , and by  $[K, {}_i H]$  the subgroup  $[[K, {}_{i-1} H], H]$  for  $i \geq 2$ . If  $G$  is a  $p$ -group, we denote by  $\Omega_1(G)$  the subgroup of  $G$  generated by its elements of order  $p$ .

**Lemma 2.1.** *(See [\[5\]](#), [Theorems 5.2.3](#), [5.2.4](#) and [5.3.6](#).) Let  $A$  and  $G$  be groups with  $(|G|, |A|) = 1$  and suppose that  $A$  acts on  $G$ . Then we have*

- (1)  $[G, A, A] = [G, A]$ ;
- (2) If  $G$  is an abelian  $p$ -group, then  $G = C_G(A) \times [G, A]$ ;
- (3) If  $G$  is an abelian  $p$ -group and  $A$  acts trivially on  $\Omega_1(G)$ , then  $A$  acts trivially on  $G$ .

**Lemma 2.2.** *Let  $G$  be an abelian  $p$ -group and  $\alpha$  a coprime automorphism of  $G$ . If  $[G, \alpha]$  is cyclic, then  $[G, \alpha] = [G, \alpha^i]$  for any integer  $i$  such that  $\alpha^i \neq 1$ .*

**Proof.** By Lemma 2.1(2) we have  $G = C_G(\alpha) \times [G, \alpha]$ . Suppose that  $\alpha^i \neq 1$  and  $[G, \alpha] \neq [G, \alpha^i]$ . Then  $C_{[G, \alpha]}(\alpha^i) \neq 1$ . Since  $[G, \alpha]$  is cyclic, we conclude that  $\Omega_1([G, \alpha]) \leq C_{[G, \alpha]}(\alpha^i)$  and therefore  $\alpha^i$  acts trivially on  $[G, \alpha]$ . This implies that  $\alpha^i = 1$ , a contradiction.  $\square$

**Lemma 2.3.** *Let  $G$  be a cyclic group faithfully acted on by a group  $A$ . The following holds.*

- (1) *The group  $A$  is abelian;*
- (2) *If  $G$  is a  $p$ -group and  $A$  is a  $p'$ -group, then  $A$  is cyclic.*

**Proof.** Both claims are immediate from the well-known fact that the group of automorphisms of the additive cyclic group  $\mathbb{Z}/n\mathbb{Z}$  is isomorphic with the multiplicative group  $(\mathbb{Z}/n\mathbb{Z})^*$ .  $\square$

**Lemma 2.4.** *Let  $j \geq 2$  and  $G$  be a group containing a normal subgroup  $N$ . If  $N \leq \delta_j^*(G)$  and  $\delta_j^*(G)/N$  is cyclic, then  $\delta_j^*(G) = N$ .*

**Proof.** We pass to the quotient  $G/N$  and without loss of generality assume that  $N = 1$ . Therefore  $\delta_j^*(G)$  is cyclic and so by Lemma 2.3(1) we have  $\delta_j^*(G) \leq Z(G')$ . It follows that  $\delta_{j-1}^*(G)$  is nilpotent and, since  $\delta_j^*(G) = \gamma_\infty(\delta_{j-1}^*(G))$ , we deduce that  $\delta_j^*(G) = 1$ . This completes the proof.  $\square$

The following lemma is well-known. The proof can be found for example in [1].

**Lemma 2.5.** *Let  $G$  be a metanilpotent group,  $P$  a Sylow  $p$ -subgroup of  $\gamma_\infty(G)$  and  $H$  a Hall  $p'$ -subgroup of  $G$ . Then  $P = [P, H]$ .*

The next lemma will be very useful.

**Lemma 2.6.** *Let  $y_1, \dots, y_{j+1}$  be powers of  $\delta_j^*$ -commutators in  $G$ . Suppose that the elements  $y_1, \dots, y_{j+1}$  normalize a subgroup  $N$  such that  $(|y_i|, |N|) = 1$  for every  $i = 1, \dots, j + 1$ . Then for every  $g \in N$  the element  $[g, y_1, \dots, y_{j+1}]$  is a  $\delta_{j+1}^*$ -commutator.*

**Proof.** We note that all elements of the form  $[g, y_1, \dots, y_i]$  are of order prime to  $|y_{i+1}|$ . An easy induction on  $i$  shows that whenever  $i \leq j$  the element  $[g, y_1, \dots, y_{i+1}]$  is a  $\delta_{i+1}^*$ -commutator. The lemma follows.  $\square$

**Lemma 2.7.** *Let  $G$  be a group,  $P$  a normal  $p$ -subgroup of  $G$  and  $x$  a  $p'$ -element in  $G$ . Let  $j \geq 1$  be an integer. Then we have*

- (1) The subgroup  $[P, x]$  is generated by  $\gamma_j^*$ -commutators.
- (2) If  $P$  is abelian, then every element of  $[P, x]$  is a  $\gamma_j^*$ -commutator.
- (3) If  $x$  is a power of a  $\delta_{j-1}^*$ -commutator, then  $[P, x]$  is generated by  $\delta_j^*$ -commutators.
- (4) If  $x$  is a power of a  $\delta_{j-1}^*$ -commutator and  $P$  is abelian, then every element of  $[P, x]$  is a  $\delta_j^*$ -commutator.

