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Quasi-adiabatic approximation for thermoelastic surface waves in orthorhombic solids

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Abstract

An asymptotic model for time-harmonic motion in fully-coupled linear thermoelastic orthorhombic materials is presented. The asymptotic approach takes advantage of the observation that the parameter expressing departure from the purely adiabatic regime is extremely small in practice. Consequently, the leading order bulk response turns out to be non-dissipative, and is governed by the usual equations of elastodynamics with adiabatic material constants. In the case of isothermal stress-free boundary conditions, it is shown that thermoelastic interaction is dominated by a thermoelastic boundary layer. Hence, effective boundary conditions may be constructed, which duly account for the influence of this boundary layer and successfully describe dispersion and dissipation of surface waves to leading order. As an illustration, in the special case of an isotropic half-space with free isothermal boundary conditions, we recover the asymptotic results by Chadwick and Windle (*Proc. R. Soc. Lond. A,* vol. 280, pp. 47–71, 1964). Numerical

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comparison of the dispersion curves for surface waves in an orthorhombic half-space shows excellent agreement between the exact fully-coupled thermoelastic problem and the corresponding quasi-adiabatic approximation, even for relatively large wavenumbers.

1. Introduction

Since the pioneering work of Danilovskaya (1950), it is known that disregard of coupling between strain and temperature fields may lead to significant inaccuracy in the prediction of the dynamic response of thermoelastic solids. Hence, after Biot (1956) provided the first consistent formulation of coupled thermoelasticity, substantial research effort has been invested to study dynamics of coupled thermoelastic media. Chadwick and Sneddon (1958) and Chadwick (1960) studied the basic properties of body waves in a thermoelastic isotropic body. Further extensions to anisotropy and pre-stress were discussed by Green (1962); Chadwick (1979); Scott (1989); Shuvalov and Chadwick (1997) and many others.

Surface waves (SWs), since their discovery by Rayleigh (1885), attracted continued interest of researchers, so it is not surprising that first studies on SWs in thermoelastic bodies appeared shortly after Biot's paper was published. Lockett (1958) showed that, to leading order, waves on the surface of a thermoelastic isotropic half-space are identical to classical Rayleigh waves appearing in an isotropic half-space, provided that adiabatic material constants are taken. Further studies by Chadwick (1960); Deresiewicz

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(1961) and, especially, Chadwick and Windle (1964), provided a more detailed characterization of thermoelastic SWs, which were shown to be both dispersive and dissipative. Various generalizations of the basic isotropic problem were also considered: Ivanov and Savova (2009) studied thermoviscoelastic SWs, Chiriţă (2013b) considered thermoelastic SWs on an exponentially graded half-space; Chakraborty and Pal (1969) studied influence of transverse anisotropy on two-dimensional thermoelastic SWs. Recently, Chiriţă (2013a) constructed the characteristic equation for a general anisotropic thermoelastic half-space by adopting the Stroh formalism.

All the afore-mentioned papers set off from the explicit secular equation for SWs, which they proceed to analyse either numerically or by using analytical tools, e.g. power series expansions. However, the problem's involved structure and lack of symmetry, even in the simplest case of isotropic thermoelasticity, entails that direct approaches are fairly complicated and lend little insight into the physics of the solution. Moreover, involved analyses are difficult to check and prone to errors: as an example, in Chadwick and Windle (1964) multiple solution branches are stated to exist, implying non-uniqueness of SWs in a thermoelastic half-space. Yet, uniqueness has been later established by Atkin and Chadwick (1981). Still, this false claim led other researchers to seek for other examples of non-uniqueness of SWs in dissipative media. For instance, Currie et al. (1977) described similar non-unique solutions for a viscoelastic half-space and again, only later, Romeo (2001) presented a proof for uniqueness of SWs in viscoelastic media, which was further generalized to the pre-stressed case by Destrade et al. (2009).

An active area of current research is devoted to strictly hyperbolic vari-

ants of thermoelasticity, as opposed to the mixed hyperbolic/parabolic equations formulated by Biot (1956), see, e.g., Chandrasekharaiah (1986, 1998). Many SW solutions have been constructed over the years for different formulations of hyperbolic thermoelasticity, e.g. Nayfeh and Nemat-Nasser (1971) for an isotropic half-space with the Lord-Shulman model, Singh and Sharma (1985) for a transversely isotropic half-space again with the Lord-Shulman model and Chandrasekharaiah (1997) for an isotropic half-space with the Green-Naghdi model. Abd-alla and Al-dawy (2001) compared SWs using classical, Lord-Shulman and Green-Lindsay models, Sharma et al. (2008) considered SWs in a piezothermoelastic rotating half-space modelled with the Lord-Shulman and Green-Lindsay models and Biswas et al. (2017) in an orthorhombic half-space using the three-phase-lag model. Interest in hyperbolic thermoelasticity is usually justified by referring to the existence of the so-called "second sound", which is an additional body wave detected in thermoelastic media at extremely low temperatures. Since accepted experimental evidence for the second sound at room temperature is currently lacking, there exits neither an accepted formulation for hyperbolic thermoelasticity, nor a set of material parameters describing a real hyperbolic thermoelastic solid. Thus, in this paper, we restrict attention to the classical hyperbolic/parabolic equations of coupled thermoelasticity given by Biot (1956), esp. in light of discussions by Fichera (1992) and Auriault (2014).

Motivation for this work lies in the desire to provide some physical insight into the dissipation mechanisms for Non-Destructive Testing (NDT), signal processing and bandpass filters applications in real devices and materials (Morgan, 2007). Modern SW devices routinely operate in the gigahertz

range. Indeed, a typical velocity figure for SWs in crystals is in the order of $3000\,\frac{\text{m}}{\text{s}}$, which amounts to operating with wavelengths scaling hundreds of nanometers. In such devices, dissipation is typically dominated by interactions with the thermal phonons (Morgan, 2007, Section 4.2). Indeed, it is possible to identify two specific physical mechanisms responsible for dissipation: on the one hand, thermoelastic coupling and on the other, inherent phonon viscosity. As an illustration, the second mechanism appears to govern propagation of bulk waves in quartz, see Maris (1969).

Importantly, dissipation caused by thermoelastic coupling is not necessarily induced by damping of the bulk waves. Instead, the dissipation can be dominated by interaction with the thermoelastic boundary layer, which is particularly relevant for waves localized in the vicinity of material boundaries. In the context of SWs, Mayer (1990) showed that thermoelastic dissipation strongly depends on the thermal boundary condition: indeed, free boundary with prescribed heat flux produced almost negligible dissipation, whereas isothermal free boundary induced dissipation an order of magnitude larger. The solution given by Mayer (1990) relies upon the projection method developed by Tiersten and Sinha (1978). In this paper, we describe an alternative, more explicit, asymptotic method capable of modelling thermoelastic dissipation of SWs in orthorhombic materials in the case of isothermal boundaries.

