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New perspectives in phase transition problems

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Abstract

Phase transitions occur in many relevant processes in physics, natural sciences and engineering: almost every industrial product involves solidification at some stage. Examples include metal casting, steel annealing, crystal growth, thermal welding, freezing of soil, freezing and melting of the earth surface water, food conservation, and others. All of these processes are characterized by two basic phenomena: heat-diffusion and exchange of latent heat of phase transition.

In this thesis, which consists of four distinct parts, we deal with phase transitions from different points of view.

The first part, titled **Control and controllability of PDEs with hysteresis with an application in phase transition modeling**, is a bridge between the master thesis work (about controllability of PDEs with hysteresis) and phase transitions. Indeed, thanks to the special link between hysteresis operators and phase transitions, the controllability results that we prove can be applied to the so-called *relaxed Stefan problem*. This is an example of a basic model of phase transition, since it simply accounts for heat-diffusion and exchange of latent heat. More complicated models, which take into account also the mechanical aspects of the process, are considered in Parts II and III.

More precisely, in the second part, titled **A viscoelastoplastic porous medium problem with phase transition**, we derive and investigate a model for filtration in porous media which takes into account the effects of freezing and melting of water in the pores. The third part, whose title is **Fatigue and phase transition in an oscillating elastoplastic beam**, is devoted to the derivation and the study of a model describing fatigue accumulation in an oscillating beam under the hypothesis that the material can partially recover by the effect of melting.

Finally, in the fourth part, titled **Regularity for double-phase variational problems**, we address the problem of the higher differentiability of solutions to the *obstacle problem*. In particular we deal with the case of *nonstandard growth conditions*, which includes the so-called *double-phase functionals*. Such functionals describe the behavior of strongly anisotropic materials whose hardening properties drastically change with the point, hence they exhibit the most dramatic phase transition. The techniques here employed are different from those used in the rest of the thesis, since they rely on the direct methods pertaining to the regularity theory in the field of Calculus of Variations.

Contents

Part I

Control and controllability of PDEs with hysteresis with an application in phase transition modeling

Introduction

In this first part we focus on the problem of controllability of the equation

$$u_t(x,t) - \Delta u(x,t) + \mathcal{F}[u](x,t) = v(x,t), \quad x \in \Omega \subset \mathbb{R}^N,\ t \in (0,T) \tag{I.1}$$

with a hysteresis operator $\mathcal{F}$ (see Section B in the Appendix for more details on hysteresis operators), a right-hand side v called the *control*, and given initial and boundary conditions. The controllability problem for equation (I.1) consists in proving that, for an arbitrary initial condition, an arbitrary final time T and an arbitrary admissible final state $\bar{u}(x)$, it is possible to choose the control v in a given class of functions of x and t in such a way that the solution satisfies $u(x,T) = \bar{u}(x)$ a. e. in Ω. The controllability problem for various kinds of linear and semilinear parabolic equations has been intensively studied in the recent decades, and a quite complete survey can be found in V. Barbu's monograph [9]. However, the main building blocks of the theory have been established earlier, probably by O. Yu. Emanuilov (Imanuvilov) and his collaborators, see, e. g., [57, 68].

First results about the *null-controllability* (that is, controllability for $\bar{u}(x) \equiv 0$) of equation (I.1) with hysteresis were obtained by F. Bagagiolo in [7]: following the techniques presented in [9] for the null-controllability of linear and semilinear parabolic equations, he proves the result performing a linearization followed by a fixed-point procedure. In my master thesis [70], carried out under F. Bagagiolo's supervision at the University of Trento, following his already mentioned paper [7] we studied the null-controllability of equation (I.1) for linearizable hysteresis operators, where the values of $\mathcal{F}[u](x,t)$ are dominated in an appropriate pointwise sense by $u(x,t)$. On the one hand, this method allows for applying the Carleman estimates to treat the case in which the control is active only on a part of the domain. On the other hand, typical hysteresis operators arising in applications are not linearizable, and a new approach motivated by M. Brokate's previous works [21, 22] on optimal control of ODEs with hysteresis needs to be developed and adapted for the PDE case.

Thus, inspired by M. Brokate's work, the project continued in cooperation with P. Krejčí from the Czech Academy of Sciences and the Czech Technical University. In our first work presented at the *MURPHYS-HSFS 2018* conference (see [74]), we discussed the null-controllability problem for equations of the form (I.1) for two classes of operators $\mathcal{F}$: either a linearizable operator $\mathcal{F}$ as in [7], or the case that $\mathcal{F}$ is the stop operator (which we recall in detail in Section B.2). The two situations are

indeed disjoint: the values of the stop are not dominated by the instantaneous input value in any sense and depend on the whole history of the process. In our second work [75] we considered substantially more general hysteresis operators which include for example operators with complex memory like, e.g., the Preisach operator (which we recall in Section B.3).
Note that Carleman estimates are based on weighted L^2-norms (see [9]), so that they are not compatible with arguments using extensively the L^1-technique, as for problems with hysteresis. Thus the price we pay is that only distributed controls are allowed if we want to deal with more general hysteresis operators. Indeed, in this case, the existence alone of the control can be deduced from the abstract topological Kakutani fixed point principle as in [9]. The main result of our contribution is, instead, to establish in Theorem 1.4 below a link between controllability of PDEs with hysteresis and an optimal control problem for a penalty approximation depending on two singular parameters.
The result has important applications in phase transition problems, as it can be used to prove the controllability of the so-called *relaxed Stefan problem in weak form*, according to the notation and the terminology used by A. Visintin in [131, 132]. Taking advantage of the special link between relaxed phase transitions and hysteresis operators of stop type, it is possible to show that the Stefan problem can be reformulated in the form (I.1).

This first part of the thesis presents the results contained in [75], and its structure is as follows. Main results about the approximation of the controllability problem by constrained and penalized minimization problems are stated in Chapter 1. Chapter 2 is devoted to the individual steps of the argument (penalization, constrained minimization, estimates independent of the singular parameters, passage to the limit). In Chapter 3 we explain the relation between our system and a simple model for phase transition in a two-phase system.

CHAPTER 1

Statement of the problem

We consider a bounded connected Lipschitzian domain $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, and fix an arbitrary final time $T > 0$. In the sequel, we denote $Q = \Omega \times (0,T)$, $\Gamma = \partial\Omega \times (0,T)$ and n is the unit outward normal vector to $\partial\Omega$. We deal with the system

$$\begin{cases} u_t - \Delta u + \mathcal{F}[u] = v & \text{in } Q, \\ u(\cdot,0) = u^0(\cdot) & \text{in } \Omega, \\ n \cdot \nabla u = 0 & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where $\mathcal{F}$ is a hysteresis operator of the form

$$\mathcal{F}[u](x,t) = \int_0^\infty f(x, r, u(x,t), s^r(x,t)) \, \mathrm{d}\mu(r) \tag{1.2}$$

with a given function $f : \Omega \times (0,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and with a Borel measure μ on $(0,\infty)$. The term s^r corresponds to the stop operator introduced in (B.10) (we refer in particular to the extension to space-dependent inputs). The function f in (1.2) is assumed to satisfy the following hypothesis.

Hypothesis 1.1. The function $f : \Omega \times (0,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and the measure μ in (1.2) have the following properties:

(i) $\mu \geq 0$, $\int_0^\infty (1 + r) \, \mathrm{d}\mu(r) < \infty$;

(ii) $f(\cdot, r, u, \sigma) : \Omega \to \mathbb{R}$ is measurable for all (r, u, σ) in the whole domain of definition;

(iii) $f(x, r, u, \sigma)$ is continuous in r and continuously differentiable in u and in σ for a. e. $x \in \Omega$;

(iv) there exists a constant $C > 0$ such that it holds $|f_u(x, r, u, \sigma)| + |f_\sigma(x, r, u, \sigma)| \leq C$ for a. e. $x \in \Omega$ and all (r, u, σ) in the whole domain of definition;

(v) there exists a constant $C > 0$ such that it holds $|f(x, r, u, \sigma) - u f_u(x, r, u, \sigma) - \sigma f_\sigma(x, r, u, \sigma)| \leq C$ for a. e. $x \in \Omega$ and all (r, u, σ) in the whole domain of definition.

Note that formula (1.2) includes the Preisach hysteresis operator (B.23) for the choice $f(x,r,u,\sigma) = \hat{f}(r,u-\sigma)$, $\hat{f}(r,v) = \int_0^v \omega(r,v')\,\mathrm{d}v'$, as well as the nonlinear stop (B.10) if μ is the Dirac measure and $f_u = 0$ (this simpler case was considered in [74]).

The data of the problem are assumed to satisfy the following hypothesis.

Hypothesis 1.2. The initial condition u^0 belongs to $W^{1,2}(\Omega) \cap L^\infty(\Omega)$ and $s^{r,0} \in L^\infty(\Omega)$ are given such that $|s^{r,0}(x)| \le r$ a. e. for all $r > 0$, the mapping $r \mapsto s^{r,0}(x)$ is Lipschitz continuous for a. e. $x \in \Omega$, $|\frac{\partial}{\partial r} s^{r,0}(x)| \le 1$ for a. e. $x \in \Omega$ and a. e. $r > 0$.

We interpret the PDE in (1.1) in variational form and state the problem (1.1)–(1.2) coupled with (B.10) as follows

$$\begin{cases} \displaystyle\int_\Omega \big(u_t\,\zeta + \nabla u \cdot \nabla\zeta + \Big(\int_0^\infty f(x,r,u,s^r)\,\mathrm{d}\mu(r)\Big)\,\zeta\big)\,\mathrm{d}x = \int_\Omega v\,\zeta\,\mathrm{d}x & \text{in } (0,T), \\ s^r_t + \partial I\left(\frac{1}{r}s^r\right) \ni u_t & \text{in } Q, \\ s^r(\cdot,0) = s^{r,0}(\cdot) & \text{in } \Omega, \\ u(\cdot,0) = u^0(\cdot) & \text{in } \Omega \end{cases} \tag{1.3}$$

for every test function $\zeta \in W^{1,2}(\Omega)$, where I is the indicator function of the interval $[-r,r]$ defined in (B.11).

1.1 Penalty approximation

Following [21, 22], we approximate the indicator function I in (B.11) by a suitable C^2-function Ψ divided by a small penalty parameter $\gamma > 0$, and replace the differential inclusion (B.10) with an ODE. More specifically, we set

$$\Psi(s) = \phi(((|s|-1)^+) = \begin{cases} \phi(s-1) & \text{for } s > 1, \\ 0 & \text{for } s \in [-1,1], \\ \phi(-s-1) & \text{for } s < -1, \end{cases} \tag{1.4}$$

with a convex C^2-function $\phi : [0,\infty) \to [0,\infty)$ with quadratic growth, for example

$$\phi(z) = \begin{cases} \frac{1}{6}z^3 & \text{for } z \in [0,1], \\ \frac{1}{2}z^2 - \frac{1}{2}z + \frac{1}{6} & \text{for } z > 1. \end{cases} \tag{1.5}$$

Then we choose a small parameter $\gamma > 0$ and replace (1.3) with a system of one PDE and a continuum of ODEs for unknown functions $(u^\gamma, s^{r\gamma})$

$$\begin{cases} \displaystyle\int_\Omega \big(u^\gamma_t\,\zeta + \nabla u^\gamma \cdot \nabla\zeta + \Big(\int_0^\infty f(x,r,u^\gamma,s^{r\gamma})\,\mathrm{d}\mu(r)\Big)\,\zeta\big)\,\mathrm{d}x = \int_\Omega v\,\zeta\,\mathrm{d}x & \text{in } (0,T), \\ \displaystyle s^{r\gamma}_t + \frac{1}{\gamma}\Psi'\left(\frac{1}{r}s^{r\gamma}\right) = u^\gamma_t & \text{in } Q \end{cases} \tag{1.6}$$

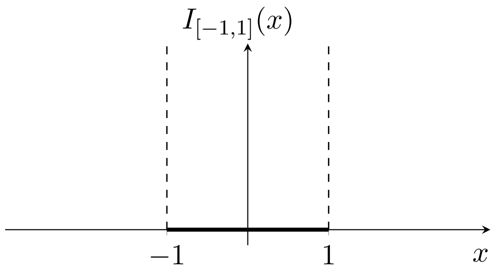

Figure 1.1. Indicator function $I_{[-1,1]}$.

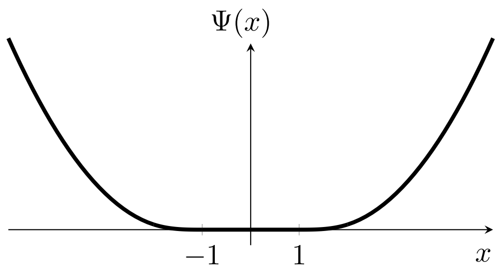

Figure 1.2. The function Ψ.

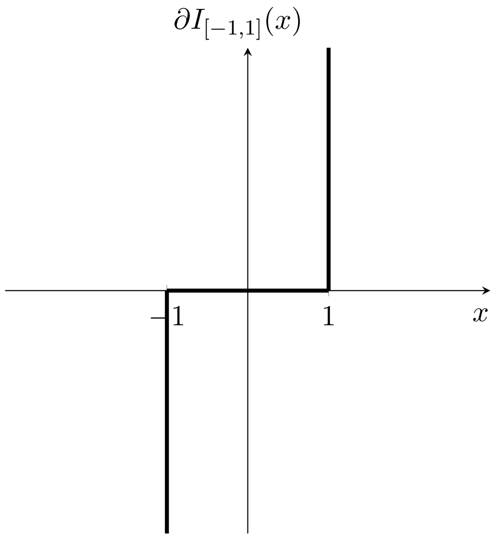

Figure 1.3. Subdifferential of the indicator function $I_{[-1,1]}$.

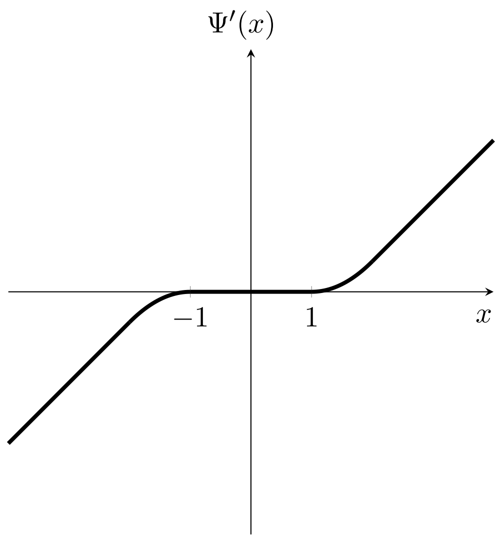

Figure 1.4. Smooth approximating function Ψ'.

for all $\zeta \in W^{1,2}(\Omega)$, with initial conditions

$$\begin{cases} s^{r\gamma}(\cdot,0) = s^{r,0}(\cdot) & \text{in } \Omega, \\ u^{\gamma}(\cdot,0) = u^{0}(\cdot) & \text{in } \Omega. \end{cases} \tag{1.7}$$

By [89, Theorem 1.12], the penalty approximation $s_t^{r\gamma} + \frac{1}{\gamma}\Psi'\left(\frac{1}{r}s^{r\gamma}\right) = u_t^{\gamma}$ of the stop operator has the same Lipschitz continuity property as the stop itself, namely the statement of Proposition B.5 holds for solutions $s_1^{r\gamma}, s_2^{r\gamma}$ corresponding to two inputs $u_1^{\gamma}, u_2^{\gamma}$, that is,

$$|s_1^{r\gamma}(t) - s_2^{r\gamma}(t)| \le |s_1^{r,0} - s_2^{r,0}| + 2 \max_{\tau\in[0,t]} |u_1^{\gamma}(\tau) - u_2^{\gamma}(\tau)| \tag{1.8}$$

for all $u_1^{\gamma}, u_2^{\gamma} \in W^{1,1}(0,T)$ and $s_1^{r,0}, s_2^{r,0} \in [-r,r]$.

1.2 Constrained minimization problem

We choose another small parameter $\varepsilon > 0$ independent of γ and define the cost functional

$$J_{\varepsilon}^{\gamma}(u^{\gamma}, \mathbf{s}^{\gamma}, v) = \frac{1}{2}\iint_Q v^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2\varepsilon}\int_{\Omega} (u^{\gamma})^2(x,T)\,\mathrm{d}x, \tag{1.9}$$

where the two summands represent the cost to implement the control and to reach the desired final state. We interpret here $\mathbf{s}^\gamma$ as a function of three variables r, x, t according to the formula

$$\mathbf{s}^\gamma(r,x,t) := s^{r\gamma}(x,t) \quad \text{for each } \ r>0,\ t\in(0,T) \text{ and a. e. } x\in\Omega.$$

For each $\gamma > 0$ we solve the following optimal control problem:

$$\text{minimize } J_\varepsilon^\gamma(u^\gamma, \mathbf{s}^\gamma, v) \text{ subject to (1.6) and (1.7).} \tag{1.10}$$

We first prove that for each $\varepsilon > 0$ and $\gamma > 0$ the minimization problem (1.10) has a solution $(u_\varepsilon^\gamma, \mathbf{s}_\varepsilon^\gamma, v_\varepsilon^\gamma)$.

Proposition 1.3. *Let $\varepsilon > 0$ and $\gamma > 0$ be given and let Hypotheses 1.1 and 1.2 hold. Then there exists $v = v_\varepsilon^\gamma \in L^2(Q)$ such that the corresponding solution $(u^\gamma, \mathbf{s}^\gamma) = (u_\varepsilon^\gamma, \mathbf{s}_\varepsilon^\gamma)$ to (1.6)–(1.7) with the regularity $u^\gamma \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;W^{1,2}(\Omega))$, $\mathbf{s}^\gamma \in W^{1,2}(0,T;L^2(\Omega))$ minimizes the value of $J_\varepsilon^\gamma(u^\gamma, \mathbf{s}^\gamma, v)$ in (1.9).*

Proof. The proof will be divided in two steps.

▶ Step 1: Existence of the solution

For each fixed $\gamma > 0$, we construct the solution $(u^\gamma, \mathbf{s}^\gamma)$ to (1.6)–(1.7) by Galerkin approximations. We choose $\mathcal{E} = \{e_k : k = 0,1,2,...\}$ in $L^2(\Omega)$ to be the complete orthonormal system of eigenfunctions defined by

$$-\Delta e_k = \lambda_k e_k \ \text{ in } \Omega, \quad \nabla e_k \cdot n\big|_{\partial\Omega} = 0,$$

with $\lambda_0 = 0$, $\lambda_k > 0$ for all $k \geq 1$. Then for $n \in \mathbb{N}$ we set

$$u_\gamma^{(n)}(x,t) = \sum_{k=0}^{n} u_k^\gamma(t)\, e_k(x)$$

with coefficients $u_k^\gamma : [0,T] \to \mathbb{R}$. Then $u_k^\gamma, s_{r\gamma}^{(n)}$ will be determined as the solution to the ODE system

$$\begin{cases} \dot{u}_k^\gamma + \lambda_k u_k^\gamma + \displaystyle\int_\Omega \left(\int_0^\infty f(x,r,u_\gamma^{(n)}, s_{r\gamma}^{(n)})\,\mathrm{d}\mu(r) \right) e_k \,\mathrm{d}x = \int_\Omega v\, e_k \,\mathrm{d}x & \text{in } (0,T), \\ (s_{r\gamma}^{(n)})_t + \dfrac{1}{\gamma}\Psi'\left(\dfrac{1}{r} s_{r\gamma}^{(n)}\right) = (u_\gamma^{(n)})_t & \text{in } Q, \end{cases} \tag{1.11}$$

with the initial conditions

$$\begin{cases} s_{r\gamma}^{(n)}(x,0) = s^{r,0}(x) & \text{in } \Omega, \\ u_k^\gamma(0) = \int_\Omega u^0(x)\, e_k(x)\,\mathrm{d}x. \end{cases} \tag{1.12}$$

The existence of a unique solution to (1.11)–(1.12) for each $v \in L^2(Q)$ is a consequence of the Lipschitz continuity (see Hypothesis 1.1 (iv) and (1.8)). Let us now prove that such a solution converges to a solution to (1.6)–(1.7). This will be done by deriving suitable estimates and then passing to the limit.

We start by testing the first equation in (1.11) by $\dot{u}_k^\gamma$ and summing up over $k = 0,1,2,\ldots$, which gives

$$\int_\Omega \left(|(u_\gamma^{(n)})_t|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla u_\gamma^{(n)}|^2 + \left(\int_0^\infty f(x,r,u_\gamma^{(n)},s_{r\gamma}^{(n)})\,\mathrm{d}\mu(r)\right)(u_\gamma^{(n)})_t\right)\mathrm{d}x = \int_\Omega v\,(u_\gamma^{(n)})_t\,\mathrm{d}x.$$

It follows from Hypothesis 1.1 (iv)–(v) that $f(x,r,u_\gamma^{(n)},s_{r\gamma}^{(n)}) \le C(1+|u_\gamma^{(n)}|+|s_{r\gamma}^{(n)}|)$ for a. e. $(x,t)\in Q$. Integrating $\int_0^\tau \mathrm{d}t$ for an arbitrary $\tau\in(0,T)$ and employing also Hypothesis 1.1 (i) we obtain

$$\begin{aligned}&\int_0^\tau\int_\Omega |(u_\gamma^{(n)})_t|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega |\nabla u_\gamma^{(n)}|^2(x,\tau)\,\mathrm{d}x - \int_\Omega |\nabla u^0|^2(x)\,\mathrm{d}x\\ &\le C\left(\int_0^\tau\int_\Omega |(u_\gamma^{(n)})_t|\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_\Omega |u_\gamma^{(n)}||(u_\gamma^{(n)})_t|\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_\Omega\left(\int_0^\infty |s_{r\gamma}^{(n)}|\,\mathrm{d}\mu(r)\right)|(u_\gamma^{(n)})_t|\,\mathrm{d}x\,\mathrm{d}t\right)\\ &+\int_0^\tau\int_\Omega v\,(u_\gamma^{(n)})_t\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

We now apply Young's inequality on the right-hand side. We recall that we are assuming $v\in L^2(Q)$, whereas the initial condition is bounded thanks to Hypothesis 1.2. Hence we obtain

$$\begin{aligned}&\int_0^\tau\int_\Omega |(u_\gamma^{(n)})_t|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega |\nabla u_\gamma^{(n)}|^2(x,\tau)\,\mathrm{d}x\\ &\le C\left(1+\int_0^\tau\int_\Omega |u_\gamma^{(n)}|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_\Omega\left(\int_0^\infty |s_{r\gamma}^{(n)}|^2(x,t)\,\mathrm{d}\mu(r)\right)\mathrm{d}x\,\mathrm{d}t\right).\end{aligned}\tag{1.13}$$

Now, testing the second equation of (1.6) by $s_{r\gamma}^{(n)}$ we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |s_{r\gamma}^{(n)}|^2\,\mathrm{d}x + \frac{1}{\gamma}\int_\Omega \Psi'\left(\frac{1}{r}s_{r\gamma}^{(n)}\right)s_{r\gamma}^{(n)}\,\mathrm{d}x = \int_\Omega (u_\gamma^{(n)})_t\,s_{r\gamma}^{(n)}\,\mathrm{d}x$$

for μ-a. e. $r>0$. Since the term $\Psi'\left(\frac{1}{r}s_{r\gamma}^{(n)}\right)s_{r\gamma}^{(n)}$ is always nonnegative (see Figure 1.4), by Young's inequality it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |s_{r\gamma}^{(n)}|^2\,\mathrm{d}x \le 2\int_\Omega (u_\gamma^{(n)})_t s_{r\gamma}^{(n)}\,\mathrm{d}x \le \int_\Omega |s_{r\gamma}^{(n)}|^2\,\mathrm{d}x + \int_\Omega |(u_\gamma^{(n)})_t|^2\,\mathrm{d}x.$$

Applying Grönwall's lemma A.1 we get

$$\int_\Omega |s_{r\gamma}^{(n)}|^2(x,\tau)\,\mathrm{d}x \le C\left(\int_\Omega |s^{r,0}|^2(x)\,\mathrm{d}x + \int_0^\tau\int_\Omega |(u_\gamma^{(n)})_t|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\right)$$

for a. e. $\tau\in(0,T)$ and for μ-a. e. $r>0$, and inserting (1.13) in the right-hand side we further obtain

$$\int_\Omega |s_{r\gamma}^{(n)}|^2(x,\tau)\,\mathrm{d}x \le C\left(1+\int_0^\tau\int_\Omega |u_\gamma^{(n)}|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_\Omega\left(\int_0^\infty |s_{r\gamma}^{(n)}|^2(x,t)\,\mathrm{d}\mu(r)\right)\mathrm{d}x\,\mathrm{d}t\right).$$

This gives, thanks to Hypothesis 1.1 (i),

$$\sup_{\mu-\text{a. e. } r>0}\int_\Omega |s_{r\gamma}^{(n)}|^2(x,\tau)\,\mathrm{d}x \le C\left(1+\int_0^\tau\int_\Omega |u_\gamma^{(n)}|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \sup_{\mu-\text{a. e. } r>0}\left(\int_\Omega |s_{r\gamma}^{(n)}|^2(x,t)\,\mathrm{d}x\right)\mathrm{d}t\right)$$

for a. e. $\tau\in(0,T)$. Applying Grönwall's lemma A.2 and coming back to (1.13) we obtain

$$\int_0^\tau\int_\Omega |(u_\gamma^{(n)})_t|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega |\nabla u_\gamma^{(n)}|^2(x,\tau)\,\mathrm{d}x \le C\left(1+\int_0^\tau\int_\Omega |u_\gamma^{(n)}|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\right).\tag{1.14}$$

Notice that it holds also

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_\gamma^{(n)}|^2\,\mathrm{d}x = 2\int_\Omega u_\gamma^{(n)}(u_\gamma^{(n)})_t\,\mathrm{d}x \le \int_\Omega |u_\gamma^{(n)}|^2\,\mathrm{d}x + \int_\Omega |(u_\gamma^{(n)})_t|^2\,\mathrm{d}x,$$

hence integrating in time $\int_0^\tau \mathrm{d}t$, inserting (1.14) in the right-hand side and using Grönwall's lemma A.2 we get

$$\operatorname*{sup\,ess}_{\tau\in(0,T)} \int_\Omega |u_\gamma^{(n)}|^2(x,\tau)\,\mathrm{d}x \le C, \tag{1.15}$$

with a constant C independent of n. Then (1.14) yields

$$\iint_Q |(u_\gamma^{(n)})_t|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \operatorname*{sup\,ess}_{\tau\in(0,T)} \int_\Omega |\nabla u_\gamma^{(n)}|^2(x,\tau)\,\mathrm{d}x \le C \tag{1.16}$$

with a constant C independent of n. We now choose a subsequence (still indexed by n for simplicity) such that

$$(u_\gamma^{(n)})_t \to u_t^\gamma \qquad \text{weakly in } L^2(Q). \tag{1.17}$$

By estimates (1.15), (1.16) we infer that the sequence $u_\gamma^{(n)}$ is uniformly bounded (independently of n) in $W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;W^{1,2}(\Omega))$, hence by Lemma A.5 we conclude that

$$u_\gamma^{(n)} \to u^\gamma \qquad \text{strongly in } L^2(\Omega;C[0,T]), \tag{1.18}$$

and by Proposition B.5 also that

$$s_{r\gamma}^{(n)} \to s^{r\gamma} \qquad \text{strongly in } L^2(\Omega;C[0,T]). \tag{1.19}$$

Note that we need the convergence of the sequences $u_\gamma^{(n)}, s_{r\gamma}^{(n)}$ to be strong in order to pass to the limit in the nonlinearity f.

Testing the second equation of (1.6) by $(s_{r\gamma}^{(n)})_t$ we obtain

$$\int_\Omega \left(|(s_{r\gamma}^{(n)})_t|^2 + \frac{1}{\gamma}\Psi'\left(\frac{1}{r}s_{r\gamma}^{(n)}\right)(s_{r\gamma}^{(n)})_t\right)\mathrm{d}x = \int_\Omega (u_\gamma^{(n)})_t (s_{r\gamma}^{(n)})_t\,\mathrm{d}x.$$

Since $\Psi'\left(\frac{1}{r}s_{r\gamma}^{(n)}\right)(s_{r\gamma}^{(n)})_t = r\frac{\mathrm{d}}{\mathrm{d}t}\Psi\left(\frac{1}{r}s_{r\gamma}^{(n)}\right)$, integrating $\int_0^\tau \mathrm{d}t$ for an arbitrary $\tau\in(0,T)$ we get

$$\begin{aligned}\int_0^\tau\int_\Omega |(s_{r\gamma}^{(n)})_t|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t &+ \frac{r}{\gamma}\int_\Omega \Psi\left(\frac{1}{r}s_{r\gamma}^{(n)}\right)(x,\tau)\,\mathrm{d}x - \frac{r}{\gamma}\int_\Omega \Psi\left(\frac{1}{r}s^{r,0}\right)(x)\,\mathrm{d}x \\ &= \int_0^\tau\int_\Omega (u_\gamma^{(n)})_t(s_{r\gamma}^{(n)})_t\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Now, since $s^{r,0}\in[-r,r]$, it follows that $\frac{1}{r}s^{r,0}\in[-1,1]$. Thus the definition of Ψ in (1.4) yields $\Psi\left(\frac{1}{r}s^{r,0}\right)=0$. Applying Young's inequality on the right-hand side and using (1.16) we finally have

$$\iint_Q |(s_{r\gamma}^{(n)})_t|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \operatorname*{sup\,ess}_{\tau\in(0,T)} \frac{r}{\gamma}\int_\Omega \Psi\left(\frac{1}{r}s_{r\gamma}^{(n)}\right)(x,\tau)\,\mathrm{d}x \le C$$

with a constant C independent of n. Hence we see that, up to a subsequence,

$$(s_{r\gamma}^{(n)})_t \to s_t^{r\gamma} \qquad \text{weakly in } L^2(Q). \tag{1.20}$$

Since the convergences (1.17)–(1.20) take place, we may pass to the limit in (1.11) and conclude that $(u^\gamma, \mathbf{s}^\gamma)$ is the solution to (1.6)–(1.7) with the desired regularity.

▶ Step 2: The solution minimizes the functional

We are now going to show that there exists $v = v_\varepsilon^\gamma \in L^2(Q)$ such that the solution $(u^\gamma, \mathbf{s}^\gamma) = (u_\varepsilon^\gamma, \mathbf{s}_\varepsilon^\gamma)$ to (1.6)–(1.7) minimizes the value of $J_\varepsilon^\gamma(u^\gamma, \mathbf{s}^\gamma, v)$. Note that the functional J_ε^γ is bounded from below by 0, and we can denote by J^* its infimum. There exists a minimizing sequence $\{v_j\}_{j\in\mathbb{N}}$, that is, $\lim_{j\to\infty} J_\varepsilon^\gamma(u_j^\gamma, \mathbf{s}_j^\gamma, v_j) = J^*$, where $(u_j^\gamma, \mathbf{s}_j^\gamma)$ are the solutions to (1.6)–(1.7) associated with the right-hand side v_j. We may assume that $J_\varepsilon^\gamma(u_j^\gamma, \mathbf{s}_j^\gamma, v_j) \le J^* + 1$ for all $j \in \mathbb{N}$, so that the $L^2(Q)$-norms of the v_j's are bounded above by $\sqrt{2(J^*+1)}$. Thus, arguing as in the previous step, we end up with the inequalities

$$\begin{aligned}
\operatorname*{sup\,ess}_{\tau\in(0,T)} \int_\Omega |u_j^\gamma|^2(x,\tau)\,\mathrm{d}x &\le C,\\
\iint_Q |(u_j^\gamma)_t|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \operatorname*{sup\,ess}_{\tau\in(0,T)} \int_\Omega |\nabla u_j^\gamma|^2(x,\tau)\,\mathrm{d}x &\le C,\\
\iint_Q |(s_j^{r\gamma})_t|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \operatorname*{sup\,ess}_{\tau\in(0,T)} \frac{r}{\gamma}\int_\Omega \Psi\left(\frac{1}{r}s_j^{r\gamma}\right)(x,\tau)\,\mathrm{d}x &\le C,
\end{aligned}$$

with constants C independent of j. Hence it is possible to choose subsequences (still indexed by j for simplicity) such that

$$\begin{aligned}
v_j \to v, \quad (u_j^\gamma)_t \to u_t^\gamma, \quad (s_j^{r\gamma})_t \to s_t^{r\gamma} \quad &\text{weakly in } L^2(Q),\\
u_j^\gamma \to u^\gamma, \quad s_j^{r\gamma} \to s^{r\gamma} \quad &\text{strongly in } L^2(\Omega; C[0,T]).
\end{aligned}$$

Thus we may pass to the limit and conclude that $(u^\gamma, \mathbf{s}^\gamma)$ is the solution to (1.6)–(1.7) corresponding to the right-hand side v.

By the weak lower semicontinuity of the norm we further have

$$\begin{aligned}
\iint_Q v^2\,\mathrm{d}x\,\mathrm{d}t &\le \liminf_{j\to\infty} \iint_Q v_j^2\,\mathrm{d}x\,\mathrm{d}t,\\
\int_\Omega (u^\gamma)^2(x,T)\,\mathrm{d}x &= \lim_{j\to\infty} \int_\Omega (u_j^\gamma)^2(x,T)\,\mathrm{d}x.
\end{aligned}$$

Hence v is the desired minimizer. This concludes the proof of Proposition 1.3. □

The main result for system (1.1) is the following.

Theorem 1.4. *Let Hypotheses 1.1 and 1.2 be satisfied. Then there exists $v \in L^2(Q)$ such that the corresponding solution $(u, \mathbf{s})$ to Problem (1.3), $\mathbf{s}(r,x,t) := s^r(x,t)$ for $r > 0$, $u \in W^{1,2}(0,T; L^2(\Omega)) \cap$*

$L^\infty(0,T;W^{1,2}(\Omega))$ and $s^r \in W^{1,2}(0,T;L^2(\Omega))$, satisfies $u(x,T)=0$ for a. e. $x\in\Omega$. Moreover, there exists a constant $C>0$ depending only on the data of the problem and sequences $\gamma_n \to 0$, $\varepsilon_n \to 0$ as $n\to\infty$, $n\in\mathbb{N}$, such that the system of minimizers $(u_n,\mathbf{s}_n,v_n) := (u^{\gamma_n}_{\varepsilon_n}, \mathbf{s}^{\gamma_n}_{\varepsilon_n}, v^{\gamma_n}_{\varepsilon_n})$ from Proposition 1.3 approximates the controllability problem in the following sense:

(i) $\displaystyle\iint_Q |(u_n)_t|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \sup_{t\in(0,T)}\mathrm{ess}\int_\Omega |\nabla u_n|^2(x,t)\,\mathrm{d}x \le C;$

(ii) $\displaystyle\lim_{n\to\infty}\int_\Omega \max_{t\in[0,T]} |u(x,t)-u_n(x,t)|^2\,\mathrm{d}x = 0;$

(iii) $\displaystyle\int_\Omega |u_n(x,T)|^2\,\mathrm{d}x \le C\varepsilon_n;$

(iv) $\displaystyle\max_{t\in[0,T]}\int_\Omega |s^r(x,t)-s^r_n(x,t)|^2\,\mathrm{d}x \le C\Big(\big(1+r^{2/3}\big)\gamma_n^{1/3} + 1/n^2\Big).$

The null-controllability condition in Theorem 1.4 can be easily extended to the general case in the following form.

Corollary 1.5. *Let the hypotheses of Theorem 1.4 hold and let $\bar u \in W^{2,2}(\Omega)\cap L^\infty(\Omega)$ be given such that $n\cdot\nabla\bar u = 0$ on Γ. Then there exists $v\in L^2(Q)$ such that the corresponding solution $u\in W^{1,2}(0,T;L^2(\Omega))\cap L^\infty(0,T;W^{1,2}(\Omega))$ to Problem* (1.3) *satisfies $u(x,T)=\bar u(x)$ for a. e. $x\in\Omega$.*

Indeed, it suffices to introduce a new unknown function $\tilde u(x,t) = u(x,t)-\bar u(x)$, replace v with $\tilde v = v+\Delta\bar u$, u^0 with $\tilde u^0 = u^0-\bar u$, and $f(x,r,u,s^r)$ with $\tilde f(x,r,\tilde u,s^r) = f(x,r,\tilde u+\bar u,s^r)$. The solution $\tilde u$ to the null-controllability problem stated in Theorem 1.4 then yields the solution to the general case.

Remark 1.6. As mentioned in the Introduction, the existence of a control for system (1.1) is a consequence of the Kakutani fixed point principle. We are going to give a short proof of this fact. Let us define for simplicity $X := W^{1,2}(0,T;L^2(\Omega))\cap L^\infty(0,T;W^{1,2}(\Omega))$, and consider the convex set $K := \Big\{z\in X : \|z\|_X \le \tilde C\Big\}$ endowed with the topology of $L^2(\Omega;C[0,T])$, which makes it compact by Lemma A.5. Then we consider the multivalued map

$$\begin{aligned}\Phi : K &\to 2^X\\ z &\mapsto u\end{aligned}$$

which to any $z\in K$ associates the set of all possible u constructed by solving the controllability problem

$$\begin{cases} u_t - \Delta u = -\mathcal{F}[z] + v & \text{in } Q,\\ u(\cdot,0) = u^0(\cdot) & \text{in } \Omega,\\ n\cdot\nabla u = 0 & \text{on } \Gamma,\end{cases}$$

with $\mathcal{F}$ as in (1.2), and taking all the possible solutions (u, v). In particular $v = \mathcal{F}[z] + w$, where w is the control that drives the system obeying to the linear equation $u_t - \Delta u = w$ from the initial state to the target in the time interval $[0, T]$ (the existence of such a w follows by a standard argument, see e.g. [9]). Moreover w is such that $\|w\|_{L^2(Q)} \le C\|u_0\|_{L^2(\Omega)}$. Hence, by the standard theory of parabolic equations, we see that, if $\tilde{C}$ is sufficiently large, then Φ maps K into itself. Moreover, for each $z \in K$, the set $\Phi(z)$ is nonempty (since the linear heat equation is controllable) and convex. In order to apply the Kakutani fixed point principle, it remains to prove that Φ has closed graph, that is, if $\{z_n\}_{n\in\mathbb{N}} \subset K$ is such that $z_n \to z$ in $L^2(\Omega; C[0, T])$ and $u_n \in \Phi(z_n)$ is such that $u_n \to u$ in $L^2(\Omega; C[0, T])$, then $u \in \Phi(z)$. But $u_n \in \Phi(z_n)$ means that

$$\begin{cases} (u_n)_t - \Delta u_n = -\mathcal{F}[z_n] + v_n & \text{in } Q, \\ u_n(\cdot,0) = u^0(\cdot) & \text{in } \Omega, \\ n \cdot \nabla u_n = 0 & \text{on } \Gamma. \end{cases} \tag{1.21}$$

If $z_n \to z$ in $L^2(\Omega; C[0, T])$, then (at least along a subsequence) $z_n \to z$ uniformly in $[0, T]$ for a.e. $x \in \Omega$. Hence by Hypothesis 1.1 and Proposition B.5 also $\mathcal{F}[z_n] \to \mathcal{F}[z]$ uniformly in $[0, T]$ for a.e. $x \in \Omega$. Note that by Hypothesis 1.1 and since $z_n \in K$

$$\begin{aligned} \iint_Q |\mathcal{F}[z_n]|^2 \,\mathrm{d}x\,\mathrm{d}t &\le C \iint_Q \left| \int_0^\infty (1 + |z_n| + |s^r|) \,\mathrm{d}\mu(r) \right|^2 \mathrm{d}x\,\mathrm{d}t \\ &\le C \left(\iint_Q \left| \int_0^\infty (1 + r) \,\mathrm{d}\mu(r) \right|^2 \mathrm{d}x\,\mathrm{d}t + \iint_Q |z_n|^2 \,\mathrm{d}x\,\mathrm{d}t \int_0^\infty \mathrm{d}\mu(r) \right) \le C, \end{aligned}$$

that is, $\mathcal{F}[z_n]$ is bounded in $L^2(Q)$ independently of n. Then (at least along a subsequence) $\mathcal{F}[z_n] \to \mathcal{F}[z]$ weakly in $L^2(Q)$, and the control $v_n = \mathcal{F}[z_n] + w_n$ is such that $\|v_n\|_{L^2(Q)} \le C\left(1 + \|u_0\|_{L^2(\Omega)}\right)$ with C independent of n. Selecting a subsequence we obtain $v_n \to v$ weakly in $L^2(Q)$. Then we can pass to the limit in (1.21), and conclude that $u \in \Phi(z)$. An infinite dimensional version of the Kakutani theorem gives the existence of a fixed point for Φ, which exactly corresponds to our controllability claim.

It is now clear that we are not interested simply in the existence of the control. Our aim is to construct a control algorithm based on passing to the limit in (1.10) as $\varepsilon \to 0$ and $\gamma \to 0$.

CHAPTER 2

Construction of the solution to the controllability problem

2.1 Necessary optimality conditions

We first derive necessary optimality conditions for problem (1.10). The classical Lagrange method consists in finding critical points of the Lagrange functional

$$\mathcal{L}(u^\gamma, \mathbf{s}^\gamma, v) = J_\varepsilon^\gamma(u^\gamma, \mathbf{s}^\gamma, v) + \langle\langle p, G_1(u^\gamma, \mathbf{s}^\gamma, v)\rangle\rangle + \int_0^\infty \langle q^r, G_2^r(u^\gamma, \mathbf{s}^\gamma, v)\rangle \, \mathrm{d}\mu(r),$$

where p, q^r are the Lagrange multipliers, the double brackets denote the duality pairing between $L^2(0,T;W^{1,2}(\Omega))$ and $L^2(0,T;W^{-1,2}(\Omega))$, the single brackets denote the canonical scalar product in $L^2(Q)$, and the constraints are

$$\begin{aligned} G_1(u^\gamma, \mathbf{s}^\gamma, v) &= u_t^\gamma - \Delta u^\gamma + \int_0^\infty f(x, r, u^\gamma, s^{r\gamma}) \, \mathrm{d}\mu(r) - v, \\ G_2^r(u^\gamma, \mathbf{s}^\gamma, v) &= s_t^{r\gamma} + \frac{1}{\gamma}\Psi'\left(\frac{1}{r}s^{r\gamma}\right) - u_t^\gamma. \end{aligned}$$

To explain the argument, let us first assume that the multipliers p, q^r possess the regularity $p \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;W^{1,2}(\Omega))$, $q^r \in W^{1,2}(0,T;L^2(\Omega))$. This assumption will be justified at the end of this section.

At a critical point, the directional derivative of $\mathcal{L}$ vanishes in every regular direction $(\hat{u}, \hat{\mathbf{s}}, \hat{v})$ such that $\hat{u}(x,0) = \hat{\mathbf{s}}(x,0) = 0$ in Ω, $n \cdot \nabla \hat{u} = 0$ on Γ. In other words, p and q^r have to be chosen in such a way that

$$\lim_{\tau \to 0} \frac{1}{\tau}\big(\mathcal{L}(u^\gamma + \tau\hat{u}, \mathbf{s}^\gamma + \tau\hat{\mathbf{s}}, v + \tau\hat{v}) - \mathcal{L}(u^\gamma, \mathbf{s}^\gamma, v)\big) = 0,$$

that is,

$$\lim_{\tau\to 0}\frac{1}{\tau}\Bigg(\frac{1}{2}\iint_Q (v+\tau\hat{v})^2\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\iint_Q v^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2\varepsilon}\int_\Omega (u^\gamma+\tau\hat{u})^2(x,T)\,\mathrm{d}x - \frac{1}{2\varepsilon}\int_\Omega (u^\gamma)^2(x,T)\,\mathrm{d}x$$
$$+\iint_Q p\left((u^\gamma+\tau\hat{u})_t + \int_0^\infty f(x,r,u^\gamma+\tau\hat{u},s^{r\gamma}+\tau\hat{s}^r)\,\mathrm{d}\mu(r) - v - \tau\hat{v}\right)\mathrm{d}x\,\mathrm{d}t$$
$$+\iint_Q \nabla p\cdot\nabla(u^\gamma+\tau\hat{u})\,\mathrm{d}x\,\mathrm{d}t$$
$$-\iint_Q p\left(u_t^\gamma + \int_0^\infty f(x,r,u^\gamma,s^{r\gamma})\,\mathrm{d}\mu(r) - v\right)\mathrm{d}x\,\mathrm{d}t - \iint_Q \nabla p\cdot\nabla u^\gamma\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^\infty\iint_Q q^r\left((s^{r\gamma}+\tau\hat{s}^r)_t + \frac{1}{\gamma}\Psi'\left(\frac{1}{r}(s^{r\gamma}+\tau\hat{s}^r)\right) - (u^\gamma+\tau\hat{u})_t\right)\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\mu(r)$$
$$-\int_0^\infty\iint_Q q^r\left(s_t^{r\gamma} + \frac{1}{\gamma}\Psi'\left(\frac{1}{r}s^{r\gamma}\right) - u_t^\gamma\right)\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\mu(r)\Bigg) = 0.$$

This implies

$$0 = \iint_Q v\hat{v}\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_\Omega u^\gamma(x,T)\hat{u}(x,T)\,\mathrm{d}x$$
$$+\iint_Q p\left(\hat{u}_t + \int_0^\infty (f_{u^\gamma}(x,r,u^\gamma,s^{r\gamma})\hat{u} + f_{s^{r\gamma}}(x,r,u^\gamma,s^{r\gamma})\hat{s}^r)\,\mathrm{d}\mu(r) - \hat{v}\right)\mathrm{d}x\,\mathrm{d}t$$
$$+\iint_Q \nabla p\cdot\nabla\hat{u}\,\mathrm{d}x\,\mathrm{d}t + \iint_Q\int_0^\infty q^r\left(\hat{s}_t^r + \frac{1}{r\gamma}\Psi''\left(\frac{1}{r}s^{r\gamma}\right)\hat{s}^r - \hat{u}_t\right)\mathrm{d}\mu(r)\,\mathrm{d}x\,\mathrm{d}t,$$

where $f_{s^{r\gamma}}(x,r,u^\gamma,s^{r\gamma})$ is to be interpreted here and in the sequel as $f_\sigma(x,r,u^\gamma,\sigma)|_{\sigma=s^{r\gamma}}$ and, similarly, $f_{u^\gamma}(x,r,u^\gamma,s^{r\gamma}) := f_u(x,r,u,s^{r\gamma})|_{u=u^\gamma}$.

Integrating the above identity by parts in time we get

$$0 = \iint_Q v\hat{v}\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_\Omega u^\gamma(x,T)\hat{u}(x,T)\,\mathrm{d}x + \int_\Omega p(x,T)\hat{u}(x,T)\,\mathrm{d}x - \int_\Omega p(x,0)\hat{u}(x,0)\,\mathrm{d}x$$
$$-\iint_Q p_t\hat{u}\,\mathrm{d}x\,\mathrm{d}t + \iint_Q p\left(\int_0^\infty (f_{u^\gamma}(x,r,u^\gamma,s^{r\gamma})\hat{u} + f_{s^{r\gamma}}(x,r,u^\gamma,s^{r\gamma})\hat{s}^r)\,\mathrm{d}\mu(r) - \hat{v}\right)\mathrm{d}x\,\mathrm{d}t$$
$$+\iint_Q \nabla p\cdot\nabla\hat{u}\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega\int_0^\infty q^r(x,T)\hat{s}^r(x,T)\,\mathrm{d}\mu(r)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega\int_0^\infty q^r(x,0)\hat{s}^r(x,0)\,\mathrm{d}\mu(r)\,\mathrm{d}x\,\mathrm{d}t$$
$$-\iint_Q\int_0^\infty \hat{q}_t^r s^r\,\mathrm{d}\mu(r)\,\mathrm{d}x\,\mathrm{d}t + \iint_Q\int_0^\infty \frac{1}{r\gamma}\Psi''\left(\frac{1}{r}s^{r\gamma}\right)q^r\hat{s}^r\,\mathrm{d}\mu(r)\,\mathrm{d}x\,\mathrm{d}t$$
$$-\int_\Omega\int_0^\infty q^r(x,T)\hat{u}(x,T)\,\mathrm{d}\mu(r)\,\mathrm{d}x + \int_\Omega\int_0^\infty q^r(x,0)\hat{u}(x,0)\,\mathrm{d}\mu(r)\,\mathrm{d}x + \iint_Q\int_0^\infty q_t^r\hat{u}\,\mathrm{d}\mu(r)\,\mathrm{d}x\,\mathrm{d}t,$$

that is, rearranging the terms and exploiting the null initial conditions for the regular directions $\hat{u}$ and $\hat{\mathbf{s}}$,

$$0 = \int_\Omega \hat{u}\left(p - \int_0^\infty q^r\,\mathrm{d}\mu(r) + \frac{1}{\varepsilon}u^\gamma\right)(x,T)\,\mathrm{d}x + \int_\Omega\int_0^\infty \hat{s}^r(x,T)q^r(x,T)\,\mathrm{d}\mu(r)\,\mathrm{d}x$$
$$+\iint_Q \nabla p\cdot\nabla\hat{u}\,\mathrm{d}x\,\mathrm{d}t - \iint_Q \hat{u}\left(p_t - \int_0^\infty (q_t^r + f_{u^\gamma}(x,r,u^\gamma,s^{r\gamma})p)\,\mathrm{d}\mu(r)\right)\mathrm{d}x\,\mathrm{d}t$$
$$-\iint_Q\int_0^\infty \hat{s}^r\left(q_t^r - \frac{1}{r\gamma}\Psi''\left(\frac{1}{r}s^{r\gamma}\right)q^r - f_{s^{r\gamma}}(x,r,u^\gamma,s^{r\gamma})p\right)\mathrm{d}\mu(r)\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \hat{v}(v-p)\,\mathrm{d}x\,\mathrm{d}t$$

for all admissible directions $(\hat{u}, \hat{\mathbf{s}}, \hat{v})$. We thus necessarily have

$$v = p \quad \text{a.e. in } Q, \tag{2.1}$$

and p, q^r are the solutions to the backward dual problem

$$\begin{cases} \int_\Omega \left(p_t\,\eta - \nabla p\cdot\nabla\eta - \left(\int_0^\infty (q_t^r + f_{u^\gamma}(x,r,u^\gamma,s^{r\gamma})\,p)\,\mathrm{d}\mu(r)\right)\eta\right)\,\mathrm{d}x = 0 & \text{in } (0,T), \\ q_t^r - \frac{1}{r\gamma}\Psi''\left(\frac{1}{r}s^{r\gamma}\right)q^r - f_{s^{r\gamma}}(x,r,u^\gamma,s^{r\gamma})\,p = 0 & \text{in } Q, \\ p(\cdot,T) = -\frac{1}{\varepsilon}u^\gamma(\cdot,T) & \text{in } \Omega, \\ q^r(\cdot,T) = 0 & \text{in } \Omega \end{cases} \tag{2.2}$$

for every test function $\eta \in W^{1,2}(\Omega)$, where the second and the fourth equations have to be fulfilled for μ-a. e. $r > 0$. This is a standard linear backward parabolic equation coupled with a linear ODE, hence it admits for every $\gamma > 0$ and every $\varepsilon > 0$ a unique solution with the regularity $p \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;W^{1,2}(\Omega))$, $q^r \in W^{1,2}(0,T;L^2(\Omega))$. This can be proved arguing as we did to prove Proposition 1.3. Actually here the situation is even simpler, since the system is linear. This justifies the above formal computations.

2.2 Derivation of estimates

In order to pass to the limits $\varepsilon \to 0, \gamma \to 0$, we first derive a series of estimates for $(u^\gamma, \mathbf{s}^\gamma, v, p, q)$ $= (u_\varepsilon^\gamma, \mathbf{s}_\varepsilon^\gamma, v_\varepsilon^\gamma, p_\varepsilon^\gamma, q_\varepsilon^\gamma)$ satisfying (1.6), (2.1), and (2.2). In what follows, we denote by C any positive constant independent of γ and ε.

We first multiply the second equation of (2.2) by $-\mathrm{sign}(q^r)$ to get

$$\begin{aligned} 0 &= -q_t^r\,\mathrm{sign}(q^r) + \frac{1}{r\gamma}\Psi''\left(\frac{1}{r}s^{r\gamma}\right)q^r\mathrm{sign}(q^r) + f_{s^{r\gamma}}(x,r,u^\gamma,s^{r\gamma})p\,\mathrm{sign}(q^r) \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}|q^r| + \frac{1}{r\gamma}\Psi''\left(\frac{1}{r}s^{r\gamma}\right)|q^r| + f_{s^{r\gamma}}(x,r,u^\gamma,s^{r\gamma})p\,\mathrm{sign}(q^r). \end{aligned}$$

Integrating from an arbitrary $t \in [0,T)$ to T we obtain

$$-|q^r(x,T)| + |q^r(x,t)| + \int_t^T \frac{1}{r\gamma}\Psi''\left(\frac{1}{r}s^{r\gamma}(x,\tau)\right)|q^r(x,\tau)|\,\mathrm{d}\tau \le \int_t^T |f_{s^{r\gamma}}(x,r,u^\gamma,s^{r\gamma})||p(x,\tau)|\,\mathrm{d}\tau.$$

But then, using the final condition for q^r in (2.2) and Hypothesis 1.1 (iv), we get that

$$|q^r(x,t)| + \int_t^T \frac{1}{r\gamma}\Psi''\left(\frac{1}{r}s^{r\gamma}(x,\tau)\right)|q^r(x,\tau)|\,\mathrm{d}\tau \le C\int_t^T |p(x,\tau)|\,\mathrm{d}\tau. \tag{2.3}$$

In the next step, we combine the first and the second equation of (2.2) to get

$$\begin{aligned} &\int_\Omega (p_t\eta - \nabla p\cdot\nabla\eta)\,\mathrm{d}x \\ &= \int_\Omega\int_0^\infty \left(\frac{1}{r\gamma}\Psi''\left(\frac{1}{r}s^{r\gamma}\right)q^r + f_{s^{r\gamma}}(x,r,u^\gamma,s^{r\gamma})p + f_{u^\gamma}(x,r,u^\gamma,s^{r\gamma})p\right)\,\mathrm{d}\mu(r)\eta\,\mathrm{d}x, \end{aligned}$$

and test the resulting equation by a Lipschitz continuous approximation $\eta = S_n(p)$ of $-\mathrm{sign}(p)$, say, $S_n(p) = -\mathrm{sign}(p)$ for $|p| \geq 1/n$, $S_n(p) = -np$ for $|p| < 1/n$.

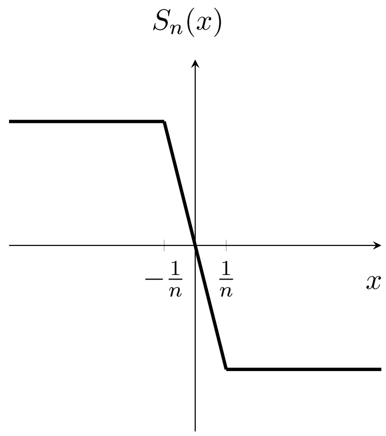

Figure 2.1. Lipschitz continuous approximation of $-\mathrm{sign}(x)$.

We get

$$\begin{aligned}&\int_\Omega \left(p_t S_n(p) - \nabla p \cdot \nabla S_n(p)\right) \mathrm{d}x \\ &= \int_\Omega \int_0^\infty \left(\frac{1}{r\gamma}\Psi''\left(\frac{1}{r}s^{r\gamma}\right) q^r + f_{s^{r\gamma}}(x,r,u^\gamma,s^{r\gamma})p + f_{u^\gamma}(x,r,u^\gamma,s^{r\gamma})p\right) \mathrm{d}\mu(r) S_n(p)\,\mathrm{d}x.\end{aligned}$$

The term $-\nabla p \cdot \nabla S_n(p) = -|\nabla p|^2 S_n'(p)$ is nonnegative for every n, hence letting n tend to infinity and using Hypothesis 1.1 (iv) we obtain for a. e. $t \in (0,T)$ that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |p(x,t)|\,\mathrm{d}x \leq C\int_\Omega |p(x,t)|\,\mathrm{d}x + \int_\Omega\int_0^\infty \frac{1}{r\gamma}\Psi''\left(\frac{1}{r}s^{r\gamma}(x,t)\right)|q^r(x,t)|\,\mathrm{d}\mu(r)\,\mathrm{d}x. \tag{2.4}$$

Integrating (2.4) $\int_0^\tau \mathrm{d}t$ yields

$$\begin{aligned}&-\int_\Omega |p(x,\tau)|\,\mathrm{d}x + \int_\Omega |p(x,0)|\,\mathrm{d}x \\ &\leq C\iint_Q |p(x,t)|\,\mathrm{d}x\,\mathrm{d}t + \iint_Q\int_0^\infty \frac{1}{r\gamma}\Psi''\left(\frac{1}{r}s^{r\gamma}(x,t)\right)|q^r(x,t)|\,\mathrm{d}\mu(r)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

We integrate again $\int_0^T \mathrm{d}\tau$ and switch the order of integration, thus getting

$$\begin{aligned}&-\int_\Omega |p(x,\tau)|\,\mathrm{d}x + \int_\Omega |p(x,0)|\,\mathrm{d}x \\ &\leq C\iint_Q |p(x,t)|\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega\int_0^\infty\int_0^T \frac{1}{r\gamma}\Psi''\left(\frac{1}{r}s^{r\gamma}(x,t)\right)|q^r(x,t)|\,\mathrm{d}t\,\mathrm{d}\mu(r)\,\mathrm{d}x.\end{aligned}$$

Using the estimate (2.3), we get a bound for $p(x,0)$ and $q^r(x,0)$, namely

$$\int_\Omega \left(|p(x,0)| + \sup_{\mu-\text{a.e. } r>0} |q^r(x,0)|\right) \mathrm{d}x \leq C\iint_Q |p(x,t)|\,\mathrm{d}x\,\mathrm{d}t. \tag{2.5}$$

Finally, we test

- the first equation in (1.6) by p,
- the second equation in (1.6) by q^r,
- the first equation in (2.2) by u^γ,
- the second equation in (2.2) by $s^{r\gamma}$,

sum up the resulting equations to get

$$\begin{aligned}&\int_\Omega \left(u_t^\gamma p + \nabla u^\gamma \cdot \nabla p + \left(\int_0^\infty f(x,r,u^\gamma,s^{r\gamma})\,\mathrm{d}\mu(r)\right)p\right)\mathrm{d}x\\ &+\int_\Omega\int_0^\infty \left(s_t^{r\gamma} + \frac{1}{\gamma}\Psi'\left(\frac{1}{r}s^{r\gamma}\right)\right)q^r\,\mathrm{d}\mu(r)\,\mathrm{d}x\\ &+\int_\Omega \left(p_t u^\gamma - \nabla p\cdot\nabla u^\gamma - \left(\int_0^\infty (q_t^r + f_{u^\gamma}(x,r,u^\gamma,s^{r\gamma})p)\,\mathrm{d}\mu(r)\right)u^\gamma\right)\mathrm{d}x\\ &+\int_\Omega\int_0^\infty \left(q_t^r - \frac{1}{r\gamma}\Psi''\left(\frac{1}{r}s^{r\gamma}\right)q^r - f_{s^{r\gamma}}(x,r,u^\gamma,s^{r\gamma})p\right)s^{r\gamma}\,\mathrm{d}\mu(r)\,\mathrm{d}x\\ &=\int_\Omega vp\,\mathrm{d}x + \int_\Omega\int_0^\infty u_t^\gamma q^r\,\mathrm{d}\mu(r)\,\mathrm{d}x.\end{aligned}$$

This entails, together with (2.1),

$$\begin{aligned}&\int_\Omega\left(u^\gamma p + \int_0^\infty (s^{r\gamma}q^r - u^\gamma q^r)\,\mathrm{d}\mu(r)\right)_t\mathrm{d}x + \int_\Omega\left(\int_0^\infty (f - u^\gamma f_{u^\gamma} - s^{r\gamma}f_{s^{r\gamma}})(x,r,u^\gamma,s^{r\gamma})\,\mathrm{d}\mu(r)\right)p\,\mathrm{d}x\\ &+\frac{1}{\gamma}\int_\Omega\int_0^\infty q^r\left(\Psi'\left(\frac{1}{r}s^{r\gamma}\right) - \frac{1}{r}s^{r\gamma}\Psi''\left(\frac{1}{r}s^{r\gamma}\right)\right)\mathrm{d}\mu(r)\,\mathrm{d}x = \int_\Omega p^2\,\mathrm{d}x.\end{aligned}$$

Integrating the above equation in time and using the initial conditions for (1.6)–(1.7) and the final conditions for (2.2) we obtain

$$\begin{aligned}&\iint_Q p^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_\Omega (u^\gamma)^2(x,T)\,\mathrm{d}x = \int_\Omega\int_0^\infty (u^0(x) - s^{r,0}(x))q^r(x,0)\,\mathrm{d}\mu(r)\,\mathrm{d}x\\ &-\int_\Omega u^0(x)p(x,0)\,\mathrm{d}x + \frac{1}{\gamma}\iint_Q\int_0^\infty q^r\left(\Psi'\left(\frac{1}{r}s^{r\gamma}\right) - \frac{1}{r}s^{r\gamma}\Psi''\left(\frac{1}{r}s^{r\gamma}\right)\right)\mathrm{d}\mu(r)\,\mathrm{d}x\,\mathrm{d}t\\ &+\iint_Q p\left(\int_0^\infty (f - u^\gamma f_{u^\gamma} - s^{r\gamma}f_{s^{r\gamma}})(x,r,u^\gamma,s^{r\gamma})\,\mathrm{d}\mu(r)\right)\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{2.6}$$

We have by Hypothesis 1.1 (v) that $|(f - u^\gamma f_{u^\gamma} - s^{r\gamma}f_{s^{r\gamma}})(x,r,u^\gamma,s^{r\gamma})| \le C$ for a. e. $(x,t)\in Q$. Moreover, the choice of Ψ guarantees that

$$|\Psi'(s) - s\Psi''(s)| \le \frac{3}{2}\Psi''(s).$$

Indeed

- if $s < -2$ then $\Psi'(s) = (-s-1)(-1) + \frac{1}{2} = s + \frac{3}{2}$ and $\Psi''(s) \equiv 1$, thus

$$|\Psi'(s) - s\Psi''(s)| = \left|s + \frac{3}{2} - s\right| = \frac{3}{2} = \frac{3}{2}\Psi''(s);$$

- if $-2 < s < -1$ then $\Psi'(s) = \frac{1}{2}(-s-1)^2(-1) = -\frac{1}{2}s^2 - s - \frac{1}{2}$ and $\Psi''(s) = -s-1$, thus

$$\begin{aligned}|\Psi'(s) - s\Psi''(s)| &= \left|-\frac{1}{2}s^2 - s - \frac{1}{2} - s(-s-1)\right| = \left|\frac{1}{2}s^2 - \frac{1}{2}\right| = \frac{1}{2}(s^2-1)\\ &= \frac{1}{2}(s-1)(s+1) \le \frac{1}{2}(-3)(s+1) = \frac{3}{2}\Psi''(s);\end{aligned}$$

- if $s \in [-1,1]$ then $\Psi'(s) = \Psi''(s) \equiv 0$;
- if $1 < s < 2$ then $\Psi'(s) = \frac{1}{2}(s-1)^2 = \frac{1}{2}s^2 - s + \frac{1}{2}$ and $\Psi''(s) = s - 1$, thus

$$|\Psi'(s) - s\Psi''(s)| = \left|\frac{1}{2}s^2 - s + \frac{1}{2} - s(s-1)\right| = \left|\frac{1}{2} - \frac{1}{2}s^2\right| = \frac{1}{2}(s^2-1)$$
$$= \frac{1}{2}(s-1)(s+1) \leq \frac{1}{2}(s-1)\,3 = \frac{3}{2}\Psi''(s);$$

- if $s > 2$ then $\Psi'(s) = (s-1) - \frac{1}{2} = s - \frac{3}{2}$ and $\Psi''(s) \equiv 1$, thus

$$|\Psi'(s) - s\Psi''(s)| = \left|s - \frac{3}{2} - s\right| = \frac{3}{2} = \frac{3}{2}\Psi''(s).$$

Hence from (2.6) and Hypothesis 1.1 (v) we infer

$$\begin{aligned}
&\iint_Q p^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_\Omega (u^\gamma)^2(x,T)\,\mathrm{d}x \\
&\leq \int_\Omega \sup_{\mu-\text{a. e. } r>0} |q^r(x,0)| \int_0^\infty |u^0(x) - s^{r,0}(x)|\,\mathrm{d}\mu(r)\,\mathrm{d}x - \int_\Omega |u^0(x)||p(x,0)|\,\mathrm{d}x \\
&\quad + \frac{1}{\gamma}\iint_Q \int_0^\infty |q^r|\,\frac{3}{2}\Psi''\left(\frac{1}{r}s^{r\gamma}\right)\mathrm{d}\mu(r)\,\mathrm{d}x\,\mathrm{d}t + \iint_Q |p| \left(\int_0^\infty C\,\mathrm{d}\mu(r)\right)\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Finally by Hypothesis 1.1 (i) and 1.2, (2.3) and (2.5) we get the estimate

$$\iint_Q p^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_\Omega (u^\gamma)^2(x,T)\,\mathrm{d}x \leq C\iint_Q |p(x,t)|\,\mathrm{d}x\,\mathrm{d}t$$

with a constant C depending on the L^∞-norm of u_0. Applying Hölder's inequality and using (2.1) again, we finally get

$$\iint_Q (v_\varepsilon^\gamma)^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_\Omega (u_\varepsilon^\gamma)^2(x,T)\,\mathrm{d}x \leq C. \tag{2.7}$$

2.3 Limit as $\varepsilon \to 0$

We first keep $\gamma > 0$ fixed. As a consequence of (2.7) and of Hypothesis 1.1 (iv)–(v) which implies that $f(x,r,u^\gamma,s^{r\gamma}) \leq C(1 + |u^\gamma| + |s^{r\gamma}|)$ for a. e. $(x,t) \in Q$, arguing as for the proof of Proposition 1.3 we see that the solutions $u_\varepsilon^\gamma, s_\varepsilon^{r\gamma}$ to the approximate system (1.6) are uniformly bounded in the spaces $W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;W^{1,2}(\Omega))$ and $W^{1,2}(0,T;L^2(\Omega))$, respectively. The first space is compactly embedded in the space $L^2(\Omega;C[0,T])$ according to Lemma A.5. For each fixed γ there exists therefore a sequence $\{\varepsilon_n(\gamma)\}$, $n \in \mathbb{N}$, $\lim_{n\to\infty}\varepsilon_n(\gamma) = 0$, such that we can conclude, using also (1.8) and (2.7), that

$$\begin{aligned}
&v^\gamma_{\varepsilon_n(\gamma)} \to v^\gamma_*,\ (s^{r\gamma}_{\varepsilon_n(\gamma)})_t \to (s^{r\gamma}_*)_t \text{ weakly in } L^2(Q) \ \text{ as } n \to \infty,\\
&\|u^\gamma_{\varepsilon_n(\gamma)} - u^\gamma_*\|_{2,\infty} \to 0,\\
&\|u^\gamma_{\varepsilon_n(\gamma)}(x,T)\|^2_{L^2(\Omega)} \leq C\varepsilon_n(\gamma),\\
&\|s^{r\gamma}_{\varepsilon_n(\gamma)} - s^{r\gamma}_*\|_{2,\infty} \leq 2\|u^\gamma_{\varepsilon_n(\gamma)} - u^\gamma_*\|_{2,\infty},
\end{aligned} \tag{2.8}$$

where $\|\cdot\|_{2,\infty}$ denotes the norm of $L^2(\Omega; C([0,T]))$ (see (A.4)) and where $u_*^\gamma, s_*^{r\gamma}$ are solutions to the system

$$\begin{cases} \displaystyle\int_\Omega \left((u_*^\gamma)_t\, \zeta + \nabla u_*^\gamma \cdot \nabla\zeta + (\int_0^\infty f(x,r,u_*^\gamma, s_*^{r\gamma})\, \mathrm{d}\mu(r))\zeta \right) \mathrm{d}x = \int_\Omega v_*^\gamma\, \zeta\, \mathrm{d}x & \text{in } (0,T), \\ (s_*^{r\gamma})_t + \dfrac{1}{\gamma}\Psi'\left(\dfrac{1}{r} s_*^{r\gamma}\right) = (u_*^\gamma)_t & \text{in } Q \end{cases} \tag{2.9}$$

for every $\zeta \in W^{1,2}(\Omega)$, with initial conditions as in (1.7). Moreover, by (2.8) the null-controllability condition $u_*^\gamma(x,T) = 0$ holds for a. e. $x \in \Omega$.