**Proof.** In view of Lemma 2.1(1)  $[P, x] = [P, \underbrace{x, \dots, x}_{j-1}]$ . Suppose first that  $P$  is abelian. Note that every element of the form  $[g, \underbrace{x, \dots, x}_{j-1}]$ , with  $g \in P$ , is a  $\gamma_j^*$ -commutator. Since  $P$  is abelian, every element of  $[P, x]$  is of the form  $[g, \underbrace{x, \dots, x}_{j-1}]$  for a suitable  $g \in P$  and therefore every element of  $[P, x]$  is a  $\gamma_j^*$ -commutator. Now drop the assumption that  $P$  is abelian. We wish to show that  $[P, x]$  is generated by  $\gamma_j^*$ -commutators. Passing to the quotient  $G/\Phi(P)$  we may assume that  $P$  is elementary abelian and use the result for the abelian case. This proves Claims (1) and (2).

The proof of Claims (3) and (4) follows a similar argument using Lemma 2.6.  $\square$

The well-known Focal Subgroup Theorem [5, Theorem 7.3.4] states that if  $G$  is a group and  $P$  a Sylow  $p$ -subgroup of  $G$ , then  $P \cap G'$  is generated by the set of commutators  $\{[g, z] \mid g \in G, z \in P, [g, z] \in P\}$ . In particular, it follows that  $P \cap G'$  can be generated by commutators lying in  $P$ . This observation led to the question on generation of Sylow subgroups of verbal subgroups of finite groups. The main result of [2] is that  $P \cap w(G)$  is generated by powers of  $w$ -values, whenever  $w$  is a multilinear commutator word. More recently an analogous result on the generation of Sylow subgroups of  $\delta_j^*(G)$  in the case where  $G$  is soluble was proved in [1]. More precisely we have the following lemma that we will need later on.

**Lemma 2.8.** (See [1], Lemma 2.6.) *Let  $j \geq 0$ . Let  $G$  be a soluble group and  $P$  a Sylow  $p$ -subgroup of  $G$ . Then  $P \cap \delta_j^*(G)$  is generated by powers of  $\delta_j^*$ -commutators.*

It is natural to conjecture that Lemma 2.8 actually holds for all finite groups. In particular, the corresponding result in [2] was proved without the assumption that  $G$  is soluble. It seems though that proving Lemma 2.8 for arbitrary groups is a complicated task. Indeed, one of the tools used in [2] is the proof of the Ore Conjecture by Liebeck, O’Brien, Shalev, and Tiep [7] that every element of any nonabelian finite simple group is a commutator. Recently it was conjectured in [11] that every element of a finite simple group is a commutator of elements of coprime orders. If this is confirmed, proving Lemma 2.8 for arbitrary groups would be easy. However the conjecture that every element of a finite simple group is a commutator of elements of coprime orders is proved only for the alternating groups [11] and the groups  $\text{PSL}(2, q)$  [9].

**Lemma 2.9.** *Let  $G$  be a noncyclic  $p$ -group that can be covered by  $m$  cyclic subgroups. Then  $|G|$  is  $m$ -bounded.*

**Proof.** To start with, we consider the case where  $G$  is abelian. We notice that the minimal number of generators of  $G$  is at most  $m$  and therefore it is sufficient to bound the exponent of  $G$ . The group  $G$  contains an elementary abelian subgroup, say  $J$ , of order  $p^2$ . One requires precisely  $p + 1$  cyclic subgroups to cover  $J$ . Hence  $p + 1 \leq m$ . Let the exponent of  $G$  be  $p^n$ . Since  $p \leq m - 1$ , it is sufficient to bound  $n$ . We assume that  $n \geq 2$ . Choose an element  $a \in G$  whose order is  $p^n$  and an element  $b \in G \setminus \langle a \rangle$  of order  $p$ . Set  $H = \langle a, b \rangle$ . It is clear that any covering of  $H$  by cyclic subgroups requires some subgroups of order  $p^n$ . Further, the element  $a^p b$  has order  $p^{n-1}$  and it is not contained in any cyclic subgroup of order  $p^n$ . Therefore any covering of  $H$  by cyclic subgroups requires also some subgroups of order  $p^{n-1}$ . Assuming that  $n \geq 3$  we now consider the element  $a^{p^2} b$ . This has order  $p^{n-2}$  and is not contained in any cyclic subgroup of order  $p^{n-1}$ . Thus any covering of  $H$  by cyclic subgroups requires some subgroups of order  $p^{n-2}$ . It now becomes clear that any covering of  $H$  by cyclic subgroups requires some subgroups of all possible orders  $p^n, p^{n-1}, \dots, p$ . It follows that  $n \leq m$  and in the case where  $G$  is abelian the lemma is proved.