More specifically, we perform dimensional analysis of the steady-state equations of thermoelasticity and identify the natural small parameters appearing in the problem (Sec.2). This allows to highlight the relevant asymptotic regimes and to show that motion is usually dominated by adiabatic interaction between the elastic and the thermal fields. Within this limit, the

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leading order regular expansion turns out to be formally equivalent to classical orthorhombic elasticity, although adiabatic parameters replace isothermal (Sec.3). Boundary layer solutions are also derived and used to construct effective boundary conditions (Sec.3.2). The latter, coupled with classical elasticity, provide a practical and efficient "quasi-adiabatic" model that incorporates thermoelastic effects without delving into the intricacies of the fully coupled problem. They also offer precious insight into the physics behind dissipation. Indeed, it is shown that the boundary layer near an isothermal boundary dominates thermoelastic dissipation and governs both dispersion and dissipation of SWs (Sec.4). The validity of our "quasi-adiabatic" model is confirmed by comparing the asymptotic dispersion curves versus the dispersion curves of the fully coupled problem (Sec.5).

2. Governing equations

The following summary of the equations of linear anisotropic thermoelasticity is mostly based upon the exposition in Nowacki (1977). We consider a rectangular Cartesian coordinate system $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ such that wave propagation occurs along $\bar{x}_1$ and the free surface is located at $\bar{x}_2 = 0$. For a homogeneous linear anisotropic thermoelastic solid, the stress components are given by the Duhamel-Neumann law (Nowacki, 1977, Eq.(2.11))

$$\bar{\sigma}_{ij} = \bar{c}_{ijkl}e_{kl} - \bar{\beta}_{ij}\bar{\theta}, \qquad i, j, k, l = 1, 2, 3, \tag{1}$$

where $\boldsymbol{e}$ and $\bar{\boldsymbol{\beta}}$ are, respectively, the rank-2 tensors of strain and thermal moduli (or temperature coefficients of stress), $\bar{\theta} = \bar{T} - \bar{T}_0$ is the local increment from the ambient temperature $\bar{T}_0$ and $\bar{\boldsymbol{c}}$ is the rank-4 isothermal stiffness tensor, whose components are denoted by $\bar{c}_{ijkl}$. Hereinafter, an overbar is used to

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denote dimensional quantities, while summation over twice repeated suffices is implied. The linear strain tensor $\boldsymbol{e}$ is defined in terms of the displacement vector field $\bar{\boldsymbol{u}}$ by the kinematical relations (Nowacki, 1977, Eq.(2.1))

$$e_{ij} = \frac{1}{2}(\bar{u}_{i,j} + \bar{u}_{j,i}), \tag{2}$$

where a comma indicates differentiation with respect to the implied spatial coordinate, i.e. $\bar{u}_{i,j} = \partial \bar{u}_i / \partial \bar{x}_j$. Motion of anisotropic linear thermoelastic solids is governed by conservation of linear momentum (Nowacki, 1977, Eq.(2.6))

$$\bar{\sigma}_{ij,j} = \bar{\rho} \ddot{\bar{u}}_i \,, \tag{3a}$$

where $\bar{\rho}$ denotes the material mass density, a superimposed dot denotes time differentiation and we assume absence of body forces. The remaining energy equation can be constructed by combining the Fourier law of heat conduction (Nowacki, 1977, Eq.(2.19))

$$-\bar{\mathsf{q}}_i = \bar{\kappa}_{ij}\bar{\theta}_{,j} \,, \tag{3b}$$

with the following identity motivated by thermodynamical considerations (Nowacki, 1977, Eqs.(2.18), (2.20)):

$$-\bar{\mathsf{q}}_{i,i} = \bar{T}_0 \bar{\beta}_{ij} \dot{e}_{ij} + \bar{\rho}\bar{C}_v \dot{\bar{\theta}}, \tag{3c}$$

where $\bar{\mathsf{q}}$ is the heat flux vector, $\bar{C}_v$ the specific heat at constant strain and $\bar{\kappa}_{ij}$ are the components of the thermal conductivity tensor $\bar{\boldsymbol{\kappa}}$.

To perform dimensional analysis, the tensor of thermal moduli, $\bar{\boldsymbol{\beta}}$, will be defined as the product of a typical (dimensional) temperature coefficient of stress, $\bar{\beta}$, and the non-dimensional tensor

$$\beta_{ij} = \bar{c}_{ijkl}\bar{a}_{kl}/\bar{\beta} \,, \tag{4}$$

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where $\bar{\boldsymbol{a}}$ is the tensor of linear expansion coefficients. Likewise, we will write $\bar{\kappa}_{ij} = \bar{\kappa}\kappa_{ij}$ for the heat conduction coefficients, where $\bar{\kappa}$ is a typical (dimensional) value for thermal conductivity. This representation will enable us to formulate a transparent non-dimensional form of the equations.

This paper is devoted to the study of two-dimensional surface waves propagating in the symmetry plane $(O, \bar{x}_1, \bar{x}_2)$ of an orthorhombic thermoelastic material (sagittal plane). In this case, both the tensor of linear expansion coefficients $\bar{\boldsymbol{a}}$ and the tensor of thermal conductivities $\bar{\kappa}\boldsymbol{\kappa}$ become diagonal (Hahn and Özişik, 2012, Chapter 15), and so does the tensor of temperature coefficients of stress $\bar{\beta}\boldsymbol{\beta}$. Also, harmonic behaviour is assumed in time, the common factor $\exp(-\mathrm{i}\bar{\omega}\bar{t})$ arising from it being omitted throughout. After specializing to the symmetry plane and the material symmetry, and after introducing the Voigt notation for components of the stiffness tensor, we can write equations (3) in explicit form

$$\bar{c}_{11}\bar{u}_{1,11} + \bar{c}_{66}\bar{u}_{1,22} + (\bar{c}_{12} + \bar{c}_{66})\bar{u}_{2,12} - \bar{\beta}\beta_{11}\bar{\theta}_{,1} = -\bar{\rho}\bar{\omega}^2\bar{u}_1 \,, \tag{5a}$$

$$\bar{c}_{66}\bar{u}_{2,11} + \bar{c}_{22}\bar{u}_{2,22} + (\bar{c}_{12} + \bar{c}_{66})\bar{u}_{1,12} - \bar{\beta}\beta_{22}\bar{\theta}_{,2} = -\bar{\rho}\bar{\omega}^2\bar{u}_2 \,, \tag{5b}$$

$$\bar{\kappa}\kappa_{11}\bar{\theta}_{,11} + \bar{\kappa}\kappa_{22}\bar{\theta}_{,22} + \mathrm{i}\bar{\omega}(\bar{T}_0\bar{\beta}\beta_{11}\bar{u}_{1,1} + \bar{T}_0\bar{\beta}\beta_{22}\bar{u}_{2,2} + \bar{\rho}\bar{C}_v\bar{\theta}) = 0 \,. \tag{5c}$$

It is worth noting that the positive definiteness of the strain energy necessitates that

$$\bar{c}_{11}, \bar{c}_{22}, \bar{c}_{66} > 0 \quad \text{and} \quad \bar{c}_{11}\bar{c}_{22} - \bar{c}_{12}^2 > 0 \,, \tag{6}$$

see Nobili and Prikazchikov (2018).