2.4 Limit as $\gamma \to 0$

The convergence $\gamma \to 0$ is more delicate. By (2.7) we have uniform bounds for v_*^γ in $L^2(Q)$. We make use again of Hypothesis 1.1 and of the inequality $f(x,r,u_*^\gamma, s_*^{r\gamma}) \le C(1 + |u_*^\gamma| + |s_*^{r\gamma}|)$ for a. e. $(x,t) \in Q$, as well as of the fact that arguing as in the proof of Proposition 1.3 we get a bound for u_*^γ in $W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;W^{1,2}(\Omega))$. Hence we find a sequence $\gamma_n \to 0$ as $n \to \infty$ and elements $v_* \in L^2(Q)$ and $u_* \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;W^{1,2}(\Omega))$ such that

$$\begin{aligned} & v_*^{\gamma_n} \to v_*,\ (u_*^{\gamma_n})_t \to (u_*)_t \text{ weakly in } L^2(Q), \\ & \|u_*^{\gamma_n} - u_*\|_{2,\infty} \to 0 \end{aligned} \tag{2.10}$$

as $n \to \infty$. Hence, the null-controllability condition $u_*(x,T) = 0$ a. e. is preserved in the limit. It remains to prove the strong convergence to $\mathfrak{s}_r[u_*, s^{r,0}]$ of the solutions $s_*^{r\gamma_n}$ to the equation

$$(s_*^{r\gamma_n})_t + \frac{1}{\gamma_n}\Psi'\left(\frac{1}{r} s_*^{r\gamma_n}\right) = (u_*^{\gamma_n})_t, \qquad s_*^{r\gamma_n}(x,0) = s^{r,0}(x).$$

Note that we need the convergence to be strong in order to pass to the limit in the nonlinearity f. To this end, we denote by $y^{r\gamma_n}$ the solution to the ODE

$$y_t^{r\gamma_n} + \frac{1}{\gamma_n}\Psi'\left(\frac{1}{r} y^{r\gamma_n}\right) = (u_*)_t, \qquad y^{r\gamma_n}(x,0) = s^{r,0}(x). \tag{2.11}$$

By (1.8) we have

$$\|s_*^{r\gamma_n} - y^{r\gamma_n}\|_{2,\infty} \le 2\|u_*^{\gamma_n} - u_*\|_{2,\infty}. \tag{2.12}$$

By (2.10), the right-hand side of (2.12) converges to 0 as $n \to \infty$. Note that we can write

$$\begin{aligned} & \|s_*^{r\gamma_n} - \mathfrak{s}_r[u_*, s^{r,0}]\|_{L^\infty(0,T;L^2(\Omega))} \\ & \quad \le \|s_*^{r\gamma_n} - y^{r\gamma_n}\|_{L^\infty(0,T;L^2(\Omega))} + \|y^{r\gamma_n} - \mathfrak{s}_r[u_*, s^{r,0}]\|_{L^\infty(0,T;L^2(\Omega))}, \end{aligned}$$

where

$$\|\cdot\|_{L^\infty(0,T;L^2(\Omega))} = \max_{t\in[0,T]} \left(\int_\Omega |\cdot|^2(x,t)\, \mathrm{d}x \right)^{1/2} \le \left(\int_\Omega \max_{t\in[0,T]} |\cdot|^2(x,t)\, \mathrm{d}x \right)^{1/2} = \|\cdot\|_{2,\infty}.$$

Hence, to prove the strong convergence of $s_*^{r\gamma_n}$ to $\mathfrak{s}_r[u_*, s^{r,0}]$ in $L^\infty(0,T;L^2(\Omega))$ it suffices to prove that $y^{r\gamma_n} \to \mathfrak{s}_r[u_*, s^{r,0}]$ strongly in $L^\infty(0,T;L^2(\Omega))$ for each $r > 0$.

First we derive some estimates for the functions $y^{r\gamma_n}$. Testing (2.11) by $y_t^{r\gamma_n}$ and integrating over $\Omega \times (0,\tau)$ for some $\tau \in (0,T)$ we obtain

$$\int_0^\tau \int_\Omega |y_t^{r\gamma_n}|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\gamma_n}\int_0^\tau \int_\Omega \Psi'\left(\frac{1}{r}y^{r\gamma_n}\right) y_t^{r\gamma_n}\,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau \int_\Omega (u_*)_t y_t^{r\gamma_n}\,\mathrm{d}x\,\mathrm{d}t,$$

that is,

$$\begin{aligned}\int_0^\tau \int_\Omega (u_*)_t y_t^{r\gamma_n}\,\mathrm{d}x\,\mathrm{d}t &= \int_0^\tau \int_\Omega |y_t^{r\gamma_n}|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\gamma_n}\int_\Omega\int_0^\tau r\,\frac{\mathrm{d}}{\mathrm{d}t}\Psi\left(\frac{1}{r}y^{r\gamma_n}\right)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^\tau \int_\Omega |y_t^{r\gamma_n}|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{r}{\gamma_n}\int_\Omega \Psi\left(\frac{1}{r}y^{r\gamma_n}\right)(x,\tau)\,\mathrm{d}x - \frac{r}{\gamma_n}\int_\Omega \Psi\left(\frac{1}{r}y^{r\gamma_n}\right)(x,0)\,\mathrm{d}x.\end{aligned}$$

By (2.11) and the definition of the stop operator (B.10), it follows that $\frac{1}{r}y^{r\gamma_n}(x,0) \in [-1,1]$, from which $\Psi\left(\frac{1}{r}y^{r\gamma_n}\right)(x,0) = 0$ by (1.4). Hence we get

$$\int_0^\tau \int_\Omega |y_t^{r\gamma_n}|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{r}{\gamma_n}\int_\Omega \Psi\left(\frac{1}{r}y^{r\gamma_n}\right)(x,\tau)\,\mathrm{d}x = \int_0^\tau \int_\Omega (u_*)_t y_t^{r\gamma_n}\,\mathrm{d}x\,\mathrm{d}t$$

which entails, using Young's inequality with exponents $1/2$ on the right-hand side and observing that $(u_*)_t$ is bounded in $L^2(Q)$ independently of τ and n,

$$\iint_Q |y_t^{r\gamma_n}|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{r}{\gamma_n}\sup_{\tau\in[0,T]}\int_\Omega \Psi\left(\frac{1}{r}y^{r\gamma_n}\right)(x,\tau)\,\mathrm{d}x \le C. \tag{2.13}$$

Up to a subsequence we thus have

$$y_t^{r\gamma_n} \to \hat{y}_t^r, \quad y^{r\gamma_n} \to \hat{y}^r, \quad \frac{1}{\gamma_n}\Psi'\left(\frac{1}{r}y^{r\gamma_n}\right) \to w^r \quad \text{weakly in } L^2(Q) \text{ as } n \to \infty. \tag{2.14}$$

Note also that by (1.5) we have for all $z \ge 0$

$$z^2 \le 8\phi(z) + 4\phi^{2/3}(z). \tag{2.15}$$

Indeed

- if $z \in [0,1]$ then $z^2 = 6^{2/3}\phi^{2/3} \le 4\phi^{2/3}$, which implies (2.15);
- if $z > 1$ then $\frac{1}{2}z^2 = \phi + \frac{1}{2}\,z - \frac{1}{6}$, and by Young's inequality

$$\frac{1}{2}\,z = \frac{1}{\delta}\,\frac{\delta}{2}\,z \le \frac{1}{2}\left(\frac{1}{\delta^2} + \frac{\delta^2}{4}\,z^2\right).$$

In particular, choosing $\delta = \sqrt{3}$ we obtain $\frac{1}{2}z^2 \le \phi + \frac{1}{6} + \frac{3}{8}z^2 - \frac{1}{6}$, which implies (2.15).

Choosing $z = \frac{1}{r}(|y^{r\gamma_n}| - r)^+$ in (2.15) we obtain

$$\left(\frac{1}{r}(|y^{r\gamma_n}| - r)^+\right)^2 \le 8\phi\left(\frac{1}{r}(|y^{r\gamma_n}| - r)^+\right) + 4\phi^{2/3}\left(\frac{1}{r}(|y^{r\gamma_n}| - r)^+\right).,$$

hence from (1.4) we also have

$$\frac{1}{r^2}\left(\left(|y^{r\gamma_n}|-r\right)^+\right)^2 \le 8\Psi\left(\frac{1}{r}\,y^{r\gamma_n}\right) + 4\Psi^{2/3}\left(\frac{1}{r}\,y^{r\gamma_n}\right).$$

Integrating over Ω and using (2.13) we get for all $\tau \in (0,T)$

$$\begin{aligned}\frac{1}{r^2}\int_\Omega \left(\left(|y^{r\gamma_n}|-r\right)^+\right)^2(x,\tau)\,\mathrm{d}x &\le 8\int_\Omega \Psi\left(\frac{1}{r}\,y^{r\gamma_n}\right)(x,\tau)\,\mathrm{d}x + 4\int_\Omega \Psi^{2/3}\left(\frac{1}{r}\,y^{r\gamma_n}\right)(x,\tau)\,\mathrm{d}x\\ &\le 8\,\frac{C\gamma_n}{r} + 4\,\frac{C\gamma_n^{2/3}}{r^{2/3}},\end{aligned}$$

from which we infer for $\gamma_n < 1$ that

$$\int_\Omega \left(\left(|y^{r\gamma_n}|-r\right)^+\right)^2(x,\tau)\,\mathrm{d}x \le C\big(1+r^{4/3}\big){\gamma_n}^{2/3} \tag{2.16}$$

with a constant C independent of τ, r and n.

We now prove that $\hat{y}^r = \mathfrak{s}_r[u_*, s^{r,0}]$. To this end, note that $\hat{y}^r$ and w^r satisfy the equation

$$\hat{y}^r_t + w^r = (u_*)_t, \qquad \hat{y}^r(x,0) = s^{r,0}(x). \tag{2.17}$$

Hence the third condition of (B.14) is verified by $\hat{y}^r$. Furthermore, for every function $b \in L^\infty(Q)$ we have

$$\iint_Q y^{r\gamma_n} b\,\mathrm{d}x\,\mathrm{d}t \le \iint_Q |y^{r\gamma_n}|\,|b|\,\mathrm{d}x\,\mathrm{d}t \le \iint_Q (|y^{r\gamma_n}|-r)^+\,|b|\,\mathrm{d}x\,\mathrm{d}t + r\iint_Q |b|\,\mathrm{d}x\,\mathrm{d}t,$$

hence, by (2.14) and (2.16), we have $\iint_Q \hat{y}^r b\,\mathrm{d}x\,\mathrm{d}t \le r$ for each function $b \in L^\infty(Q)$ such that $\iint_Q |b|\,\mathrm{d}x\,\mathrm{d}t \le 1$, which in turn implies that $|\hat{y}^r(x,t)| \le r$ a. e. Indeed, if there exists a set A of positive measure and some $\delta > 0$ such that $y(x,t) \ge r+\delta$ or $y(x,t) \le -r-\delta$ for $(x,t) \in A$, by choosing $b(x,t) = \pm\frac{1}{|A|}\chi_A(x,t)$ (where by $|A|$ and χ_A we denote respectively the Lebesgue measure and the characteristic function of the set A) we obtain a contradiction. Hence $|\hat{y}^r(x,t)| \le r$ a. e., and the first condition in the definition of the stop (B.14) is satisfied by $\hat{y}^r$.

In order to verify that also the second condition holds true, we multiply (2.11) by $y^{r\gamma_n}$ and (2.17) by $\hat{y}^r$ and integrate over Q. We get, exploiting the initial conditions in (2.11) and (2.17),

$$\begin{cases}\dfrac{1}{2}\displaystyle\int_\Omega (y^{r\gamma_n})^2(x,T)\,\mathrm{d}x - \dfrac{1}{2}\displaystyle\int_\Omega (s^{r,0})^2(x)\,\mathrm{d}x = -\dfrac{1}{\gamma_n}\displaystyle\iint_Q \Psi'\left(\frac{1}{r}y^{r\gamma_n}\right) y^{r\gamma_n}\,\mathrm{d}x\,\mathrm{d}t + \displaystyle\iint_Q (u_*)_t\,y^{r\gamma_n}\,\mathrm{d}x\,\mathrm{d}t,\\ \dfrac{1}{2}\displaystyle\int_\Omega (\hat{y}^r)^2(x,T)\,\mathrm{d}x - \dfrac{1}{2}\displaystyle\int_\Omega (s^{r,0})^2(x)\,\mathrm{d}x = -\displaystyle\iint_Q w^r\hat{y}^r\,\mathrm{d}x\,\mathrm{d}t + \displaystyle\iint_Q (u_*)_t\,\hat{y}^r\,\mathrm{d}x\,\mathrm{d}t.\end{cases}$$

Hence for all $\gamma_n > 0$ we have the equality

$$\begin{aligned}&\frac{1}{2}\int_\Omega (y^{r\gamma_n})^2(x,T)\,\mathrm{d}x + \frac{1}{\gamma_n}\iint_Q \Psi'\left(\frac{1}{r}y^{r\gamma_n}\right) y^{r\gamma_n}\,\mathrm{d}x\,\mathrm{d}t - \iint_Q (u_*)_t\,y^{r\gamma_n}\,\mathrm{d}x\,\mathrm{d}t\\ &= \frac{1}{2}\int_\Omega (\hat{y}^r)^2(x,T)\,\mathrm{d}x + \iint_Q w^r\hat{y}^r\,\mathrm{d}x\,\mathrm{d}t - \iint_Q (u_*)_t\,\hat{y}^r\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Passing to the $\liminf_{\gamma_n\to 0}$ we obtain

$$\begin{aligned}&\frac{1}{2}\liminf_{\gamma_n\to 0}\int_\Omega (y^{r\gamma_n})^2(x,T)\,\mathrm{d}x+\liminf_{\gamma_n\to 0}\frac{1}{\gamma_n}\iint_Q \Psi'\left(\frac{1}{r}y^{r\gamma_n}\right)y^{r\gamma_n}\,\mathrm{d}x\,\mathrm{d}t\\&=\frac{1}{2}\int_\Omega(\hat{y}^r)^2(x,T)\,\mathrm{d}x+\iint_Q w^r\hat{y}^r\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{2.18}$$

Now, since the norm is weakly lower semicontinuous, by the weak convergence (2.14) we have $\int_\Omega(\hat{y}^r)^2(x,T)\,\mathrm{d}x\le\liminf_{\gamma_n\to 0}\int_\Omega(y^{r\gamma_n})^2(x,T)\,\mathrm{d}x$. Hence for equality (2.18) to hold we necessarily have

$$\liminf_{\gamma_n\to 0}\frac{1}{\gamma_n}\iint_Q\Psi'\left(\frac{1}{r}y^{r\gamma_n}\right)y^{r\gamma_n}\,\mathrm{d}x\,\mathrm{d}t\le\iint_Q w^r\hat{y}^r\,\mathrm{d}x\,\mathrm{d}t.\tag{2.19}$$

Note that since Ψ' is monotone and vanishes in $[-1{,}1]$, it holds

$$\iint_Q\Psi'\left(\frac{1}{r}y^{r\gamma_n}\right)(y^{r\gamma_n}-\rho)\,\mathrm{d}x\,\mathrm{d}t\ge 0\tag{2.20}$$

for every measurable function ρ such that $|\rho(x,t)|\le r$ a. e. Thus, putting together (2.19) and (2.20), for every such ρ we have

$$0\le\iint_Q w^r(\hat{y}^r-\rho)\,\mathrm{d}x\,\mathrm{d}t=\iint_Q((u_*)_t-\hat{y}^r_t)(\hat{y}^r-\rho)\,\mathrm{d}x\,\mathrm{d}t,\tag{2.21}$$

where in the equality we used (2.17). Hence $\hat{y}^r$ verifies also the second condition of (B.14), and this proves that $\hat{y}^r=\mathfrak{s}_r[u_*,s^{r,0}]$.

It remains to prove that the convergence $y^{r\gamma_n}\to\hat{y}^r$ in (2.14) is strong in the space $L^\infty(0,T;L^2(\Omega))$. Indeed, by choosing in (2.21) a test function

$$\tilde{\rho}(x,t)=\begin{cases}\hat{y}^r(x,t) & \text{for } t\ge\tau,\\ \rho(x,t) & \text{for } t\le\tau\end{cases}$$

with an arbitrary $\tau\in(0,T)$, we obtain from (2.21) that the inequality

$$\int_0^\tau\int_\Omega((u_*)_t-\hat{y}^r_t)(\hat{y}^r-\rho)\,\mathrm{d}x\,\mathrm{d}t\ge 0\tag{2.22}$$

holds for all τ. On the other hand, (2.11) and inequality (2.20) imply that we have

$$\int_0^\tau\int_\Omega((u_*)_t-y^{r\gamma_n}_t)(y^{r\gamma_n}-\rho)\,\mathrm{d}x\,\mathrm{d}t\ge 0\tag{2.23}$$

for all $n\in\mathbb{N}$ and all test functions ρ such that $|\rho(x,t)|\le r$ a. e. We now set $\rho=\hat{y}^r$ in (2.23) and $\rho=P_r(y^{r\gamma_n})$ in (2.22), where $P_r:\mathbb{R}\to[-r,r]$ is the projection onto the interval $[-r,r]$, that is,

$$P_r(z)=\max\{-r,\min\{x,r\}\}.$$

Summing up (2.22) and (2.23) yields

$$0\le\int_0^\tau\int_\Omega((u_*)_t-\hat{y}^r_t)(\hat{y}^r-P_r(y^{r\gamma_n}))\,\mathrm{d}x\,\mathrm{d}t+\int_0^\tau\int_\Omega((u_*)_t-y^{r\gamma_n}_t)(y^{r\gamma_n}-\hat{y}^r)\,\mathrm{d}x\,\mathrm{d}t,$$

that is

$$\begin{aligned}
0 &\le \int_0^\tau \int_\Omega (u_*)_t(\hat{y}^r - P_r(y^{r\gamma_n}) + y^{r\gamma_n} - \hat{y}^r)\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega \hat{y}^r(-\hat{y}_t^r + y_t^{r\gamma_n})\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega (\hat{y}_t^r P_r(y^{r\gamma_n}) - y_t^{r\gamma_n} y^{r\gamma_n} \pm \hat{y}_t^r y^{r\gamma_n})\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\tau \int_\Omega (u_*)_t(y^{r\gamma_n} - P_r(y^{r\gamma_n}))\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega (y_t^{r\gamma_n} - \hat{y}_t^r)\hat{y}^r\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^\tau \int_\Omega \hat{y}_t^r(y^{r\gamma_n} - P_r(y^{r\gamma_n}))\,\mathrm{d}x\,\mathrm{d}t - \int_0^\tau \int_\Omega (y_t^{r\gamma_n} - \hat{y}_t^r)y^{r\gamma_n}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

This can be rewritten as

$$\int_0^\tau \int_\Omega (y_t^{r\gamma_n} - \hat{y}_t^r)(y^{r\gamma_n} - \hat{y}^r)\,\mathrm{d}x\,\mathrm{d}t \le \int_0^\tau \int_\Omega |(u_*)_t - \hat{y}_t^r||y^{r\gamma_n} - P_r(y^{r\gamma_n})|\,\mathrm{d}x\,\mathrm{d}t. \tag{2.24}$$

Notice that we have

$$|y^{r\gamma_n} - P_r(y^{r\gamma_n})| = (|y^{r\gamma_n}| - r)^+.$$

Thus (2.24) is equivalent to

$$\frac{1}{2}\int_0^\tau \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}(y^{r\gamma_n} - \hat{y}^r)^2\,\mathrm{d}x\,\mathrm{d}t \le \int_0^\tau \int_\Omega |(u_*)_t - \hat{y}_t^r|(|y^{r\gamma_n}| - r)^+\,\mathrm{d}x\,\mathrm{d}t,$$

that is, using the fact that $y^{r\gamma_n}(x,0) = \hat{y}^r(x,0) = s^{r,0}(x)$ on the left-hand side and Hölder's inequality on the right-hand side,

$$\frac{1}{2}\int_\Omega |y^{r\gamma_n} - \hat{y}^r|^2(x,\tau)\,\mathrm{d}x \le C\left(\iint_Q \left((|y^{r\gamma_n}| - r)^+\right)^2\mathrm{d}x\,\mathrm{d}t\right)^{1/2}.$$

Hence from (2.16) it follows for every $\tau \in (0,T)$ that

$$\int_\Omega |y^{r\gamma_n} - \hat{y}^r|^2(x,\tau)\,\mathrm{d}x \le C\big(1 + r^{2/3}\big){\gamma_n}^{1/3} \tag{2.25}$$

with a constant C independent of n, r and τ. This finally implies that $y^{r\gamma_n} \to \mathfrak{s}_r[u_*, s^{r,0}]$ strongly in $L^\infty(0,T;L^2(\Omega))$ for each $r > 0$, which is what was left to prove in order to pass to the limit in the system (2.9).

To conclude the proof of Theorem 1.4, it suffices to pass to a subsequence if necessary and choose γ_n in (2.10) in such a way that

$$\|u_*^{\gamma_n} - u_*\|_{2,\infty} < \frac{1}{n}, \tag{2.26}$$

$\varepsilon_n := \varepsilon_n(\gamma_n)$ in (2.8) such that $\varepsilon_n \to 0$ as $n \to \infty$ and

$$\|u_{\varepsilon_n}^{\gamma_n} - u_*^{\gamma_n}\|_{2,\infty} < \frac{1}{n} \tag{2.27}$$

for each $n \in \mathbb{N}$. The four assertions of Theorem 1.4 now easily follows:

(i) In Proposition 1.3 we proved that $u_\varepsilon^\gamma \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;W^{1,2}(\Omega))$ for every $\varepsilon, \gamma > 0$, from which the inequality immediately follows.

(ii) We have that

$$\|u_n - u_*\|_{2,\infty} = \|u_{\varepsilon_n}^{\gamma_n} - u_*\|_{2,\infty} \leq \|u_{\varepsilon_n}^{\gamma_n} - u_*^{\gamma_n}\|_{2,\infty} + \|u_*^{\gamma_n} - u_*\|_{2,\infty} \to 0$$

by (2.8) and (2.10).

(iii) This inequality was obtained in (2.8).

(iv) We have that, according to the notation introduced above,

$$\begin{aligned}
&\max_{t\in[0,T]} \int_\Omega |s_n^r(x,t) - \hat{y}^r(x,t)|^2 \,\mathrm{d}x = \max_{t\in[0,T]} \int_\Omega |s_{\varepsilon_n}^{r\gamma_n}(x,t) - \hat{y}^r(x,t)|^2 \,\mathrm{d}x \\
&\leq \max_{t\in[0,T]} \int_\Omega |s_{\varepsilon_n}^{r\gamma_n}(x,t) - s_*^{r\gamma_n}(x,t)|^2 \,\mathrm{d}x + \max_{t\in[0,T]} \int_\Omega |s_*^{r\gamma_n}(x,t) - y^{r\gamma_n}(x,t)|^2 \,\mathrm{d}x \\
&\quad + \max_{t\in[0,T]} \int_\Omega |y^{r\gamma_n}(x,t) - \hat{y}^r(x,t)|^2 \,\mathrm{d}x \\
&\leq \|s_{\varepsilon_n}^{r\gamma_n} - s_*^{r\gamma_n}\|_{2,\infty}^2 + \|s_*^{r\gamma_n} - y^{r\gamma_n}\|_{2,\infty}^2 + \max_{t\in[0,T]} \int_\Omega |y^{r\gamma_n}(x,t) - \hat{y}^r(x,t)|^2 \,\mathrm{d}x \\
&\leq 2\|u_{\varepsilon_n}^{\gamma_n} - u_*^{\gamma_n}\|_{2,\infty}^2 + 2\|u_*^{\gamma_n} - u_*\|_{2,\infty}^2 + \max_{t\in[0,T]} \int_\Omega |y^{r\gamma_n}(x,t) - \hat{y}^r(x,t)|^2 \,\mathrm{d}x \\
&\leq \frac{2}{n^2} + \frac{2}{n^2} + C\big(1 + r^{2/3}\big)\gamma_n^{1/3}
\end{aligned}$$

by (2.8), (2.12) and (2.25)–(2.27). Since we showed that the limit $\hat{y}^r$ corresponds to the stop operator, the inequality is proved.

CHAPTER 3

Controllability and phase transitions

In this chapter we are going to show how our controllability result can be applied to a simple two-phase system, namely, the *relaxed Stephan problem.* This is an example of a basic model of solid-liquid phase transition, since it simply accounts for heat-diffusion and exchange of latent heat in terms of partial differential equations.

It is natural to recognize this phenomenon as an example of free boundary problem, since the evolution of the domains occupied by the phases is not known a priori. Many mathematicians addressed (and still address) the Stefan problem from this point of view.

However, phase transitions may also be regarded from a different perspective. Heat diffusion and exchange of latent heat may also be formulated in weak form, since they are accounted for by the energy balance equation. This leads to the formulation of an initial- and boundary-value problem in a fixed space–time domain for a nonlinear parabolic equation. This nonlinearity is expressed via a *maximal monotone graph*, and the problem may thus be reduced to a *variational inequality.*

The two approaches above are known as the classical and the weak formulation of the Stefan problem. However, rather than being two formulations of the same problem, these represent two alternative models of phase transitions, that turn out to be equivalent only in special cases. The classical model is a genuine free boundary problem, since it is based on the assumption that the phases are separated by an (unknown) smooth interface that also evolves smoothly. On the other hand, the weak formulation makes no direct reference to any phase interface: this may or may not exist, anyway it does not explicitly occur in the statement of the model. Solid and liquid phases may actually be separated by a set having nonempty interior, a so-called *mushy region.* In this respect, the weak formulation is more general than the classical one.

Here we follow this second approach, and consider the Stefan problem in weak form. We shall represent the phase transition in an especially simplified way, focusing upon the thermal aspects, that is, heat-diffusion and exchange of latent heat, neglecting stress and deformation in the solid. More

complicated models of phase transition, which take into account also the mechanical aspects of the process, will be considered in Parts II and III of this thesis.
We will also assume that a form of *relaxation* occurs during the process. This assumption will be crucial in order to reformulate the problem as a semilinear PDE containing a hysteresis operator (more precisely, the stop operator from Section B.2), which will allow us to apply the controllability result from the previous chapters.
The link between differential inclusions describing (relaxed) phase transitions and the stop operator (B.10) will be of fundamental importance also in Parts II and III. The regularity and continuity results contained in Section B.2 will be frequently used when dealing with the phase parameter.

Let us now derive the equations of the relaxed Stephan problem in weak form. Assume that a domain $\Omega \subset \mathbb{R}^N$ (in practice, we choose $N = 3$) is filled with a material substance in which two phases may coexist: solid and liquid. The state variables are the following functions of the space variable $x \in \Omega$ and time $t \in [0, T]$:

$s(x,t) \in [-1,1]$	phase fraction: $s = -1$ solid, $s = 1$ liquid, $s \in (-1,1)$ mixture;
$\theta(x,t) > 0$	absolute temperature.

The process of transition between the two phases is governed by the first and the second principle of thermodynamics:

1. First Principle: There exists a state function U called *internal energy* which is conserved in the sense that its increase rate equals the sum of the power supplied to the system and the heat flux through the boundary;
2. Second Principle: There exists a state function S called the *entropy* which is nondecreasing in the sense that its increase rate is greater than or equal to the sum of the external entropy source and the entropy flux through the boundary.

In other words, $U = U(\theta, s)$ and $S = S(\theta, s)$ have to satisfy the energy balance equation

$$U_t + \mathrm{div} q = h \tag{3.1}$$

and the Clausius-Duhem inequality

$$S_t + \mathrm{div}\left(\frac{q}{\theta}\right) \geq \frac{h}{\theta} \tag{3.2}$$

for all processes. Here we denote by q the heat flux vector and by h the heat source density. We further introduce the *free energy* $F = F(\theta, s)$ by the formula

$$F(\theta, s) = U(\theta, s) - \theta S(\theta, s), \tag{3.3}$$

so that in terms of F the second principle (3.2) can be equivalently stated as

$$F_t + \theta_t S + \frac{1}{\theta}\langle q, \nabla\theta\rangle \leq 0. \tag{3.4}$$

The above inequality must hold for every thermodynamic process, including slow nonhomogeneous processes (where the time derivatives are negligible compared to the gradient of the temperature $\nabla\theta$), as well as fast homogeneous processes (where $\nabla\theta$ is negligible compared to the time derivatives). In other words, the time and space derivatives appear at different size scales. Therefore, the inequality (3.4) implies that both $\langle q, \nabla\theta\rangle$ and $F_t + \theta_t S$ must be nonpositive for all processes. Assuming for the heat flux the Fourier law

$$q = -\kappa\nabla\theta \tag{3.5}$$

with a constant heat conductivity $\kappa > 0$, the nonpositivity of $\langle q, \nabla\theta\rangle$ is obvious. We now compare the remaining inequality $F_t + \theta_t S \leq 0$ for all processes with the chain rule identity $F_t = F_\theta\theta_t + F_s s_t$ and obtain another formally (in the sense that we need to give a meaning to the partial derivatives) equivalent reformulation of the second principle, namely

$$S = -F_\theta\,, \tag{3.6}$$

$$F_s s_t \leq 0\,. \tag{3.7}$$

From (3.3) and (3.6) we deduce the following differential equation for F

$$F - \theta F_\theta = U, \tag{3.8}$$

which can be solved if we know the internal energy U. For simplicity, we assume the internal energy in the form

$$U = c\,\theta + L(s), \tag{3.9}$$

where $c > 0$ is the specific heat capacity which we assume constant, and L is an increasing C^1-function representing the latent heat. Solutions to (3.8)–(3.9) can be explicitly found and they all differ only by an additive "integration" constant which may depend on s. A "minimal" choice in the sense that no unphysical constants are involved and all values of the phase fraction s outside the admissible interval $[-1,1]$ are excluded is given by the formula

$$F(\theta, s) = -c\,\theta\log\left(\frac{\theta}{\theta_c}\right) + L(s)\left(1 - \frac{\theta}{\theta_c}\right) + I(s),$$

where $I(s)$ is the indicator function of the interval $[-1,1]$, and $\theta_c > 0$ is a fixed reference temperature (the melting temperature). Thus we can compute

$$F_s(\theta, s) = \partial F(\theta, s) = \partial I(s) + L'(s)\left(1 - \frac{\theta}{\theta_c}\right) + \partial I(s),$$

where the derivative of F with respect to the variable s contains components that are not Fréchet differentiable but are convex, so that it can be interpreted as the subdifferential. Therefore, by (B.16), condition (3.7) reads for $s \in (-1,1)$ (note that L is an increasing function)

$$\frac{L'(s)}{\theta_c}(\theta - \theta_c)\,s_t \geq 0\,,$$

which has a clear meaning: the substance has the tendency to melt for low temperatures, and the tendency to solidify for high temperatures. A natural choice for the phase dynamics equation is then (see also Remark B.2)

$$\rho s_t \in -\partial F(\theta, s) = -\partial I(s) + \frac{L'(s)}{\theta_c}(\theta - \theta_c) \tag{3.10}$$

with phase relaxation time $\rho > 0$. It means that the system tends to move towards local minima of the free energy with speed proportional to $1/\rho$, hence the smaller ρ is, the faster the phase transition takes place. When $\rho \to 0$ the phase transition becomes instantaneous, which corresponds to the classical Stefan problem. The above differential inclusion and the energy balance equation

$$(c\theta + L(s))_t - \kappa\Delta\theta = h \tag{3.11}$$

resulting from (3.1), (3.5), and (3.9) give rise to the full system describing the relaxed Stefan problem, see [131].

We now show that the energy balance equation (3.11) can be transformed into the form (I.1). Indeed, we define a new unknown u by the formula

$$u_t = \frac{L'(s)}{\rho\theta_c}(\theta - \theta_c).$$

Then the phase dynamics equation in (3.10) reads

$$s_t + \partial I(s) \ni u_t,$$

which is nothing but the definition of the stop operator with threshold 1

$$s = \mathfrak{s}_1[u],$$

see (B.10) in the Appendix. This enables us to rewrite (3.11) in the form

$$cu_{tt} + \frac{1}{\rho\theta_c}L(\mathfrak{s}_1[u])_t - \kappa\Delta u_t = \frac{1}{\rho\theta_c}h.$$

Integrating the above equation in time leads to

$$cu_t + \frac{1}{\rho\theta_c}L(\mathfrak{s}_1[u]) - \kappa\Delta u = v,$$

with v containing the time integral of h and additional terms coming from the initial conditions. Up to the physical constants, this is precisely equation (I.1) with $\mathcal{F}[u] = L(\mathfrak{s}_1[u])$. The homogeneous Neumann boundary condition for θ (and therefore for u in (1.1)) has the physical meaning of a thermally insulated body.

Part II

A viscoelastoplastic porous medium problem with phase transition

Introduction

In this second part we study a model describing fluid diffusion in an unsaturated deformable porous medium, assuming that the fluid may undergo phase transition and that two sources of hysteresis are observed: the solid itself is subject to *irreversible plastic deformations*, and the fluid flow exhibits *capillary hysteresis*, which is often explained by the surface tension on the interfaces between water and air.

Much attention has been paid to phenomena related to this problem, and understanding the mechanism of the solid-liquid interaction is the goal of many existing models in the engineering literature. In [14], M. A. Biot proposed to describe an elastic partially saturated porous medium as a continuum in Lagrangian framework and derived balance equations of conservation of mass and conservation of momentum which have become a basis for further studies. Let us mention, for example, a mathematical theory including plasticity developed by R. E. Showalter and U. Stefanelli in [125, 126].

We state the problem in Lagrangian coordinates, too. Studies about the Eulerian fluid flow interacting with a moving solid exist in the literature, but either the solid is assumed to be rigid such as, e. g., in [3], or the fluid domain is two-dimensional and the moving part of the boundary is represented by a smooth curve described by a hyperviscoelastic constitutive equation ([85]). Eulerian flow in rigid porous materials has also been studied ([60, 61]). Here, we focus on the description of deformations of the porous solid produced by the fluid diffusion through the pores, so that the Lagrangian formalism seems to be a natural choice.

Even within the linear elasticity theory, the interaction between fluid and a porous solid is a nonlinear phenomenon. The mass conservation principle is expressed by the Darcy's law which states that the fluid mass saturation increment in a control volume V is compensated by the mass flux through the boundary of V, and that the mass flux vector is proportional to the pressure gradient. The pressure-saturation curve is, however, necessarily bounded, as it ranges between 0 (i. e., empty pores) and 1 (full saturation). Moreover, the wetting and the drying curves are typically not the same, and the phenomenon is called capillary hysteresis, see [4, 64, 65, 84]. This produces a degeneracy in the mass balance, as we lose immediate control of the time derivative of the pressure. Methods have been developed in [5, 40, 94] to prove the solvability of the system. They are all based on a variant of the

Moser iteration technique which allows one to establish a pointwise upper bound for the pressure, so that the process stays away from the degeneracy.

The theory is robust in the sense that other physical effects, such as temperature dependence or solid-liquid phase transitions with volume changes can be taken into account without affecting the thermodynamic consistency of the model, see [40, 100]. As a drawback, the pressure-volume relation becomes more complicated and additional nonlinearities occur in the system. The solvability of the resulting system was established in [100] only under the assumption that inertial effects and shear stresses are neglected.

Here we still neglect inertia, but our main goal is to propose a method for dealing with shear stresses in the case of strongly nonlinear pressure-volume interactions including plasticity of the matrix material, still assuming that phase transition may occur as in [100]. A first step in this direction was made in the submitted paper [76], where the isothermal case is considered and the effects of temperature and phase transitions are simulated by including a nonlinearity in the pressure-volume relation. This makes the construction of a solution much more complicated than in [5], where g is linear. Starting from these results, here we take a step forward including also the effects of freezing and melting. This second part of the thesis is then completely original, since the simultaneous occurrence of shear stresses and phase transition has never been considered in the literature. Here it is assumed that the pores in the matrix material contain a mixture of H_2O and gas, and H_2O itself is a mixture of the liquid (water) and the solid phase (ice). That is, in addition to the other physical quantities like capillary pressure, displacement and absolute temperature, we need to consider the evolution of a phase parameter χ representing the relative proportion of water in the H_2O part and its influence on pressure changes due to the different mass densities of water and ice.

Typical examples in which such situations arise are related to groundwater flows and to the freezing–melting cycles of water sucked into the pores of concrete. Note that the latter process forms one of the main reasons for the degradation of concrete in buildings, bridges, and roads. However, many of the governing effects in concrete like the multi-component microstructure, the breaking of pores and chemical reactions are still neglected in our model.

As it will be detailed in Chapter 4, we assume that the deformations are small, so that $\mathrm{div} u$ is the relative local volume change, where u represents the displacement vector. Moreover, we assume that the volume of the matrix material does not change during the process, and thus the volume and mass balance equations with Darcy's law for the water flux lead to a nonlinear degenerate parabolic equation for the capillary pressure. In the equation of motion, we take into account the pressure components due to phase transition and temperature changes, and we further simplify the system in order to make it mathematically tractable by assuming that the process is quasistatic. Finally, we use the

balance of internal energy and the entropy inequality to derive the dynamics for absolute temperature and phases; they turn out to be, respectively, a parabolic equation for the temperature with highly nonlinear right-hand side (quadratic in the derivatives) and an ordinary differential inclusion for the phase parameter χ, which represents the relative proportion of water in the H_2O part. The freezing and melting phenomena in the pores is modeled according to [100], which follows the ideas contained in the earlier publications on freezing and melting in containers filled with water with rigid, elastic, or elastoplastic boundaries ([92, 96–99]). It was shown there how important it is to account for the difference in specific volumes of water and of ice. Actually, only few publications take into account that the mass densities and specific volumes of the phases differ.
The main difficulties related to this problem arise from the low regularity of the temperature field, mainly due to the presence of high order dissipative terms in the internal energy balance. Since the test of the internal energy balance by the temperature θ is not allowed, we test by a suitable negative power of θ and use the growth condition of the heat conductivity κ. Another key point in our proof is the L^∞ estimate we get on the pressure, which entails a bound in a proper negative Sobolev space for the time derivative of the absolute temperature, which turns out to be another fundamental ingredient in order to pass to the limit in our approximation scheme.

This part is structured as follows. In Chapter 4 we derive the model in full generality from the basic principles of continuum thermodynamics. In Chapter 5 we state the mathematical problem, the main assumptions on the data and the main Theorem 5.3, the proof of which is split into Sections 5.1–5.3.

CHAPTER 4

A model for unsaturated porous media flow

4.1 Derivation of the model

Consider a bounded connected domain $\Omega \subset \mathbb{R}^3$ of class $C^{1,1}$ filled with an elastoplastic solid matrix material with pores containing a mixture of H_2O and gas, where we assume that H_2O may appear in one of the two phases: water or ice. We also assume that the volume of the solid matrix remains constant during the process. We state the balance laws in referential (Lagrangian) coordinates, assume the deformations small and denote for $x \in \Omega$ and time $t \in [0, T]$

$A(x,t) \in [0,1]$	relative amount of air in the total pore volume;
$W(x,t) \in [0,1]$	relative amount of H_2O in the total pore volume;
$\chi(x,t) \in [0,1]$	relative amount of water in the H_2O part;
$\xi(x,t)$	mass flux vector;
$p(x,t)$	capillary pressure;
$u(x,t)$	displacement vector in the solid;
$\varepsilon(x,t)$	linear strain tensor, $\varepsilon = \nabla_s u := \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$;
$\sigma(x,t)$	stress tensor;
$\theta(x,t)$	absolute temperature.

Then χW represents the relative proportion of water in the total pore volume, and $(1-\chi)W$ represents the relative proportion of ice in the total pore volume.

To explain the meaning of W and A, consider first an arbitrary control volume $V_0 \subset \Omega$ in the reference state and set

$$V(t) = \left\{ y \in \mathbb{R}^3 : y = x + u(x,t),\ x \in V_0 \right\}.$$

Then denote by $V_A(t)$, $V_W(t)$, $V_S(t)$ the subdomains of $V(t)$ occupied at time t by air, H_2O and solid, respectively. Then $V_A(t) \cup V_W(t) \cup V_S(t) = V(t)$, and denoting by $|V|$ the Lebesgue measure of a set V we assume that the porosity

$$\pi := \frac{|V_A(t) \cup V_W(t)|}{|V(t)|} \in (0,1)$$

remains constant and independent of the choice of V_0 and t. Let $J(x,t)$ be the Jacobian of the transformation $x \mapsto x + u(x,t)$. Under the small deformation hypothesis, we may consider $J(x,t) \approx 1 + \mathrm{div}u(x,t)$; hence $\mathrm{div}u$ represents the relative local volume increment. Indeed, we have

$$|V(t)| = \int_{V(t)} \mathrm{d}y = \int_{V_0} J(x,t)\,\mathrm{d}x,$$

so that

$$\lim_{|V_0|\to 0,\ x\in V_0} \frac{|V(t)|}{|V_0|} = J(x,t) \approx 1 + \mathrm{div}u(x,t).$$

By hypothesis, the volume of the matrix material does not change, so

$$c_S := \frac{|V_S(t)|}{|V_0|}$$

is a constant independent of V_0 and t. Setting

$$W(x,t) := \lim_{|V_0|\to 0,\ x\in V_0} \frac{|V_W(t)|}{|V_0|}, \qquad A(x,t) := \lim_{|V_0|\to 0,\ x\in V_0} \frac{|V_A(t)|}{|V_0|},$$

we obtain, under the small deformation hypothesis, the volume balance equation in Lagrange coordinates:

$$W(x,t) + A(x,t) + c_S = \lim_{|V_0|\to 0,\ x\in V_0} \frac{|V(t)|}{|V_0|} \approx 1 + \mathrm{div}u(x,t). \tag{4.1}$$

From the work of D. Flynn [64, 65], between the capillary pressure and the air content there exists a Preisach-type hysteresis relation

$$b := 1 - c_S - A = \mathcal{G}[p],$$

where $\mathcal{G}$ is the *Preisach hysteresis operator*. Indeed, the pressure-saturation wetting and drying curves are typically not the same (see Figure 4.1), and the phenomenon is called *capillary hysteresis*. This hysteretic behavior has been shown experimentally, and can be explained essentially by the surface tension on the contact between water and air in the pores. We refer to Section B.3 in the Appendix for more details on this hysteresis operator. Then for A we assume the functional relation

$$A = 1 - c_S - \mathcal{G}[p]. \tag{4.2}$$

Combining (4.1) and (4.2) we get that

$$W = \mathcal{G}[p] + \mathrm{div}u. \tag{4.3}$$

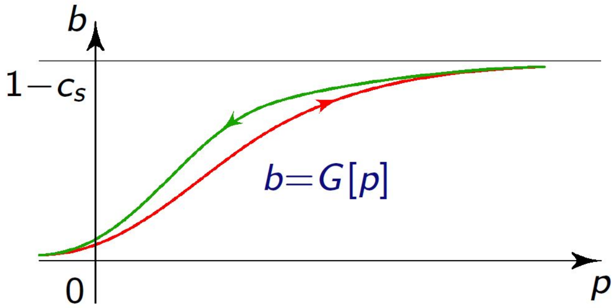

Figure 4.1. Pressure-saturation hysteresis diagram.

4.1.1 Mass balance

Consider an arbitrary control volume $V \subset \Omega$. The water content in V is given by $\int_V \rho_W \chi W \,\mathrm{d}x$, where ρ_W is the water mass density, and the ice content is $\int_V \rho_E (1-\chi) W \,\mathrm{d}x$, where ρ_E is the ice mass density. The mass conservation principle then reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho_W \chi W \,\mathrm{d}x + \int_{\partial V} \xi \cdot n \,\mathrm{d}s(x) = -\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho_E (1-\chi) W \,\mathrm{d}x,$$

where n is the unit outward normal vector to ∂V. In differential form we get

$$\rho_W (\chi W)_t + \mathrm{div}\xi = -\rho_E \left((1-\chi)W\right)_t . \tag{4.4}$$

The right-hand side of (4.4) is the positive or negative liquid water source due to the solidification or melting of the ice. The liquid mass flux vector ξ is assumed to obey Darcy's law

$$\xi = -\mu(p)\nabla p, \tag{4.5}$$

with a proportionality factor $\mu(p) > 0$ (the permeability coefficient). Using (4.3) and (4.5), we rewrite the mass balance equation (4.4) as

$$\big((\rho_W \chi + \rho_E(1-\chi))(\mathcal{G}[p] + \mathrm{div} u)\big)_t - \mathrm{div}(\mu(p)\nabla p) = 0.$$

Then, setting $\rho^* = \rho_E/\rho_W \in (0{,}1)$, we finally get a partial differential equation with hysteresis of the form

$$\big((\chi + \rho^*(1-\chi))(\mathcal{G}[p] + \mathrm{div} u)\big)_t - \frac{1}{\rho_W}\mathrm{div}(\mu(p)\nabla p) = 0. \tag{4.6}$$

Note that $\mathcal{G}[p] \in (0\,,\,1-c_S)$ for all p, so that the above equation is degenerate in the sense that we do not control a priori the time derivatives of p.

4.1.2 Momentum balance

The equation of motion of a deformable body is, in classical continuum mechanics (see [104]),

$$\rho_S u_{tt} = \mathrm{div}\sigma + g, \tag{4.7}$$

where ρ_S is the solid mass density, σ is the stress tensor and g is a volume force acting on the body (e. g. gravity). For σ we prescribe the constitutive equation

$$\sigma = \mathcal{P}[\varepsilon] + \mathbf{B}\varepsilon_t + \big((\chi + \rho^*(1-\chi))(\lambda \mathrm{div} u - p) - \beta(\theta - \theta_c)\big)\delta, \tag{4.8}$$

where δ is the Kronecker tensor, $\mathcal{P}$ is a hysteresis operator describing the elastoplastic response of the solid (see Section B.1 in the Appendix), $\mathbf{B}$ is a symmetric positive definite viscosity tensor, $\lambda > 0$ is the bulk elasticity modulus of water, $\beta \in \mathbb{R}$ is the relative solid-liquid thermal expansion coefficient and $\theta_c > 0$ is a fixed reference temperature. The term $(\chi + \rho^*(1-\chi))(\lambda \mathrm{div} u - p)$ represents the pressure component due to the phase transition.

4.1.3 Energy balance

We assume that both hysteresis operators $\mathcal{G}$ (capillarity) and $\mathcal{P}$ (elastoplasticity) admit hysteresis potentials $U_{\mathcal{G}}$, $U_{\mathcal{P}}$ and dissipation operators $D_{\mathcal{G}}$, $D_{\mathcal{P}}$ such that the energy identities (B.8) and (B.27) hold. For more details and the explicit formulas of all these operators see Sections B.1 and B.3 in the Appendix.

The goal of this subsection is to derive formulas for the densities of internal energy U and entropy S such that the energy balance equation and the Clausius-Duhem inequality hold for all processes.

Let q be the heat flux vector, and let $V \subset \Omega$ be again an arbitrary control volume. The total internal energy in V is $\int_V U \,\mathrm{d}x$, and the total mechanical power $Q(V)$ supplied to V equals

$$Q(V) = \int_V \sigma : \varepsilon_t \,\mathrm{d}x - \int_{\partial V} \frac{1}{\rho_W} p\, \xi \cdot n \,\mathrm{d}s(x).$$

Thus, from the first principle of thermodynamics we have that the internal energy U must be conserved in the following sense:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V U \,\mathrm{d}x + \int_{\partial V} q \cdot n \,\mathrm{d}s(x) = \int_V \sigma : \varepsilon_t \,\mathrm{d}x - \int_{\partial V} \frac{1}{\rho_W} p\, \xi \cdot n \,\mathrm{d}s(x).$$

Again, by the Gauss formula and by Darcy's law (4.5) we get the energy balance equation in differential form

$$U_t + \mathrm{div} q = \sigma : \varepsilon_t + \frac{1}{\rho_W}\mathrm{div}(p\mu(p)\nabla p). \tag{4.9}$$

According to the second principle of thermodynamics, the entropy S must be nondecreasing in the sense of the Clausius-Duhem inequality

$$S_t + \mathrm{div}\left(\frac{q}{\theta}\right) \geq 0.$$

Developing the second summand the inequality takes the form

$$S_t + \frac{\theta \mathrm{div} q - q \cdot \nabla\theta}{\theta^2} \geq 0,$$

that is, multiplying by θ and isolating $\mathrm{div} q$,

$$\mathrm{div} q \geq -\theta S_t + \frac{q \cdot \nabla\theta}{\theta}.$$

Thus, taking into account the energy balance (4.9), we get the inequality

$$U_t - \theta S_t + \frac{q \cdot \nabla\theta}{\theta} \leq \sigma : \varepsilon_t + \frac{1}{\rho_W} \mathrm{div}(p\mu(p)\nabla p).$$

Hence, arguing as in Chapter 3, two inequalities have to hold separately for all processes, namely

$$q \cdot \nabla\theta \leq 0, \qquad U_t - \theta S_t \leq \sigma : \varepsilon_t + \frac{1}{\rho_W} \mathrm{div}(p\mu(p)\nabla p). \tag{4.10}$$

The first condition is certainly satisfied if we assume Fourier law for the heat flux

$$q = -\kappa(\theta)\nabla\theta, \tag{4.11}$$

with the heat conductivity coefficient $\kappa = \kappa(\theta) > 0$. We further introduce the free energy F by the formula $F = U - \theta S$ so that, in terms of F, the second inequality in (4.10) takes the form

$$F_t + \theta_t S \leq \sigma : \varepsilon_t + \frac{1}{\rho_W} \mathrm{div}(p\mu(p)\nabla p). \tag{4.12}$$

Combining (4.12) with the chain rule for F_t and the mass balance (4.6), one can "formally" prove that a formula for the internal energy F is obtained by integrating the constitutive relation (4.8). More precisely, the mechanical part of F is obtained by integration with respect to ε, whereas the capillarity part is obtained integrating with respect to $\mathcal{G}[p]$. Finally, the caloric part is obtained by thermodynamics similarly as in Chapter 3. The procedure is formal in the sense that we cannot give a precise meaning to the integration with respect to $\mathcal{G}[p]$, since hysteresis operators are not differentiable in the usual sense. Hence we are going to follow a different approach. More precisely, we use the constitutive relation and the energy balance for $\mathcal{G}$ to prove that for (4.12) to be satisfied, a "minimal" choice for F (in the sense that no unphysical constants are involved and all values of the phase fraction χ outside the admissible interval [0,1] are excluded) is given by, under the assumption of constant latent heat L,

$$\begin{aligned} F = {} & U_{\mathcal{P}}[\varepsilon] + (\chi + \rho^*(1-\chi))\left(U_{\mathcal{G}}[p] + \frac{\lambda}{2}(\mathrm{div} u)^2\right) - \beta(\theta - \theta_c)\,\mathrm{div} u \\ & + F_0(\theta) + L\chi\left(1 - \frac{\theta}{\theta_c}\right) + I_{[0,1]}(\chi), \end{aligned} \tag{4.13}$$

where $I_{[0,1]}$ is the indicator function of the interval [0,1], provided that the phase dynamics equation is chosen in the form

$$\gamma\chi_t + \partial I_{[0,1]}(\chi) \ni (1-\rho^*)\left(p\,\mathcal{G}[p] - U_{\mathcal{G}}[p] + p\,\mathrm{div}u - \frac{\lambda}{2}(\mathrm{div}u)^2\right) + L\left(\frac{\theta}{\theta_c} - 1\right) \tag{4.14}$$

with a relaxation coefficient $\gamma > 0$ possibly depending on the state variables $\mathrm{div}u, \theta, \chi$. The function $F_0(\theta)$ appearing in (4.13) is related to the caloric component $\mathcal{C}_V(\theta)$ of the internal energy by the formula

$$\mathcal{C}_V(\theta) = F_0(\theta) - \theta F_0'(\theta). \tag{4.15}$$

If the specific heat capacity c_V is constant, that is, $\mathcal{C}_V(\theta) = c_V\theta$, we find the classical formula $F_0(\theta) = c_V\theta\,(1 - \log(\theta/\theta_c))$. Note that from (4.13) we obtain also

$$S = -\frac{\partial F}{\partial \theta} = \beta\mathrm{div}u - F_0'(\theta) + \frac{L}{\theta_c}\chi. \tag{4.16}$$

In order to prove that under these choices (4.12) holds, we are going to develop the three summands of this inequality.

- By (4.13)

$$\begin{aligned} F_t = U_{\mathcal{P}}[\varepsilon]_t &+ (1-\rho^*)\chi_t\left(U_{\mathcal{G}}[p] + \frac{\lambda}{2}(\mathrm{div}u)^2\right) + (\chi + \rho^*(1-\chi))(U_{\mathcal{G}}[p]_t + \lambda\mathrm{div}u\,\mathrm{div}u_t) \\ &- \beta\theta_t\,\mathrm{div}u - \beta(\theta-\theta_c)\,\mathrm{div}u_t + F_0'(\theta)\,\theta_t + L\chi_t\left(1 - \frac{\theta}{\theta_c}\right) - \frac{L}{\theta_c}\chi\theta_t + \partial I_{[0,1]}(\chi)\chi_t, \end{aligned}$$

  where the summand $\partial I_{[0,1]}(\chi)\chi_t$ vanishes (see Remark B.2). Hence employing (4.16) we obtain

$$\begin{aligned} F_t + \theta_t S = U_{\mathcal{P}}[\varepsilon]_t &+ (1-\rho^*)\chi_t\left(U_{\mathcal{G}}[p] + \frac{\lambda}{2}(\mathrm{div}u)^2\right) \\ &+ (\chi + \rho^*(1-\chi))(U_{\mathcal{G}}[p]_t + \lambda\mathrm{div}u\,\mathrm{div}u_t) - \beta(\theta-\theta_c)\,\mathrm{div}u_t + L\chi_t\left(1 - \frac{\theta}{\theta_c}\right). \end{aligned}$$

- By (4.8)

$$\sigma : \varepsilon_t = \mathcal{P}[\varepsilon] : \varepsilon_t + \mathbf{B}\varepsilon_t : \varepsilon_t + \big((\chi + \rho^*(1-\chi))(\lambda\mathrm{div}u - p) - \beta(\theta - \theta_c)\big)\delta : \varepsilon_t.$$

  Note that

$$\delta : \varepsilon_t = \delta : \nabla_s u_t = \mathrm{div}u_t.$$

  Hence, using also the energy identity (B.8), we obtain

$$\sigma : \varepsilon_t = U_{\mathcal{P}}[\varepsilon]_t + \|D_{\mathcal{P}}[\varepsilon]_t\|_* + \mathbf{B}\varepsilon_t : \varepsilon_t + (\chi + \rho^*(1-\chi))(\lambda\mathrm{div}u\,\mathrm{div}u_t - p\,\mathrm{div}u_t) - \beta(\theta-\theta_c)\,\mathrm{div}u_t.$$

- By (4.6)

$$\frac{1}{\rho_W}\mathrm{div}(p\mu(p)\nabla p) = \frac{1}{\rho_W}\mu(p)|\nabla p|^2 + p(\chi + \rho^*(1-\chi))(\mathcal{G}[p]_t + \mathrm{div}u_t) + p(1-\rho^*)\chi_t(\mathcal{G}[p] + \mathrm{div}u).$$

Hence, coming back to (4.12) and using the energy identity (B.27) we get

$$\begin{aligned} F_t + \theta_t S - \sigma : \varepsilon_t - \frac{1}{\rho_W}\mathrm{div}(p\mu(p)\nabla p) = -\mathbf{B}\varepsilon_t : \varepsilon_t - \|D_{\mathcal{P}}[\varepsilon]_t\|_* - (\chi + \rho^*(1-\chi))|D_{\mathcal{G}}[p]_t| \\ - \frac{1}{\rho_W}\mu(p)|\nabla p|^2 - \chi_t\left(L\left(\frac{\theta}{\theta_c} - 1\right) + (1-\rho^*)\left(-U_{\mathcal{G}}[p] - \frac{\lambda}{2}(\mathrm{div}u)^2 + p\,\mathcal{G}[p] + p\,\mathrm{div}u\right)\right) \end{aligned} \tag{4.17}$$

from which we deduce, by virtue of (4.14),

$$\begin{aligned} F_t + \theta_t S - \sigma : \varepsilon_t - \frac{1}{\rho_W}\mathrm{div}(p\mu(p)\nabla p) = -\mathbf{B}\varepsilon_t : \varepsilon_t - \|D_{\mathcal{P}}[\varepsilon]_t\|_* - (\chi + \rho^*(1-\chi))|D_{\mathcal{G}}[p]_t| \\ - \frac{1}{\rho_W}\mu(p)|\nabla p|^2 - \gamma\chi_t^2 \le 0 \end{aligned}$$

so that (4.12) holds.

Now we are going to obtain the equation for the temperature rewriting (4.9) in a more suitable form. From (4.11), (4.16) and (4.17) we have

$$\begin{aligned} 0 &= U_t + \mathrm{div}q - \sigma : \varepsilon_t - \frac{1}{\rho_W}\mathrm{div}(p\mu(p)\nabla p) \\ &= F_t + \theta_t S + \theta S_t + \mathrm{div}q - \sigma : \varepsilon_t - \frac{1}{\rho_W}\mathrm{div}(p\mu(p)\nabla p) \\ &= -\mathbf{B}\varepsilon_t : \varepsilon_t - \|D_{\mathcal{P}}[\varepsilon]_t\|_* - (\chi + \rho^*(1-\chi))|D_{\mathcal{G}}[p]_t| - \frac{1}{\rho_W}\mu(p)|\nabla p|^2 \\ &\quad - \gamma\chi_t^2 + \theta\left(\beta\mathrm{div}u_t - F_0''(\theta)\theta_t + \frac{L}{\theta_c}\chi_t\right) - \mathrm{div}(\kappa(\theta)\nabla\theta), \end{aligned}$$

from which, since (4.15) yields $\mathcal{C}_V(\theta)_t = -\theta F_0''(\theta)\theta_t$,

$$\begin{aligned} \mathcal{C}_V(\theta)_t - \mathrm{div}(\kappa(\theta)\nabla\theta) = \mathbf{B}\varepsilon_t : \varepsilon_t + \frac{1}{\rho_W}\mu(p)|\nabla p|^2 + \|D_{\mathcal{P}}[\varepsilon]_t\|_* + (\chi + \rho^*(1-\chi))|D_{\mathcal{G}}[p]_t| \\ + \gamma\chi_t^2 - \frac{L}{\theta_c}\theta\chi_t - \beta\theta\mathrm{div}u_t. \end{aligned} \tag{4.18}$$

4.2 The mathematical problem

For mathematical reasons that will be clearer later, we assume that the relaxation coefficient γ of the phase transition explicitly depends on both θ and $\mathrm{div}u$. Gathering together (4.6)–(4.8), (4.14) and (4.18), we obtain that, in terms of the unknown functions p, u, θ, χ, our model system of equations has the form

$$\big((\chi + \rho^*(1-\chi))(\mathcal{G}[p] + \mathrm{div}u)\big)_t = \frac{1}{\rho_W}\mathrm{div}(\mu(p)\nabla p), \tag{4.19}$$

$$\rho_S u_{tt} = \mathrm{div}\sigma + g, \tag{4.20}$$

$$\sigma = \mathcal{P}[\nabla_s u] + \mathbf{B}\nabla_s u_t + \big((\chi + \rho^*(1-\chi))(\lambda\mathrm{div}u - p) - \beta(\theta - \theta_c)\big)\delta, \tag{4.21}$$

$$\begin{aligned} \mathcal{C}_V(\theta)_t - \mathrm{div}(\kappa(\theta)\nabla\theta) = \mathbf{B}\nabla_s u_t : \nabla_s u_t + \frac{1}{\rho_W}\mu(p)|\nabla p|^2 + \|D_{\mathcal{P}}[\nabla_s u]_t\|_* \\ + (\chi + \rho^*(1-\chi))|D_{\mathcal{G}}[p]_t| + \gamma(\theta, \mathrm{div}u)\chi_t^2 - \frac{L}{\theta_c}\theta\chi_t - \beta\theta\mathrm{div}u_t, \end{aligned} \tag{4.22}$$

$$\gamma(\theta, \mathrm{div} u)\chi_t + \partial I_{[0,1]}(\chi) \ni (1-\rho^*)\left(p\,\mathcal{G}[p] - U_{\mathcal{G}}[p] + p\,\mathrm{div} u - \frac{\lambda}{2}(\mathrm{div} u)^2\right) + L\left(\frac{\theta}{\theta_c} - 1\right). \tag{4.23}$$

On $\partial\Omega$ we prescribe boundary conditions

$$\left.\begin{aligned} u &= 0, \\ \frac{1}{\rho_W}\mu(p)\nabla p \cdot n &= \alpha(x)(p^* - p), \\ \kappa(\theta)\nabla\theta \cdot n &= \omega(x)(\theta^* - \theta), \end{aligned}\right\} \tag{4.24}$$

where p^* is a given outer pressure, θ^* is a given outer temperature, $\alpha(x) \geq 0$ is the permeability of the boundary and $\omega(x) \geq 0$ is the heat conductivity of the boundary.

We simplify the problem by assuming that:

1. water is incompressible: this corresponds to the choice $\lambda = 0$;
2. inertial effects are negligible. This hypothesis is justified by the expectation that the processes are slow.

Whence system (4.19)–(4.23) becomes

$$\big((\chi + \rho^*(1-\chi))(\mathcal{G}[p] + \mathrm{div} u)\big)_t = \frac{1}{\rho_W}\mathrm{div}(\mu(p)\nabla p), \tag{4.25}$$

$$-\mathrm{div}\left(\mathbf{B}\nabla_s u_t + \mathcal{P}[\nabla_s u]\right) + \nabla\big(p(\chi + \rho^*(1-\chi)) + \beta(\theta - \theta_c)\big) = g, \tag{4.26}$$

$$\begin{aligned} \mathcal{C}_V(\theta)_t - \mathrm{div}(\kappa(\theta)\nabla\theta) &= \mathbf{B}\nabla_s u_t : \nabla_s u_t + \frac{1}{\rho_W}\mu(p)|\nabla p|^2 + \|D_{\mathcal{P}}[\nabla_s u]_t\|_* \\ &+ (\chi + \rho^*(1-\chi))|D_{\mathcal{G}}[p]_t| + \gamma(\theta, \mathrm{div} u)\chi_t^2 - \frac{L}{\theta_c}\theta\chi_t - \beta\theta\mathrm{div} u_t, \end{aligned} \tag{4.27}$$

$$\gamma(\theta, \mathrm{div} u)\chi_t + \partial I_{[0,1]}(\chi) \ni (1-\rho^*)\left(p\,\mathcal{G}[p] - U_{\mathcal{G}}[p] + p\,\mathrm{div} u\right) + L\left(\frac{\theta}{\theta_c} - 1\right), \tag{4.28}$$

with initial conditions

$$\left.\begin{aligned} p(x,0) &= p^0(x), \\ u(x,0) &= u^0(x), \\ \theta(x,0) &= \theta^0(x), \\ \chi(x,0) &= \chi^0(x), \end{aligned}\right\} \tag{4.29}$$

and boundary conditions (4.24).

Remark 4.1. As we have already seen in Chapter 3, there is a strict link between evolutions of "phase-relaxation" type and the hysteresis operator of stop type. Observing the inclusion (B.10) defining the stop operator, we can interpret (4.28) in an equivalent way in the form

$$\chi(x,t) = \mathfrak{s}_{[0,1]}[F(x,\cdot), \chi^0(x)](t), \tag{4.30}$$

where

$$F(x,t) := \int_0^t \frac{(1-\rho^*)\left(p\,\mathcal{G}[p] - U_{\mathcal{G}}[p] + p\,\mathrm{div} u\right) + L\left(\theta/\theta_c - 1\right)}{\gamma(\theta, \mathrm{div} u)}(x,\tau)\,\mathrm{d}\tau \tag{4.31}$$

and where $\mathfrak{s}_{[0,1]}$ is the (shifted) stop operator with thresholds 0 and 1. The advantage of this representation is that now χ is defined by a formula involving, by virtue of Proposition B.5, an operator with good Lipschitz continuity properties.

CHAPTER 5

Solvability of the problem

We introduce the spaces

$$X = W^{1,2}(\Omega), \quad X_0 = \{\psi \in W^{1,2}(\Omega;\mathbb{R}^3) : \psi\big|_{\partial\Omega} = 0\}, \quad X_{q^*} = W^{1,q^*}(\Omega)$$

for some $q^* > 2$ that will be specified below in Theorem 5.3. Taking into account the boundary conditions (4.24), we consider (4.25)–(4.28) in variational form

$$\begin{aligned}&\int_\Omega \big((\chi + \rho^*(1-\chi))(f(p) + \mathcal{G}_0[p] + \mathrm{div}u)\big)_t \,\phi \,\mathrm{d}x + \int_\Omega \frac{1}{\rho_W}\mu(p)\nabla p \cdot \nabla\phi \,\mathrm{d}x \\ &= \int_{\partial\Omega} \alpha(x)(p^* - p)\,\phi \,\mathrm{d}s(x),\end{aligned} \tag{5.1}$$

$$\int_\Omega (\mathcal{P}[\nabla_s u] + \mathbf{B}\nabla_s u_t) : \nabla_s\psi \,\mathrm{d}x - \int_\Omega \big(p(\chi + \rho^*(1-\chi)) + \beta(\theta - \theta_c)\big)\,\mathrm{div}\psi \,\mathrm{d}x = \int_\Omega g\cdot\psi \,\mathrm{d}x, \tag{5.2}$$

$$\begin{aligned}&\int_\Omega \Bigg(\mathcal{C}_V(\theta)_t - \mathbf{B}\nabla_s u_t : \nabla_s u_t - \|D_{\mathcal{P}}[\nabla_s u]_t\|_* - \frac{1}{\rho_W}\mu(p)|\nabla p|^2 - (\chi + \rho^*(1-\chi))|D_0[p]_t| \\ &\quad -\gamma(\theta,\mathrm{div}u)\chi_t^2 + \left(\frac{L}{\theta_c}\chi_t + \beta\mathrm{div}u_t\right)\theta\Bigg)\zeta \,\mathrm{d}x + \int_\Omega \kappa(\theta)\nabla\theta\cdot\nabla\zeta \,\mathrm{d}x = \int_{\partial\Omega}\omega(x)(\theta^* - \theta)\,\zeta \,\mathrm{d}x,\end{aligned} \tag{5.3}$$

$$\gamma(\theta,\mathrm{div}u)\chi_t + \partial I_{[0,1]}(\chi) \ni (1-\rho^*)\left(\Phi(p) + p\,\mathcal{G}_0[p] - U_0[p] + p\,\mathrm{div}u\right) + L\left(\frac{\theta}{\theta_c} - 1\right) \quad \text{a.e.} \tag{5.4}$$

for a. e. $t \in (0,T)$ and all test functions $\phi \in X$, $\psi \in X_0$ and $\zeta \in X_{q^*}$. Note that we split the capillary hysteresis terms in hysteretic and nonhysteretic part according to (B.23), (B.28) and (B.29) in view of the regularization performed in Section 5.1. Indeed, only the nonhysteretic part will be affected by the cut-off.

We assume the following hypothesis holds.

Hypothesis 5.1. There exist constants $A^\flat > 0$, $B^\flat > 0$, $P^\sharp > 0$, $\bar\theta > 0$ such that

(i) $\mathbf{A}_e$, $\mathbf{A}_h$, $\mathbf{B}$ are constant symmetric positive definite fourth order tensors such that $\mathbf{A}_e\,\xi : \xi \geq A^\flat|\xi|^2$, $\mathbf{A}_h\,\xi : \xi \geq A^\flat|\xi|^2$, $\mathbf{B}\,\xi : \xi \geq B^\flat|\xi|^2$ for all $\xi \in \mathbb{R}^{3\times3}$;

(ii) $g \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)) \cap W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3))$ is a given function and there exists a function $G \in L^4(\Omega \times (0,T))$ such that $g = -\nabla G$;

(iii) $\alpha \in W^{1,\infty}(\partial\Omega)$, $\alpha(x) \geq 0$ a.e. and $\int_{\partial\Omega} \alpha(x)\,\mathrm{d}s(x) > 0$; $\omega \in L^\infty(\partial\Omega)$, $\omega(x) \geq 0$ a.e. and $\int_{\partial\Omega} \omega(x)\,\mathrm{d}s(x) > 0$;

(iv) $p^* \in L^\infty((0,T) \times \partial\Omega)$ and $p^*_t \in L^2(\partial\Omega \times (0,T))$, $|p^*(x,t)| \leq P^\sharp$ a.e.; $\theta^* \in L^\infty(\partial\Omega \times (0,T))$, $\theta^*_t \in L^2(\partial\Omega \times (0,T))$, $\theta^*(x,t) \geq \bar{\theta}$ a.e.;

(v) $p^0 \in L^\infty(\Omega) \cap W^{2,2}(\Omega)$, $|p^0(x)| \leq P^\sharp$ a.e., $u^0 \in X_0 \cap W^{1,4}(\Omega;\mathbb{R}^3)$, $\theta^0 \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$, $\theta^0(x) \geq \bar{\theta}$ a.e., $\chi^0 \in L^\infty(\Omega)$, $\chi^0(x) \in [0,1]$ a.e..

