Even cycles and even 2-factors in the line graph of a simple graph.

Arrigo Bonisoli*

Dipartimento di Scienze Fisiche, Informatiche e Matematiche
Università di Modena e Reggio Emilia
Italy

arrigo.bonisoli@unimore.it

Simona Bonvicini

Dipartimento di Scienze Fisiche, Informatiche e Matematiche
Università di Modena e Reggio Emilia
Italy

simona.bonvicini@unimore.it

Submitted: Oct 27, 2015; Accepted: Oct 11, 2017; Published: Oct 20, 2017
Mathematics Subject Classifications: 05B40, 05C20, 05C38

Abstract

Let G be a connected graph with an even number of edges. We show that if the subgraph of G induced by the vertices of odd degree has a perfect matching, then the line graph of G has a 2-factor whose connected components are cycles of even length (an even 2-factor). For a cubic graph G, we also give a necessary and sufficient condition so that the corresponding line graph $L(G)$ has an even cycle decomposition of index 3, i.e., the edge-set of $L(G)$ can be partitioned into three 2-regular subgraphs whose connected components are cycles of even length. The more general problem of the existence of even cycle decompositions of index m in $2d$-regular graphs is also addressed.

Keywords: cycle decomposition; 2-factor; oriented graphs; line graph

1 Introduction

The graphs considered in this paper are simple and finite. We refer to [3] for graph theory notation and terminology which are not introduced explicitly here. In particular, we shall

*Research performed within the activity of INdAM–GNSAGA with the financial support of the Italian Ministry MIUR, project "Strutture Geometriche, Combinatoria e loro Applicazioni"

use the following terminology: a *cycle* is a connected 2-regular graph; the number of edges in a cycle is called its *length*; a cycle is *even* if it has even length. The vertex-disjoint union of cycles, that are contained in a graph G, is a 2-regular subgraph of G. A spanning 2-regular subgraph of G is a 2-factor of G. A 2-regular subgraph (in particular, a 2-factor) is *even* if its connected components are even cycles. A *cycle decomposition* of a graph G is a partition of the edge-set of G into cycles. It is known that a graph G possesses a cycle decomposition if and only if every vertex of G has even degree (see [20]). Cycle decompositions satisfying additional conditions are widely studied (see for instance the survey paper [12]). In this paper we consider *even cycle decompostions*, i.e., cycle decompositions whose elements are even cycles. A necessary condition for the existence of an even cycle decomposition of a graph G is that every vertex has even degree and every block of G has an even number of edges. The existence of even cycle decompositions in planar graphs is completely settled. It is known that for planar graphs the necessary condition is also sufficient (see [17]). For a non-planar graph G a result by Zhang [24] holds, namely, if G satisfies the necessary condition and has no K_5-minor, then G possesses an even cycle decomposition.

Given a graph G with a cycle decomposition, we can color the cycles of the decomposition in such a way that cycles sharing a vertex receive distinct colors. Each colored class is then a 2-regular subgraph of G. If m is the minimum number of colors that are required in such a coloring, then we say that the decomposition is a *cycle decomposition of index* m. Since each colored class is a 2-regular subgraph of G, a cycle decomposition of index m provides a partition of the edge-set of G into m 2-regular subgraphs, i.e., a 2-regular subgraph decomposition of G of cardinality m. If each colored class is a 2-factor of G, then a 2-regular subgraph decomposition of G is known as a 2*-factorization* of G. There might be more than one way to color the cycles of a decomposition by m colors, i.e., a cycle decomposition of index m might provide more than one 2-regular subgraph decomposition of cardinality m. Obviously, a cycle decomposition of index m consisting of c cycles provides a 2-regular subgraph decomposition of cardinality m', for every $m \leqslant m' \leqslant c$. By Petersen's Theorem [16], a $2d$-regular graph possesses a 2-factorization. Therefore, every $2d$-regular graph has at least one cycle decomposition of index d. An arbitrary cycle decomposition of a $2d$-regular graph has index $m \geqslant d$. If a cycle decomposition of index m satisfies additional properties, then these properties may well be inherited by the corresponding 2-regular subgraph decomposition of cardinality m. In particular, an even cycle decomposition of index m provides an even 2-regular subgraph decomposition of cardinality m. We are interested in determining the minimum number m of even 2-regular subgraphs which partition the edge-set of a graph or, equivalently, the minimum value of m taken over all even cycle decompositions of index m. Our motivation for formulating this problem is explained in Section 1.1. By the previous remarks, the index of an even cycle decomposition of a $2d$-regular graph satisfies the inequality $m \geqslant d$. A class 1 regular graph of degree $2d$ possesses an even cycle decomposition of index d (since the edge-set of a class 1 regular graph of degree $2d$ can be partitioned into $2d$ perfect matchings, we can pair the matchings and find d even 2-factors). It is easy to show that the converse is also true. Hence, a $2d$-regular graph has an even cycle decomposition of index d if and only if it is class 1. If the graph is class 2, then every even cycle decomposition has index $m \geqslant d+1$ (see [7] for the definition of graphs of class 1 and 2 according to Vizing's Theorem).

We restrict our attention to 4-regular graphs and in particular to 4-regular line graphs. We study the existence/non-existence of even 2-factors and even cycle decompositions in these graphs. We recall that the vertices of the line graph $L(G)$ of a graph G correspond to the edges of G. Two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G share a vertex. Hence, if G is 3-regular, then $L(G)$ is 4-regular. The reason of our interest in 4-regular graphs arises from the study of a chromatic parameter (the palette index) and will be explained in Section 1.1. We consider line graphs because these graphs are completely characterized by a list of nine forbidden subgraphs (see [1]) which reduces to seven forbidden subgraphs if the graphs are regular of degree 4. We shall see that some properties holding for line graphs can be used to determine the structure of the original graphs and conversely (see for instance Proposition 5, 6, 8). Our results involve other decompositions, namely, *even star decompositions* and P_2*-decompositions*. An *even star decomposition* $\mathcal{D}$ of a simple graph G is a partition of the edge-set of G into stars $K_{1,2h}$ whose centre has even degree $2h \geqslant 2$ (even stars). We do not require that the stars in $\mathcal{D}$ are pairwise isomorphic. If each star in $\mathcal{D}$ is a path P_2 (a path with two edges), then we say that $\mathcal{D}$ is a P_2*-decomposition* of G.

We briefly summerize the contents of this paper. Section 2.2 is devoted to the existence of an even 2-factor in the line graph of an arbitrary graph G. We note that the line graph of a connected graph G with an even number of edges has an even 2-factor if and only if the graph G has a pair of disjoint P_2-decompositions ("disjoint" here is understood in a set-theoretical sense, i.e., the disjoint decompositions have no common member). We prove the following sufficient condition: if the subgraph K induced by the vertices of odd degree has a perfect matching, then G has a pair of disjoint P_2-decompositions. In Section 2.1 we always assume that G is cubic and prove that the existence of an even cycle decomposition in $L(G)$ is equivalent to the existence of three subgraphs of G satisfying certain conditions (see Proposition 8). Every class 1 cubic graph possesses such subgraphs. An almost straighforward argument shows that the line graph of a class 1 cubic graph has an even cycle decomposition of index 3 (see Corollary 9). By the results in [13], the line graph of a class 1 cubic graph with an even number of edges is a 4-regular graph of class 1. Hence, in this case, we can find an even cycle decomposition of smallest index, namely, of index 2 (see the previous remarks on graphs of class 1).

The existence of the three subgraphs for a class 2 cubic graph does not seem to be an immediate consequence of the definition: we namely give a sufficient condition for the existence of such subgraphs (see Proposition 11 and 13). We use these conditions to show that the line graphs of some class 2 cubic graphs (flower snarks, Blanuša snarks, Goldberg snarks and others) admit even cycle decompositions of index 3. The numerous examples of 4-regular line graphs with an even cycle decomposition of index 3 seem to confirm a conjecture by Markström [15] stating that a 4-regular graph on $2n+1$ vertices asymptotically almost surely decomposes into one cycle of length $2n$ and two further even cycles, i.e., it has an even cycle decomposition of index 3. We note that our results hold for graphs on an even number of vertices, as well.

Our constructions are described in terms of "even orientations" (more details in Section 2). In particular, in Proposition 3, we obtain a refinement of a classical result of Kotzig [14] stating that a connected graph has an even orientation if and only if it has an even number of edges.

1.1 A related question.

A (proper) edge-coloring of a graph G defines at each vertex v the set of colors of its incident edges, the so called *palette* of the vertex v. The minimum number of distinct palettes taken over all proper edge-colorings of G is the palette index of G and is denoted by $\check{s}(G)$ (see [9]). As remarked in [9], the palette index of a regular graph is different from 2 and satisfies the inequalities $1 \leqslant \check{s}(G) \leqslant \chi'(G)$ where $\chi'(G)$ denotes the chromatic index. Moreover, $\check{s}(G) = 1$ if and only if the graph G is class 1. Consequently, the possible values for the palette index of a class 2 cubic graph are 3 and 4. By the results in [9], a class 2 cubic graph has palette index 3 if and only if it has a perfect matching, otherwise the palette index is 4. Hence, for every admissible value it is possible to find an example of a class 2 cubic graph with the required palette index.