Equations (5) must be solved in conjunction with the appropriate mechanical and thermal boundary conditions on the free surface $\bar{x}_2 = 0$. The

vanishing of tangential and normal stress is achieved when

$$\bar{u}_{1,2} + \bar{u}_{2,1} = 0\,, \quad \text{at} \quad \bar{x}_2 = 0\,, \tag{7a}$$

$$\bar{c}_{12}\bar{u}_{1,1} + \bar{c}_{22}\bar{u}_{2,2} - \bar{\beta}\beta_{22}\bar{\theta} = 0\,, \quad \text{at} \quad \bar{x}_2 = 0\,. \tag{7b}$$

Lastly, we enforce an isothermal boundary

$$\bar{\theta} = 0\,, \quad \text{at} \quad \bar{x}_2 = 0\,; \tag{7c}$$

the physical significance, limitations and reasoning behind this specific boundary condition being addressed later in the paper.

Given that spotlight is set on how elastic waves are affected by coupling with the thermal field, we introduce the characteristic length $\bar{h}$, which is the typical wavelength of the elastic disturbance. From this, we move to the non-dimensional form of equations (5) and of the boundary conditions (7), which brings out the natural small parameters that govern the problem. More specifically, we use $\bar{h}$ as our unit of length and $\bar{\mathfrak{T}} = \bar{h}/\sqrt{\bar{c}_{66}/\bar{\rho}}$ as our unit of time and introduce dimensionless quantities accordingly

$$x_i = \bar{h}^{-1}\bar{x}_i\,, \quad \bar{u}_i = \bar{h}^{-1}\bar{u}_i\,, \quad \omega = \bar{\mathfrak{T}}\bar{\omega}\,, \quad \theta = \frac{\bar{\rho}\bar{C}_v}{\bar{\beta}\bar{T}_0}\bar{\theta}\,. \tag{8}$$

This re-scaling transforms equations (5) into non-dimensional form

$$\frac{\bar{c}_{11}}{\bar{c}_{66}}u_{1,11} + u_{1,22} + \left(\frac{\bar{c}_{12}}{\bar{c}_{66}} + 1\right)u_{2,12} - \varepsilon\beta_{11}\theta_{,1} = -\omega^2 u_1\,, \tag{9a}$$

$$u_{2,11} + \frac{\bar{c}_{22}}{\bar{c}_{66}}u_{2,22} + \left(\frac{\bar{c}_{12}}{\bar{c}_{66}} + 1\right)u_{1,12} - \varepsilon\beta_{22}\theta_{,2} = -\omega^2 u_2\,, \tag{9b}$$

$$\alpha(\kappa_{11}\theta_{,11} + \kappa_{22}\theta_{,22}) + \mathrm{i}\omega(\beta_{11}u_{1,1} + \beta_{22}u_{2,2} + \theta) = 0\,, \tag{9c}$$

within which

$$\varepsilon = \frac{\bar{\beta}^2\bar{T}_0}{\bar{\rho}\bar{C}_v\bar{c}_{66}} \quad \text{and} \quad \alpha = \bar{\ell}/\bar{h}, \quad \text{where} \quad \bar{\ell} = \bar{\kappa}/(\bar{C}_v\sqrt{\bar{\rho}\bar{c}_{66}})\,. \tag{10}$$

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Similarly, the boundary conditions (7) re-scale as

$$u_{1,2} + u_{2,1} = 0, \quad \text{at} \quad x_2 = 0, \tag{11a}$$

$$\frac{\bar{c}_{12}}{\bar{c}_{66}}u_{1,1} + \frac{\bar{c}_{22}}{\bar{c}_{66}}u_{2,2} - \varepsilon\beta_{22}\theta = 0, \quad \text{at} \quad x_2 = 0, \tag{11b}$$

$$\theta = 0, \quad \text{at} \quad x_2 = 0. \tag{11c}$$

The newly introduced non-dimensional parameters ε and α characterize fundamental aspects of the interaction between the thermal and the elastic fields. The parameter ε, characterizing the strength of coupling between the fields, differs by the constant factor $\bar{c}_{11}/\bar{c}_{66}$ from the identically named parameter given in Chadwick and Sneddon (1958). Since the thermoelastic effect is not particularly strong, numerical values for ε are typically around a few per cent, see e.g. Chadwick and Sneddon (1958, Table 1).

The parameter α expresses the ratio between the characteristic lengths $\bar{\ell}$ and $\bar{h}$, where $\bar{\ell}$ is proportional to the mean free path of the thermal phonons and is measured on the scale of nanometers, see Kittel (1996). It is, effectively, an equivalent of the Knudsen number in fluid mechanics. In the vast majority of devices using surface acoustic waves, the parameter α is going to be small or very small. In fact, the equations of classical thermoelasticity can be formulated without any phenomenological assumptions, by a formal limiting procedure that moves from the microscopic description provided by Boltzmann's equation. The key assumption in such a procedure is that strains and rates of strain should vary slowly, compared to the mean free path of the thermal phonons (Maris, 1969, eqns. (28) and (29)), i.e., in our terms, when $\bar{\ell} \ll \bar{h}$ or, equivalently, $\alpha \ll 1$. The use of thermoelasticity in situations where the parameter α is not small cannot be similarly justified,

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see also Auriault (2014).

3. Quasi-adiabatic approximation

The smallness of α can be put to considerable advantage by constructing an essentially simplified version of our boundary value problem. We begin by making the usual asymptotic assumption that constant factors in front of the terms in equations (9) indicate their relative contribution and that differentiation of all relevant fields does not alter their asymptotic order. It is then clear that the first term on the left hand side of equation (9c) is vanishingly small. The remaining terms can be rearranged into

$$\theta^a = -\beta_{11} u^a_{1,1} - \beta_{22} u^a_{2,2} \,, \tag{12}$$

which can be recognized as a type of adiabatic condition, because it says that local temperature is a function of the local deformation (which reduces to volume deformation in the isotropic case). Here, we added the superscript a to indicate adiabatic fields that satisfy (12) and, effectively, are the leading order terms in the regular expansion in α. We can now treat local temperature as a dependent parameter, which can be determined from the displacement field $\bar{\boldsymbol{u}}$. Consequently, we use equation (12) to eliminate θ^a from the equations of motion (9a), (9b), which yields

$$\frac{\tilde{c}_{11}}{\tilde{c}_{66}} u^a_{1,11} + u^a_{1,22} + \left(\frac{\tilde{c}_{12}}{\tilde{c}_{66}} + 1 \right) u^a_{2,12} = -\omega^2 u^a_1 \,, \tag{13a}$$

$$u^a_{2,11} + \frac{\tilde{c}_{22}}{\bar{c}_{66}} u^a_{2,22} + \left(\frac{\tilde{c}_{12}}{\tilde{c}_{66}} + 1 \right) u^a_{1,12} = -\omega^2 u^a_2 \,, \tag{13b}$$