We now drop the assumption that  $G$  is abelian. Let  $N$  be a maximal normal abelian subgroup. Then  $N = C_G(N)$ . If  $N$  is noncyclic, then by the previous argument  $|N|$  is  $m$ -bounded and, since  $G/N$  embeds in  $\text{Aut } N$ , the order of  $G$  is  $m$ -bounded, too. Hence we assume that  $N$  is cyclic of order  $p^n$ . The quotient  $G/G'$  is abelian and noncyclic. Hence  $G/G'$  contains an elementary abelian subgroup of order  $p^2$ . We have remarked in the previous paragraph that the existence of such a subgroup implies that  $p \leq m - 1$  and so now it is sufficient to bound  $n$ . Let  $y$  be an element of least order in  $G \setminus N$ . In view of [5, Theorem 5.4.4] the order of  $y$  is either  $p$  or 4. Let  $P = N\langle y \rangle$ . Since  $C_P(y)$  is abelian, the previous paragraph shows that  $|C_P(y)|$  is  $m$ -bounded. Hence, it is sufficient to bound the index of  $C_N(y)$  in  $N$ . This is precisely the order of the subgroup  $[N, y]$ . Observe that all elements in the coset  $[N, y]y^{-1}$  are conjugate to  $y^{-1}$  and so  $P$  contains at least  $|[N, y]|$  elements of order  $|y|$  (which is either  $p$  or 4). Any nontrivial cyclic  $p$ -group contains exactly  $p - 1$  elements of order  $p$  and at most two elements of order 4. Therefore one requires at least  $\lceil |[N, y]|/p \rceil$  cyclic subgroups in  $P$  to cover the coset  $[N, y]y^{-1}$ . Hence  $\lceil |[N, y]|/p \rceil \leq m$  and since  $p \leq m - 1$ , we deduce that  $|[N, y]| \leq m(m - 1)$ . The proof is complete.  $\square$

We close this preliminary section with the following results about coprime actions.

**Lemma 2.10.** *Let  $j$  be a positive integer,  $P$  a  $p$ -group of class  $c$  and  $\alpha$  a  $p'$ -automorphism of  $P$ . Suppose that  $P$  has  $m$  cyclic subgroups whose union contains all elements of the form  $[x, \underbrace{\alpha, \dots, \alpha}_j]$ , with  $x \in P$ . If  $[P, \alpha]$  is noncyclic, then the order of  $[P, \alpha]$  is  $(c, m)$ -bounded.*

**Proof.** By Lemma 2.1(1) we have  $P = [P, \alpha] = [P, \underbrace{\alpha, \dots, \alpha}_j]$ . We argue by induction on the nilpotency class  $c$ . If  $c = 1$ , then  $P$  is abelian and it consists of elements of the form  $[x, \underbrace{\alpha, \dots, \alpha}_j]$ . It follows that  $P$  can be covered by  $m$  cyclic subgroups and by Lemma 2.9 the order of  $P$  is  $m$ -bounded.

Assume  $c \geq 2$  and pass to the quotient  $\bar{P} = P/P'$ . Of course  $\bar{P}$  is not cyclic and abelian. Hence by the argument in the previous paragraph the order of  $\bar{P}$  is  $m$ -bounded and since  $P$  is nilpotent of class  $c$ , it follows that  $|P|$  is  $(c, m)$ -bounded, as desired.  $\square$

**Lemma 2.11.** *Let  $A$  be a noncyclic  $p'$ -group of automorphisms of a noncyclic abelian  $p$ -group  $G$ . Then there exists  $a \in A$  such that  $[G, a]$  is noncyclic.*

**Proof.** Suppose that the lemma is false and  $[G, a]$  is cyclic for every  $a$  in  $A$ .

Firstly we consider the case where  $A$  is abelian. Choose a nontrivial element  $a_1 \in A$ . The cyclic subgroup  $[G, a_1]$  is  $A$ -invariant and, by Lemma 2.3, the quotient  $A/C_A([G, a_1])$  is cyclic. In particular  $C_A([G, a_1]) \neq 1$  so we choose a nontrivial element  $a_2 \in C_A([G, a_1])$ . Since  $a_2$  centralizes  $[G, a_1]$ , it follows that  $[G, a_1][G, a_2]$  is not cyclic. Moreover, it is clear that  $a_1$  centralizes  $[G, a_2]$ . Hence,  $[G, a_1][G, a_2] \leq [G, a_1 a_2]$  and this is a contradiction. Thus, in the case where  $A$  is abelian the result follows.

Suppose now that  $A$  is nilpotent. If  $A$  contains a noncyclic abelian subgroup, then the result follows from the previous paragraph. Hence, without loss of generality, we suppose that every abelian subgroup of  $A$  is cyclic. It follows (see for example [5, Theorem 4.10(ii), p. 199]) that  $A$  is isomorphic to  $Q \times C$ , where  $Q$  is the generalized quaternion group and  $C$  is a cyclic group of odd order. By Lemma 2.2 for any  $a$  in  $A$  and any integer  $i$  such that  $a^i \neq 1$  we have  $[G, a] = [G, a^i]$ . Let  $a_0$  be the unique involution of  $A$ . It is clear that  $a_0$  is contained in all maximal cyclic subgroups of  $A$ . It follows that  $[G, a] = [G, a_0]$  for all  $a$  in  $A$ . Hence we conclude that  $[G, A] = [G, a_0]$  which is cyclic. By Lemma 2.1  $A$  acts faithfully on  $[G, a_0]$  and, in view of Lemma 2.3(2), the group  $A$  must be cyclic. This is a contradiction.