What is the behavior of d-regular graphs with $d > 3$ in this respect? In other words, given an integer r, $3 \leqslant r \leqslant d+1$, is it possible to find a d-regular graph with palette index r? The case $d = 4$ is studied in [4] and it is shown that for every integer r, $3 \leqslant r \leqslant 5$, it is possible to find a 4-regular graph with palette index r.

While studying the palette index of 4-regular graphs we observed the following phenomenon: a 4-regular graph of class 2 has palette index 3 if and only if it has an even 2-factor or an even cycle decomposition of index 3. We can exhibit instances of 4-regular graphs with an even 2-factor and/or an even cycle decomposition of index 3 (see [4]). We can also give examples of 4-regular graphs with palette index 4 and 5, but none of them admits an even cycle decomposition. We do not know examples of graphs, with palette index larger than 3, possessing an even cycle decomposition. If such a graph exists, then it has no even 2-factor and every even cycle decomposition has index $m > 3$ (otherwise the palette index should not exceed 3). The problem of finding a 4-regular graph all of whose even cycle decompositions have index larger than 3 was another motivation for this paper, as we found no result concerning the index of an even cycle decomposition in the literature.

2 Even orientations.

An *orientation* of a simple graph $G = (V(G), E(G))$ is a directed graph $\overrightarrow{G} = (V(G), D(G))$ obtained from G by specifying, for each edge $[u, v]$ of G, an order on its end-vertices. If $a = (u, v)$ is an arc of $\overrightarrow{G}$, then the vertex u is the *tail* of a and v is the *head* of a. The *indegree* of a vertex v in $V(\overrightarrow{G})$ is the number of arcs with head v (*incoming arcs* in v); the *outdegree* of a vertex v in $V(\overrightarrow{G})$ is the number of arcs with tail v (*outgoing arcs* in

v). We say that $\overrightarrow{G}$ is an *even orientation* of G if for every vertex $v \in V(\overrightarrow{G})$ the number of incoming arcs $(u, v) \in \overrightarrow{G}$ in v is even. In [14] Kotzig proved the following theorems.

Theorem 1. *[14, Theorem 1] Let $\overrightarrow{G}$ be an arbitrary orientation of a graph G and let n denote the number of vertices of $\overrightarrow{G}$ with odd indegree. Then $n \equiv |E(G)| \pmod 2$.*

Theorem 2. *[14, Theorem 3] If G is a connected graph with an even number of edges, then G has an even orientation.*

Combining Theorem 1 and Theorem 2, one can see that a connected graph has an even orientation if and only if it has an even number of edges. This result is also known as follows: "a connected graph has a P_2-decomposition if and only if it has an even number of edges". In the next section we analyse the relationship between even orientations and P_2-decompositions in greater detail. We give some remarks that will be used to prove our results.

2.1 Even orientations and even decompositions.

Let $\overrightarrow{G}$ be an even orientation of a graph G. If v is a vertex of $\overrightarrow{G}$ with indegree $2h > 0$, then the underlying edges of the incoming arcs in v form a subgraph of G isomorphic to the star $K_{1,2h}$ (a path P_2 for $h = 1$). If we consider all the vertices of $\overrightarrow{G}$ with positive even indegree, then we obtain an even star decomposition of G. Obviously, also the converse is true: an even star decomposition of G gives rise to an even orientation of $\overrightarrow{G}$ (for every star $K_{1,2h}$ with centre v and vertices u_i, $1 \leqslant i \leqslant 2h$, we declare the arcs (u_i, v) to be in $\overrightarrow{G}$). Therefore there exists a one-to-one correspondence between even orientations and even star decompositions. If G is a graph with an even number of edges and maximum degree 3, then the one-to-one correspondence is between even orientations and P_2-decompositions of G (for graphs with maximum degree 3 a star $K_{1,2h}$ is a path P_2). For an arbitrary graph G, with an even number of edges, if the star decomposition $\mathcal{D}$ corresponding to the even orientation of $\overrightarrow{G}$ contains at least one star $D \cong K_{1,2h}$ with $2h > 2$, then we can pair in an arbitrary way the edges of D and find more than one P_2-decomposition of G arising from the same even orientation.

Let $\mathcal{D}_1$, $\mathcal{D}_2$ be a pair of disjoint even star decompositions of a graph G. Let $\overrightarrow{G}_1$, $\overrightarrow{G}_2$ be the corresponding even orientations. Since $\mathcal{D}_1$ and $\mathcal{D}_2$ are disjoint, i.e., $E(D) \neq E(D')$ for every $D \in \mathcal{D}_1$, $D' \in \mathcal{D}_2$, the corresponding even orientations $\overrightarrow{G}_1$ and $\overrightarrow{G}_2$ satisfy the following property: for every vertex $v \in V(G)$ the set of arcs $(u, v) \in D(\overrightarrow{G}_1)$ incoming in v is distinct from the set of arcs $(u, v) \in D(\overrightarrow{G}_2)$ incoming in v. We say that a pair of even orientations of the same graph G are *disjoint* if they satisfy the aforementioned property. A pair of disjoint even star decompositions gives rise to a pair of disjoint even orientations and conversely. We can now prove the following result.

Proposition 3. *Let G be a connected graph with an even number of edges. Let K be the subgraph of G induced by the vertices of odd degree. If K has a perfect matching M, then* *there exists a pair of disjoint even orientations of* G*, say* $\overrightarrow{G}_1$*,* $\overrightarrow{G}_2$*, whose arc-sets* $D(\overrightarrow{G}_1)$*,* $D(\overrightarrow{G}_2)$ *share only the arcs whose underlying edge belongs to* M*.*

Proof. By Theorem 2, the graph G has an even orientation $\overrightarrow{G}_1$. We change the orientation of all arcs whose underlying edges are not in M. Parity is preserved. □

Corollary 4. *Let* G *be a connected graph with an even number of edges. Let* K *be the subgraph of* G *induced by the vertices of odd degree. If* K *has a perfect matching, then there exists a pair of disjoint even star decompositions of* G *and also a pair of disjoint* P_2*-decompositions of* G*.*

Proof. The existence of a pair of disjoint even star decompositions, say $\mathcal{D}_1$, $\mathcal{D}_2$, follows from Proposition 3, since a pair of disjoint even orientations of G corresponds to a pair of disjoint even star decompositions of G. If G is a graph with maximum degree $\leqslant 3$, then an even-star decomposition is nothing but a P_2-decomposition, hence the assertion follows in this case.

We consider G with maximum degree larger than 3 and prove that G has a pair of disjoint P_2-decompositions. Let $\overrightarrow{G}_1$, $\overrightarrow{G}_2$ be the disjoint even orientations of G arising from Proposition 3 and corresponding to the disjoint even star decompositions $\mathcal{D}_1$, $\mathcal{D}_2$, respectively. By Proposition 3, the arc-sets $D(\overrightarrow{G}_1)$, $D(\overrightarrow{G}_2)$ share only the arcs whose underlying edges belong to M. Hence, for every $D \in \mathcal{D}_1$ and $D' \in \mathcal{D}_2$, the cardinality of $E(D) \cap E(D')$ is at most 1 ($E(D) \cap E(D')$ is either empty or consists of a single edge in M). Consequently, if $D \in \mathcal{D}_1$ and $D' \in \mathcal{D}_2$ are stars with the same centre v, then by pairing the edges in D and in D' we obtain a set A and a set A' containing exactly $|E(D)|/2$ and $|E(D')|/2$ paths P_2, respectively. The sets A and A' share no path P_2, since the cardinality of $E(D) \cap E(D')$ is at most 1. Hence, $\mathcal{D}_1$ and $\mathcal{D}_2$ provide a pair of disjoint P_2-decompositions of G. □

2.2 Even 2-factors.

A P_2-decomposition of a graph G corresponds to a perfect matching in the line graph $L(G)$. More generally, a set of k paths P_2 in G, which are pairwise edge-disjoint, corresponds to a matching of cardinality k in $L(G)$. Conversely: since an edge of $L(G)$ corresponds to a pair of adjacent edges in G, a matching of cardinality k in $L(G)$ (respectively, a perfect matching in $L(G)$) corresponds to a set of k paths P_2 in G which are pairwise edge-disjoint (respectively, to a P_2-decomposition in G). We can prove the following results.

Proposition 5. *The line graph* $L(G)$ *of a graph* G *has an even* 2*-factor if and only if the graph* G *has a pair of disjoint* P_2*-decompositions.*

Proof. Assume that $L(G)$ has an even 2-factor F, then we can alternately color the edges of F and obtain two edge-disjoint perfect matchings, say M_1 and M_2, of $L(G)$. Each perfect matching M_i, $i = 1, 2$, corresponds to a P_2-decomposition of G, say $\mathcal{D}_i$. Since M_1 and M_2 are edge-disjoint, the corresponding P_2-decompositions share no path P_2, i.e., $\mathcal{D}_1$ and $\mathcal{D}_2$ are disjoint. It is easy to see that the converse is true as well. □

Proposition 6. *Let G be a connected graph with an even number of edges. Let K be the subgraph of G induced by the vertices of odd degree. If K has a perfect matching, then the line graph $L(G)$ has an even 2-factor.*

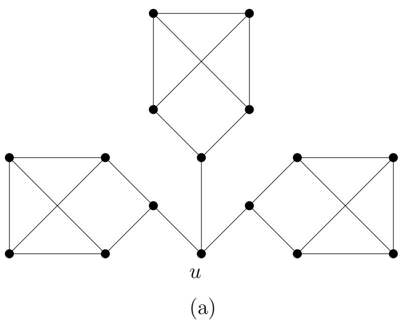
Proof. It follows from Corollary 4 and Proposition 5. □

The following property is a straightforward consequence of Proposition 6.