where we introduced the adiabatic stiffness constants $\tilde{c}_{ij}$ (no summation implied over repeated indices)

$$\tilde{c}_{ij} = \bar{c}_{ij} + \varepsilon \bar{c}_{66} \beta_{ii} \beta_{jj} \,, \quad i,j = 1,2 \,, \quad \text{and} \quad \tilde{c}_{66} = \bar{c}_{66} \,, \tag{14}$$

see e.g. (Shuvalov and Scott, 2000, p. 3). Equations (13) are formally equivalent to the usual equations of elastodynamics for an orthorhombic solid, written in the time-harmonic regime (Nobili and Prikazchikov, 2018). Nowacki demonstrates a similar reduction for isotropic materials (Nowacki, 1975, Section 1.16); he refers to the resulting effective equations as equations of *elastokinetics*. These equations reflect the fact that thermoelastic wave motion is governed, to the leading order in α, by equations formally equivalent to those of isothermal elasticity, although with adiabatic material constants. Indeed, the distinction between static and dynamic elastic modulae is put forward in the engineering literature for the same reason.

The parameter α is often so small that equations (13) are perfectly adequate for describing many thermoelastic problems. However, it is important to recognize that elimination of the $O(\alpha)$ terms disregards some of the qualitative features specific of the thermoelastic response. For example, equations (13) cannot predict bulk wave dissipation, which occurs owing to the coupling with the thermal field. In the case of isotropic elasticity, it is possible to construct the higher-order version of equations (13) with truncation error $O(\alpha^2)$, see Pichugin (2010). The resulting equations remain formally equivalent to these of isothermal elasticity, although material constants become complex-valued and frequency dependent. Consequently, a viscoelastic Voigt-like solid is retrieved, which can be interpreted as an effective model for thermoelastic dissipation in the bulk.

Importantly, thermoelastic dissipation is not always dominated by dissipation of bulk waves. Indeed, similarly to Konstantinov's effect in acoustics, see Legusha (1984), dissipation can be induced by the near-surface boundary

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layer. In fact, reduction of the three equations of thermoelasticity (9) to the pair of equations of elastokinetics (13) expresses the result that temperature is no longer an independent field. As a result, the thermal boundary condition cannot be accommodated without a thermoelastic boundary layer.

3.1. Boundary layer

Boundary layer solutions may be constructed by introducing the scaled coordinates

$$x_1 = \xi, \quad x_2 = \sqrt{\alpha}\zeta, \tag{15}$$

which allow to zoom into the vicinity of boundary. Furthermore, we assume the following asymptotic structure for our fields:

$$u_1^{bl} = \alpha u_1^{(0)} + O(\alpha^2), \tag{16a}$$

$$u_2^{bl} = \sqrt{\alpha}u_2^{(0)} + O(\alpha^{3/2}), \tag{16b}$$

$$\theta^{bl} = \theta^{(0)} + O(\alpha). \tag{16c}$$

After inserting the ansatz (16) into the governing equations (9), we obtain the leading order problem

$$u_{1,\zeta\zeta}^{(0)} + \left(\frac{\bar{c}_{12}}{\bar{c}_{66}} + 1\right) u_{2,\xi\zeta}^{(0)} - \varepsilon\beta_{11}\theta_{,\xi}^{(0)} = 0, \tag{17a}$$

$$\frac{\bar{c}_{22}}{\bar{c}_{66}} u_{2,\zeta\zeta}^{(0)} - \varepsilon\beta_{22}\theta_{,\zeta}^{(0)} = 0, \tag{17b}$$

$$\kappa_{22}\theta_{,\zeta\zeta}^{(0)} + i\omega\left(\theta^{(0)} + \beta_{22}u_{2,\zeta}^{(0)}\right) = 0. \tag{17c}$$

Integrating equation (17b) yields the "anti-adiabatic" relationship (in the sense that it provides displacement gradients with respect to local temperature)

$$u_{2,\zeta}^{(0)} = \varepsilon\beta_{22}\frac{\bar{c}_{66}}{\bar{c}_{22}}\theta^{(0)}, \tag{18}$$

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which, upon substitution into equation (17c), lends an explicit constant-coefficient ordinary differential equation for the boundary layer temperature

$$\kappa_{22}\theta^{(0)}_{,\zeta\zeta} + \mathrm{i}\omega\delta^2\theta^{(0)} = 0\,, \tag{19}$$

where $\delta^2 = 1 + \varepsilon\beta^2_{22}\frac{\bar{c}_{66}}{\bar{c}_{22}}$ is a material parameter assuming the role of a decay index. This equation admits the solution

$$\theta^{(0)} = \Theta^{(0)}\exp\left(\pm\frac{(1-\mathrm{i})\sqrt{\omega}}{\sqrt{2\kappa_{22}}}\delta\zeta\right)\,, \tag{20}$$

where $\Theta^{(0)} = \Theta^{(0)}(\xi)$ and the sign in the exponent is chosen as to ensure decay as $\zeta \to \mp\infty$ (corresponding to the solid occupying the lower or the upper half-plane). This solution represents an oscillatory boundary layer, rapidly decaying into the interior of half-space, very similar to thermal boundary layers in acoustics and fluid mechanics. The associated displacement field is established integrating the remaining governing equation (17a)

$$u^{(0)}_1 = \mathrm{i}\varepsilon\kappa_{22}\omega^{-1}\delta^{-2}\left(\beta_{11} - \beta_{22}\frac{\bar{c}_{12}+\bar{c}_{66}}{\bar{c}_{22}}\right)\theta^{(0)}_{,\xi}\,, \tag{21}$$

and accounting for (18,20)

$$u^{(0)}_2 = \pm(1+\mathrm{i})\varepsilon\frac{\beta_{22}\bar{c}_{66}\sqrt{\kappa_{22}}}{\sqrt{2}\bar{c}_{22}\sqrt{\omega}\delta}\theta^{(0)}. \tag{22}$$

Thus, it is possible to recover the leading order boundary layer solution in terms of the amplitude function $\Theta^{(0)} = \Theta^{(0)}(\xi)$.