We also assume that there exist constants $f^\sharp > f^\flat > 0$, $\nu \in (0,1/2]$, $\mu^\flat > 0$, $c^\sharp > c^\flat > 0$, $1/2 \leq b < \hat{b} < 1$, $\kappa^\sharp > \kappa^\flat > 0$, $0 < a < 1-b$, $a < \hat{a} < \frac{(8+3a+2b)(1+b)}{7-2b}$, $\gamma^\sharp > \gamma^\flat > 0$ such that the nonlinearities satisfy the following conditions:

(vi) $f : \mathbb{R} \to (0,1)$ is a continuously differentiable function, $f^\flat(1+|p|)^{-1-\nu} \leq f'(p) \leq f^\sharp$ for all $p \in \mathbb{R}$;

(vii) $\mu : \mathbb{R} \to \mathbb{R}$ is a continuous function, $\mu(p) \geq \mu^\flat$ for all $p \in \mathbb{R}$;

(viii) $\mathcal{C}_V : [0,\infty) \to [0,\infty)$ is a continuously differentiable function, $\mathcal{C}_V'(\theta) =: c_V(\theta)$ is such that $c^\flat(1+\theta^b) \leq c_V(\theta) \leq c^\sharp(1+\theta^{\hat{b}})$ for all $\theta \geq 0$;

(ix) $\kappa : [0,\infty) \to [0,\infty)$ is a continuous function, $\kappa^\flat(1+\theta^{1+a}) \leq \kappa(\theta) \leq \kappa^\sharp(1+\theta^{1+\hat{a}})$ for all $\theta \geq 0$;

(x) $\gamma : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function, $\gamma^\flat(1+\theta+|\mathrm{div} u|^2) \leq \gamma(\theta, \mathrm{div} u) \leq \gamma^\sharp(1+\theta+|\mathrm{div} u|^2)$ for all $\theta \geq 0$, $u \in \mathbb{R}^3$;

(xi) $\mathcal{G}_0$ is the Preisach operator from Section B.3 in the Appendix with an initial memory state $\lambda_{-1} \in \Lambda_K$ for some $K \geq P^\sharp$, with density function satisfying Hypothesis B.10 and with potential U_0 and dissipation operator D_0 defined in (B.26);

(xii) $\mathcal{P} : C([0,T];\mathbb{R}^{3\times 3}_{\mathrm{sym}}) \to C([0,T];\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ is the constitutive operator of elastoplasticity from Section B.1 in the Appendix, with dissipation operator $D_{\mathcal{P}}$ defined in (B.7). Here and in the sequel $\mathbb{R}^{3\times 3}_{\mathrm{sym}}$ denotes the space of symmetric 3×3 tensors.

Remark 5.2. In this remark we comment on some of the above hypotheses.

(ii) It is not restrictive to assume that there exists G such that $g = -\nabla G$, since in our case the volume force g represents gravity, and thus it certainly admits a potential G.

(iii) The hypothesis $\alpha \in W^{1,\infty}(\partial\Omega)$ is requested in order to apply the result from [95, Theorem 4.1] about the spatial $W^{2,2}$-regularity for parabolic equations with nonlinear boundary conditions on $C^{1,1}$ domains.

(vi) The growth condition for f is purely technical and plays a substantial role in the Moser iteration argument in Subsection 5.2.7.

(viii) The growth condition for c_V will be of fundamental importance in Subsection 5.2.6 where, in order to estimate $\mathrm{div} u_t$ in $L^q(0,T;L^2(\Omega))$ with an exponent $q > 4$, we need a higher integrability (in space) for the temperature than simply $L^\infty(0,T;L^1(\Omega))$.

(ix) The tangled bound

$$\hat{a} < \frac{(8+3a+2b)(1+b)}{7-2b}$$

for the growth exponent of the function κ comes from Subsection 5.2.10, where we apply an iterative method in order to derive higher order estimates for the temperature.

The main result is the following existence theorem.

Theorem 5.3. *Let Hypothesis 5.1 hold. Then there exists a solution* (p,u,θ,χ) *to the system* (5.1)–(5.4) *with initial conditions* (4.29) *with the regularity*

- $p \in L^\infty(\Omega \times (0,T))$, $p_t \in L^2(\Omega \times (0,T))$, $M(p) \in L^2(0,T;W^{2,2}(\Omega))$ *with* $M(p)$ *given by* (5.119)*;*
- $u_t \in L^q(0,T;X_0 \cap W^{1,q}(\Omega;\mathbb{R}^3))$ *for all* $q < \frac{(8+3a+2b)(4+b)}{7-2b}$, $\nabla_s u \in L^2(\Omega;C([0,T];\mathbb{R}^{3\times 3}_{\mathrm{sym}}))$*;*
- $\theta \in L^q(\Omega \times (0,T))$ *for all* $q < \frac{(8+3a+2b)(4+b)}{7-2b}$, $\nabla\theta \in L^2(\Omega \times (0,T);\mathbb{R}^3)$, $\theta_t \in L^2(0,T;W^{-1,q^*}(\Omega))$ *with* $q^* > 2$ *given by* (5.142)*;*
- $\chi \in L^q(\Omega;C[0,T])$, $\chi_t \in L^q(\Omega \times (0,T))$ *for all* $q \in [1,\infty)$.

The proof of Theorem 5.3 will be divided into several steps. In order to eliminate possible degeneracy of the functions f and μ, we start by regularizing the problem by means of a large parameter R. Then we prove that this regularized problem admits a solution by the standard Faedo-Galerkin method: here the parameter R will be of fundamental importance in order to gain some regularity. Once we have derived suitable estimates, we pass to the limit in the Faedo-Galerkin scheme. The second part of the proof will consist in the derivation of a priori estimates independent of R, which will allow us to pass to the limit in the regularized system and infer the existence of a solution with the desired regularity.

In what follows, we denote by C any positive constant depending only on the data and by C_R any constant depending on the data and on R, all independent of the dimension n of the Galerkin approximation.

Moreover, we will systematically use the pointwise inequality

$$|\mathrm{div} u|^2(x,t) \le 3\,|\nabla_s u|^2(x,t) \quad \text{for all } (x,t) \in \Omega \times (0,T). \tag{5.5}$$

5.1 Cut-off system

We choose a regularizing parameter $R > 1$, and first solve a cut-off system with the intention to let $R \to \infty$.

For $z \in \mathbb{R}$ we denote by

$$Q_R(z) = \max\{-R, \min\{z, R\}\} \tag{5.6}$$

the projection onto $[-R, R]$. Then we cut-off some nonlinearities by setting

$$f_R(p) = \begin{cases} f(p) & \text{for } |p| \le R \\ f(R) + f'(R)(p-R) & \text{for } p > R \\ f(-R) + f'(-R)(p+R) & \text{for } p < -R \end{cases}, \tag{5.7}$$

$$\Phi_R(p) = \int_0^p f_R(z)\,\mathrm{d}z, \qquad V_R(p) = p f_R(p) - \Phi_R(p) = \int_0^p f_R'(z) z\,\mathrm{d}z, \tag{5.8}$$

$$\mu_R(p) = \begin{cases} \mu(p) & \text{for } |p| \le R \\ \mu(R) & \text{for } p > R \\ \mu(-R) & \text{for } p < -R \end{cases}, \tag{5.9}$$

$$\gamma_R(p, \theta, \mathrm{div} u) = \gamma(Q_R(\theta^+) + (p^2 - R^2)^+, \mathrm{div} u) \tag{5.10}$$

for $p, \theta, \mathrm{div} u \in \mathbb{R}$. Note that by Hypothesis 5.1 (vi) we deduce that $|f_R(p)| \le |f(0)| + f^\sharp |p|$, from which

$$|f_R(p)| \le C\,(1 + |p|)\,, \quad |\Phi_R(p)| \le C\left(1 + p^2\right), \quad C\left(|p|^{1-\nu} - 1\right) \le V_R(p) \le C\,p^2, \tag{5.11}$$

and also, from Hypothesis 5.1 (x),

$$\gamma^\flat\left(1 + Q_R(\theta^+) + (p^2 - R^2)^+ + |\mathrm{div} u|^2\right) \le \gamma_R(p, \theta, \mathrm{div} u) \le \gamma^\sharp\left(1 + Q_R(\theta^+) + (p^2 - R^2)^+ + |\mathrm{div} u|^2\right). \tag{5.12}$$

We replace (5.1)–(5.4) by the cut-off system

$$\begin{aligned}&\int_\Omega \big((\chi+\rho^*(1-\chi))(f_R(p)+\mathcal{G}_0[p]+\mathrm{div}u)\big)_t\,\phi\,\mathrm{d}x+\int_\Omega \frac{1}{\rho_W}\mu_R(p)\nabla p\cdot\nabla\phi\,\mathrm{d}x\\&=\int_{\partial\Omega}\alpha(x)(p^*-p)\,\phi\,\mathrm{d}s(x),\end{aligned}\tag{5.13}$$

$$\begin{aligned}&\int_\Omega \big((\mathcal{P}[\nabla_s u]+\mathbf{B}\nabla_s u_t):\nabla_s\psi\big)\,\mathrm{d}x\\&-\int_\Omega\big(p(\chi+\rho^*(1-\chi))+\beta(Q_R(\theta^+)-\theta_c)\big)\,\mathrm{div}\psi\,\mathrm{d}x=\int_\Omega g\cdot\psi\,\mathrm{d}x,\end{aligned}\tag{5.14}$$

$$\begin{aligned}&\int_\Omega\bigg(\mathcal{C}_V(\theta)_t-\mathbf{B}\nabla_s u_t:\nabla_s u_t-\|D_{\mathcal{P}}[\nabla_s u]_t\|_*-\frac{1}{\rho_W}\mu_R(p)\,Q_R(|\nabla p|^2)-(\chi+\rho^*(1-\chi))|D_0[p]_t|\\&-\gamma_R(p,\theta,\mathrm{div}u)\chi_t^2+\left(\frac{L}{\theta_c}\chi_t+\beta\mathrm{div}u_t\right)Q_R(\theta^+)\bigg)\zeta\,\mathrm{d}x+\int_\Omega\kappa(Q_R(\theta^+))\nabla\theta\cdot\nabla\zeta\,\mathrm{d}x\\&=\int_{\partial\Omega}\omega(x)(\theta^*-\theta)\,\zeta\,\mathrm{d}s(x),\end{aligned}\tag{5.15}$$

$$\begin{aligned}&\gamma_R(p,\theta,\mathrm{div}u)\chi_t+\partial I_{[0,1]}(\chi)\\&\ni(1-\rho^*)\,(\Phi_R(p)+p\,\mathcal{G}_0[p]-U_0[p]+p\,\mathrm{div}u)+L\left(\frac{Q_R(\theta^+)}{\theta_c}-1\right)\ \text{a. e.}\end{aligned}\tag{5.16}$$

for all test functions $\phi,\zeta\in X$ and $\psi\in X_0$. For the system (5.13)–(5.16) the following result holds true.

Proposition 5.4. *Let Hypothesis 5.1 hold and let $R>1$ be given. Then there exists a solution (p,u,θ,χ) to (5.13)–(5.16), (4.29) with the regularity*

- $p\in L^q(\Omega;C[0,T])$ *for all* $q\in[1.6)$, $\nabla p\in L^2(\Omega\times(0,T);\mathbb{R}^3)$, $p_t\in L^2(\Omega\times(0,T))$;
- $u_t\in L^2(0,T;X_0)$, $\nabla_s u_t\in L^4(\Omega\times(0,T);\mathbb{R}^{3\times3}_{\mathrm{sym}})$;
- $\theta\in L^2(\Omega\times(0,T))$, $\nabla\theta\in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$, $\theta_t\in L^2(\Omega\times(0,T))$;
- $\chi\in L^q(\Omega;C[0,T])$, $\chi_t\in L^q(\Omega\times(0,T))$ *for all* $q\in[1,\infty)$.

We split the proof of Proposition 5.4 in two steps. First, in Subsection 5.1.1, we further regularize the system by means of a small parameter $\eta>0$ in order to obtain some extra-regularity for the gradient of the capillary pressure. Then, in Subsection 5.1.2, we solve this new problem by Galerkin approximations. Here the extra-regularization will be of fundamental importance in order to pass to the limit in the nonlinearity $Q_R(|\nabla p^{(n)}|^2)$, where n is the dimension of the Galerkin scheme. As a last step, we let $\eta\to0$.

5.1.1 $W^{2,2}$-regularization of the capillary pressure

We define the functions

$$M_R(p):=\int_0^p\mu_R(z)\,\mathrm{d}z,\qquad K_R(\theta):=\int_0^\theta\kappa(Q_R(z^+))\,\mathrm{d}z\tag{5.17}$$

for $p, \theta \in \mathbb{R}$, and introduce the new variables $v = M_R(p)$, $z = K_R(\theta)$. We then choose another regularizing parameter $\eta \in (0,1)$ and consider the following system in the unknowns v, u, z, χ:

$$\begin{aligned}&\int_\Omega \big((\chi + \rho^*(1-\chi))(f_R(M_R^{-1}(v)) + \mathcal{G}_0[M_R^{-1}(v)] + \operatorname{div} u)\big)_t \,\phi \,\mathrm{d}x \\ &+ \int_\Omega \frac{1}{\rho_W} \left(\nabla v \cdot \nabla \phi + \eta \Delta v \Delta \phi\right) \mathrm{d}x = \int_{\partial\Omega} \alpha(x)(p^* - M_R^{-1}(v))\,\phi \,\mathrm{d}s(x),\end{aligned} \tag{5.18}$$

$$\begin{aligned}&\int_\Omega \big((\mathcal{P}[\nabla_s u] + \mathbf{B}\nabla_s u_t) : \nabla_s \psi\big) \,\mathrm{d}x \\ &- \int_\Omega \big(M_R^{-1}(v)(\chi + \rho^*(1-\chi)) + \beta(Q_R((K_R^{-1}(z))^+) - \theta_c)\big) \operatorname{div}\psi \,\mathrm{d}x = \int_\Omega g\cdot\psi \,\mathrm{d}x,\end{aligned} \tag{5.19}$$

$$\begin{aligned}&\int_\Omega \Bigg(\mathcal{C}_V(K_R^{-1}(z))_t - \mathbf{B}\nabla_s u_t : \nabla_s u_t - \|D_{\mathcal{P}}[\nabla_s u]_t\|_* - \frac{1}{\rho_W}\mu_R(M_R^{-1}(v))Q_R(|\nabla(M_R^{-1}(v))|^2) \\ &- (\chi + \rho^*(1-\chi))|D_0[M_R^{-1}(v)]_t| - \gamma_R(M_R^{-1}(v), K_R^{-1}(z), \operatorname{div} u)\chi_t^2 \\ &+ \left(\frac{L}{\theta_c}\chi_t + \beta \operatorname{div} u_t\right) Q_R((K_R^{-1}(z))^+)\Bigg)\zeta \,\mathrm{d}x + \int_\Omega \nabla z \cdot \nabla \zeta \,\mathrm{d}x \\ &= \int_{\partial\Omega} \omega(x)(\theta^* - K_R^{-1}(z))\,\zeta \,\mathrm{d}s(x),\end{aligned} \tag{5.20}$$

$$\begin{aligned}&\gamma_R(M_R^{-1}(v), K_R^{-1}(z), \operatorname{div} u)\chi_t + \partial I_{[0,1]}(\chi) \\ &\ni (1-\rho^*)\left(\Phi_R(M_R^{-1}(v)) + M_R^{-1}(v)\,\mathcal{G}_0[M_R^{-1}(v)] - U_0[M_R^{-1}(v)] + M_R^{-1}(v) \operatorname{div} u\right) \\ &+ L\left(\frac{Q_R((K_R^{-1}(z))^+)}{\theta_c} - 1\right) \text{ a.e.}\end{aligned} \tag{5.21}$$

with test functions $\phi \in W^{2,2}(\Omega)$, $\zeta \in X$ and $\psi \in X_0$. Here we imposed the additional boundary condition

$$\nabla(\Delta v)\cdot n = 0 \quad \text{on } \partial\Omega. \tag{5.22}$$

5.1.2 Galerkin approximations

For each fixed $R > 1$, system (5.18)–(5.21) will be solved by Faedo-Galerkin approximations. To this end, let $\mathcal{W} = \{\phi_i : i = 0,1,2,\dots\} \subset L^2(\Omega)$ and $\mathcal{Z} = \{\zeta_k : k = 0,1,2,\dots\} \subset L^2(\Omega)$ be the complete orthonormal systems of eigenfunctions defined by

$$\begin{aligned}-\Delta\phi_i &= \lambda_i\phi_i \quad \text{in } \Omega, \qquad \nabla\phi_i\cdot n\big|_{\partial\Omega} = 0, \\ -\Delta\zeta_k &= \nu_k\zeta_k \quad \text{in } \Omega, \qquad \nabla\zeta_k\cdot n\big|_{\partial\Omega} = 0,\end{aligned}$$

with $\lambda_0 = \nu_0 = 0$, $\lambda_i, \nu_k > 0$ for $i, k \ge 1$. Given $n \in \mathbb{N}$, we approximate v and z by the finite sums

$$v^{(n)}(x,t) = \sum_{i=0}^n v_i(t)\phi_i(x), \quad z^{(n)}(x,t) = \sum_{k=0}^n z_k(t)\zeta_k(x)$$

where the coefficients $v_i, z_k : [0,T] \to \mathbb{R}$ and $u^{(n)}, \chi^{(n)}$ will be determined as the solution of the system

$$\begin{aligned}&\int_\Omega \big((\chi^{(n)} + \rho^*(1-\chi^{(n)}))(f_R(p^{(n)}) + \mathcal{G}_0[p^{(n)}] + \mathrm{div} u^{(n)})\big)_t \,\phi_i \,\mathrm{d}x + \frac{1}{\rho_W}\left(\lambda_i + \eta\lambda_i^2\right) v_i \\ &= \int_{\partial\Omega} \alpha(x)(p^* - p^{(n)})\,\phi_i \,\mathrm{d}s(x),\end{aligned} \tag{5.23}$$

$$\begin{aligned}&\int_\Omega \big((\mathcal{P}[\nabla_s u^{(n)}] + \mathbf{B}\nabla_s u_t^{(n)}\big) : \nabla_s\psi \,\mathrm{d}x \\ &- \int_\Omega \Big(p^{(n)}(\chi^{(n)} + \rho^*(1-\chi^{(n)})) + \beta(Q_R((\theta^{(n)})^+) - \theta_c)\Big)\,\mathrm{div}\psi \,\mathrm{d}x = \int_\Omega g\cdot\psi \,\mathrm{d}x,\end{aligned} \tag{5.24}$$

$$\begin{aligned}&\int_\Omega \Bigg(\mathcal{C}_V(\theta^{(n)})_t - \mathbf{B}\nabla_s u_t^{(n)} : \nabla_s u_t^{(n)} - \|D_{\mathcal{P}}[\nabla_s u^{(n)}]_t\|_* - \frac{1}{\rho_W}\mu_R(p^{(n)})Q_R(|\nabla p^{(n)}|^2) \\ &- (\chi^{(n)} + \rho^*(1-\chi^{(n)}))|D_0[p^{(n)}]_t| - \gamma_R(p^{(n)}, \theta^{(n)}, \mathrm{div} u^{(n)})|\chi_t^{(n)}|^2 \\ &+ \left(\frac{L}{\theta_c}\chi_t^{(n)} + \beta\,\mathrm{div} u_t^{(n)}\right) Q_R((\theta^{(n)})^+)\Bigg)\,\zeta_k \,\mathrm{d}x + \nu_k z_k = \int_{\partial\Omega} \omega(x)(\theta^* - \theta^{(n)})\,\zeta_k \,\mathrm{d}s(x),\end{aligned} \tag{5.25}$$

$$\begin{aligned}&\gamma_R(p^{(n)}, \theta^{(n)}, \mathrm{div} u^{(n)})\chi_t^{(n)} + \partial I_{[0,1]}(\chi^{(n)}) \\ &\ni (1-\rho^*)\Big(\Phi_R(p^{(n)}) + p^{(n)}\mathcal{G}_0[p^{(n)}] - U_0[p^{(n)}] + p^{(n)}\,\mathrm{div} u^{(n)}\Big) + L\left(\frac{Q_R((\theta^{(n)})^+)}{\theta_c} - 1\right) \text{ a. e.}\end{aligned} \tag{5.26}$$

for $i, k = 0,1,\dots,n$ and for all $\psi \in X_0$, and with $p^{(n)} := M_R^{-1}(v^{(n)})$, $\theta^{(n)} := K_R^{-1}(z^{(n)})$. We prescribe the initial conditions

$$\left.\begin{aligned} v_i(0) &= \int_\Omega M_R(p^0(x))\,\phi_i(x)\,\mathrm{d}x, \\ u^{(n)}(x,0) &= u^0(x), \\ z_k(0) &= \int_\Omega K_R(\theta^0(x))\,\zeta_k(x)\,\mathrm{d}x, \\ \chi^{(n)}(x,0) &= \chi^0(x). \end{aligned}\right\} \tag{5.27}$$

This is an ODE system coupled with a nonlinear PDE (5.24). It is nontrivial to prove that such a system admits a unique strong solution. We proceed as follows. For a given function $w \in L^p(\Omega\times(0,T))$ consider the equation

$$\int_\Omega \mathbf{B}\nabla_s u_t(x,t) : \nabla_s\psi(x)\,\mathrm{d}x + \int_\Omega \mathcal{P}[\nabla_s u](x,t) : \nabla_s\psi(x)\,\mathrm{d}x = \int_\Omega w(x,t)\,\mathrm{div}\psi(x)\,\mathrm{d}x, \tag{5.28}$$

which is to be satisfied for every $\psi \in X_0$ a. e. in $(0,T)$ together with an initial condition $u(x,0) = u^0(x)$, $u^0 \in X_0 \cap W^{1,p}(\Omega;\mathbb{R}^n)$ and boundary condition $u = 0$ on $\partial\Omega$.

Step 1. By the L^p-regularity for elliptic systems in divergence form (see e. g. [6, Theorem 15.12]), for every $w \in L^p(\Omega\times(0,T))$ with some $p \in [2,\infty)$ the problem

$$\int_\Omega \mathbf{B}\nabla_s u_t(x,t) : \nabla_s\psi(x)\,\mathrm{d}x = \int_\Omega w(x,t)\,\mathrm{div}\psi(x)\,\mathrm{d}x$$

has a unique solution such that $\nabla_s u_t \in L^p(\Omega\times(0,T);\mathbb{R}^{3\times3}_{\mathrm{sym}})$, and it holds

$$\int_\Omega |\nabla_s u_t|^p(x,t)\,\mathrm{d}x \le C\int_\Omega |w|^p(x,t)\,\mathrm{d}x \quad \text{a. e.}$$

Step 2. Let $\hat{u} \in L^p(0,T;X_0)$ be such that $\nabla_s \hat{u}_t \in L^p(\Omega \times (0,T);\mathbb{R}^{3\times3}_{\mathrm{sym}})$, $\hat{u}(x,0) = u^0(x)$ a. e., and let u be the solution of the equation

$$\int_\Omega \mathcal{P}[\nabla_s \hat{u}](x,t) : \nabla_s \psi(x)\,\mathrm{d}x + \int_\Omega \mathbf{B}\nabla_s u_t(x,t) : \nabla_s \psi(x)\,\mathrm{d}x = \int_\Omega w(x,t)\,\mathrm{div}\psi(x)\,\mathrm{d}x,$$

the existence of which follows from Step 1. We prove that the mapping $\hat{u}_t \mapsto u_t$ is a contraction with respect to a suitable norm.

Indeed, let $\hat{u}_1, \hat{u}_2$ be given, and let u_1, u_2 be the corresponding solutions. The difference $\bar{u} = u_1 - u_2$ is the solution of the equation

$$\int_\Omega \mathbf{B}\nabla_s \bar{u}_t(x,t) : \nabla_s \psi(x)\,\mathrm{d}x = -\int_\Omega (\mathcal{P}[\nabla_s \hat{u}_1] - \mathcal{P}[\nabla_s \hat{u}_2])(x,t) : \nabla_s \psi(x)\,\mathrm{d}x.$$

According to Step 1, we have

$$\int_\Omega |\nabla_s \bar{u}_t|^p(x,t)\,\mathrm{d}x \le C \int_\Omega |\mathcal{P}[\nabla_s \hat{u}_1] - \mathcal{P}[\nabla_s \hat{u}_2]|^p(x,t)\,\mathrm{d}x \quad \text{a. e.} \tag{5.29}$$

By inequality (B.4) in the Appendix we have for a. e. (x,t)

$$|\mathcal{P}[\nabla_s \hat{u}_1] - \mathcal{P}[\nabla_s \hat{u}_2]|(x,t) \le C \int_0^t |\nabla_s (\hat{u}_1 - \hat{u}_2)_t|(x,\tau)\,\mathrm{d}\tau$$

with a constant $C > 0$. Hence, by Hölder's inequality,

$$\begin{aligned}\int_\Omega |\mathcal{P}[\nabla_s \hat{u}_1] - \mathcal{P}[\nabla_s \hat{u}_2]|^p(x,t)\,\mathrm{d}x &\le C \int_\Omega \left(\int_0^t |\nabla_s (\hat{u}_1 - \hat{u}_2)_t|(x,\tau)\,\mathrm{d}\tau\right)^p \mathrm{d}x \\ &\le C t^{p-1} \int_0^t \int_\Omega |\nabla_s (\hat{u}_1 - \hat{u}_2)_t|^p(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned} \tag{5.30}$$

Now, set

$$W(t) = \int_\Omega |\nabla_s (u_1 - u_2)_t|^p(x,\tau)\,\mathrm{d}x, \qquad \hat{W}(t) = \int_\Omega |\nabla_s (\hat{u}_1 - \hat{u}_2)_t|^p(x,\tau)\,\mathrm{d}x.$$

It follows from (5.29) and (5.30) that

$$W(t) \le C t^{p-1} \int_0^t \hat{W}(\tau)\,\mathrm{d}\tau.$$

We now multiply both sides of the above inequality by e^{-Ct^p}, and after an integration over $t \in [0,T]$ we obtain from the Fubini Theorem

$$\begin{aligned}\int_0^T \mathrm{e}^{-Ct^p} W(t)\,\mathrm{d}t &\le \int_0^T \left(-\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-Ct^p} \int_0^t \hat{W}(\tau)\,\mathrm{d}\tau\right) \mathrm{d}t \\ &= -\frac{1}{p}\int_0^T \left(\int_\tau^T \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-Ct^p}\,\mathrm{d}t\right) \hat{W}(\tau)\,\mathrm{d}\tau \\ &= \frac{1}{p}\int_0^T \left(\mathrm{e}^{-C\tau^p} - \mathrm{e}^{-CT^p}\right) \hat{W}(\tau)\,\mathrm{d}\tau \le \frac{1}{p}\int_0^T \mathrm{e}^{-Ct^p} \hat{W}(t)\,\mathrm{d}t.\end{aligned}$$

This means that the mapping $\hat{u}_t \mapsto u_t$ is a contraction in $L^p(0,T;X_0 \cap W^{1,p}(\Omega;\mathbb{R}^3))$ with respect to the weighted norm

$$|||u_t||| = \left(\int_0^T \mathrm{e}^{-Ct^p} \int_\Omega |\nabla_s u_t|^p(x,t)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/p},$$

hence it has a unique fixed point which is a solution of (5.28).

Step 3. The mapping which with a right-hand side $w \in L^p(\Omega \times (0,T))$ associates the solution $u_t \in L^p(0,T;X_0 \cap W^{1,p}(\Omega;\mathbb{R}^3))$ of (5.28) is Lipschitz continuous. Indeed, consider w_1, w_2 and the corresponding solutions u_1, u_2, and set as before $\bar{w} = w_1 - w_2$, $\bar{u} = u_1 - u_2$. As a counterpart of (5.29) we get

$$\int_\Omega |\nabla_s \bar{u}_t|^p(x,t)\,\mathrm{d}x \le C\int_\Omega \left(|\mathcal{P}[\nabla_s u_1] - \mathcal{P}[\nabla_s u_2]|^p + |\bar{w}|^p\right)(x,t)\,\mathrm{d}x \quad \text{a.e.},$$

and the computations as in (5.30) yield

$$\int_\Omega |\nabla_s \bar{u}_t|^p(x,t)\,\mathrm{d}x \le Ct^{p-1}\int_0^t \int_\Omega |\nabla_s \bar{u}_t|^p(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau + C\int_\Omega |\bar{w}|^p(x,t)\,\mathrm{d}x \quad \text{a.e.}$$

We obtain the Lipschitz continuity result when we test by $\mathrm{e}^{-\frac{C}{p}t^p}$ and integrate over $t \in [0,T]$, similarly as in Step 2.

Now, coming back to our equation (5.24), we see that it is of the form (5.28) with $w(x,t) = w^{(n)}(x,t) := p^{(n)}(\chi^{(n)} + \rho^*(1-\chi^{(n)})) + \beta(Q_R((\theta^{(n)})^+) - \theta_c)(x,t) + G(x,t)$. Therefore, denoting by $\mathcal{S} : w^{(n)} \mapsto u_t^{(n)}$ its associated solution operator, we conclude that (5.23), (5.25) and (5.26) give rise to a system of ODEs with a locally Lipschitz continuous right-hand side containing the operator $\mathcal{S}$. Here the inclusion (5.26) can be interpreted as in Remark 4.1 and handled with Proposition B.5.

Thus system (5.23)–(5.26) has a unique strong solution in a maximal interval of existence $[0,T_n] \subset [0,T]$. This interval coincides with the whole $[0,T]$, provided we prove that the solution remains bounded in $[0,T_n)$.

We now derive a series of estimates. Note that we decompose the auxiliary variables v and z instead of p and θ into a Fourier series with respect to the basis $\mathcal{W}$ and $\mathcal{Z}$ because we are going to test equations (5.23), (5.25) by nonlinear expressions of p and θ, namely, by their Kirchhoff transforms (5.17). Indeed, the Galerkin method allows one to test only by linear functions and their derivatives. Moreover, we do not discretize the momentum equation because considering the full PDE is the only way to deduce compactness of the sequence $\{\nabla_s u_t^{(n)}\}$, which is needed in order to pass to the limit in some nonlinear terms. Indeed, we will not be able to control the second derivatives, and this will prevent us from applying the usual embedding theorems.

Estimates independent of n

▶ **Estimate 1**

We test (5.23) by v_i and sum up over $i = 0,1,\dots,n$, and (5.24) by $\psi = u_t^{(n)}$. Then we sum up the two equations to obtain

$$\begin{aligned}
&\int_\Omega \big((\chi^{(n)} + \rho^*(1-\chi^{(n)}))(f_R(p^{(n)}) + \mathcal{G}_0[p^{(n)}] + \mathrm{div} u^{(n)})\big)_t M_R(p^{(n)})\,\mathrm{d}x \\
&+ \int_\Omega \frac{1}{\rho_W}\Big(|\nabla v^{(n)}|^2 + \eta\,|\Delta v^{(n)}|^2\Big)\,\mathrm{d}x + \int_\Omega (\mathbf{B}\nabla_s u_t^{(n)} + \mathcal{P}[\nabla_s u^{(n)}]) : \nabla_s u_t^{(n)}\,\mathrm{d}x \\
&- \int_\Omega \Big(p^{(n)}(\chi^{(n)} + \rho^*(1-\chi^{(n)})) + \beta(Q_R((\theta^{(n)})^+) - \theta_c)\Big)\,\mathrm{div} u_t^{(n)}\,\mathrm{d}x \\
&= \int_{\partial\Omega} \alpha(x)(p^* - p^{(n)})M_R(p^{(n)})\,\mathrm{d}s(x) + \int_\Omega g\cdot u_t^{(n)}\,\mathrm{d}x,
\end{aligned}$$

that is, computing the time derivative in the first summand and exploiting the energy identity (B.8),

$$\begin{aligned}
&\int_\Omega (\chi^{(n)} + \rho^*(1-\chi^{(n)}))f_R(p^{(n)})_t M_R(p^{(n)})\,\mathrm{d}x + \int_\Omega (\chi^{(n)} + \rho^*(1-\chi^{(n)}))\mathcal{G}_0[p^{(n)}]_t M_R(p^{(n)})\,\mathrm{d}x \\
&+ \int_\Omega \Big(\mathbf{B}\nabla_s u_t^{(n)} : \nabla_s u_t^{(n)} + U_{\mathcal{P}}[\nabla_s u^{(n)}]_t + \|D_{\mathcal{P}}[\nabla_s u^{(n)}]_t\|_*\Big)\,\mathrm{d}x \\
&+ \int_\Omega \frac{1}{\rho_W}\Big(|\nabla v^{(n)}|^2 + \eta\,|\Delta v^{(n)}|^2\Big)\,\mathrm{d}x + \int_{\partial\Omega} \alpha(x)(p^{(n)} - p^*)M_R(p^{(n)})\,\mathrm{d}s(x) \\
&= -\int_\Omega (1-\rho^*)\chi_t^{(n)}\Big(f_R(p^{(n)}) + \mathcal{G}_0[p^{(n)}]\Big)M_R(p^{(n)})\,\mathrm{d}x - \int_\Omega (1-\rho^*)\chi_t^{(n)}\mathrm{div} u^{(n)} M_R(p^{(n)})\,\mathrm{d}x \\
&- \int_\Omega (\chi^{(n)} + \rho^*(1-\chi^{(n)}))\,\mathrm{div} u_t^{(n)} M_R(p^{(n)})\,\mathrm{d}x + \int_\Omega p^{(n)}(\chi^{(n)} + \rho^*(1-\chi^{(n)}))\,\mathrm{div} u_t^{(n)}\,\mathrm{d}x \\
&+ \int_\Omega \beta(Q_R((\theta^{(n)})^+) - \theta_c)\,\mathrm{div} u_t^{(n)}\,\mathrm{d}x + \int_\Omega g\cdot u_t^{(n)}\,\mathrm{d}x.
\end{aligned} \tag{5.31}$$

We now define

$$V_{M,R}(p) := \int_0^p f_R'(z)M_R(z)\,\mathrm{d}z$$

so that

$$\begin{aligned}
&\int_\Omega (\chi^{(n)} + \rho^*(1-\chi^{(n)}))f_R(p^{(n)})_t M_R(p^{(n)})\,\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\chi^{(n)} + \rho^*(1-\chi^{(n)}))V_{M,R}(p^{(n)})\,\mathrm{d}x - \int_\Omega (1-\rho^*)\chi_t^{(n)} V_{M,R}(p^{(n)})\,\mathrm{d}x,
\end{aligned}$$

and introduce the modified Preisach potential

$$U_{M,R}[p] := \int_0^\infty \int_0^{\mathfrak{p}_r[p]} M_R(v)\,\psi(r,v)\,\mathrm{d}v\,\mathrm{d}r > 0$$

which satisfies

$$\mathcal{G}_0[p]_t\, M_R(p) - U_{M,R}[p]_t \geq 0 \quad \text{a. e.}$$

according to (B.32) and (B.33). Note that (5.7) and (5.9) together with Hypothesis 5.1 (vi) and (vii) yield

$$c_R\, p^2 \leq V_{M,R}(p) \leq C_R\, p^2 \tag{5.32}$$

for all $p \in \mathbb{R}$, with some positive constants c_R, C_R depending only on R. Moreover, the estimates

$$U_{M,R}[p] \leq C_R\,(1+|p|)\,, \tag{5.33}$$

$$U_{M,R}[p](x{,}0) = \int_0^\infty \int_0^{\mathfrak{p}_r[p^0](x)} M_R(v)\,\psi(r,v)\,\mathrm{d}v\,\mathrm{d}r \leq C_R \max\{|p^0(x)|, K\} \tag{5.34}$$

hold as a counterpart of (B.30) and (B.31). We rewrite (5.31) as

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big((\chi^{(n)} + \rho^*(1-\chi^{(n)}))(V_{M,R}(p^{(n)}) + U_{M,R}[p^{(n)}]) + U_{\mathcal{P}}[\nabla_s u^{(n)}]\Big)\,\mathrm{d}x \\
&+ \int_\Omega \left(\frac{1}{\rho_W}\left(|\nabla v^{(n)}|^2 + \eta\,|\Delta v^{(n)}|^2\right) + \mathbf{B}\nabla_s u_t^{(n)} : \nabla_s u_t^{(n)} + \|D_{\mathcal{P}}[\nabla_s u^{(n)}]_t\|_*\right)\mathrm{d}x \\
&+ \int_{\partial\Omega} \alpha(x)(p^{(n)} - p^*) M_R(p^{(n)})\,\mathrm{d}s(x) \\
&\leq \int_\Omega (1-\rho^*)\chi_t^{(n)}\left(V_{M,R}(p^{(n)}) + U_{M,R}[p^{(n)}]\right)\mathrm{d}x - \int_\Omega (1-\rho^*)\chi_t^{(n)}\left(f_R(p^{(n)}) + \mathcal{G}_0[p^{(n)}]\right) M_R(p^{(n)})\,\mathrm{d}x \\
&- \int_\Omega (1-\rho^*)\chi_t^{(n)} \mathrm{div} u^{(n)} M_R(p^{(n)})\,\mathrm{d}x + \int_\Omega (\chi^{(n)} + \rho^*(1-\chi^{(n)}))\,\mathrm{div} u_t^{(n)}\left(p^{(n)} - M_R(p^{(n)})\right)\mathrm{d}x \\
&+ \int_\Omega \beta(Q_R((\theta^{(n)})^+) - \theta_c)\,\mathrm{div} u_t^{(n)}\,\mathrm{d}x + \int_\Omega g\cdot u_t^{(n)}\,\mathrm{d}x.
\end{aligned} \tag{5.35}$$

By the definition of $v^{(n)}$ and Hypothesis 5.1 (vii) we deduce

$$\int_\Omega \frac{1}{\rho_W}|\nabla v^{(n)}|^2\,\mathrm{d}x = \int_\Omega \frac{1}{\rho_W}|\mu_R(p)|^2|\nabla p^{(n)}|^2\,\mathrm{d}x \geq \frac{(\mu^\flat)^2}{\rho_W}\int_\Omega |\nabla p^{(n)}|^2\,\mathrm{d}x. \tag{5.36}$$

Moreover, thanks again to Hypothesis 5.1 (vii), the boundary term is such that

$$\begin{aligned}
&\int_{\partial\Omega} \alpha(x)(p^{(n)} - p^*) M_R(p^{(n)})\,\mathrm{d}s(x) \\
&= \int_{\partial\Omega} \alpha(x)|p^{(n)}|^2 M_R^*(p^{(n)})\,\mathrm{d}s(x) - \int_{\partial\Omega} \alpha(x) p^* p^{(n)} M_R^*(p^{(n)})\,\mathrm{d}s(x) \\
&\geq \mu^\flat \int_{\partial\Omega} \alpha(x)|p^{(n)}|^2\,\mathrm{d}s(x) - C_R\int_{\partial\Omega} \alpha(x)|p^* p^{(n)}|\,\mathrm{d}s(x),
\end{aligned}$$

where for $p \in \mathbb{R}$ we set

$$M_R^*(p) := \begin{cases} M_R(p)/p & \text{for } p \neq 0, \\ M_R'(0) & \text{for } p = 0. \end{cases}$$

Young's inequality and Hypothesis 5.1 (iii), (iv) give

$$\int_{\partial\Omega} \alpha(x)(p^{(n)} - p^*) M_R(p^{(n)})\,\mathrm{d}s(x) \geq \frac{\mu^\flat}{2}\int_{\partial\Omega} \alpha(x)|p^{(n)}|^2\,\mathrm{d}s(x) - C_R. \tag{5.37}$$

Moreover, by Hölder's inequality and Hypothesis 5.1 (ii),

$$\begin{aligned}
\int_\Omega (g\cdot u_t^{(n)})(x,t)\,\mathrm{d}x &\leq C\left(\int_\Omega |u_t^{(n)}|^2(x,t)\,\mathrm{d}x\right)^{1/2} \\
&\leq \frac{2C}{\sqrt{c\,B^\flat}}\frac{\sqrt{B^\flat}}{2}\left(\int_\Omega |\nabla_s u_t^{(n)}|^2\,\mathrm{d}x\right)^{1/2} \leq C + \frac{B^\flat}{8}\int_\Omega |\nabla_s u_t^{(n)}|^2\,\mathrm{d}x
\end{aligned} \tag{5.38}$$

where in the last line we used first Korn's inequality (A.11) and then Young's inequality. Neglecting some lower order positive terms on the left-hand side, exploiting estimates (5.11), (5.32) and (5.33) and the fact that $\rho^* \le (\chi + \rho^*(1-\chi)) \le 1$ for all $\chi \in [0,1]$, from (5.35) and the subsequent computations we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big((\chi^{(n)} + \rho^*(1-\chi^{(n)}))(V_{M,R}(p^{(n)}) + U_{M,R}[p^{(n)}]) + U_{\mathcal{P}}[\nabla_s u^{(n)}]\Big)\,\mathrm{d}x \\
&+ \int_\Omega \left(|\nabla p^{(n)}|^2 + \eta\,|\Delta v^{(n)}|^2 + \frac{B^\flat}{2}|\nabla_s u_t^{(n)}|^2\right)\mathrm{d}x + \int_{\partial\Omega} \alpha(x)|p^{(n)}|^2\,\mathrm{d}s(x) \\
&\le C_R\left(1 + \int_\Omega \left(|\chi_t^{(n)}||p^{(n)}|^2 + |\chi_t^{(n)}||\mathrm{div} u^{(n)}||p^{(n)}| + |p^{(n)}|^2\right)\mathrm{d}x\right),
\end{aligned} \tag{5.39}$$

where we used also Young's inequality and the pointwise inequality (5.5) to absorb $|\mathrm{div} u_t^{(n)}|$ on the left-hand side together with the term coming from (5.38).

We need to control $|\chi_t^{(n)}|$. To this aim we consider the phase dynamics in the formulation (4.30)–(4.31), and employ the identity (B.21) for the stop (see also Remark B.2) to deduce that $|\chi_t^{(n)}| \le |F_t^{(n)}|$. This yields, thanks to (5.11), (5.12) and (B.30),

$$|\chi_t^{(n)}(x,t)| \le \frac{C\,(1+|p^{(n)}|^2) + |\mathrm{div} u^{(n)}|^2/2 + L\left|Q_R((\theta^{(n)})^+)/\theta_c - 1\right|}{\gamma^\flat\left(1 + (Q_R((\theta^{(n)})^+) + (|p^{(n)}|^2 - R^2)^+ + |\mathrm{div} u^{(n)}|^2\right)} \le C_R \tag{5.40}$$

for a. e. $(x,t) \in \Omega \times (0,T_n)$. We now come back to (5.39) and integrate in time $\int_0^\tau \mathrm{d}t$ for some $\tau \in [0,T_n]$. The initial conditions are kept under control thanks to (5.32), (5.34), (B.2) and (B.7) and Hypothesis 5.1 (v). Hence Young's inequality yields

$$\begin{aligned}
&\int_\Omega \left(|p^{(n)}|^2 + |\nabla_s u^{(n)}|^2\right)(x,t)\,\mathrm{d}x + \int_0^\tau \int_\Omega \left(|\nabla p^{(n)}|^2 + \eta\,|\Delta v^{(n)}|^2 + |\nabla_s u_t^{(n)}|^2\right)(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau \int_{\partial\Omega} \alpha(x)|p^{(n)}|^2(x,t)\,\mathrm{d}s(x)\,\mathrm{d}t \le C_R\left(1 + \int_0^\tau \int_\Omega \left(|p^{(n)}|^2 + |\mathrm{div} u^{(n)}|^2\right)(x,t)\,\mathrm{d}x\,\mathrm{d}t\right).
\end{aligned}$$

Using (5.5) and Grönwall's lemma A.2, we see that the approximate solution remains bounded in the maximal interval of existence $[0,T_n]$. Hence the solution exists globally, and for every $n \in \mathbb{N}$ we have $T_n = T$. We thus have obtained

$$\sup_{\tau\in[0,T]}\mathrm{ess} \int_\Omega \left(|p^{(n)}|^2 + |\nabla_s u^{(n)}|^2\right)(x,\tau)\,\mathrm{d}x \le C_R, \tag{5.41}$$

$$\int_0^T \Big(\int_\Omega \left(|\nabla p^{(n)}|^2 + |\nabla_s u_t^{(n)}|^2\right)(x,t)\,\mathrm{d}x + \int_{\partial\Omega} \alpha(x)|p^{(n)}|^2(x,t)\,\mathrm{d}s(x)\Big)\,\mathrm{d}t \le C_R, \tag{5.42}$$

and also

$$\int_0^T \int_\Omega |\Delta v^{(n)}|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \le \frac{C_R}{\eta}\,. \tag{5.43}$$

Now, in terms of the variable $v^{(n)} = M_R(p^{(n)})$, the boundary condition is nonlinear. By the spatial $W^{2,2}$-regularity result for parabolic equations with nonlinear boundary conditions on $C^{1,1}$ domains stated and proved in [95, Theorem 4.1], we finally see that

$$\|M_R(p^{(n)})\|^2_{L^2(0,T;W^{2,2}(\Omega))} \le \frac{C_R}{\eta}\,. \tag{5.44}$$

▶ **Estimate 2**

We test (5.23) by $\dot{v}_i$ and sum up over $i = 0,1,\ldots,n$. We get

$$\begin{aligned}&\int_\Omega \big((\chi^{(n)} + \rho^*(1-\chi^{(n)}))(f_R(p^{(n)}) + \mathcal{G}_0[p^{(n)}] + \mathrm{div}u^{(n)})\big)_t\, M_R(p^{(n)})_t \,\mathrm{d}x \\ &+ \int_\Omega \frac{1}{\rho_W}\left(\nabla v^{(n)}\cdot\nabla v_t^{(n)} + \eta\,\Delta v^{(n)}\Delta v_t^{(n)}\right)\,\mathrm{d}x = \int_{\partial\Omega}\alpha(x)(p^*-p^{(n)})M_R(p^{(n)})_t\,\mathrm{d}s(x).\end{aligned} \tag{5.45}$$

Defining

$$\hat{\mu}_R(p) := \int_0^p M_R'(z)z\,\mathrm{d}z = \int_0^p \mu_R(z)z\,\mathrm{d}z$$

for $p\in\mathbb{R}$, we can rewrite

$$\begin{aligned}&\int_{\partial\Omega}\alpha(x)(p^{(n)}-p^*)M_R(p^{(n)})_t\,\mathrm{d}s(x) = \int_{\partial\Omega}\alpha(x)\big(p^{(n)}M_R(p^{(n)})_t - p^*M_R(p^{(n)})_t\big)\,\mathrm{d}s(x)\\ &= \int_{\partial\Omega}\alpha(x)\big(\hat{\mu}_R(p^{(n)})_t - (p^*M_R(p^{(n)}))_t + p_t^*M_R(p^{(n)})\big)\,\mathrm{d}s(x)\\ &= \int_{\partial\Omega}\alpha(x)\big(\hat{\mu}_R(p^{(n)}) - p^*M_R(p^{(n)})\big)_t\,\mathrm{d}s(x) + \int_{\partial\Omega}\alpha(x)p_t^*M_R(p^{(n)})\,\mathrm{d}s(x).\end{aligned}$$

Hence, computing the time derivative in the first summand and rearranging the terms, we can rewrite (5.45) as

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \frac{1}{2\rho_W}\left(|\nabla v^{(n)}|^2 + \eta\,|\Delta v^{(n)}|^2\right)\,\mathrm{d}x + \int_{\partial\Omega}\alpha(x)(\hat{\mu}_R(p^{(n)}) - p^*M_R(p^{(n)}))\,\mathrm{d}s(x)\right)\\ &+\int_\Omega(\chi^{(n)}+\rho^*(1-\chi^{(n)}))\left(f_R(p^{(n)})_t + \mathcal{G}_0[p^{(n)}]_t\right)M_R(p^{(n)})_t\,\mathrm{d}x\\ &= -\int_\Omega(1-\rho^*)\chi_t^{(n)}\mathrm{div}u^{(n)}M_R(p^{(n)})_t\,\mathrm{d}x - \int_\Omega(1-\rho^*)\chi_t^{(n)}\left(f_R(p^{(n)}) + \mathcal{G}_0[p^{(n)}]\right)M_R(p^{(n)})_t\,\mathrm{d}x\\ &-\int_\Omega(\chi^{(n)}+\rho^*(1-\chi^{(n)}))\,\mathrm{div}u_t^{(n)}M_R(p^{(n)})_t\,\mathrm{d}x - \int_{\partial\Omega}\alpha(x)p_t^*M_R(p^{(n)})\,\mathrm{d}s(x).\end{aligned} \tag{5.46}$$

Combining (B.24) with the identity (B.21) for the play, we see that it holds

$$\mathcal{G}_0[p^{(n)}]_t\,M_R(p^{(n)})_t = \mathcal{G}_0[p^{(n)}]_t\,p_t^{(n)}\mu_R(p^{(n)}) \geq 0.$$

Hence by (5.7), (5.9) and Hypothesis 5.1 (vi), (vii) we obtain the pointwise lower bound

$$(\chi^{(n)}+\rho^*(1-\chi^{(n)}))\left(f_R(p^{(n)})_t + \mathcal{G}_0[p^{(n)}]_t\right)M_R(p^{(n)})_t \geq \rho^*f_R'(p^{(n)})\,\mu_R(p^{(n)})\,|p_t^{(n)}|^2 \geq C_R\,|p_t^{(n)}|^2.$$

We now integrate (5.46) in time $\int_0^\tau \mathrm{d}t$ for some $\tau\in[0,T]$. Note that $\hat{\mu}_R(p)\geq \mu^\flat\,p^2/2$ for all $p\in\mathbb{R}$. Hence, arguing as for estimate (5.37) with $p^{(n)}M_R(p^{(n)})$ replaced by $\hat{\mu}_R(p^{(n)})$, we obtain that the boundary term on the left-hand side is such that

$$\int_{\partial\Omega}\alpha(x)(\hat{\mu}_R(p^{(n)}) - p^*M_R(p^{(n)}))(x,\tau)\,\mathrm{d}s(x) \geq \frac{\mu^\flat}{4}\int_{\partial\Omega}\alpha(x)|p^{(n)}|^2(x,\tau)\,\mathrm{d}s(x) - C_R. \tag{5.47}$$

Concerning the initial conditions, we employ Hypothesis 5.1 (iv) and (v) together with the following computations

$$\begin{aligned}\int_{\partial\Omega}\alpha(x)\hat{\mu}_R(p^{(n)})(x,0)\,\mathrm{d}s(x) &= \int_{\partial\Omega}\alpha(x)\hat{\mu}_R(M_R^{-1}(v^{(n)}))(x,0)\,\mathrm{d}s(x)\\ &\le \frac{C_R}{(\mu^\flat)^2}\int_{\partial\Omega}\alpha(x)|v^{(n)}|^2(x,0)\,\mathrm{d}s(x),\\ \int_{\partial\Omega}\alpha(x)|p^*|^2(x,0)\,\mathrm{d}s(x) &= \int_{\partial\Omega}\alpha(x)|p^*|^2(x,\tau)\,\mathrm{d}s(x) - 2\int_0^\tau\int_{\partial\Omega}\alpha(x)(p^*p_t^*)(x,t)\,\mathrm{d}s(x)\,\mathrm{d}\tau,\\ \int_{\partial\Omega}\alpha(x)|M_R(p^{(n)})|^2(x,0)\,\mathrm{d}s(x) &= \int_{\partial\Omega}\alpha(x)|v^{(n)}|^2(x,0)\,\mathrm{d}s(x).\end{aligned}$$

Hence, exploiting also (5.36) and (5.40), we get

$$\begin{aligned}&\int_\Omega\left(|\nabla p^{(n)}|^2+\eta\,|\Delta v^{(n)}|^2\right)(x,\tau)\,\mathrm{d}x+\int_{\partial\Omega}\alpha(x)|p^{(n)}|^2(x,\tau)\,\mathrm{d}s(x)+\int_0^\tau\int_\Omega|p_t^{(n)}|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &\le C_R\left(1+\int_0^\tau\int_\Omega\left(|\mathrm{div}u^{(n)}||p_t^{(n)}|+|p^{(n)}||p_t^{(n)}|+|\mathrm{div}u_t^{(n)}||p_t^{(n)}|\right)\,\mathrm{d}x\,\mathrm{d}t\right.\\ &\quad\left.+\int_0^\tau\int_{\partial\Omega}\alpha(x)|p_t^*||p^{(n)}|\,\mathrm{d}s(x)\,\mathrm{d}t\right).\end{aligned}$$

Young's inequality and Hypothesis 5.1 (iii) and (iv) give

$$\begin{aligned}&\int_\Omega\left(|\nabla p^{(n)}|^2+\eta\,|\Delta v^{(n)}|^2\right)(x,\tau)\,\mathrm{d}x+\int_{\partial\Omega}\alpha(x)|p^{(n)}|^2(x,\tau)\,\mathrm{d}s(x)+\int_0^\tau\int_\Omega|p_t^{(n)}|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &\le C_R\left(1+\int_0^\tau\int_\Omega\left(|\mathrm{div}u^{(n)}|^2+|p^{(n)}|^2+|\mathrm{div}u_t^{(n)}|^2\right)(x,t)\,\mathrm{d}x\,\mathrm{d}t+\int_0^\tau\int_{\partial\Omega}\alpha(x)|p^{(n)}|^2(x,t)\,\mathrm{d}s(x)\,\mathrm{d}t\right).\end{aligned}$$

Using estimates (5.5), (5.41) and (5.42) we deduce

$$\sup_{\tau\in[0,T]}\mathrm{ess}\left(\int_\Omega|\nabla p^{(n)}|^2(x,\tau)\,\mathrm{d}x+\int_{\partial\Omega}\alpha(x)|p^{(n)}|^2(x,\tau)\,\mathrm{d}s(x)\right)\le C_R, \tag{5.48}$$

$$\sup_{\tau\in[0,T]}\mathrm{ess}\int_\Omega|\Delta v^{(n)}|^2(x,\tau)\,\mathrm{d}x\le\frac{C_R}{\eta}, \tag{5.49}$$

$$\int_0^T\int_\Omega|p_t^{(n)}|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\le C_R. \tag{5.50}$$

▶ **Estimate 3**

We test (5.25) by $\dot{z}_k$ and sum over $k=0,1,\dots,n$. We obtain

$$\begin{aligned}&\int_\Omega\Bigg(\mathcal{C}_V(\theta^{(n)})_t-\mathbf{B}\nabla_s u_t^{(n)}:\nabla_s u_t^{(n)}-\|D_{\mathcal{P}}[\nabla_s u^{(n)}]_t\|_*-\frac{1}{\rho_W}\mu_R(p^{(n)})Q_R(|\nabla p^{(n)}|^2)\\ &\quad-(\chi^{(n)}+\rho^*(1-\chi^{(n)}))|D_0[p^{(n)}]_t|-\gamma_R(p^{(n)},\theta^{(n)},\mathrm{div}u^{(n)})|\chi_t^{(n)}|^2\\ &\quad+\left(\frac{L}{\theta_c}\chi_t^{(n)}+\beta\mathrm{div}u_t^{(n)}\right)Q_R((\theta^{(n)})^+)\Bigg)K_R(\theta^{(n)})_t\,\mathrm{d}x+\int_\Omega\nabla z^{(n)}\cdot\nabla z_t^{(n)}\,\mathrm{d}x\\ &=\int_{\partial\Omega}\omega(x)(\theta^*-\theta^{(n)})K_R(\theta^{(n)})_t\,\mathrm{d}s(x).\end{aligned} \tag{5.51}$$

Defining

$$\hat{\kappa}_R(\theta):=\int_0^\theta K_R'(z)z\,\mathrm{d}z=\int_0^\theta\kappa(Q_R(z^+))z\,\mathrm{d}z$$

for $\theta \in \mathbb{R}$, we can rewrite

$$\int_{\partial\Omega} \omega(x)(\theta^{(n)} - \theta^*)K_R(\theta^{(n)})_t \,\mathrm{d}s(x) = \int_{\partial\Omega} \omega(x)\big(\theta^{(n)}K_R(\theta^{(n)})_t - \theta^* K_R(\theta^{(n)})_t\big)\,\mathrm{d}s(x)$$
$$= \int_{\partial\Omega} \omega(x)\big(\hat{\kappa}_R(\theta^{(n)})_t - (\theta^* K_R(\theta^{(n)}))_t + \theta^*_t K_R(\theta^{(n)})\big)\,\mathrm{d}s(x)$$
$$= \int_{\partial\Omega} \omega(x)\big(\hat{\kappa}_R(\theta^{(n)}) - \theta^* K_R(\theta^{(n)})\big)_t \,\mathrm{d}s(x) + \int_{\partial\Omega} \omega(x)\theta^*_t K_R(\theta^{(n)})\,\mathrm{d}s(x).$$

Hence, rearranging the terms in (5.51), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega |\nabla z^{(n)}|^2\,\mathrm{d}x + \int_{\partial\Omega} \omega(x)\big(\hat{\kappa}_R(\theta^{(n)}) - \theta^* K_R(\theta^{(n)})\big)\,\mathrm{d}s(x)\right)$$
$$+ \int_\Omega \mathcal{C}'_V(\theta^{(n)})\,\kappa(Q_R((\theta^{(n)})^+))\,|\theta^{(n)}_t|^2\,\mathrm{d}x$$
$$= \int_\Omega \Bigg(\mathbf{B}\nabla_s u^{(n)}_t : \nabla_s u^{(n)}_t + \|D_{\mathcal{P}}[\nabla_s u^{(n)}]_t\|_* + \frac{1}{\rho_W}\mu_R(p^{(n)})Q_R(|\nabla p^{(n)}|^2)$$
$$+ (\chi^{(n)} + \rho^*(1-\chi^{(n)}))|D_0[p^{(n)}]_t| + \gamma_R(p^{(n)}, \theta^{(n)}, \mathrm{div} u^{(n)})|\chi^{(n)}_t|^2$$
$$- \left(\frac{L}{\theta_c}\chi^{(n)}_t + \beta \mathrm{div} u^{(n)}_t\right) Q_R((\theta^{(n)})^+)\Bigg)\kappa(Q_R((\theta^{(n)})^+))\theta^{(n)}_t\,\mathrm{d}x - \int_{\partial\Omega} \omega(x)\theta^*_t K_R(\theta^{(n)})\,\mathrm{d}s(x).$$

Note that Hypothesis 5.1 (ix) yields

$$\kappa^\flat \le \kappa^\flat\left(1 + (Q_R((\theta^{(n)})^+))^a\right) \le \kappa(Q_R((\theta^{(n)})^+)) \le \kappa^\sharp\left(1 + (Q_R((\theta^{(n)})^+))^a\right) \le C_R. \tag{5.52}$$

We now integrate (5.51) in time $\int_0^\tau \mathrm{d}t$ for some $\tau \in [0,T]$. By (5.52) and Hypothesis 5.1 (viii) it holds

$$\int_\Omega |\nabla z^{(n)}|^2\,\mathrm{d}x = \int_\Omega |\kappa(Q_R((\theta^{(n)})^+))|^2|\nabla\theta^{(n)}|^2\,\mathrm{d}x \ge (\kappa^\flat)^2 \int_\Omega |\nabla\theta^{(n)}|^2\,\mathrm{d}x,$$

$$\int_0^\tau\int_\Omega \mathcal{C}'_V(\theta^{(n)})\,\kappa(Q_R((\theta^{(n)})^+))\,|\theta^{(n)}_t|^2\,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau\int_\Omega c_V(\theta^{(n)})\,\kappa(Q_R((\theta^{(n)})^+))\,|\theta^{(n)}_t|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\ge c^\flat\kappa^\flat \int_0^\tau\int_\Omega |\theta^{(n)}_t|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Note also that $\hat{\kappa}_R(\theta) \ge \kappa^\flat\,\theta^2/2$ for all $\theta \in \mathbb{R}$. Hence, using Young's inequality as in (5.37) we obtain

$$\int_{\partial\Omega} \omega(x)\big(\hat{\kappa}_R(\theta^{(n)}) - \theta^* K_R(\theta^{(n)})\big)\,\mathrm{d}s(x) \ge \frac{\kappa^\flat}{2}\int_{\partial\Omega} \omega(x)|\theta^{(n)}|^2\,\mathrm{d}s(x) - C_R\int_{\partial\Omega} \omega(x)|\theta^*\theta^{(n)}|^2\,\mathrm{d}s(x)$$
$$\ge \frac{\kappa^\flat}{4}\int_{\partial\Omega} \omega(x)|\theta^{(n)}|^2\,\mathrm{d}s(x) - C_R.$$

Concerning the initial conditions, we employ Hypothesis 5.1 (iv) and (v) together with the following computations

$$\int_{\partial\Omega} \omega(x)\hat{\kappa}_R(\theta^{(n)})(x,0)\,\mathrm{d}s(x) = \int_{\partial\Omega} \omega(x)\hat{\kappa}_R(K_R^{-1}(z^{(n)}))(x,0)\,\mathrm{d}s(x)$$
$$\le C_R\int_{\partial\Omega} \omega(x)|z^{(n)}|^2(x,0)\,\mathrm{d}s(x),$$

$$\int_{\partial\Omega} \omega(x)|\theta^*|^2(x,0)\,\mathrm{d}s(x) = \int_{\partial\Omega} \omega(x)|\theta^*|^2(x,\tau)\,\mathrm{d}s(x) - 2\int_0^\tau\int_{\partial\Omega} \omega(x)(\theta^*\theta^*_t)(x,t)\,\mathrm{d}s(x)\,\mathrm{d}\tau,$$

$$\int_{\partial\Omega} \omega(x)|K_R(\theta^{(n)})|^2(x,0)\,\mathrm{d}s(x) = \int_{\partial\Omega} \omega(x)|z^{(n)}|^2(x,0)\,\mathrm{d}s(x).$$

Thus, exploiting also (5.12), (5.40), (5.52), (B.9) and (B.30), we get

$$\begin{aligned}&\int_\Omega |\nabla\theta^{(n)}|^2(x,\tau)\,\mathrm{d}x + \int_{\partial\Omega}\omega(x)|\theta^{(n)}|^2(x,\tau)\,\mathrm{d}s(x) + \int_0^\tau\int_\Omega |\theta_t^{(n)}|^2(x,t)\,\mathrm{d}t\\ &\le C_R\bigg(1+\int_0^\tau\int_\Omega\Big(|\nabla_s u_t^{(n)}|^2+|p_t^{(n)}|+|p^{(n)}|^2+|\mathrm{div}u^{(n)}|^2\Big)|\theta_t^{(n)}|\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_0^\tau\int_{\partial\Omega}\omega(x)|\theta_t^*||\theta^{(n)}|\,\mathrm{d}s(x)\,\mathrm{d}t\bigg).\end{aligned}$$

Young's inequality, Hypothesis 5.1 (iii), (iv) and estimate (5.50) give

$$\begin{aligned}&\int_\Omega |\nabla\theta^{(n)}|^2(x,\tau)\,\mathrm{d}x + \int_{\partial\Omega}\omega(x)|\theta^{(n)}|^2(x,\tau)\,\mathrm{d}s(x) + \int_0^\tau\int_\Omega |\theta_t^{(n)}|^2(x,t)\,\mathrm{d}t\\ &\le C_R\bigg(1+\int_0^\tau\int_\Omega\Big(|\nabla_s u_t^{(n)}|^4+|p^{(n)}|^4+|\mathrm{div}u^{(n)}|^4\Big)\,\mathrm{d}x\,\mathrm{d}t+\int_0^\tau\int_{\partial\Omega}\omega(x)|\theta^{(n)}|^2\,\mathrm{d}s(x)\,\mathrm{d}t\bigg).\end{aligned}\tag{5.53}$$

We now need to estimate the terms $\nabla_s u_t^{(n)}$, $p^{(n)}$, $\mathrm{div}u^{(n)}$ in the norm of $L^4(\Omega\times(0,T))$. Note that (5.48) and (5.50) entail $\nabla p^{(n)}\in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$, $p_t^{(n)}\in L^2(\Omega\times(0,T))$ independently of n. Thus, applying Lemma A.3 with $p_0=q_0=q_1=2$, $p_1=\infty$, $p_2=q_2=4$ we see that

$$\int_0^T\int_\Omega |p^{(n)}|^4(x,t)\,\mathrm{d}x\,\mathrm{d}t\le C_R.\tag{5.54}$$

Now, let us consider (5.24) rewritten in the form

$$\int_\Omega \mathcal{P}[\nabla_s u^{(n)}](x,t):\nabla_s\psi(x)\,\mathrm{d}x+\int_\Omega \mathbf{B}\nabla_s u_t^{(n)}(x,t):\nabla_s\psi(x)\,\mathrm{d}x=\int_\Omega w^{(n)}(x,t)\,\mathrm{div}\psi(x)\,\mathrm{d}x,$$

where

$$w^{(n)}(x,t):=p^{(n)}(\chi^{(n)}+\rho^*(1-\chi^{(n)}))+\beta(Q_R((\theta^{(n)})^+)-\theta_c)(x,t)+G(x,t)$$

according to Hypothesis 5.1 (ii). By the already mentioned L^p-regularity (with some $p\in[2,\infty)$) for elliptic systems in divergence form and by (B.6) we deduce, arguing as for (5.30),

$$\begin{aligned}\int_\Omega|\nabla_s u_t^{(n)}|^p(x,t)\,\mathrm{d}x &\le C\int_\Omega|\nabla_s u^{(n)}|^p(x,0)\,\mathrm{d}x+Ct^{p-1}\int_0^t\int_\Omega|\nabla_s u_t^{(n)}|^p(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\\ &\quad+C\int_\Omega|w^{(n)}|^p(x,t)\,\mathrm{d}x\quad\text{a. e.}\end{aligned}\tag{5.55}$$

By (5.54) and Hypothesis 5.1 (ii) we see that

$$\int_0^\tau\int_\Omega|w^{(n)}|^4(x,t)\,\mathrm{d}x\,\mathrm{d}t\le C_R\bigg(\int_0^\tau\int_\Omega|p^{(n)}|^4(x,t)\,\mathrm{d}x\,\mathrm{d}t+1\bigg)\le C_R.$$

Therefore, choosing $p=4$ in (5.55) and using also Hypothesis 5.1 (v), by Grönwall's lemma A.2 we obtain

$$\int_0^\tau\int_\Omega|\nabla_s u_t^{(n)}|^4(x,t)\,\mathrm{d}x\,\mathrm{d}t\le C_R,$$

from which we deduce a bound also for the term $\mathrm{div} u^{(n)}$ since $\nabla_s u_t^{(n)}$ is dominant. Thus, coming back to (5.53), we finally obtain

$$\int_\Omega |\nabla\theta^{(n)}|^2(x,\tau)\,\mathrm{d}x + \int_{\partial\Omega} \omega(x)|\theta^{(n)}|^2(x,\tau)\,\mathrm{d}s(x) + \int_0^\tau \int_\Omega |\theta_t^{(n)}|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t$$
$$\le C_R \left(1 + \int_0^\tau \int_{\partial\Omega} \omega(x)|\theta^{(n)}|^2(x,t)\,\mathrm{d}s(x)\,\mathrm{d}t\right)$$

from which, by Poincaré's inequality (A.10),

$$\int_\Omega \left(|\theta^{(n)}|^2 + |\nabla\theta^{(n)}|^2\right)(x,\tau)\,\mathrm{d}x + \int_{\partial\Omega} \omega(x)|\theta^{(n)}|^2(x,\tau)\,\mathrm{d}s(x) + \int_0^\tau \int_\Omega |\theta_t^{(n)}|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t$$
$$\le C_R \left(1 + \int_0^\tau \int_{\partial\Omega} \omega(x)|\theta^{(n)}|^2(x,t)\,\mathrm{d}s(x)\,\mathrm{d}t\right).$$