Proposition 7. *Let G be a connected cubic graph with an even number of edges. If G has a perfect matching, then the line graph $L(G)$ has an even 2-factor.* □

The existence of a perfect matching, in a connected graph with an even number of edges and no vertex of even degree, is a sufficient condition for the existence of an even 2-factor in the corresponding line graph. The condition is not necessary: for instance, the graph G in Figure 1(a) is a cubic graph with 24 edges and no perfect matching whose line graph has an even 2-factor. More specifically, the graph G in Figure 1(a) has 3 subgraphs, say N_1, N_2 and N_3, that contain the vertex u and are isomorphic to the graph N in Figure 1(b); the graph N has 8 edges and a perfect matching $M = \{[u, v_1], [v_2, v_3], [v_4, v_5]\}$; by Proposition 3, the graph N has a pair of disjoint P_2-decompositions; therefore, each subgraph N_i has a pair of disjoint P_2-decompositions, say $\mathcal{D}_{i,1}$ and $\mathcal{D}_{i,2}$ with $i = 1$, 2, 3; the sets $\mathcal{D}_1 = \cup_{i=1}^{3}\mathcal{D}_{i,1}$ and $\mathcal{D}_2 = \cup_{i=1}^{3}\mathcal{D}_{i,2}$ are disjoint P_2-decompositions of G; by Proposition 5, the line graph $L(G)$ has an even 2-factor.

We can also give some examples of graphs with no perfect matching whose line graph has no even 2-factor. For instance, the graph G in Figure 2 is a cubic graph with 42 edges and no perfect matching whose line graph has no even 2-factor. More specifically, every even orientation of G contains the arcs (u, v), (w, v), i.e., every P_2-decomposition of G contains the P_2-path (u, v, w); hence G cannot have a pair of disjoint P_2-decompositions; by Proposition 5, the line graph $L(G)$ cannot have an even 2-factor.



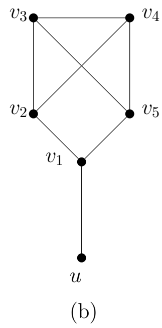


Figure 1: (a) A cubic graph with no perfect matching whose line graph has an even 2-factor; (b) the graph N.

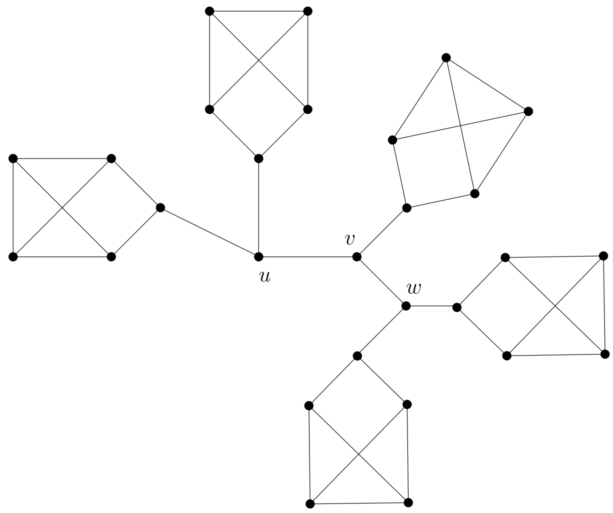


Figure 2: A cubic graph with no perfect matching whose line graph has no even 2-factor.

3 Even cycle decompositions for the line graph of a cubic graph.

The definition of disjoint even orientations of a cubic graph G can be read as follows: two even orientations of G, say $\overrightarrow{G}_1$, $\overrightarrow{G}_2$, are disjoint if for every $v \in V(G)$ with positive indegree in $\overrightarrow{G}_1$ or in $\overrightarrow{G}_2$, the pair of arcs in $D(\overrightarrow{G}_1)$ incoming in v is distinct from the pair of arcs in $D(\overrightarrow{G}_2)$ incoming in v. For our purposes, we extend the definition of disjoint even orientations to the subgraphs of a cubic graph G. More specifically, if H and K are subgraphs of a cubic graph G possessing an even orientation $\overrightarrow{H}$ and $\overrightarrow{K}$, respectively, we will say that $\overrightarrow{H}$ and $\overrightarrow{K}$ are *disjoint on the vertex* $v \in V(H) \cap V(K)$ if v has indegree 0 in at least one of the orientations $\overrightarrow{H}$, $\overrightarrow{K}$, or the pair of arcs in $D(\overrightarrow{H})$ incoming in v is distinct from the pair of arcs in $D(\overrightarrow{K})$ incoming in v.

By the one-to-one correspondence between even orientations and P_2-decompositions we can write the following statement in terms of even orientations or P_2-decompositions.

Proposition 8. *Let G be a cubic graph. The corresponding line graph $L(G)$ has an even cycle decomposition of index 3 if and only if the graph G has three subgraphs, say H_1, H_2, H_3, such that:*

(a) *each edge of G is contained in exactly two of the subgraphs H_i, $i = 1,2, 3$;*

(b) *each subgraph H_i has a pair of disjoint even orientations, say $\overrightarrow{H}_{i,1}$, $\overrightarrow{H}_{i,2}$(or, equivalently, a pair of disjoint P_2-decompositions);*

(c) *for every vertex $v \in V(H_i) \cap V(H_j)$ the even orientations $\overrightarrow{H}_{i,r}$ and $\overrightarrow{H}_{j,s}$, with $i, j = 1$, 2, 3, $i \neq j$, $r, s = 1$, 2, are disjoint on v.*

Proof. Assume that G has three subgraphs H_i, $i = 1, 2, 3$, verifying (a), (b) and (c). The pair of disjoint even orientations $\overrightarrow{H}_{i,1}$, $\overrightarrow{H}_{i,2}$ of H_i gives rise to a pair of edge-disjoint matchings $M_{i,1}$, $M_{i,2}$ of $L(G)$ whose cardinality is $|E(H_i)|/2$ and whose union is an even 2-regular subgraph D_i of $L(G)$ (D_i is an even 2-factor of $L(H_i)$). Since an edge in $M_{i,r}$, $i = 1, 2, 3$, $r = 1, 2$, corresponds to a pair of arcs of $\overrightarrow{H}_{i,r}$ incoming in a vertex v of G, assumption (c) ensures that the matchings $M_{i,r}$ are pairwise edge-disjoint. Therefore, the graph $D_1 \cup D_2 \cup D_3$ contains exactly $|E(D_1)| + |E(D_2)| + |E(D_3)| =|E(H_1)| + |E(H_2)| + |E(H_3)|$ edges of $L(G)$. Assumption (a) implies $|E(H_1)| + |E(H_2)| + |E(H_3)| = 2|E(G)|$, i.e., $D_1 \cup D_2 \cup D_3$ contains the edge-set of $L(G)$. Hence $D_1 \cup D_2 \cup D_3 = L(G)$, i.e., $L(G)$ has an even cycle decomposition of index 3.

Conversely: assume that $L(G)$ has an even cycle decomposition $\mathcal{D} = \{D_1, D_2, D_3\}$ of index 3. In each even 2-regular subgraph D_i, $i = 1, 2, 3$, we alternately color the edges of each even cycle and obtain a pair of edge-disjoint matchings, say $M_{i,1}$, $M_{i,2}$, whose cardinality is $|V(D_i)|/2$. The edges in $M_{i,j}$, with $j = 1, 2$, yield a set $\mathcal{D}_{i,j}$ containing exactly $|V(D_i)|/2$ paths P_2 of G that are pairwise edge-disjoint. The paths in $\mathcal{D}_{i,j}$ form a subgraph $H_{i,j}$ of G admitting $\mathcal{D}_{i,j}$ as a P_2-decomposition. Since the matchings $M_{i,1}$, $M_{i,2}$ have the same vertex-set, the subgraphs $H_{i,1}$ and $H_{i,2}$ of G coincide and we set $H_i = H_{i,1} = H_{i,2}$. Since $M_{i,1}$, $M_{i,2}$ are edge-disjoint, the sets $\mathcal{D}_{i,1}$ and $\mathcal{D}_{i,2}$ are disjoint P_2-decompositions of H_i, i.e., H_i has a pair of disjoint even orientations $\overrightarrow{H}_{i,1}$, $\overrightarrow{H}_{i,2}$. Hence condition (b) is verified. Since $\mathcal{D}$ is an even cycle decomposition of $L(G)$ and $L(G)$ is regular of degree 4, each vertex of $L(G)$ belongs to exactly two of the even 2-regular subgraphs D_1, D_2, D_3, hence condition (a) is verified. Moreover, the even orientations $\overrightarrow{H}_{i,r}$, with $i = 1, 2, 3$, $r = 1, 2$, verify condition (c). ☐