3.2. Effective boundary conditions

The regular expansion

$$\theta = \theta^a + \alpha\theta^{(1)} + \dots,$$

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reduced the system of three partial differential equations of thermoelasticity (9) to the system of two quasi-adiabatic equations (13). We can now use the newly constructed explicit solution for thermoelastic boundary layer to perform a similar reduction for the boundary conditions (11). This is done using the Vishik-Lyusternik method, see Višik and Lyusternik (1962). Specifically, we seek our solution in the form of the composite expansions

$$u_1 = u_1^a + u_1^{bl}, \quad u_2 = u_2^a + u_2^{bl}, \quad \theta = \theta^a + \theta^{bl}. \tag{23}$$

In view of these, our isothermal boundary condition (11c) can be used to recover the boundary layer amplitude

$$\Theta^{(0)} = \theta^{bl}|_{x_2=0} = -\theta^a|_{x_2=0} = \left[\beta_{11}u_{1,1}^a + \beta_{22}u_{2,2}^a\right]_{x_2=0}, \tag{24}$$

where we have employed the adiabatic relationship (12). This completely defines the leading order boundary layer solution in terms of the leading order adiabatic displacements. Upon recalling (15,16,22,23), we may recast the stress boundary conditions (11a) and (11b) purely in terms of the adiabatic solution

$$u_{1,2}^a + u_{2,1}^a + \sqrt{\alpha}\frac{\varepsilon D}{\sqrt{\omega}}\left(\beta_{11}u_{1,11}^a + \beta_{22}u_{2,12}^a\right) = 0, \quad \text{at} \quad x_2 = 0, \tag{25a}$$

$$\frac{\tilde{c}_{12}}{\tilde{c}_{66}}u_{1,1}^a + \frac{\tilde{c}_{22}}{\tilde{c}_{66}}u_{2,2}^a = 0, \quad \text{at} \quad x_2 = 0, \tag{25b}$$

where the non-dimensional parameter D is defined by

$$D = \pm\frac{1+\mathrm{i}}{\sqrt{2}}\frac{\sqrt{\kappa_{22}}\left(\beta_{11} - \beta_{22}\frac{\tilde{c}_{12}}{\tilde{c}_{22}}\right)}{\sqrt{1 - \varepsilon\beta_{22}^2\frac{\tilde{c}_{66}}{\tilde{c}_{22}}}}. \tag{26}$$

Disregarding $O(\sqrt{\alpha})$ terms, the effective boundary conditions (25) are formally equivalent to the classical stress-free boundary conditions of plain

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elasticity, provided that these are written in terms of adiabatic material constants. Thus, we conclude that the leading order contribution coming from coupling thermal and elastic fields is most simply obtained replacing isothermal with adiabatic material constants, similarly to what occurs for isotropic media, see Lockett (1958). Next, the $O(\sqrt{\alpha})$ contribution to the shear stress (25a) describes the leading order term arising from thermoelastic diffusion. Remarkably, for the specific case of isothermal boundary condition considered here, this contribution is larger than the diffusion contribution arising from bulk waves, which appears through $O(\alpha)$ terms. As a result, in the vicinity of isothermal boundaries, the thermoelastic boundary layer dominates diffusive interactions.

A similar asymptotic analysis may be performed considering the complementary boundary condition for temperature, which is that of perfectly insulated (or zero thermal flux) boundary. In this case, it turns out that the first boundary correction term is $O(\alpha)$, which is of the same asymptotic order as the first omitted term from the governing equations of elastokinetics (13). As a consequence, a consistent description of this condition demands producing higher-order correction terms for both the governing equations as well as for the boundary conditions. However, this rests outside the scope of this paper, also in consideration of the fact that the corresponding thermoelastic contributions are much weaker. This is independently confirmed in the analysis by Mayer (1990).

4. Surface waves on isothermal boundary

We are now in a position to exploit the quasi-adiabatic governing equations (13) and the effective boundary conditions (25) to describe fully-coupled thermoelastic surface waves. This is done by seeking travelling wave solutions

$$u^a_{1,2} = U^a_{1,2} \exp\left(kqx_2\right) \exp\left(\mathrm{i}kx_1\right),\tag{27}$$

where $U^a_{1,2}$ are constant amplitudes, $k \in \mathbb{R}$ is the dimensionless wavenumber and q the attenuation index. We observe that, in light of the implied factor $\exp(-\mathrm{i}\bar{\omega}\bar{t}) \equiv \exp(-\mathrm{i}\omega t)$, positive values of k correspond to right-propagating waves moving with phase velocity $v = \omega/k$.

Substitution of (27) into the quasi-adiabatic governing equations (13) yields a homogeneous algebraic system that is linear in the unknowns U^a_1 and U^a_2. Non-trivial solutions of this system exist when its determinant vanishes, whence we get the secular (or frequency) equation

$$q^4 + d_2 q^2 + d_0 = 0,\tag{28}$$

that is a bi-quadratic polynomial in q with coefficients

$$d_2 = \left(1 + \frac{\tilde{c}_{66}}{\tilde{c}_{22}}\right)v^2 + \frac{2\tilde{c}_{12}\tilde{c}_{66} - \tilde{c}_{11}\tilde{c}_{22} + \tilde{c}_{12}^2}{\tilde{c}_{66}\tilde{c}_{22}},\tag{29a}$$

$$d_0 = (1 - v^2)\frac{\tilde{c}_{11} - \tilde{c}_{66}v^2}{\tilde{c}_{22}}.\tag{29b}$$

Equation (28) sets the condition for existence of bulk waves. These are generally inhomogeneous, although, in the special situation $v = 1$, homogeneous bulk waves moving with the longitudinal wavespeed $\sqrt{\tilde{c}_{66}/\bar{\rho}}$ are retrieved. Again, this wavespeed formally corresponds to the classical result, except for the use of adiabatic material parameters.

We now move to surface waves and assume, to fix ideas, that the half-space is located at $x_2 \leq 0$, whence decay occurs as $x_2 \to -\infty$. For this, the pair of admissible solutions, q_1 and q_2, of the bi-quadratic secular equation (28) must have positive real part, i.e. $\Re(q_1) > 0$ and $\Re(q_2) > 0$. To each solution q_i we attach the wave amplitudes $U_{1,i}^a$ and $U_{2,i}^a$ that form the mode vector. They are related by

$$
U_{2,i}^a = \mathrm{i}q_i \frac{\tilde{c}_{12} + \tilde{c}_{66}}{\tilde{c}_{66}\left(1 - v^2\right) - \tilde{c}_{22}q_i^2} U_{1,i}^a, \quad i = 1, 2. \tag{30}
$$

Also, in light of Vieta's theorem, we have

$$
q_1^2 + q_2^2 = -d_2, \quad \text{and} \quad q_1^2 q_2^2 = d_0. \tag{31}
$$

Substituting the general wave solutions (27), written for each decay index q_i, into the effective boundary conditions (25), computing the determinant of the resulting linear system and employing relations (31,30), one obtains, after lengthy algebraic manipulations, the following dispersion relation

$$
\begin{aligned}
v^2 &+ \frac{\tilde{c}_{12}^2 - \tilde{c}_{11}\tilde{c}_{22}}{\tilde{c}_{22}\tilde{c}_{66}} + \frac{q_1 q_2 v^2}{1 - v^2} \\
&+ \varepsilon D \frac{(\beta_{11}\tilde{c}_{22} - \beta_{22}\tilde{c}_{12})}{\tilde{c}_{22}\sqrt{v}}(q_1 + q_2)\sqrt{k}\sqrt{\alpha} = 0,
\end{aligned} \tag{32}
$$

where terms of the order $O(\alpha)$ and higher are omitted. The leading order terms of the dispersion relation are equivalent to the secular equation for surface waves in orthorhombic bodies, see Chadwick (1976), Royer and Dieulesaint (1984) or Nobili and Prikazchikov (2018). The remaining $O(\sqrt{\alpha})$ term encapsulates the leading order influence of thermoelastic coupling, which results in both dispersion and dissipation of the surface waves.