Applying Grönwall's lemma A.2 we finally obtain for all $t \in (0,T)$ the estimates

$$\sup_{\tau\in[0,T]} \mathrm{ess} \left(\int_\Omega \left(|\theta^{(n)}|^2 + |\nabla\theta^{(n)}|^2\right)(x,\tau)\,\mathrm{d}x + \int_{\partial\Omega} \omega(x)|\theta^{(n)}|^2(x,\tau)\,\mathrm{d}s(x)\right) \le C_R, \tag{5.56}$$

$$\int_0^T \int_\Omega |\theta_t^{(n)}|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \le C_R. \tag{5.57}$$

Limit as $n \to \infty$

For the moment we keep the regularization parameters η and R fixed, and let $n \to \infty$ in (5.23)–(5.26). From estimates (5.41), (5.42), (5.44), (5.48), (5.50), (5.56) and (5.57) we see that there exists a subsequence of $\{(p^{(n)}, \theta^{(n)}) : n \in \mathbb{N}\}$, which is again indexed by n, and functions p, θ such that

$$\begin{aligned} &p_t^{(n)} \to p_t, \quad \theta_t^{(n)} \to \theta_t && \text{weakly in } L^2(\Omega\times(0,T)),\\ &\nabla\theta^{(n)} \to \nabla\theta && \text{weakly-star in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)). \end{aligned}$$

By the compact embeddings established in Theorem A.3, Theorem A.4 and Corollary A.9 we also have

$$\begin{aligned} p^{(n)} &\to p && \text{strongly in } L^q(\Omega;C[0,T]) \text{ for } q\in[1,6) \text{ and in } L^2(\partial\Omega\times(0,T)),\\ \nabla p^{(n)} &\to \nabla p && \text{strongly in } L^2(\Omega\times(0,T);\mathbb{R}^3),\\ \theta^{(n)} &\to \theta && \text{strongly in } L^2(\Omega\times(0,T)) \text{ and in } L^2(\partial\Omega\times(0,T)). \end{aligned}$$

We also need strong convergence of the sequences $\{\nabla_s u^{(n)}\}$ and $\{\nabla_s u_t^{(n)}\}$ in order to pass to the limit in some nonlinear terms. Note that, arguing as for (5.55), we obtain for $p = 2$

$$\int_0^T \int_\Omega |\nabla_s u_t^{(n)} - \nabla_s u_t^{(m)}|^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \le C \int_0^T \int_\Omega |w^{(n)} - w^{(m)}|^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\le C_R \left(1 + \int_0^T \int_\Omega \left(|p^{(n)} - p^{(m)}|^2 + |\chi^{(n)} - \chi^{(m)}|^2\right)(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\right)$$

a. e. in $(0,T)$. Remark 4.1 and Proposition B.5 yield

$$|\chi^{(n)}-\chi^{(m)}|(x,\tau)\le\int_0^\tau|\chi_t^{(n)}-\chi_t^{(m)}|(x,t)\,\mathrm{d}t\le 2\int_0^\tau|F_t^{(n)}-F_t^{(m)}|(x,t)\,\mathrm{d}t, \tag{5.58}$$

from which, by Proposition B.12, Remark B.13 and the pointwise inequality (5.5),

$$\begin{aligned}
&\int_0^T\int_\Omega|\nabla_s u_t^{(n)}-\nabla_s u_t^{(m)}|^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\\
&\le C_R\Bigg(1+\int_0^T\int_\Omega\Bigg(|p^{(n)}-p^{(m)}|^2(x,\tau)\\
&\qquad+\int_0^\tau\Big(\sup_{t\in[0,\tau]}\mathrm{ess}\,|p^{(n)}-p^{(m)}|^2(x,t)+|\mathrm{div}u^{(n)}-\mathrm{div}u^{(m)}|^2(x,t)\Big)\,\mathrm{d}t\Bigg)\,\mathrm{d}x\,\mathrm{d}\tau\Bigg)\\
&\le C_R\Bigg(1+\int_0^T\int_\Omega\Bigg(\sup_{t\in[0,\tau]}\mathrm{ess}\,|p^{(n)}-p^{(m)}|^2(x,t)+\int_0^\tau\int_0^t|\nabla_s u_t^{(n)}-\nabla_s u_t^{(m)}|^2(x,s)\,\mathrm{d}s\,\mathrm{d}t\Bigg)\,\mathrm{d}x\,\mathrm{d}\tau\Bigg).
\end{aligned}$$

Using Fubini Theorem[1] and Grönwall's lemma A.2 we obtain

$$\begin{aligned}
&\int_\Omega\sup_{\tau\in[0,T]}\mathrm{ess}\,|\nabla_s u^{(n)}-\nabla_s u^{(m)}|^2(x,\tau)\,\mathrm{d}x\le\int_0^T\int_\Omega|\nabla_s u_t^{(n)}-\nabla_s u_t^{(m)}|^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\\
&\le C_R\left(1+\int_\Omega\sup_{\tau\in[0,T]}\mathrm{ess}\,|p^{(n)}-p^{(m)}|^2(x,\tau)\,\mathrm{d}x\right).
\end{aligned}$$

The sequence $\{p^{(n)}\}$ is Cauchy in $L^2(\Omega;C[0,T])$, hence $\{\nabla_s u^{(n)}\}$ and $\{\nabla_s u_t^{(n)}\}$ are also Cauchy sequences in $L^2(\Omega;C([0,T];\mathbb{R}^{3\times3}_{\mathrm{sym}}))$ and in $L^2(\Omega\times(0,T);\mathbb{R}^{3\times3}_{\mathrm{sym}})$, respectively. Thus we conclude

$$\begin{aligned}
\nabla_s u^{(n)}&\to\nabla_s u &&\text{strongly in } L^2(\Omega;C([0,T];\mathbb{R}^{3\times3}_{\mathrm{sym}})),\\
\nabla_s u_t^{(n)}&\to\nabla_s u_t &&\text{strongly in } L^2(\Omega\times(0,T);\mathbb{R}^{3\times3}_{\mathrm{sym}}).
\end{aligned}$$

We are now ready to pass to the limit in the nonlinearities. By Theorem A.7, Hypothesis 5.1 (viii) and estimates (5.11), (5.12) we obtain

$$\begin{aligned}
&f_R(p^{(n)})\to f_R(p),\quad \Phi_R(p^{(n)})\to\Phi_R(p) &&\text{strongly in } L^2(\Omega\times(0,T)),\\
&\mathcal{C}_V(\theta^{(n)})\to\mathcal{C}_V(\theta) &&\text{strongly in } L^q(\Omega\times(0,T)) \text{ for all } q\in\left[1,\tfrac{2}{1+\hat{b}}\right],\\
&\mathbf{B}\nabla_s u_t^{(n)}:\nabla_s u_t^{(n)}\to\mathbf{B}\nabla_s u_t:\nabla_s u_t &&\text{strongly in } L^1(\Omega\times(0,T)),\\
&\left.\begin{aligned}
Q_R(|\nabla p^{(n)}|^2)&\to Q_R(|\nabla p|^2)\\
\mu_R(p^{(n)})&\to\mu_R(p)\\
Q_R((\theta^{(n)})^+)&\to Q_R(\theta^+)\\
\frac{1}{\gamma_R(p^{(n)},\theta^{(n)},\mathrm{div}u^{(n)})}&\to\frac{1}{\gamma_R(p,\theta,\mathrm{div}u)}
\end{aligned}\right\} &&\text{strongly in } L^q(\Omega\times(0,T)) \text{ for all } q\in[1,\infty),
\end{aligned}$$

---

[1]We use Fubini Theorem as follows

$$\int_0^\tau\left(\int_0^t f(s)\,\mathrm{d}s\right)\,\mathrm{d}s=\int_0^\tau\left(\int_s^\tau\mathrm{d}t\right)f(s)\,\mathrm{d}s=\int_0^\tau(\tau-s)f(s)\,\mathrm{d}s.$$

The above is also called *Cauchy formula for repeated integrals.* It will be frequently used in the whole thesis.

from which

$$
\begin{aligned}
f_R(p^{(n)})_t \to f_R(p)_t, \quad \mathcal{C}_V(\theta^{(n)})_t \to \mathcal{C}_V(\theta)_t \qquad & \text{weakly in } L^2(\Omega \times (0,T)),\\
\gamma_R(p^{(n)}, \theta^{(n)}, \mathrm{div} u^{(n)}) \;\to\; \gamma_R(p, \theta, \mathrm{div} u) \qquad & \text{strongly in } L^q(\Omega \times (0,T)) \text{ for all } q \in [1,\infty).
\end{aligned}
$$

Concerning the hysteresis terms, from Proposition B.1 (i) and Proposition B.12 we deduce

$$
\begin{aligned}
\mathcal{P}[\nabla_s u^{(n)}] &\to \mathcal{P}[\nabla_s u] && \text{strongly in } L^2(\Omega; C([0,T]; \mathbb{R}^{3\times 3}_{\mathrm{sym}})),\\
\mathcal{G}_0[p^{(n)}] &\to \mathcal{G}_0[p] && \text{strongly in } L^2(\Omega; C[0,T]),\\
\mathcal{G}_0[p^{(n)}]_t &\to \mathcal{G}_0[p]_t && \text{weakly in } L^2(\Omega \times (0,T)).
\end{aligned}
$$

Looking at the definition of $U_{\mathcal{P}}$ in (B.7) and at Remark B.13 for U_0, we see that

$$
U_{\mathcal{P}}[\nabla_s u^{(n)}] \to U_{\mathcal{P}}[\nabla_s u], \quad U_0[p^{(n)}] \to U_0[p] \qquad \text{strongly in } L^1(\Omega; C[0,T]).
$$

These imply

$$
U_{\mathcal{P}}[\nabla_s u^{(n)}]_t \to U_{\mathcal{P}}[\nabla_s u]_t, \quad U_0[p^{(n)}]_t \to U_0[p]_t \qquad \text{weakly in } L^2(\Omega \times (0,T)),
$$

which together with the energy identities (B.8), (B.25) gives

$$
\|D_{\mathcal{P}}[\nabla_s u^{(n)}]_t\|_* \to \|D_{\mathcal{P}}[\nabla_s u]_t\|_*\,, \quad |D_0[p^{(n)}]_t| \to |D_0[p]_t| \qquad \text{weakly in } L^2(\Omega \times (0,T)).
$$

We now prove that the sequences $\{\chi^{(n)}\}$, $\{\chi^{(n)}_t\}$ converge strongly in appropriate function spaces. Inequality (5.58) yields

$$
\int_\Omega \sup_{\tau\in[0,t]} |\chi^{(n)} - \chi^{(m)}|(x,\tau)\,\mathrm{d}x \le \int_0^\tau \int_\Omega |\chi^{(n)}_t - \chi^{(m)}_t|(x,t)\,\mathrm{d}x\,\mathrm{d}t \le 2\int_0^\tau \int_\Omega |F^{(n)}_t - F^{(m)}_t|(x,t)\,\mathrm{d}x\,\mathrm{d}t
$$

where, due to the above convergences, $F^{(n)}_t \to F_t$ strongly in $L^1(\Omega \times (0,T))$. Hence we conclude that $\{\chi^{(n)}(x,t)\}$ and $\{\chi^{(n)}_t(x,t)\}$ are Cauchy sequences in $L^1(\Omega; C[0,T])$ and in $L^1(\Omega\times(0,T))$, respectively. Moreover, since both $|\chi^{(n)}|$ and $|\chi^{(n)}_t|$ admit a uniform pointwise upper bound (see (5.40)), we can use the Lebesgue dominated convergence theorem to conclude that

$$
\begin{aligned}
\chi^{(n)} &\to \chi && \text{strongly in } L^q(\Omega; C[0,T]) \text{ for all } q \in [1,\infty),\\
\chi^{(n)}_t &\to \chi_t && \text{strongly in } L^q(\Omega \times (0,T)) \text{ for all } q \in [1,\infty).
\end{aligned}
$$

Passing to the limit as $n \to \infty$ in (5.23)–(5.27) we see that (p, u, θ, χ) is a solution to (5.18)–(5.21), (4.29) with the regularity stated in Proposition 5.4.

Limit as $\eta \to 0$

Let us denote by $(v^{(\eta)}, u^{(\eta)}, z^{(\eta)}, \chi^{(\eta)})$ the solution to (5.18)–(5.21). Note that estimate (5.49) is preserved in the limit as $n \to \infty$, hence the limit function $v^{(\eta)}$ satisfies the inequality

$$
\sup_{\tau\in[0,T]} \mathrm{ess} \int_\Omega |\Delta v^{(\eta)}|^2(x,t)\,\mathrm{d}x \le \frac{C_R}{\eta}\,. \tag{5.59}
$$

This allows us to integrate by parts in equation (5.18) taking into account the boundary condition (5.22), obtaining

$$\begin{aligned}&\int_\Omega \big((\chi^{(\eta)}+\rho^*(1-\chi^{(\eta)}))(f_R(M_R^{-1}(v^{(\eta)}))+\mathcal{G}_0[M_R^{-1}(v^{(\eta)})]+\mathrm{div}u^{(\eta)})\big)_t\,\phi\,\mathrm{d}x\\&+\int_\Omega \frac{1}{\rho_W}\Big(-\Delta v^{(\eta)}\,\phi+\eta\Delta v^{(\eta)}\Delta\phi\Big)\,\mathrm{d}x=\int_{\partial\Omega}\alpha(x)(p^*-M_R^{-1}(v^{(\eta)}))\,\phi\,\mathrm{d}s(x).\end{aligned}\tag{5.60}$$

Choosing $\phi\in W^{2,2}(\Omega)\cap X_0$ and introducing the new variable $\hat{v}^{(\eta)}=\frac{1}{\rho_W}\Delta v^{(\eta)}$, we rewrite (5.60) in the form

$$\int_\Omega \hat{v}^{(\eta)}\,(\phi-\eta\Delta\phi)\,\mathrm{d}x=\int_\Omega h^{(\eta)}\phi\,\mathrm{d}x\tag{5.61}$$

where

$$h^{(\eta)}=\big((\chi^{(\eta)}+\rho^*(1-\chi^{(\eta)}))(f_R(M_R^{-1}(v^{(\eta)}))+\mathcal{G}_0[M_R^{-1}(v^{(\eta)})]+\mathrm{div}u^{(\eta)})\big)_t\,.$$

Note that the term $\mathcal{G}_0[p^{(\eta)}]_t$ is of order $p_t^{(\eta)}$ by (B.21) and (B.24). Hence by estimates (5.40)–(5.42) and (5.50) we see that $h^{(\eta)}\in L^2(\Omega\times(0,T))$, and its L^2-norm is bounded independently of η.

Consider now the system $\{e_k : k\in\mathbb{N}\}$ of eigenfunctions of the negative Laplace operator with zero Dirichlet boundary conditions

$$-\Delta e_k=\lambda_k e_k\,,\quad e_k\big|_{\partial\Omega}=0\,,\quad \int_\Omega |e_k(x)|^2\,\mathrm{d}x=1.$$

They form a complete orthonormal system in $L^2(\Omega)$ with $0<\lambda_1\le\lambda_2\le\lambda_3\le\dots$. The functions $h^{(\eta)},\hat{v}^{(\eta)}$ admit the expansions

$$h^{(\eta)}(x,t)=\sum_{k=1}^\infty h_k^{(\eta)}(t)\,e_k(x)\,,\quad \hat{v}^{(\eta)}(x,t)=\sum_{k=1}^\infty \hat{v}_k^{(\eta)}(t)\,e_k(x)\,,$$

with coefficients $h_k^{(\eta)}:[0,T]\to\mathbb{R}$, $\hat{v}_k^{(\eta)}:[0,T]\to\mathbb{R}$. Choosing $\phi=e_k$ in (5.61) we obtain

$$\hat{v}_k^{(\eta)}(t)=\frac{h_k^{(\eta)}(t)}{1+\eta\lambda_k}\,,$$

hence

$$\begin{aligned}\int_0^T\int_\Omega |\hat{v}^{(\eta)}(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t&=\int_0^T\sum_{k=1}^\infty|\hat{v}_k^{(\eta)}(t)|^2\,\mathrm{d}t\\&\le\int_0^T\sum_{k=1}^\infty|h_k^{(\eta)}(t)|^2\,\mathrm{d}t=\int_0^T\int_\Omega|h^{(\eta)}(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t\le C_R\end{aligned}\tag{5.62}$$

for some positive constant C_R independent of η. Thus we get, as $\eta\to0$,

$$\begin{aligned}\nabla v^{(\eta)}&\;\to\;\nabla v\qquad \text{strongly in } L^2(\Omega\times(0,T);\mathbb{R}^3),\\ \eta\,\Delta v^{(\eta)}&\;\to\;0\qquad\ \ \text{strongly in } L^2(\Omega\times(0,T)),\end{aligned}$$

from which, by definition of M_R in (5.17) and Hypothesis 5.1 (vii),

$$\begin{aligned}\nabla M_R^{-1}(v^{(\eta)})&\;\to\;\nabla M_R^{-1}(v)\qquad\text{strongly in } L^2(\Omega\times(0,T);\mathbb{R}^3),\\ Q_R(|\nabla M_R^{-1}(v^{(\eta)})|^2)&\;\to\;Q_R(|\nabla M_R^{-1}(v)|^2)\qquad\text{strongly in } L^q(\Omega\times(0,T)) \text{ for all } q\in[1,\infty).\end{aligned}$$

Estimates (5.41), (5.42), (5.48), (5.50), (5.56) and (5.57) are preserved when $n \to \infty$. Since they are independent of η, by letting $\eta \to 0$ we obtain the same convergences as before. This completes the proof of Proposition 5.4.

As a side product, from (5.62) we get that the estimate

$$\int_0^T \int_\Omega |\Delta M_R(p)|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \le C_R$$

holds also in the limit as $\eta \to 0$. Hence, arguing as for (5.44) we get

$$\|M_R(p)\|^2_{L^2(0,T;W^{2,2}(\Omega))} \le C_R.$$

This, together with the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, yields

$$\int_0^T \left(\int_\Omega |\nabla p|^6(x,t)\,\mathrm{d}x\right)^{1/3} \mathrm{d}t \le C_R. \tag{5.63}$$

5.2 Estimates independent of the cut-off parameter

We now come back to our cut-off system (5.13)–(5.16). We are going to derive a series of estimates independent of R. More precisely, after proving that the temperature stays away from zero, we will perform the energy estimate and the Dafermos estimate in order to gain some regularity for the temperature. Subsequently, a key-step will be the derivation of a bound for p in an anisotropic Lebesgue space. Then an analogous estimate based on the particular structure of equation (5.14) is obtained for $\nabla_s u_t$. We finally show that this is sufficient for starting the Moser iteration and obtain an L^∞ bound for p. After deriving some higher order estimates for the capillary pressure and for the temperature, we will be ready to let R tend to ∞ in (5.13)–(5.16).

5.2.1 Positivity of the temperature

For every nonnegative test function $\zeta \in X$ we have, by virtue of (5.15),

$$\begin{aligned}
&\int_\Omega \big(\mathcal{C}_V(\theta)_t\zeta + \kappa(Q_R(\theta^+))\nabla\theta\cdot\nabla\zeta\big)\,\mathrm{d}x + \int_{\partial\Omega} \omega(x)(\theta-\theta^*)\zeta\,\mathrm{d}s(x)\\
&= \int_\Omega \Bigg(\mathbf{B}\nabla_s u_t : \nabla_s u_t + \|D_{\mathcal{P}}[\nabla_s u]_t\|_* + \frac{1}{\rho_W}\mu_R(p)Q_R(|\nabla p|^2)\\
&\quad + (\chi + \rho^*(1-\chi))|D_0[p]_t| + \gamma_R(p,\theta,\mathrm{div}u)\chi_t^2 - \left(\frac{L}{\theta_c}\chi_t + \beta\mathrm{div}u_t\right)Q_R(\theta^+)\Bigg)\zeta\,\mathrm{d}x\\
&\ge \int_\Omega \left(\frac{B^\flat}{3}|\mathrm{div}u_t|^2 + \gamma^\flat\chi_t^2 - \left(\frac{L}{\theta_c}\chi_t + \beta\mathrm{div}u_t\right)Q_R(\theta^+)\right)\zeta\,\mathrm{d}x,
\end{aligned}$$

where in the last line we used Hypothesis 5.1 (i) together with inequality (5.5), and also estimate (5.12). Note that, by Young's inequality,

$$\frac{B^\flat}{3}|\mathrm{div}u_t|^2 - \beta\mathrm{div}u_t\, Q_R(\theta^+) = \frac{B^\flat}{3}|\mathrm{div}u_t|^2 - \sqrt{\frac{B^\flat}{2}}\mathrm{div}u_t\,\frac{\sqrt{2}\beta}{\sqrt{B^\flat}}Q_R(\theta^+) \geq \frac{B^\flat}{12}|\mathrm{div}u_t|^2 - C(Q_R(\theta^+))^2,$$

$$\gamma^\flat\chi_t^2 - \frac{L}{\theta_c}\chi_t\, Q_R(\theta^+) = \gamma^\flat\chi_t^2 - \sqrt{\gamma^\flat}\chi_t\,\frac{L}{\theta_c\sqrt{\gamma^\flat}}Q_R(\theta^+) \geq \frac{1}{2}\gamma^\flat\chi_t^2 - C(Q_R(\theta^+))^2.$$

Hence we get

$$\int_\Omega \big(\mathcal{C}_V(\theta)_t\zeta + \kappa(Q_R(\theta^+))\nabla\theta\cdot\nabla\zeta\big)\,\mathrm{d}x + \int_{\partial\Omega}\omega(x)(\theta-\theta^*)\zeta\,\mathrm{d}s(x) \geq -C\int_\Omega (Q_R(\theta^+))^2\zeta\,\mathrm{d}x$$

with a constant C depending on L, θ_c, β, $\mathbf{B}$, $\gamma^\flat$. Let now $\varphi(t)$ be the solution of the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}_V(\varphi(t)) + C\varphi^2(t) = 0, \quad \varphi(0) = \bar\theta$$

with $\bar\theta$ from Hypothesis 5.1. Then φ is

- nonincreasing: this immediately follows from the fact that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}_V(\varphi(t)) \leq 0$ for all $t \geq 0$;
- positive: since $\varphi(0) > 0$ and φ is decreasing, we need to prove that there is no $t > 0$ such that $\varphi(t) = 0$. Let us assume by contradiction that at a certain time $t^* > 0$ it happens that $\varphi(t^*) = 0$. Now, defining $y := \mathcal{C}_V(\varphi)$, we have that y is a solution to the following Cauchy problem

$$\dot y(t) + C\left((\mathcal{C}_V)^{-1}(y(t))\right)^2 = 0, \quad y(0) = \mathcal{C}_V(\bar\theta).$$

For equations of this form, it is possible to deduce a formula for computing the vanishing time t^*. Let us consider $t \in [0, t^*)$, so that $(\mathcal{C}_V)^{-1}(y(t)) > 0$. Rewriting

$$\frac{\dot y(t)}{C\left((\mathcal{C}_V)^{-1}(y(t))\right)^2} = -1,$$

integrating $\int_0^{t^*}\,\mathrm{d}t$ and operating the change of variable $y = y(t)$ we obtain

$$\int_{y^0}^{y^*}\frac{1}{C\left((\mathcal{C}_V)^{-1}(y)\right)^2}\,\mathrm{d}y = t^*,$$

where $y^0 = y(0)$ and $y^* = y(t^*)$. Coming back to the original variable $\varphi = (\mathcal{C}_V)^{-1}(y)$, the above integral can be rewritten as

$$t^* = \int_0^{\bar\theta}\frac{\mathcal{C}_V'(\varphi)}{C\varphi^2}\,\mathrm{d}\varphi = \int_0^{\bar\theta}\frac{c_V(\varphi)}{C\varphi^2}\,\mathrm{d}w \geq \frac{c^\flat}{C}\int_0^{\bar\theta}\frac{1}{\varphi^{3/2}}\,\mathrm{d}\varphi = \infty,$$

where we used Hypothesis 5.1 (viii). Thus we have proved that such a time t^* does not exist.

Taking into account the fact that $\mathcal{C}_V(\varphi)_t = -C\varphi^2$ and $\nabla\varphi = 0$, for every nonnegative test function $\zeta \in X$ we have in particular

$$\int_\Omega \big(\left(\mathcal{C}_V(\varphi) - \mathcal{C}_V(\theta)\right)_t\zeta + \kappa(Q_R(\theta^+))\nabla(\varphi-\theta)\cdot\nabla\zeta\big)\,\mathrm{d}x + \int_{\partial\Omega}\omega(x)(\theta-\theta^*)\zeta\,\mathrm{d}s(x)$$
$$\leq C\int_\Omega\left((Q_R(\theta^+))^2 - \varphi^2\right)\zeta\,\mathrm{d}x.$$

Consider now the following regularization of the Heaviside function

$$H_\lambda(z) = \begin{cases} 0 & \text{for } z \le 0, \\ \dfrac{z}{\lambda} & \text{for } 0 < z \le \lambda, \\ 1 & \text{for } z > \lambda, \end{cases}$$

and set $\zeta(x,t) = \varphi(t) - H_\lambda(\theta(x,t))$, which is an admissible test function. For all values of x and t it holds that

- $\diamond$ $0 \le \nabla(\varphi - \theta) \cdot \nabla H_\lambda(\varphi - \theta) = \begin{cases} 0 & \text{for } z < 0 \text{ and for } z > \lambda, \\ \dfrac{|\nabla(\varphi - \theta)|^2}{\lambda} & \text{for } 0 < z < \lambda; \end{cases}$
- $\diamond$ $(\theta^* - \theta)\, H_\lambda(\varphi - \theta) \ge 0$ since
  - if $\theta \le \theta^*$ then it is trivial,
  - if $\theta > \theta^*$ then by Hypothesis 5.1 (iv) $\theta > \bar{\theta}$, and from the monotonicity of φ we get $\theta > \varphi$, which implies $H_\lambda(\varphi - \theta) = 0$;
- $\diamond$ $\left((Q_R(\theta^+))^2 - \varphi^2\right) H_\lambda(\varphi - \theta) = (Q_R(\theta^+) - \varphi)(Q_R(\theta^+) + \varphi) H_\lambda(\varphi - \theta) \le 0$ since
  - if $(Q_R(\theta^+) - \varphi)(Q_R(\theta^+) + \varphi) \le 0$ then it is trivial,
  - if $(Q_R(\theta^+) - \varphi)(Q_R(\theta^+) + \varphi) > 0$, that is, $0 < \varphi < Q_R(\theta^+)$, then $\varphi < \theta$ which implies $H_\lambda(\varphi - \theta) = 0$.

This yields

$$\int_\Omega \left(\mathcal{C}_V(\varphi) - \mathcal{C}_V(\theta)\right)_t H_\lambda(\varphi - \theta)\,\mathrm{d}x \le 0.$$

By the Lebesgue Dominated Convergence Theorem we can pass to the limit in the above inequality for $\lambda \to 0$, getting

$$\int_\Omega \left(\mathcal{C}_V(\varphi) - \mathcal{C}_V(\theta)\right)_t H(\varphi - \theta)\,\mathrm{d}x \le 0,$$

that is, by the monotonicity of $\mathcal{C}_V$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega (\mathcal{C}_V(\varphi) - \mathcal{C}_V(\theta))^+ \,\mathrm{d}x \le 0, \qquad (\mathcal{C}_V(\varphi) - \mathcal{C}_V(\theta))^+(x{,}0) = 0$$

which implies $(\mathcal{C}_V(\varphi) - \mathcal{C}_V(\theta))^+ \equiv 0$. Owing again to the monotonicity of $\mathcal{C}_V$ and φ, we conclude that, independently of R,

$$\theta(x,t) \ge \varphi(t) \ge \varphi(T) =: \theta_T > 0 \quad \text{for all } x \text{ and } t. \tag{5.64}$$

We now pass to a series of estimates independent of R.

5.2.2 Energy estimate

Since we proved that the temperature stays positive, from now on we will write $Q_R(\theta^+) = Q_R(\theta)$. We test (5.13) by $\phi = p$, (5.14) by $\psi = u_t$ and (5.15) by $\zeta = 1$. Summing up the three resulting equations we obtain

$$\begin{aligned}
&\int_\Omega \big((\chi + \rho^*(1-\chi))(f_R(p) + \mathcal{G}_0[p] + \mathrm{div}u)\big)_t\, p \,\mathrm{d}x + \int_\Omega \frac{1}{\rho_W}\mu_R(p)|\nabla p|^2 \,\mathrm{d}x \\
&+ \int_\Omega (\mathcal{P}[\nabla_s u] + \mathbf{B}\nabla_s u_t) : \nabla_s u_t \,\mathrm{d}x - \int_\Omega \Big(p\,(\chi + \rho^*(1-\chi)) + \beta(Q_R(\theta) - \theta_c)\Big)\,\mathrm{div}u_t \,\mathrm{d}x \\
&+ \int_\Omega \Big(\mathcal{C}_V(\theta)_t - \mathbf{B}\nabla_s u_t : \nabla_s u_t - \|D_{\mathcal{P}}[\nabla_s u]_t\|_* - \frac{1}{\rho_W}\mu_R(p) Q_R(|\nabla p|^2) \\
&\quad - (\chi + \rho^*(1-\chi))|D_0[p]_t| - \gamma_R(p,\theta,\mathrm{div}u)\chi_t^2\Big)\,\mathrm{d}x + \int_\Omega \Big(\frac{L}{\theta_c}\chi_t + \beta\,\mathrm{div}u_t\Big) Q_R(\theta)\,\mathrm{d}x \\
&= \int_{\partial\Omega} \alpha(x)(p^* - p)p \,\mathrm{d}s(x) + \int_\Omega g\cdot u_t \,\mathrm{d}x + \int_{\partial\Omega} \omega(x)(\theta^* - \theta)\,\mathrm{d}s(x).
\end{aligned}$$

Note that some of the terms cancel out. Moreover, recalling the notation introduced in (5.8) and the energy balance (B.25), the identities

$$\int_\Omega \big((\chi + \rho^*(1-\chi))f_R(p)\big)_t\, p\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\chi + \rho^*(1-\chi))V_R(p)\,\mathrm{d}x + \int_\Omega (1-\rho^*)\chi_t \Phi_R(p)\,\mathrm{d}x, \tag{5.65}$$

$$\begin{aligned}
&\int_\Omega \big((\chi + \rho^*(1-\chi))\mathcal{G}_0[p]\big)_t\, p\,\mathrm{d}x - \int_\Omega (\chi + \rho^*(1-\chi))|D_0[p]_t|\,\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\chi + \rho^*(1-\chi))U_0(p)\,\mathrm{d}x + \int_\Omega (1-\rho^*)\chi_t\,(p\,\mathcal{G}_0[p] - U_0[p])\,\mathrm{d}x
\end{aligned} \tag{5.66}$$

hold true. Hence we obtain, using also (B.8),

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big(\mathcal{C}_V(\theta) + \beta\theta_c\,\mathrm{div}u + (\chi + \rho^*(1-\chi))(V_R(p) + U_0[p]) + U_{\mathcal{P}}[\nabla_s u]\Big)\,\mathrm{d}x \\
&+ \int_\Omega \Big(\frac{1}{\rho_W}\mu_R(p)\Big(|\nabla p|^2 - Q_R(|\nabla p|^2)\Big) + \frac{L}{\theta_c}\,Q_R(\theta)\chi_t\Big)\,\mathrm{d}x \\
&- \int_\Omega \chi_t\,\Big(\gamma_R(p,\theta,\mathrm{div}u)\chi_t - (1-\rho^*)(\Phi_R(p) + p\mathcal{G}_0[p] - U_0[p] + p\,\mathrm{div}u)\Big)\,\mathrm{d}x \\
&= \int_\Omega g\cdot u_t\,\mathrm{d}x + \int_{\partial\Omega}\Big(\alpha(x)(p^* - p)p + \omega(x)(\theta^* - \theta)\Big)\,\mathrm{d}s(x),
\end{aligned}$$

and by (5.16) (see Remark B.2)

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big(\mathcal{C}_V(\theta) + L\chi + \beta\theta_c\,\mathrm{div}u + (\chi + \rho^*(1-\chi))(V_R(p) + U_0[p]) + U_{\mathcal{P}}[\nabla_s u]\Big)\,\mathrm{d}x \\
&+ \frac{1}{\rho_W}\int_\Omega \mu_R(p)\Big(|\nabla p|^2 - Q_R(|\nabla p|^2)\Big)\,\mathrm{d}x + \int_{\partial\Omega}\Big(\alpha(x)(p - p^*)p + \omega(x)(\theta - \theta^*)\Big)\,\mathrm{d}s(x) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega g\cdot u\,\mathrm{d}x - \int_\Omega g_t\cdot u\,\mathrm{d}x.
\end{aligned} \tag{5.67}$$

We now integrate in time $\int_0^\tau \mathrm{d}t$. On the left-hand side Young's inequality, (B.9) and (5.5) entail

$$\begin{aligned}
U_{\mathcal{P}}[\nabla_s u] + \beta\theta_c\,\mathrm{div}u &= U_{\mathcal{P}}[\nabla_s u] + \frac{2\beta\theta_c}{\sqrt{A^\flat}}\,\frac{\sqrt{A^\flat}}{2}\mathrm{div}u \\
&\geq \frac{A^\flat}{2}|\nabla_s u|^2 - \frac{2\beta^2\theta_c^2}{A^\flat} - \frac{A^\flat}{8}|\mathrm{div}u|^2 \geq \frac{A^\flat}{8}|\nabla_s u|^2 - C.
\end{aligned} \tag{5.68}$$

By the definition of Q_R in (5.6), it holds $|\nabla p|^2 \geq Q_R(|\nabla p|^2)$. The boundary term is such that

$$\begin{aligned}
&\int_0^\tau \int_{\partial\Omega} \Big(\alpha(x)(p-p^*)p + \omega(x)(\theta-\theta^*)\Big) \,\mathrm{d}s(x)\,\mathrm{d}t \\
&\geq \int_0^\tau \int_{\partial\Omega} \left(\alpha(x)\left(p^2 - \frac{|p^*|^2}{2} - \frac{p^2}{2}\right) + \omega(x)(\theta-\theta^*)\right) \,\mathrm{d}s(x)\,\mathrm{d}t \\
&\geq \int_0^\tau \int_{\partial\Omega} \left(\alpha(x)\frac{p^2}{2} + \omega(x)\theta\right) \,\mathrm{d}s(x)\,\mathrm{d}t - C
\end{aligned} \tag{5.69}$$

thanks to Hypothesis 5.1 (iii) and (iv). Concerning the right-hand side of (5.67), the time integration gives

$$\int_\Omega (g\cdot u)(x,\tau)\,\mathrm{d}x - \int_\Omega g(x,0)\cdot u^0(x)\,\mathrm{d}x - \int_0^\tau\int_\Omega (g_t\cdot u)(x,t)\,\mathrm{d}x\,\mathrm{d}t,$$

where the term containing the initial conditions is controlled by using Hölder's inequality and observing that

$$\int_\Omega |g|^2(x,0)\,\mathrm{d}x = \int_\Omega |g|^2(x,\tau)\,\mathrm{d}x - 2\int_0^\tau\int_\Omega (g\cdot g_t)(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Hence by Young's inequality and Hypothesis 5.1 (ii), (v) we deduce

$$\begin{aligned}
&\int_\Omega (g\cdot u)(x,\tau)\,\mathrm{d}x - \int_\Omega g(x,0)\cdot u^0(x)\,\mathrm{d}x - \int_0^\tau\int_\Omega (g_t\cdot u)(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\leq \frac{2\sqrt{2}}{\sqrt{A^\flat c}}\left(\int_\Omega |g|^2(x,\tau)\,\mathrm{d}x\right)^{1/2} \frac{\sqrt{A^\flat c}}{2\sqrt{2}}\left(\int_\Omega |u|^2(x,\tau)\,\mathrm{d}x\right)^{1/2} + C\left(1 + \left(\int_0^\tau\int_\Omega |u|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\right) \\
&\leq \frac{A^\flat}{16}\int_\Omega |\nabla_s u|^2(x,\tau)\,\mathrm{d}x + C\left(1 + \int_0^\tau\int_\Omega |\nabla_s u|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\right),
\end{aligned}$$

where in the last line we used Korn's inequality (A.11). The first term in the last line is absorbed by (5.68). Finally, the initial conditions are kept under control thanks to (5.11), (B.7), (B.31) and Hypothesis 5.1 (v). Hence what we eventually get is

$$\begin{aligned}
&\int_\Omega \left(\mathcal{C}_V(\theta) + V_R(p) + |\nabla_s u|^2\right)(x,\tau)\,\mathrm{d}x + \int_0^\tau\int_{\partial\Omega}\Big(\alpha(x)p^2 + \omega(x)\theta\Big)(x,t)\,\mathrm{d}s(x)\,\mathrm{d}t \\
&\leq C\left(1 + \int_0^\tau\int_\Omega |\nabla_s u|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\right),
\end{aligned}$$

and applying Grönwall's lemma A.2 we finally obtain the estimates

$$\sup_{\tau\in[0,T]}\operatorname{ess}\int_\Omega \left(\mathcal{C}_V(\theta) + V_R(p) + |\nabla_s u|^2\right)(x,\tau)\,\mathrm{d}x \leq C, \tag{5.70}$$

$$\int_0^T\int_{\partial\Omega}\Big(\alpha(x)p^2 + \omega(x)\,\theta\Big)(x,t)\,\mathrm{d}s(x)\,\mathrm{d}t \leq C. \tag{5.71}$$

Estimate (5.70) also gives

$$\sup_{\tau\in[0,T]}\operatorname{ess}\int_\Omega |\theta|^{1+b}(x,\tau)\,\mathrm{d}x \leq C, \tag{5.72}$$

where b is from Hypothesis 5.1 (viii).

5.2.3 Dafermos estimate

We set $\hat{\theta} := Q_R(\theta)$ and test (5.15) by $\zeta = -\hat{\theta}^{-a}$, with a from Hypothesis 5.1. This yields the identity

$$\begin{aligned}
&\int_\Omega \Big(\mathcal{C}_V(\theta)_t - \mathbf{B}\nabla_s u_t : \nabla_s u_t - \|D_{\mathcal{P}}[\nabla_s u]_t\|_* - \frac{1}{\rho_W}\mu_R(p) Q_R(|\nabla p|^2) - (\chi + \rho^*(1-\chi))|D_0[p]_t| \\
&- \gamma_R(p,\theta,\mathrm{div} u)\chi_t^2 + \left(\frac{L}{\theta_c}\chi_t + \beta \mathrm{div} u_t\right)\hat{\theta}\Big)(-\hat{\theta}^{-a})\,\mathrm{d}x + \int_\Omega \kappa(\hat{\theta})\nabla\theta\cdot\nabla(-\hat{\theta}^{-a})\,\mathrm{d}x \\
&= \int_{\partial\Omega} \omega(x)(\theta^* - \theta)\,(-\hat{\theta}^{-a})\,\mathrm{d}s(x).
\end{aligned} \tag{5.73}$$

It holds

$$\int_\Omega \mathcal{C}_V(\theta)_t\,(-\hat{\theta}^{-a})\,\mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega F_a(\theta)\,\mathrm{d}x$$

where

$$F_a(\theta) := \int_0^\theta \frac{c_V(s)}{(Q_R(s))^a}\,\mathrm{d}s,$$

and by Hypothesis 5.1 (ix) also

$$\int_\Omega \kappa(\hat{\theta})\nabla\theta\cdot\nabla(-\hat{\theta}^{-a})\,\mathrm{d}x = \int_\Omega \kappa(\hat{\theta})\,a\,\hat{\theta}^{-a-1}\nabla\theta\cdot\nabla\hat{\theta}\,\mathrm{d}x \ge a\kappa^\flat \int_\Omega |\nabla\hat{\theta}|^2\,\mathrm{d}x.$$

Hence from (5.73) we get, using also Hypothesis 5.1 (i) and inequalities (5.5), (5.12),

$$\begin{aligned}
&\int_\Omega \left(\frac{B^\flat}{3}|\mathrm{div} u_t|^2 + \gamma^\flat\chi_t^2\right)\hat{\theta}^{-a}\,\mathrm{d}x + a\kappa^\flat\int_\Omega |\nabla\hat{\theta}|^2\,\mathrm{d}x \\
&\le \int_\Omega \left(\frac{L}{\theta_c}\chi_t + \beta\mathrm{div} u_t\right)\hat{\theta}^{1-a}\,\mathrm{d}x + \int_{\partial\Omega}\omega(x)(\theta-\theta^*)\,\hat{\theta}^{-a}\,\mathrm{d}s(x) + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega F_a(\theta)\,\mathrm{d}x,
\end{aligned} \tag{5.74}$$

where we neglected some positive terms on the left-hand side. Young's inequality yields

$$\begin{aligned}
\left(\frac{L}{\theta_c}\chi_t + \beta\mathrm{div} u_t\right)\hat{\theta}^{1-a} &= \sqrt{\gamma^\flat}\chi_t\,\hat{\theta}^{-a/2}\,\frac{L}{\theta_c\sqrt{\gamma^\flat}}\hat{\theta}^{1-a/2} + \sqrt{\frac{B^\flat}{2}}\mathrm{div} u_t\,\hat{\theta}^{-a/2}\,\frac{\sqrt{2}\beta}{\sqrt{B^\flat}}\hat{\theta}^{1-a/2} \\
&\le \left(\frac{\gamma^\flat}{2}|\chi_t|^2 + \frac{B^\flat}{4}|\mathrm{div} u_t|^2\right)\hat{\theta}^{-a} + C\hat{\theta}^{2-a},
\end{aligned}$$

with a constant C depending only on L, θ_c, β, $\mathbf{B}$, $\gamma^\flat$, whereas the boundary term is such that

$$\begin{aligned}
\int_{\partial\Omega}\omega(x)(\theta-\theta^*)\,\hat{\theta}^{-a}\,\mathrm{d}s(x) &\le \frac{1}{\theta_T^a}\left(\int_{\partial\Omega}\omega(x)\theta\,\mathrm{d}s(x) + \int_{\partial\Omega}\omega(x)\theta^*\,\mathrm{d}s(x)\right) \\
&\le C\left(1 + \int_{\partial\Omega}\omega(x)\theta\,\mathrm{d}s(x)\right)
\end{aligned}$$

by (5.64) and Hypothesis 5.1 (iii), (iv). Note also that $F_a(\theta) \le \mathcal{C}_V(\theta)$ for all $\theta \ge 0$. Thus integrating (5.74) in time $\int_0^\tau\,\mathrm{d}t$ for some $\tau \in [0,T]$ and neglecting some other positive terms on the left-hand side we obtain

$$\begin{aligned}
&\int_0^\tau\int_\Omega |\nabla\hat{\theta}|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\le C\left(1 + \int_0^\tau\int_\Omega \hat{\theta}^{2-a}(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_{\partial\Omega}\omega(x)\theta(x,t)\,\mathrm{d}s(x)\,\mathrm{d}t + \int_\Omega \mathcal{C}_V(\theta)(x,\tau)\,\mathrm{d}x\right),
\end{aligned}$$

and from estimates (5.70), (5.71)

$$\int_0^\tau \int_\Omega |\nabla\hat{\theta}|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \le C\left(1+\int_0^\tau \int_\Omega \hat{\theta}^{2-a}(x,t)\,\mathrm{d}x\,\mathrm{d}t\right). \tag{5.75}$$

Now, owing to estimate (5.72), we can apply the Gagliardo-Nirenberg inequality (A.8) with the choices $s=1+b$, $p=2$ and $N=3$ obtaining, for $t\in(0,T)$,

$$|\hat{\theta}(t)|_q \le C\left(|\hat{\theta}(t)|_{1+b}+|\hat{\theta}(t)|_{1+b}^{1-\varrho}|\nabla\hat{\theta}(t)|_2^{\varrho}\right) \le C\left(1+|\nabla\hat{\theta}(t)|_2^{\varrho}\right)$$

with $\varrho=\frac{6(q-1-b)}{(5-b)q}$ and for every $1+b<q<6$. In particular, since $\varrho\cdot\frac{(5-b)q}{3(q-1-b)}=2$, this and (5.75) yield

$$\int_0^T |\hat{\theta}(t)|_q^{(5-b)q/3(q-1-b)}\,\mathrm{d}t \le C\left(1+\int_0^T |\nabla\hat{\theta}(t)|_2^2\,\mathrm{d}t\right) \le C\left(1+\int_0^T |\hat{\theta}(t)|_{2-a}^{2-a}\,\mathrm{d}t\right). \tag{5.76}$$

Let us now choose $q=2-a$, which is admissible in the sense that $1+b<2-a$ thanks to Hypothesis 5.1. Since $\frac{5-b}{3(1-a-b)}>1$, we can apply Young's inequality on the right-hand side getting

$$\begin{aligned}\int_0^T \left(|\hat{\theta}(t)|_{2-a}^{2-a}\right)^{(5-b)/3(1-a-b)}\,\mathrm{d}t &\le C\left(1+\int_0^T |\hat{\theta}(t)|_{2-a}^{2-a}\,\mathrm{d}t\right)\\ &\le C+\frac{3(1-a-b)}{5-b}\int_0^T \left(|\hat{\theta}(t)|_{2-a}^{2-a}\right)^{(5-b)/3(1-a-b)}\,\mathrm{d}t,\end{aligned}$$

from which

$$\int_0^T |\hat{\theta}(t)|_{2-a}^{2-a}\,\mathrm{d}t \le C.$$

Substituting in (5.76) entails

$$\int_0^T \int_\Omega |\nabla\hat{\theta}|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \le C. \tag{5.77}$$

Coming back to (5.76) again and choosing $q=8/3+2b/3$, we also get

$$\int_0^T \int_\Omega \hat{\theta}^{8/3+2b/3}(x,t)\,\mathrm{d}x\,\mathrm{d}t \le C. \tag{5.78}$$

5.2.4 Mechanical energy estimate

In order to estimate the capillary pressure in a suitable anisotropic Lebesgue space, we first need to find a bound for $\mathrm{div} u_t$ in $L^2(\Omega\times(0,T))$, independently of R. To this purpose, we test (5.13) by $\phi=p$, (5.14) by $\psi=u_t$ and sum up to obtain, with the notation of the previous subsection,

$$\begin{aligned}&\int_\Omega \big((\chi+\rho^*(1-\chi))(f_R(p)+\mathcal{G}_0[p]+\mathrm{div}u)\big)_t\,p\,\mathrm{d}x+\int_\Omega \frac{1}{\rho_W}\mu_R(p)|\nabla p|^2\,\mathrm{d}x\\ &+\int_\Omega (\mathcal{P}[\nabla_s u]+\mathbf{B}\nabla_s u_t):\nabla_s u_t\,\mathrm{d}x-\int_\Omega \big(p(\chi+\rho^*(1-\chi))+\beta(\hat{\theta}-\theta_c)\big)\,\mathrm{div}u_t\,\mathrm{d}x\\ &=\int_{\partial\Omega}\alpha(x)(p^*-p)\,p\,\mathrm{d}s(x)+\int_\Omega g\cdot u_t\,\mathrm{d}x.\end{aligned}$$

Note that some terms cancel out. Owing to (5.65) and (5.66) and exploiting also the energy identity (B.8), what we eventually get is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big((\chi+\rho^*(1-\chi))\,\left(V_R(p)+U_0[p]\right)+U_{\mathcal{P}}[\nabla_s u]\Big)\,\mathrm{d}x+\int_\Omega \frac{1}{\rho_W}\mu_R(p)|\nabla p|^2\,\mathrm{d}x+\int_\Omega \mathbf{B}\nabla_s u_t:\nabla_s u_t\,\mathrm{d}x$$
$$+\int_\Omega (1-\rho^*)\chi_t\left(\Phi_R(p)+p\,\mathcal{G}_0[p]+p\,\mathrm{div}u-U_0[p]\right)\mathrm{d}x+\int_{\partial\Omega}\alpha(x)(p-p^*)\,p\,\mathrm{d}s(x)$$
$$\le \int_\Omega \beta(\hat\theta-\theta_c)\,\mathrm{div}u_t\,\mathrm{d}x+\int_\Omega g\cdot u_t\,\mathrm{d}x.$$

Now, (5.16) yields (see Remark B.2)

$$(1-\rho^*)\chi_t\left(\Phi_R(p)+p\,\mathcal{G}_0[p]+p\,\mathrm{div}u-U_0[p]\right)=\gamma_R(p,\theta,\mathrm{div}u)\chi_t^2-L\chi_t\left(\frac{\hat\theta}{\theta_c}-1\right)$$
$$=\gamma_R(p,\theta,\mathrm{div}u)\chi_t^2-\sqrt{\gamma_R(p,\theta,\mathrm{div}u)}\chi_t\,\frac{L}{\sqrt{\gamma_R(p,\theta,\mathrm{div}u)}}\left(\frac{\hat\theta}{\theta_c}-1\right)$$
$$\ge \frac{1}{2}\gamma_R(p,\theta,\mathrm{div}u)\chi_t^2-C(1+\hat\theta) \tag{5.79}$$

where in the last line we used Young's inequality and (5.12), and where the constant C is independent of R. Moreover, from the pointwise inequality (5.5) and arguing as for (5.38) we get

$$\int_\Omega \beta(\hat\theta-\theta_c)\,\mathrm{div}u_t\,\mathrm{d}x=\int_\Omega \frac{\beta\sqrt{6}}{\sqrt{B^\flat}}(\hat\theta-\theta_c)\sqrt{\frac{B^\flat}{6}}\mathrm{div}u_t\,\mathrm{d}x\le C\left(1+\int_\Omega \hat\theta^2\,\mathrm{d}x\right)+\frac{B^\flat}{4}\int_\Omega |\nabla_s u_t|^2\,\mathrm{d}x,$$
$$\int_\Omega g\cdot u_t\,\mathrm{d}x\le C+\frac{B^\flat}{4}\int_\Omega |\nabla_s u_t|^2\,\mathrm{d}x.$$

Hence we obtain, exploiting also Hypothesis 5.1 (i) to absorb the terms coming from the two estimates above,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big((\chi+\rho^*(1-\chi))\,\left(V_R(p)+U_0[p]\right)+U_{\mathcal{P}}[\nabla_s u]\Big)\,\mathrm{d}x+\frac{\mu^\flat}{\rho_W}\int_\Omega |\nabla p|^2\,\mathrm{d}x+\frac{B^\flat}{2}\int_\Omega |\nabla_s u_t|^2\,\mathrm{d}x$$
$$+\frac{1}{2}\int_\Omega \gamma_R(p,\theta,\mathrm{div}u)\chi_t^2\,\mathrm{d}x+\frac{1}{2}\int_{\partial\Omega}\alpha(x)p^2\,\mathrm{d}s(x)\le C\left(1+\int_\Omega \hat\theta^2\,\mathrm{d}x\right)$$

where the boundary term was handled as in (5.69). We now integrate in time $\int_0^\tau \mathrm{d}t$ for some $\tau\in[0,T]$. The right-hand side is bounded thanks to estimate (5.78), whereas the initial conditions are kept under control thanks to (5.11), (B.2), (B.7), (B.31) and Hypothesis 5.1 (v). Hence, neglecting some already estimated positive terms, we finally obtain

$$\int_0^T\int_\Omega \left(|\nabla p|^2+|\nabla_s u_t|^2\right)(x,t)\,\mathrm{d}x\,\mathrm{d}t\le C \tag{5.80}$$

independently of R. This, together with (5.71) and Poincaré's inequality (A.10), yields

$$\|p\|^2_{L^2(0,T;W^{1,2}(\Omega))}\,\mathrm{d}t\le C. \tag{5.81}$$

5.2.5 Estimate for the capillary pressure

We choose an even function $b : \mathbb{R} \to (0, \infty)$ such that $b'(p) \geq 0$ for $p > 0$ and $pb(p) \in X$. Then we test (5.13) by $\phi = pb(p)$. We obtain

$$\begin{aligned}&\int_\Omega \big((\chi + \rho^*(1-\chi))(f_R(p) + \mathcal{G}_0[p] + \mathrm{div} u)\big)_t\, pb(p)\,\mathrm{d}x + \int_\Omega \frac{1}{\rho_W}\mu_R(p)(b(p) + pb'(p))|\nabla p|^2\,\mathrm{d}x \\ &= \int_{\partial\Omega} \alpha(x)(p^* - p)\, pb(p)\,\mathrm{d}s(x).\end{aligned} \tag{5.82}$$

The term under the time derivative has the form

$$\begin{aligned}&\int_\Omega \big((\chi + \rho^*(1-\chi))(f_R(p) + \mathcal{G}_0[p] + \mathrm{div} u)\big)_t\, pb(p)\,\mathrm{d}x \\ &= \int_\Omega (1-\rho^*)\chi_t\,(f_R(p) + \mathcal{G}_0[p] + \mathrm{div} u)\, pb(p)\,\mathrm{d}x + \int_\Omega (\chi + \rho^*(1-\chi)) f_R'(p) p_t\, pb(p)\,\mathrm{d}x \\ &+ \int_\Omega (\chi + \rho^*(1-\chi))\,\mathcal{G}_0[p]_t\, pb(p)\,\mathrm{d}x + \int_\Omega (\chi + \rho^*(1-\chi))\,\mathrm{div} u_t\, pb(p)\,\mathrm{d}x.\end{aligned} \tag{5.83}$$

We now define

$$V_{b,R}(p) := \int_0^p f_R'(z) z b(z)\,\mathrm{d}z$$

so that

$$\int_\Omega (\chi + \rho^*(1-\chi)) f_R'(p) p_t\, pb(p)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\chi + \rho^*(1-\chi)) V_{b,R}(p)\,\mathrm{d}x - \int_\Omega (1-\rho^*)\chi_t V_{b,R}(p)\,\mathrm{d}x,$$

and introduce the modified Preisach potential

$$U_b[p] := \int_0^\infty \int_0^{\mathfrak{p}_r[p]} vb(v)\,\psi(r,v)\,\mathrm{d}v\,\mathrm{d}r$$

which satisfies

$$\mathcal{G}_0[p]_t\, pb(p) - U_b[p]_t \geq 0 \quad \text{a.e.}$$

according to (B.32) and (B.33). Note that $V_{b,R}(p) > 0$ and $U_b[p] \geq 0$ for all $p \neq 0$. Then (5.83) can be rewritten as

$$\begin{aligned}&\int_\Omega \big((\chi + \rho^*(1-\chi))(f_R(p) + \mathcal{G}_0[p] + \mathrm{div} u)\big)_t\, pb(p)\,\mathrm{d}x \\ &\geq \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\chi + \rho^*(1-\chi))\,(V_{b,R}(p) + U_b[p])\,\mathrm{d}x + \int_\Omega (\chi + \rho^*(1-\chi))\,\mathrm{div} u_t\, pb(p)\,\mathrm{d}x \\ &+ \int_\Omega (1-\rho^*)\chi_t \Big((pf_R(p) + p\,\mathcal{G}_0[p] + p\,\mathrm{div} u)\, b(p) - V_{b,R}(p) - U_b[p]\Big)\,\mathrm{d}x.\end{aligned} \tag{5.84}$$

Defining

$$\Psi_{b,R}(p) := V_R(p) b(p) - V_{b,R}(p),$$

we see that from (5.8) and (5.16) it holds

$$\begin{aligned}&(1-\rho^*)\chi_t \Big((pf_R(p) + p\,\mathcal{G}_0[p] + p\,\mathrm{div} u)\, b(p) - V_{b,R}(p) - U_b[p]\Big) \\ &= (1-\rho^*)\chi_t \Big((\Phi_R(p) + p\,\mathcal{G}_0[p] - U_0[p] + p\,\mathrm{div} u)\, b(p) + \Psi_{b,R}(p) + U_0[p]b(p) - U_b[p]\Big) \\ &= \left(\gamma_R(p,\theta,\mathrm{div} u)\chi_t^2 - L\chi_t\left(\frac{\hat\theta}{\theta_c} - 1\right)\right) b(p) + (1-\rho^*)\chi_t\,(\Psi_{b,R}(p) + U_0[p]b(p) - U_b[p])\,.\end{aligned}$$

Now, using Young's inequality as in (5.79) we obtain

$$\left(\gamma_R(p,\theta,\mathrm{div}u)\chi_t^2 - L\chi_t\left(\frac{\hat\theta}{\theta_c}-1\right)\right)b(p) \ge \frac{1}{2}\gamma_R(p,\theta,\mathrm{div}u)\chi_t^2\, b(p) - C(1+\hat\theta)\, b(p),$$

and similarly

$$\begin{aligned}&\left|(1-\rho^*)\chi_t\left(\Psi_{b,R}(p)+U_0[p]b(p)-U_b[p]\right)\right|\\ &\le \frac{1}{4}\gamma_R(p,\theta,\mathrm{div}u)\chi_t^2\, b(p) + C\,\frac{\left(\Psi_{b,R}(p)+U_0[p]b(p)-U_b[p]\right)^2}{\gamma_R(p,\theta,\mathrm{div}u)\, b(p)},\end{aligned}$$

so that (5.84) entails

$$\begin{aligned}&\int_\Omega \left((\chi+\rho^*(1-\chi))(f_R(p)+\mathcal{G}_0[p]+\mathrm{div}u)\right)_t pb(p)\,\mathrm{d}x\\ &\ge \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(\chi+\rho^*(1-\chi))\left(V_{b,R}(p)+U_b[p]\right)\,\mathrm{d}x + \int_\Omega(\chi+\rho^*(1-\chi))\,\mathrm{div}u_t\, pb(p)\,\mathrm{d}x\\ &+\frac{1}{4}\int_\Omega \gamma_R(p,\theta,\mathrm{div}u)\chi_t^2\, b(p)\,\mathrm{d}x - C\int_\Omega\left((1+\hat\theta)\, b(p) + \frac{\left(\Psi_{b,R}(p)+U_0[p]b(p)-U_b[p]\right)^2}{\gamma_R(p,\theta,\mathrm{div}u)\, b(p)}\right)\,\mathrm{d}x.\end{aligned} \tag{5.85}$$

We are going to prove that the quantity

$$\frac{\left(\Psi_{b,R}(p)+U_0[p]b(p)-U_b[p]\right)^2}{\gamma_R(p,\theta,\mathrm{div}u)\, b(p)}$$

is bounded by $(1+p^2)b(p)$ independently of R. To this aim, let us show that the inequality

$$V_R(p) \le V(p) + \frac{f^\sharp}{2}(p^2-R^2)^+ \tag{5.86}$$

holds for all $p\in\mathbb{R}$. Indeed, by the definitions of f_R, V_R and V in (5.7), (5.8) and (B.28), we can argue as follows:

- if $|p|\le R$ then
$$V_R(p) = \int_0^p f_R'(z)z\,\mathrm{d}z = \int_0^p f'(z)z\,\mathrm{d}z = V(p)$$
and (5.86) immediately follows;
- if $p>R$ then
$$V_R(p) = \int_0^R f'(z)z\,\mathrm{d}z + \int_R^p f_R'(z)z\,\mathrm{d}z = V(R) + \frac{1}{2}f'(R)(p^2-R^2).$$
Now,
$$V(R)+\frac{1}{2}f'(R)(p^2-R^2) \le V(p) + \frac{f^\sharp}{2}(p^2-R^2)$$
if and only if
$$\begin{aligned}0\le V(p)-V(R)+\frac{1}{2}(f^\sharp-f'(R))(p^2-R^2) &= \int_R^p f'(z)z\,\mathrm{d}z + \int_R^p (f^\sharp - f'(R))z\,\mathrm{d}z\\ &= \int_R^p (f'(z)+f^\sharp-f'(R))z\,\mathrm{d}z,\end{aligned}$$
which is certainly true since $f'(z)\ge 0$ for all $z\in\mathbb{R}$ and $f^\sharp\ge f'(R)$ by Hypothesis 5.1 (vi);

- if $p < -R$ then we argue as in the previous case.

Hence inequality (5.86), together with estimates (5.12), (B.30), the definition of $V(p)$ in (B.28) and Hypothesis 5.1 (vi) on f, entails

$$\frac{(\Psi_{b,R}(p)+U_0[p]b(p)-U_b[p])^2}{(1+p^2)\,\gamma_R(p,\theta,\mathrm{div}u)\,b^2(p)} \le \frac{(V_R(p)+U_0[p])^2}{(1+p^2)\,\gamma_R(p,\theta,\mathrm{div}u)} \le \frac{\Big(V(p)+\frac{f^\sharp}{2}(p^2-R^2)^+ + U_0[p]\Big)^2}{(1+p^2)\,(1+(p^2-R^2)^+)}$$
$$\le \frac{\Big(1+p^2+\frac{f^\sharp}{2}(p^2-R^2)^+\Big)^2}{(1+p^2)\,(1+(p^2-R^2)^+)} \le C$$

since $1+p^2+(p^2-R^2)^+ + p^2(p^2-R^2)^+ \ge 1+p^2+\big((p^2-R^2)^+\big)^2$. Thus we have proved that

$$\frac{(\Psi_{b,R}(p)+U_0[p]b(p)-U_b[p])^2}{\gamma_R(p,\theta,\mathrm{div}u)\,b(p)} \le C(1+p^2)b(p)$$

independently of R. From (5.85) we conclude

$$\begin{aligned}
&\int_\Omega \big((\chi+\rho^*(1-\chi))(f_R(p)+\mathcal{G}_0[p]+\mathrm{div}u)\big)_t\, pb(p)\,\mathrm{d}x \\
&\ge \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\chi+\rho^*(1-\chi))\,(V_{b,R}(p)+U_b[p])\,\mathrm{d}x + \frac{1}{4}\int_\Omega \gamma_R(p,\theta,\mathrm{div}u)\chi_t^2\, b(p)\,\mathrm{d}x \\
&- C\int_\Omega \Big(1+|p||\mathrm{div}u_t|+\hat\theta+p^2\Big)\, b(p)\,\mathrm{d}x,
\end{aligned}$$

so that (5.82) yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\chi+\rho^*(1-\chi))\,(V_{b,R}(p)+U_b[p])\,\mathrm{d}x + \int_\Omega \frac{1}{\rho_W}\mu_R(p)(b(p)+pb'(p))|\nabla p|^2\,\mathrm{d}x \\
&+\int_{\partial\Omega}\alpha(x)(p-p^*)\,pb(p)\,\mathrm{d}s(x) \le C\int_\Omega \Big(1+|p||\mathrm{div}u_t|+\hat\theta+p^2\Big)\, b(p)\,\mathrm{d}x.
\end{aligned}$$

We now integrate in time $\int_0^\tau \mathrm{d}t$ for some $\tau\in[0,T]$, and obtain

$$\begin{aligned}
&\int_\Omega (\chi+\rho^*(1-\chi))\,(V_{b,R}(p)+U_b[p])(x,\tau)\,\mathrm{d}x + \int_0^\tau\int_\Omega \frac{1}{\rho_W}\mu_R(p)(b(p)+pb'(p))|\nabla p|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&+\int_0^\tau\int_{\partial\Omega}\alpha(x)(p-p^*)\,pb(p)(x,t)\,\mathrm{d}s(x)\,\mathrm{d}t \\
&\le C\int_0^\tau\int_\Omega \Big(1+|p||\mathrm{div}u_t|+\hat\theta+p^2\Big)\, b(p)(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega (V_{b,R}(p)+U_b[p])(x,0)\,\mathrm{d}x.
\end{aligned} \tag{5.87}$$

Now that we got rid of χ_t and derived a manageable estimate, we choose $b(p)=|p|^{2k}$ with $k\ge\nu/2$ which will be specified later. Here ν is as in Hypothesis 5.1. Note that this is an admissible choice, that is, $pb(p)\in X$. Indeed, by estimates (5.50), (5.63) and Theorem A.3 with $p_0=q_0=p_1=2$, $q_1=6$, $q_2=\infty$ we have

$$p\in L^q(0,T;C(\bar\Omega)) \tag{5.88}$$

for any $q\in[1,4)$. The bound also depends on R, but for a fixed R and each $k>0$ the function $p|p|^{2k}(\cdot,t)$ belongs to X for a. e. $t\in(0,T)$.

With this choice (5.87) takes the form

$$\begin{aligned}&\int_\Omega (\chi+\rho^*(1-\chi))\big(V_R^k(p)+U^k[p]\big)(x,\tau)\,\mathrm{d}x+(1+2k)\int_0^\tau\int_\Omega \frac{1}{\rho_W}\mu_R(p)|p|^{2k}|\nabla p|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\\&+\int_0^\tau\int_{\partial\Omega}\alpha(x)(p-p^*)\,p|p|^{2k}(x,t)\,\mathrm{d}s(x)\,\mathrm{d}t\\&\le C\int_0^\tau\int_\Omega\Big(|\mathrm{div}u_t||p|^{1+2k}+(1+\hat\theta)|p|^{2k}+|p||p|^{1+2k}\Big)(x,t)\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega\Big(V_R^k(p)+U^k[p]\Big)(x,0)\,\mathrm{d}x\end{aligned}$$

where, from Hypothesis 5.1 (vi),

$$\begin{aligned}V_R^k(p)(x,t)&:=\int_0^{p(x,t)} f_R'(z)z|z|^{2k}\,\mathrm{d}z\ge\int_0^p\frac{f^\flat}{(1+|z|)^{1+\nu}}\,z|z|^{2k}\,\mathrm{d}z\ge\int_0^p\frac{f^\flat}{2\max\{1,|z|\}^{1+\nu}}\,z|z|^{2k}\,\mathrm{d}z\\&=\frac{f^\flat}{2}\left(\int_0^1 z|z|^{2k}\,\mathrm{d}z+\int_1^p z|z|^{2k-1-\nu}\,\mathrm{d}z\right)\ge\frac{f^\flat}{1+2k-\nu}|p|^{1+2k-\nu}-C,\\U^k[p](x,t)&:=\int_0^\infty\int_0^{\mathfrak{p}_r[p](x,t)} v|v|^{2k}\,\psi(r,v)\,\mathrm{d}v\,\mathrm{d}r\ge 0.\end{aligned}$$

Note also that, from Hypothesis 5.1 (vi) and an analogous version of (B.31),

$$\begin{aligned}V_R^k(p)(x,0)&=\int_0^{p^0(x)} f_R'(z)z|z|^{2k}\,\mathrm{d}z\le\frac{f^\sharp}{2+2k}|p^0(x)|^{2+2k},\\U^k[p](x,0)&=\int_0^\infty\int_0^{\mathfrak{p}_r[p^0](x)} v|v|^{2k}\,\psi(r,v)\,\mathrm{d}v\,\mathrm{d}r\le C_\psi^*\max\{|p^0(x)|,K\}^{1+2k}.\end{aligned}$$

Moreover

$$|p|^{2k}|\nabla p|^2=\frac{1}{(1+k)^2}|\nabla(p|p|^k)|^2.$$

Finally, by Young's inequality with conjugate exponents $\left(\frac{2+2k}{1+2k}\,,\,2+2k\right)$, we see that the boundary term is such that

$$\begin{aligned}&\int_0^\tau\int_{\partial\Omega}\alpha(x)(p-p^*)\,p|p|^{2k}\,\mathrm{d}s(x)\,\mathrm{d}t\\&=\int_0^\tau\int_{\partial\Omega}\alpha(x)|p|^{2+2k}\,\mathrm{d}s(x)\,\mathrm{d}t-\int_0^\tau\int_{\partial\Omega}\alpha(x)p^*p|p|^{2k}(x,t)\,\mathrm{d}s(x)\,\mathrm{d}t\\&\ge\int_0^\tau\int_{\partial\Omega}\alpha(x)|p|^{2+2k}\,\mathrm{d}s(x)\,\mathrm{d}t-\frac{1+2k}{2+2k}\int_0^\tau\int_{\partial\Omega}\alpha(x)|p|^{2+2k}\,\mathrm{d}s(x)\,\mathrm{d}t\\&-\frac{1}{2+2k}\int_0^\tau\int_{\partial\Omega}\alpha(x)|p^*|^{2+2k}\,\mathrm{d}s(x)\,\mathrm{d}t\\&=\frac{1}{2+2k}\int_0^\tau\int_{\partial\Omega}\alpha(x)|p|^{2+2k}\,\mathrm{d}s(x)\,\mathrm{d}t-\frac{1}{2+2k}\int_0^\tau\int_{\partial\Omega}\alpha(x)|p^*|^{2+2k}\,\mathrm{d}s(x)\,\mathrm{d}t.\end{aligned}$$

Hence, using also Hypothesis 5.1 (vii), we obtain

$$\begin{aligned}&\frac{f^\flat}{1+2k-\nu}\int_\Omega|p(x,\tau)|^{1+2k-\nu}\,\mathrm{d}x+\mu^\flat\frac{1+2k}{(1+k)^2}\int_0^\tau\int_\Omega|\nabla(p|p|^k)|^2\,\mathrm{d}x\,\mathrm{d}t\\&+\frac{1}{2+2k}\int_0^\tau\int_{\partial\Omega}\alpha(x)|p|^{2+2k}\,\mathrm{d}s(x)\,\mathrm{d}t\\&\le\frac{1}{2+2k}\int_0^\tau\int_{\partial\Omega}\alpha(x)|p^*|^{2+2k}\,\mathrm{d}s(x)\,\mathrm{d}t+\frac{f^\sharp}{2+2k}\int_\Omega|p^0(x)|^{2+2k}\,\mathrm{d}x+C_\psi^*\int_\Omega\max\{|p^0(x)|,K\}^{1+2k}\,\mathrm{d}x\\&+C\int_0^\tau\int_\Omega\Big(|\mathrm{div}u_t||p|^{1+2k}+(1+\hat\theta)\,(1+|p|)^{1+2k}+|p||p|^{1+2k}\Big)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Multiplying by $(1+k)$ and using Hypothesis 5.1 (iv) and (v) yield, for all $\tau \in [0,T]$, that there exists a function $h \in L^2(\Omega \times (0,T))$ such that

$$\|h\|_{L^2(\Omega\times(0,T))} \le C$$

independently of R, as well as

$$\begin{aligned} &\int_\Omega |p(x,\tau)|^{1+2k-\nu}\,\mathrm{d}x + \int_0^\tau\int_\Omega |\nabla(p|p|^k)|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_{\partial\Omega} \alpha(x)|p|^{2+2k}\,\mathrm{d}s(x)\,\mathrm{d}t \\ &\qquad \le C(1+k)\left(C^k + \int_0^\tau\int_\Omega |h||p|^{1+2k}\,\mathrm{d}x\,\mathrm{d}t\right) \end{aligned} \tag{5.89}$$

with a constant $C \ge 1$ independent of R and k. The existence of such a function h comes from estimates (5.5) and (5.80) for $\mathrm{div} u_t$, (5.78) for $\hat\theta$ and (5.81) for p.