Finally we can drop the assumption that  $A$  is nilpotent. If  $A$  contains at least one noncyclic nilpotent subgroup, we use the previous case. Thus, we assume that all nilpotent subgroups in  $A$  are cyclic and in this case  $A$  is soluble. Let  $F = F(A)$  be the Fitting subgroup of  $A$ . Of course we can assume that  $A$  is not nilpotent and so we can choose a subgroup  $Q$  of  $F$  of prime order  $q$  such that  $Q$  is not contained in  $Z(A)$ . Then there exists a  $q'$ -element  $a$  in  $A$  such that  $[Q, a] = Q$ . The element  $a$  acts on  $[G, Q]$ , which is a cyclic  $p$ -group. Thus  $Q\langle a \rangle$  acts on  $[G, Q]$ , but this leads to a contradiction since by Lemma 2.3(1) the group of automorphisms of a cyclic group is abelian.  $\square$

### 3. Theorem 1.3

Turull introduced in [12] the concept of an irreducible  $B$ -tower and showed that a soluble group  $G$  has Fitting height  $h$  if and only if  $h$  is maximal such that there exists an irreducible tower of height  $h$  consisting of subgroups of  $G$  (see Lemmas 1.4 and 1.9(3) in [12]). We will now remind the reader some of the properties of subgroups forming an irreducible tower (we require only the case  $B = 1$  and refer to these objects simply as “towers”).

Let  $P_i$ , where  $i = 1, \dots, h$  be subgroups of  $G$  forming a tower of height  $h$ . Then we have

- (1)  $P_i$  is a  $p_i$ -group ( $p_i$  a prime) for  $i = 1, \dots, h$ .
- (2)  $P_i$  normalizes  $P_j$  for  $i < j$ .
- (3)  $p_i \neq p_{i+1}$  for  $i = 1, \dots, h - 1$ .
- (4)  $[P_i, P_{i-1}] = P_i$  for  $i = 2, \dots, h$ .
- (5) Let  $\bar{P}_i = P_i/C_{P_i}(\bar{P}_{i+1})$  for  $i = 1, \dots, h - 1$  and  $\bar{P}_h = P_h$ . Then  $\phi(\phi(\bar{P}_i)) = 1$ ,  $\phi(\bar{P}_i) \leq Z(\bar{P}_i)$ . Moreover  $P_{i-1}$  centralizes  $\phi(\bar{P}_i)$  for  $i = 2, \dots, h$ . Here  $\phi$  denotes the Frattini subgroup.

In the next few lemmas we will assume that  $\delta_{j+1}^*(G) = 1$ . Therefore  $\delta_j^*(G)$  is nilpotent and so any Sylow subgroups of  $\delta_j^*(G)$  is normal in  $G$ .

**Lemma 3.1.** *Let  $p$  be a prime,  $j$  a positive integer and  $G$  a group such that  $\delta_{j+1}^*(G) = 1$ . Suppose that  $\delta_j^*(G)$  is a nontrivial abelian  $p$ -group. Then either there exists a  $p'$ -element  $x$  which is a power of a  $\delta_{j-1}^*$ -commutator with the property that  $[\delta_j^*(G), x]$  is noncyclic, or  $\delta_j^*(G)$  is cyclic and  $j = 1$ .*

**Proof.** For simplicity denote  $\delta_j^*(G)$  by  $P$ . Suppose first that  $P$  is cyclic. If  $j \geq 2$ , then in view of Lemma 2.4 we deduce that  $P = 1$ , a contradiction. Hence, if  $P$  is cyclic, we have  $j = 1$ . Now assume that  $P$  is noncyclic.

Consider the case where  $j = 1$ . We wish to show that there exists a  $p'$ -element  $x \in G$  with the property that  $[P, x]$  is noncyclic. Let  $L$  be a Hall  $p'$ -subgroup in  $G$  and suppose that  $[P, x]$  is cyclic for every  $x \in L$ . If  $L/C_L(P)$  is not cyclic, we obtain a contradiction with Lemma 2.11. Therefore assume that  $L/C_L(P)$  is cyclic. Let  $a$  be an element of  $L$  such that  $\langle a, C_L(P) \rangle = L$ . We have  $P = [P, L] = [P, a]$ , which is again a contradiction since  $[P, a]$  is cyclic.

Hence we may assume that  $j \geq 2$ . Moreover we assume that  $G$  is a counter-example with minimal possible order. Since  $\delta_{j+1}^*(G) = 1$ , it follows that  $G$  is soluble and the Fitting height precisely  $j + 1$ . By [12]  $G$  possesses a tower of height  $j + 1$ , i.e., a subgroup  $P_0 \dots P_{j-2}P_{j-1}P_j$ , where  $P_j \leq P$ . Again  $P_j$  is noncyclic and therefore, in view of minimality of  $|G|$ , we have  $G = P_0 \dots P_{j-2}P_{j-1}P_j$  and  $P_j = P$ .



By [11, Lemma 2.5], each subgroup  $P_i$  of the tower is generated by  $\delta_{i-1}^*$ -commutators contained in  $P_i$ . Set  $H = P_{j-1}$ . We know that  $P = [P, H]$ . Let  $B$  be the set of all elements of  $H$  which can be written as powers of  $\delta_{j-1}^*$ -commutators and assume that  $[P, b]$  is cyclic for any  $b$  in  $B$ . First we consider the case where  $H$  has odd order.