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The relative simplicity of the dispersion relation (32) suggests to formulate an explicit asymptotic expansion for its solution:

$$v_R^2 = v_0^2(1 + v_1^2\sqrt{\alpha} + O(\alpha)).\tag{33}$$

Here v_0 is the surface wave speed in the orthorhombic half-space with adiabatic material constants, i.e. the solution of

$$v_0^2 + \frac{\tilde{c}_{12}^2 - \tilde{c}_{11}\tilde{c}_{22}}{\tilde{c}_{22}\tilde{c}_{66}} + v_0^2\sqrt{\frac{\tilde{c}_{11} - \tilde{c}_{66}v_0^2}{\tilde{c}_{22}(1 - v_0^2)}} = 0\,,\tag{34}$$

and we get the correction

$$v_1^2 = \frac{2\varepsilon\tilde{c}_{22}Dv_0^{-1/2}(\beta_{11}\tilde{c}_{22} - \beta_{22}\tilde{c}_{12})\left[q_1^2 q_2^2(q_1 + q_2)\right]_{v=v_0}\sqrt{k}}{(\tilde{c}_{11}\tilde{c}_{22} - \tilde{c}_{12}^2)(\tilde{c}_{66}v_0^4 - (\tilde{c}_{11} + 3\tilde{c}_{66})v_0^2 + 2\tilde{c}_{11}) + \tilde{c}_{66}v_0^4(\tilde{c}_{22}\tilde{c}_{66} - \tilde{c}_{12}^2)}\,.\tag{35}$$

The key take away message of this section is not really about constructing an asymptotic description for dispersion and dissipation of surface waves in orthorhombic media, even though expansion (33)–(35) is new and remarkably straightforward. The real importance of this result is that it was obtained by solving the effective "quasi-adiabatic" boundary value problem (13), (25) instead of the much more complex full boundary value problem (9), (11). The additional effort spent on deriving effective boundary conditions (25) may be recuperated by noting that they can now be used to solve other boundary value problems, for example, the problem of reflection and/or conversion of plane waves at the free surface, see e.g. Nobili et al. (2020).

4.1. Specialization to isotropic materials

The asymptotic expansion (33) assumes a particularly simple form for isotropic materials, for which

$$\tilde{c}_{11} = \tilde{c}_{22} = \tilde{\lambda} + 2\tilde{\mu}\,,\quad \tilde{c}_{12} = \tilde{\lambda}\,,\quad \tilde{c}_{66} = \tilde{\mu}\,,\quad \beta_{11} = \beta_{22} = \kappa_{11} = \kappa_{22} = 1\,,\tag{36}$$

where $\tilde{\lambda}$ and $\tilde{\mu}$ are the adiabatic values of the standard Lamé parameters. The associated secular equation (34) can be re-written as

$$v_0^2 \left(1 + \sqrt{\frac{1 - \tilde{\Lambda} v_0^2}{1 - v_0^2}} \right) = 4(1 - \tilde{\Lambda}) \,, \quad \text{where} \quad \tilde{\Lambda} = \frac{\tilde{\mu}}{\tilde{\lambda} + 2\tilde{\mu}} \,. \qquad (37)$$

Even though equation (37) may look unfamiliar, Chadwick (1976) describes a sequence of steps to transform (37) into the standard form for the Rayleigh secular equation. The corresponding expression for the correction v_1^2 is also much simpler in the isotropic case

$$v_1^2 = \frac{\epsilon \tilde{\Lambda} D (2 - v_0^2)^4 \sqrt{2 v_0^4 - 4(3 + \tilde{\Lambda}) v_0^2 + 16}}{8 \sqrt{v_0} (1 - \tilde{\Lambda})((1 - 8\tilde{\Lambda}) v_0^4 + (4 + 12\tilde{\Lambda}) v_0^2 - 8)} \sqrt{k} \,, \qquad (38)$$

where $D = (1 + \mathrm{i}) \sqrt{2} \tilde{\Lambda} / \sqrt{1 - \epsilon \tilde{\Lambda}}$.

An essentially equivalent asymptotic expansion for the thermoelastic surface wave speed in isotropic materials was given by Chadwick and Windle (1964). Direct comparison of this expression with (33) is difficult, because the implicit nature of the Rayleigh equation (37) for v_0 makes the expressions for all of the higher-order coefficients non-unique. However, we can still revert to numerical comparison. Table 1 lists the material parameters for worked pure copper, which were used for all numerical examples in Chadwick and Windle (1964)[1]. The constant factor c_1 for the first correction to

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[1]The parameters given by Chadwick and Windle (1964) are incomplete, i.e. they cannot be used to provide a full description of surface waves. Hence, in order to provide the full picture, we use literature values for pure copper, see Davis (2001, pp. 446–452), and work out the components of the stiffness and of the thermal conductivity tensors to match the results reported by Chadwick and Windle (1964).

Worked pure copper (isotropic)	$\bar{\rho} = 8940$ kg/m^3; $\bar{c}_{11} = \bar{c}_{22} = 199.5$ GPa, $\bar{c}_{12} = \bar{c}_{13} = \bar{c}_{23} = 101.346$ GPa, $\bar{c}_{66} = 49.077$ GPa; $\bar{T}_0 = 293$ K; $\bar{a}_{11} = \bar{a}_{22} = \bar{a}_{33} = 16.7 \cdot 10^{-6}$ K^{-1}; $\bar{C}_v = 386$ J/(kg·K) and $\bar{\kappa} = 383$ W/(m·K), $\kappa_{11} = \kappa_{22} = 1$, see Davis (2001, pp. 446–452). Equations (4) and (10)$_1$ yield $\bar{\beta} = 6.717$ MPa K^{-1}, $\beta_{11} = \beta_{22} = 1$ and $\varepsilon = 0.07805$. For $\bar{h} = 1 \cdot 10^{-3}$ m one has $\alpha = 0.00004737$.

Table 1: Material parameters which are selected to reproduce the response of the worked pure copper, as used by Chadwick and Windle (1964).

surface wave velocity in the paper by Chadwick and Windle (1964) can be written in terms of our constant v_1^2 as follows

$$c_1 = -\frac{(1-\mathrm{i})\sqrt{1-\epsilon\tilde{\Lambda}}}{\sqrt{2v_0 k}}v_1^2\,. \tag{39}$$

Evaluating this expression using the parameters of Table 1 for pure worked copper gives $c_1 \approx 0.004896593$, which fully agrees with the corresponding value given in Chadwick and Windle (1964, p. 58).