For the sake of simplicity, in the rest of the Section we will denote by $|v|_r$ the $L^r(\Omega)$-norm of a function $v \in L^r(\Omega)$ and $v \in L^r(\Omega;\mathbb{R}^3)$ for $r \in [0,\infty]$, by $|v|_{1;r}$ the norm of a function $v \in W^{1,r}(\Omega)$ and by $\|v\|_r$ the norm of a function $v \in L^r(\Omega\times(0,T))$. Moreover, since we deal with anisotropic spaces $L^q(0,T;L^r(\Omega))$, $q \ne r$, it is convenient to introduce for the norm of a function $v \in L^q(0,T;L^r(\Omega))$ the symbol

$$\|v\|_{r\to q} := \left(\int_0^T |v(t)|_r^q\,\mathrm{d}t\right)^{1/q}. \tag{5.90}$$

For the function

$$w_k(x,t) = p(x,t)|p(x,t)|^k \tag{5.91}$$

we obtain from (5.89) using Hölder's inequality and Poincaré's inequality (A.10) that

$$\sup_{\tau\in[0,T]}\mathrm{ess}\, |w_k(\tau)|_{s_k}^{s_k} + \int_0^\tau |w_k(t)|_{1;2}^2\,\mathrm{d}t \le C(1+k)\left(C^k + \left(\int_0^T |w_k(t)|_{q_k}^{q_k}\,\mathrm{d}t\right)^{1/2}\right) \tag{5.92}$$

for all $\tau \in [0,T]$, with

$$s_k = \frac{1+2k-\nu}{1+k}, \qquad q_k = \frac{2+4k}{1+k}$$

and with a constant C independent of τ, R and k. We now show that for a suitably chosen k, the right-hand side of (5.92) is dominated by the left-hand side, which will imply a bound for the left-hand side. By the Gagliardo-Nirenberg inequality (A.9) with $q = q_k$, $s = s_k$, $p = 2$ and $N = 3$ we have

$$|w_k(t)|_{q_k} \le C|w_k(t)|_{s_k}^{1-\varrho_k}|w_k(t)|_{1;2}^{\varrho_k}, \qquad \varrho_k = \frac{\frac{1}{s_k}-\frac{1}{q_k}}{\frac{1}{s_k}-\frac{1}{6}}. \tag{5.93}$$

We now choose k in such a way that $\varrho_k q_k = 2$, that is, $3q_k = 6 + 2s_k$, which yields

$$k = 1-\nu, \quad s_k = \frac{3(1-\nu)}{2-\nu}, \quad q_k = \frac{6-4\nu}{2-\nu}, \quad q_k(1-\varrho_k) = q_k - 2 = \frac{2(1-\nu)}{2-\nu} = \frac{2}{3}s_k.$$

By Hypothesis 5.1 we have $s_k \ge 1$. Hence, by (5.93),

$$\int_0^T |w_k(t)|_{q_k}^{q_k}\,\mathrm{d}t \le C \sup_{\tau\in[0,T]}\mathrm{ess}\, |w_k(\tau)|_{s_k}^{(2/3)s_k}\int_0^T |w_k(t)|_{1;2}^2\,\mathrm{d}t.$$

Since $k < 1$, we conclude from (5.92) that there exists a constant C independent of R such that, in particular,

$$\sup_{\tau\in[0,T]}\text{ess}\,|w_k(\tau)|_{s_k} \le C\,, \qquad \int_0^T |w_k(t)|_{q_k}^{q_k}\,\mathrm{d}t \le C\,.$$

Invoking (5.91), we obtain for p the estimates

$$\sup_{\tau\in[0,T]}\text{ess}\,|p(\tau)|_{3(1-\nu)} \le C\,, \qquad \int_0^T |p(t)|_{6-4\nu}^{6-4\nu}\,\mathrm{d}t \le C\,. \tag{5.94}$$

We now distinguish two cases: $\nu \le 1/3$ and $\nu > 1/3$. For $\nu \le 1/3$ (that is, $3(1-\nu) \ge 2$) we have

$$\sup_{\tau\in[0,T]}\text{ess}\,|p(\tau)|_2 \le C\,. \tag{5.95}$$

For $\nu > 1/3$ (that is, $3(1-\nu) < 2$) we use again the Gagliardo-Nirenberg inequality (A.9) with $q = 2$, $s = 3(1-\nu)$, $p = 2$ and $N = 3$, obtaining

$$|p(t)|_2 \le C|p(t)|_{3(1-\nu)}^{1-\varrho}|p(t)|_{1;2}^{\varrho}\,, \qquad \varrho = \frac{3\nu-1}{1+\nu}\,,$$

so that for

$$q_\nu = \frac{2(1+\nu)}{3\nu-1}$$

we have

$$\left(\int_0^T |p(t)|_2^{q_\nu}\,\mathrm{d}t\right)^{1/q_\nu} \le C\sup_{t\in[0,T]}\text{ess}\,|p(t)|_{3(1-\nu)}^{1-\varrho}\left(\int_0^T |p(t)|_{1;2}^2\,\mathrm{d}t\right)^{1/q_\nu}.$$

Hence by virtue of (5.81) and (5.94) we obtain

$$\|p\|_{2\to q_\nu} \le C \tag{5.96}$$

with a constant $C > 0$ independent of R, according to the notation (5.90). Note that $q_\nu \ge 6$ thanks to Hypothesis 5.1.

5.2.6 Estimate for the displacement

Now that we have obtained a suitable estimate for the capillary pressure in (5.96), we derive an analogous estimate for $\text{div}u_t$. To this aim we test (5.14) by $\psi = u_t$, which yields

$$\int_\Omega (\mathbf{B}\nabla_s u_t : \nabla_s u_t)(x,t)\,\mathrm{d}x \le \int_\Omega \left(-\mathcal{P}[\nabla_s u] : \nabla_s u_t + |p||\text{div}u_t| + \beta|\hat\theta - \theta_c||\text{div}u_t| + |g||u_t|\right)(x,t)\,\mathrm{d}x.$$

By (B.6), Hypothesis 5.1 (v) and (5.80) we have

$$\int_\Omega |\mathcal{P}[\nabla_s u]|^2(x,t)\,\mathrm{d}x \le C\left(1 + \int_0^T\int_\Omega |\nabla_s u_t|^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\right) \le C,$$

hence using Hypothesis 5.1 (i) and Young's inequality as in Subsection 5.2.4 we conclude that the estimate

$$\int_\Omega |\nabla_s u_t|^2(x,t)\,\mathrm{d}x \le C\left(1 + \int_\Omega \left(p^2 + \hat\theta^2\right)(x,t)\,\mathrm{d}x\right) \tag{5.97}$$

holds for a. e. $t \in (0,T)$ with a constant $C > 0$ independent of R. We want to find and estimate for $\nabla_s u_t$ in the norm of $L^q(0,T;L^2(\Omega))$ for a suitable q. To this aim we apply the Gagliardo-Nirenberg inequality (A.9) to $\hat\theta$ with the choices $q = p = 2$, $s = 1 + b$ (with b from Hypothesis 5.1) and $N = 3$. We obtain that, for $t \in (0,T)$,

$$|\hat\theta(t)|_2 \le C|\hat\theta(t)|_{1+b}^{1-\varrho}|\nabla\hat\theta(t)|_{1;2}^{\varrho}, \qquad \varrho = \frac{3-3b}{5-b},$$

so that for

$$q_b = \frac{2(5-b)}{3-3b}$$

we have

$$\left(\int_0^T |\hat\theta(t)|_2^{q_b}\,\mathrm{d}t\right)^{1/q_b} \le C \operatorname*{sup\,ess}_{t\in[0,T]} |\hat\theta(t)|_{1+b}^{1-\varrho} \left(\int_0^T |\nabla\hat\theta(t)|_{1;2}^2\,\mathrm{d}t\right)^{1/q_b}.$$

Hence by virtue of (5.72) and (5.77) we get

$$\|\hat\theta\|_{2\to q_b} \le C \tag{5.98}$$

independently of R. Note that our hypotheses on b imply $q_b \ge 6$. Thus, coming back to (5.97) we have obtained that there exists $q := \min\{q_\nu, q_b\} \ge 6$ such that, thanks to (5.95) or (5.96) and (5.97),

$$\|\nabla_s u_t\|_{2\to q} \le C \tag{5.99}$$

independently of R, according to the notation (5.90).

5.2.7 Anisotropic Moser iterations

The starting point of our analysis is the inequality (5.89). Unlike in Subsection 5.2.5, we do not keep the exponent k bounded, but we let $k \to \infty$ in a controlled way. The key step is the following modification of [94, Lemma 1.3.1].

Lemma 5.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitzian domain, and let $\bar q \ge 1, \bar r \ge 1$ be such that*

$$\frac{1}{\bar r} + \frac{2}{\bar q N} = 1, \quad \frac{1}{\bar q} \ge \frac{1}{\bar r} > 1 - \frac{2}{N}. \tag{5.100}$$

Let a, b be real numbers satisfying the inequalities

$$\frac12 \le a \le b \le \frac{N+2a}{N+2} \le 1\,. \tag{5.101}$$

Then there exists a constant $C > 0$ independent of a, b such that for every $v \in L^2(0,T;W^{1,2}(\Omega))$ we have

$$\|v\|_{2b\bar r\to 2b\bar q}^{2b} \le C \max\left\{1, \operatorname*{sup\,ess}_{t\in(0,T)} |v(t)|_{2a}^{2a} + \int_0^T |v(t)|_{1;2}^2\,\mathrm{d}t\right\}. \tag{5.102}$$

Proof. By virtue of the Gagliardo-Nirenberg inequality (A.9) with $q = 2b\bar{r}$, $s = 2a$ and $p = 2$, there exists a constant $C > 0$ independent of a, b and of $v \in L^2(0,T;W^{1,2}(\Omega))$ such that

$$|v(t)|_{2b\bar{r}} \le C|v(t)|_{2a}^{1-\varrho}|v(t)|_{1;2}^{\varrho} \quad \text{for } t \in (0,T) \tag{5.103}$$

with

$$\varrho = \frac{\frac{1}{a} - \frac{1}{b\bar{r}}}{\frac{1}{a} + \frac{2}{N} - 1}, \quad 1 - \varrho = \frac{\frac{1}{b\bar{r}} + \frac{2}{N} - 1}{\frac{1}{a} + \frac{2}{N} - 1}.$$

Indeed, the fact that $1 \le 2a \le 2, 1 \le 2b \le 2$ makes the constant C range in a bounded interval, from which we deduce that such a constant is independent of both a and b. Moreover, we easily check that $\varrho \in (0,1)$. Raising (5.103) to the power $2b\bar{q}$ and integrating in t yields

$$\int_0^T |v(t)|_{2b\bar{r}}^{2b\bar{q}}\,\mathrm{d}t \le C^{2b\bar{q}} \operatorname*{sup\,ess}_{t\in(0,T)} |v(t)|_{2a}^{(1-\varrho)2b\bar{q}} \int_0^T |v(t)|_{1;2}^{2\varrho b\bar{q}}\,\mathrm{d}t. \tag{5.104}$$

We claim that $\varrho b\bar{q} \le 1$. Indeed,

$$\begin{aligned}\left(\frac{1}{a} + \frac{2}{N} - 1\right)(1-\varrho b\bar{q}) &= \frac{1}{a} + \frac{2}{N} - 1 - \frac{b\bar{q}}{a} + \frac{\bar{q}}{\bar{r}} \\ &= \left(\frac{b}{a} - 1\right)\left(1 - \frac{\bar{q}}{\bar{r}}\right) + \frac{1}{a} + \frac{2}{N} - \frac{b}{a}\left(1 + \frac{2}{N}\right) \\ &\ge \frac{1}{aN}(N + 2a - b(N+2)) \ge 0\end{aligned}$$

by virtue of (5.100) and (5.101). We can therefore use Hölder's inequality in (5.104) and obtain

$$\int_0^T |v(t)|_{2b\bar{r}}^{2b\bar{q}}\,\mathrm{d}t \le T^{1-\varrho b\bar{q}} C^{2b\bar{q}} \operatorname*{sup\,ess}_{t\in(0,T)} |v(t)|_{2a}^{(1-\varrho)2b\bar{q}} \left(\int_0^T |v(t)|_{1,2}^2\,\mathrm{d}t\right)^{\varrho b\bar{q}},$$

that is,

$$\begin{aligned}\|v\|_{2b\bar{r}\to 2b\bar{q}}^{2b} &= \left(\int_0^T |v(t)|_{2b\bar{r}}^{2b\bar{q}}\,\mathrm{d}t\right)^{1/\bar{q}} \\ &\le T^{(1/\bar{q})-\varrho b} C^{2b} \left(\operatorname*{sup\,ess}_{t\in(0,T)} |v(t)|_{2a}^{2a}\right)^{(1-\varrho)b/a} \left(\int_0^T |v(t)|_{1,2}^2\,\mathrm{d}t\right)^{\varrho b}.\end{aligned} \tag{5.105}$$

We check that

$$\hat{\rho} := (1-\varrho)\frac{b}{a} + \varrho b \le 1.$$

This follows from the relation

$$\begin{aligned}\left(\frac{1}{a} + \frac{2}{N} - 1\right)(1 - \hat{\rho}) &= \left(\frac{1}{a} - 1\right)\left(1 - \frac{1}{\bar{r}}\right) - \frac{2}{N}\left(\frac{b}{a} - 1\right) \\ &\ge \frac{2}{N+2}\left(\frac{1}{a} - 1\right) - \frac{2}{N}\left(\frac{b}{a} - 1\right) \ge 0.\end{aligned}$$

We have used hypothesis (5.101) and the fact that $\bar{r} \ge 1 + 2/N$, which follows from (5.100). Furthermore, for all positive numbers λ, μ, c, d we have as a consequence of Young's inequality that

$$c^\lambda d^\mu \le \left(\frac{\lambda}{\lambda+\mu}c + \frac{\mu}{\lambda+\mu}d\right)^{\lambda+\mu},$$

so that (5.105) can be rewritten as

$$\|v\|^{2b}_{2b\bar r\to 2b\bar q} \le T^{(1/\bar q)-\varrho b} C^{2b}\left(\sup_{t\in(0,T)}\operatorname{ess}|v(t)|^{2a}_{2a} + \int_0^T |v(t)|^2_{1,2}\,\mathrm{d}t\right)^{\hat\rho}$$

which implies (5.102). □

We apply Lemma 5.5 to (5.89) first in a general space dimension $N \ge 2$. It is interesting to note that a condition similar to (5.100) appears also in [102, Chapter VII], where the authors prove an L^∞-bound for solutions to linear parabolic equations. The measure α of degeneracy of the function f thus does not seem to play any substantial role here.

As in (5.91), we define auxiliary functions $w_k = p|p|^k$ and rewrite (5.89) for $k \ge 1-\nu$ as

$$\begin{aligned}\int_\Omega |w_k(x,\tau)|^{2a_k}\,\mathrm{d}x + \int_0^\tau\int_\Omega |\nabla w_k|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_{\partial\Omega}\gamma(x)|w_k|^2\,\mathrm{d}s(x)\,\mathrm{d}t\\ \le (1+k)\max\left\{C^{1+k}, \int_0^\tau\int_\Omega |h||w_k|^{2b_k}\,\mathrm{d}x\,\mathrm{d}t\right\}\end{aligned} \tag{5.106}$$

with a constant $C \ge 1$ independent of k, and with

$$a_k = \frac{1+2k-\nu}{2+2k}\,, \quad b_k = \frac{1+2k}{2+2k}\,.$$

Assume now that there exist $q \ge r$ such that

$$h \in L^q(0,T;L^r(\Omega))\,, \qquad \frac{1}{q} + \frac{N}{2r} =: \epsilon < 1\,. \tag{5.107}$$

From (5.106), Poincaré's inequality (A.10) and Hölder's inequality it follows that

$$\sup_{t\in(0,T)}\operatorname{ess}|w_k(\tau)|^{2a_k}_{2a_k}\,\mathrm{d}x + \int_0^T\int_\Omega |w_k(t)|^2_{1;2}\,\mathrm{d}t \le (1+k)\max\left\{C^{1+k}, H\|w_k\|^{2b_k}_{2b_k r'\to 2b_k q'}\right\}, \tag{5.108}$$

where

$$H = \max\{1, \|h\|_{r\to q}\}\,, \quad \frac{1}{r'} = 1-\frac{1}{r}\,, \quad \frac{1}{q'} = 1-\frac{1}{q}\,. \tag{5.109}$$

We now check that the conditions (5.100) and (5.101) are fulfilled with the choice

$$\bar r = \sigma r', \quad \bar q = \sigma q', \quad \sigma = 1 + \frac{2}{N}(1-\epsilon) > 1\,, \quad a = a_k, \quad b = b_k\,. \tag{5.110}$$

Indeed, the inequality $\bar r \ge \bar q$ is obvious. Furthermore,

$$\sigma\left(\frac{1}{\bar r} + \frac{2}{N}\right) = 1 - \frac{1}{r} + \frac{2\sigma}{N} \ge 1 - \frac{2\epsilon}{N} + \frac{2\sigma}{N} = \sigma + \frac{2}{N}(\sigma-1) > \sigma,$$

hence $1/\bar r > 1 - 2/N$. The first identity of (5.100) is obtained from the following computation:

$$\sigma\left(\frac{1}{\bar r} + \frac{2}{\bar q N}\right) = \frac{1}{r'} + \frac{2}{q'N} = 1 + \frac{2}{N} - \frac{2\epsilon}{N} = \sigma,$$

which is precisely the desired result. To check that (5.101) holds, we just notice that the inequalities $1/2 \le a_k \le b_k$ are obvious for $k \ge 1-\nu$. We also have

$$\frac{1}{b_k}\,\frac{N+2a_k}{N+2} = \frac{(1+2k)(N+2)+N-2\nu}{(1+2k)(N+2)} > 1,$$

and the assumptions of Lemma 5.5 are thus verified. Therefore, taking into account also (5.108), we obtain for every $k \ge 1-\nu$ the inequality

$$\|w_k\|^{2b_k}_{2b_k\sigma r' \to 2b_k\sigma q'} \le (1+k)H \max\left\{A^{1+k}, \|w_k\|^{2b_k}_{2b_k r' \to 2b_k q'}\right\}, \tag{5.111}$$

with constants $\sigma > 1$ given by (5.110), $H \ge 1$ given by (5.109) and $A \ge 1$ depending only on the data of the problem, all independent of k. We are now ready to prove the following result.

Proposition 5.6. *Let Hypothesis 5.1 hold and let (p,u,θ,χ) be a solution of (5.13)–(5.16) with the regularity from Proposition 5.4. Then the function p admits an L^∞-bound independent of R, more precisely,*

$$|p(x,t)| \le C\left((\bar\nu H)^{\sigma/(\bar\nu(\sigma-1))}\sigma^{\sigma/(\bar\nu(\sigma-1)^2)}\right) =: R_\sigma \tag{5.112}$$

for a. e. $(x,t) \in \Omega \times (0,T)$ with $\sigma = 19/18$, $H = \max\left\{1, \|p\|_{2\to 6}, \|\mathrm{div} u_t\|_{2\to 6}, \|\hat\theta\|_{2\to 6}\right\}$, and with positive constants $\bar\nu, C$ depending only on the data.

Proof. By (5.96), (5.98) and (5.99) we are in the situation of (5.107) with $r = 2$, $q = 6$, $N = 3$. Hence $r' = 2$, $q' = 6/5$ and by (5.107), (5.110) we have

$$\epsilon = \frac{11}{12}, \quad \sigma = \frac{19}{18},$$

and (5.111) holds for all $k \ge 1-\nu$. We have $|w_k| = |p|^{1+k}$, hence (5.111) can be interpreted as

$$\|p\|^{1+2k}_{(1+2k)\sigma r' \to (1+2k)\sigma q'} \le (1+k)H \max\left\{A^{1+k}, \|p\|^{1+2k}_{(1+2k)r' \to (1+2k)q'}\right\},$$

or

$$\max\left\{A, \|p\|_{(1+2k)\sigma r' \to (1+2k)\sigma q'}\right\} \le ((1+k)H)^{1/(1+2k)} \max\left\{A, \|p\|_{(1+2k)r' \to (1+2k)q'}\right\}. \tag{5.113}$$

We now define an increasing sequence $\{k_j; j = 0,1,...\}$ starting at $k_0 = 1-\nu$ and such that $2k_j+1 = \bar\nu\sigma^j$ for all $j \in \mathbb{N}\cup\{0\}$ with a suitable $\bar\nu > 0$. A straightforward computation yields $\bar\nu = 3-2\nu$. We also set

$$X_j := \max\left\{A, \|p\|_{\bar\nu\sigma^j r' \to \bar\nu\sigma^j q'}\right\}.$$

It follows from (5.113) that for each $j \in \mathbb{N}$ we have

$$X_j \le ((1+k_{j-1})H)^{1/(2k_{j-1}+1)} X_{j-1}.$$

For $Y_j = \log X_j$ this yields

$$Y_j - Y_{j-1} \leq \frac{\sigma^{-j+1}}{\bar{\nu}} \left(\log \bar{\nu} H + (j-1)\log \sigma\right).$$

The right-hand side is the general term of a convergent series, more precisely,

$$\sum_{j=1}^{\infty} \frac{\sigma^{-j+1}}{\bar{\nu}} \left(\log \bar{\nu} H + (j-1)\log \sigma\right) = \frac{\sigma \log \bar{\nu} H}{\bar{\nu}(\sigma-1)} + \frac{\sigma \log \sigma}{\bar{\nu}(\sigma-1)^2}.$$

Hence,

$$X_j \leq X_0 \left((\bar{\nu} H)^{\sigma/(\bar{\nu}(\sigma-1))} \sigma^{\sigma/(\bar{\nu}(\sigma-1)^2)} \right). \tag{5.114}$$

By (5.94) we have

$$X_0 = \max\left\{A, \|p\|_{\bar{\nu}r' \to \bar{\nu}q'}\right\} = \max\left\{A, \|p\|_{6-4\nu \to \frac{18}{5} - \frac{12}{5}\nu}\right\} \leq C$$

since $18/5 - 12\nu/5 \leq 6 - 4\nu$ being $\nu \leq 1/2$, and the assertion follows. □

We want to drive the reader's attention on the fact that the above computations show that L^2-regularity in space is sufficient for starting Moser in 3D if regularity in time is big enough. In particular, (5.107) shows that in our case $\epsilon < 1$ if and only if $q > 4$, hence regularity in time must be more than L^4.

The main consequence of Proposition 5.6 is that, since we aim at taking the limit as $R \to \infty$ in (5.13)–(5.16), we can restrict ourselves to parameter values $R > R_\sigma$, with R_σ from (5.112), so that the cut-off (5.7)–(5.9) is never active and $\gamma_R(p, \theta, \mathrm{div} u) = \gamma(\hat{\theta}, \mathrm{div} u)$. Hence we can rewrite (5.13)–(5.16) in the form

$$\begin{aligned}&\int_\Omega \big((\chi + \rho^*(1-\chi))(f(p) + \mathcal{G}_0[p] + \mathrm{div} u)\big)_t \,\phi \,\mathrm{d}x + \int_\Omega \frac{1}{\rho_W}\mu(p)\nabla p \cdot \nabla \phi \,\mathrm{d}x \\ &= \int_{\partial\Omega} \alpha(x)(p^* - p)\,\phi \,\mathrm{d}s(x),\end{aligned} \tag{5.115}$$

$$\int_\Omega (\mathcal{P}[\nabla_s u] + \mathbf{B}\nabla_s u_t) : \nabla_s \psi \,\mathrm{d}x - \int_\Omega \big(p(\chi + \rho^*(1-\chi)) + \beta(\hat{\theta} - \theta_c)\big)\,\mathrm{div}\psi \,\mathrm{d}x = \int_\Omega g \cdot \psi \,\mathrm{d}x, \tag{5.116}$$

$$\begin{aligned}&\int_\Omega \Big(\mathcal{C}_V(\theta)_t \,\zeta + \kappa(\hat{\theta})\nabla\theta \cdot \nabla\zeta\Big)\,\mathrm{d}x + \int_{\partial\Omega} \omega(x)(\theta - \theta^*)\,\zeta \,\mathrm{d}s(x) \\ &= \int_\Omega \Bigg(\mathbf{B}\nabla_s u_t : \nabla_s u_t + \|D_{\mathcal{P}}[\nabla_s u]_t\|_* + \frac{1}{\rho_W}\mu(p)\,Q_R(|\nabla p|^2) + (\chi + \rho^*(1-\chi))|D_0[p]_t| \\ &+ \gamma(\hat{\theta}, \mathrm{div} u)\chi_t^2 - \left(\frac{L}{\theta_c}\chi_t + \beta \mathrm{div} u_t\right)\hat{\theta}\Bigg)\zeta \,\mathrm{d}x,\end{aligned} \tag{5.117}$$

$$\gamma(\hat{\theta}, \mathrm{div} u)\chi_t + \partial I_{[0,1]}(\chi) \ni (1-\rho^*)\left(\Phi(p) + p\,\mathcal{G}_0[p] - U_0[p] + p\,\mathrm{div} u\right) + L\left(\frac{\hat{\theta}}{\theta_c} - 1\right) \quad \text{a.e.} \tag{5.118}$$

for all test functions $\phi, \zeta \in X$, $\psi \in X_0$, with $\hat{\theta} = Q_R(\theta)$ and with initial conditions (4.29). In order to pass to the limit as $R \to \infty$, we still need to derive some higher order estimates.

5.2.8 Higher order estimates for the capillary pressure

Before we proceed, let us point out that estimate (5.112) yields the existence of a constant $\mu^\sharp > \mu^\flat > 0$ such that $\mu(p) \le \mu^\sharp$ for all $p \in [-R_\sigma, R_\sigma]$, since from Hypothesis 5.1 (vii) it turns out that μ is continuous on this compact set.

Let us define

$$M(p) := \int_0^p \mu(z)\,\mathrm{d}z \tag{5.119}$$

for $p \in \mathbb{R}$, so that $\mu(p)\nabla p = \nabla M(p)$. We would like to test (5.115) by $\phi = M(p)_t = \mu(p)p_t$ which, however, is not an admissible test function since $p_t \notin X$. Hence we choose a small $h > 0$ and test by $\phi = \frac{1}{h}\left(M(p)(t) - M(p)(t-h)\right)$, where

$$\phi(x,t) = \frac{1}{h}\left(M(p)(t) - M(p)(t-h)\right)(x) := \frac{1}{h}\left(M(p(x,t)) - M(p(x,t-h))\right),$$

with the intention to let $h \to 0$. We obtain

$$\begin{aligned}
&\int_\Omega \big((\chi + \rho^*(1-\chi))(f(p) + \mathcal{G}_0[p] + \mathrm{div}u)\big)_t \frac{1}{h}\left(M(p)(t) - M(p)(t-h)\right)\mathrm{d}x \\
&+ \int_\Omega \frac{1}{\rho_W}\nabla M(p)\cdot\nabla\left(\frac{1}{h}\left(M(p)(t) - M(p)(t-h)\right)\right)\mathrm{d}x \\
&= \int_{\partial\Omega} \alpha(x)(p^* - p)\,\frac{1}{h}\left(M(p)(t) - M(p)(t-h)\right)\mathrm{d}s(x).
\end{aligned} \tag{5.120}$$

Concerning the second summand on the left-hand side of (5.120), note that

$$\nabla M(p)(x,t)\cdot\nabla\left(\frac{1}{h}\left(M(p)(x,t) - M(p)(x,t-h)\right)\right) \ge \frac{1}{2h}\left(|\nabla M(p)|^2(x,t) - |\nabla M(p)|^2(x,t-h)\right).$$

The boundary term is such that

$$\begin{aligned}
&\int_{\partial\Omega} \alpha(x)(p - p^*)(x,t)\,\frac{1}{h}\left(M(p)(x,t) - M(p)(x,t-h)\right)\mathrm{d}s(x) \\
&= \int_{\partial\Omega} \alpha(x)p(x,t)\,\frac{1}{h}\left(M(p)(x,t) - M(p)(x,t-h)\right)\mathrm{d}s(x) \\
&- \int_{\partial\Omega} \alpha(x)p^*(x,t)\,\frac{1}{h}\left(M(p)(x,t) - M(p)(x,t-h)\right)\mathrm{d}s(x),
\end{aligned} \tag{5.121}$$

where

$$\begin{aligned}
&p^*(x,t)\,\frac{1}{h}\left(M(p)(x,t) - M(p)(x,t-h)\right) \\
&= \frac{1}{h}\big(p^*(x,t)\,M(p)(x,t) - p^*(x,t-h)M(p)(x,t-h)\big) - \frac{1}{h}\left(p^*(x,t) - p^*(x,t-h)\right)M(p)(x,t-h).
\end{aligned}$$

To handle the term $p\,\frac{1}{h}\left(M(p)(t) - M(p)(t-h)\right)$ in (5.121), we use the inequality $F(y) - F(z) \le F'(y)(y-z)$ which holds for every convex function F and every y, z. We interpret $M(p)(t)$ as y, $M(p)(t-h)$ as z and $F'(y) = M^{-1}(M(p(t))) = M^{-1}(y)$. The function M^{-1} is increasing, hence its antiderivative F is convex. Thus

$$p\left(M(p)(t) - M(p)(t-h)\right) \ge \int_{M(p)(t-h)}^{M(p)(t)} M^{-1}(z)\,\mathrm{d}z = \int_{p(t-h)}^{p(t)} \xi\,M'(\xi)\,\mathrm{d}\xi.$$

Defining

$$\hat{\mu}(p) := \int_0^p z\mu(z)\,\mathrm{d}z$$

for $p \in \mathbb{R}$, we obtain

$$p(x,t)\,\frac{1}{h}\left(M(p)(x,t) - M(p)(x,t-h)\right) \geq \frac{1}{h}\left(\hat{\mu}(p)(x,t) - \hat{\mu}(p)(x,t-h)\right).$$

Thus (5.120) and the above estimates entail

$$\begin{aligned}
&\int_\Omega (\chi + \rho^*(1-\chi))(f(p)_t + \mathcal{G}_0[p]_t)(x,t)\,\frac{1}{h}\left(M(p)(x,t) - M(p)(x,t-h)\right)\,\mathrm{d}x \\
&+ \frac{1}{2\rho_W}\int_\Omega \frac{1}{h}\left(|\nabla M(p)|^2(x,t) - |\nabla M(p)|^2(x,t-h)\right)\mathrm{d}x \\
&+ \int_{\partial\Omega} \alpha(x)\,\frac{1}{h}\left(\hat{\mu}(p)(x,t) - \hat{\mu}(p)(x,t-h)\right)\,\mathrm{d}s(x) \\
&- \int_{\partial\Omega} \alpha(x)\,\frac{1}{h}\left(p^*(x,t)\,M(p)(x,t) - p^*(x,t-h)M(p)(x,t-h)\right)\mathrm{d}s(x) \\
&\leq -\int_\Omega (1-\rho^*)\chi_t\,(f(p) + \mathcal{G}_0[p] + \mathrm{div}u)(x,t)\,\frac{1}{h}\left(M(p)(x,t) - M(p)(x,t-h)\right)\,\mathrm{d}x \\
&- \int_\Omega (\chi + \rho^*(1-\chi))\,\mathrm{div}u_t(x,t)\,\frac{1}{h}\left(M(p)(x,t) - M(p)(x,t-h)\right)\,\mathrm{d}x \\
&- \int_{\partial\Omega} \alpha(x)\,\frac{1}{h}\left(p^*(x,t) - p^*(x,t-h)\right)M(p)(x,t-h)\,\mathrm{d}s(x).
\end{aligned}$$

We are now ready to integrate in time from h to some $\tau \in (0,T)$ and then let $h \to 0$. Note that estimate (5.50) entails that the function $M(p)_t = \mu(p)p_t$ is in L^2, so that the convergence is strong in L^2. We obtain

$$\begin{aligned}
&\int_0^\tau \int_\Omega (\chi + \rho^*(1-\chi))(f(p)_t + \mathcal{G}_0[p]_t)\,\mu(p)p_t\,\mathrm{d}x\,\mathrm{d}t \\
&+ \frac{1}{2\rho_W}\int_\Omega |\nabla M(p)|^2(x,\tau)\,\mathrm{d}x - \frac{1}{2\rho_W}\int_\Omega |\nabla M(p^0)|^2(x)\,\mathrm{d}x \\
&+ \int_{\partial\Omega} \alpha(x)\left(\hat{\mu}(p) - p^*M(p)\right)(x,\tau)\,\mathrm{d}s(x) - \int_{\partial\Omega} \alpha(x)\left(\hat{\mu}(p^0)(x) - p^*(x{,}0)M(p^0)(x)\right)\,\mathrm{d}s(x) \\
&\leq -\int_0^\tau \int_\Omega (1-\rho^*)\chi_t\,(f(p) + \mathcal{G}_0[p] + \mathrm{div}u)\,\mu(p)p_t\,\mathrm{d}x\,\mathrm{d}t - \int_0^\tau \int_\Omega (\chi + \rho^*(1-\chi))\,\mathrm{div}u_t\,\mu(p)p_t\,\mathrm{d}x\,\mathrm{d}t \\
&- \int_0^\tau \int_{\partial\Omega} \alpha(x)\,p_t^*M(p)\,\mathrm{d}s(x)\,\mathrm{d}t.
\end{aligned}$$

Combining (B.24) with the identity (B.21) for the play, we see that it holds $\mu(p)\,\mathcal{G}_0[p]_t\,p_t \geq 0$, thus

$$(\chi + \rho^*(1-\chi))\,(f(p)_t + \mathcal{G}_0[p]_t)\,\mu(p)p_t \geq \rho^*\mu^\flat \frac{f^\flat}{2\max\{1, R_\sigma\}^{1+\nu}}\,|p_t|^2$$

thanks to Hypothesis 5.1 (vi), (vii) and estimate (5.112). Concerning the initial conditions, we employ Hypothesis 5.1 (iv) and (v) together with the following computations

$$\int_\Omega |\nabla M(p^0)|^2(x)\,\mathrm{d}x = \int_\Omega \mu^2(p^0)|\nabla p^0|^2(x)\,\mathrm{d}x,$$

$$\int_{\partial\Omega} \alpha(x)|p^*|^2(x{,}0)\,\mathrm{d}s(x) = \int_{\partial\Omega} \alpha(x)|p^*|^2(x,\tau)\,\mathrm{d}s(x) - 2\int_0^\tau \int_{\partial\Omega} \alpha(x)(p^*p_t^*)(x,t)\,\mathrm{d}s(x)\,\mathrm{d}\tau.$$

Note also that $\mu^\flat p^2/2 \le \hat\mu(p) \le \mu^\sharp p^2/2$ for all $p \in \mathbb{R}$. Hence there exists a constant $c > 0$ such that for every $t \in (0,T)$ we have

$$
\begin{aligned}
&c\int_0^\tau \int_\Omega |p_t|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \frac{(\mu^\flat)^2}{2\rho_W}\int_\Omega |\nabla p|^2(x,\tau)\,\mathrm{d}x + \frac{\mu^\flat}{2}\int_{\partial\Omega}\alpha(x)p^2(x,\tau)\,\mathrm{d}s(x)\\
&\le C\left(1 + \int_0^\tau\int_\Omega \left(|\chi_t|(1+|\mathrm{div}u|)|p_t| + |\mathrm{div}u_t||p_t|\right)(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_{\partial\Omega}\alpha(x)|p_t^*||p|(x,t)\,\mathrm{d}s(x)\,\mathrm{d}t\right)
\end{aligned}
$$

thanks to Hypothesis 5.1 (vi) and (vii), where we handled the boundary term on the left-hand side as in (5.47). Arguing as for estimate (5.40), we obtain

$$
|\chi_t(x,t)| \le \left|\frac{(1-\rho^*)\left(\Phi(p) + p\,\mathcal{G}_0[p] - U_0[p] + p\,\mathrm{div}u\right) + L\left(\hat\theta/\theta_c - 1\right)}{\gamma(\hat\theta, \mathrm{div}u)}\right| \le C \tag{5.122}
$$

for a. e. $(x,t) \in \Omega\times(0,T)$, this time independently of R thanks to (5.112). Thus, employing also Young's inequality, estimates (5.70), (5.71), (5.80) and Hypothesis 5.1 (iii) and (iv), we conclude that

$$
\sup_{\tau\in[0,T]}\mathrm{ess}\left(\int_\Omega |\nabla p|^2(x,\tau)\,\mathrm{d}x + \int_{\partial\Omega}\alpha(x)\,p^2(x,\tau)\,\mathrm{d}s(x)\right) \le C, \tag{5.123}
$$

$$
\int_0^T\int_\Omega p_t^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \le C. \tag{5.124}
$$

By (5.70), (5.80), (5.81) and (5.124) and by comparison in equation (5.115), we see that the term $\Delta M(p)$ is bounded in $L^2(\Omega\times(0,T))$ independently of R. In terms of the new variable $\tilde p = M(p)$, the boundary condition in (6.29) is nonlinear, and from considerations similar to those used in the proof of [95, Theorem 4.1] it follows

$$
\|M(p)\|^2_{L^2(0,T;W^{2,2}(\Omega))} \le C. \tag{5.125}
$$

We thus may employ the Gagliardo-Nirenberg inequality (A.8) with $s = p = 2$, $N = 3$ obtaining

$$
|\nabla M(p)(t)|_q \le C\left(|\nabla M(p)(t)|_2 + |\nabla M(p)(t)|_2^{1-\varrho}|\Delta M(p)(t)|_2^{\varrho}\right), \qquad \varrho = 3\left(\frac{1}{2} - \frac{1}{q}\right),
$$

which holds for all $2 < q < 6$. This yields, elevating to some power s and integrating in time,

$$
\int_0^T |\nabla M(p)(t)|_q^s\,\mathrm{d}t \le C\left(\int_0^T |\nabla M(p)(t)|_2^s\,\mathrm{d}t + \int_0^T |\nabla M(p)(t)|_2^{(1-\varrho)s}|\Delta M(p)(t)|_2^{\varrho s}\,\mathrm{d}t\right).
$$

Choosing ϱ in such a way that $\varrho s = 2$, using Hölder's inequality in time and estimates (5.123), (5.125) we obtain a uniform bound for the right-hand side, which by Hypothesis 5.1 (vii) yields

$$
\int_0^T |\nabla p(t)|_q^s\,\mathrm{d}t \le C \qquad \text{for } q \in (2,6) \quad \text{and} \quad \frac{1}{q} + \frac{2}{3s} = \frac{1}{2}.
$$

In particular, for $s = 4$, $q = 3$ and $s = q = \frac{10}{3}$ we obtain, respectively,

$$
\|\nabla p\|_{3 \ast 4} \le C, \qquad\qquad \|\nabla p\|_{10/3} \le C, \tag{5.126}
$$

according to the notation (5.90).

5.2.9 Higher order estimates for the displacement

Let us consider equation (5.116). Setting

$$w(x,t) := p(\chi + \rho^*(1-\chi)) + \beta(\hat{\theta} - \theta_c)(x,t) + G(x,t)$$

and arguing as for (5.55) we deduce

$$\begin{aligned}\int_\Omega |\nabla_s u_t|^p(x,t)\,\mathrm{d}x \le C\int_\Omega |\nabla_s u^0|^p(x)\,\mathrm{d}x + Ct^{p-1}\int_0^t\int_\Omega |\nabla_s u_t|^p(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\ + C\int_\Omega |w|^p(x,t)\,\mathrm{d}x \quad \text{a. e.}\end{aligned} \tag{5.127}$$

Thus, by choosing $p = 8/3 + 2b/3$ (with $b \in [1/2,1)$ from Hypothesis 5.1) in the above inequality we obtain from (5.78), (5.112), Hypothesis 5.1 (ii) and Grönwall's lemma A.2

$$\|\nabla_s u_t\|_{8/3+2b/3} \le C. \tag{5.128}$$

We now derive an estimate for $\nabla_s u_t$ in a suitable anisotropic Lebesgue space. To this aim we need to derive first an additional estimate for $\hat{\theta}$. We use Gagliardo-Nirenberg inequality (A.8) with the choices $s = 1+b$, $p = 2$ and $N = 3$ obtaining, for $t \in (0,T)$,

$$|\hat{\theta}(t)|_q \le C\left(|\hat{\theta}(t)|_{1+b} + |\hat{\theta}(t)|_{1+b}^{1-\varrho}|\nabla\hat{\theta}(t)|_2^{\varrho}\right), \qquad \varrho = \left(\frac{1}{1+b} - \frac{1}{q}\right)\frac{6(1+b)}{5-b},$$

which holds for all $1+b < q < 6$. This yields, elevating to some power s and integrating in time,

$$\int_0^T |\hat{\theta}(t)|_q^s\,\mathrm{d}t \le C\left(\int_0^T |\hat{\theta}(t)|_{1+b}^s\,\mathrm{d}t + \int_0^T |\hat{\theta}(t)|_{1+b}^{(1-\rho)s}|\nabla\hat{\theta}(t)|_2^{\varrho s}\,\mathrm{d}t\right).$$

Choosing ϱ in such a way that $\varrho s = 2$, using Hölder's inequality in time and estimates (5.72), (5.77) we obtain

$$\int_0^T |\hat{\theta}(t)|_q^s\,\mathrm{d}t \le C \qquad \text{for } q \in (1+b\,,\,6) \quad \text{and} \quad \frac{1}{q} + \frac{5-b}{3s(1+b)} = \frac{1}{1+b}.$$

In particular, for $s = 4$, $q = \frac{12(1+b)}{7+b}$ we obtain

$$\|\hat{\theta}\|_{12(1+b)/(7+b)\ast 4} \le C. \tag{5.129}$$

Note that

$$\frac{12(1+b)}{7+b} < \frac{8}{3} + \frac{2b}{3} \;\Leftrightarrow\; -7 < b < 2 \;\vee\; b > 5,$$

which is certainly true under our hypotheses. Therefore, choosing $p = \frac{12(1+b)}{7+b}$ in (5.127) we obtain, thanks to (5.128) and Hypothesis 5.1 (v),

$$\int_\Omega |\nabla_s u_t|^{12(1+b)/(7+b)}(x,t)\,\mathrm{d}x \le C\left(1 + \int_\Omega |w|^{12(1+b)/(7+b)}(x,t)\,\mathrm{d}x\right)$$

for a. e. $t \in (0,T)$. Hypothesis 5.1 (ii) and estimates (5.112), (5.129) then yield

$$\|\nabla_s u_t\|_{12(1+b)/(7+b)\ast 4} \le C \tag{5.130}$$

independently of R.

5.2.10 Higher order estimates for the temperature

Note that (5.116) with $\psi = u_t$ and (5.118) entail, respectively,

$$\int_\Omega \mathbf{B}\nabla_s u_t : \nabla_s u_t \,\mathrm{d}x = -\int_\Omega \mathcal{P}[\nabla_s u] : \nabla_s u_t \,\mathrm{d}x + \int_\Omega \big(p(\chi + \rho^*(1-\chi)) + \beta(\hat{\theta} - \theta_c)\big)\,\mathrm{div} u_t \,\mathrm{d}x$$
$$+ \int_\Omega g \cdot u_t \,\mathrm{d}x,$$
$$\gamma(\hat{\theta}, \mathrm{div} u)\chi_t^2 = (1-\rho^*)\chi_t \Big(\Phi(p) + p\,\mathcal{G}_0[p] - U_0[p] + p\,\mathrm{div} u\Big) + L\chi_t \left(\frac{\hat{\theta}}{\theta_c} - 1\right).$$

Plugging these identities into (5.117) we obtain

$$\int_\Omega \Big(\mathcal{C}_V(\theta)_t\,\zeta + \kappa(\hat{\theta})\nabla\theta\cdot\nabla\zeta\Big)\,\mathrm{d}x + \int_{\partial\Omega} \omega(x)(\theta - \theta^*)\,\zeta\,\mathrm{d}s(x)$$
$$= \int_\Omega \Big(-\mathcal{P}[\nabla_s u] : \nabla_s u_t + \big(p(\chi + \rho^*(1-\chi)) - \beta\theta_c\big)\,\mathrm{div} u_t + g \cdot u_t$$
$$+ \|D_{\mathcal{P}}[\nabla_s u]_t\|_* + \frac{1}{\rho_W}\mu(p)\,Q_R(|\nabla p|^2) + (\chi + \rho^*(1-\chi))|D_0[p]_t|$$
$$+ \chi_t\Big((1-\rho^*)\,\big(\Phi(p) + p\,\mathcal{G}_0[p] - U_0[p] + p\,\mathrm{div} u\big) - L\Big)\Big)\zeta\,\mathrm{d}x$$
$$=: \int_\Omega \Gamma(x,t)\,\zeta\,\mathrm{d}x \tag{5.131}$$

for every $\zeta \in X$, where $\Gamma(x,t)$ has the regularity of the worst term. Estimates (5.112), (5.122), (A.11), (B.6), (B.9) and (B.30) yield

$$|\Gamma| \lesssim |\nabla_s u|^2 + |\nabla_s u_t|^2 + |\nabla p|^2 + |p_t|,$$

which from (5.124), (5.126), (5.128) and (5.130) implies

$$\|\Gamma\|_{4/3+b/3} \le C, \qquad \|\Gamma\|_{6(1+b)/(7+b)\to 2} \le C \tag{5.132}$$

independently of R, with b as in Hypothesis 5.1.

Assume now that for some $p_0 \ge 8/3 + 2b/3$ we have proved

$$\|\hat{\theta}\|_{p_0} \le C. \tag{5.133}$$

We know that this is true for $p_0 = 8/3 + 2b/3$ by virtue of (5.78). Set

$$r_0 = \frac{1+b}{4+b}\,p_0, \tag{5.134}$$

and set $\zeta = \hat{\theta}^{r_0}$ in (5.131). We obtain

$$\int_\Omega \Big(\mathcal{C}_V(\theta)_t\,\hat{\theta}^{r_0} + \kappa(\hat{\theta})\nabla\theta\cdot\nabla\hat{\theta}^{r_0}\Big)\,\mathrm{d}x + \int_{\partial\Omega}\omega(x)(\theta - \theta^*)\,\hat{\theta}^{r_0}\,\mathrm{d}s(x) = \int_\Omega \Gamma\,\hat{\theta}^{r_0}\,\mathrm{d}x. \tag{5.135}$$

It holds

$$\int_\Omega \mathcal{C}_V(\theta)_t\,\hat{\theta}^{r_0}\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega F_{r_0}(\theta)\,\mathrm{d}x$$

where

$$F_{r_0}(\theta) := \int_0^\theta c_V(s)(Q_R(s))^{r_0}\,\mathrm{d}s,$$

and by Hypothesis 5.1 (ix) also

$$\int_\Omega \kappa(\hat\theta)\nabla\theta\cdot\nabla\hat\theta^{r_0}\,\mathrm{d}x = \int_\Omega \kappa(\hat\theta)\,r_0\,\hat\theta^{r_0-1}\nabla\theta\cdot\nabla\hat\theta\,\mathrm{d}x \ge r_0\kappa^\flat\int_\Omega \hat\theta^{r_0+a}|\nabla\hat\theta|^2\,\mathrm{d}x.$$

Concerning the boundary term, we use Young's inequality with exponents $\left(\frac{r_0+1}{r_0}\,,\,r_0+1\right)$, and obtain

$$\begin{aligned}&\int_{\partial\Omega}\omega(x)(\theta-\theta^*)\hat\theta^{r_0}\,\mathrm{d}s(x) \ge \int_{\partial\Omega}\omega(x)\hat\theta^{r_0+1}\,\mathrm{d}s(x) - \int_{\partial\Omega}\omega(x)\theta^*\hat\theta^{r_0}\,\mathrm{d}s(x)\\ &\ge \int_{\partial\Omega}\omega(x)\hat\theta^{r_0+1}\,\mathrm{d}s(x) - \frac{r_0}{r_0+1}\int_{\partial\Omega}\omega(x)\hat\theta^{r_0+1}\,\mathrm{d}s(x) - \frac{1}{r_0+1}\int_{\partial\Omega}\omega(x)(\theta^*)^{r_0+1}\,\mathrm{d}s(x)\\ &\ge \frac{1}{r_0+1}\int_{\partial\Omega}\omega(x)\hat\theta^{r_0+1}\,\mathrm{d}s(x) - C\end{aligned}$$

by Hypothesis 5.1 (iii) and (iv). We now integrate (5.135) in time $\int_0^\tau \mathrm{d}t$ for some $\tau\in[0,T]$. Observe that $F_{r_0}(\theta)\ge\frac{\hat\theta^{r_0+1+b}}{r_0+1+b}$ by Hypothesis 5.1 (viii). Moreover, thanks to the choice (5.134) and Hölder's inequality with exponents $\left(\frac{4+b}{3}\,,\,\frac{4+b}{1+b}\right)$, the right-hand side is such that

$$\int_0^\tau\int_\Omega \Gamma\,\hat\theta^{r_0}\,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau\int_\Omega \Gamma\,(\hat\theta^{p_0})^{(1+b)/(4+b)}\,\mathrm{d}x\,\mathrm{d}t \le \|\Gamma\|_{(4+b)/3}\,\|\hat\theta\|_{p_0}\le C$$

by estimates (5.132) and (5.133). Hence we have obtained

$$\begin{aligned}&\frac{1}{r_0+1+b}\int_\Omega \hat\theta^{r_0+1+b}(x,\tau)\,\mathrm{d}x + r_0\int_0^\tau\int_\Omega \hat\theta^{r_0+a}|\nabla\hat\theta|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &+\frac{1}{r_0+1}\int_0^\tau\int_{\partial\Omega}\omega(x)\hat\theta^{r_0+1}(x,t)\,\mathrm{d}s(x)\,\mathrm{d}t\le C.\end{aligned}\tag{5.136}$$

We now denote

$$p = 1+\frac{r_0+a}{2}\,,\qquad s=\frac{r_0+1+b}{p}\,,$$

and rewrite (5.136) as

$$\frac{1}{r_0+1+b}\int_\Omega \hat\theta^{ps}(x,\tau)\,\mathrm{d}x + \frac{1}{r_0+1+b}\frac{r_0(r_0+1+b)}{p^2}\int_\Omega p^2\hat\theta^{2p-2}|\nabla\hat\theta|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\le C$$

where we neglected the positive boundary term. Introducing the new variable $v=\hat\theta^p$ this is equivalent to

$$\int_\Omega |v|^s(x,\tau)\,\mathrm{d}x + \int_0^\tau\int_\Omega |\nabla v|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\le C(r_0+1+b),\tag{5.137}$$

since

$$\frac{r_0(r_0+1+b)}{p^2}\ge\frac{2r_0(2r_0+3)}{(r_0+3)^2}\ge C$$

for some $C>0$ when r_0 is far away from zero. For $s<q<6$ and $t\in(0,T)$ we have, by virtue of the Gagliardo-Nirenberg inequality (A.8) with $p=2$ and $N=3$,

$$|v(t)|_q\le C\left(|v(t)|_s + |v(t)|_s^{1-\varrho}|\nabla v(t)|_2^\varrho\right),\qquad \varrho=\frac{\frac{1}{s}-\frac{1}{q}}{\frac{1}{s}-\frac{1}{6}}.$$

If q is chosen in such a way that $\varrho q = 2$, that is,

$$q = \frac{2}{3}s + 2,$$

then integrating in time from 0 to T and employing Young's inequality with exponents $\left(\frac{q}{q-2}\,,\,\frac{q}{2}\right)$ we obtain

$$\|v\|_q \le C\left(\sup_{t\in[0,T]}|v(t)|_s + \sup_{t\in[0,T]}|v(t)|_s^{(q-2)/q}\|\nabla v\|_2^{2/q}\right) \le C\left(\sup_{t\in[0,T]}|v(t)|_s + \|\nabla v\|_2\right).$$

Estimate (5.137) yields

$$\sup_{t\in[0,T]}|v(t)|_s \le C(r_0+1+b)^{1/s},$$

$$\|\nabla v\|_2 \le C(r_0+1+b)^{1/2},$$

so that $\|v\|_q \le C(r_0+1+b)$. Coming back to the variable $\hat\theta$, we have proved that

$$\|\hat\theta\|_{p_1} \le C(r_0+1+b) \qquad \text{for } p_1 = pq = p\left(\frac{2}{3}s+2\right) = \frac{2}{3}(r_0+1+b)+2+r_0+a$$
$$= \frac{5r_0}{3}+\frac{8}{3}+a+\frac{2b}{3} = \frac{5(1+b)p_0}{3(4+b)}+\frac{8}{3}+a+\frac{2b}{3}.$$

We now proceed by induction according to the rule

$$p_{j+1} = \frac{5(1+b)p_j}{3(4+b)}+\frac{8}{3}+a+\frac{2b}{3}, \qquad r_j = \frac{(1+b)p_j}{(4+b)}.$$

We have $\lim_{j\to\infty} p_j = \frac{(8+3a+2b)(4+b)}{7-2b}$. After finitely many iterations we obtain, using also (5.136),

$$\|\hat\theta\|_{\bar p} + \sup_{t\in[0,T]}\mathrm{ess}\,|\hat\theta(t)|_{\bar r+1+b} \le C, \qquad \text{for every } \bar p < \frac{(8+3a+2b)(4+b)}{7-2b}, \quad \bar r = \frac{(1+b)\bar p}{(4+b)} > \hat a, \tag{5.138}$$

with the constant $\hat a$ introduced in Hypothesis 5.1 (ix). We now come back to (5.131), which we test by $\zeta = \theta$ (note that this is an admissible choice by Proposition 5.4). It holds

$$\int_\Omega \mathcal{C}_V(\theta)_t\theta(x,t)\,\mathrm{d}x = \int_\Omega c_V(\theta)\theta\theta_t(x,t)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\int_0^{\theta(x,t)} c_V(s)s\,\mathrm{d}s\right)\mathrm{d}x,$$

hence from Hypothesis 5.1 (ix) and (5.132) we obtain, after a time integration,

$$\begin{aligned}&\int_\Omega \theta^{2+b}(x,\tau)\,\mathrm{d}x + \int_0^T\int_\Omega \kappa(\hat\theta)|\nabla\theta|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^T\int_{\partial\Omega}\omega(x)\theta^2(x,t)\,\mathrm{d}s(x)\,\mathrm{d}t \le C\|\theta\|_{(4+b)/(1+b)}\end{aligned} \tag{5.139}$$

where we handled the boundary term by means of Young's inequality and Hypothesis 5.1 (iii) and (iv). Using the Gagliardo-Nirenberg inequality (A.8) again with $q = \frac{4+b}{1+b}$, $s = 1+b$ (note that $1+b < \frac{4+b}{1+b} < 6$ under our hypotheses), $p = 2$ and $N = 3$ we have that, for each fixed $t \in (0,T)$,

$$|\theta(t)|_{(4+b)/(1+b)} \le C\left(|\theta(t)|_{1+b} + |\theta(t)|_{1+b}^{1-\varrho}|\nabla\theta(t)|_2^{\varrho}\right) \le C\left(1+|\nabla\theta(t)|_2^{\varrho}\right)$$

with $\varrho = \frac{6(3-b^2-b)}{(5-b)(4+b)}$ and where we used estimate (5.72). Raising to the power $(4+b)/(1+b)$, integrating $\int_0^T \mathrm{d}t$ and using Hölder's inequality in time (note that $\frac{2(1+b)}{\rho(4+b)} \geq 1$ under our hypotheses) we get

$$\begin{aligned}
\|\theta\|_{(4+b)/(1+b)} &\leq C\left(1+\int_0^T\left(\int_\Omega |\nabla\theta|^2\,\mathrm{d}x\right)^{\rho(4+b)/2(1+b)}\mathrm{d}t\right)^{(1+b)/(4+b)}\\
&\leq C\left(1+\left(\int_0^T\int_\Omega |\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\rho(4+b)/2(1+b)}\right)^{(1+b)/(4+b)}\\
&\leq C\left(1+\left(\int_0^T\int_\Omega \kappa(\hat\theta)|\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\rho/2}\right).
\end{aligned}$$

Plugging this back into (5.139) and using Young's inequality we deduce

$$\int_\Omega \theta^{2+b}(x,\tau)\,\mathrm{d}x + \int_0^T\int_\Omega \kappa(\hat\theta)|\nabla\theta|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\partial\Omega}\omega(x)\theta^2(x,t)\,\mathrm{d}s(x)\,\mathrm{d}t \leq C. \tag{5.140}$$

This enables us to derive an upper bound for the integral $\int_\Omega \kappa(\hat\theta)\nabla\theta\cdot\nabla\zeta\,\mathrm{d}x$, which we need for getting an estimate for θ_t from equation (5.131). By Hölder's inequality and Hypothesis 5.1 (ix) we have that

$$\begin{aligned}
\int_\Omega |\kappa(\hat\theta)\nabla\theta\cdot\nabla\zeta|\,\mathrm{d}x &= \int_\Omega |\kappa^{1/2}(\hat\theta)\nabla\theta\cdot\kappa^{1/2}(\hat\theta)\nabla\zeta|\,\mathrm{d}x\\
&\leq C\left(\int_\Omega \kappa(\hat\theta)|\nabla\theta|^2\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega \max\{1,\hat\theta^{1+\hat a}\}|\nabla\zeta|^2\,\mathrm{d}x\right)^{1/2}.
\end{aligned} \tag{5.141}$$

Let us now choose $\hat q > 1$ such that $(1+\hat a)\hat q = 1+\bar r+b$, where $\bar r$ is defined in (5.138). Note that such a $\hat q$ exists since $1+\bar r+b > 1+\hat a+b > 1+\hat a$. Defining

$$q^* := \frac{2\hat q}{\hat q-1} = 2+\frac{2}{\hat q-1} > 2\,, \tag{5.142}$$

we get from Hölder's inequality with conjugate exponents $\left(\hat q\,,\,\frac{q^*}{2}\right)$ that

$$\int_\Omega \hat\theta^{1+\hat a}|\nabla\zeta|^2\,\mathrm{d}x \leq \left(\int_\Omega \hat\theta^{1+\bar r+b}\,\mathrm{d}x\right)^{1/\hat q}\left(\int_\Omega |\nabla\zeta|^{q^*}\,\mathrm{d}x\right)^{2/q^*} \leq C\left(\int_\Omega |\nabla\zeta|^{q^*}\,\mathrm{d}x\right)^{2/q^*}$$

by virtue of (5.138). Inequality (5.141) then yields the bound

$$\int_\Omega |\kappa(\hat\theta)\nabla\theta\cdot\nabla\zeta|\,\mathrm{d}x \leq C\left(\int_\Omega \kappa(\hat\theta)|\nabla\theta|^2\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega |\nabla\zeta|^{q^*}\,\mathrm{d}x\right)^{1/q^*}.$$

Hence, by (5.140),

$$\int_0^T\int_\Omega |\kappa(\hat\theta)\nabla\theta\cdot\nabla\zeta|\,\mathrm{d}x\,\mathrm{d}t \leq C\|\zeta\|_{L^2(0,T;W^{1,q^*}(\Omega))}.$$

From (5.132) it follows that testing with $\zeta \in L^2(0,T;W^{1,q^*}(\Omega))$ is admissible, in the sense that the term $\Gamma\zeta$ is integrable. Indeed, the Sobolev exponent of q^* is

$$q^*_S = \frac{3q^*}{3-q^*} = \frac{6\hat q}{\hat q-3}\,,$$

from which

$$(q_S^*)' = \frac{1}{1-\frac{1}{q_S^*}} = \frac{6\hat{q}}{5\hat{q}+3} = \frac{6}{5} - \frac{18}{5(5\hat{q}+3)} < \frac{6}{5} \leq \frac{6(1+b)}{7+b}$$

for all $b \in [1/2{,}1)$. We thus obtain from (5.131) that

$$\int_0^T \int_\Omega \theta_t \zeta \,\mathrm{d}x\,\mathrm{d}t \leq C\|\zeta\|_{L^2(0,T;W^{1,q^*}(\Omega))}. \tag{5.143}$$

5.3 Passage to the limit

In this section we conclude the proof of Theorem 5.3 by passing to the limit in (5.115)–(5.118) as $R \to \infty$. Most of the convergences can be handled as at the end of Subsection 5.1.2, hence we focus here on the main differences.

Let $R_i \nearrow \infty$ be a sequence such that $R_i > R_\sigma$, with R_σ as in (5.112), and let $(p, u, \chi, \theta) = (p^{(i)}, u^{(i)}, \chi^{(i)}, \theta^{(i)})$ be solutions of (5.115)–(5.118) corresponding to $R = R_i$, with $\hat{\theta} = \hat{\theta}^{(i)} = Q_{R_i}(\theta^{(i)})$ and test functions $\phi, \zeta \in X$, $\psi \in X_0$. Our aim is to check that at least a subsequence converges as $i \to \infty$ to a solution of (5.1)–(5.4) with test functions $\phi \in X$, $\psi \in X_0$ and $\zeta \in X_{q^*}$.

First, for the capillary pressure $p = p^{(i)}$ we have the estimates (5.112) and (5.123)–(5.126), which imply that, passing to a subsequence if necessary,

$$\begin{aligned}
p_t^{(i)} &\to p_t && \text{weakly in } L^2(\Omega \times (0,T)),\\
p^{(i)} &\to p && \text{strongly in } L^q(\Omega; C[0,T]) \text{ for all } q \in [1,\infty),\\
\nabla p^{(i)} &\to \nabla p && \text{strongly in } L^q(\Omega \times (0,T); \mathbb{R}^3) \text{ for all } q \in \left[1, \tfrac{10}{3}\right),
\end{aligned}$$

where we used also Theorem A.4. We easily show that

$$Q_{R_i}(|\nabla p^{(i)}|^2) \to |\nabla p|^2 \quad \text{strongly in } L^q(\Omega \times (0,T); \mathbb{R}^3) \text{ for all } q \in \left[1\,, \frac{5}{3}\right). \tag{5.144}$$

Indeed, let $\Omega_T^{(i)} \subset \Omega \times (0,T)$ be the set of all $(x,t) \in \Omega \times (0,T)$ such that $|\nabla p^{(i)}(x,t)|^2 > R_i$. By (5.126) we have

$$C \geq \int_0^T \int_\Omega |\nabla p^{(i)}(x,t)|^{10/3} \,\mathrm{d}x\,\mathrm{d}t \geq \iint_{\Omega_T^{(i)}} |\nabla p^{(i)}(x,t)|^{10/3} \,\mathrm{d}x\,\mathrm{d}t \geq |\Omega_T^{(i)}| R_i^{5/3},$$

hence $|\Omega_T^{(i)}| \leq C R_i^{-5/3}$. For $q < \frac{5}{3}$ we use Hölder's inequality to get the estimate

$$\begin{aligned}
\int_0^T \int_\Omega \left| Q_{R_i}(|\nabla p^{(i)}|^2) - |\nabla p^{(i)}|^2 \right|^q \mathrm{d}x\,\mathrm{d}t &= \iint_{\Omega_T^{(i)}} \left| R_i - |\nabla p^{(i)}|^2 \right|^q \mathrm{d}x\,\mathrm{d}t \leq \iint_{\Omega_T^{(i)}} |\nabla p^{(i)}|^{2q} \,\mathrm{d}x\,\mathrm{d}t\\
&\leq \left(\iint_{\Omega_T^{(i)}} |\nabla p^{(i)}|^{10/3} \,\mathrm{d}x\,\mathrm{d}t \right)^{3q/5} |\Omega_T^{(i)}|^{1-3q/5} \leq C R^{-(5-3q)/3},
\end{aligned}$$

and (5.144) follows.

For the temperature $\theta = \theta^{(i)}$ we proceed in a similar way. By estimates (5.140) and (5.143) we obtain

$$\begin{array}{rcll} \nabla\theta^{(i)} & \to & \nabla\theta & \text{weakly in } L^2(\Omega\times(0,T);\mathbb{R}^3), \\ \theta_t^{(i)} & \to & \theta_t & \text{weakly in } L^2(0,T;W^{-1,q*}(\Omega)), \\ \theta^{(i)} & \to & \theta & \text{strongly in } L^2(\Omega\times(0,T)), \end{array}$$

where for the last estimate we exploited Lemma A.6 with $B_0 = W^{1,2}(\Omega)$, $B = L^2(\Omega)$, $B_1 = W^{-1,2}(\Omega)$, $p_0 = p_1 = 2$ and the embedding $W^{-1,q^*}(\Omega) \hookrightarrow W^{-1,2}(\Omega)$ (recall that $q^* > 2$). Furthermore, estimate (5.138) entails that $\hat{\theta}^{(i)}$ are uniformly bounded in $L^q(\Omega\times(0,T))$ for every $q < \frac{(8+3a+2b)(4+b)}{7-2b}$. Hence a similar argument as above yields that

$$\hat{\theta}^{(i)} = Q_{R_i}(\theta^{(i)}) \to \theta \quad \text{strongly in } L^q(\Omega\times(0,T)) \text{ for all } q \in \left[1\,,\, \frac{(8+3a+2b)(4+b)}{7-2b}\right).$$

The strong convergences $\nabla_s u^{(i)} \to \nabla_s u$, $\nabla_s u_t^{(i)} \to \nabla_s u_t$, $\chi^{(i)} \to \chi$, $\chi_t^{(i)} \to \chi_t$ follow as at the end of Subsection 5.1.2, as well as the convergence of the hysteresis terms.

Therefore the limit as $i \to \infty$ yields a solution to (5.1)–(5.4), and the proof of Theorem 5.3 is completed.

Part III

Fatigue and phase transition in an oscillating elastoplastic beam

Introduction

This third part deals with a model for the cyclic fatigue accumulation in a transversally oscillating elastoplastic beam. Temperature and phase transition effects, which usually accompany the process, are taken into account.

The model we present here is part of a project which started some years ago from the scientific collaboration between M. Eleuteri, P. Krejčí and J. Kopfová, with the aim of deriving and studying models for the cyclic fatigue accumulation in transversally oscillating bodies (beams and plates).

Elastoplastic materials subject to cyclic loading exhibit increasing fatigue, which is manifested by material softening and material failure in finite time with strong heat release. The analysis of the so-called *rainflow method of cyclic fatigue accumulation* in elastoplastic materials carried out in [23] has shown a qualitative and quantitative relationship between accumulated fatigue and dissipated energy, similarly as in [63]. Starting from these facts, a first step in the direction of investigating the well-posedness of the system describing nonisothermal fatigue accumulation in a transversally oscillating elastoplastic beam was made in the papers [48, 50]. Here the main modeling hypothesis is that the fatigue accumulation rate is proportional to the dissipation rate. The model is also based on the results contained in [101], where by means of the Kirchhoff-Love method it was shown that the 3D problem of transversal oscillations of a solid elastoplastic beam with the single-yield von Mises plasticity law can be transformed, after dimensional reduction, into the beam equation with a multi-yield hysteresis *Prandtl-Ishlinskiĭ constitutive operator* (see Section B.4 in the Appendix). This can be explained by the fact that in the 1D model only deformations of longitudinal fibres parameterized by the transversal coordinate are taken into account, and the individual fibres do not switch from the elastic to the plastic regime at the same time. More precisely, eccentric layers are subject to larger deformations than the central ones, so that plastic yielding propagates gradually from the outer surface towards the midsurface (see Figure 5.1). This is translated into the mathematical language by means of the Prandtl–Ishlinskiĭ combination of elastic–perfectly plastic elements with different yield limits that are not all simultaneously activated.