Let  $b_1, b_2$  be elements of  $B$  and  $B_0 = \langle b_1, b_2 \rangle$ . We have  $[P, B_0] = [P, b_1][P, b_2]$ . Consider now the subgroup  $\Omega_1([P, B_0])$ . Obviously,  $\Omega_1([P, B_0])$  can be viewed as a linear space of dimension at most two over the field with  $p$  elements. It is well-known that the nilpotent subgroups of odd order of  $GL(2, p)$  are abelian. Hence, we conclude that the derived group of  $B_0$  centralizes  $\Omega_1([P, B_0])$  and, by Lemma 2.1(3), also centralizes  $P$ . Recall that  $B_0$  is a subgroup generated by two arbitrarily chosen elements  $b_1, b_2 \in B$ . By Lemma 2.8 we have  $H = \langle B \rangle$ , and so we conclude that  $H'$  centralizes  $P$ . Let  $\bar{G} = G/C_G(P)$ . There is a natural action of  $\bar{G}$  on  $P$  and so we will view  $\bar{G}$  as a group of automorphisms of  $P$ . We already know that  $\bar{H}$  is abelian and it is clear that  $\delta_j^*(\bar{G}) = 1$ .

Suppose first that  $\bar{H}$  is cyclic and choose an element  $b \in B$  such that  $\bar{H}$  is generated by  $bC_G(P)$ . We have  $P = [P, \bar{H}] = [P, b]$  which is cyclic, a contradiction. Hence,  $\bar{H}$  is not cyclic. Let  $q$  be the prime such that  $H$  has  $q$ -power order. By induction the group  $\bar{G}$  contains a  $q'$ -element  $y$  which is a power of  $\delta_{j-2}^*$ -commutator with the property that  $[\bar{Q}, y]$  is noncyclic. Moreover, Lemma 2.7(4) shows that  $[\bar{Q}, y]$  consists entirely of  $\delta_{j-1}^*$ -commutators. For any element  $t \in [\bar{Q}, y]$  we can choose  $b_t \in B$  such that  $[P, t] = [P, b_t]$ . Therefore  $[P, t]$  is cyclic for each  $t \in [\bar{H}, y]$ . In view of Lemma 2.11 this leads to a contradiction.

Now consider the case where  $H$  is a 2-subgroup. In this case the properties of towers listed before the lemma play an important role in our arguments. As before we have  $[P, H] = P$  and we wish to show that  $H$  contains a  $\delta_{j-1}^*$ -commutator  $x$  with the property that  $[P, x]$  is noncyclic. We can pass to the quotient  $G/C_H(P)$  and assume that  $H$  acts on  $P$  faithfully. Choose a  $\delta_{j-2}^*$ -commutator  $b \in P_{j-2}$ . Suppose that  $b$  normalizes an abelian subgroup  $A$  in  $H$ . If  $[A, b] \neq 1$ , then  $[A, b]$  is a noncyclic abelian subgroup which, by Lemma 2.7(4), entirely consists of  $\delta_{j-1}^*$ -commutators. By Lemma 2.11  $[P, x]$  is noncyclic for some  $x \in [A, b]$  and we are done. Therefore  $[A, b] = 1$  for every abelian subgroup  $A$  of  $H$  which is normalized by  $b$ .

We know that  $[h, \underbrace{b, \dots, b}_{j-1}]$  is a  $\delta_{j-1}^*$ -commutator for every  $h \in H$ . Therefore we can choose  $a \in H$  such that  $a$  and  $[a, b]$  are both nontrivial  $\delta_{j-1}^*$ -commutators. If both  $a$  and  $[a, b]$  have order 2, then the subgroup  $\langle a, [a, b] \rangle$  is abelian and consists of  $\delta_{j-1}^*$ -commutators. By Lemma 2.11  $[P, x]$  is noncyclic for some  $x \in \langle a, [a, b] \rangle$  and we are done. Therefore we can choose  $a \in H$  such that  $a$  and  $[a, b]$  are both nontrivial  $\delta_{j-1}^*$ -commutators, the element  $[a, b]$  being of order four. Since  $a^2 \in Z(H)$  and since  $[Z(H), b] = 1$ , we have  $[a^2, b] = 1$ . So we have

$$1 = [a^2, b] = [a, b][a, b]^a$$

and in particular  $a$  inverts  $[a, b]$ . It follows that  $a$  normalizes  $[P, [a, b]]$  which is a cyclic subgroup. Now consider the action of the subgroup  $D = \langle a, [a, b] \rangle$  on  $[P, [a, b]]$ . By Lemma 2.3

$D'$  centralizes  $[P, [a, b]]$ . So in particular  $[a, b]^2$  is nontrivial and it centralizes the cyclic subgroup  $[P, [a, b]]$ . Thus we get a contradiction by Lemma 2.2. The proof is now complete.  $\square$

**Lemma 3.2.** *Let  $p$  be a prime,  $j$  a positive integer and  $G$  a group such that  $\delta_{j+1}^*(G) = 1$ . Let  $P$  be the Sylow  $p$ -subgroup of  $\delta_j^*(G)$  and assume that  $[P, x]$  is cyclic for every  $p'$ -element  $x$  which is a power of a  $\delta_{j-1}^*$ -commutator. Then  $P$  is cyclic.*

**Proof.** By passing to the quotient  $G/O_{p'}(\delta_j^*(G))$  we may assume that  $\delta_j^*(G)$  is a  $p$ -group and that  $P = \delta_j^*(G)$ . If  $P$  is abelian, the result is immediate from Lemma 3.1. Thus, we assume that  $P$  is not abelian and use induction on the nilpotency class of  $P$ . We consider the quotient  $G/Z(P)$  and by induction we conclude that  $P/Z(P)$  is cyclic. However this implies that  $P$  is abelian and we get a contradiction.  $\square$