Corollary 9. *Let G be a class 1 cubic graph. The corresponding line graph $L(G)$ has an even cycle decomposition of index 3.*

Proof. Since G is class 1, the edge-set of G can be partitioned into three perfect matchings M_1, M_2, M_3. We set $H_1 = M_1 \cup M_2$, $H_2 = M_1 \cup M_3$ and $H_3 = M_2 \cup M_3$. The subgraphs H_i, $i = 1, 2, 3$, are even 2-regular subgraphs of G. They satisfy Proposition 8, hence the assertion follows. ☐

Lemma 10. *Let G be a connected cubic graph possessing a subgraph consisting of a cycle $C =(u_0,u_1,\ldots, u_n, u_0)$ of length $n + 1 \equiv |E(G)| \pmod 2$ with a pendant edge $[u_0, v_0]$. Let H be the subgraph of G obtained by deleting the edges $[u_i, u_{i+1}]$, with $1 \leqslant i \leqslant n - 1$, and let K be the subgraph of H induced by the vertices of degree 3 and 2 in H. If every connected component of H has an even number of edges and the connected component of K containing the vertex u_n is 2-connected, then H admits an even orientation $\overrightarrow{H}_0$ such that the vertices u_1 and u_n have the same indegree in $\overrightarrow{H}_0$.*

Proof. By Theorem 2, the graph H has an even orientation $\overrightarrow{H}$. If the vertices u_1 and u_n have the same indegree in $\overrightarrow{H}$, then the assertion follows. Consider u_1 and u_n with different indegree in $\overrightarrow{H}$, i.e., u_1 has indegree 2 whereas u_n has indegree 0 (or vice versa). Let S be the connected component of K containing the vertex u_n (it may well happen that $S = K$). The vertices u_0 and u_1 belong to S. Since S is 2-connected, the edge $[u_1, u_n]$ does not belong to $E(G)$. Moreover, there exist two internally disjoint paths of S connecting the vertices u_n and v_0, i.e., the vertices u_n and v_0 belong to a cycle C_1 of S. We can always assume that C_1 does not contain the vertex u_1, i.e., C_1 does not contain the edge $[u_0, u_1]$ (if C_1 contains the edge $[u_0, u_1]$ we can take the chord $[u_0, v_0]$ and find a cycle of S containing the vertices u_n, v_0, but not the vertex u_1). We use C_1 to construct a new even orientation $\overrightarrow{H}_0$ of H: starting from $\overrightarrow{H}$, we leave unchanged the direction on the arcs in $\overrightarrow{H}$ whose underlying edges do not belong to C_1; we reverse the direction on the arcs in $\overrightarrow{H}$ with underlying edge belonging to C_1. The new orientation $\overrightarrow{H}_0$ is even: the vertices not in C_1 do not change their set of incoming arcs; the vertices in C_1 change the direction on exactly two of their arcs, hence the number of incoming arcs is always even. Moreover, u_n and u_1 have the same indegree in $\overrightarrow{H}_0$ and the assertion follows. □

Proposition 11. *Let G be a cubic graph with an odd number of edges possessing a subgraph consisting of a cycle $C = (u_0, u_1, \ldots, u_n, u_0)$ of odd length with a pendant edge $[u_0, v_0]$. Denote by H the subgraph of G obtained by deleting the edges $[u_i, u_{i+1}]$, with $1 \leqslant i \leqslant n-1$. Denote by K_i the subgraph of H induced by the vertices of degree 3 and i, with $i = 1, 2$. If the connected components of H have an even number of edges, the connected component of K_2 containing u_n is 2-connected and the subgraphs K_1, K_2 have edge-disjoint perfect matchings M_1, M_2, respectively, then the line graph $L(G)$ has an even cycle decomposition of index 3.*

Proof. We prove that Proposition 8 is verified. We set $H_1 = H$ and denote by H_2 the graph obtained from G by deleting the vertex u_0. We denote by H_3 the graph given by the cycle $C = (u_0, u_1, \ldots, u_n, u_0)$ with the pendant edge $[u_0, v_0]$. The graphs H_1, H_2, H_3 verify condition (i) of Proposition 8. We show that they also verify condition (ii). The graph H_3 has a pair of disjoint even orientations $\overrightarrow{H}_{3,1}$ and $\overrightarrow{H}_{3,2}$ that can be defined as in Figure 3. By the assumptions, every connected component of H_1 has an even number of edges. Therefore, by Theorem 2, the graph H_1 has an even orientation $\overrightarrow{H}_{1,1}$. By Proposition 3, there exists an even orientation $\overrightarrow{H}_{1,2}$ which is disjoint from $\overrightarrow{H}_{1,1}$. Moreover, the arc-sets $D(\overrightarrow{H}_{1,1})$, $D(\overrightarrow{H}_{1,2})$ share only the arcs whose underlying edges belong to M_1. By Lemma 10, the vertices u_1, u_n have the same indegree in $\overrightarrow{H}_{1,1}$. The vertices u_1, u_2 have indegree 2 in $\overrightarrow{H}_{1,1}$ and indegree 0 in $\overrightarrow{H}_{1,2}$ (or vice versa), as $D(\overrightarrow{H}_{1,1})$, $D(\overrightarrow{H}_{1,2})$ share only the arcs whose underlying edges belong to M_1 and u_1, u_n are unmatched in M_1. Without loss of generality, we can assume that u_1, u_n have indegree 2 in $\overrightarrow{H}_{1,1}$ and indegree 0 in $\overrightarrow{H}_{1,2}$. Figure 4 shows the arcs in $D(\overrightarrow{H}_{1,1})$ and $D(\overrightarrow{H}_{1,2})$ with at least one vertex belonging to $V(C)$. Note that (u_0, v_0) is in $D(\overrightarrow{H}_{1,1})$ and $D(\overrightarrow{H}_{1,2})$. These properties will be used to define a pair of disjoint even orientations in H_2. The edge-disjoint matchings M_1 and M_2 in K_1 and K_2, respectively give rise to two edge-disjoint matchings $M_1' = M_1 - [u_0, v_0]$ and $M_2' = M_2 - [u_0, u_1]$, respectively, in H_2. The vertices v_0, u_1, $u_n \in V(H_2)$ are unmatched in M_1', whereas all other vertices of H_2 have a mate in M_1' (see the bold and dashed edges in Figure 4 and 5). Hence M_1' is a perfect matching in the subgraph of H_2 induced by the vertices of degree 3 and 1. The vertices of H_2 in $V(C) - \{u_n\}$ are unmatched in M'_2, whereas all other vertices of H_2 have a mate in M'_2. We use M'_2 and the even orientation $\overrightarrow{H}_{1,1}$ of H_1 to construct an even orientation $\overrightarrow{H}_{2,1}$ of H_2 as follows: we leave unchanged the direction on the arcs in $D(\overrightarrow{H}_{1,1})$ whose underlying edges belong to M'_2; we reverse the direction on the arcs in $D(\overrightarrow{H}_{1,1})$ whose underlying edges belong to $(H_1 \cap H_2) - M'_2$. Since we are assuming that u_1 and u_n have indegree 2 in $\overrightarrow{H}_{1,1}$, we add the arcs (u_i, u_{i+1}), with $1 \leqslant i \leqslant n-1$ (see Figure 5). One can easily verify that $\overrightarrow{H}_{2,1}$ is an even orientation of H_2. Since M'_1 is a perfect matching on the subgraph of H_2 induced by the vertices of degree 3 and 1, we can apply Proposition 3 and find an even orientation $\overrightarrow{H}_{2,2}$ of H_2 which is disjoint from $\overrightarrow{H}_{2,1}$ (see Figure 5). By the same proposition, the arc-sets $D(\overrightarrow{H}_{2,1})$, $D(\overrightarrow{H}_{2,2})$ share only the arcs whose underlying edges belong to M'_1.

We show that condition (*iii*) of Proposition 8 is verified. The orientations $\overrightarrow{H}_{1,r}$ and $\overrightarrow{H}_{3,s}$, with $r, s = 1, 2$ are disjoint on every $v \in V(H_1) \cap V(H_3)$, since $E(H_1)$, $E(H_3)$ share only the edges that are incident to u_0 (see Figure 3 and 4). From Figure 3 and 5, one can see that also the orientations $\overrightarrow{H}_{2,r}$ and $\overrightarrow{H}_{3,s}$, with $r, s = 1, 2$ are disjoint on every $v \in V(H_2) \cap V(H_3)$. Consider a vertex $v \in V(H_1) \cap V(H_2)$ of degree 3 in H_1 and H_2. The matching M_1 contains exactly one edge, say $[u, v]$, incident to v. The edge $[u, v]$ also belongs to M'_1, since $M'_1 = M_1 - [u_0, v_0]$ and $v \neq u_0, v_0$. By the construction of $\overrightarrow{H}_{2,1}$ from $\overrightarrow{H}_{1,1}$, the arcs incident to v in $\overrightarrow{H}_{1,r}$ and $\overrightarrow{H}_{2,s}$, with $r, s = 1, 2$, are oriented as in Figure 6. Hence the orientations $\overrightarrow{H}_{1,r}$ and $\overrightarrow{H}_{2,s}$ are disjoint on v. Analogously, if v is a vertex of degree different from 3 in H_1 or H_2, i.e., if $v = v_0$ or $v \in V(C) - \{u_0\}$. It is thus proved that Proposition 8 holds, hence the assertion follows. $\square$

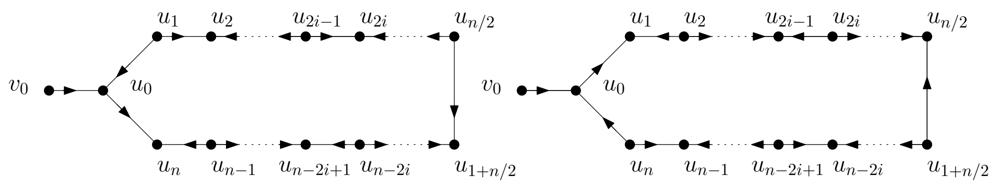


Figure 3: A pair of disjoint even orientations of the graph H_3.