In the more recent work [52], still relying on the Prandtl-Ishlinskiĭ formalism, the previous models for transversally oscillating beams were extended. The fatigue accumulation law is still based on the

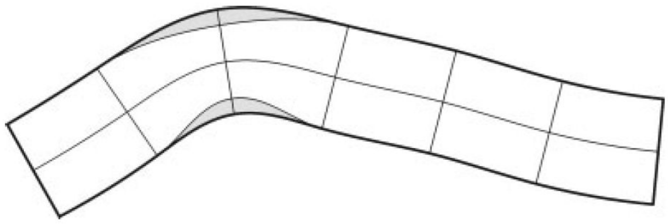

Figure 5.1. Deformed beam with grey plastified zone. Picture taken from [101].

observation that there exists a proportionality between accumulated fatigue and dissipated energy. However, unlike in [50], it is assumed that out of all dissipative components in the energy balance, only the purely plastic dissipation produces damage. This makes the mathematical problem easier: the system of equations then does not develop singularities in finite time and a unique regular solution is proved to exist on every bounded time interval. On the other hand an additional difficulty is here considered, namely, a fatigue-dependent weight function in the definition of the Prandtl-Ishlinskii operator. Furthermore, it is assumed that the material can partially recover by the effect of melting when a solid-liquid phase transition takes place. Thus a differential inclusion for the phase dynamics completes the system of equations, for which existence and uniqueness of a strong solution are proved to hold.

Results have been obtained correspondingly also for the plate. We mention the papers [49], which deals with the simplified situation of fixed temperature, and [51], where existence of solutions is proved for the nonisothermal model. In [47] the dependence of the plastic dissipation on both the temperature and the fatigue parameter was considered in the 2D case. It was also mentioned that in principle it makes sense from the point of view of modeling and applications to allow a further dependence of the Prandlt-Ishlinskiĭ density on the phase parameter, and it was shown, assuming a new special flow rule for the phase variable, that the resulting model is still thermodynamically consistent.

Here we start the mathematical analysis of this new problem in the 1D case. We take into account the possibility of partial fatigue recovery by the effect of melting as in [52], and additionally allow the dependence of the Prandtl-Ishlinskiĭ density function not only on the fatigue parameter, but also on the phase variable. The aim is to show existence and uniqueness of a strong solution of the underlying system of equations, which brings nontrivial mathematical difficulties. The results are contained in the paper [42].

The plan is the following: in Chapter 6 we derive the model; Chapter 7 contains the statement of our main result, whose existence part is proved in Sections 7.1–7.3, whereas Section 7.4 deals with the uniqueness.

CHAPTER 6

A model for an oscillating beam

6.1 Derivation of the model

In this chapter we deal with a transversally inhomogeneous beam of length 1. We let $x \in [0,1]$ be the longitudinal variable and $t \in [0,T]$ the time variable, and denote by

$m(x,t)$	fatigue accumulation parameter;
$\chi(x,t) \in [0,1]$	phase fraction: $\chi = 0$ solid, $\chi = 1$ liquid, $\chi \in (-1,1)$ mixture;
$w(x,t)$	transversal displacement of the point x at time t;
$\varepsilon(x,t) = w_{xx}(x,t)$	linearized curvature;
$\sigma(x,t)$	bending moment;
$\theta(x,t) > 0$	absolute temperature.

As we have already outlined, the main novelty is the dependence on χ of the Prandtl-Ishlinskiĭ constitutive operator of elastoplasticity. Starting from the basic model presented in Section B.4 in the Appendix and given $m, \chi, \varepsilon : \Omega \times (0,T) \to \mathbb{R}$ such that for a. e. $x \in \Omega$ it holds $\varepsilon(x,\cdot) \in W^{1,1}(0,T)$, we set

$$P_0[m,\chi,\varepsilon](x,t) = P_0[\tilde{\gamma}_{(x)}(\cdot,r), \varepsilon(x,\cdot)](t) \qquad \forall\, t \in [0,T],$$

where $\tilde{\gamma}_{(x)} : [0,T] \times (0,\infty) \to [0,\infty)$ is defined by

$$\tilde{\gamma}_{(x)}(t,r) = \gamma(m(x,t), \chi(x,t), r). \tag{6.1}$$

In other words, we define the fatigue and phase dependent Prantdl-Ishlinskiĭ operator by the integral

$$P_0[m,\chi,\varepsilon] = \int_0^\infty \gamma(m,\chi,r)\, \mathfrak{s}_r[\varepsilon]\, \mathrm{d}r. \tag{6.2}$$

Here $\mathfrak{s}_r$ denotes the stop operator from (B.14)–(B.15) with thresholds $-r, r$ and with the canonical initial condition $s^{r,0} = s^r(0) = Q_r(\varepsilon(0))$, see Remark B.4. In this context, the energy balance (B.34) becomes

$$\varepsilon_t P_0[m,\chi,\varepsilon] = \frac{\mathrm{d}}{\mathrm{d}t} V[m,\chi,\varepsilon] + D[m,\chi,\varepsilon]$$
$$- \frac{1}{2}\int_0^\infty \Big(\gamma_m(m,\chi,r)\, m_t + \gamma_\chi(m,\chi,r)\,\chi_t\Big)\, \mathfrak{s}_r^2[\varepsilon](t)\,\mathrm{d}r \qquad \text{a.e. in } \Omega\times(0,T),$$

where

$$V[m,\chi,\varepsilon] = \frac{1}{2}\int_0^\infty \gamma(m,\chi,r)\,\mathfrak{s}_r^2[\varepsilon]\,\mathrm{d}r \tag{6.3}$$
$$D[m,\chi,\varepsilon] = \int_0^\infty r\gamma(m,\chi,r)|\mathfrak{p}_r[\varepsilon]_t|\,\mathrm{d}r \tag{6.4}$$

are the new fatigue and phase dependent Prandtl-Ishlinskiĭ potential and dissipation operators.

6.1.1 Momentum balance

We assume a thermo-visco-elastoplastic scalar constitutive relation in the form

$$\sigma = B\varepsilon + P_0[m,\chi,\varepsilon] + \nu\varepsilon_t - \beta(\theta-\theta_{\mathrm{ref}}), \tag{6.5}$$

with $B>0$ elastic modulus, ν viscosity coefficient and β thermal expansion coefficient related to a layered structure of the beam. Moreover θ_{ref} is the melting temperature, which is considered as a fixed referential temperature.

Following [101], Newton's law of motion is formally written as

$$\mu w_{tt} - \alpha w_{xxtt} + \sigma_{xx} = F(x,t), \tag{6.6}$$

where $\alpha = \mu l^2/12$ with $l>0$ thickness of the beam and μ mass density, assumed to be constant, and where F is the external load.

6.1.2 Phase and fatigue evolution

The evolution of the phase variable χ is assumed to be of "phase-relaxation" type

$$-\rho\chi_t \in \partial I_{[0,1]}(\chi) - \frac{L}{\theta_{\mathrm{ref}}}(\theta-\theta_{\mathrm{ref}}) + \frac{1}{2}\int_0^\infty \gamma_\chi(m,\chi,r)\,\mathfrak{s}_r^2[\varepsilon]\,\mathrm{d}r \tag{6.7}$$

where $\rho>0$ is the relaxation coefficient, $L>0$ is the latent heat of the process and $\partial I_{[0,1]}$ is the subdifferential of the indicator function $I_{[0,1]}$. Indeed, we necessarily have $\chi\in[0,1]$, and we interpret $\chi=0$ as the solid phase, $\chi=1$ as liquid, and the intermediate values correspond to the relative liquid content in a mixture of the two.

We now turn our attention to the derivation of a law for the evolution of the fatigue parameter m. As already mentioned, there is a close relationship between accumulated fatigue and dissipated energy. Here, following [52], we assume that out of all dissipative components in the energy balance, only the purely plastic dissipation produces damage. This prevents the system of equations from developing singularities in finite time, see [50]. Still following [52], we assume in addition that partial recovery of the damaged material is possible under strong local melting. Mathematically, this is expressed in terms of the evolution equation for the fatigue variable m

$$m_t \in -\partial I_{[0,\infty)}(m)(x,t) - h(\chi_t(t)) + \int_0^1 \lambda(x-y) D[m,\chi,\varepsilon](y,t)\,\mathrm{d}y, \tag{6.8}$$

where h is a nonnegative nondecreasing function vanishing for negative arguments, see Hypothesis 7.1, λ is a nonnegative smooth function with (small) compact support and $D[m,\chi,\varepsilon]$ is the dissipation operator defined in (6.4) associated with the Prandtl-Ishlinskiĭ operator $P_0[m,\chi,\varepsilon]$. The subdifferential $\partial I_{[0,\infty)}$ of the indicator function $I_{[0,\infty)}$ ensures that the fatigue parameter remains nonnegative.
The meaning of (6.8) is simple. If no phase transition takes place or if the material solidifies, that is, $\chi_t \le 0$, then fatigue at a point x increases proportionally to the energy dissipated in a neighborhood of the point x. On the other hand, under strong melting if χ grows faster than the plastic dissipation rate, the fatigue may decrease until it possibly reaches the unperturbed state $m = 0$.
As we have already seen in Chapters 3 and 4, we can interpret (6.7) and (6.8) for the phase variable χ and fatigue variable m, with a choice $\chi^0(x) \in [0,1]$, $m^0(x) \ge 0$ of initial conditions, in an equivalent way in the form

$$\chi(x,t) = \mathfrak{s}_{[0,1]}[A(x,\cdot),\chi^0(x)](t), \tag{6.9}$$

$$m(x,t) = \mathfrak{s}_{[0,\infty)}[S(x,\cdot),m^0(x)](t), \tag{6.10}$$

where

$$A(x,t) := \int_0^t \frac{1}{\rho}\left(\frac{L}{\theta_{\mathrm{ref}}}(\theta-\theta_{\mathrm{ref}}) - \frac{1}{2}\int_0^\infty \gamma_\chi(m,\chi,r)\mathfrak{s}_r^2[\varepsilon](t)\,\mathrm{d}r\right)(x,\tau)\,\mathrm{d}\tau, \tag{6.11}$$

$$S(x,t) := \int_0^t \left(-h(\chi_t(\tau)) + \int_0^1 \lambda(x-y)D[m,\chi,\varepsilon](y,\tau)\,\mathrm{d}y\right)(x,\tau)\,\mathrm{d}\tau. \tag{6.12}$$

The advantage of this representation is that now χ and m are defined by equations involving, by virtue of Proposition B.5, only operators that are Lipschitz continuous in $W^{1,1}(0,T)$.

6.1.3 Energy balance

By the first principle of thermodynamics we have that the internal energy U of the system must be conserved in the following sense

$$U(m,\chi,\varepsilon)_t + q_x = \sigma\varepsilon_t + g, \tag{6.13}$$

where $q = -\kappa\theta_x$ is the heat flux with a constant heat conductivity $\kappa > 0$, and g is the heat source density. Moreover, according to the second principle of thermodynamics, there exists a state function

S called the entropy which is nondecreasing in the sense of the Clausius-Duhem inequality

$$S(m,\chi,\varepsilon)_t + \left(\frac{q}{\theta}\right)_x \geq \frac{g}{\theta}\,. \tag{6.14}$$

We claim that, if the phase dynamics is chosen in the form (6.7) and the fatigue variable m satisfies the evolution equation (6.8), then the right choice of the free energy $F = U - \theta S$ for the system to be thermodynamically consistent, under the assumption of constant heat capacity c and latent heat L, is given by

$$F(m,\chi,\varepsilon) = c\theta\left(1 - \log\left(\frac{\theta}{\theta_{\text{ref}}}\right)\right) + \frac{B}{2}\varepsilon^2 - \beta(\theta-\theta_{\text{ref}})\varepsilon + V[m,\chi,\varepsilon] - \frac{L}{\theta_{\text{ref}}}(\theta-\theta_{\text{ref}})\chi + I_{[0,1]}(\chi) \tag{6.15}$$

where $V[m,\chi,\varepsilon]$ is the Prandtl-Ishlinskiĭ potential introduced in (6.3). As already explained in Subsection 4.1.3, a formula for the free energy can be derived from the constitutive relation (6.5) by a "formal" integration. However, we prefer to follow an "a posteriori" (and more rigorous) approach and use the constitutive relation and the energy balance for P_0 to prove that for F as in (6.15) the Clausius-Duhem inequality (6.14) is satisfied.

From (6.15) it follows that the entropy operator S and internal energy operator U have the form

$$S(m,\chi,\varepsilon) = -\frac{\partial F}{\partial \theta} = c\log\left(\frac{\theta}{\theta_{\text{ref}}}\right) + \beta\varepsilon + \frac{L}{\theta_{\text{ref}}}\chi, \tag{6.16}$$

$$\begin{aligned} U(m,\chi,\varepsilon) &= F(m,\chi,\varepsilon) + \theta S(m,\chi,\varepsilon) \\ &= c\theta + \frac{B}{2}\varepsilon^2 + \beta\theta_{\text{ref}}\varepsilon + V[m,\chi,\varepsilon] + L\chi + I_{[0,1]}(\chi). \end{aligned} \tag{6.17}$$

Note that

$$\left(\frac{q}{\theta}\right)_x = \frac{q_x\theta - q\theta_x}{\theta^2} = \frac{q_x}{\theta} + \frac{\kappa\theta_x^2}{\theta^2},$$

hence

$$S(m,\chi,\varepsilon)_t + \left(\frac{q}{\theta}\right)_x - \frac{g}{\theta} = \frac{c\theta_t}{\theta} + \beta\varepsilon_t + \frac{L}{\theta_{\text{ref}}}\chi_t + \frac{q_x}{\theta} + \frac{\kappa\theta_x^2}{\theta^2} - \frac{g}{\theta}. \tag{6.18}$$

By (6.13) it follows

$$\frac{q_x}{\theta} - \frac{g}{\theta} = \frac{1}{\theta}\big(\sigma\varepsilon_t - U(m,\chi,\varepsilon)_t\big)$$

which, together with the constitutive relation (6.5) and the energy balance (6.3), yields

$$\begin{aligned} \frac{q_x}{\theta} - \frac{g}{\theta} &= \frac{1}{\theta}\bigg(B\varepsilon\varepsilon_t + \varepsilon_t P_0[m,\chi,\varepsilon] + \nu\varepsilon_t^2 - \beta(\theta-\theta_{\text{ref}})\varepsilon_t - U(m,\chi,\varepsilon)_t\bigg) \\ &= \frac{1}{\theta}\bigg(B\varepsilon\varepsilon_t + \nu\varepsilon_t^2 - \beta(\theta-\theta_{\text{ref}})\varepsilon_t - U(m,\chi,\varepsilon)_t + V[m,\chi,\varepsilon]_t + D[m,\chi,\varepsilon] \\ &\quad - \frac{1}{2}\int_0^\infty \Big(\gamma_m(m,\chi,r)\,m_t + \gamma_\chi(m,\chi,r)\,\chi_t\Big)\mathfrak{s}_r^2[\varepsilon]\,\mathrm{d}r\bigg). \end{aligned}$$

Plugging it into (6.18) gives

$$\begin{aligned}
&S(m,\chi,\varepsilon)_t + \Big(\frac{q}{\theta}\Big)_x - \frac{g}{\theta} = \frac{c\theta_t}{\theta} + \beta\varepsilon_t + \frac{L}{\theta_{\mathrm{ref}}}\chi_t + \frac{\kappa\theta_x^2}{\theta^2} \\
&+ \frac{1}{\theta}\Bigg(B\varepsilon\varepsilon_t + \nu\varepsilon_t^2 - \beta(\theta-\theta_{\mathrm{ref}})\varepsilon_t - U(m,\chi,\varepsilon)_t + V[m,\chi,\varepsilon]_t + D[m,\chi,\varepsilon] \\
&- \frac{1}{2}\int_0^\infty \Big(\gamma_m(m,\chi,r)\, m_t + \gamma_\chi(m,\chi,r)\,\chi_t\Big)\mathfrak{s}_r^2[\varepsilon]\,\mathrm{d}r\Bigg).
\end{aligned}$$

A direct computation of $U(m,\chi,\varepsilon)_t$, with U from (6.17), yields

$$U(m,\chi,\varepsilon)_t = c\theta_t + B\varepsilon\varepsilon_t + \beta\theta_{\mathrm{ref}}\varepsilon_t + V[m,\chi,\varepsilon]_t + L\chi_t + \partial I_{[0,1]}\chi_t, \tag{6.19}$$

where $\partial I_{[0,1]}(\chi)\chi_t = 0$ (see Remark B.2). Hence inserting (6.19) in (6.19) we obtain, after some cancellations,

$$\begin{aligned}
&S(m,\chi,\varepsilon)_t + \Big(\frac{q}{\theta}\Big)_x - \frac{g}{\theta} \\
&= \frac{\kappa\theta_x^2}{\theta^2} + \frac{\nu\varepsilon_t^2}{\theta} + \frac{1}{\theta}\Bigg(\frac{L}{\theta_{\mathrm{ref}}}(\theta-\theta_{\mathrm{ref}})\,\chi_t + D[m,\chi,\varepsilon] - \frac{1}{2}\int_0^\infty \Big(m_t\gamma_m(m,\chi,r) + \chi_t\gamma_\chi(m,\chi,r)\Big)\mathfrak{s}_r^2[\varepsilon]\,\mathrm{d}r\Bigg).
\end{aligned}$$

Multiplying the differential inclusion (6.7) by χ_t we obtain (see Remark B.2)

$$-\rho\chi_t^2 = -\frac{L}{\theta_{\mathrm{ref}}}(\theta-\theta_{\mathrm{ref}})\chi_t + \frac{1}{2}\chi_t\int_0^\infty \gamma_\chi(m,\chi,r)\,\mathfrak{s}_r^2[\varepsilon]\,\mathrm{d}r,$$

thus we get

$$S(m,\chi,\varepsilon)_t + \Big(\frac{q}{\theta}\Big)_x - \frac{g}{\theta} = \frac{\kappa\theta_x^2}{\theta^2} + \frac{\nu\varepsilon_t^2}{\theta} + \frac{1}{\theta}\Bigg(\rho\chi_t^2 + D[m,\chi,\varepsilon] - \frac{1}{2}m_t\int_0^\infty \gamma_m(m,\chi,r)\,\mathfrak{s}_r^2[\varepsilon]\,\mathrm{d}r\Bigg). \tag{6.20}$$

In the next chapter we will put suitable hypotheses on the function h appearing in the inclusion (6.8) and on the Prantdl-Ishlinskiĭ density γ, and we will check that the absolute temperature θ stays positive. This will allow us to conclude that the Clausius-Duhem inequality (6.13) holds.

We now rewrite the energy conservation law (6.13) in a more suitable form. Using (6.19), the constitutive law (6.5) and the expression for q, (6.13) becomes

$$c\theta_t + B\varepsilon\varepsilon_t + \beta\theta_{\mathrm{ref}}\varepsilon_t + V[m,\chi,\varepsilon]_t + L\chi_t - \kappa\theta_{xx} = B\varepsilon\varepsilon_t + \varepsilon_t P_0[m,\chi,\varepsilon] + \nu\varepsilon_t^2 - \beta(\theta-\theta_{\mathrm{ref}})\varepsilon_t + g.$$

Note that some terms cancel out. Thus, employing the energy balance (6.3), we obtain

$$\begin{aligned}
c\theta_t - \kappa\theta_{xx} &= -\beta\theta\varepsilon_t + \nu\varepsilon_t^2 + D[m,\chi,\varepsilon] \\
&- \frac{1}{2}\int_0^\infty \Big(m_t\gamma_m(m,\chi,r) + \chi_t\gamma_\chi(m,\chi,r)\Big)\mathfrak{s}_r^2[\varepsilon]\,\mathrm{d}r - L\chi_t + g.
\end{aligned} \tag{6.21}$$

6.2 The mathematical problem

For any $T > 0$ we set

$$\Omega_T = (0,1) \times (0,T)\,, \qquad u(x,t) = \int_0^t \sigma(x,\tau)\,\mathrm{d}\tau\,,$$
$$f(x,t) = \int_0^t F(x,\tau)\,\mathrm{d}\tau + \mu w_t(x,0) - \alpha w_{xxt}(x,0).$$

We rewrite the equations (6.5), (6.6), (6.7), (6.8), (6.21) as the system

$$u_t = B w_{xx} + P_0[m,\chi,w_{xx}] + \nu w_{xxt} - \beta(\theta - \theta_{\mathrm{ref}}), \tag{6.22}$$

$$\mu w_t - \alpha w_{xxt} = -u_{xx} + f(x,t), \tag{6.23}$$

$$c\theta_t - \kappa\theta_{xx} = -\beta\theta\varepsilon_t + \nu\varepsilon_t^2 + D[m,\chi,\varepsilon] \tag{6.24}$$

$$- \frac{1}{2}\int_0^\infty \Big(m_t\gamma_m(m,\chi,r) + \chi_t\gamma_\chi(m,\chi,r)\Big)\mathfrak{s}_r^2[\varepsilon]\,\mathrm{d}r - L\chi_t + g, \tag{6.25}$$

$$-\rho\chi_t \in \partial I_{[0,1]}(\chi) - \frac{L}{\theta_{\mathrm{ref}}}(\theta - \theta_{\mathrm{ref}}) + \frac{1}{2}\int_0^\infty \gamma_\chi(m,\chi,r)\,\mathfrak{s}_r^2[\varepsilon]\,\mathrm{d}r, \tag{6.26}$$

$$m_t \in -\partial I_{[0,\infty)}(m) - h(\chi_t) + \int_0^1 \lambda(x-y)D[m,\chi,\varepsilon](y,t)\,\mathrm{d}y, \tag{6.27}$$

for unknown functions u, w, θ, m, χ, with initial and boundary conditions

$$\left.\begin{aligned} w(x,0) &= u(x,0) = 0,\\ m(x,0) &= m^0(x) = 0,\\ \theta(x,0) &= \theta^0(x),\\ \chi(x,0) &= \chi^0(x), \end{aligned}\right\} \tag{6.28}$$

$$\left.\begin{aligned} &w(0,t) = u(0,t) = w(1,t) = u(1,t) = 0,\\ &\theta_x(0,t) = \theta_x(1,t) = 0. \end{aligned}\right\} \tag{6.29}$$

The zero initial conditions for w and m are motivated by the fact that it is difficult to determine the initial degree of fatigue for a material with unknown loading history, and the most transparent hypothesis consists in assuming that no deformation (and therefore no fatigue) has taken place prior to the time $t = 0$.

CHAPTER 7

Solvability of the problem

In this chapter we are going to state and prove the main result for problem (6.22)–(6.29). The data are required to fulfill the following hypotheses:

Hypothesis 7.1.

(i) P_0 is a Prandtl-Ishlinskiĭ operator (6.2) with a measurable density function $\gamma : [0,\infty) \times [0,1] \times (0,\infty) \to [0,\infty)$, which is locally Lipschitz continuous in the first two variables, with γ_m and γ_χ locally Lipschitz continuous as well, and such that there exist $\tilde{\gamma}, \gamma^* \in L^1(0,\infty)$ with $\gamma(m,\chi,r) \le \tilde{\gamma}(r)$, $0 \le -\gamma_m(m,\chi,r) \le \gamma^*(r)$, $|\gamma_\chi(m,\chi,r)| \le \gamma^*(r)$, $|\gamma_{mm}(m,\chi,r)| \le \gamma^*(r)$, $|\gamma_{\chi m}(m,\chi,r)| \le \gamma^*(r)$, $|\gamma_{\chi\chi}(m,\chi,r)| \le \gamma^*(r)$ a. e. Moreover, $M := \int_0^\infty r\tilde{\gamma}(r)\,\mathrm{d}r < \infty$ and also $\int_0^\infty (1+r^2)\gamma^*(r)\,\mathrm{d}r < 2L$, being L the latent heat of the process.

(ii) $B, \nu, \beta, \theta_{\mathrm{ref}}, \mu, \alpha, c, \kappa, L, \rho$ are given positive constants.

(iii) $\lambda : \mathbb{R} \to [0,\infty)$ is a C^1 function with compact support, and we set $\Lambda := \max\{\lambda(x) + |\lambda'(x)|\,,\ x \in \mathbb{R}\}$.

(iv) $f \in L^2(\Omega_T)$ is a given function for some fixed $T > 0$, such that $f_x, f_{tt}, f_{xt} \in L^2(\Omega_T)$.

(v) $\theta^0 \in L^\infty(0,1)$ and $\chi^0 \in W^{1,2}(0,1)$ are such that $\theta^0 \ge \theta_*$ with $0 < \theta_* < 1$, $\theta^0_x \in L^2(0,1)$, $\chi^0(x) \in [0,1]$ for all $x \in [0,1]$.

(vi) $h : \mathbb{R} \to [0,\infty)$ is a nondecreasing Lipschitz continuous function such that $0 \le h'(z) \le a$ a. e. and $h(z) \le bz^2$ for $z \in \mathbb{R}$, where a, b are positive constants; in particular b is such that $bL \le \rho$, where L is the latent heat of the process and ρ is the relaxation coefficient from (6.26).

(vii) $g : [0,\infty) \times \Omega_T \to \mathbb{R}$ is a Carathéodory function such that $g_0(x,t) := g(0,x,t) \ge 0$, $g_0 \in L^2(\Omega_T)$, and $|g_\theta(\theta,x,t)| \le g_1$ a. e. with g_1 a constant.

Remark 7.2. In this remark we comment on some of the above hypotheses.

(i) The assumption that $\gamma(m,\chi,r)$ decreases with increasing fatigue m corresponds to the observation that the stiffness of the material decreases with increasing fatigue. Moreover, the assumptions on γ together with the definition of the stop operator in (B.14) implies

$$|P_0[m,\chi,\varepsilon]| = \left|\int_0^\infty \gamma(m,\chi,r)\mathfrak{s}_r[\varepsilon]\,\mathrm{d}r\right| \le \int_0^\infty \tilde{\gamma}(r)r\,\mathrm{d}r \le M,$$

whereas the assumptions on γ_m and γ_χ allow us to conclude that

$$0 \le M[m,\chi,\varepsilon] = -\frac{1}{2}\int_0^\infty \gamma_m(m,\chi,\theta,r)\mathfrak{s}_r^2[\varepsilon_k]\,\mathrm{d}r \le \frac{1}{2}\int_0^\infty \gamma^*(r)r^2\,\mathrm{d}r \le L,$$
$$|K[m,\chi,\varepsilon]| = \left|-\frac{1}{2}\int_0^\infty \gamma_\chi(m,\chi,\theta,r)\mathfrak{s}_r^2[\varepsilon_k]\,\mathrm{d}r\right| \le \frac{1}{2}\int_0^\infty \gamma^*(r)r^2\,\mathrm{d}r \le L.$$

This means that hysteresis effects vanish far away from the equilibrium. Furthermore, estimate (B.35) yields the following upper bound for the dissipation in terms of the input velocity

$$D[m,\chi,\varepsilon](t) \le M|\dot{\varepsilon}(t)|. \tag{7.1}$$

(iv) In what follows we will frequently use the fact that $f_t \in L^2(\Omega_T)$, which is a direct consequence of the assumption $f, f_{tt} \in L^2(\Omega_T)$. This can be proved for example by comparing the Fourier series.

(vi) The assumptions on the function h are needed for the proof of the thermodynamic consistency of the model. Coming back to (6.20), by the positivity of the dissipation operator and by the inclusion (6.8) for the fatigue, we infer

$$\rho\chi_t^2 + D[m,\chi,\varepsilon] - \frac{1}{2}m_t\int_0^\infty \gamma_m(m,\chi,r)\,\mathfrak{s}_r^2[\varepsilon]\,\mathrm{d}r \ge \rho\chi_t^2 - Lh(\chi_t) \ge \rho\chi_t^2 - Lb\chi_t^2 \ge 0.$$

Hence the Clausius-Duhem inequality (6.13) holds, provided that the temperature stays positive. In particular the boundedness of h' is used to show the positivity of the temperature, see the first part of Subsection 7.2.

(vii) The assumption that $g_0(x,t) \ge 0$ makes sense since g is the heat source density, so at zero temperature we cannot remove heat from the system.

The main result reads as follows.

Theorem 7.3. *Let Hypothesis 7.1 hold. Then there exists a unique solution to the system* (6.22)–(6.29) *in Ω_T such that $\theta(x,t) > 0$ for all $(x,t) \in \Omega_T$, and with the regularity*

- $w_{xxxt}, w_{xxtt}, \theta_t, \theta_{xx}, u_{tt}, u_{xxt} \in L^2(\Omega_T)$,
- $\theta, m_t, \chi_t \in L^\infty(\Omega_T)$.

The proof of the existence result is carried out in three steps: approximation, a priori estimates and passage to the limit.

7.1 Approximation

From now on the values of all physical constants are set to 1 for simplicity, with the exception of $\rho, L, \theta_{\mathrm{ref}}$ in order to emphasize the role of the phase transition.

Let us choose an integer $n \in \mathbb{N}$ and consider an equidistant partition of the interval [0,1]. Let us take the space discrete approximations of (6.22)–(6.27) for $k = 1, \ldots, n-1$:

$$\dot{u}_k = \varepsilon_k + P_0[m_k, \chi_k, \varepsilon_k] + \dot{\varepsilon}_k - \theta_k + \theta_{\mathrm{ref}}, \tag{7.2}$$

$$\dot{w}_k - \dot{\varepsilon}_k = -n^2(u_{k+1} - 2u_k + u_{k-1}) + f_k, \tag{7.3}$$

$$\varepsilon_k = n^2(w_{k+1} - 2w_k + w_{k-1}), \tag{7.4}$$

$$\dot{\theta}_k = n^2(\theta_{k+1} - 2\theta_k + \theta_{k-1}) - \theta_k\dot{\varepsilon}_k + \dot{\varepsilon}_k^2 + D_k + \dot{m}_k\, M_k + \dot{\chi}_k\,(K_k - L) + g_k(\theta_k, t), \tag{7.5}$$

$$\rho\,\chi_k = \mathfrak{s}_{[0,1]}[\chi_k^0, A_k], \qquad A_k(t) = \int_0^t \left(\frac{L}{\theta_{\mathrm{ref}}}(\theta_k - \theta_{\mathrm{ref}}) + K_k\right)(\tau)\,\mathrm{d}\tau, \tag{7.6}$$

$$m_k = \mathfrak{s}_{[0,\infty)}[0, S_k], \qquad S_k(t) = \int_0^t (-h(\dot{\chi}_k) + D_k^*)(\tau)\,\mathrm{d}\tau, \tag{7.7}$$

where

$$\begin{aligned}
M_k(t) &= -\frac{1}{2}\int_0^\infty \gamma_m(m_k(t), \chi_k(t), r)\mathfrak{s}_r^2[\varepsilon_k](t)\,\mathrm{d}r,\\
K_k(t) &= -\frac{1}{2}\int_0^\infty \gamma_\chi(m_k(t), \chi_k(t), r)\mathfrak{s}_r^2[\varepsilon_k](t)\,\mathrm{d}r,\\
D_k(t) &= \int_0^\infty \gamma(m_k(t), \chi_k(t), r)\,\mathfrak{s}_r[\varepsilon_k](t)(\varepsilon_k - \mathfrak{s}_r[\varepsilon_k])_t(t)\,\mathrm{d}r,\\
D_k^*(t) &= \frac{1}{n}\sum_{j=1}^{n-1}\lambda_{k-j}D_j(t),\\
\lambda_i &= \lambda(i/n),\\
f_k(t) &= n\int_{(k-1)/n}^{k/n} f(x,t)\,\mathrm{d}x,\\
g_k(\theta, t) &= \begin{cases} n\displaystyle\int_{(k-1)/n}^{k/n} g(\theta, x, t)\,\mathrm{d}x & \text{for } \theta \ge 0,\\[2ex] g_k(0,t) & \text{for } \theta < 0.\end{cases}
\end{aligned}$$

Remark 7.4. By Hypothesis 7.1 (i) we have $M_k(t) \in [0, L]$, $|K_k(t)| \le L$ for a. e. $t \in [0, T]$. In addition, by (7.1) and Hypothesis 7.1 (i) we deduce $0 \le D_k(t) \le M|\dot{\varepsilon}_k(t)|$ for a. e. $t \in [0, T]$. This, together with Hypothesis 7.1 (iii), yields $0 \le D_k^*(t) \le \Lambda M \left(\frac{1}{n}\sum_{k=1}^{n-1}|\dot{\varepsilon}_k(t)|\right)$ for a. e. $t \in [0, T]$.

We prescribe initial conditions for $k = 1, \dots, n-1$

$$\left.\begin{aligned} &w_k(0) = u_k(0) = 0, \\ &\theta_k(0) = \theta_k^0 := \theta^0(k/n), \\ &m_k(0) = 0, \\ &\chi_k(0) = \chi_k^0 := n \int_{(k-1)/n}^{k/n} \chi^0(x)\,\mathrm{d}x, \end{aligned}\right\} \tag{7.8}$$

and "boundary conditions"

$$\left.\begin{aligned} &w_0 = w_n = u_0 = u_n = 0, \\ &\theta_0 = \theta_1,\ \theta_n = \theta_{n-1}. \end{aligned}\right\} \tag{7.9}$$

Remark 7.5. Note that by equation (7.4) and by the initial condition for w_k we deduce also

$$\varepsilon_k(0) = 0 \quad \text{for } k = 1, \dots, n-1.$$

By Remark B.4 this implies $\mathfrak{s}_r[\varepsilon_k](0) = 0$, which in turn gives

$$P_0[m_k, \chi_k, \varepsilon_k](0) = V[m_k, \chi_k, \varepsilon_k](0) = D_k(0) = M_k(0) = K_k(0) = 0 \quad \text{for } k = 1, \dots, n-1.$$

Problem (7.2)–(7.7) turns out to be a system of ODEs for the unknowns $u_k, w_k, \theta_k, \chi_k, m_k$, which admits a $W^{1,\infty}$ solution in an interval $[0, T_n]$. Indeed, denoting by $\mathbf{w}$ the vector $(w_1, \dots, w_{n-1})$, and $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_{n-1})$, we have by (7.4) $-\boldsymbol{\varepsilon} = \mathcal{S}\mathbf{w}$ with a positive definite matrix $\mathcal{S}$, which has the form

$$\mathcal{S} = n^2 \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Hence the left-hand side of (7.3) reads $(I + \mathcal{S})\dot{\mathbf{w}}$. Hence, by Proposition B.7 together with Hypothesis 7.1 (i) and (vi), we see that (7.7) defines a locally Lipschitz continuous mapping that with $\boldsymbol{\varepsilon}$, $\dot{\boldsymbol{\varepsilon}}$ and $\dot{\boldsymbol{\chi}} = (\dot{\chi}_1, \dots, \dot{\chi}_{n-1})$ (and therefore $\boldsymbol{\theta} = (\theta_1, \dots, \theta_{n-1})$ by (7.6)) associates the solution m_k. By (7.3), $\dot{\boldsymbol{\varepsilon}}$ is itself a Lipschitz continuous mapping of $\mathbf{u} = (u_1, \dots, u_{n-1})$. Thus (7.2)–(7.5) can be considered as an ODE system in u_k, w_k, θ_k with a right-hand side which is locally Lipschitz in $L^1(0,t)$ for $t \in [0,T]$, and the existence and uniqueness of a local absolutely continuous solution in an interval $[0, T_n]$ follows from the standard theory of ODEs. Consequently, the right hand side is bounded, and we conclude that the solution belongs to $W^{1,\infty}(0, T_n)$. We will show in Subsection 7.2.2 that the solution exists globally, and actually $T_n = T$.

7.2 A priori estimates

This subsection is divided into 4 parts:

- *Positivity of the temperature:* it is important for the thermodynamic consistency of the model;
- *Discrete energy estimate:* it constitutes the basic a priori estimate giving the first minimal regularity (which is not enough however to pass to the limit);
- *Discrete Dafermos estimates:* it provides additional regularity for the temperature, necessary to deduce further a priori estimates;
- *Higher order estimates:* this last part provides the necessary estimates in order to proceed with the passage to the limit.

In what follows the operation of "testing by a function" a discretized equation will consist in multiplying the equation by the function, summing up over $k = 1, \dots, n-1$ (or in some cases over $k = 1, \dots, n$) and then dividing by n.

Moreover, we will systematically use the "summation by parts formula"

$$\sum_{k=1}^{n-1} \xi_k(\eta_{k+1} - 2\eta_k + \eta_{k-1}) + \sum_{k=1}^{n} (\xi_k - \xi_{k-1})(\eta_k - \eta_{k-1}) = \xi_n(\eta_n - \eta_{n-1}) - \xi_0(\eta_1 - \eta_0) \tag{7.10}$$

for all vectors $(\xi_0, \dots, \xi_n)$, $(\eta_0, \dots, \eta_n)$.

We will denote by C any generic positive constant independent of n.

7.2.1 Positivity of the temperature

First of all it is important to prove that θ_k remains positive in the whole range of existence. For this purpose we test (7.5) by $-\theta_k^-$, where θ_k^- is the negative part of θ_k, getting

$$\begin{aligned} -\frac{1}{n}\sum_{k=1}^{n-1} \dot{\theta}_k \theta_k^- = &- n \sum_{k=1}^{n-1} (\theta_{k+1} - 2\theta_k + \theta_{k-1})\, \theta_k^- - \frac{1}{n}\sum_{k=1}^{n-1} \left(D_k + \dot{\varepsilon}_k^2 + g_k(\theta_k, t)\right) \theta_k^- \\ &- \frac{1}{n}\sum_{k=1}^{n-1} \dot{\chi}_k\, (K_k - L)\, \theta_k^- - \frac{1}{n}\sum_{k=1}^{n-1} \dot{m}_k\, M_k\, \theta_k^- + \frac{1}{n}\sum_{k=1}^{n-1} \theta_k \dot{\varepsilon}_k \theta_k^- . \end{aligned} \tag{7.11}$$

The left-hand side is such that

$$-\frac{1}{n}\sum_{k=1}^{n-1} \dot{\theta}_k \theta_k^- = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2n}\sum_{k=1}^{n-1} (\theta_k^-)^2\right).$$

Let us now focus on the right-hand side of (7.11). By (7.10) with $\xi_k = \theta_k^-$ and $\eta_k = \theta_k$ we also deduce

$$-n \sum_{k=1}^{n-1} \theta_k^- (\theta_{k+1} - 2\theta_k + \theta_{k-1}) = n \sum_{k=1}^{n} (\theta_k^- - \theta_{k-1}^-)(\theta_k - \theta_{k-1}) - n\left[\theta_n^-(\theta_n - \theta_{n-1}) - \theta_0^-(\theta_1 - \theta_0)\right]$$

The second summand vanishes by (7.9), whereas the first is such that

$$n\sum_{k=1}^{n}(\theta_k^- - \theta_{k-1}^-)(\theta_k - \theta_{k-1}) = n\sum_{k=1}^{n}(\theta_k^- - \theta_{k-1}^-)\left[(\theta_k^+ - \theta_{k-1}^+) - (\theta_k^- - \theta_{k-1}^-)\right] = -n\sum_{k=1}^{n}(\theta_k^- - \theta_{k-1}^-)^2.$$

Hence

$$-n\sum_{k=1}^{n-1}\theta_k^-(\theta_{k+1} - 2\theta_k + \theta_{k-1}) = -n\sum_{n=1}^{n}(\theta_k^- - \theta_{k-1}^-)^2 \le 0.$$

Moreover, being $D_k(t) \ge 0$ by definition and $g_k(\theta,t) \ge 0$ for $\theta \le 0$ by Hypothesis 7.1 (vii), it follows that

$$-\left(D_k(t) + \dot\varepsilon_k^2(t) + g_k(\theta_k,t)\right)\theta_k^- \le 0.$$

Now we deal with the phase term. By (7.6) and (B.21) (see also Remark B.2) we have

$$\left.\begin{aligned} &\dot\chi_k(t)\,\theta_k^-(t) = 0 && \text{if } \dot\chi_k(t) = 0,\\ &\dot\chi_k(t)\,\theta_k^-(t) = \frac{L}{\theta_{\mathrm{ref}}\rho}(\theta_k(t) - \theta_{\mathrm{ref}})\,\theta_k^-(t) + \frac{K_k}{\rho}\theta_k^-(t) && \text{otherwise.}\end{aligned}\right\} \tag{7.12}$$

Note that

$$\frac{L}{\theta_{\mathrm{ref}}\rho}(\theta_k(t) - \theta_{\mathrm{ref}})\,\theta_k^-(t) + \frac{K_k}{\rho}\theta_k^-(t) = -\frac{L}{\theta_{\mathrm{ref}}\rho}(\theta_k^-(t))^2 + \frac{1}{\rho}(K_k - L)\,\theta_k^-(t) \le 0$$

by Remark 7.4. Hence (7.12) entails

$$\dot\chi_k(t)\,\theta_k^-(t) \le 0, \tag{7.13}$$

and again by Remark 7.4 we deduce

$$-(K_k - L)\dot\chi_k(t)\,\theta_k^-(t) \le 0.$$

Finally, by (7.7) and (B.21) (see also Remark B.2) we have

$$\left.\begin{aligned} &-\dot m_k(t)M_k(t)\,\theta_k^-(t) = 0 && \text{if } \dot m_k(t) = 0,\\ &-\dot m_k(t)M_k(t)\,\theta_k^-(t) = (h(\dot\chi_k(t)) - D_k^*(t))\,M_k(t)\,\theta_k^-(t) && \text{otherwise.}\end{aligned}\right\} \tag{7.14}$$

Hypothesis 7.1 (vi) (in particular the Lipschitzianity of h in 0 with Lipschitz constant a) and Remark 7.4 allow us to show that, by (7.13),

$$(h(\dot\chi_k(t)) - D_k^*(t))\,M_k(t)\,\theta_k^-(t) \le h(\dot\chi_k(t))M_k(t)\,\theta_k^-(t) \le 0.$$

Hence (7.14) entails

$$-\dot m_k(t)M_k(t)\,\theta_k^-(t) \le 0.$$

The last term on the right-hand side of (7.11) is such that

$$\frac{1}{n}\sum_{k=1}^{n-1}\theta_k\dot\varepsilon_k\theta_k^- = \frac{1}{n}\sum_{k=1}^{n-1}\left(\theta_k^+ - \theta_k^-\right)\dot\varepsilon_k\theta_k^- = -\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k(\theta_k^-)^2.$$

Summarizing the above computations we come to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2n}\sum_{k=1}^{n-1}(\theta_k^-)^2\right) \leq \frac{1}{n}\sum_{k=1}^{n-1}|\dot\varepsilon_k|(\theta_k^-)^2 \leq \frac{C_{\varepsilon,n}}{n}\sum_{k=1}^{n-1}(\theta_k^-)^2,$$

where we set

$$C_{\varepsilon,n} := \max\{|\dot\varepsilon_k(t)| : k=1,\dots,n-1,\;\; t\in[0,T_n]\}.$$

Grönwall's lemma A.1 now yields

$$(\theta_k^-)^2(t) \leq \mathrm{e}^{\int_0^t 2C_{\varepsilon,n}\,\mathrm{d}\tau}(\theta_k^-)^2(0),$$

and since by (7.8) and Hypothesis 7.1 (v) it holds $\theta_k^-(0) = \big(\theta^0(k/n)\big)^- = 0$, we finally obtain that $\theta_k^-(t) = 0$ for all k and $t\in[0,T_n]$.

At this point we prove that in addition $\theta_k(t)$ is bounded away from 0 for all k and all $t\in[0,T_n]$. The idea is to compare the decay of θ_k with the solution to the differential equation

$$\dot p + \psi(p) = 0\,,\quad p(0) = \theta_*, \tag{7.15}$$

with $\theta_* > 0$ from Hypothesis 7.1 (v) and ψ suitable function of polynomial kind that we will choose later.

First of all, if $\dot\chi_k \neq 0$ then by Remark 7.4

$$-\rho\dot\chi_k = -\frac{L}{\theta_{\mathrm{ref}}}(\theta_k-\theta_{\mathrm{ref}}) - K_k \geq -\frac{L\,\theta_k}{\theta_{\mathrm{ref}}}. \tag{7.16}$$

Thus

$$\dot\chi_k(K_k - L) \geq \frac{L\,\theta_k}{\theta_{\mathrm{ref}}\,\rho}(K_k-L) \geq -\frac{2L^2}{\theta_{\mathrm{ref}}\rho}\theta_k, \tag{7.17}$$

where the second inequality follows again from Remark 7.4. On the other hand by Hypothesis 7.1 (vi) and (7.16)

$$\dot m_k \geq -h(\dot\chi_k) \geq -a\dot\chi_k \geq -\frac{a\,L\,\theta_k}{\theta_{\mathrm{ref}}\rho},$$

from which we deduce thanks to Remark 7.4

$$\dot m_k M_k \geq -\frac{a\,L^2}{\theta_{\mathrm{ref}}\rho}\theta_k. \tag{7.18}$$

Moreover, since we already proved that $\theta_k > 0$, by Hypothesis 7.1 (vii) it follows

$$\begin{aligned} g_k(\theta_k,t) &= n\int_{(k-1)/n}^{k/n} g(\theta_k,x,t)\,\mathrm{d}x \geq n\int_{(k-1)/n}^{k/n}\left(g(\theta_k,x,t)-g(0,x,t)\right)\,\mathrm{d}x \\ &= n\int_{(k-1)/n}^{k/n}\left(\int_0^{\theta_k} g_\theta(\theta,x,t)\,\mathrm{d}\theta\right)\,\mathrm{d}x \geq n\int_{(k-1)/n}^{k/n}(-g_1)\,\theta_k\,\mathrm{d}x = -g_1\theta_k. \end{aligned} \tag{7.19}$$

By the elementary inequality $a^2 - ab + \frac{b^2}{4} \geq 0\;\forall\, a,b\in\mathbb{R}$ we additionally have

$$\dot\varepsilon_k^2 - \theta_k\dot\varepsilon_k \geq -\frac{1}{4}\theta_k^2. \tag{7.20}$$

Inserting (7.17)–(7.20) in (7.5) we deduce

$$\dot{\theta}_k - n^2(\theta_{k+1} - 2\theta_k + \theta_{k-1}) \geq -\psi(\theta_k), \tag{7.21}$$

where

$$\psi(z) := -\frac{1}{4}z^2 - \left[\frac{L^2}{\theta_{\text{ref}}\rho}(2+a) + g_1\right] z.$$

Coming back to (7.15), it is not difficult to check that for such a choice for ψ we obtain

$$p(t) = \frac{\mu\theta_* e^{-\mu t}}{\delta\theta_*(1-e^{-\mu t}) + \mu} \qquad \text{with} \quad \delta = \frac{1}{4}, \quad \mu = \frac{L^2}{\theta_{\text{ref}}\gamma}(2+a) + g_1.$$

Note that p is nonnegative and nonincreasing in $[0,\infty)$. We are going to show that $\theta_k(t) \geq p(t) > 0$ for all k and all $t \in [0, T_n]$. To this aim we compare θ_k and p. By (7.21) and (7.15) it holds

$$(\dot{p} - \dot{\theta}_k) - n^2\big((p-\theta_{k+1}) - 2(p-\theta_k) + (p-\theta_{k-1})\big) + \psi(p) - \psi(\theta_k) \leq 0.$$

We now test by $(p-\theta_k)^+$, getting

$$\frac{1}{n}\sum_{k=1}^{n-1}(\dot{p}-\dot{\theta}_k)(p-\theta_k)^+ - n\sum_{k=1}^{n-1}\big((p-\theta_{k+1}) - 2(p-\theta_k) + (p-\theta_{k-1})\big)(p-\theta_k)^+$$
$$+\frac{1}{n}\sum_{k=1}^{n-1}\left(\psi(p) - \psi(\theta_k)\right)(p-\theta_k)^+ \leq 0.$$

It holds

◇ $$\frac{1}{n}\sum_{k=1}^{n-1}(\dot{p}-\dot{\theta}_k)(p-\theta_k)^+ = \begin{cases} \dfrac{1}{n}\displaystyle\sum_{k=1}^{n-1}(\dot{p}-\dot{\theta}_k)(p-\theta_k) & \text{if } p > \theta_k \\ 0 & \text{if } p \leq \theta_k \end{cases} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2n}\sum_{k=1}^{n-1}\left((p-\theta_k)^+\right)^2\right);$$

◇ for the second term

$$\begin{aligned}
&-n\sum_{k=1}^{n-1}\big((p-\theta_{k+1}) - 2(p-\theta_k) + (p-\theta_{k-1})\big)(p-\theta_k)^+ \\
&\overset{(7.10)}{=} n\sum_{k=1}^{n}\left((p-\theta_k)^+ - (p-\theta_{k-1})^+\right)\left((p-\theta_k) - (p-\theta_{k-1})\right) \\
&\qquad - n\left[\big((p-\theta_n) - (p-\theta_{n-1})\big)(p-\theta_n)^+ - \big((p-\theta_1) - (p-\theta_0)\big)(p-\theta_0)^+\right] \\
&\overset{(7.9)}{=} n\sum_{k=1}^{n}\left((p-\theta_k)^+ - (p-\theta_{k-1})^+\right)\left((p-\theta_k) - (p-\theta_{k-1})\right) \\
&= n\sum_{k=1}^{n}\left((p-\theta_k)^+ - (p-\theta_{k-1})^+\right)\left((p-\theta_k)^+ - (p-\theta_k)^- - (p-\theta_{k-1})^+ + (p-\theta_{k-1})^-\right) \\
&\geq n\sum_{k=1}^{n}\left((p-\theta_k)^+ - (p-\theta_{k-1})^+\right)^2;
\end{aligned}$$

◇ $$\frac{1}{n}\sum_{k=1}^{n-1}\left(\psi(p) - \psi(\theta_k)\right)(p-\theta_k)^+ = \begin{cases} \dfrac{1}{n}\displaystyle\sum_{k=1}^{n-1}\left(\psi(p) - \psi(\theta_k)\right)(p-\theta_k) & \text{if } p > \theta_k \\ 0 & \text{if } p \leq \theta_k \end{cases}$$

hence it is always nonnegative since ψ is nondecreasing for positive arguments.

Thus
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2n}\sum_{k=1}^{n-1}\left((p-\theta_k)^+\right)^2\right)\le 0,$$
from which we obtain
$$\frac{1}{n}\sum_{k=1}^{n-1}\left((p-\theta_k)^+\right)^2(t)\le\frac{1}{n}\sum_{k=1}^{n-1}\left((p-\theta_k)^+\right)^2(0)=\frac{1}{n}\sum_{k=1}^{n-1}\left((\theta_*-\theta^0(k/n))^+\right)^2=0$$
since $\theta^0\ge\theta_*$ by Hypothesis 7.1 (v). Hence $(p-\theta_k)^+(t)=0$ for all k and all $t\in[0,T_n]$, so that $\theta_k(t)\ge p(t)>0$ for all k and all $t\in[0,T_n]$, which is the desired result. Note that the positive lower bound is independent of the discretization parameter, and therefore is preserved in the limit and implies the positivity of the temperature.

7.2.2 Discrete energy estimate

We test (7.2) by $\dot\varepsilon_k$, then we differentiate (7.3) in time, test it by $\dot w_k$ and sum up the two equations. We start by testing (7.2) by $\dot\varepsilon_k$. We obtain
$$\frac{1}{n}\sum_{k=1}^{n-1}\dot u_k\dot\varepsilon_k=\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k\left(\varepsilon_k+P_0[m_k,\chi_k,\varepsilon_k]+\dot\varepsilon_k-\theta_k+\theta_{\mathrm{ref}}\right).$$
Using (7.4) on the left-hand side this is equivalent to
$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k\left(\varepsilon_k+P_0[m_k,\chi_k,\varepsilon_k]+\dot\varepsilon_k-\theta_k+\theta_{\mathrm{ref}}\right)&=n\sum_{k=1}^{n-1}\dot u_k(\dot w_{k+1}-2\dot w_k+\dot w_{k-1})\\ &\overset{(7.10)}{=}-n\sum_{k=1}^{n}\left(\dot u_k-\dot u_{k-1}\right)\left(\dot w_k-\dot w_{k-1}\right)+n\left[\dot u_n\left(\dot w_n-\dot w_{n-1}\right)-\dot u_0\left(\dot w_1-\dot w_0\right)\right]\\ &\overset{(7.9)}{=}-n\sum_{k=1}^{n}\left(\dot u_k-\dot u_{k-1}\right)\left(\dot w_k-\dot w_{k-1}\right).\end{aligned}\tag{7.22}$$
Then we differentiate (7.3) in time. This yields
$$\ddot w_k-\ddot\varepsilon_k=-n^2(\dot u_{k+1}-2\dot u_k+\dot u_{k-1})+\dot f_k.$$
Testing it by $\dot w_k$ we get
$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n-1}\ddot w_k\dot w_k-\frac{1}{n}\sum_{k=1}^{n-1}\ddot\varepsilon_k\dot w_k&=-n\sum_{k=1}^{n-1}(\dot u_{k+1}-2\dot u_k+\dot u_{k-1})\dot w_k+\frac{1}{n}\sum_{k=1}^{n-1}\dot f_k\dot w_k\\ &\overset{(7.10)}{=}n\sum_{k=1}^{n}\left(\dot u_k-\dot u_{k-1}\right)\left(\dot w_k-\dot w_{k-1}\right)-n\left[\left(\dot u_n-\dot u_{n-1}\right)\dot w_n-\left(\dot u_1-\dot u_0\right)\dot w_0\right]+\frac{1}{n}\sum_{k=1}^{n-1}\dot f_k\dot w_k\\ &\overset{(7.9)}{=}n\sum_{k=1}^{n}\left(\dot u_k-\dot u_{k-1}\right)\left(\dot w_k-\dot w_{k-1}\right)+\frac{1}{n}\sum_{k=1}^{n-1}\dot f_k\dot w_k.\end{aligned}\tag{7.23}$$
Summing up (7.22) and (7.23) we obtain
$$\frac{1}{n}\sum_{k=1}^{n-1}\left(\ddot w_k\dot w_k-\ddot\varepsilon_k\dot w_k\right)+\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k\left(\varepsilon_k+P_0[m_k,\chi_k,\varepsilon_k]+\dot\varepsilon_k-\theta_k+\theta_{\mathrm{ref}}\right)=\frac{1}{n}\sum_{k=1}^{n-1}\dot f_k\dot w_k.$$

Note that by (7.4) we also have

$$
\begin{aligned}
-\frac{1}{n}\sum_{k=1}^{n-1}\ddot{\varepsilon}_k\dot{w}_k &= -n\sum_{k=1}^{n-1}(\ddot{w}_{k+1}-2\ddot{w}_k+\ddot{w}_{k-1})\dot{w}_k\\
&\overset{(7.10)}{=} n\sum_{k=1}^{n}\left(\ddot{w}_k-\ddot{w}_{k-1}\right)\left(\dot{w}_k-\dot{w}_{k-1}\right)-n\left[\left(\ddot{w}_n-\ddot{w}_{n-1}\right)\dot{w}_n-\left(\ddot{w}_1-\ddot{w}_0\right)\dot{w}_0\right]\\
&\overset{(7.9)}{=} n\sum_{k=1}^{n}\left(\ddot{w}_k-\ddot{w}_{k-1}\right)\left(\dot{w}_k-\dot{w}_{k-1}\right),
\end{aligned}
$$

hence we get

$$
\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n-1}\ddot{w}_k\dot{w}_k+n\sum_{k=1}^{n}\left(\ddot{w}_k-\ddot{w}_{k-1}\right)\left(\dot{w}_k-\dot{w}_{k-1}\right)\\
&+\frac{1}{n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k\left(\varepsilon_k+P_0[m_k,\chi_k,\varepsilon_k]+\dot{\varepsilon}_k-\theta_k+\theta_{\mathrm{ref}}\right)=\frac{1}{n}\sum_{k=1}^{n-1}\dot{f}_k\dot{w}_k.
\end{aligned}
$$

This entails

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2n}\sum_{k=1}^{n-1}\left(\dot{w}_k^2+\varepsilon_k^2\right)+\frac{1}{n}\sum_{k=1}^{n-1}\theta_{\mathrm{ref}}\varepsilon_k+\frac{n}{2}\sum_{k=1}^{n}(\dot{w}_k-\dot{w}_{k-1})^2\right)\\
&+\frac{1}{n}\sum_{k=1}^{n-1}\dot{\varepsilon}_kP_0[m_k,\chi_k,\varepsilon_k]=\frac{1}{n}\sum_{k=1}^{n-1}(-\dot{\varepsilon}_k^2+\theta_k\dot{\varepsilon}_k+\dot{f}_k\dot{w}_k).
\end{aligned}
\tag{7.24}
$$

Now we test (7.5) by 1 obtaining

$$
\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n-1}\dot{\theta}_k &= n\sum_{k=1}^{n-1}(\theta_{k+1}-2\theta_k+\theta_{k-1})\\
&\quad+\frac{1}{n}\sum_{k=1}^{n-1}\left(-\theta_k\dot{\varepsilon}_k+\dot{\varepsilon}_k^2+D_k+\dot{m}_k\,M_k+\dot{\chi}_k\left(K_k-L\right)+g_k(\theta_k,t)\right).
\end{aligned}
\tag{7.25}
$$

Adding (7.24) to (7.25) allows us to obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2n}\sum_{k=1}^{n-1}\left(\dot{w}_k^2+\varepsilon_k^2\right)+\frac{1}{n}\sum_{k=1}^{n-1}\theta_{\mathrm{ref}}\varepsilon_k+\frac{n}{2}\sum_{k=1}^{n}(\dot{w}_k-\dot{w}_{k-1})^2+\frac{1}{n}\sum_{k=1}^{n-1}\theta_k\right)+\frac{1}{n}\sum_{k=1}^{n-1}\dot{\varepsilon}_kP_0[m_k,\chi_k,\varepsilon_k]\\
&=n\sum_{k=1}^{n-1}(\theta_{k+1}-2\theta_k+\theta_{k-1})+\frac{1}{n}\sum_{k=1}^{n-1}\left(D_k+\dot{m}_k\,M_k+\dot{\chi}_k\left(K_k-L\right)+g_k+\dot{f}_k\dot{w}_k\right).
\end{aligned}
$$

Then, by virtue of the discrete version of (B.34) it holds also

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2n}\sum_{k=1}^{n-1}\left(\dot{w}_k^2+\varepsilon_k^2\right)+\frac{1}{n}\sum_{k=1}^{n-1}\theta_{\mathrm{ref}}\varepsilon_k+\frac{n}{2}\sum_{k=1}^{n}(\dot{w}_k-\dot{w}_{k-1})^2+\sum_{k=1}^{n-1}\theta_k\right)\\
&+\frac{1}{n}\sum_{k=1}^{n-1}\left(V[m_k,\chi_k,\varepsilon_k]_t+D_k+\dot{m}_k\,M_k+\dot{\chi}_k\,K_k\right)\\
&=n\sum_{k=1}^{n-1}(\theta_{k+1}-2\theta_k+\theta_{k-1})+\frac{1}{n}\sum_{k=1}^{n-1}\left(D_k+\dot{m}_k\,M_k+\dot{\chi}_k\left(K_k-L\right)+g_k+\dot{f}_k\dot{w}_k\right),
\end{aligned}
$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{1}{2}\dot{w}_k^2+\frac{1}{2}\varepsilon_k^2+\theta_{\mathrm{ref}}\varepsilon_k+\theta_k+V[m_k,\chi_k,\varepsilon_k]+L\chi_k\right)+\frac{n}{2}\sum_{k=1}^{n}(\dot{w}_k-\dot{w}_{k-1})^2\right)$$
$$=n\sum_{k=1}^{n-1}(\theta_{k+1}-2\theta_k+\theta_{k-1})+\frac{1}{n}\sum_{k=1}^{n-1}\left(g_k+\dot{f}_k\dot{w}_k\right).$$

Note that the summation by parts formula (7.10) with $\xi_k \equiv 1$ and $\eta_k = \theta_k$ implies

$$n\sum_{k=1}^{n-1}(\theta_{k+1}-2\theta_k+\theta_{k-1})=(\theta_n-\theta_{n-1})-(\theta_1-\theta_0)\overset{(7.9)}{=}0.$$

Hence we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{1}{2}\dot{w}_k^2+\frac{1}{2}\varepsilon_k^2+\theta_{\mathrm{ref}}\varepsilon_k+\theta_k+V[m_k,\chi_k,\varepsilon_k]+L\chi_k\right)+\frac{n}{2}\sum_{k=1}^{n}(\dot{w}_k-\dot{w}_{k-1})^2\right)$$
$$=\frac{1}{n}\sum_{k=1}^{n-1}\left(g_k+\dot{f}_k\dot{w}_k\right).$$

We now integrate in time $\int_0^\tau \mathrm{d}t$ for some $\tau\in[0,T_n]$. Note that it holds

$$\theta_{\mathrm{ref}}\varepsilon_k=2\theta_{\mathrm{ref}}\cdot\frac{1}{2}\varepsilon_k\geq-\frac{1}{2}(2\theta_{\mathrm{ref}})^2-\frac{1}{2}\left(\frac{1}{2}\varepsilon_k\right)^2=-2\theta_{\mathrm{ref}}^2-\frac{1}{8}\varepsilon_k^2, \tag{7.26}$$

hence we obtain

$$\frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{1}{2}\dot{w}_k^2(\tau)+\frac{3}{8}\varepsilon_k^2(\tau)+\theta_k(\tau)+V[m_k,\chi_k,\varepsilon_k](\tau)+L\chi_k(\tau)\right)+\frac{n}{2}\sum_{k=1}^{n}(\dot{w}_k-\dot{w}_{k-1})^2(\tau)$$
$$\leq\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\left(g_k(\theta_k,t)+\dot{f}_k(t)\dot{w}_k(t)\right)\,\mathrm{d}t+2\theta_{\mathrm{ref}}^2 \tag{7.27}$$
$$+\frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{1}{2}\dot{w}_k^2(0)+\frac{3}{8}\varepsilon_k^2(0)+\theta_k(0)+V[m_k,\chi_k,\varepsilon_k](0)+L\chi_k(0)\right)+\frac{n}{2}\sum_{k=1}^{n}(\dot{w}_k-\dot{w}_{k-1})^2(0).$$

Since we have already proved that $\theta_k>0$ for all k, by Hypothesis 7.1 (vii) the first summand on the right-hand side is such that

$$\begin{aligned}\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}g_k(\theta_k,t)\,\mathrm{d}t&=\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\left(n\int_{(k-1)/n}^{k/n}g(\theta,x,t)\,\mathrm{d}x\right)\,\mathrm{d}t\\&=\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\left(n\int_{(k-1)/n}^{k/n}\left(\int_0^{\theta_k}g_\theta(\theta,x,t)\,\mathrm{d}\theta+g_0(x,t)\right)\,\mathrm{d}x\right)\,\mathrm{d}t\\&\leq\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\left(g_1\theta_k(t)+n\int_{(k-1)/n}^{k/n}g_0(x,t)\,\mathrm{d}x\right)\,\mathrm{d}t\\&\leq\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}g_1\theta_k(t)\,\mathrm{d}t+\iint_{\Omega_T}g_0(x,t)\,\mathrm{d}x\,\mathrm{d}t\\&\leq C\left(1+\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\theta_k(t)\,\mathrm{d}t\right).\end{aligned} \tag{7.28}$$

We now estimate the second summand using the discrete Hölder's inequality (see Remark A.10) and then Young's inequality. It holds

$$\begin{aligned}\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1} \dot f_k(t)\dot w_k(t)\,\mathrm{d}t &\le \int_0^\tau \left(\frac{1}{n}\sum_{k=1}^{n-1}\dot f_k^2(t)\right)^{1/2}\left(\frac{1}{n}\sum_{k=1}^{n-1}\dot w_k^2(t)\right)^{1/2}\mathrm{d}t\\ &\le \int_0^\tau \frac{1}{2n}\sum_{k=1}^{n-1}\dot f_k^2(t)\,\mathrm{d}t + \int_0^\tau \frac{1}{2n}\sum_{k=1}^{n-1}\dot w_k^2(t)\,\mathrm{d}t,\end{aligned}$$

where

$$\begin{aligned}\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\dot f_k^2(t)\,\mathrm{d}t &= \int_0^\tau n\sum_{k=1}^{n-1}\left(\int_{(k-1)/n}^{k/n} f_t(x,t)\,\mathrm{d}x\right)^2\mathrm{d}t\\ &\le \int_0^\tau n\sum_{k=1}^{n-1}\left(\left(\int_{(k-1)/n}^{k/n} f_t^2(x,t)\,\mathrm{d}x\right)^{1/2}\left(\int_{(k-1)/n}^{k/n} 1^2\,\mathrm{d}x\right)^{1/2}\right)^2\mathrm{d}t\\ &= \int_0^\tau n\sum_{k=1}^{n-1}\int_{(k-1)/n}^{k/n} f_t^2(x,t)\,\mathrm{d}x\cdot\frac{1}{n}\,\mathrm{d}t \le \iint_{\Omega_T} f_t^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \le C\end{aligned} \tag{7.29}$$

by Hypothesis 7.1 (iv) and Remark 7.2 (iv). Hence from (7.27) we obtain

$$\begin{aligned}&\frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{1}{2}\dot w_k^2(\tau)+\frac{3}{8}\varepsilon_k^2(\tau)+\theta_k(\tau)+V[m_k,\chi_k,\varepsilon_k](\tau)+L\chi_k(\tau)\right)+\frac{n}{2}\sum_{k=1}^{n}(\dot w_k-\dot w_{k-1})^2(\tau)\\ &\le C\left(1+\int_0^\tau\frac{1}{2n}\sum_{k=1}^{n-1}\dot w_k^2(t)\,\mathrm{d}t+\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\theta_k(t)\,\mathrm{d}t\right)\\ &+\frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{1}{2}\dot w_k^2(0)+\frac{3}{8}\varepsilon_k^2(0)+\theta_k(0)+V[m_k,\chi_k,\varepsilon_k](0)+L\chi_k(0)\right)+\frac{n}{2}\sum_{k=1}^{n}(\dot w_k-\dot w_{k-1})^2(0).\end{aligned} \tag{7.30}$$

We need to estimate the initial data. To bound the terms $\frac{1}{n}\sum_{k=1}^{n-1}\dot w_k^2(0)$ and $n\sum_{k=1}^{n}(\dot w_k-\dot w_{k-1})^2(0)$ we write equation (7.3) for $t=0$

$$\dot w_k(0)-\dot\varepsilon_k(0)=f_k(0), \tag{7.31}$$

where we exploited also the initial condition (7.8) for u_k. Then we test it by $\dot w_k(0)$ to get

$$\frac{1}{n}\sum_{k=1}^{n-1}\dot w_k^2(0)-\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k(0)\dot w_k(0)=\frac{1}{n}\sum_{k=1}^{n-1}f_k(0)\dot w_k(0). \tag{7.32}$$

It holds

$$\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k(0)\dot w_k(0)\overset{(7.4)}{=}n\sum_{k=1}^{n-1}(\dot w_{k+1}-2\dot w_k+\dot w_{k-1})(0)\dot w_k(0)\overset{(7.10)}{=}-n\sum_{k=1}^{n}\left(\dot w_k-\dot w_{k-1}\right)^2(0) \tag{7.33}$$

where we used also the zero initial condition in (7.8) for $\dot w_k$. Thus we obtain, by Young's inequality,

$$\frac{1}{2n}\sum_{k=1}^{n-1}\dot w_k^2(0)+n\sum_{k=1}^{n}\left(\dot w_k-\dot w_{k-1}\right)^2(0)\le\frac{1}{2n}\sum_{k=1}^{n-1}f_k^2(0). \tag{7.34}$$

We need to estimate the term on the right-hand side. It holds

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n-1} f_k^2(0) &= \frac{1}{n}\sum_{k=1}^{n-1} f_k^2(\tau) - 2\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1} f_k(t)\dot{f}_k(t)\,\mathrm{d}t \\ &\le \frac{1}{n}\sum_{k=1}^{n-1} f_k^2(\tau) + 2\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1} |f_k(t)||\dot{f}_k(t)|\,\mathrm{d}t.\end{aligned}$$

Arguing as in (7.29) we get

$$\frac{1}{n}\sum_{k=1}^{n-1} f_k^2(\tau) \le \int_0^1 f^2(x,\tau)\,\mathrm{d}x. \tag{7.35}$$

Moreover, Hölder's inequality in space yields

$$\begin{aligned}\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1} |f_k(t)||\dot{f}_k(t)|\,\mathrm{d}t &= \int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1} \left| n\int_{(k-1)/n}^{k/n} f(x,t)\,\mathrm{d}x \right| \left| n\int_{(k-1)/n}^{k/n} f_t(x,t)\,\mathrm{d}x \right| \mathrm{d}t \\ &\le n\int_0^\tau \sum_{k=1}^{n-1} \left(\int_{(k-1)/n}^{k/n} |f(x,t)|\,\mathrm{d}x \right) \left(\int_{(k-1)/n}^{k/n} |f_t(x,t)|\,\mathrm{d}x \right) \mathrm{d}t \\ &\le n\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1} \left(\int_{(k-1)/n}^{k/n} f^2(x,t)\,\mathrm{d}x \right)^{1/2} \left(\int_{(k-1)/n}^{k/n} f_t^2(x,t)\,\mathrm{d}x \right)^{1/2} \mathrm{d}t\end{aligned}$$

which entails, applying first the discrete Hölder's inequality and then Hölder's inequality in time,

$$\begin{aligned}&\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1} |f_k(t)||\dot{f}_k(t)|\,\mathrm{d}t \\ &\le n\int_0^\tau \left(\frac{1}{n}\sum_{k=1}^{n-1} \int_{(k-1)/n}^{k/n} f^2(x,t)\,\mathrm{d}x \right)^{1/2} \left(\frac{1}{n}\sum_{k=1}^{n-1} \int_{(k-1)/n}^{k/n} f_t^2(x,t)\,\mathrm{d}x \right)^{1/2} \mathrm{d}t \\ &\le \left(\int_0^\tau \int_0^1 f^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} \left(\int_0^\tau \int_0^1 f_t^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \right)^{1/2}.\end{aligned}$$

Hence we obtain

$$\frac{1}{n}\sum_{k=1}^{n-1} f_k^2(0) \le \int_0^1 f^2(x,\tau)\,\mathrm{d}x + 2\left(\int_0^\tau \int_0^1 f^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} \left(\int_0^\tau \int_0^1 f_t^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \right)^{1/2}.$$

Integrating now in time $\int_0^T \mathrm{d}\tau$ we get

$$\frac{T}{n}\sum_{k=1}^{n-1} f_k^2(0) \le \iint_{\Omega_T} f^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau + 2T\left(\iint_{\Omega_T} f^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} \left(\iint_{\Omega_T} f_t^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \right)^{1/2},$$

that is, by Hypothesis 7.1 (iv),

$$\frac{1}{n}\sum_{k=1}^{n-1} f_k^2(0) \le C. \tag{7.36}$$

Coming back to (7.34) this implies

$$\frac{1}{n}\sum_{k=1}^{n-1} \dot{w}_k^2(0) + n\sum_{k=1}^{n} \left(\dot{w}_k - \dot{w}_{k-1}\right)^2(0) \le C. \tag{7.37}$$

From Remark 7.5 we deduce

$$\varepsilon_k^2(0) = V[m_k, \chi_k, \varepsilon_k](0) = 0, \tag{7.38}$$

whereas by (7.8) and Hypothesis 7.1 (v)

$$\theta_k(0) = \theta^0(k/n) \leq \sup_{(0,1)} \operatorname{ess} \theta^0 \leq C, \tag{7.39}$$

$$L\chi_k(0) = L\, n \int_{(k-1)/n}^{k/n} \chi^0(x)\, \mathrm{d}x \leq L, \tag{7.40}$$

for all $k = 1, \ldots, n-1$. Hence, exploiting also the positivity of $V[m_k, \chi_k, \varepsilon_k]$ (see (6.3) and Hypothesis 7.1 (i)) and of $L\chi_k$, from (7.30) we deduce

$$\frac{1}{n}\sum_{k=1}^{n-1} \left(\dot{w}_k^2(\tau) + \varepsilon_k^2(\tau) + \theta_k(\tau)\right) + \frac{n}{2}\sum_{k=1}^{n} (\dot{w}_k - \dot{w}_{k-1})^2(\tau) \leq C\left(1 + \int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1} \left(\dot{w}_k^2(t) + \theta_k(t)\right)\, \mathrm{d}t\right).$$

Using Grönwall's lemma A.2, we see that the approximate solution remains bounded in the maximal interval of existence $[0, T_n]$. Hence the solution exists globally, and for every $n \in \mathbb{N}$ we have $T_n = T$. We thus have obtained

$$\frac{1}{n}\sum_{k=1}^{n-1} \left(\dot{w}_k^2 + \varepsilon_k^2 + \theta_k\right)(\tau) + \frac{n}{2}\sum_{k=1}^{n} (\dot{w}_k - \dot{w}_{k-1})^2(\tau) \leq C. \tag{7.41}$$

for all $\tau \in [0, T]$. In particular the approximate solutions exist globally, and $T_n = T$.