**Lemma 3.3.** *Let  $p$  be a prime,  $j$  a positive integer and  $G$  a group such that  $\delta_{j+1}^*(G) = 1$ . Suppose that  $G$  possesses  $m$  cyclic subgroups whose union contains all  $\delta_j^*$ -commutators of  $G$  and that the Sylow  $p$ -subgroup  $P$  of  $\delta_j^*(G)$  is nilpotent of class  $c$ . Let  $x$  be a  $p'$ -element which is a power of  $\delta_{j-1}^*$ -commutator in  $G$  such that  $[P, x]$  is noncyclic. Then the order of  $[P, x]$  is  $(c, m)$ -bounded.*

**Proof.** The conjugation by the element  $x$  induces a  $p'$ -automorphism of  $P$ . Since every element of the form  $[y, \underbrace{x, \dots, x}_j]$ , with  $y \in P$  is a  $\delta_j^*$ -commutator, Lemma 2.10 shows that the order of  $[P, x]$  is  $(c, m)$ -bounded, as desired.  $\square$

**Lemma 3.4.** *Let  $j$  be a non-negative integer and  $G$  a group such that  $\delta_j^*(G)$  is nilpotent of class  $c$ . Suppose that  $G$  possesses  $m$  cyclic subgroups whose union contains all  $\delta_j^*$ -commutators of  $G$ . Then  $\delta_j^*(G)$  contains a subgroup  $\Delta$  of  $(c, m)$ -bounded order which is normal in  $G$  and has the property that  $\delta_j^*(G)/\Delta$  is cyclic. If  $j \geq 2$ , then  $\delta_j^*(G) = \Delta$ .*

**Proof.** We argue by induction on  $j$ . Suppose first that  $j = 0$ . In this case  $G$  is nilpotent of class  $c$  and it is covered by  $m$  cyclic subgroups. The result is rather straightforward applying Lemma 2.9 to each Sylow subgroup of  $G$ .

So we assume that  $j \geq 1$ . Let  $P$  be a Sylow  $p$ -subgroup of  $\delta_j^*(G)$  for some prime  $p$ . Denote by  $\Delta_p$  the subgroup generated by all subgroups of the form  $[P, y]$ , where  $y$  ranges over the set of all  $p'$ -elements which are powers of  $\delta_{j-1}^*$ -commutators in  $G$  such that  $[P, y]$  is noncyclic. By Lemma 3.3 the orders of all such subgroups  $[P, y]$  have a common bound, which depends only on  $c$  and  $m$ . We observe that  $\Delta_p$  is a group which is nilpotent of class at most  $c$  and is generated by elements of  $(c, m)$ -bounded order. Hence the exponent of  $\Delta_p$  is  $(c, m)$ -bounded. Moreover, by Lemma 2.7(3)  $\Delta_p$  is generated by  $\delta_j^*$ -commutators that are all contained in  $m$  cyclic subgroups, and so we conclude that  $\Delta_p$  has at most  $m$  generators. It follows that the order of  $\Delta_p$  is  $(c, m)$ -bounded. We further observe that

since the bound on the order of  $\Delta_p$  does not depend on  $p$ , it follows that  $\Delta_p = 1$  for all primes  $p$  which are bigger than certain number depending only on  $c$  and  $m$ .

Let  $\Delta$  be the product of the subgroups  $\Delta_p$  over all prime divisors of  $|\delta_j^*(G)|$ . It is clear that  $|\Delta|$  is  $(c, m)$ -bounded. Consider the quotient  $G/\Delta$ . For simplicity, we just assume that  $\Delta = 1$ . Then  $[P, x]$  is cyclic for every  $p'$ -element  $x$  which is a power of a  $\delta_{j-1}^*$ -commutator. Then, by Lemma 3.2,  $P$  is cyclic. Thus all Sylow subgroups of  $\delta_j^*(G)$  are cyclic. It follows that  $\delta_j^*(G)/\Delta$  is cyclic. Of course, if  $j \geq 2$ , then by Lemma 2.4 we have  $\delta_j^*(G) = \Delta$ .  $\square$

We are now ready to complete the proof of Theorem 1.3.

**Proof.** Recall that  $j \geq 2$  and  $G$  possesses  $m$  cyclic subgroups whose union contains all  $\delta_j^*$ -commutators of  $G$ . We wish to show that the order of  $\delta_j^*(G)$  is  $m$ -bounded. Let  $C_1, \dots, C_m$  be the cyclic subgroups whose union contains all  $\delta_j^*$ -commutators of  $G$ . Without loss of generality we assume that each subgroup  $C_i$  is generated by  $\delta_j^*$ -commutators (not necessarily by a single  $\delta_j^*$ -commutator). Thus,  $\delta_j^*(G) = \langle C_1, \dots, C_m \rangle$  and in particular it follows that  $\delta_j^*(G)$  can be generated by  $m$  elements. Let  $x \in G$  be a  $\delta_j^*$ -commutator. For any  $g \in G$  the conjugate  $x^g$  is again a  $\delta_j^*$ -commutator and so  $x^g \in C_i$  for some  $i$ . Since  $C_i$  is cyclic, it contains only at most one subgroup of any given order and we conclude that the cyclic subgroup  $\langle x \rangle$  has at most  $m$  conjugates. Therefore the index of the normalizer of  $\langle x \rangle$  in  $G$  is at most  $m$ . Let  $N$  be the intersection of all normalizers of cyclic subgroups generated by a  $\delta_j^*$ -commutator and set  $K = \delta_j^*(G) \cap N$ . Since  $\delta_j^*(G)$  is  $m$ -generated, it follows that the number of subgroups of  $\delta_j^*(G)$  whose index is at most  $m$  is  $m$ -bounded [6, Theorem 7.2.9] and so we deduce that the index of  $K$  in  $\delta_j^*(G)$  is  $m$ -bounded as well. It is clear that  $K$  normalizes each of the subgroups  $C_1, \dots, C_m$ . This implies that  $K$  is nilpotent of class at most 2. Indeed, since  $\text{Aut } C_i$  is abelian for every  $i = 1, \dots, m$ , we deduce that  $K/C_K(C_i)$  is abelian. So  $K'$  centralizes  $\delta_j^*(G)$  and therefore  $K' \leq Z(K)$ .