As an application of Proposition 11, consider the following example.

Example 12. The Petersen graph, denoted by $\boldsymbol{GP(5,2)}$, is a class 2 cubic graph with 15 edges. We denote by $V = \{u_i, v_i : 0 \leqslant i \leqslant 4\}$ the vertex-set and by $E = \{[u_i, u_{i+1}], [u_i, v_i], [v_i, v_{i+2}] : 0 \leqslant i \leqslant 4\}$ the edge-set of $GP(5,2)$ (the subscripts are taken modulo 5). We denote by H the subgraph of $GP(5,2)$ obtained by deleting the edges $[u_i, u_{i+1}]$, with $1 \leqslant i \leqslant 3$, of the cycle $C = (u_0, u_1, \dots, u_4, u_0)$. The subgraph K_1 of H has a perfect matching $M_1 = \{[u_i, v_i] : i = 0, 2, 3\} \cup \{[v_1, v_4]\}$ (see the bold edges in Figure 7); the subgraph K_2 of H is 2-connected and has a perfect matching $M_2 = \{[u_0, u_1], [u_4, v_4], [v_0, v_2], [v_1, v_3]\}$ (see the dashed edges in Figure 7). Following the

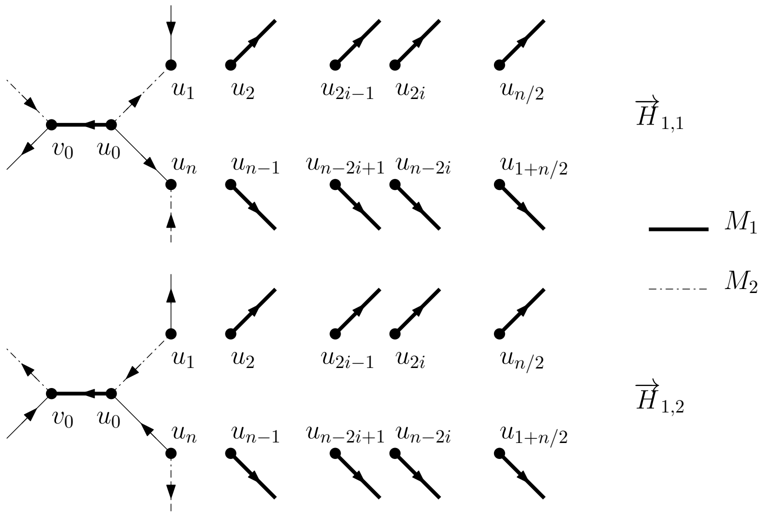


Figure 4: The arcs in the even orientations of H_1 possessing at least one vertex in $V(C)$. Note that the edge $[u_1, u_n]$ does not belong to the graph G, since the connected component of H_1 containing u_n is 2-connected. The vertex v_0 is adjacent to at most one of the vertices u_1, u_n, since each connected component of H_1 has an even number of edges. For the same reason, the cycle C contains no chord. The arcs whose underlying edges belong to M_1 (respectively, to M_2) are depicted with a bold line (respectively, with a dashed line).

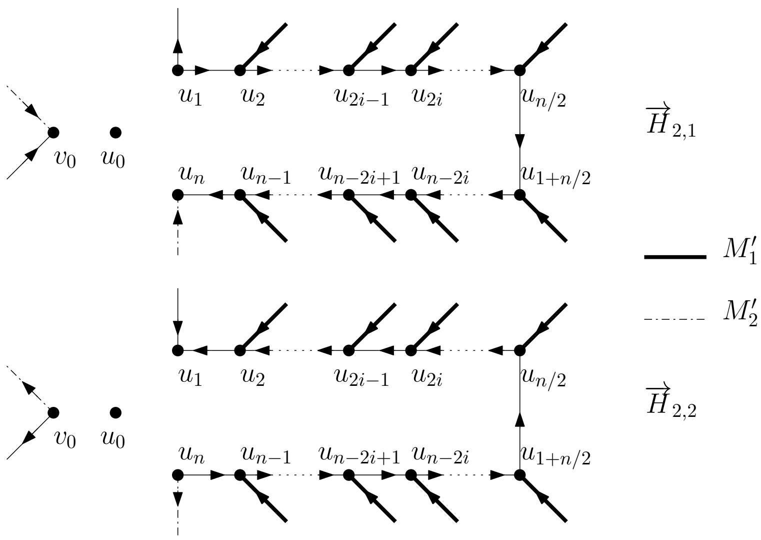


Figure 5: The pair of disjoint even orientations of H_2 defined in the proof of Proposition 11.

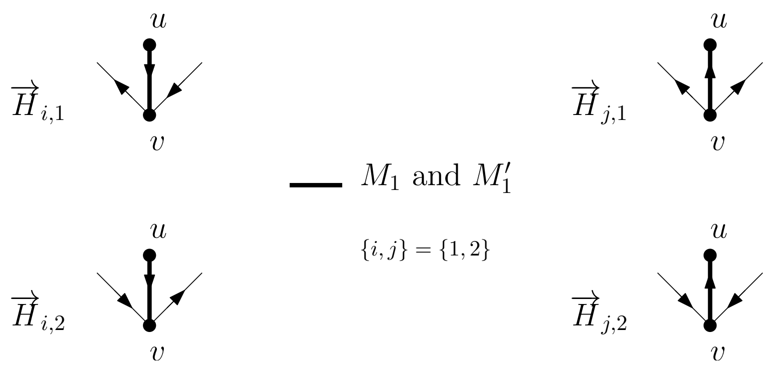


Figure 6: The incoming arcs in a vertex v of degree 3 in the subgraphs H_1 and H_2 of Proposition 11.

proof of Proposition 11, we can construct the even orientations $\overrightarrow{H}_{i,r}$ with $i = 1, 2, 3$ and $r = 1, 2$ in Figure 7. Hence the line graph of $GP(5, 2)$ has an even cycle decomposition of index 3.

Proposition 13. *Let G be a cubic graph with an even number of edges possessing a subgraph consisting of a cycle $C = (u_0, u_1, \ldots, u_n, u_0)$ of odd length with a pendant edge $[u_0, v_0]$. Let H be the subgraph of G obtained by deleting the edges $[u_0, v_0]$ and $[u_i, u_{i+1}]$ with $1 \leqslant i \leqslant n-1$. Denote by K_i the subgraph of H induced by the vertices of degree 3 and i, with $i = 1$, 2. If every connected component of H has an even number of edges and K_1, K_2 have edge-disjoint perfect matchings M_1, M_2, respectively, then the line graph $L(G)$ has an even cycle decomposition of index* 3.

Proof. We set $H_1 = H$ and denote by H_2 the subgraph of G obtained by deleting the edges $[u_0, u_1]$, $[u_0, u_n]$, by H_3 the subgraph of G consisting of the cycle C with the pendant edge $[u_0, v_0]$. The proof is similar to the proof of Proposition 11 (see also Example 14). $\square$

As an application of Proposition 13, consider the flower snark J_5.

Example 14. The flower snark J_5 is a class 2 cubic graph with 30 edges. We denote by $V = \{x_i, y_i, v_i, u_i : 0 \leqslant i \leqslant 4\}$ the vertex-set of J_5. The edge-set of J_5 can be defined as follows: the vertices x_i, y_i, v_i, u_i induce a star $K_{1,3}$ with centre v_i; the vertices $u_0, u_2, \ldots, u_4$ induce a cycle $(u_0, u_1, \ldots, u_4, u_0)$ of length 5; the vertices x_i, y_i, with $0 \leqslant i \leqslant 4$, induce a cycle $(x_0, x_1, \ldots, x_4, y_0, y_1, \ldots, y_4, x_0)$ of length 10.