7.2.3 Discrete Dafermos estimate

We test (7.5) by $\theta_k^{-1/3}$ and obtain

$$\begin{aligned} 0 = {} & \frac{1}{n}\sum_{k=1}^{n-1}\left(-\dot{\theta}_k\theta_k^{-1/3}\right) + n\sum_{k=1}^{n-1}(\theta_{k-1} - 2\theta_k + \theta_{k-1})\,\theta_k^{-1/3} + \frac{1}{n}\sum_{k=1}^{n-1}\dot{m}_k\, M_k\, \theta_k^{-1/3} + \frac{1}{n}\sum_{k=1}^{n-1} D_k\theta_k^{-1/3} \\ & + \frac{1}{n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k^2\,\theta_k^{-1/3} - \frac{1}{n}\sum_{k=1}^{n-1}\theta_k\dot{\varepsilon}_k\theta_k^{-1/3} + \frac{1}{n}\sum_{k=1}^{n-1}\dot{\chi}_k(K_k - L)\,\theta_k^{-1/3} + \frac{1}{n}\sum_{k=1}^{n-1} g_k\,\theta_k^{-1/3} \\ & =: T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8. \end{aligned}$$

Observe that $T_4 \geq 0$ since $D_k \geq 0$, and we already proved that the temperature stays positive. Hence we can rewrite the previous equality as an inequality of the form

$$T_1 + T_2 + T_3 + T_5 + T_6 + T_7 + T_8 \leq 0. \tag{7.42}$$

We estimate the remaining terms. The term T_1 can be rewritten as

$$T_1 := \frac{1}{n}\sum_{k=1}^{n-1}\left(-\dot{\theta}_k\theta_k^{-1/3}\right) = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3}{2n}\sum_{k=1}^{n-1}\theta_k^{2/3}\right).$$

Concerning the term T_2, using the summation by parts formula (7.10) and the elementary inequality

$$-(x-y)(x^{-1/3}-y^{-1/3}) \geq 3(x^{1/3}-y^{1/3})^2 \quad \forall\, x,y>0$$

with the choice $x=\theta_k$, $y=\theta_{k-1}$, we deduce

$$\begin{aligned}
T_2 &:= n\sum_{k=1}^{n-1}(\theta_{k-1}-2\theta_k+\theta_{k-1})\,\theta_k^{-1/3} \\
&= -n\sum_{k=1}^{n}(\theta_k-\theta_{k-1})(\theta_k^{-1/3}-\theta_{k-1}^{-1/3}) + n[(\theta_n-\theta_{n-1})\theta_n^{-1/3}-(\theta_1-\theta_0)\theta_0^{-1/3}] \\
&\geq 3n\sum_{k=1}^{n}(\theta_k^{1/3}-\theta_{k-1}^{1/3})^2,
\end{aligned}$$

where the term $n[\theta_n^{-1/3}(\theta_n-\theta_{n-1})-\theta_0^{-1/3}(\theta_1-\theta_0)]$ vanishes due to (7.9).

Concerning the term T_3, we observe that by (7.18)

$$T_3 := \frac{1}{n}\sum_{k=1}^{n-1}\dot{m}_k\, M_k\, \theta_k^{-1/3} \geq -\frac{a\,L^2}{\theta_{\mathrm{ref}}\rho}\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{2/3}.$$

The same happens for the term T_7 noticing that, by (7.17),

$$T_7 := \frac{1}{n}\sum_{k=1}^{n-1}\dot{\chi}_k(K_k-L)\theta_k^{-1/3} \geq -\frac{2L^2}{\theta_{\mathrm{ref}}\rho}\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{2/3}.$$

Concerning the term T_8, by (7.19) we have

$$T_8 := \frac{1}{n}\sum_{k=1}^{n-1}g_k\theta_k^{-1/3} \geq -g_1\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{2/3}.$$

Hence from (7.42) we deduce

$$3n\sum_{k=1}^{n}(\theta_k^{1/3}-\theta_{k-1}^{1/3})^2 + \frac{1}{n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k^2\,\theta_k^{-1/3} \leq C_1\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{2/3} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3}{2n}\sum_{k=1}^{n-1}\theta_k^{2/3}\right) + \frac{1}{n}\sum_{k=1}^{n-1}|\dot{\varepsilon}_k|\theta_k^{2/3},$$

where we set

$$C_1 := \frac{L^2}{\theta_{\mathrm{ref}}\rho}(2+a)+g_1.$$

Integrating in time we obtain for all $\tau\in[0,T]$

$$\begin{aligned}
&\int_0^\tau\left(3n\sum_{k=1}^{n}(\theta_k^{1/3}-\theta_{k-1}^{1/3})^2 + \frac{1}{n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k^2\,\theta_k^{-1/3}\right)\mathrm{d}t + \frac{3}{2n}\sum_{k=1}^{n-1}(\theta_k^0)^{2/3} \\
&\leq C_1\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{2/3}\,\mathrm{d}t + \frac{3}{2n}\sum_{k=1}^{n-1}\theta_k^{2/3}(\tau) + \int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}|\dot{\varepsilon}_k|\theta_k^{2/3}\,\mathrm{d}t.
\end{aligned} \tag{7.43}$$

The first two terms on the right-hand side of (7.43) are bounded due to (7.41). We estimate third term by the discrete Hölder's inequality (see Remark A.10) as follows. We have

$$\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}|\dot{\varepsilon}_k|\theta_k^{2/3}\,\mathrm{d}t = \int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}|\dot{\varepsilon}_k|\theta_k^{5/6}\theta_k^{-1/6}\,\mathrm{d}t \leq \left(\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{5/3}\,\mathrm{d}t\right)^{1/2}\left(\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{-1/3}\dot{\varepsilon}_k^2\,\mathrm{d}t\right)^{1/2},$$

thus Young's inequality and (7.43) yield

$$\int_0^\tau \left(n\sum_{k=1}^{n} (\theta_k^{1/3}-\theta_{k-1}^{1/3})^2 + \frac{1}{n}\sum_{k=1}^{n-1} \dot\varepsilon_k^2\,\theta_k^{-1/3} \right) \mathrm{d}t \le C\left(1+\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{5/3}\,\mathrm{d}t \right). \tag{7.44}$$

We now apply the discrete Gagliardo-Nirenberg inequality (see Remark A.11) with $v=\theta_k^{1/3}$, $s=3$, $q=5$, $p=2$, $\varrho=4/25$. By (7.9) we have

$$\begin{aligned}\left(\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{5/3}\right)^{1/5} &\le C\left[\left(\frac{1}{n}\sum_{k=1}^{n-1}\theta_k\right)^{1/3} + \left(\frac{1}{n}\sum_{k=1}^{n-1}\theta_k\right)^{21/75}\left(n\sum_{k=1}^{n}(\theta_k^{1/3}-\theta_{k-1}^{1/3})^2\right)^{2/25}\right]\\ &\le C\left[1+\left(n\sum_{k=1}^{n}(\theta_k^{1/3}-\theta_{k-1}^{1/3})^2\right)^{2/25}\right],\end{aligned}$$

where the second inequality follows from (7.41). Raising to the power 5, integrating in time and inserting the resulting inequality into (7.44) we obtain

$$\begin{aligned}\int_0^\tau \left(n\sum_{k=1}^{n} (\theta_k^{1/3}-\theta_{k-1}^{1/3})^2 + \frac{1}{n}\sum_{k=1}^{n-1} \dot\varepsilon_k^2\,\theta_k^{-1/3} \right) \mathrm{d}t &\le C\left[1+\int_0^\tau\left(n\sum_{k=1}^{n}(\theta_k^{1/3}-\theta_{k-1}^{1/3})^2\right)^{2/5}\mathrm{d}t\right]\\ &\le C+\frac{3}{5}C^{5/3}+\frac{2}{5}\int_0^\tau n\sum_{k=1}^{n}(\theta_k^{1/3}-\theta_{k-1}^{1/3})^2\,\mathrm{d}t,\end{aligned}$$

where in the last inequality we used Young's inequality with conjugate exponents $(5/3\,,\,5/2)$. This gives

$$\int_0^\tau \left(n\sum_{k=1}^{n} (\theta_k^{1/3}-\theta_{k-1}^{1/3})^2 + \frac{1}{n}\sum_{k=1}^{n-1} \dot\varepsilon_k^2\,\theta_k^{-1/3} \right) \mathrm{d}t \le C. \tag{7.45}$$

Applying again the discrete Gagliardo-Nirenberg inequality with the choices $v=\theta_k^{1/3}$, $s=3$, $q=8$, $p=2$, $N=1$, $\varrho=1/4$, by (7.9) we obtain that

$$\left(\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{8/3}\right)^{1/8} \le C\left[\left(\frac{1}{n}\sum_{k=1}^{n-1}\theta_k\right)^{1/3} + \left(\frac{1}{n}\sum_{k=1}^{n-1}\theta_k\right)^{1/4}\left(n\sum_{k=1}^{n}(\theta_k^{1/3}-\theta_{k-1}^{1/3})^2\right)^{1/8}\right].$$

This, after a time integration, together with (7.41) and (7.45) brings the estimate

$$\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{8/3}(t)\,\mathrm{d}t \le C. \tag{7.46}$$

Now that we have more regularity for the temperature, we come back to (7.24) and derive a further estimate. We start by rewriting it as

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2n}\sum_{k=1}^{n-1}\left(\dot w_k^2+\varepsilon_k^2\right)+\frac{1}{n}\sum_{k=1}^{n-1}\theta_{\mathrm{ref}}\varepsilon_k+\frac{n}{2}\sum_{k=1}^{n}(\dot w_k-\dot w_{k-1})^2\right)+\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^2\\ &=\frac{1}{n}\sum_{k=1}^{n-1}\Big(\left(\theta_k-P_0[m_k,\chi_k,\varepsilon_k]\right)\dot\varepsilon_k+\dot f_k\dot w_k\Big).\end{aligned}$$

We now integrate in time $\int_0^\tau \,\mathrm{d}t$ for some $\tau \in [0,T]$. Concerning the initial conditions, we have already pointed out in (7.38) that the term containing $\varepsilon_k(0)$ vanishes, whereas the other two terms involving the initial data are bounded thanks to (7.37). Hence Young's inequality and (7.26) give

$$\begin{aligned}
&\frac{1}{2n}\sum_{k=1}^{n-1}\left(\dot{w}_k^2(\tau)+\frac{3}{4}\varepsilon_k^2(\tau)\right)+\frac{n}{2}\sum_{k=1}^{n}(\dot{w}_k-\dot{w}_{k-1})^2(\tau)+\int_0^\tau \frac{1}{2n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k^2\,\mathrm{d}t\\
&\le C\left(1+\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\left(\theta_k^2+|P_0[m_k,\chi_k,\varepsilon_k]|^2+\dot{f}_k^2+\dot{w}_k^2\right)\,\mathrm{d}t\right).
\end{aligned}$$

Then, by Remark 7.2 (i) and estimates (7.29), (7.41) and (7.46) we infer

$$\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k^2(t)\,\mathrm{d}t \le C, \tag{7.47}$$

where we neglected the first two terms on the left-hand side since they were estimated in (7.41).

7.2.4 Higher order estimates

Before continuing, we need to extend the validity of (7.2), (7.6), (7.7) to $k=0$ and $k=n$. For this purpose for $k=0$ we solve the system of these three equations in the unknowns $\varepsilon_0, \chi_0, m_0$, and for $k=n$ in the unknowns $\varepsilon_n, \chi_n, m_n$. Let us deal with the case $k=0$, for $k=n$ the argument is the same. By (7.9) we have

$$0 = \varepsilon_0 + P_0[m_0,\chi_0,\varepsilon_0] + \dot{\varepsilon}_0 - \theta_0 + \theta_{\mathrm{ref}}, \tag{7.48}$$

$$\rho\,\chi_0 = \mathfrak{s}_{[0,1]}[\chi_0^0, A_0], \qquad A_0(t) = \int_0^t \left(\frac{L}{\theta_{\mathrm{ref}}}(\theta_0-\theta_{\mathrm{ref}})+M_0\right)(\tau)\,\mathrm{d}\tau, \tag{7.49}$$

$$m_0 = \mathfrak{s}_{[0,\infty)}[0,S_0], \qquad S_0(t) = \int_0^t (-h(\dot{\chi}_0)+D_0^*)(\tau)\,\mathrm{d}\tau, \tag{7.50}$$

where

$$\begin{aligned}
M_0(t) &= -\frac{1}{2}\int_0^\infty \gamma_\chi(m_0(t),\chi_0(t),r)\mathfrak{s}_r^2[\varepsilon_0](t)\,\mathrm{d}r,\\
D_0(t) &= \int_0^\infty \gamma(m_0(t),\chi_0(t),r)\,\mathfrak{s}_r[\varepsilon_0](t)(\varepsilon_0-\mathfrak{s}_r[\varepsilon_0])_t(t)\,\mathrm{d}r,\\
D_0^*(t) &= \frac{1}{n}\sum_{j=1}^{n-1}\lambda_{-j}D_j(t),\\
\lambda_i &= \lambda(i/n),
\end{aligned}$$

and with initial conditions

$$\left.\begin{aligned}
&\varepsilon_0(0)=0,\\
&m_0(0)=0,\\
&\chi_0(0)=\chi_0^0 := \chi^0(0).
\end{aligned}\right\} \tag{7.51}$$

By Proposition B.5 system (7.48)–(7.50) admits a Lipschitz continuous right-hand side, which implies that $\varepsilon_0, \chi_0, m_0$ are determined uniquely. The same holds for $\varepsilon_n, \chi_n, m_n$.
In this way (7.2) makes sense for all $k = 0, \dots, n$, so that we can take space increments as follows

$$n\,(\dot u_k - \dot u_{k-1}) = n\,(\varepsilon_k - \varepsilon_{k-1} + P_0[m_k, \chi_k, \varepsilon_k] - P_0[m_{k-1}, \chi_{k-1}, \varepsilon_{k-1}] + \dot\varepsilon_k - \dot\varepsilon_{k-1} - \theta_k + \theta_{k-1})$$

for all $k = 1, \dots, n$. We now take the square of both sides and test by 1, obtaining

$$\begin{aligned} & n\sum_{k=1}^{n} (\dot u_k - \dot u_{k-1})^2 \\ &= n\sum_{k=1}^{n} (\varepsilon_k - \varepsilon_{k-1} + P_0[m_k, \chi_k, \varepsilon_k] - P_0[m_{k-1}, \chi_{k-1}, \varepsilon_{k-1}] + \dot\varepsilon_k - \dot\varepsilon_{k-1} - \theta_k + \theta_{k-1})^2 . \end{aligned}$$

The summation by parts formula (7.10) and the boundary conditions (7.9) give on the left-hand side

$$n\sum_{k=1}^{n}(\dot u_k - \dot u_{k-1})^2 = -n\sum_{k=1}^{n-1} \dot u_k(\dot u_{k+1} - 2\dot u_k + \dot u_{k-1}),$$

and equations (7.2)–(7.3) imply

$$-n\sum_{k=1}^{n-1} \dot u_k\,(\dot u_{k+1} - 2\dot u_k + \dot u_{k-1}) = \frac{1}{n}\sum_{k=1}^{n-1} (\varepsilon_k + P_0[m_k, \chi_k, \varepsilon_k] + \dot\varepsilon_k - \theta_k + \theta_{\mathrm{ref}})\,(\ddot w_k - \ddot\varepsilon_k - \dot f_k).$$

Then we get

$$\begin{aligned} & n\sum_{k=1}^{n}(\varepsilon_k - \varepsilon_{k-1} + P_0[m_k, \chi_k, \varepsilon_k] - P_0[m_{k-1}, \chi_{k-1}, \varepsilon_{k-1}] + \dot\varepsilon_k - \dot\varepsilon_{k-1} - \theta_k + \theta_{k-1})^2 \\ &= \frac{1}{n}\sum_{k=1}^{n-1}(\ddot w_k - \ddot\varepsilon_k - \dot f_k)(\varepsilon_k + P_0[m_k, \chi_k, \varepsilon_k] + \dot\varepsilon_k - \theta_k + \theta_{\mathrm{ref}}). \end{aligned}$$

By the elementary inequality $(a+b)^2 \geq \frac{a^2}{2} - b^2 \ \ \forall\, a, b \in \mathbb{R}$ we obtain

$$\begin{aligned} & \frac{n}{2}\sum_{k=1}^{n} \Big((\varepsilon_k - \varepsilon_{k-1}) + (\dot\varepsilon_k - \dot\varepsilon_{k-1}) \Big)^2 \\ & \leq n\sum_{k=1}^{n} \Big((P_0[m_k, \chi_k, \varepsilon_k] - P_0[m_{k-1}, \chi_{k-1}, \varepsilon_{k-1}])^2 + (\theta_k - \theta_{k-1})^2 \Big) \\ & + \frac{1}{n}\sum_{k=1}^{n-1}(\ddot w_k - \ddot\varepsilon_k - \dot f_k)(\varepsilon_k + P_0[m_k, \chi_k, \varepsilon_k] + \dot\varepsilon_k - \theta_k + \theta_{\mathrm{ref}}) \end{aligned}$$

or, equivalently,

$$\begin{aligned} & \frac{n}{2}\sum_{k=1}^{n} \Big((\varepsilon_k - \varepsilon_{k-1})^2 + (\dot\varepsilon_k - \dot\varepsilon_{k-1})^2 \Big) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{n}{2}\sum_{k=1}^{n}(\varepsilon_k - \varepsilon_{k-1})^2 \right) \\ & \leq n\sum_{k=1}^{n} \Big((P_0[m_k, \chi_k, \varepsilon_k] - P_0[m_{k-1}, \chi_{k-1}, \varepsilon_{k-1}])^2 + (\theta_k - \theta_{k-1})^2 \Big) \\ & + \frac{1}{n}\sum_{k=1}^{n-1}(\ddot w_k - \ddot\varepsilon_k - \dot f_k)(\varepsilon_k + P_0[m_k, \chi_k, \varepsilon_k] + \dot\varepsilon_k - \theta_k + \theta_{\mathrm{ref}}). \end{aligned} \tag{7.52}$$

We briefly focus on the last summand. We have

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n-1}(\ddot{w}_k - \ddot{\varepsilon}_k - \dot{f}_k)(\varepsilon_k + P_0[m_k,\chi_k,\varepsilon_k] + \dot{\varepsilon}_k - \theta_k + \theta_{\text{ref}})\\
&= \frac{1}{n}\sum_{k=1}^{n-1}(\ddot{w}_k - \ddot{\varepsilon}_k)\dot{\varepsilon}_k + \frac{1}{n}\sum_{k=1}^{n-1}(\ddot{w}_k - \ddot{\varepsilon}_k)(\varepsilon_k + P_0[m_k,\chi_k,\varepsilon_k] - \theta_k + \theta_{\text{ref}})\\
&- \frac{1}{n}\sum_{k=1}^{n-1}\dot{f}_k(\varepsilon_k + P_0[m_k,\chi_k,\varepsilon_k] + \dot{\varepsilon}_k - \theta_k + \theta_{\text{ref}}).
\end{aligned} \tag{7.53}$$

Concerning the first term on the right-hand side, using (7.4) and (7.10) we deduce

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n-1}(\ddot{w}_k - \ddot{\varepsilon}_k)\dot{\varepsilon}_k &= n\sum_{k=1}^{n-1}\dot{w}_k(\dot{w}_{k+1} - 2\dot{w}_k + \dot{w}_{k-1}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k^2\right)\\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{n}{2}\sum_{k=1}^{n}(\dot{w}_k - \dot{w}_{k-1})^2 + \frac{1}{2n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k^2\right),
\end{aligned} \tag{7.54}$$

whereas the second term is such that

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n-1}(\ddot{w}_k - \ddot{\varepsilon}_k)(\varepsilon_k + P_0[m_k,\chi_k,\varepsilon_k] - \theta_k + \theta_{\text{ref}})\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{n}\sum_{k=1}^{n-1}(\dot{w}_k - \dot{\varepsilon}_k)(\varepsilon_k + P_0[m_k,\chi_k,\varepsilon_k] - \theta_k + \theta_{\text{ref}})\right)\\
&- \frac{1}{n}\sum_{k=1}^{n-1}(\dot{w}_k - \dot{\varepsilon}_k)(\dot{\varepsilon}_k + P_0[m_k,\chi_k,\varepsilon_k]_t - \dot{\theta}_k).
\end{aligned} \tag{7.55}$$

Inserting (7.54) and (7.55) in (7.53), from (7.52) we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{n}{2}\sum_{k=1}^{n}(\dot{w}_k - \dot{w}_{k-1})^2 + \frac{1}{2n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k^2 + \frac{n}{2}\sum_{k=1}^{n}(\varepsilon_k - \varepsilon_{k-1})^2\right) + \frac{n}{2}\sum_{k=1}^{n}\left((\varepsilon_k - \varepsilon_{k-1})^2 + (\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})^2\right)\\
&\le n\sum_{k=1}^{n}\left((P_0[m_k,\chi_k,\varepsilon_k] - P_0[m_{k-1},\chi_{k-1},\varepsilon_{k-1}])^2 + (\theta_k - \theta_{k-1})^2\right)\\
&+ \frac{1}{n}\sum_{k=1}^{n-1}|\dot{f}_k|\,|\varepsilon_k + P_0[m_k,\chi_k,\varepsilon_k] + \dot{\varepsilon}_k - \theta_k + \theta_{\text{ref}}|\\
&+ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{n}\sum_{k=1}^{n-1}(\dot{w}_k - \dot{\varepsilon}_k)(\varepsilon_k + P_0[m_k,\chi_k,\varepsilon_k] - \theta_k + \theta_{\text{ref}})\right) + \frac{1}{n}\sum_{k=1}^{n-1}|\dot{w}_k - \dot{\varepsilon}_k|\,|\dot{\varepsilon}_k + P_0[m_k,\chi_k,\varepsilon_k]_t - \dot{\theta}_k|.
\end{aligned} \tag{7.56}$$

We now integrate in time $\int_0^\tau$ for some $\tau \in [0,T]$. The initial conditions

$$\frac{n}{2}\sum_{k=1}^{n}(\dot{w}_k - \dot{w}_{k-1})^2(0) + \frac{1}{2n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k^2(0) + \frac{n}{2}\sum_{k=1}^{n}(\varepsilon_k - \varepsilon_{k-1})^2(0)$$

can be controlled in the following way. For the first term we use (7.37), whereas the third term vanishes by Remark 7.5. It remains to derive a bound for the second term. To this aim we test (7.31) by $\dot{\varepsilon}_k(0)$, getting

$$\frac{1}{n}\sum_{k=1}^{n-1}\dot{w}_k(0)\dot{\varepsilon}_k(0) - \frac{1}{n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k^2(0) = \frac{1}{n}\sum_{k=1}^{n-1}f_k(0)\dot{\varepsilon}_k(0).$$

By (7.33) and Young's inequality it holds

$$n\sum_{k=1}^{n}(\dot w_k-\dot w_{k-1})^2(0)+\frac{1}{2n}\sum_{k=1}^{n-1}\dot\varepsilon_k^2(0)\le\frac{1}{2n}\sum_{k=1}^{n-1}f_k^2(0),$$

hence by (7.36) we finally get

$$\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^2(0)\le C. \tag{7.57}$$

Concerning the initial conditions coming from the right-hand side of (7.56), by (7.37), (7.39) and (7.57) we obtain the bound

$$\left|\frac{1}{n}\sum_{k=1}^{n-1}(\dot w_k-\dot\varepsilon_k)(0)(\varepsilon_k+P_0[m_k,\chi_k,\varepsilon_k]-\theta_k+\theta_{\mathrm{ref}})(0)\right|\le C$$

recalling that $\varepsilon_k(0)=P_0[m_k,\chi_k,\varepsilon_k](0)=0$ by Remark 7.5. Thus we get

$$\begin{aligned}
&\frac{n}{2}\sum_{k=1}^{n}(\dot w_k-\dot w_{k-1})^2(\tau)+\frac{1}{2n}\sum_{k=1}^{n-1}\dot\varepsilon_k^2(t\tau)+\frac{n}{2}\sum_{k=1}^{n}(\varepsilon_k-\varepsilon_{k-1})^2(\tau)\\
&+\int_0^\tau\frac{n}{2}\sum_{k=1}^{n}\Big((\varepsilon_k-\varepsilon_{k-1})^2(t)+(\dot\varepsilon_k-\dot\varepsilon_{k-1})^2(t)\Big)\,\mathrm{d}t\\
&\le\int_0^\tau n\sum_{k=1}^{n}\Big((P_0[m_k,\chi_k,\varepsilon_k]-P_0[m_{k-1},\chi_{k-1},\varepsilon_{k-1}])^2(t)+(\theta_k-\theta_{k-1})^2(t)\Big)\,\mathrm{d}t\\
&+\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}|\dot f_k(t)|\,|\varepsilon_k+P_0[m_k,\chi_k,\varepsilon_k]+\dot\varepsilon_k-\theta_k+\theta_{\mathrm{ref}}|(t)\,\mathrm{d}t\\
&+\frac{1}{n}\sum_{k=1}^{n-1}(\dot w_k-\dot\varepsilon_k)(t)(\varepsilon_k+P_0[m_k,\chi_k,\varepsilon_k]-\theta_k+\theta_{\mathrm{ref}})(\tau)\\
&+\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}|\dot w_k-\dot\varepsilon_k|(t)|\dot\varepsilon_k+P_0[m_k,\chi_k,\varepsilon_k]_t-\dot\theta_k|(t)\,\mathrm{d}t+C\\
&=:H_1+H_2+H_3+H_4+C.
\end{aligned}\tag{7.58}$$

Let us deal first with the last three terms. By the discrete Hölder's inequality (see Remark A.10) and by Hölder's inequality in time it holds

$$\begin{aligned}
H_2&:=\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}|\dot f_k|\,|\varepsilon_k+P_0[m_k,\chi_k,\varepsilon_k]+\dot\varepsilon_k-\theta_k+\theta_{\mathrm{ref}}|\,\mathrm{d}t\\
&\le\int_0^\tau\left(\frac{1}{n}\sum_{k=1}^{n-1}\dot f_k^2\right)^{1/2}\left(\frac{1}{n}\sum_{k=1}^{n-1}|\varepsilon_k+P_0[m_k,\chi_k,\varepsilon_k]+\dot\varepsilon_k-\theta_k+\theta_{\mathrm{ref}}|^2\right)^{1/2}\mathrm{d}t\\
&\le\left(\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\dot f_k^2\,\mathrm{d}t\right)^{1/2}\left(\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}|\varepsilon_k+P_0[m_k,\chi_k,\varepsilon_k]+\dot\varepsilon_k-\theta_k+\theta_{\mathrm{ref}}|^2\,\mathrm{d}t\right)^{1/2}\le C
\end{aligned}\tag{7.59}$$

where for the final bound we used (7.29), Remark 7.2 (i) and the previous a priori estimates (7.41), (7.46), (7.47). Arguing analogously we also obtain

$$
\begin{aligned}
H_3 &:= \frac{1}{n}\sum_{k=1}^{n-1}(\dot w_k - \dot\varepsilon_k)(\varepsilon_k + P_0[m_k,\chi_k,\varepsilon_k] - \theta_k + \theta_{\mathrm{ref}}) \\
&\le \left(\frac{1}{n}\sum_{k=1}^{n-1}(\dot w_k - \dot\varepsilon_k)^2\right)^{1/2}\left(\frac{1}{n}\sum_{k=1}^{n-1}(\varepsilon_k + P_0[m_k,\chi_k,\varepsilon_k] - \theta_k + \theta_{\mathrm{ref}})^2\right)^{1/2} \\
&\le C\left(1 + \frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^2\right)^{1/2}\left(1 + \frac{1}{n}\sum_{k=1}^{n-1}\theta_k^2\right)^{1/2} = 2C\left(\frac{1}{4} + \frac{1}{4n}\sum_{k=1}^{n-1}\dot\varepsilon_k^2\right)^{1/2}\left(1 + \frac{1}{n}\sum_{k=1}^{n-1}\theta_k^2\right)^{1/2} \\
&\le \frac{1}{8n}\sum_{k=1}^{n-1}\dot\varepsilon_k^2 + C\left(1 + \frac{1}{n}\sum_{k=1}^{n-1}\theta_k^2\right),
\end{aligned}
$$

where in the last line we used Young's inequality. Note that applying first the discrete Hölder's inequality and then Hölder's inequality in time we obtain

$$
\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^2(\tau) &= \frac{1}{n}\sum_{k=1}^{n-1}(\theta_k(0))^2 + 2\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\theta_k\dot\theta_k\,\mathrm{d}t \\
&\le C\left(1 + \left(\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\theta_k^2\,\mathrm{d}t\right)^{1/2}\left(\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\dot\theta_k^2\,\mathrm{d}t\right)^{1/2}\right),
\end{aligned}
$$

where we used also (7.39). Then estimate (7.47) yields

$$
H_3 \le \frac{1}{8n}\sum_{k=1}^{n-1}\dot\varepsilon_k^2 + C\left(1 + \left(\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\dot\theta_k^2\,\mathrm{d}t\right)^{1/2}\right). \tag{7.60}
$$

Concerning the term H_4, note that

$$
\begin{aligned}
|P_0[m_k,\chi_k,\varepsilon_k]_t| &\le \left|\dot m_k\int_0^\infty \gamma_m(m_k,\chi_k,r)\mathfrak{s}_r[\varepsilon_k]\,\mathrm{d}r\right| \\
&\quad + \left|\dot\chi_k\int_0^\infty \gamma_\chi(m_k,\chi_k,r)\mathfrak{s}_r[\varepsilon_k]\,\mathrm{d}r\right| + \left|\int_0^\infty \gamma(m_k,\chi_k,r)(\mathfrak{s}_r[\varepsilon_k])_t\,\mathrm{d}r\right|.
\end{aligned}
$$

Then

- by (B.14) we have $|\mathfrak{s}_r[\varepsilon_k]| \le r$;
- by (B.21) we have $|\mathfrak{s}_r[\varepsilon_k]_t| \le |\dot\varepsilon_k|$;
- equation (7.6) and identity (B.21) (see also Remark B.2) yield

$$
|\dot\chi_k| \le \left|\frac{L}{\theta_{\mathrm{ref}}}(\theta_k - \theta_{\mathrm{ref}}) + K_k\right|,
$$

  from which by Remark 7.4

$$
|\dot\chi_k| \le C(1+\theta_k); \tag{7.61}
$$

- equation (7.7), Hypothesis 7.1 (vi) and Remark 7.4 yield

$$
|\dot m_k| \le |-h(\dot\chi_k) + D_k^*| \le C\left(|\dot\chi_k| + \frac{1}{n}\sum_{k=1}^{n-1}|\dot\varepsilon_k|\right),
$$

from which by (7.61)

$$|\dot{m}_k| \leq C \left(1 + \theta_k + \frac{1}{n} \sum_{j=1}^{n-1} |\dot{\varepsilon}_j| \right). \tag{7.62}$$

From Hypothesis 7.1 (i) we thus obtain the bound

$$|P_0[m_k, \chi_k, \varepsilon_k]_t| \leq C \left(1 + \theta_k + |\dot{\varepsilon}_k| + \frac{1}{n} \sum_{j=1}^{n-1} |\dot{\varepsilon}_j| \right) \quad \text{a. e.} \tag{7.63}$$

and this, together with the discrete Hölder's inequality and Hölder's inequality in time employed as above, enables us to get

$$\begin{aligned} H_4 &:= \int_0^\tau \frac{1}{n} \sum_{k=1}^{n-1} |\dot{w}_k - \dot{\varepsilon}_k| |\dot{\varepsilon}_k + P_0[m_k, \chi_k, \varepsilon_k]_t - \dot{\theta}_k| \, \mathrm{d}t \\ &\leq C \left(1 + \int_0^\tau \frac{1}{n} \sum_{k=1}^{n-1} \dot{\varepsilon}_k^2 \, \mathrm{d}t \right)^{1/2} \left(\int_0^\tau \frac{1}{n} \sum_{k=1}^{n-1} \left(1 + \dot{\varepsilon}_k^2 + \theta_k^2 + \frac{1}{n} \sum_{j=1}^{n-1} \dot{\varepsilon}_j^2 + \dot{\theta}_k^2 \right) \, \mathrm{d}t \right)^{1/2} \\ &\leq C \left(1 + \left(\int_0^\tau \frac{1}{n} \sum_{k=1}^{n-1} \dot{\theta}_k^2 \, \mathrm{d}t \right)^{1/2} \right) \end{aligned} \tag{7.64}$$

where we used also the previous a priori estimates (7.41), (7.46) and (7.47). It is a bit more complicated to deal with the term H_1. First of all we have by Hypothesis 7.1 (i) and (B.14)

$$\begin{aligned} &|P_0[m_k, \chi_k, \varepsilon_k] - P_0[m_{k-1}, \chi_{k-1}, \varepsilon_{k-1}]| \\ &= \left| \int_0^\infty \left(\gamma(m_k, \chi_k, r) \mathfrak{s}_r[\varepsilon_k] - \gamma(m_{k-1}, \chi_{k-1}, r) \mathfrak{s}_r[\varepsilon_{k-1}] \right) \, \mathrm{d}r \right| \\ &\leq \int_0^\infty \Big(|\gamma(m_k, \chi_k, r) - \gamma(m_{k-1}, \chi_k, r)| \, |\mathfrak{s}_r[\varepsilon_k]| + |\gamma(m_{k-1}, \chi_k, r) - \gamma(m_{k-1}, \chi_{k-1}, r)| \, |\mathfrak{s}_r[\varepsilon_k]| \, \mathrm{d}r \Big) \\ &+ \int_0^\infty |\gamma(m_{k-1}, \chi_{k-1}, r)| |\mathfrak{s}_r[\varepsilon_k] - \mathfrak{s}_r[\varepsilon_{k-1}]| \, \mathrm{d}r \\ &\leq \int_0^\infty \Big(\gamma^*(r) |m_k - m_{k-1}| r + \gamma^*(r) |\chi_k - \chi_{k-1}| r \, \mathrm{d}r \Big) + \int_0^\infty |\gamma(m_{k-1}, \chi_{k-1}, r)| |\mathfrak{s}_r[\varepsilon_k] - \mathfrak{s}_r[\varepsilon_{k-1}]| \, \mathrm{d}r. \end{aligned}$$

Then by Hypothesis 7.1 (i) and Proposition B.5 (note that $\mathfrak{s}_r[\varepsilon_k](0) = \mathfrak{s}_r[\varepsilon_{k-1}](0) = 0$, see Remark 7.5)

$$\begin{aligned} &|P_0[m_k, \chi_k, \varepsilon_k] - P_0[m_{k-1}, \chi_{k-1}, \varepsilon_{k-1}]| \, (t) \\ &\leq C \left(|m_k - m_{k-1}|(t) + |\chi_k - \chi_{k-1}|(t) + \int_0^t |\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1}|(s) \, \mathrm{d}s \right). \end{aligned} \tag{7.65}$$

By (7.7) and again Proposition B.5 (note that $m_k(0) = m_{k-1}(0) = 0$ by (7.8)), it holds

$$\begin{aligned}\int_0^t |\dot m_k - \dot m_{k-1}|(s)\,\mathrm{d}s &\le \int_0^t |\dot S_k - \dot S_{k-1}|(s)\,\mathrm{d}s \\ &\le \int_0^t \left(|h(\dot\chi_{k-1}) - h(\dot\chi_k)| + \left|D_k^* - D_{k-1}^*\right|\right)\,\mathrm{d}s \\ &\le \int_0^t \left(a\,|\dot\chi_k - \dot\chi_{k-1}| + \frac{1}{n}\sum_{j=1}^{n-1} |\lambda_{k-j} - \lambda_{k-j-1}| D_j(s)\right)\,\mathrm{d}s \\ &\le \int_0^t \left(a\,|\dot\chi_k - \dot\chi_{k-1}| + \frac{1}{n}\sum_{j=1}^{n-1} \frac{\Lambda}{n} M|\dot\varepsilon_j|\right)\,\mathrm{d}s,\end{aligned}$$

where we used the lipschitzianity of h (see Hypothesis 7.1 (vi)), Remark 7.4 and Hypothesis 7.1 (iii) together with Lagrange's theorem. Thus, by the discrete Hölder's inequality and estimate (7.47), we obtain

$$\begin{aligned}\int_0^t |\dot m_k - \dot m_{k-1}|\,\mathrm{d}s &\le C\left(\int_0^t |\dot\chi_k - \dot\chi_{k-1}|\,\mathrm{d}s + \frac{1}{n}\int_0^t \left(\frac{1}{n}\sum_{j=1}^{n-1}\dot\varepsilon_j^2\right)^{1/2}\left(\frac{1}{n}\sum_{j=1}^{n-1} 1^2\right)^{1/2}\mathrm{d}s\right) \\ &\le C\left(\int_0^t |\dot\chi_k - \dot\chi_{k-1}|\,\mathrm{d}s + \frac{1}{n}\right). \qquad (7.66)\end{aligned}$$

We now estimate the terms $|\chi_k - \chi_{k-1}|$ in (7.65) and $\int_0^t |\dot\chi_k - \dot\chi_{k-1}|\,\mathrm{d}s$ in (7.66). By (7.6) and Proposition B.5 we obtain

$$\begin{aligned}\int_0^t |\dot\chi_k - \dot\chi_{k-1}|\,\mathrm{d}s &\le 2\left(|\chi_k^0 - \chi_{k-1}^0| + \int_0^t |\dot A_k - \dot A_{k-1}|\,\mathrm{d}s\right) \\ &\le C\left(|\chi_k^0 - \chi_{k-1}^0| + \int_0^t \left(|\theta_k - \theta_{k-1}| + |K_k - K_{k-1}|\right)\mathrm{d}s\right),\end{aligned}$$

where by Hypothesis 7.1 (i)

$$\begin{aligned}|K_k - K_{k-1}| \le \int_0^\infty \Big(&|\gamma_\chi(m_k,\chi_k,r) - \gamma_\chi(m_{k-1},\chi_k,r)|\mathfrak{s}_r^2[\varepsilon_k] \\ &+ |\gamma_\chi(m_{k-1},\chi_k,r) - \gamma_\chi(m_{k-1},\chi_{k-1},r)|\mathfrak{s}_r^2[\varepsilon_k] + |\gamma_\chi(m_{k-1},\chi_{k-1},r)||\mathfrak{s}_r^2[\varepsilon_k] - \mathfrak{s}_r^2[\varepsilon_{k-1}]|\Big)\,\mathrm{d}r \\ \le \int_0^\infty \Big(&\gamma^*(r)|m_k - m_{k-1}|r^2 + \gamma^*(r)|\chi_k - \chi_{k-1}|r^2 + \gamma^*(r)|\mathfrak{s}_r^2[\varepsilon_k] - \mathfrak{s}_r^2[\varepsilon_{k-1}]|\Big)\,\mathrm{d}r.\end{aligned}$$

Note that by (B.14) and Proposition B.5 (recall that $\mathfrak{s}_r[\varepsilon_k](0) = \mathfrak{s}_r[\varepsilon_{k-1}](0) = 0$, see Remark 7.5) it follows

$$|\mathfrak{s}_r^2[\varepsilon_k] - \mathfrak{s}_r^2[\varepsilon_{k-1}]| = |\mathfrak{s}_r[\varepsilon_k] + \mathfrak{s}_r[\varepsilon_{k-1}]||\mathfrak{s}_r[\varepsilon_k] - \mathfrak{s}_r[\varepsilon_{k-1}]| \le 2r\cdot 2\int_0^s |\dot\varepsilon_k - \dot\varepsilon_{k-1}|(s')\,\mathrm{d}s'.$$

Hence by Hypothesis 7.1 (i) we infer

$$\begin{aligned}&\int_0^t |\dot\chi_k - \dot\chi_{k-1}|\,\mathrm{d}s \\ &\le C\bigg(|\chi_k^0 - \chi_{k-1}^0| + \int_0^t \left(|\theta_k - \theta_{k-1}| + |m_k - m_{k-1}| + |\chi_k - \chi_{k-1}| + \int_0^s |\dot\varepsilon_k - \dot\varepsilon_{k-1}|\,\mathrm{d}s'\right)\mathrm{d}s\bigg) \\ &\le C\bigg(|\chi_k^0 - \chi_{k-1}^0| + \int_0^t \left(|\theta_k - \theta_{k-1}| + \int_0^s \left(|\dot m_k - \dot m_{k-1}| + |\dot\chi_k - \dot\chi_{k-1}| + |\dot\varepsilon_k - \dot\varepsilon_{k-1}|\right)\mathrm{d}s'\right)\mathrm{d}s\bigg).\end{aligned}$$

Grönwall's lemma A.2 then yields

$$\begin{aligned}&\int_0^t |\dot\chi_k - \dot\chi_{k-1}|\,\mathrm{d}s\\&\le C\left(|\chi_k^0 - \chi_{k-1}^0| + \int_0^t \left(|\theta_k - \theta_{k-1}| + \int_0^s \left(|\dot m_k - \dot m_{k-1}| + |\dot\varepsilon_k - \dot\varepsilon_{k-1}|\right)\mathrm{d}s'\right)\mathrm{d}s\right).\end{aligned} \tag{7.67}$$

Plugging this back into (7.66) we obtain, using Grönwall's lemma A.2 again together with Fubini Theorem,

$$\begin{aligned}|m_k - m_{k-1}|(t) + |\chi_k - \chi_{k-1}|(t) &\le \int_0^t \left(|\dot m_k - \dot m_{k-1}| + |\dot\chi_k - \dot\chi_{k-1}|\right)\mathrm{d}s\\&\le C\left(|\chi_k^0 - \chi_{k-1}^0| + \int_0^t \left(|\theta_k - \theta_{k-1}| + |\dot\varepsilon_k - \dot\varepsilon_{k-1}|\right)\mathrm{d}s + \frac{1}{n}\right).\end{aligned}$$

Coming back to (7.65) we get

$$\begin{aligned}&\left|P_0[m_k,\chi_k,\varepsilon_k] - P_0[m_{k-1},\chi_{k-1},\varepsilon_{k-1}]\right|(t)\\&\le C\left(|\chi_k^0 - \chi_{k-1}^0| + \int_0^t \left(|\theta_k - \theta_{k-1}| + |\dot\varepsilon_k - \dot\varepsilon_{k-1}|^2\right)\mathrm{d}s + \frac{1}{n}\right),\end{aligned}$$

thus

$$\begin{aligned}H_1 &:= \int_0^\tau n\sum_{k=1}^n \left((P_0[m_k,\chi_k,\varepsilon_k] - P_0[m_{k-1},\chi_{k-1},\varepsilon_{k-1}])^2(t) + (\theta_k - \theta_{k-1})^2(t)\right)\mathrm{d}t\\&\le C\left(1 + n\sum_{k=1}^n (\chi_k^0 - \chi_{k-1}^0)^2 + \int_0^\tau n\sum_{k=1}^n (\theta_k - \theta_{k-1})^2\,\mathrm{d}t + \int_0^\tau\int_0^t n\sum_{k=1}^n (\dot\varepsilon_k - \dot\varepsilon_{k-1})^2\,\mathrm{d}s\,\mathrm{d}t\right).\end{aligned}$$

Note that the term $n\sum_{k=1}^n (\chi_k^0 - \chi_{k-1}^0)^2$ is nothing but the L^2-norm (squared) of the discrete space derivative of term χ_k^0, see Remarks A.10 and A.11. We are going to show that this term is bounded, provided that $\chi_x^0 \in L^2(0,1)$ (as stated in Hypothesis 7.1 (v)). Indeed by definition in (7.8)

$$n\sum_{k=1}^n (\chi_k^0 - \chi_{k-1}^0)^2 = n\sum_{k=1}^n \left(n\int_{(k-1)/n}^{k/n} \chi^0(x)\,\mathrm{d}x - n\int_{(k-2)/n}^{(k-1)/n} \chi^0(x)\,\mathrm{d}x\right)^2.$$

Performing the change of variable $\xi = x + 1/n$ in the second integral we get

$$\begin{aligned}&n\sum_{k=1}^n \left(n\int_{(k-1)/n}^{k/n} \chi^0(x)\,\mathrm{d}x - n\int_{(k-1)/n}^{k/n} \chi^0(\xi - 1/n)\,\mathrm{d}\xi\right)^2\\&= n^3\sum_{k=1}^n \left(\int_{(k-1)/n}^{k/n} \left(\chi^0(x) - \chi^0(x-1/n)\right)\mathrm{d}x\right)^2\\&= n^3\sum_{k=1}^n \left(\int_{(k-1)/n}^{k/n}\int_{x-1/n}^{x} \chi_x^0(\xi)\,\mathrm{d}\xi\,\mathrm{d}x\right)^2.\end{aligned}$$

Since $x \in [(k-1)/n, k/n]$ it holds $x - 1/n \geq (k-2)/n$ and $x \leq k/n$, thus

$$\begin{aligned} n\sum_{k=1}^{n}(\chi_k^0 - \chi_{k-1}^0)^2 &= n^3 \sum_{k=1}^{n}\left(\int_{(k-1)/n}^{k/n}\int_{x-1/n}^{x}\chi_x^0(\xi)\,\mathrm{d}\xi\,\mathrm{d}x\right)^2 \\ &\leq n^3 \sum_{k=1}^{n}\left(\int_{(k-1)/n}^{k/n}\int_{(k-2)/n}^{k/n}\chi_x^0(\xi)\,\mathrm{d}\xi\,\mathrm{d}x\right)^2 \\ &= n^3 \sum_{k=2}^{n}\left(\frac{1}{n}\int_{(k-2)/n}^{k/n}\chi_x^0(\xi)\,\mathrm{d}\xi\right)^2 \\ &\leq n \sum_{k=2}^{n}\left[\left(\frac{2}{n}\right)^{1/2}\left(\int_{(k-2)/n}^{k/n}|\chi_x^0(\xi)|^2\,\mathrm{d}\xi\right)^{1/2}\right]^2 \\ &= 2\sum_{k=2}^{n}\int_{(k-2)/n}^{k/n}|\chi_x^0(\xi)|^2\,\mathrm{d}\xi \leq 2\cdot 2\int_0^1 |\chi_x^0(\xi)|^2\,\mathrm{d}\xi \leq C \end{aligned} \tag{7.68}$$

by Hypothesis 7.1 (v), and where we used also Hölder's inequality in space. Then it holds

$$H_1 \leq C\left(1 + \int_0^\tau n\sum_{k=1}^{n}(\theta_k - \theta_{k-1})^2\,\mathrm{d}t + \int_0^\tau\int_0^t n\sum_{k=1}^{n}(\dot\varepsilon_k - \dot\varepsilon_{k-1})^2\,\mathrm{d}s\,\mathrm{d}t\right). \tag{7.69}$$

Inserting (7.59), (7.60), (7.64) and (7.69) in (7.58) we obtain

$$\begin{aligned} &\frac{n}{2}\sum_{k=1}^{n}(\dot w_k - \dot w_{k-1})^2(\tau) + \frac{3}{8n}\sum_{k=1}^{n-1}\dot\varepsilon_k^2(\tau) + \frac{n}{2}\sum_{k=1}^{n}(\varepsilon_k - \varepsilon_{k-1})^2(\tau) \\ &+ \int_0^\tau \frac{n}{2}\sum_{k=1}^{n}(\varepsilon_k - \varepsilon_{k-1})^2(t)\,\mathrm{d}t + \int_0^\tau \frac{n}{2}\sum_{k=1}^{n}(\dot\varepsilon_k - \dot\varepsilon_{k-1})^2(t)\,\mathrm{d}t \\ &\leq C\left(1 + \left(\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\dot\theta_k^2(t)\,\mathrm{d}t\right)^{1/2} + \int_0^\tau n\sum_{k=1}^{n}(\theta_k - \theta_{k-1})^2(t)\,\mathrm{d}t \right. \\ &\left. + \int_0^\tau\int_0^t n\sum_{k=1}^{n}(\dot\varepsilon_k - \dot\varepsilon_{k-1})^2(s)\,\mathrm{d}s\,\mathrm{d}t\right). \end{aligned}$$

Grönwall's lemma A.2 yields

$$\begin{aligned} &\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^2(\tau) + \int_0^\tau n\sum_{k=1}^{n}(\dot\varepsilon_k - \dot\varepsilon_{k-1})^2(t)\,\mathrm{d}t \\ &\leq C\left(1 + \left(\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\dot\theta_k^2(t)\,\mathrm{d}t\right)^{1/2} + \int_0^\tau n\sum_{k=1}^{n}(\theta_k - \theta_{k-1})^2(t)\,\mathrm{d}t\right), \end{aligned} \tag{7.70}$$

where we neglected the terms $\frac{n}{2}\sum_{k=1}^{n}(\dot w_k - \dot w_{k-1})^2$ because we already have the estimate (7.41), and $n\sum_{k=1}^{n}(\varepsilon_k - \varepsilon_{k-1})^2$, $\int_0^\tau n\sum_{k=1}^{n}(\varepsilon_k - \varepsilon_{k-1})^2\,\mathrm{d}t$ because the term $\int_0^\tau n\sum_{k=1}^{n}(\dot\varepsilon_k - \dot\varepsilon_{k-1})^2\,\mathrm{d}t$ is dominant. We need now more regularity for the temperature, since we have to estimate the $W^{1,2}$-norm of θ_k in both space and time. To this aim we now test (7.5) by θ_k and obtain

$$\frac{1}{n}\sum_{k=1}^{n-1}\dot\theta_k\theta_k = n\sum_{k=1}^{n-1}(\theta_{k+1}-2\theta_k+\theta_{k-1})\,\theta_k + \frac{1}{n}\sum_{k=1}^{n-1}\Big(-\theta_k\dot\varepsilon_k+\dot\varepsilon_k^2+D_k+\dot m_k\,M_k+\dot\chi_k\,(K_k-L)+g_k\Big)\theta_k.$$

By (7.9) and (7.10) it holds

$$n\sum_{k=1}^{n-1}(\theta_{k+1}-2\theta_k+\theta_{k-1})\,\theta_k = -n\sum_{k=1}^{n}(\theta_k-\theta_{k-1})^2,$$

then we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^2\right)+n\sum_{k=1}^{n}(\theta_k-\theta_{k-1})^2 \le \frac{1}{n}\sum_{k=1}^{n-1}\Big(\theta_k|\dot\varepsilon_k|+\dot\varepsilon_k^2+D_k+|\dot m_k|\,M_k+|\dot\chi_k||K_k-L|+|g_k|\Big)\theta_k.$$

We now estimate the right-hand side. By Remark 7.4 and estimates (7.61)–(7.62) for $|\dot\chi_k|$ and $|\dot m_k|$ we infer

$$\begin{aligned}&\frac{1}{n}\sum_{k=1}^{n-1}\Big(\theta_k|\dot\varepsilon_k|+\dot\varepsilon_k^2+D_k+|\dot m_k|\,M_k+|\dot\chi_k||K_k-L|+|g_k|\Big)\theta_k\\ &\le C\left(1+\frac{1}{n}\sum_{k=1}^{n-1}\left(\theta_k|\dot\varepsilon_k|+\dot\varepsilon_k^2+|\dot\varepsilon_k|+\theta_k+\frac{1}{n}\sum_{j=1}^{n-1}|\dot\varepsilon_j|+|g_k|\right)\theta_k\right).\end{aligned}$$

Note that

$$\frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{1}{n}\sum_{j=1}^{n-1}|\dot\varepsilon_j|\right)\theta_k = \left(\frac{1}{n}\sum_{k=1}^{n-1}\theta_k\right)\left(\frac{1}{n}\sum_{j=1}^{n-1}|\dot\varepsilon_j|\right)\le C\left(\frac{1}{n}\sum_{k=1}^{n-1}|\dot\varepsilon_k|\right)$$

by estimate (7.41). Hence we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^2\right)+n\sum_{k=1}^{n}(\theta_k-\theta_{k-1})^2 \le C\left(1+\frac{1}{n}\sum_{k=1}^{n-1}\Big(\dot\varepsilon_k^2\,\theta_k+\theta_k^2+|\dot\varepsilon_k|\theta_k+|\dot\varepsilon_k|\theta_k^2+|\dot\varepsilon_k|+|g_k|\theta_k\Big)\right).$$

Integrating in time $\int_0^\tau \mathrm{d}t$ and using the discrete Hölder's inequality yields

$$\begin{aligned}&\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^2+\int_0^\tau n\sum_{k=1}^{n}(\theta_k-\theta_{k-1})^2\,\mathrm{d}t\\ &\le C\left(1+\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^2\,\theta_k\,\mathrm{d}t+\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^2\,\mathrm{d}t+\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^2\,\mathrm{d}t+\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}|\dot\varepsilon_k|\theta_k^2\,\mathrm{d}t+\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}g_k^2\,\mathrm{d}t\right),\end{aligned}$$

where the initial condition can be controlled as in (7.39). Arguing as for (7.28) and using Hölder's inequality in space we deduce

$$\begin{aligned}\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1} g_k^2(\theta_k,\tau)\,\mathrm{d}\tau &= \int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\left(n\int_{(k-1)/n}^{k/n} g(\theta,x,t)\,\mathrm{d}x\right)^2 \mathrm{d}t\\ &\le C\left(\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1} g_1^2\theta_k^2(t) + \left(n\int_{(k-1)/n}^{k/n} g_0(x,t)\,\mathrm{d}x\right)^2 \mathrm{d}t\right)\\ &\le C\left(\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1} g_1^2\theta_k^2(t)\,\mathrm{d}t + \iint_{\Omega_T} |g_0(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t\right)\\ &\le C\left(1+\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\theta_k^2(t)\,\mathrm{d}t\right). \qquad (7.71)\end{aligned}$$

Hence by (7.46)–(7.47) we obtain

$$\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^2 + \int_0^\tau n\sum_{k=1}^{n}(\theta_k-\theta_{k-1})^2\,\mathrm{d}t \le C\left(1+\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^2\,\theta_k\,\mathrm{d}t + \int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}|\dot\varepsilon_k|\theta_k^2\,\mathrm{d}t\right). \qquad (7.72)$$

Now, by (7.46)

$$\begin{aligned}\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^2\,\theta_k\,\mathrm{d}t &\le \int_0^\tau \left(\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^4\right)^{1/2}\left(\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^2\right)^{1/2}\mathrm{d}t\\ &\le \left(\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^4\,\mathrm{d}t\right)^{1/2}\left(\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\theta_k^2\,\mathrm{d}t\right)^{1/2} \le C\left(\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^4\,\mathrm{d}t\right)^{1/2},\end{aligned}$$

$$\begin{aligned}\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}|\dot\varepsilon_k|\theta_k^2\,\mathrm{d}t &\le \int_0^\tau \left(\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^4\right)^{1/4}\left(\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{8/3}\right)^{3/4}\mathrm{d}t\\ &\le \left(\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^4\,\mathrm{d}t\right)^{1/4}\left(\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{8/3}\,\mathrm{d}t\right)^{3/4} \le C\left(1+\left(\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^4\,\mathrm{d}t\right)^{1/2}\right).\end{aligned}$$

Plugging these two estimates into (7.72) we get

$$\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^2 + \int_0^\tau n\sum_{k=1}^{n}(\theta_k-\theta_{k-1})^2\,\mathrm{d}t \le C\left(1+\left(\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^4\,\mathrm{d}t\right)^{1/2}\right). \qquad (7.73)$$

We need to estimate the term on the right-hand side. To this aim we test (7.5) by $\dot\theta_k$, and obtain

$$\frac{1}{n}\sum_{k=1}^{n-1}\dot\theta_k^2 = n\sum_{k=1}^{n-1}(\theta_{k+1}-2\theta_k+\theta_{k-1})\,\dot\theta_k + \frac{1}{n}\sum_{k=1}^{n-1}\Big(-\theta_k\dot\varepsilon_k + \dot\varepsilon_k^2 + D_k + \dot m_k\,M_k + \dot\chi_k\,(K_k-L) + g_k\Big)\dot\theta_k.$$

By (7.9) and (7.10) we infer

$$n\sum_{k=1}^{n-1}(\theta_{k+1}-2\theta_k+\theta_{k-1})\,\dot\theta_k = -n\sum_{k=1}^{n}(\theta_k-\theta_{k-1})(\dot\theta_k-\dot\theta_{k-1}) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(n\sum_{k=1}^{n}(\theta_k-\theta_{k-1})^2\right),$$

then by Young's inequality

$$\frac{1}{2n}\sum_{k=1}^{n-1}\dot\theta_k^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(n\sum_{k=1}^{n}(\theta_k-\theta_{k-1})^2\right) \le \frac{1}{2n}\sum_{k=1}^{n-1}\Big(\theta_k|\dot\varepsilon_k| + \dot\varepsilon_k^2 + D_k + |\dot m_k|\,K_k + |\dot\chi_k||M_k-L| + |g_k|\Big)^2.$$

Integrating in time $\int_0^{\tau'} \mathrm{d}t$ for some $\tau' < \tau$ gives

$$\begin{aligned}
&\int_0^{\tau'} \frac{1}{n}\sum_{k=1}^{n-1} \dot\theta_k^2 \,\mathrm{d}t + n\sum_{k=1}^{n}(\theta_k-\theta_{k-1})^2(\tau') \\
&\le C\left(\int_0^{\tau'} \frac{1}{n}\sum_{k=1}^{n-1}\Big(\theta_k|\dot\varepsilon_k| + \dot\varepsilon_k^2 + D_k + |\dot m_k|\, M_k + |\dot\chi_k||K_k - L| + |g_k|\Big)^2 \mathrm{d}t\right) + n\sum_{k=1}^{n}(\theta_k^0-\theta_{k-1}^0)^2.
\end{aligned}$$

Concerning the initial condition, arguing as for (7.68) we see that it holds

$$\begin{aligned}
n\sum_{k=1}^{n}(\theta_k^0-\theta_{k-1}^0)^2 &= n\sum_{k=1}^{n}\left(\theta^0\left(\frac{k}{n}\right) - \theta^0\left(\frac{k-1}{n}\right)\right)^2 \\
&= n\sum_{k=1}^{n}\left(\int_{(k-1)/n}^{k/n}\theta_x^0(\xi)\,\mathrm{d}x\right)^2 \le \int_0^1 |\theta_x^0(\xi)|^2\,\mathrm{d}\xi \le C
\end{aligned} \tag{7.74}$$

thanks to Hypothesis 7.1 (v). We estimate the first term on the right-hand side similarly as we did to obtain (7.72), getting

$$\begin{aligned}
&\int_0^{\tau'} \frac{1}{n}\sum_{k=1}^{n-1}\Big(\theta_k|\dot\varepsilon_k| + \dot\varepsilon_k^2 + D_k + |\dot m_k|\, M_k + |\dot\chi_k||K_k - L| + |g_k|\Big)^2 \mathrm{d}t \\
&\le C\left(1 + \int_0^{\tau'}\frac{1}{n}\sum_{k=1}^{n-1}\left(\theta_k^2\dot\varepsilon_k^2 + \dot\varepsilon_k^4 + \theta_k^2 + \left(\frac{1}{n}\sum_{k=1}^{n-1}|\dot\varepsilon_k|\right)^2 + g_k^2\right)\mathrm{d}t\right) \\
&\le C\left(1 + \int_0^{\tau'}\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^4\,\mathrm{d}t + \int_0^{\tau'}\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^4\,\mathrm{d}t\right)
\end{aligned} \tag{7.75}$$

thanks to estimates (7.46), (7.47) and (7.71). Hence

$$\int_0^{\tau'} \frac{1}{n}\sum_{k=1}^{n-1} \dot\theta_k^2 \,\mathrm{d}t + n\sum_{k=1}^{n}(\theta_k-\theta_{k-1})^2(\tau') \le C\left(1 + \int_0^{\tau'}\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^4\,\mathrm{d}t + \int_0^{\tau'}\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^4\,\mathrm{d}t\right)$$

for some $\tau' \in [0,\tau]$. Passing to the $\max_{\tau'\in[0,\tau]}$ yields

$$\int_0^{\tau} \frac{1}{n}\sum_{k=1}^{n-1} \dot\theta_k^2 \,\mathrm{d}t + \max_{\tau'\in[0,\tau]}\left(n\sum_{k=1}^{n}(\theta_k-\theta_{k-1})^2(\tau')\right) \le C\left(1 + \int_0^{\tau}\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^4\,\mathrm{d}t + \int_0^{\tau}\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^4\,\mathrm{d}t\right). \tag{7.76}$$

The last term on the right-hand side can be estimated using Hölder's inequality and estimate (7.46) as follows

$$\begin{aligned}
\int_0^{\tau}\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^4\,\mathrm{d}t &= \int_0^{\tau}\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{4/3}\,\theta_k^{8/3}\,\mathrm{d}t \le \int_0^{\tau}\left(\max_{k=1,\dots,n-1}\theta_k^{4/3}\right)\left(\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{8/3}\right)\mathrm{d}t \\
&\le \left(\max_{t\in[0,\tau]}\max_{k=1,\dots,n-1}\theta_k^{4/3}(t)\right)\int_0^{\tau}\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{8/3}\,\mathrm{d}t \le C\max_{t\in[0,\tau]}\max_{k=1,\dots,n-1}\theta_k^{4/3}(t).
\end{aligned} \tag{7.77}$$

Moreover the discrete Gagliardo-Nirenberg inequality (see Remark A.11) with the choices $v = \theta_k$,

$s = 1$, $q = \infty$, $p = 2$, $N = 1$, $\varrho = 2/3$ yields, taking into account (7.9),

$$\max_{k=1,\dots,n-1} \theta_k(t) \le C \left(\frac{1}{n} \sum_{k=1}^{n-1} \theta_k + \left(\frac{1}{n} \sum_{k=1}^{n-1} \theta_k \right)^{1/3} \left(n \sum_{k=1}^{n} (\theta_k - \theta_{k-1})^2 \right)^{1/3} \right)$$
$$\le C \left(1 + \left(n \sum_{k=1}^{n} (\theta_k - \theta_{k-1})^2(t) \right)^{1/3} \right), \tag{7.78}$$

where the last inequality follows from estimate (7.41) and Hypothesis 7.1 (vii). Hence it holds also

$$\max_{k=1,\dots,n-1} \theta_k^{4/3}(t) = \left(\max_{k=1,\dots,n-1} \theta_k(t) \right)^{4/3} \le C \left(1 + \left(n \sum_{k=1}^{n} (\theta_k - \theta_{k-1})^2(t) \right)^{4/9} \right).$$

Substituting in (7.77) we deduce

$$\int_0^\tau \frac{1}{n} \sum_{k=1}^{n-1} \theta_k^4 \, \mathrm{d}t \le C \left(1 + \max_{t \in [0,\tau]} \left(n \sum_{k=1}^{n} (\theta_k - \theta_{k-1})^2(t) \right)^{4/9} \right). \tag{7.79}$$

Inserting the previous inequality in (7.76) we get

$$\int_0^\tau \frac{1}{n} \sum_{k=1}^{n-1} \dot\theta_k^2 \, \mathrm{d}t + \max_{t \in [0,\tau]} \left(n \sum_{k=1}^{n} (\theta_k - \theta_{k-1})^2(t) \right)$$
$$\le C \left(1 + \int_0^\tau \frac{1}{n} \sum_{k=1}^{n-1} \dot\varepsilon_k^4 \, \mathrm{d}t + \max_{t \in [0,\tau]} \left(n \sum_{k=1}^{n} (\theta_k - \theta_{k-1})^2(t) \right)^{4/9} \right).$$