5. Numerical examples

To better illustrate the effectiveness of the quasi-adiabatic approximation, we compare solution (33) of the asymptotic dispersion relation (28) against the numerical solutions of the exact dispersion relation for the fully coupled thermoelastic problem. In order to derive such dispersion relation, we consider a travelling wave solution for the governing equations (9)

$$\{u_1, u_2, \theta\} = \{U_1, U_2, \Theta\} \exp\left(kqx_2\right)\exp\left(\mathrm{i}kx_1\right)\,, \tag{40}$$

and obtain a homogeneous linear algebraic system in the amplitudes $U_{1,2}$ and Θ. Just as in Section 4, a common factor $\exp(-\mathrm{i}\bar{\omega}\bar{t}) \equiv \exp(-\mathrm{i}\omega t)$ is assumed, see $(8)_3$, and $k > 0$ corresponds to the right-propagating waves with the velocity $v = \omega/k$. The requirement that a non-trivial solution exists provides the secular equation, in the form of a bi-cubic equation for the attenuation index q:

$$b_6 q^6 + b_4 q^4 + b_2 q^2 + b_0 = 0\,, \tag{41}$$

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where

$$b_6 = \kappa_{22} \frac{\bar{c}_{22}}{\bar{c}_{66}} \alpha, \quad b_4 = \frac{iv}{k} \left(\frac{\bar{c}_{22}}{\bar{c}_{66}} + \varepsilon \beta_{22}^2 \right) - \left(\kappa_{11} \frac{\bar{c}_{22}}{\bar{c}_{66}} - \kappa_{22} B_1 \right) \alpha, \qquad (42a)$$

$$b_2 = \frac{iv}{k} \left(B_1 - \varepsilon \beta_{11}^2 \frac{\bar{c}_{22}}{\bar{c}_{66}} + 2\varepsilon \beta_{11} \beta_{22} \left(\frac{\bar{c}_{12}}{\bar{c}_{66}} + 1 \right) + \varepsilon \beta_{22}^2 B_2 \right)$$

$$- \left(\kappa_{11} B_1 + \kappa_{22}(1 - v^2) B_2 \right) \alpha, \qquad (42b)$$

$$b_0 = (1 - v^2) \left(\frac{iv}{k} (\varepsilon \beta_{11}^2 - B_2) + \kappa_{11} B_2 \alpha \right), \qquad (42c)$$

as well as

$$B_1 = v^2 + \frac{\bar{c}_{12}^2}{\bar{c}_{66}^2} + 2\frac{\bar{c}_{12}}{\bar{c}_{66}} + \frac{\bar{c}_{22}}{\bar{c}_{66}} B_2 \quad \text{and} \quad B_2 = v^2 - \frac{\bar{c}_{11}}{\bar{c}_{66}}. \qquad (43)$$

We observe that the decay requirement as $x_2 \to \mp\infty$ demands that $\Re(q_i) \gtrless 0$ for the roots q_i, $i = 1, 2, 3$, of (41), which provides the correct sign for the square root. Alongside the secular equation, we obtain relations for the ratios of component amplitudes U_2/U_1 and Θ/U_1 (explicit expressions for those are omitted for the sake of brevity).

Enforcing now the boundary conditions (11) provides the sought-for dispersion relation, in the form of a polynomial in the attenuation indices q_i

$$\sum_{i,j,k=0}^{2} a_{ijk} q_1^i q_2^j q_3^k = 0, \qquad (44)$$

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Rochelle salt NaKC$_4$H$_4$O$_6$ · 4H$_2$O (orthorhombic)	$\bar{\rho} = 1767$ kg/m^3 (Berlincourt et al., 1964, p. 180); $\bar{c}_{11} = 25.5$ GPa, $\bar{c}_{12} = 14.1$ GPa, $\bar{c}_{13} = 11.6$ GPa, $\bar{c}_{22} = 38.1$ GPa, $\bar{c}_{23} = 14.6$ GPa, $\bar{c}_{66} = 9.79$ GPa (Huntington, 1958, p. 72); $\bar{T}_0 = 300$ K; $\bar{C}_v = 1390$ J/(kg·K) (Helwig, 1974, p. 226); $\bar{a}_{11} = 58 \cdot 10^{-6}$ K^{-1}, $\bar{a}_{22} = 33.53 \cdot 10^{-6}$ K^{-1}, $\bar{a}_{33} = 43.33 \cdot 10^{-6}$ K^{-1} (Moina et al., 2011, p. 8); $\bar{\kappa} = 0.5$ W/(m·K), $\kappa_{11} = 1$, $\kappa_{22} = 1.22$ (Newnham, 2005, p. 206). Equations (4) and (10)$_1$ yield $\bar{\beta} = 2.45$ MPa K^{-1}, $\beta_{11} = 1$, $\beta_{22} = 1.11$ and $\varepsilon = 0.0749$. For $\bar{h} = 1 \cdot 10^{-6}$ m one has $\alpha = 0.0000865$.

Table 2: Material parameters for the Rochelle salt.

where $a_{ijk} = 0$ unless $i + j + k \in \{0, 2, 4, 6\}$. The nonzero coefficients are

$$a_{222} = \bar{c}_{22}\bar{c}_{66}\bar{A}_2(\bar{A}_2 - \bar{c}_{66}\beta_{22}v^2) \,, \tag{45a}$$

$$a_{220} = \bar{c}_{22}\bar{c}_{66}\bar{A}_2(\bar{c}_{66}\beta_{22}v^2 - \bar{A}_1)(1 - v^2) \,, \tag{45b}$$

$$a_{200} = \bar{c}_{22}\bar{c}_{66}(\bar{A}_1 - \bar{A}_6 v^2)(\bar{A}_1 - \bar{c}_{66}\beta_{22}v^2)(1 - v^2) \,, \tag{45c}$$

$$a_{211} = \bar{c}_{22}\bar{c}_{66}v^2(\bar{c}_{66}\beta_{22}(\bar{A}_6 - \bar{A}_2)v^2 + \bar{A}_1\bar{A}_2 - \bar{c}_{66}\bar{A}_2\beta_{11} - \bar{c}_{66}\bar{A}_1\beta_{22}) \,, \tag{45d}$$

$$a_{110} = a_{211}(1 - v^2)(\bar{c}_{12}^2 - \bar{c}_{11}\bar{c}_{22} + \bar{c}_{22}\bar{c}_{66}v^2)/(\bar{c}_{22}\bar{c}_{66}v^2) \,, \tag{45e}$$

$$a_{000} = (\bar{A}_6 v^2 - \bar{A}_1)(1 - v^2) \left[\bar{c}_{22}\bar{c}_{66}^2\beta_{22}v^4 - (\bar{c}_{11}\bar{A}_2 + 2\bar{c}_{12}\bar{A}_1)\bar{c}_{66} \right. \tag{45f}$$

$$\left. + \bar{c}_{66}(\bar{c}_{12}\bar{A}_6 - 2\bar{c}_{22}\bar{A}_1 - \bar{c}_{12}\bar{A}_2 + \bar{c}_{22}\bar{c}_{66}\beta_{11})v^2 - (\bar{c}_{12}^2 - \bar{c}_{11}\bar{c}_{22})\bar{A}_1 \right] \,, \tag{45g}$$

where

$$\bar{A}_1 = \bar{c}_{11}\beta_{22} - \bar{c}_{12}\beta_{11} \,, \quad \bar{A}_2 = \bar{c}_{22}\beta_{11} - \bar{c}_{12}\beta_{22} \,, \quad \bar{A}_6 = \bar{c}_{66}(\beta_{11} + \beta_{22}) \,, \tag{45h}$$

and the cyclic permutation property $a_{ijk} = a_{jki} = a_{kij}$ stands, namely

$$a_{200} = a_{020} = a_{002} \,, \qquad a_{011} = a_{101} = a_{110} \,, \tag{45i}$$

$$a_{211} = a_{121} = a_{112} \,, \qquad a_{022} = a_{202} = a_{220} \,. \tag{45j}$$