We consider the subgraph of J_5 consisting of the cycle $C = (u_0, u_1, \ldots, u_4, u_0)$ with the pendant edge $[u_0, v_0]$. Following the proof of Proposition 13, we denote by H_1 the subgraph of J_5 obtained by deleting the edges $[u_0, v_0]$, $[u_i, u_{i+1}]$ with $1 \leqslant i \leqslant 3$. By Theorem 2, the graph H_1 has an even orientation $\overrightarrow{H}_{1,1}$. Since the subgraph K_1 of H_1 has a perfect matching M_1 (see the bold edges in Figure 8), we can apply Proposition 3 and find an even orientation $\overrightarrow{H}_{1,2}$ which is disjoint from $\overrightarrow{H}_{1,1}$. Moreover, the arc-sets $D(\overrightarrow{H}_{1,1})$

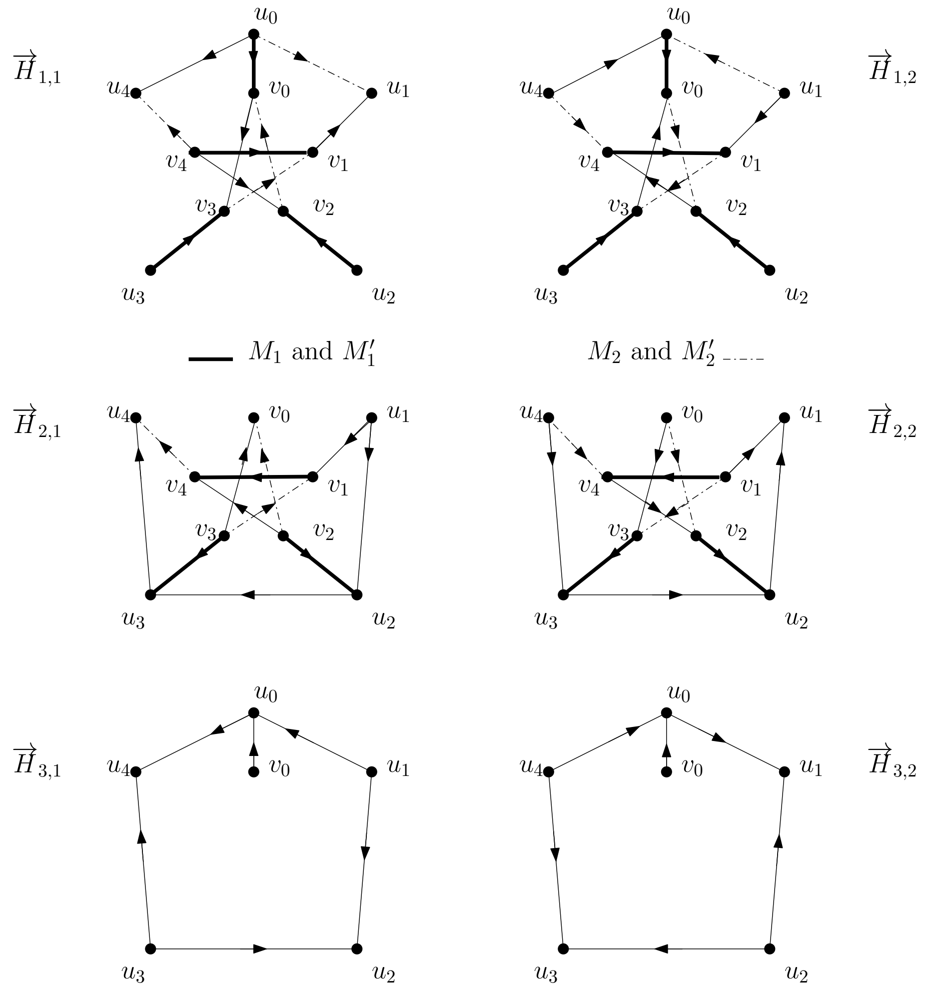


Figure 7: An application of Proposition 11 to the Petersen graph $GP(5, 2)$.

and $D(\overrightarrow{H}_{1,2})$ share only the arcs whose underlying edges belong to M_1 (see for instance Figure 8). The subgraph K_2 of H_1 has a perfect matching M_2 which is edge-disjoint from M_1 (see the dashed edges in Figure 8). The matchings M_1 and M_2 give rise to the matchings $M_1' = M_1 \cup \{[u_0, v_0]\}$ and $M_2' = M_2 - [u_0, u_1]$ of the subgraph H_2 obtained from J_5 by deleting the edges $[u_0, u_1]$, $[u_0, u_4]$. The even orientation $\overrightarrow{H}_{1,1}$ and M_2' can be used to construct an even orientation $\overrightarrow{H}_{2,1}$ of H_2 as follows: we leave unchanged the direction on the arcs of $\overrightarrow{H}_{1,1}$ whose underlying edges belong to M_2'; we reverse the direction on the arcs of $\overrightarrow{H}_{1,1}$ whose underlying edges belong to $H_2 - M_2'$; we add the arcs (u_0, v_0), (u_4, u_3), (u_3, u_2), (u_2, u_1). Since M_1' is a perfect matching of the subgraph of H_2 induced by the vertices of degree 3 and 1 in H_2, we can apply Proposition 3 and find an even orientation $\overrightarrow{H}_{2,2}$ of H_2 which is disjoint form $\overrightarrow{H}_{2,1}$; moreover, the arc-sets $D(\overrightarrow{H}_{2,1})$ and $D(\overrightarrow{H}_{2,2})$ share only the arcs whose underlying edges belong to M_1' (see Figure 9). Finally, we denote by H_3 the subgraph of J_5 consisting of the cycle $C = (u_0,\ u_1, \ldots, u_4, u_0)$ with the pendant edge $[u_0, v_0]$; H_3 has a pair of disjoint even orientations (see for instance Figure 7). The subgraphs H_1, H_2, H_3 verify Proposition 8. Hence the line graph of J_5 has an even cycle decomposition of index 3.

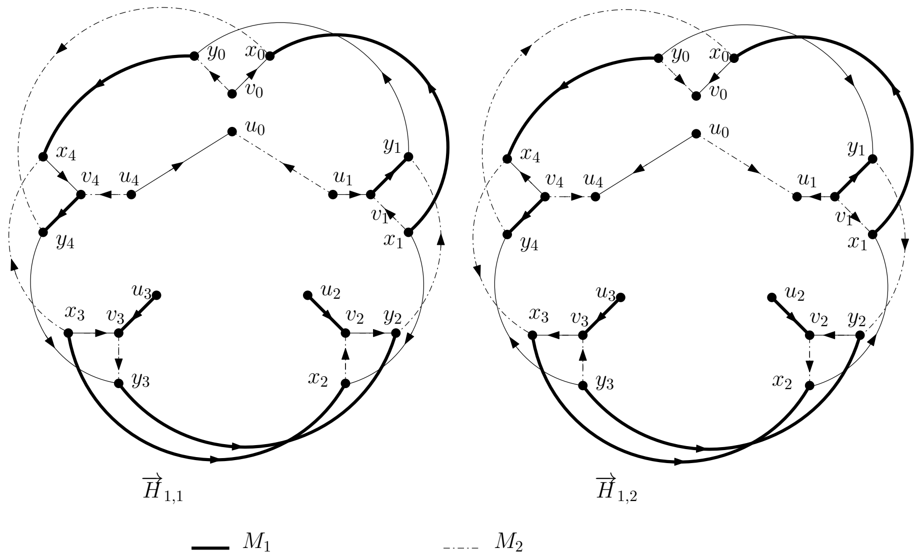


Figure 8: A pair of disjoint even orientations of the subgraph H_1 in J_5.

There exist graphs with an even number of edges that do not satisfy the condition in Proposition 13.

Example 15. The Zamfirescu snark [23] is a class 2 cubic graph with 36 vertices and 54 edges. The reader can verify that this snark has no cycle C satisfying Proposition 13. Nevertheless, it has an even cycle decomposition of index 3, since it satisfies Proposition 8. More specifically, from Figure 10 one can see that each edge of the Zamfirescu snark is contained in exactly two of the three subgraphs H_1, H_2, H_3. Each subgraph H_i, $i = 1$, 2, 3, has a pair of disjoint even orientations $\overrightarrow{H}_{i,1}$, $\overrightarrow{H}_{i,2}$, since it satisfies Proposition 3. A direct inspection of Figure 10 shows that for every $v \in V(H_i) \cap V(H_j)$, with $i, j = 1, 2, 3$, $i \neq j$, the even orientations $\overrightarrow{H}_{i,r}$, $\overrightarrow{H}_{i,s}$, with $r, s = 1, 2$, are disjoint on v.

4 The line graph of a snark.

Does a class 2 cubic graph always satisfy Proposition 8? A direct answer to this question seems far from obvious, so it is rather natural to test the situation in some known classes of such graphs. In the previous section we have already seen three examples for which the answer is affirmative.