Recall that given a group  $G$ , the last term of the upper central series of  $G$  is called the hypercenter of  $G$ . It will be denoted by  $Z_\infty(G)$ . Let us show that  $K \leq Z_\infty(\delta_j^*(G))$ . Choose a Sylow  $p$ -subgroup  $P$  of  $K$ . It is clear that  $P$  is normal in  $G$ . If all the subgroups  $C_i$  have  $p$ -power order, then all  $\delta_j^*$ -commutators of  $G$  are  $p$ -elements and by [11, Theorem 2.4]  $G$  is soluble and  $\delta_j^*(G)$  is a  $p$ -subgroup. Thus  $\delta_j^*(G)$  is nilpotent and so, we have  $Z_\infty(\delta_j^*(G)) = \delta_j^*(G)$  and the inclusion  $P \leq Z_\infty(\delta_j^*(G))$  is clear. Otherwise, choose a  $p'$ -element  $x \in C_i$  for some  $i$  which is a power of a  $\delta_j^*$ -commutator. Since  $P$  normalizes  $\langle x \rangle$ , it follows that  $x$  centralizes  $P$ . Therefore  $\delta_j^*(G)/C_{\delta_j^*(G)}(P)$  is a  $p$ -group and again the inclusion  $P \leq Z_\infty(\delta_j^*(G))$  follows. Thus,  $P \leq Z_\infty(\delta_j^*(G))$  for every prime  $p$  and hence indeed  $K \leq Z_\infty(\delta_j^*(G))$ .

Therefore the index of  $Z_\infty(\delta_j^*(G))$  in  $\delta_j^*(G)$  is  $m$ -bounded. Thus, by Baer’s Theorem [10, Corollary 2, p. 113],  $\gamma_\infty(\delta_j^*(G))$  has  $m$ -bounded order. Passing to the quotient  $G/\gamma_\infty(\delta_j^*(G))$  we can assume that  $\delta_j^*(G)$  is nilpotent. Hence  $\delta_j^*(G)$  is the direct product of its Sylow subgroups. It is sufficient to show that any Sylow subgroup of  $\delta_j^*(G)$  has

bounded order. Let us choose  $p$  a prime that divides  $|\delta_j^*(G)|$  and pass to the quotient  $G/O_{p'}(\delta_j^*(G))$ . So we assume that  $\delta_j^*(G)$  is a  $p$ -group. In view of [Lemma 3.4](#) it is now sufficient to bound the nilpotency class of  $\delta_j^*(G)$ . It has already been mentioned that  $K'$  centralizes  $\delta_j^*(G)$  and therefore we can pass to the quotient  $G/K'$  and, without loss of generality, assume that  $K$  is abelian. Choose generators  $x_1, \dots, x_m$  of the subgroups  $C_1, \dots, C_m$  and let  $t$  be the index of  $K$  in  $\delta_j^*(G)$ . Since each subgroup  $K\langle x_i \rangle$  is nilpotent of class at most 2 and since  $x_i^t \in K$ , it follows that  $K^t$  centralizes  $x_i$  for each  $i = 1, \dots, m$ . In other words  $K^t \leq Z(\delta_j^*(G))$ . Passing again to the quotient  $G/Z(\delta_j^*(G))$  we can assume that  $K^t = 1$ . Since the index  $t$  of  $K$  in  $\delta_j^*(G)$  is  $m$ -bounded and since  $\delta_j^*(G)$  can be generated by  $m$  elements, we conclude that the minimal number of generators for  $K$  is  $m$ -bounded. Combining this with the fact that  $K^t = 1$ , we immediately deduce that the order of  $K$  and therefore that of  $\delta_j^*(G)$  are  $m$ -bounded. Of course, this implies that so is the nilpotency class of  $\delta_j^*(G)$ . The proof is complete.  $\square$

#### 4. Theorem 1.1

In this section we will deal with [Theorem 1.1](#). The proof is similar to that of [Theorem 1.3](#) but in fact it is easier. Therefore we will not give a detailed proof here but rather describe only some steps.

The next lemma is similar to [Lemma 3.1](#).