Figure 1 illustrates a sample numerical solution of the dispersion relation (44) for the Rochelle salt, which is an anisotropic material widely used for piezoelectric transducers. The crystal symmetry of the Rochelle salt is orthorhombic, and a representative set of material parameters is gathered in Table 2. Superposed onto the dispersion curve for the fully coupled system is the asymptotic approximation (33), that is plotted in dotted style. It appears that the latter reproduces the exact curve over a wide range of wave numbers, especially in terms of dispersion. Overall, an excellent agreement can be observed for (dimensionless) wavenumbers k up to 100. For

higher wavenumbers, the exact solution exhibits a peak of dissipation, see Figure 1(b), which is not captured by the asymptotic solution. This happens because dissipation peaks for wavelengths comparable to the mean free path of thermal phonons, which is precisely where our natural small parameter ceases to be small. Still, it is worth reiterating that the validity of the very equations of the thermoelasticity in this regime is very questionable, see Fichera (1992) and Auriault (2014).

6. Conclusions

Thermo-mechanical coupling drives design of modern microelectronics systems, on account of the outstanding growth in the number of integrated sub-systems which can now be fitted in a single packaging. Besides, thermo-mechanical interaction is a key factor in understanding dispersion and dissipation at high frequencies, which are typical of modern signal processing and passband filter devices. However, the fully coupled equations of thermoelasticity pose a formidable obstacle to analytical approaches, while numerical solutions suffer from the usual flaw of lending little insight on the role of the problem's parameters.

In this paper, the equations of linear thermoelasticity are considered for orthorhombic materials. The aim is to develop better physical insight into the way in which thermoelastic coupling contributes to dispersive and dissipative properties of the media. This is achieved by an asymptotic approach which moves from the observation that the non-dimensional parameter α, equivalent to the Knudsen number in fluid mechanics, is very small in practice. An asymptotic model is proposed which reduces the role of temperature to that

of a dependent field. Consequently, the leading order governing equations are expressed entirely in terms of displacements and formally correspond to those in classical linear elastodynamics, provided that isothermal material constants are replaced by their adiabatic counterparts. Such equations are known in the literature as equations of elastokinetics. Since the temperature field is now a dependent quantity, a boundary layer is required to satisfy thermal boundary conditions, in a manner similar to the edge-effect of shell theory. Consideration of this boundary layer leads to the formulation of effective boundary conditions for the equations of elastokinetics, through which the leading order contribution of the correct thermal boundary conditions is reproduced. Interestingly, even though the equations of thermoelasticity exhibit dissipative behaviour, it is worth observing that the thus-constructed leading order asymptotic model admits a time-dependent Hamiltonian structure.

Remarkably, in the case of isothermal boundary conditions, the viscoelastic character is produced, to leading order, by the boundary layer. This is in contrast to common sense expectation, which ascribes dissipation to bulk properties. In fact, dissipation is well described by the newly derived effective boundary conditions. On the other hand, adiabatic boundary conditions induce dissipation at a asymptotically higher order, and, to it, boundary and bulk thermal effects equally contribute.

As an illustration of the effectiveness of this approach, surface wave propagation in an orthorhombic thermoelastic half-plane is considered. The dispersion relations for the asymptotic as well as for the fully coupled problem are given explicitly and then compared numerically for a reference ma-

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terial, namely potassium sodium tartrate tetrahydrate (the Rochelle salt), that is widely used in piezoelectric transducers. Excellent agreement, for both dispersion and dissipation, is found up to relatively high dimensionless wavenumbers. The special case of isotropy is also considered, as it can be compared with existing results in the literature.

Overall, this paper shows that dealing with fully-coupled thermoelastic problems does not need to be much more cumbersome analytically than dealing with the purely elastic boundary value problems. We are confident that similar techniques are applicable to modelling of even more complex inhomogeneous waves in thermoelastic media, including waves guided by plate edges (Pichugin and Rogerson, 2012) or other topographical features (Pichugin, 2013). Our results are likely to be particularly relevant for modelling micro and nano-scale structures, because their characteristic dimensions can become comparable with the mean free path of the thermal phonons, and consequently thermo-elastic coupling plays a significant role.

7. Acknowledgements

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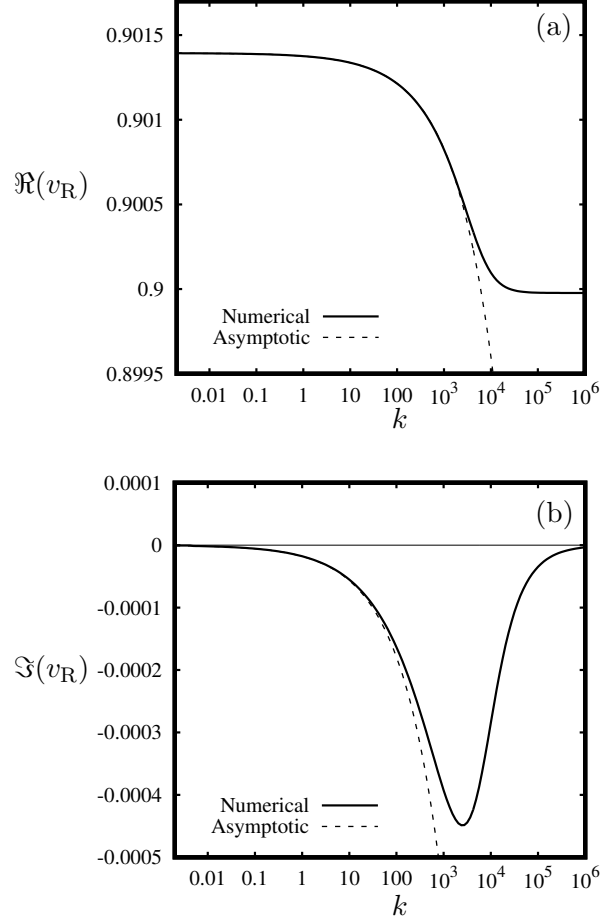
Figure 1: Dispersion and dissipation of thermoelastic surface waves propagating along the Ox_1-axis in the Rochelle salt: respectively, real (a) and imaginary part (b) of the phase velocity v_R. The solid curve, representing the numerical solution of the fully coupled thermoelastic system, is superposed onto the asymptotic approximation (33), that appears in dotted style.