To our knowledge, the earliest class 2 cubic graphs that have been discovered are: the Petersen graph [16], the first and second Blanuša snark [2], the Descartes snark [6] and

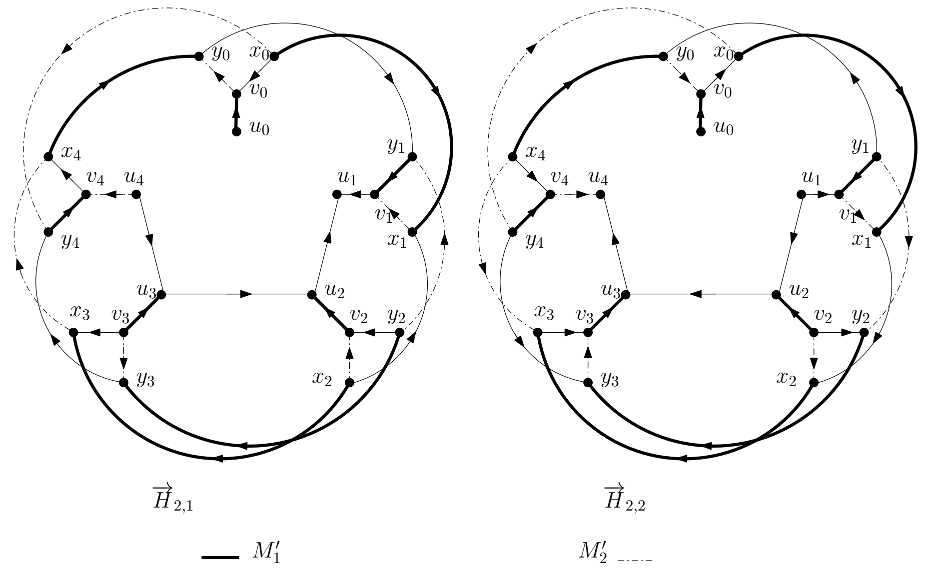


Figure 9: A pair of disjoint even orientations of the subgraph H_2 in J_5.

the Szekeres snark [18]. In [10] Isaacs found new class 2 cubic graphs: the BDS class, the flower snarks and the double star. The snarks of Blanuša, Descartes and Szekeres inspired the construction of the BDS class and are contained in it. The BDS class also contains other snarks that were constructed by other authors. See for instance the Celmins-Swart snarks [5] and the Watkins snark [21]. Watkins also generlized the construction of the two Blanuša snarks [22]. Further known families of class 2 cubic graphs are the two families of snarks found by Loupekine and described by Isaacs in [11] and the Goldberg snarks [8].

The sufficient conditions in Proposition 11 and 13 are satisfied for the following graphs: the double star, the generalized Blanuša snarks, the flower snarks, the Goldberg snarks and for the snarks of Szekeres, Descartes, Celmins-Swart, Watkins, Loupekine LP_0 [19]. Hence, the line graphs of these snarks have an even cycle decomposition of index 3. We checked the same sufficient conditions on some class 2 cubic graphs on 12, 14, 28 and 30, respectively (they are usually considered to be "trivial" snarks since their girth is < 5). Some of them do not verify these sufficient conditions, but verify the necessary and sufficient condition in Proposition 8. Therefore, also in this case we can find an even cycle decomposition of index 3 in the corresponding line graph. We give a detailed proof of the existence of an even cycle decomposition of index 3 for the generalized Blanuša snarks and for the Goldberg snarks. For the flower snarks, one can generalize the construction in Example 14. For the remaining cases we prefer to omit the proof.

The generalized Blanuša snarks are divided into two families: the generalized Blanuša snarks of type 1 and those of type 2. A generalized Blanuša snark of type 1, denoted by B_n^1, is constructed as follows: consider $n-1$ copies $B_1, B_2, \ldots, B_{n-1}$ of the block B in Figure 11 and exactly one copy of the graph A_1; label the vertices a, a', b, b' of each copy B_i by a_i, a'_i, b_i, b'_i, respectively; construct the edges $[a'_1, a]$, $[b'_1, b]$, $[a', a_{n-1}]$, $[b', b_{n-1}]$,

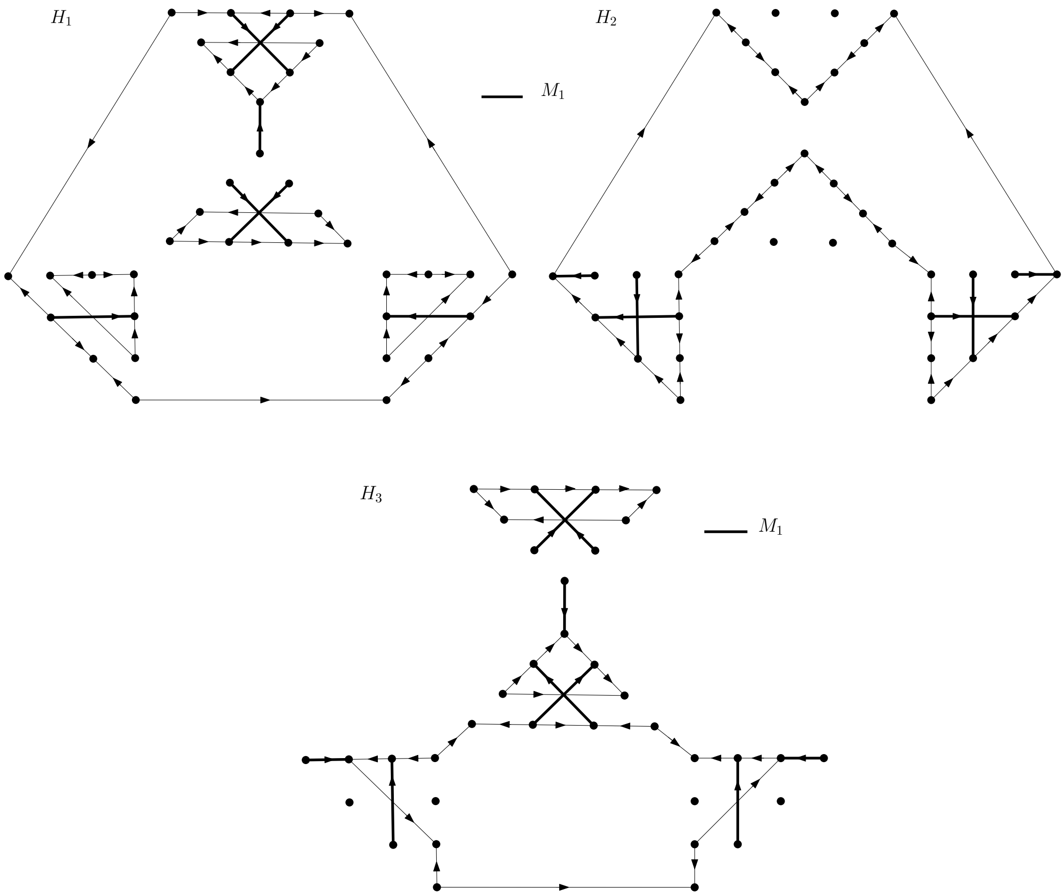


Figure 10: The subgraphs H_1, H_2, H_3 of the Zamfirescu snark; in each H_i, $i = 1, 2, 3$, the subgraph K induced by the vertices of odd degree has a perfect matching M_1 (bold edges), hence each H_i satisfies Proposition 3.

$[a_i, a'_{i+1}]$, $[b_i, b'_{i+1}]$ with $1 \leqslant i \leqslant n-2$. In the construction of a Blanuša snark of type 2, denoted by B_n^2, the block A_1 is replaced by the block A_2. The generalized Blanuša snarks B_2^1 and B_2^2 are known as the first Blanuša snark and the second Blanuša snark, respectively. A generalized Blanuša snark has $3(4n+1)$ edges.

Proposition 16. *The line graph of a generalized Blanuša snark has an even cycle decomposition of index* 3*.*

Proof. The proof is based on Proposition 11. The block B in Figure 11 has a cycle $C = (a, y, x, b, z, a)$ of length 5 with a pendant edge. We delete the edges $[a, y]$, $[x, y]$, $[x, b]$ in the block B_1 of each generalized Blanuša snark and obtain a subgraph H of B_n^j, $j = 1, 2$. The subgraphs K_1 and K_2 of H have a perfect matching M_1 and M_2, respectively, that can be defined as in Figure 12 (bold edges belong to M_1, dashed edges belong to M_2) and are edge-disjoint. The graph H is connected, the subgraph K_2 is 2-connected. We can apply Proposition 11 and find an even cycle decomposition of index

3 in the corresponding line graph. □

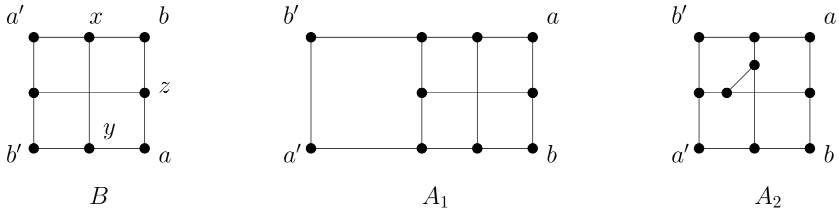


Figure 11: The basic blocks in the construction of the Blanuša snarks.