**Lemma 4.1.** *Let  $p$  be a prime and  $G$  a metanilpotent group. Suppose that the Sylow  $p$ -subgroup  $P$  of  $\gamma_2^*(G)$  is abelian and noncyclic. Then there exists a  $p'$ -element  $x$  with the property that  $[P, x]$  is noncyclic.*

**Proof.** By [Lemma 2.5](#) there is a Hall  $p'$ -subgroup  $H$  of  $G$  such that  $P = [P, H]$ . Now we consider the quotient  $H/C_H(P)$  which acts faithfully on  $P$ . If  $H/C_H(P)$  is noncyclic, then by [Lemma 2.11](#) there exists an element  $x$  in  $H$  such that  $[P, x]$  is noncyclic. Therefore we assume that  $H/C_H(P)$  is cyclic and let  $x$  be an element in  $H$  such that  $xC_H(P)$  generates  $H/C_H(P)$ . Then  $P = [P, x]$  is noncyclic and  $x$  is the required element.  $\square$

The proof of the next lemma follows word-by-word that of [Lemma 3.2](#). Therefore we omit the details.

**Lemma 4.2.** *Let  $p$  be a prime and  $G$  a metanilpotent group. Let  $P$  be the Sylow  $p$ -subgroup of  $\gamma_2^*(G)$  and assume that  $[P, x]$  is cyclic for every  $p'$ -element  $x$ . Then  $P$  is cyclic.*

The next results are similar to [Lemmas 3.3 and 3.4](#). Their proofs can be obtained in the same way as those of [Lemmas 3.3 and 3.4](#) with only obvious changes required.

**Lemma 4.3.** *Let  $p$  be a prime,  $j$  a positive integer and  $G$  a metanilpotent group. Suppose that  $G$  possesses  $m$  cyclic subgroups whose union contains all  $\gamma_j^*$ -commutators of  $G$ , and*

that the Sylow  $p$ -subgroup  $P$  of  $\gamma_j^*(G)$  is nilpotent of class  $c$ . Let  $x$  be a  $p'$ -element in  $G$  such that  $[P, x]$  is noncyclic. Then the order of  $[P, x]$  is  $(c, m)$ -bounded.

**Lemma 4.4.** *Let  $j$  be a positive integer and  $G$  a group such that  $\gamma_j^*(G)$  is nilpotent of class  $c$ . Suppose that  $G$  possesses  $m$  cyclic subgroups whose union contains all  $\gamma_j^*$ -commutators of  $G$ . Then  $\gamma_j^*(G)$  contains a subgroup  $\Delta$ , of  $(c, m)$ -bounded order, which is normal in  $G$  and has the property that  $\gamma_j^*(G)/\Delta$  is cyclic.*

From this we deduce our [Theorem 1.1](#).

**Proof of Theorem 1.1.** Recall that  $G$  has  $m$  cyclic subgroups whose union contains all  $\gamma_j^*$ -commutators of  $G$ . We wish to prove that  $\gamma_j^*(G)$  contains a subgroup  $\Delta$ , of  $m$ -bounded order, which is normal in  $G$  and has the property that  $\gamma_j^*(G)/\Delta$  is cyclic. Let  $C_1, \dots, C_m$  be the cyclic subgroups whose union contains all  $\gamma_j^*$ -commutators of  $G$ . We assume that each subgroup  $C_i$  is generated by  $\gamma_j^*$ -commutators. Thus,  $\gamma_j^*(G) = \langle C_1, \dots, C_m \rangle$  and in particular it follows that  $\gamma_j^*(G)$  can be generated by  $m$  elements. Let  $N$  be the intersection of all normalizers of cyclic subgroups generated by a  $\gamma_j^*$ -commutator and  $K = \gamma_j^*(G) \cap N$ . Arguing as in the proof of [Theorem 1.3](#) we deduce that the index of  $K$  in  $\gamma_j^*(G)$  is  $m$ -bounded. It is clear that  $K$  is nilpotent of class at most 2 and  $K \leq Z_\infty(\gamma_j^*(G))$ . By Baer's Theorem  $\gamma_\infty(\gamma_j^*(G))$  has  $m$ -bounded order. Passing to the quotient  $G/\gamma_\infty(\gamma_j^*(G))$  we can assume that  $\gamma_j^*(G)$  is nilpotent and, with further reductions, that  $\gamma_j^*(G)$  is a  $p$ -group. In view of [Lemma 4.4](#) it is now sufficient to bound the nilpotency class of  $\gamma_j^*(G)$ . Since  $K'$  centralizes  $\gamma_j^*(G)$ , we can pass to the quotient  $G/K'$  and without loss of generality assume that  $K$  is abelian. Choose generators  $x_1, \dots, x_m$  of the subgroups  $C_1, \dots, C_m$  and let  $t$  be the index of  $K$  in  $\gamma_j^*(G)$ . Since each subgroup  $K\langle x_i \rangle$  is nilpotent of class at most 2 and since  $x_i^t \in K$ , it follows that  $K^t$  centralizes  $x_i$  for each  $i = 1, \dots, m$ . In other words  $K^t \leq Z(\gamma_j^*(G))$ . Passing again to the quotient  $G/Z(\gamma_j^*(G))$  we can assume that  $K^t = 1$ . Since the index  $t$  of  $K$  in  $\gamma_j^*(G)$  is  $m$ -bounded and  $\gamma_j^*(G)$  can be generated by  $m$  elements, we conclude that the minimal number of generators for  $K$  is  $m$ -bounded. Combining this with the fact that  $K^t = 1$ , we deduce that the order of  $K$  and therefore that of  $\gamma_j^*(G)$  are  $m$ -bounded. Of course, this implies that so is the nilpotency class of  $\gamma_j^*(G)$ . The proof is complete.  $\square$

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