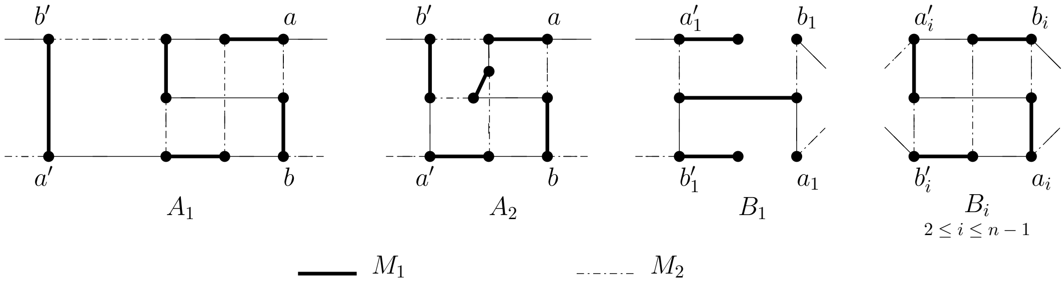
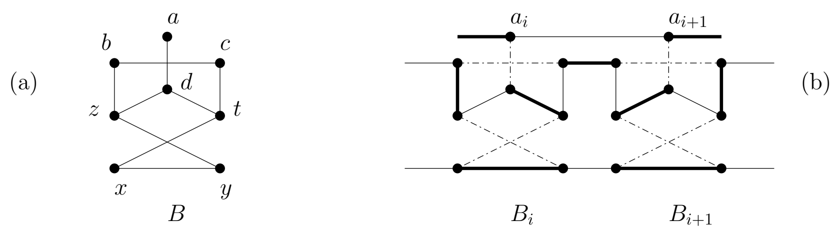


Figure 12: The matchings M_1 and M_2 in the subgraph H of a generalized Blanuša snark.

A Goldberg snark G_n is a cubic graph with $12n$ edges and can be defined as follows: consider and odd integer $n \geqslant 5$ and n copies B_1, $B_2, \ldots, B_n$, of the block B in Figure 13(a); label the vertices a, b, c, d, x, y, z, t of each copy B_i by a_i, b_i, c_i, d_i, x_i, y_i, z_i, t_i, respectively; add the edges $[a_i, a_{i+1}]$, $[c_i, b_{i+1}]$, $[y_i, x_{i+1}]$ with $1 \leqslant i \leqslant n$ (the subscripts are taken modulo n). We have the following property.

Proposition 17. *The line graph of the Goldberg snark G_n has an even cycle decomposition of index* 3*.*

Proof. We show that Proposition 13 holds. The block B in Figure 13(a) has a subgraph given by a cycle $C = (x, y, z, d, t, x)$ of length 5 and the pendant edge $[a, d]$. We denote by H the subgraph of G_n obtained by deleting the edges $[a, d]$, $[z, y]$, $[y, x]$, $[x, t]$ of the block B_1. The subgraphs K_1 and K_2 of H have a perfect matching M_1 and M_2, respectively, that are defined as follows: for $n \geqslant 5$, color the edges of the blocks B_1, B_2, B_{n-2}, B_{n-1}, B_n as in Figure 13(c); for $n > 5$ and for every odd index i, $3 \leqslant i \leqslant n-4$, color the edges of the blocks B_i, B_{i+1} as in Figure 13(b). The matchings M_1 and M_2 are edge-disjoint, hence the result follows from Proposition 13. □



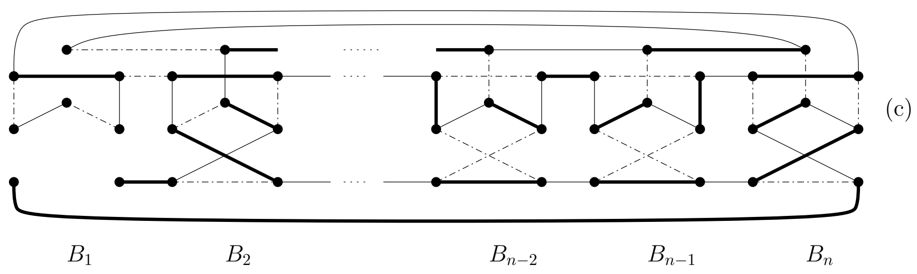


Figure 13: The definition of the matchings M_1 and M_2 in the subgraph H of a Goldberg snark.

5 Final remarks.

The existence of a 2-connected 4-regular graph possessing only even cycle decompositions of index larger than 3 remains an open problem. Futhermore, in view of the results obtained for 4-regular line graphs, we can add another related problem, namely, the existence of a 2-connected cubic graph that does not verify the necessary and sufficient condition in Proposition 8. The line graph of such a graph would admit only even cycle decompositions of index larger than 3 or no even cycle decomposition at all. As remarked in Section 1, our results seem to confirm a conjecture in [15] stating that a 4-regular graph on an odd number of vertices asymptotically almost surely has an even cycle decomposition of index 3. Moreover, our results hold for graphs on an even number of vertices as well. In [15], it is also conjectured that the edge-set of the line graph of a 2-connected cubic graph decomposes into cycles of even length. Finding a 2-connected cubic graph not fulfilling Proposition 8 appears thus to be a hard problem.

As remarked in Section 1, the existence of an even cycle decomposition of index 3 in a 4-regular graph of class 2 is a particular case of the more general problem about the existence of an even cycle decomposition of minimum index m in a $2d$-regular graph of class 2 for which the inequality $m \geqslant d+1$ holds. We note that the necessary and sufficient condition in Proposition 8 can be generalized to $(2d+1)$-regular graphs by taking $3d \geqslant 3$ subgraphs H_i, $1 \leqslant i \leqslant 3d$, verifying the three conditions in Proposition 8. This generalization provides an even cycle decomposition of index $3d$ for the corresponding $4d$-regular line graph. Therefore, the minimum value m for the index of an even cycle decomposition of a $4d$-regular line graph satisfies the inequalities $2d+1 \leqslant m \leqslant 3d$.

For 4-regular graphs of class 2, the minimum value m for the index of an even cycle decomposition is 3 and Proposition 8 describes how to find 4-regular graphs possessing an even cycle decomposition whose index is as small as possible. For $2d$-regular graphs of class 2, $d > 2$, a generalization of Proposition 8 (which would be quite straightforward) would not yield a description of how to find $2d$-regular graphs possessing an even cycle decomposition whose index is as small as possible. It would only give an upper bound for the minimum value m in the case of $4d$-regular line graphs.

References

[1] L. W. Beineke. Characterization of derived graphs. *J. Combin. Theory Ser. B*, 9:129–135, 1970.

[2] D. Blanuša. Problem ceteriju boja (The problem of four colors). *Hrvatsko Prirodoslovno Društvo Glasnik Mat.-Fiz. Astr. Ser. II*, 1:31–42, 1946.

[3] J. A. Bondy, and U. S. R. Murty. *Graph Theory.* Springer, 2008.

[4] S. Bonvicini, and G. Mazzuoccolo. Edge-colorings of 4-regular graphs with the minimum number of palettes. *Graphs Combin.*, 32:1293–1311, 2016.

[5] U. A. Celmins, and E. R. Swart. The construction of snarks. Research Report CORR 79-18, Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada.

[6] B. Descartes. Network-colourings. *Math. Gazette*, 32:67–69, 1948.

[7] S. Fiorini, and R. J. Wilson. *Edge-colorings of graphs.* Research Notes in Mathematics, no. 16. Pitman, London, 1977.

[8] M. K. Goldberg. Construction of class 2 graphs with maximum vertex degree 3. *J. Combin. Theory Ser. B*, 31:282–291, 1981.

[9] M. Horňák, R. Kalinowski, M. Meszka, and M. Woźniak. Minimum number of palettes in edge colorings. *Graphs Combin.*, 30:619–626, 2014.

[10] R. Isaacs. Infinite families of nontrivial graphs which are not Tait colorable. *Amer. Math. Monthly*, 82:221–239, 1975.

[11] R. Isaacs. Loupekine's snarks: a bifamily on non-Tait-colorable graphs. Technical Report 263, Department of Mathematical Sciences, The Johns Hopkins University, 1976.

[12] B. Jackson. On circuit covers, circuit decompositions and Euler tours of graphs. In *Surveys in Combinatorics*, volume 187 of *London Math. Soc. Lecture Note Series*, pages 191-210. Cambridge University Press, 1993.

[13] F. Jaeger. Sur l'indice chromatique du graphe représentatif des arétes d'un graphe regulier. *Discrete Math.*, 9:161–172, 1974.

[14] A. Kotzig. From the theory of finite regular graphs of degree three and four. *Časopis Pěst. Mat.*, 82:76–92, 1957, (in Czech).

[15] K. Markström. Even cycle decompositions of 4-regular graphs and line graphs. *Discrete Math.*, 312:2676–2681, 2012.

[16] J. Petersen. Die Theorie der regulären Graphen. *Acta Math.*, 15:193–220, 1891.

[17] P. D. Seymour. Even circuits in planar graphs. *J. Combin. Theory Ser. B*, 31:327–338, 1982.

[18] G. Szekeres. Polyhedral decompositions of cubic graphs. *Bull. Austral. Math. Soc.*, 8:367–387, 1973.

[19] L. Vaux. Théorie des graphes: flots circulaire. Technical report, ÉNS Lyon, Université Lyon 1, 2002.

[20] O. Veblen. An application of modular equations in analysis situs. *Ann. Math.*, 14:86–94, 1912.

[21] J. J. Watkins. On the construction of snarks. *Ars Combin.*, 16-B:111–123, 1983.

[22] J. J. Watkins. Snarks. *Ann. N. Y. Acad. Sci.*, 576:606–622, 1989.

[23] T. Zamfirescu. On longest paths and circuits in graphs. *Math. Scand.*, 38:211–239, 1976.

[24] C. Q. Zhang. On even circuit decompositions of Eulerian graphs. *J. Graph Theory*, 18:51–57, 1994.