Young's inequality with conjugate exponents (9/5 , 9/4) gives

$$\int_0^\tau \frac{1}{n} \sum_{k=1}^{n-1} \dot\theta_k^2 \, \mathrm{d}t + n \sum_{k=1}^{n} (\theta_k - \theta_{k-1})^2(\tau) \le C \left(1 + \int_0^\tau \frac{1}{n} \sum_{k=1}^{n-1} \dot\varepsilon_k^4 \, \mathrm{d}t \right). \tag{7.80}$$

This and (7.73) in turn give, substituting in (7.70),

$$\frac{1}{n} \sum_{k=1}^{n-1} \dot\varepsilon_k^2(\tau) + \int_0^\tau n \sum_{k=1}^{n} (\dot\varepsilon_k - \dot\varepsilon_{k-1})^2(t) \, \mathrm{d}t \le C \left(1 + \left(\int_0^\tau \frac{1}{n} \sum_{k=1}^{n-1} \dot\varepsilon_k^4 \, \mathrm{d}t \right)^{1/2} \right). \tag{7.81}$$

The term on the right-hand side can be estimated using once more the discrete Gagliardo-Nirenberg inequality (see Remark A.11). Note that, however, the "Neumann boundary conditions" (7.9) for θ_k made the previous applications of Gagliardo-Nirenberg inequality easier. This time, instead, we have to consider the whole vectors $\dot{\boldsymbol{\varepsilon}}$ and $\mathbf{D}\dot{\boldsymbol{\varepsilon}}$ (see the vector notation in Remark A.11). Using equation (7.48) and the "boundary condition" (7.9) we see that it holds

$$|\dot{\boldsymbol{\varepsilon}}(\tau)|_2^2 = \frac{1}{n} \sum_{k=1}^{n-1} \dot\varepsilon_k^2(\tau) + \frac{C}{n} \left(1 + \dot\varepsilon_0^2(\tau) + \dot\varepsilon_n^2(\tau) \right) \le \frac{1}{n} \sum_{k=1}^{n-1} \dot\varepsilon_k^2(\tau) + \frac{C}{n} \left(1 + \int_0^\tau \sum_{k=1}^{n-1} \theta_k^2(t) \, \mathrm{d}t \right)$$
$$\le C + \frac{1}{n} \sum_{k=1}^{n-1} \dot\varepsilon_k^2(\tau),$$

where in the last line we used (7.46). Thus (7.81) for the whole vector reads

$$\max_{\tau\in[0,\tau']} |\dot{\boldsymbol{\varepsilon}}(\tau)|_2^2 + \int_0^\tau |\mathbf{D}\dot{\boldsymbol{\varepsilon}}(t)|_2^2 \,\mathrm{d}t \le C\left(1+\left(\int_0^\tau |\dot{\boldsymbol{\varepsilon}}(t)|_4^4 \,\mathrm{d}t\right)^{1/2}\right) \tag{7.82}$$

form some $\tau' > \tau$. We are now ready to estimate the right-hand side by using the discrete Gagliardo-Nirenberg inequality with the choices $v = \dot{\varepsilon}_k$, $q = 4$, $p = s = 2$, $N = 1$ $\varrho = 1/4$. This yields

$$|\dot{\boldsymbol{\varepsilon}}(t)|_4 \le C\left(|\dot{\boldsymbol{\varepsilon}}(t)|_2 + |\dot{\boldsymbol{\varepsilon}}(t)|_2^{3/4}|\mathbf{D}\dot{\boldsymbol{\varepsilon}}(t)|_2^{1/4}\right).$$

The above computations and estimates (7.46)–(7.47) entail

$$\int_0^\tau |\dot{\boldsymbol{\varepsilon}}(t)|_2^2 \,\mathrm{d}t \le \int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1} \dot{\varepsilon}_k^2(t)\,\mathrm{d}t + C\left(1+\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\theta_k^2(t)\,\mathrm{d}t\right) \le C.$$

Hence, applying twice Hölder's inequality in time,

$$\begin{aligned}
\int_0^\tau |\dot{\boldsymbol{\varepsilon}}(t)|_4^4 \,\mathrm{d}t &\le C\int_0^\tau |\dot{\boldsymbol{\varepsilon}}(t)|_2^2\left(|\dot{\boldsymbol{\varepsilon}}(t)|_2^2 + |\dot{\boldsymbol{\varepsilon}}(t)|_2|\mathbf{D}\dot{\boldsymbol{\varepsilon}}(t)|_2\right)\,\mathrm{d}t\\
&\le C\max_{t\in[0,\tau]}|\dot{\boldsymbol{\varepsilon}}(t)|_2^2\left(\int_0^\tau \left(|\dot{\boldsymbol{\varepsilon}}(t)|_2^2 + |\dot{\boldsymbol{\varepsilon}}(t)|_2|\mathbf{D}\dot{\boldsymbol{\varepsilon}}(t)|_2\right)\,\mathrm{d}\tau\right)\\
&\le C\max_{t\in[0,\tau]}|\dot{\boldsymbol{\varepsilon}}(t)|_2^2\left(\int_0^\tau |\dot{\boldsymbol{\varepsilon}}(t)|_2^2\,\mathrm{d}t + \left(\int_0^\tau |\dot{\boldsymbol{\varepsilon}}(t)|_2^2\,\mathrm{d}t\right)^{1/2}\left(\int_0^\tau |\mathbf{D}\dot{\boldsymbol{\varepsilon}}(t)|_2^2\,\mathrm{d}t\right)^{1/2}\right)\\
&\le C\max_{t\in[0,\tau]}|\dot{\boldsymbol{\varepsilon}}(t)|_2^2\left(1+\int_0^\tau |\mathbf{D}\dot{\boldsymbol{\varepsilon}}(t)|_2^2\,\mathrm{d}t\right)^{1/2},
\end{aligned}$$

which combined with (7.82) by Young's inequality yields

$$\max_{t\in[0,\tau]}|\dot{\boldsymbol{\varepsilon}}(t)|_2^2 + \int_0^\tau |\mathbf{D}\dot{\boldsymbol{\varepsilon}}(t)|_2^2\,\mathrm{d}t \le C.$$

Therefore there exists a constant $C > 0$ such that

$$\frac{1}{n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k^2(\tau) + \int_0^\tau n\sum_{k=1}^{n}(\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})^2(t)\,\mathrm{d}t + \int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}(\dot{\varepsilon}_k^4 + \varepsilon_k^4)(t)\,\mathrm{d}t \le C, \tag{7.83}$$

and substituting in (7.80) also

$$\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\dot{\theta}_k^2(t)\,\mathrm{d}t + n\sum_{k=1}^{n}(\theta_k - \theta_{k-1})^2(\tau) \le C. \tag{7.84}$$

Then (7.78) and (7.79) yield

$$\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\theta_k^4\,\mathrm{d}t \le C, \tag{7.85}$$

$$\max_{t\in[0,\tau]}\max_{k=1,\dots,n-1}\theta_k(t) \le C. \tag{7.86}$$

This last estimate together with (7.61) gives

$$\max_{t\in[0,\tau]}\max_{k=1,\dots,n-1}|\dot{\chi}_k(t)| \le C, \tag{7.87}$$

whereas together with (7.62) and (7.83) gives

$$\max_{t\in[0,\tau]}\max_{k=1,\dots,n-1}|\dot m_k(t)|\le C. \tag{7.88}$$

We now derive a higher order estimate for the temperature. Note now that by (7.5), arguing as in (7.75) we also have

$$\begin{aligned}&\int_0^\tau n^3\sum_{k=1}^{n-1}(\theta_{k+1}-2\theta_k+\theta_{k-1})^2(t)\,\mathrm{d}t\\&=\int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\Big(\dot\theta_k-\dot m_k\,M_k-D_k-\dot\varepsilon_k^2+\theta_k\dot\varepsilon_k+\dot\chi_k\,(K_k-L)-g_k\Big)^2\,\mathrm{d}t\\&\le C\left(1+\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\dot\theta_k^2\,\mathrm{d}t+\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^4\,\mathrm{d}t+\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^4\,\mathrm{d}t\right),\end{aligned}$$

thus it holds by (7.83), (7.84) and (7.86)

$$\int_0^\tau n^3\sum_{k=1}^{n-1}(\theta_{k+1}-2\theta_k+\theta_{k-1})^2(t)\,\mathrm{d}t\le C. \tag{7.89}$$

Finally, we differentiate (7.2) once in t and test by $\ddot\varepsilon_k$, (7.3) twice in t and test by $\dddot w_k$, and sum up the two equations. We obtain

$$\frac{1}{n}\sum_{k=1}^{n-1}\ddot u_k\ddot\varepsilon_k=\frac{1}{n}\sum_{k=1}^{n-1}\Big(\dot\varepsilon_k+P_0[m_k,\chi_k,\varepsilon_k]_t+\dddot\varepsilon_k-\dot\theta_k\Big)\ddot\varepsilon_k$$

added to

$$\frac{1}{n}\sum_{k=1}^{n-1}\ddddot w_k\dddot w_k-\frac{1}{n}\sum_{k=1}^{n-1}\dddot\varepsilon_k\dddot w_k=-n\sum_{k=1}^{n-1}(\ddot u_{k+1}-2\ddot u_k+\ddot u_{k-1})\dddot w_k+\frac{1}{n}\sum_{k=1}^{n-1}\ddot f_k\dddot w_k,$$

which gives

$$\begin{aligned}&\frac{1}{n}\sum_{k=1}^{n-1}\ddddot w_k\dddot w_k-\frac{1}{n}\sum_{k=1}^{n-1}\dddot\varepsilon_k\dddot w_k+\frac{1}{n}\sum_{k=1}^{n-1}\Big(\dot\varepsilon_k+P_0[m_k,\chi_k,\varepsilon_k]_t+\dddot\varepsilon_k-\dot\theta_k\Big)\ddot\varepsilon_k\\&=\frac{1}{n}\sum_{k=1}^{n-1}\ddot u_k\ddot\varepsilon_k-n\sum_{k=1}^{n-1}(\ddot u_{k+1}-2\ddot u_k+\ddot u_{k-1})\dddot w_k+\frac{1}{n}\sum_{k=1}^{n-1}\ddot f_k\dddot w_k.\end{aligned} \tag{7.90}$$

Note that

$$\begin{aligned}&\frac{1}{n}\sum_{k=1}^{n-1}\ddot u_k\ddot\varepsilon_k-n\sum_{k=1}^{n-1}(\ddot u_{k+1}-2\ddot u_k+\ddot u_{k-1})\dddot w_k\\&\overset{(7.4)}{=}n\sum_{k=1}^{n-1}\ddot u_k(\dddot w_{k+1}-2\dddot w_k+\dddot w_{k-1})-n\sum_{k=1}^{n-1}(\ddot u_{k+1}-2\ddot u_k+\ddot u_{k-1})\dddot w_k\overset{(7.10)}{=}0.\end{aligned}$$

Moreover

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n-1}\dddot\varepsilon_k\dddot w_k&\overset{(7.4)}{=}n\sum_{k=1}^{n-1}(\ddddot w_{k+1}-2\ddddot w_k+\ddddot w_{k-1})\dddot w_k\\&\overset{(7.10)}{=}-n\sum_{k=1}^{n}(\ddddot w_k-\ddddot w_{k-1})(\dddot w_k-\dddot w_{k-1})=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(n\sum_{k=1}^{n}(\dddot w_k-\dddot w_{k-1})^2\right).\end{aligned}$$

Hence (7.90) becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{n}\sum_{k=1}^{n-1}\ddot{w}_k^2+n\sum_{k=1}^{n}(\ddot{w}_k-\ddot{w}_{k-1})^2\right)+\frac{1}{n}\sum_{k=1}^{n-1}\dddot{\varepsilon}_k^2$$
$$=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k^2\right)-\frac{1}{n}\sum_{k=1}^{n-1}\left(P_0[m_k,\chi_k,\varepsilon_k]_t-\dot{\theta}_k\right)\ddot{\varepsilon}_k+\frac{1}{n}\sum_{k=1}^{n-1}\ddot{f}_k\ddot{w}_k.$$

Integrating in time $\int_0^\tau \mathrm{d}t$ yields

$$\frac{1}{2n}\sum_{k=1}^{n-1}\ddot{w}_k^2(\tau)+\frac{n}{2}\sum_{k=1}^{n}(\ddot{w}_k-\ddot{w}_{k-1})^2(\tau)+\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\dddot{\varepsilon}_k^2\,\mathrm{d}t$$
$$=-\frac{1}{2n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k^2(\tau)+\frac{1}{2n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k^2(0)-\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\left(P_0[m_k,\chi_k,\varepsilon_k]_t-\dot{\theta}_k\right)\ddot{\varepsilon}_k\,\mathrm{d}t$$
$$+\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\ddot{f}_k\ddot{w}_k\,\mathrm{d}t+\frac{1}{2n}\sum_{k=1}^{n-1}\ddot{w}_k^2(0)+\frac{n}{2}\sum_{k=1}^{n}(\ddot{w}_k-\ddot{w}_{k-1})^2(0).$$

We now use Young's inequality on the right-hand side. To estimate the term $P_0[m_k,\chi_k,\varepsilon_k]_t$ we employ (7.63), whereas for the term $\ddot{f}_k$ we argue as in (7.29) obtaining

$$\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\ddot{f}_k^2(t)\,\mathrm{d}t\le\iint_{\Omega_T}f_{tt}^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\le C$$

by Hypothesis 7.1 (iv). Employing estimates (7.46)–(7.47), (7.57) and (7.83)–(7.84) we get

$$\begin{aligned}&\frac{1}{n}\sum_{k=1}^{n-1}\ddot{w}_k^2(\tau)+n\sum_{k=1}^{n}(\ddot{w}_k-\ddot{w}_{k-1})^2(\tau)+\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\dddot{\varepsilon}_k^2(t)\,\mathrm{d}t\\&\le C\left(1+\frac{1}{n}\sum_{k=1}^{n-1}\ddot{w}_k^2(0)+n\sum_{k=1}^{n}(\ddot{w}_k-\ddot{w}_{k-1})^2(0)+\int_0^\tau\frac{1}{n}\sum_{k=1}^{n-1}\ddot{w}_k^2(t)\,\mathrm{d}t\right).\end{aligned}\tag{7.91}$$

We need to estimate the initial values

$$\frac{1}{n}\sum_{k=1}^{n-1}\ddot{w}_k^2(0)+n\sum_{k=1}^{n}(\ddot{w}_k-\ddot{w}_{k-1})^2(0).$$

To this aim we consider equation (7.2) for $t=0$, where $\varepsilon_k(0)=P_0[m_k,\chi_k,\varepsilon_k](0)=0$ by Remark 7.5. Hence we have

$$\dot{u}_k(0)=\dot{\varepsilon}_k(0)-\theta_k(0)+\theta_{\mathrm{ref}},$$

from which also

$$n(\dot{u}_k-\dot{u}_{k-1})(0)=n(\dot{\varepsilon}_k-\dot{\varepsilon}_{k-1})(0)-n(\theta_k-\theta_{k-1})(0).\tag{7.92}$$

On the other hand by equation (7.3) and (7.8) it holds

$$n(\dot{w}_k-\dot{w}_{k-1})(0)-n(\dot{\varepsilon}_k-\dot{\varepsilon}_{k-1})(0)=n(f_k-f_{k-1})(0)\tag{7.93}$$

and also

$$\ddot{w}_k(0)-\ddot{\varepsilon}_k(0)=-n^2(\dot{u}_{k+1}-2\dot{u}_k+\dot{u}_{k-1})(0)+\dot{f}_k(0).\tag{7.94}$$

We now test (7.93) by $n(\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})(0)$ obtaining

$$n\sum_{k=1}^{n}(\dot{w}_k - \dot{w}_{k-1})(\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})(0) - n\sum_{k=1}^{n}(\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})^2(0) = n\sum_{k=1}^{n}(f_k - f_{k-1})(\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})(0).$$

By Young's inequality we infer

$$\frac{n}{2}\sum_{k=1}^{n}(\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})^2(0) \leq n\sum_{k=1}^{n}(\dot{w}_k - \dot{w}_{k-1})^2(0) + n\sum_{k=1}^{n}(f_k - f_{k-1})^2(0). \tag{7.95}$$

The first term on the right-hand side is bounded thanks to (7.37). We are going to derive a bound for the second term. Note that it holds

$$\begin{aligned}
&n\sum_{k=1}^{n}(f_k - f_{k-1})^2(0) = n\sum_{k=1}^{n}(f_k - f_{k-1})^2(\tau) - 2\int_0^\tau n\sum_{k=1}^{n}(f_k - f_{k-1})(\dot{f}_k - \dot{f}_{k-1})\,\mathrm{d}t \\
&= n\sum_{k=1}^{n}\left(n\int_{(k-1)/n}^{k/n} f(x,\tau)\,\mathrm{d}x - n\int_{(k-2)/n}^{(k-1)/n} f(x,\tau)\,\mathrm{d}x\right)^2 \\
&- 2\int_0^\tau n\sum_{k=1}^{n}\left(n\int_{(k-1)/n}^{k/n} f(x,t)\,\mathrm{d}x - n\int_{(k-2)/n}^{(k-1)/n} f(x,t)\,\mathrm{d}x\right) \\
&\quad\cdot\left(n\int_{(k-1)/n}^{k/n} f_t(x,t)\,\mathrm{d}x - n\int_{(k-2)/n}^{(k-1)/n} f_t(x,t)\,\mathrm{d}x\right)\mathrm{d}t \\
&= n\sum_{k=1}^{n}\left(n\int_{(k-1)/n}^{k/n}\left(f(x,\tau) - f(x-1/n,\tau)\right)\,\mathrm{d}x\right)^2 \\
&- 2\int_0^\tau n\sum_{k=1}^{n}\left(n\int_{(k-1)/n}^{k/n}\left(f(x,t) - f(x-1/n,t)\right)\,\mathrm{d}x\right)\left(n\int_{(k-1)/n}^{k/n}\left(f_t(x,t) - f_t(x-1/n,t)\right)\,\mathrm{d}x\right)\mathrm{d}t \\
&= n^3\sum_{k=1}^{n}\left(\int_{(k-1)/n}^{k/n}\int_{x-1/n}^{x} f_x(y,\tau)\,\mathrm{d}y\,\mathrm{d}x\right)^2 \\
&- 2\int_0^\tau n^3\sum_{k=1}^{n}\left(\int_{(k-1)/n}^{k/n}\int_{x-1/n}^{x} f_x(y,t)\,\mathrm{d}y\,\mathrm{d}x\right)\left(\int_{(k-1)/n}^{k/n}\int_{x-1/n}^{x} f_{xt}(y,t)\,\mathrm{d}y\,\mathrm{d}x\right)\mathrm{d}t.
\end{aligned}$$

Following the same steps leading to (7.68) and using Hölder's inequality in time in the last summand, what we eventually get is

$$n\sum_{k=1}^{n}(f_k - f_{k-1})^2(0) \leq C\left(\int_0^1 f_x^2(y,\tau)\,\mathrm{d}y + \left(\int_0^\tau\int_0^1 f_x^2(y,t)\,\mathrm{d}y\,\mathrm{d}t\right)^{1/2}\left(\int_0^\tau\int_0^1 f_{xt}^2(y,t)\,\mathrm{d}y\,\mathrm{d}t\right)^{1/2}\right).$$

Integrating in time $\int_0^T \mathrm{d}\tau$ and using Hypothesis 7.1 (iv) then yield

$$n\sum_{k=1}^{n}(f_k - f_{k-1})^2(0) \leq C.$$

Coming back to (7.95) we thus obtain

$$\frac{n}{2}\sum_{k=1}^{n}(\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})^2(0) \leq C. \tag{7.96}$$

Testing now (7.92) by $n(\dot u_k - \dot u_{k-1})(0)$ and using Young's inequality we get

$$\frac{n}{2}\sum_{k=1}^{n}(\dot u_k - \dot u_{k-1})^2(0) \le n\sum_{k=1}^{n}(\dot\varepsilon_k - \dot\varepsilon_{k-1})^2(0) + n\sum_{k=1}^{n}(\theta_k - \theta_{k-1})^2(0),$$

hence by (7.74) and (7.96)

$$n\sum_{k=1}^{n}(\dot u_k - \dot u_{k-1})^2(0) \le C. \tag{7.97}$$

We finally test (7.94) by $\ddot w_k(0)$ obtaining

$$\frac{1}{n}\sum_{k=1}^{n-1}\ddot w_k^2(0) - \frac{1}{n}\sum_{k=1}^{n-1}\ddot\varepsilon_k\ddot w_k(0) = -n\sum_{k=1}^{n-1}(\dot u_{k+1} - 2\dot u_k + \dot u_{k-1})\ddot w_k(0) + \frac{1}{n}\sum_{k=1}^{n-1}\dot f_k\ddot w_k(0).$$

Note that formula (7.10) entails

$$-\frac{1}{n}\sum_{k=1}^{n-1}\ddot\varepsilon_k\ddot w_k(0) \overset{(7.4)}{=} -n\sum_{k=1}^{n-1}(\ddot w_{k+1} - 2\ddot w_k + \ddot w_{k-1})\ddot w_k(0) = n\sum_{k=1}^{n}(\ddot w_k - \ddot w_{k-1})^2(0),$$

$$-n\sum_{k=1}^{n-1}(\dot u_{k+1} - 2\dot u_k + \dot u_{k-1})\ddot w_k(0) = n\sum_{k=1}^{n}(\dot u_k - \dot u_{k-1})(\ddot w_k - \ddot w_{k-1})(0).$$

Hence we obtain

$$\frac{1}{n}\sum_{k=1}^{n-1}\ddot w_k^2(0) + n\sum_{k=1}^{n}(\ddot w_k - \ddot w_{k-1})^2(0) = n\sum_{k=1}^{n}(\dot u_k - \dot u_{k-1})(\ddot w_k - \ddot w_{k-1})(0) + \frac{1}{n}\sum_{k=1}^{n-1}\dot f_k\ddot w_k(0),$$

and by Young's inequality

$$\frac{1}{2n}\sum_{k=1}^{n-1}\ddot w_k^2(0) + \frac{n}{2}\sum_{k=1}^{n}(\ddot w_k - \ddot w_{k-1})^2(0) \le \frac{n}{2}\sum_{k=1}^{n}(\dot u_k - \dot u_{k-1})^2(0) + \frac{1}{2n}\sum_{k=1}^{n-1}\dot f_k^2(0).$$

Arguing as for (7.36) we obtain

$$\frac{1}{n}\sum_{k=1}^{n-1}\dot f_k^2(0) \le \frac{1}{T}\iint_{\Omega_T} f_t^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + 2\left(\iint_{\Omega_T} f_t^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\left(\iint_{\Omega_T} f_{tt}^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \le C$$

thanks to Hypothesis 7.1 (iv) and Remark 7.2 (iv). This, together with estimate (7.97), finally gives

$$\frac{1}{n}\sum_{k=1}^{n-1}\ddot w_k^2(0) + n\sum_{k=1}^{n}(\ddot w_k - \ddot w_{k-1})^2(0) \le C.$$

Coming back to (7.91), by Grönwall's lemma A.2 we obtain the estimate

$$\frac{1}{n}\sum_{k=1}^{n-1}\ddot w_k^2(\tau) + n\sum_{k=1}^{n}(\ddot w_k - \ddot w_{k-1})^2(\tau) + \int_0^\tau \frac{1}{n}\sum_{k=1}^{n-1}\ddot\varepsilon_k^2(t)\,\mathrm{d}t \le C \tag{7.98}$$

which, combined with (7.29) by comparison in (7.3) gives

$$\int_0^\tau n^3\sum_{k=1}^{n-1}(\dot u_{k+1} - 2\dot u_k + \dot u_{k-1})^2(t)\,\mathrm{d}t \le C. \tag{7.99}$$

Another higher order estimate can be deduced by (7.47) and by comparison in equation 7.4, namely

$$\int_0^\tau n^3\sum_{k=1}^{n-1}(w_{k+1} - 2w_k + w_{k-1})^2(t)\,\mathrm{d}t \le C. \tag{7.100}$$

7.3 Proof of the existence

We are now ready to prove our main Theorem 7.3. The existence part will be carried out by passing to the limit in the approximating system (7.2)–(7.7), whose solutions satisfy the a priori estimates we have derived in the previous subsection.
Here and in the next section we will denote by $|\cdot|_p$ the norm in $L^p(0,1)$, and by $\|\cdot\|_p$ the norm in $L^p(\Omega_T)$.

For a generic sequence $\{\varphi_k : k = 0,1,\dots,n\}$ we set $\Delta_k\varphi = n(\varphi_k - \varphi_{k-1})$ and $\Delta_k^2\varphi = n^2(\varphi_{k+1} - 2\varphi_k + \varphi_{k-1})$, with the aim to define correspondingly piecewise constant, piecewise linear and piecewise quadratic interpolations

$$\bar\varphi^{(n)}(x) = \begin{cases} \varphi_k & \text{for } x \in \left[\frac{k-1}{n}, \frac{k}{n}\right),\ k = 1,\dots,n-1, \\ \varphi_{n-1} & \text{for } x \in \left[\frac{n-1}{n}, 1\right], \end{cases} \tag{7.101}$$

$$\hat\varphi^{(n)}(x) = \varphi_{k-1} + \left(x - \tfrac{k-1}{n}\right)\Delta_k\varphi \quad \text{for } x \in \left[\tfrac{k-1}{n}, \tfrac{k}{n}\right), \qquad k = 1,\dots,n, \tag{7.102}$$

$$\tilde\varphi^{(n)}(x) = \begin{cases} \frac12(\varphi_{k-1} + \varphi_k) + \left(x - \frac{k-1}{n}\right)\Delta_k\varphi + \frac12\left(x - \frac{k-1}{n}\right)^2\Delta_k^2\varphi \\ \text{for } x \in \left[\frac{k-1}{n}, \frac{k}{n}\right), \quad k = 1,\dots,n-1\,, \\ \frac12(\varphi_{n-1} + \varphi_n) + \left(x - \frac{n-1}{n}\right)\Delta_n\varphi + \frac12\left(x - \frac{n-1}{n}\right)^2\Delta_{n-1}^2\varphi \\ \text{for } x \in \left[\frac{n-1}{n}, 1\right]. \end{cases} \tag{7.103}$$

We also define

$$\lambda^{(n)}(x,y) = \lambda_{k-j} \qquad \text{for } (x,y) \in \left[\frac{k-1}{n}, \frac{k}{n}\right) \times \left[\frac{j-1}{n}, \frac{j}{n}\right). \tag{7.104}$$

The estimates we have derived in the previous section can be rewritten by using this notation for functions $\bar\varepsilon^{(n)}$, $\bar\theta^{(n)}$, $\bar u^{(n)}$, $\bar w^{(n)}$, $\bar\chi^{(n)}$, $\bar m^{(n)}$, $\hat\varepsilon^{(n)}$, $\hat\theta^{(n)}$, $\hat w^{(n)}$, $\tilde\theta^{(n)}$, $\tilde u^{(n)}$, $\tilde w^{(n)}$. In particular

- by (7.83)
$$\left|\bar\varepsilon_t^{(n)}(\tau)\right|_2^2 + \int_0^\tau \left|\hat\varepsilon_{xt}^{(n)}(t)\right|_2^2 \mathrm{d}t + \int_0^\tau \left(\left|\bar\varepsilon_t^{(n)}(t)\right|_4^4 + \left|\bar\varepsilon^{(n)}(t)\right|_4^4\right) \mathrm{d}t \le C, \tag{7.105}$$

- by (7.84)–(7.85)
$$\int_0^\tau \left(\left|\bar\theta_t^{(n)}(t)\right|_2^2 + \left|\bar\theta^{(n)}(t)\right|_4^4\right) \mathrm{d}t + \left|\hat\theta_x^{(n)}(\tau)\right|_2^2 \le C, \tag{7.106}$$

- by (7.89)
$$\int_0^\tau \left|\tilde\theta_{xx}^{(n)}(t)\right|_2^2 \mathrm{d}t \le C, \tag{7.107}$$

- by (7.98)
$$\left|\bar w_{tt}^{(n)}(\tau)\right|_2^2 + \left|\hat w_{xtt}^{(n)}(\tau)\right|_2^2 + \int_0^\tau \left|\bar\varepsilon_{tt}^{(n)}(t)\right|_2^2 \mathrm{d}t \le C, \tag{7.108}$$

- by (7.86)–(7.88)
$$|\bar\theta^{(n)}(\tau)|_\infty + |\bar\chi_t^{(n)}(\tau)|_\infty + |\bar m_t^{(n)}(\tau)|_\infty \le C, \tag{7.109}$$

- by (7.99)–(7.100)

$$\int_0^\tau \left(\left|\tilde{w}^{(n)}_{xxt}(t)\right|_2^2 + \left|\tilde{u}^{(n)}_{xxt}(t)\right|_2^2\right) \mathrm{d}t \le C. \tag{7.110}$$

With the above notation system (7.2)–(7.7) can be expressed in the form

$$\bar{u}^{(n)}_t = \bar{\varepsilon}^{(n)} + P_0[\bar{m}^{(n)}, \bar{\chi}^{(n)}, \bar{\varepsilon}^{(n)}] + \bar{\varepsilon}^{(n)}_t - (\bar{\theta}^{(n)} - \theta_{\mathrm{ref}}), \tag{7.111}$$

$$\bar{w}^{(n)}_t - \bar{\varepsilon}^{(n)}_t = -\tilde{u}^{(n)}_{xx} + \bar{f}^{(n)}, \tag{7.112}$$

$$\bar{\varepsilon}^{(n)} = \tilde{w}^{(n)}_{xx}, \tag{7.113}$$

$$\bar{\theta}^{(n)}_t = \tilde{\theta}^{(n)}_{xx} + \bar{m}^{(n)}_t \bar{K}^{(n)} + \bar{\chi}^{(n)}_t \bar{M}^{(n)} + \bar{D}^{(n)} + (\bar{\varepsilon}^{(n)}_t)^2 - \bar{\theta}^{(n)}\bar{\varepsilon}^{(n)}_t + \bar{g}^{(n)}(\bar{\theta}^{(n)}) - L\bar{\chi}^{(n)}_t, \tag{7.114}$$

$$\bar{\chi}^{(n)}(x,t) = \mathfrak{s}_{[0,1]}[\bar{\chi}^{(n)}(0), \bar{A}^{(n)}(x,\cdot)](t), \tag{7.115}$$

$$\bar{m}^{(n)}(x,t) = \mathfrak{s}_{[0,\infty)}[0, \bar{S}^{(n)}(x,\cdot)](t), \tag{7.116}$$

with $\bar{\chi}^{(n)}(0)$ chosen according to (7.8) and where

$$\begin{aligned}
\bar{A}^{(n)}(x,t) &= \int_0^t \frac{1}{\gamma}\left(\frac{L}{\theta_{\mathrm{ref}}}(\bar{\theta}^{(n)} - \theta_{\mathrm{ref}}) + \bar{M}^{(n)}\right)(x,\tau)\,\mathrm{d}\tau,\\
\bar{S}^{(n)}(x,t) &= \int_0^t \left(-h(\bar{\chi}^{(n)}_t(x,\tau)) + \int_0^1 \lambda^{(n)}(x,y)\bar{D}^{(n)}(y,\tau)\,\mathrm{d}y\right)\mathrm{d}\tau,\\
\bar{K}^{(n)}(x,t) &= -\frac{1}{2}\int_0^\infty \gamma_m(\bar{m}^{(n)}, \bar{\chi}^{(n)}, r)\mathfrak{s}_r^2[\bar{\varepsilon}^{(n)}]\,\mathrm{d}r,\\
\bar{M}^{(n)}(x,t) &= -\frac{1}{2}\int_0^\infty \gamma_\chi(\bar{m}^{(n)}, \bar{\chi}^{(n)}, r)\mathfrak{s}_r^2[\bar{\varepsilon}^{(n)}](x,t)\,\mathrm{d}r,\\
\bar{D}^{(n)}(x,t) &= \int_0^\infty \gamma(\bar{m}^{(n)}, \bar{\chi}^{(n)}, r)\mathfrak{s}_r[\bar{\varepsilon}^{(n)}](\bar{\varepsilon}^{(n)} - \mathfrak{s}_r[\bar{\varepsilon}^{(n)}])_t(x,t)\,\mathrm{d}r.
\end{aligned}$$

It holds

$$\begin{aligned}
|\hat{\varepsilon}^{(n)}_{tt}(t)|_2^2 &= \int_0^1 |\hat{\varepsilon}^{(n)}_{tt}(x,t)|^2\,\mathrm{d}x\\
&= \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \left|\ddot{\varepsilon}_{k-1}(t) + \left(x - \frac{k-1}{n}\right) n\left(\ddot{\varepsilon}_k - \ddot{\varepsilon}_{k-1}\right)(t)\right|^2 \mathrm{d}x\\
&= \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \left|\left(x - \frac{k-1}{n}\right) n\,\ddot{\varepsilon}_k(t) - \left(x - \frac{k}{n}\right) n\,\ddot{\varepsilon}_{k-1}(t)\right|^2 \mathrm{d}x\\
&\le \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \left|\frac{1}{n}\cdot n\,|\ddot{\varepsilon}_k(t)| + \frac{1}{n}\cdot n\,|\ddot{\varepsilon}_{k-1}(t)|\right|^2 \mathrm{d}x\\
&\le \frac{1}{n}\sum_{k=1}^n \ddot{\varepsilon}_k^2(t) + \frac{1}{n}\sum_{k=1}^n \ddot{\varepsilon}_{k-1}^2(t) \le \frac{2}{n}\sum_{k=0}^n \ddot{\varepsilon}_k^2(t),
\end{aligned}$$

hence employing equation (7.48) and (7.9) we further have

$$\begin{aligned}
\int_0^\tau \left|\hat{\varepsilon}^{(n)}_{tt}(t)\right|_2^2 \mathrm{d}t &\le \int_0^\tau \frac{2}{n}\sum_{k=0}^n \ddot{\varepsilon}_k^2(t)\,\mathrm{d}t \le \int_0^\tau \left(\frac{2}{n}\sum_{k=1}^{n-1}\ddot{\varepsilon}_k^2(t) + \frac{2}{n}(\ddot{\varepsilon}_0^2(t) + \ddot{\varepsilon}_n^2(t))\right)\mathrm{d}t\\
&\le C\left(1 + \int_0^\tau \frac{2}{n}\sum_{k=1}^{n-1}\left(\ddot{\varepsilon}_k^2 + \dot{\theta}_k^2\right)(t)\,\mathrm{d}t\right).
\end{aligned}$$

From (7.84) and (7.98) we finally get

$$\int_0^\tau \left|\hat{\varepsilon}_{tt}^{(n)}(t)\right|_2^2 \mathrm{d}t \le C. \tag{7.117}$$

Now, looking at estimates (7.105) and (7.117), we see that we can apply the embedding theorem A.3 with $p_0' = q_0 = q_1 = 2$, $p_1 = \infty$, $N = 1$ and Rellich-Kondrakov theorem, getting the existence of $\varepsilon \in W^{1,2}(\Omega_T)$ such that $\varepsilon_{xt}, \varepsilon_{tt} \in L^2(\Omega_T)$, and a subsequence of $\{\hat{\varepsilon}^{(n)}\}$, still indexed by n, such that

$$\begin{array}{lcll} \hat{\varepsilon}^{(n)} & \to & \varepsilon & \text{strongly in } C(\overline{\Omega}_T)\,, \\ \hat{\varepsilon}_t^{(n)} & \to & \varepsilon_t & \text{strongly in } L^p(\Omega_T) \text{ for all } p \in [1,\infty). \end{array}$$

Furthermore

$$\begin{aligned} |\bar{\varepsilon}_t^{(n)} - \hat{\varepsilon}_t^{(n)}|^2(x,t) &\le \left|\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1} - \left(x - \frac{k-1}{n}\right) n(\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})\right|^2 (t) \\ &= |(k-nx)(\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})|^2(t) \le |\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1}|^2(t) \end{aligned}$$

for $x \in [(k-1)/n, k/n]$, so that from estimate (7.83)

$$\int_0^\tau \int_0^1 |\bar{\varepsilon}_t^{(n)} - \hat{\varepsilon}_t^{(n)}|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \le \int_0^\tau \frac{1}{n}\sum_{k=1}^n (\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})^2(t)\,\mathrm{d}t \le \frac{C}{n^2}.$$

This yields, together with (7.3),

$$\bar{\varepsilon}_t^{(n)} \to \varepsilon_t \;\text{ strongly in } L^2(\Omega_T). \tag{7.118}$$

Similarly, from estimate (7.83)

$$\begin{aligned} |\bar{\varepsilon}^{(n)} - \hat{\varepsilon}^{(n)}|^2(x,t) &\le \left|\varepsilon_k - \varepsilon_{k-1} - \left(x - \frac{k-1}{n}\right) n(\varepsilon_k - \varepsilon_{k-1})\right|^2 (t) = |(k-nx)(\varepsilon_k - \varepsilon_{k-1})|^2(t) \\ &\le \max_{k=1,\dots,n} |\varepsilon_k - \varepsilon_{k-1}|^2(t) \le \sum_{k=1}^n (\varepsilon_k - \varepsilon_{k-1})^2(t) \le \frac{C}{n}, \end{aligned}$$

hence by (7.3)

$$\bar{\varepsilon}^{(n)} \to \varepsilon \;\text{ strongly in } L^\infty(\Omega_T). \tag{7.119}$$

Using (7.113), (7.118) we find that

$$\tilde{w}_{xxt}^{(n)} = \bar{\varepsilon}_t^{(n)} \to \varepsilon_t = w_{xxt} \;\text{ strongly in } L^2(\Omega_T).$$

Arguing as above, we deduce the existence of $w \in C(\overline{\Omega}_T)$ such that

$$\begin{array}{lcll} \hat{w}_t^{(n)} & \to & w_t & \text{strongly in } C(\overline{\Omega}_T), \\ \bar{w}_t^{(n)} & \to & w_t & \text{strongly in } L^\infty(\Omega_T). \end{array}$$

This, together with Hypothesis 7.1 (iv), equation (7.112) and (7.118), entails

$$\tilde{u}_{xx}^{(n)} = \bar{\varepsilon}_t^{(n)} - \bar{w}_t^{(n)} + \bar{f}^{(n)} \to \varepsilon_t - w_t + f = u_{xx} \;\text{ strongly in } L^2(\Omega_T)$$

where $u \in C(\overline{\Omega}_T)$. From estimates (7.106) and (7.107) we immediately obtain

$$\bar{\theta}_t^{(n)} \to \theta_t,\ \hat{\theta}_x^{(n)} \to \theta_x,\ \tilde{\theta}_{xx}^{(n)} \to \theta_{xx} \ \text{ weakly in } L^2(\Omega_T)$$

from which $\theta \in C(\overline{\Omega}_T)$ and, arguing as above,

$$\bar{\theta}^{(n)} \to \theta \ \text{ strongly in } L^\infty(\Omega_T). \tag{7.120}$$

The main complications come when dealing with the phase term. Using Proposition B.5 as for (7.67) we obtain for all $n, l \in \mathbb{N}$

$$\begin{aligned} &|\bar{\chi}^{(n)} - \bar{\chi}^{(l)}|(x,\tau) \leq \int_0^\tau |\bar{\chi}_t^{(n)} - \bar{\chi}_t^{(l)}|(x,t)\,\mathrm{d}t \\ &\leq C\bigg(|\bar{\chi}^{(n)} - \bar{\chi}^{(l)}|(x,0) + \int_0^\tau (|\bar{\theta}^{(n)} - \bar{\theta}^{(l)}| + |\bar{m}^{(n)} - \bar{m}^{(l)}|)(x,t)\,\mathrm{d}t + \max_{t\in[0,\tau]} |\bar{\varepsilon}^{(n)} - \bar{\varepsilon}^{(l)}|(x,t) \bigg). \end{aligned} \tag{7.121}$$

In an analogous way, with $\bar{A}^{(n)}$ replaced by $\bar{S}^{(n)}$, we also get

$$\begin{aligned} &|\bar{m}^{(n)} - \bar{m}^{(l)}|(x,\tau) \leq \int_0^\tau |\bar{m}_t^{(n)} - \bar{m}_t^{(l)}|(x,t)\,\mathrm{d}t \\ &\leq C \int_0^\tau |\bar{\chi}_t^{(n)} - \bar{\chi}_t^{(l)}|(x,t)\,\mathrm{d}t \\ &+ \int_0^\tau \int_0^1 \int_0^\infty \Big| \lambda^{(n)}(x,y)\gamma(\bar{m}^{(n)}, \bar{\chi}^{(n)}, r)\delta^{(n)}(y,t,r) - \lambda^{(l)}(x,y)\gamma(\bar{m}^{(l)}, \bar{\chi}^{(l)}, r)\delta^{(l)}(y,t,r) \Big|\,\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}t \end{aligned} \tag{7.122}$$

where we denote

$$\delta^{(n)} = \delta^{(n)}(y,t,r) = \mathfrak{s}_r[\bar{\varepsilon}^{(n)}](\bar{\varepsilon}^{(n)} - \mathfrak{s}_r[\bar{\varepsilon}^{(n)}])_t(y,t) \overset{\text{(B.18)}}{=} r|\mathfrak{p}_r[\bar{\varepsilon}^{(n)}]_t(y,t)|.$$

Now (7.122) implies

$$\begin{aligned} &|\bar{m}^{(n)} - \bar{m}^{(l)}|(x,\tau) \leq C \int_0^\tau |\bar{m}_t^{(n)} - \bar{m}_t^{(l)}|(x,t)\,\mathrm{d}t \\ &\leq C \left(\int_0^\tau (|\bar{\theta}^{(n)} - \bar{\theta}^{(l)}| + |\bar{m}^{(n)} - \bar{m}^{(l)}|)(x,t)\,\mathrm{d}t + \max_{t\in[0,\tau]} |\bar{\varepsilon}^{(n)} - \bar{\varepsilon}^{(l)}|(x,t) \right) + |\bar{\chi}^{(n)} - \bar{\chi}^{(l)}|(x,0) \\ &+ \int_0^\tau \int_0^1 \int_0^\infty \lambda^{(n)}(x,y)\gamma(\bar{m}^{(n)}, \bar{\chi}^{(n)}, r)|\delta^{(n)} - \delta^{(l)}|(y,t,r)\,\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}t \\ &+ \int_0^\tau \int_0^1 \int_0^\infty \lambda^{(n)}(x,y)|\gamma(\bar{m}^{(n)}, \bar{\chi}^{(n)}, r) - \gamma(\bar{m}^{(l)}, \bar{\chi}^{(l)}, r)|\delta^{(l)}(y,t,r)\,\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}t \\ &+ \int_0^\tau \int_0^1 \int_0^\infty |\lambda^{(n)}(x,y) - \lambda^{(l)}(x,y)|\gamma(\bar{m}^{(l)}, \bar{\chi}^{(l)}, r)|\delta^{(l)}(y,t,r)\,\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}t. \end{aligned} \tag{7.123}$$

Proposition B.7 allows us to conclude that

$$\int_0^\tau |\delta^{(n)} - \delta^{(l)}|(y,t)\,\mathrm{d}t \leq r \int_0^\tau |\bar{\varepsilon}_t^{(n)} - \bar{\varepsilon}_t^{(l)}|(y,t)\,\mathrm{d}t,$$

hence

$$\begin{aligned} \int_0^\tau \int_0^1 \int_0^\infty \lambda^{(n)}(x,y)\gamma(\bar{m}^{(n)}, \bar{\chi}^{(n)}, r)|\delta^{(n)} - \delta^{(l)}|\,\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}t &\leq \Lambda \int_0^\tau \int_0^1 \int_0^\infty \tilde{\gamma}(r)|\delta^{(n)} - \delta^{(l)}|\,\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}t \\ &\leq C \int_0^\tau \int_0^1 |\bar{\varepsilon}_t^{(n)} - \bar{\varepsilon}_t^{(l)}|(y,t)\,\mathrm{d}y\,\mathrm{d}t \end{aligned} \tag{7.124}$$

by Hypothesis 7.1 (i), (iii). In a similar way, exploiting (B.21) and Hypothesis 7.1 (i) we have

$$\begin{aligned}
&\int_0^\tau \int_0^1 \int_0^\infty \delta^{(l)} \lambda^{(n)}(x,y) |\gamma(\bar m^{(n)}, \bar\chi^{(n)}, r) - \gamma(\bar m^{(l)}, \bar\chi^{(l)}, r)| \,\mathrm{d}r \,\mathrm{d}y \,\mathrm{d}t \\
&\le \Lambda \int_0^\tau \int_0^1 \int_0^\infty r |\bar\varepsilon_t^{(l)}(y,t)| \Big(|\gamma(\bar m^{(n)}, \bar\chi^{(n)}, r) - \gamma(\bar m^{(l)}, \bar\chi^{(n)}, r)| \\
&\qquad + |\gamma(\bar m^{(l)}, \bar\chi^{(n)}, r) - \gamma(\bar m^{(l)}, \bar\chi^{(l)}, r)| \Big) \,\mathrm{d}r \,\mathrm{d}y \,\mathrm{d}t \\
&\le \Lambda \int_0^\tau \int_0^1 \int_0^\infty r |\bar\varepsilon_t^{(l)}(y,t)| \Big(\gamma^*(r) |\bar m^{(n)} - \bar m^{(l)}| + \gamma^*(r) |\bar\chi^{(n)} - \bar\chi^{(l)}| \Big) \,\mathrm{d}r \,\mathrm{d}y \,\mathrm{d}t,
\end{aligned}$$

thus employing Hypothesis 7.1 (i) again we end up with

$$\begin{aligned}
&\int_0^\tau \int_0^1 \int_0^\infty \delta^{(l)} \lambda^{(n)}(x,y) |\gamma(\bar m^{(n)}, \bar\chi^{(n)}, r) - \gamma(\bar m^{(l)}, \bar\chi^{(l)}, r)| \,\mathrm{d}r \,\mathrm{d}y \,\mathrm{d}t \\
&\le C \int_0^\tau \left(\int_0^1 |\bar\varepsilon_t^{(l)}(y,t)| \,\mathrm{d}y \right) \max_{x\in(0,1)} \Big(|m^{(n)} - m^{(l)}|(x,t) + |\chi^{(n)} - \chi^{(l)}|(x,t) \Big) \,\mathrm{d}t.
\end{aligned} \tag{7.125}$$

Finally, for every $(x,y) \in \big([(k-1)/n, k/n) \times [(j-1)/n, j/n)\big) \cap \big([(w-1)/l, w/l) \times [(v-1)/l, v/l)\big)$ by Hypothesis 7.1 (iii) we have the pointwise upper bound

$$\begin{aligned}
|\lambda^{(n)}(x,y) - \lambda^{(l)}(x,y)| = |\lambda_{k-j} - \lambda_{w-v}| &= \left| \lambda\left(\frac{k-j}{n}\right) - \lambda\left(\frac{w-v}{l}\right) \right| \\
&\le \Lambda \left(\left| \frac{k}{n} - \frac{w}{l} \right| + \left| \frac{j}{n} - \frac{v}{l} \right| \right) \le \frac{4\Lambda}{\min\{n,l\}} \,.
\end{aligned} \tag{7.126}$$

Combining (7.121)–(7.126) gives the inequality

$$\begin{aligned}
&\max_{x\in(0,1)} \Big(|\bar m^{(n)} - \bar m^{(l)}|(x,\tau) + |\bar\chi^{(n)} - \bar\chi^{(l)}|(x,\tau) \Big) \\
&\le \max_{x\in(0,1)} \int_0^\tau \Big(|\bar m_t^{(n)} - \bar m_t^{(l)}|(x,t) + |\bar\chi_t^{(n)} - \bar\chi_t^{(l)}|(x,t) \Big) \,\mathrm{d}t \\
&\le q_{nl} + C \int_0^\tau \left(\int_0^1 |\bar\varepsilon_t^{(l)}(y,t)| \,\mathrm{d}y \right) \max_{x\in(0,1)} \Big(|\bar m^{(n)} - \bar m^{(l)}|(x,t) + |\bar\chi^{(n)} - \bar\chi^{(l)}|(x,t) \Big) \,\mathrm{d}t,
\end{aligned} \tag{7.127}$$

with

$$q_{nl} = C \left(\frac{1}{\min\{n,l\}} + |\bar\chi^{(n)}(\cdot,0) - \bar\chi^{(l)}(\cdot,0)|_\infty + \|\bar\theta^{(n)} - \bar\theta^{(l)}\|_\infty + \|\bar\varepsilon_t^{(n)} - \bar\varepsilon_t^{(l)}\|_1 + \|\bar\varepsilon^{(n)} - \bar\varepsilon^{(l)}\|_\infty \right).$$

Inequality (7.127) can be interpreted as an inequality of the form

$$q(\tau) \le q_{nl} + \int_0^\tau s^{(l)}(t) q(t) \,\mathrm{d}t,$$

with

$$\begin{aligned}
q(t) &= \max_{x\in(0,1)} \Big(|\bar m^{(n)} - \bar m^{(l)}| + |\bar\chi^{(n)} - \bar\chi^{(l)}| \Big)(x,t), \\
s^{(l)}(t) &= C \int_0^1 |\bar\varepsilon_t^{(l)}(y,t)| \,\mathrm{d}y,
\end{aligned}$$

where $s^{(l)}$ is uniformly bounded in $L^1(0,T)$. We obtain using Grönwall's lemma A.2 that

$$q(\tau) \leq q_{nl}\, \mathrm{e}^{\int_0^\tau s^{(l)}(t)\,\mathrm{d}t} \leq C q_{nl}.$$

Note that the sequence $\bar{\chi}^{(n)}(\cdot,0)$ is uniformly convergent to $\chi^0(\cdot)$ by Hypothesis 7.1 (v). Indeed, by definition of $\bar{\chi}(x)$ in (7.101) for $x \in [(k-1)/n, k/n)$, $k = 1, \ldots, n-1$ it holds

$$|\chi^0(x) - \bar{\chi}^{(n)}(x,0)| = |\chi^0(x) - \chi_k(0)| = \left|\chi^0(x) - n\int_{(k-1)/n}^{k/n} \chi^0(y)\,\mathrm{d}y\right| = \left|n\int_{(k-1)/n}^{k/n}\int_y^x \chi^0_x(\xi)\,\mathrm{d}\xi\,\mathrm{d}y\right|$$
$$\leq n\int_{(k-1)/n}^{k/n}\int_{(k-1)/n}^{k/n} |\chi^0_x(\xi)|\,\mathrm{d}\xi\,\mathrm{d}y \leq \frac{1}{\sqrt{n}}\left(\int_{(k-1)/n}^{k/n} |\chi^0_x(\xi)|^2\,\mathrm{d}\xi\right)^{1/2},$$

and the same for $x \in [(n-1)/n,1]$ since $|\chi^0(x) - \bar{\chi}^{(n)}(x,0)| = |\chi^0(x) - \chi_{n-1}(0)|$. Hence $\bar{\chi}^{(n)}(\cdot,0)$ is a Cauchy sequence in $L^\infty(0,1)$. This, together with the convergences (7.118), (7.119) and (7.120), implies that q_{nl} is small if n, l are large. Hence, $\bar{m}^{(n)}$ and $\bar{\chi}^{(n)}$ are Cauchy sequences in $L^\infty(\Omega_T)$, and this gives

$$\bar{m}^{(n)} \to m, \quad \bar{\chi}^{(n)} \to \chi \qquad \text{strongly in } L^\infty(\Omega_T), \tag{7.128}$$
$$\bar{m}^{(n)}_t \to m_t, \quad \bar{\chi}^{(n)}_t \to \chi_t \quad \text{strongly in } L^\infty(0,1; L^1(0,T)). \tag{7.129}$$

By virtue of (5.80) $\bar{m}^{(n)}_t$ and $\bar{\chi}^{(n)}_t$ are uniformly bounded in $L^\infty(\Omega_T)$, hence

$$\bar{m}^{(n)}_t \to m_t, \quad \bar{\chi}^{(n)}_t \to \chi_t \qquad \text{weakly-star in } L^\infty(\Omega_T).$$

Concerning the hysteretic terms, arguing as for (7.65) and using Proposition B.5 we get

$$\int_0^\tau |P_0[\bar{m}^{(n)}, \bar{\chi}^{(n)}, \bar{\varepsilon}^{(n)}] - P_0[m,\chi,\varepsilon]|^2(x,t)\,\mathrm{d}t$$
$$\leq C\left(\int_0^\tau \left(|\bar{m}^{(n)} - m|^2 + |\bar{\chi}^{(n)} - \chi|^2\right)(x,t)\,\mathrm{d}t + \max_{t\in[0,\tau]} |\bar{\varepsilon}^{(n)} - \varepsilon|^2(x,t)\right),$$

whereas by Hypothesis 7.1 (i), identity (B.21) for the play and Proposition B.7 we obtain for all $x \in [0,1]$

$$\int_0^\tau |\bar{D}^{(n)} - D[m,\chi,\varepsilon]|^2(x,t)\,\mathrm{d}t$$
$$= \int_0^t \left|\int_0^\infty \gamma(\bar{m}^{(n)}, \bar{\chi}^{(n)}, r) r |\mathfrak{p}_r[\bar{\varepsilon}^{(n)}]_t|\,\mathrm{d}r - \int_0^\infty \gamma(m,\chi,r) r |\mathfrak{p}_r[\varepsilon]_t|\,\mathrm{d}r\right|^2 \mathrm{d}\tau$$
$$= \int_0^\tau \Bigg| \int_0^\infty r(\gamma(\bar{m}^{(n)}, \bar{\chi}^{(n)}, r) - \gamma(m, \bar{\chi}^{(n)}, r))|\mathfrak{p}_r[\bar{\varepsilon}^{(n)}]_t|\,\mathrm{d}r$$
$$+ r(\gamma(m, \bar{\chi}^{(n)}, r) - \gamma(m,\chi,r))|\mathfrak{p}_r[\bar{\varepsilon}^{(n)}]_t|\,\mathrm{d}r + r\gamma(m,\chi,r)(|\mathfrak{p}_r[\bar{\varepsilon}^{(n)}]_t| - |\mathfrak{p}_r[\varepsilon]_t|)\,\mathrm{d}r\Bigg|^2 \mathrm{d}t$$
$$\leq C\left(\int_0^\tau \left(|\bar{m}^{(n)} - m|^2 + |\bar{\chi}^{(n)} - \chi|^2 + |\bar{\varepsilon}^{(n)}_t|^2\right)(x,t)\,\mathrm{d}t + \max_{t\in[0,\tau]} |\bar{\varepsilon}^{(n)} - \varepsilon|^2(x,t)\right),$$

and arguing analogously

$$
\begin{aligned}
&\int_0^\tau |\bar{K}^{(n)} - K[m,\chi,\varepsilon]|^2(x,t)\,\mathrm{d}t \\
&= \int_0^\tau \left| -\frac{1}{2}\int_0^\infty \gamma_m(\bar{m}^{(n)},\bar{\chi}^{(n)},r)\mathfrak{s}_r^2[\bar{\varepsilon}^{(n)}]\,\mathrm{d}r + \frac{1}{2}\int_0^\infty \gamma_m(m,\chi,r)\mathfrak{s}_r^2[\varepsilon]\,\mathrm{d}r\right|^2\,\mathrm{d}t \\
&\le C\Big(\int_0^\tau \Big(|\bar{m}^{(n)} - m|^2 + |\bar{\chi}^{(n)} - \chi|^2\Big)(x,t)\,\mathrm{d}t + \max_{t\in[0,\tau]} |\bar{\varepsilon}^{(n)} - \varepsilon|^2(x,t)\Big),
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^\tau |\bar{M}^{(n)} - M[m,\chi,\varepsilon]|^2(x,t)\,\mathrm{d}t \\
&= \int_0^\tau \left| -\frac{1}{2}\int_0^\infty \gamma_\chi(\bar{m}^{(n)},\bar{\chi}^{(n)},r)\mathfrak{s}_r^2[\bar{\varepsilon}^{(n)}]\,\mathrm{d}r + \frac{1}{2}\int_0^\infty \gamma_\chi(m,\chi,r)\mathfrak{s}_r^2[\varepsilon]\,\mathrm{d}r\right|^2\,\mathrm{d}t \\
&\le C\left(\int_0^\tau \Big(|\bar{m}^{(n)} - m|^2 + |\bar{\chi}^{(n)} - \chi|^2\Big)(x,t)\,\mathrm{d}t + \max_{t\in[0,\tau]} |\bar{\varepsilon}^{(n)} - \varepsilon|^2(x,t)\right)
\end{aligned}
$$

since for $t \in [0,\tau]$

$$
|\mathfrak{s}_r^2[\bar{\varepsilon}^{(n)}] - \mathfrak{s}_r^2[\varepsilon]|(x,t) = |\mathfrak{s}_r[\bar{\varepsilon}^{(n)}] + \mathfrak{s}_r[\varepsilon]||\mathfrak{s}_r[\bar{\varepsilon}^{(n)}] - \mathfrak{s}_r[\varepsilon]|(x,t) \le 2r\cdot 2\max_{t\in[0,\tau]}|\bar{\varepsilon}^{(n)} - \varepsilon|(x,t)
$$

by Proposition B.5. Hence using the convergences (7.118), (7.119) and (7.128) we obtain, for all $x \in (0,1)$,

$$
\left.
\begin{aligned}
P_0[\bar{m}^{(n)},\bar{\chi}^{(n)},\bar{\varepsilon}^{(n)}](x,\cdot) &\;\to\; P_0[m,\chi,\varepsilon](x,\cdot) \\
\bar{D}^{(n)}(x,\cdot) &\;\to\; D[m,\chi,\varepsilon](x,\cdot) \\
\bar{K}^{(n)}(x,\cdot) &\;\to\; K[m,\chi,\varepsilon](x,\cdot) \\
\bar{M}^{(n)}(x,\cdot) &\;\to\; M[m,\chi,\varepsilon](x,\cdot)
\end{aligned}
\right\}\quad \text{strongly in } L^2(0,T).
$$

We can then pass to the limit in (7.111)–(7.116), and conclude that (u,w,θ,m,χ) is a strong solution to (6.22)–(6.26) with the regularity stated in Theorem 7.3 and satisfying the initial conditions (6.28). It remains to check that the boundary conditions (6.29) hold. By (7.9) we have $w_n(t) = 0$, hence

$$
\begin{aligned}
|\tilde{w}^{(n)}(1,t)| &= \left|\frac{1}{2}(w_n(t) - w_{n-1}(t)) + \left(1 - \frac{n-1}{n}\right)n(w_n(t) - w_{n-1}(t))\right. \\
&\quad \left. + \frac{1}{2}\left(1 - \frac{n-1}{n}\right)^2 n^2(w_n - 2w_{n-1} + w_{n-2})\right| \\
&= \left|w_n(t) - w_{n-1}(t) - \frac{1}{2}(w_{n-1}(t) - w_{n-2}(t))\right| \\
&\le 2\Big(\sum_{k=1}^n |w_k - w_{k-1}|^2(t)\Big)^{1/2} \le \frac{C}{\sqrt{n}}\,,
\end{aligned}
$$

where the last inequality follows from estimate (7.41) for the dominant term $\sum_{k=1}^n(\dot{w}_k - \dot{w}_{k-1})^2$. A similar argument holds also for $w(0,t)$, $u(1,t)$, $u(0,t)$. To complete the existence proof, we only need to check the homogeneous Neumann boundary condition for θ. In other words, we have to check that for every $\tilde{\psi} \in C^1(\overline{\Omega}_T)$ we have

$$
\int_0^T\int_0^1 (\theta_x\tilde{\psi}_x + \theta_{xx}\tilde{\psi})(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0\,. \tag{7.130}
$$

Integrating by parts in space and using the boundary conditions (7.9) yields

$$
\begin{aligned}
&\int_0^T \int_0^1 (\tilde\theta_x^{(n)} \tilde\psi_x + \tilde\theta_{xx}^{(n)} \tilde\psi)(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \Bigg(\sum_{k=1}^{n-1} \int_{(k-1)/n}^{k/n} \left(n(\theta_k - \theta_{k-1}) + \left(x - \frac{k-1}{n} \right) n^2 (\theta_{k+1} - 2\theta_k + \theta_{k-1}) \right) \tilde\psi_x \,\mathrm{d}x \\
&\quad + \int_{(n-1)/n}^{1} \left(n(\theta_n - \theta_{n-1}) + \left(x - \frac{n-1}{n} \right) n^2 (\theta_n - 2\theta_{n-1} + \theta_{n-2}) \right) \tilde\psi_x \,\mathrm{d}x \\
&\quad + \sum_{k=1}^{n-1} \int_{(k-1)/n}^{k/n} n^2 (\theta_{k+1} - 2\theta_k + \theta_{k-1})\, \tilde\psi \,\mathrm{d}x + \int_{(n-1)/n}^{1} n^2 (\theta_n - 2\theta_{n-1} + \theta_{n-2})\, \tilde\psi \,\mathrm{d}x \Bigg) \,\mathrm{d}t \\
&= \int_0^T \Bigg(\sum_{k=1}^{n-1} \left(n(\theta_{k+1} - \theta_k)(t)\, \tilde\psi\left(\frac{k}{n}, t \right) - n(\theta_k - \theta_{k-1})(t)\, \tilde\psi\left(\frac{k-1}{n}, t \right) \right) \\
&\quad - n(\theta_{n-1} - \theta_{n-2})(t)\, \tilde\psi(1,t) \Bigg) \,\mathrm{d}t \\
&= \int_0^T \Big(n(\theta_n - \theta_{n-1})(t)\, \tilde\psi(1,t) - n(\theta_1 - \theta_0)(t)\, \tilde\psi(0,t) - n(\theta_{n-1} - \theta_{n-2})(t)\, \tilde\psi(1,t) \Big) \,\mathrm{d}t,
\end{aligned}
$$

hence using (7.9) one more time we obtain

$$
\int_0^T \int_0^1 (\tilde\theta_x^{(n)} \tilde\psi_x + \tilde\theta_{xx}^{(n)} \tilde\psi)(x,t)\,\mathrm{d}x\,\mathrm{d}t = - \int_0^T n(\theta_{n-1} - \theta_{n-2})(t)\, \tilde\psi(1,t)\,\mathrm{d}t\,.
$$

We have

$$
\begin{aligned}
\int_0^T n^2 (\theta_{n-1} - \theta_{n-2})^2(t)\,\mathrm{d}t &= \int_0^T n^2 (\theta_n - 2\theta_{n-1} + \theta_{n-2})^2(t)\,\mathrm{d}t \\
&\le \int_0^T n^2 \sum_{k=1}^{n-1} (\theta_{k+1} - 2\theta_k + \theta_{k-1})^2(t)\,\mathrm{d}t \le \frac{C}{n}
\end{aligned}
$$

by (7.89), hence

$$
\lim_{n\to\infty} \int_0^T \int_0^1 (\tilde\theta_x^{(n)} \tilde\psi_x + \tilde\theta_{xx}^{(n)} \tilde\psi)(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0
$$

and (7.130) follows.

7.4 Proof of the uniqueness

Let (u, w, θ, χ, m), $(\tilde u, \tilde w, \tilde\theta, \tilde\chi, \tilde m)$ be two solutions to (6.22)–(6.27), (6.29) with the regularity as in Theorem 7.3, and with the same initial conditions and the same right-hand sides.

We start by integrating in time $\int_0^t \mathrm{d}\tau$ the difference of (6.25) for θ and $\tilde\theta$ obtaining

$$
\begin{aligned}
&(\theta-\tilde\theta)(x,t) - \int_0^t \kappa(\theta_{xx}-\tilde\theta_{xx})\,\mathrm{d}\tau \\
&= -\int_0^t \beta(\theta w_{xxt}-\tilde\theta\tilde w_{xxt})\,\mathrm{d}\tau + \int_0^t \nu(w_{xxt}^2-\tilde w_{xxt}^2)\,\mathrm{d}\tau \\
&+ \int_0^t (D[m,\chi,w_{xx}]-D[\tilde m,\tilde\chi,\tilde w_{xx}])\,\mathrm{d}\tau + \int_0^t (m_t\,K[m,\chi,w_{xx}]-\tilde m_t\,K[\tilde m,\tilde\chi,\tilde w_{xx}])\,\mathrm{d}\tau \\
&+ \int_0^t (\chi_t\,M[m,\chi,w_{xx}]-\tilde\chi_t\,M[\tilde m,\tilde\chi,\tilde w_{xx}])\,\mathrm{d}\tau - \int_0^t L(\chi_t-\tilde\chi_t)\,\mathrm{d}\tau + \int_0^t (g(\theta,x,\tau)-g(\tilde\theta,x,\tau))\,\mathrm{d}\tau.
\end{aligned}
$$

Next we test by $(\theta(x,t)-\tilde\theta(x,t))$ and integrate in space $\int_0^1 \mathrm{d}x$, getting

$$
\begin{aligned}
&\int_0^1 (\theta-\tilde\theta)^2(x,t)\,\mathrm{d}x - \int_0^1 \left(\int_0^t \kappa(\theta_{xx}-\tilde\theta_{xx})\,\mathrm{d}\tau\right)(\theta-\tilde\theta)(x,t)\,\mathrm{d}x \\
&= -\int_0^1 \left(\int_0^t \beta(\theta w_{xxt}-\tilde\theta\tilde w_{xxt})\,\mathrm{d}\tau\right)(\theta-\tilde\theta)(x,t)\,\mathrm{d}x + \int_0^1 \left(\int_0^t \nu(w_{xxt}^2-\tilde w_{xxt}^2)\,\mathrm{d}\tau\right)(\theta-\tilde\theta)(x,t)\,\mathrm{d}x \\
&+ \int_0^1 \left(\int_0^t (D[m,\chi,w_{xx}]-D[\tilde m,\tilde\chi,\tilde w_{xx}])\,\mathrm{d}\tau\right)(\theta-\tilde\theta)(x,t)\,\mathrm{d}x \\
&+ \int_0^1 \left(\int_0^t (m_t\,K[m,\chi,w_{xx}]-\tilde m_t\,K[\tilde m,\tilde\chi,\tilde w_{xx}])\,\mathrm{d}\tau\right)(\theta-\tilde\theta)(x,t)\,\mathrm{d}x \\
&+ \int_0^1 \left(\int_0^t (\chi_t\,M[m,\chi,w_{xx}]-\tilde\chi_t\,M[\tilde m,\tilde\chi,\tilde w_{xx}])\,\mathrm{d}\tau\right)(\theta-\tilde\theta)(x,t)\,\mathrm{d}x \\
&- \int_0^1 \left(\int_0^t L(\chi_t-\tilde\chi_t)\,\mathrm{d}\tau\right)(\theta-\tilde\theta)(x,t)\,\mathrm{d}x + \int_0^1 \left(\int_0^t (g(\theta,x,\tau)-g(\tilde\theta,x,\tau))\,\mathrm{d}\tau\right)(\theta-\tilde\theta)(x,t)\,\mathrm{d}x.
\end{aligned}
\tag{7.131}
$$

Concerning the second summand on the left-hand side, note that integrating by parts in space and exploiting (6.29) we obtain

$$
\begin{aligned}
-\int_0^1 \left(\int_0^t (\theta_{xx}-\tilde\theta_{xx})\,\mathrm{d}\tau\right)(\theta-\tilde\theta)(x,t)\,\mathrm{d}x &= \int_0^1 \left(\int_0^t (\theta_x-\tilde\theta_x)\,\mathrm{d}\tau\right)(\theta_x-\tilde\theta_x)(x,t)\,\mathrm{d}x \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(\int_0^t (\theta_x-\tilde\theta_x)(x,\tau)\,\mathrm{d}\tau\right)^2 \mathrm{d}x.
\end{aligned}
$$

We now estimate the terms on the right-hand side, where we are going to apply Hölder's inequality in space. It holds

$$
\begin{aligned}
&\int_0^t |\theta w_{xxt}-\tilde\theta\tilde w_{xxt}|(x,\tau)\,\mathrm{d}\tau \le \int_0^t |w_{xxt}|\,|\theta-\tilde\theta|(x,\tau)\,\mathrm{d}\tau + \int_0^t |\tilde\theta|\,|w_{xxt}-\tilde w_{xxt}|(x,\tau)\,\mathrm{d}\tau \\
&\le \left(\int_0^t |w_{xxt}|^2(x,\tau)\,\mathrm{d}\tau\right)^{1/2} \left(\int_0^t |\theta-\tilde\theta|^2(x,\tau)\,\mathrm{d}\tau\right)^{1/2} + \|\tilde\theta\|_\infty \int_0^t |w_{xxt}-\tilde w_{xxt}|(x,\tau)\,\mathrm{d}\tau.
\end{aligned}
$$

The continuous embedding $W^{1,2}(0,1;L^2(0,T)) \hookrightarrow L^\infty(0,1;L^2(0,T))$ yields

$$
\max_{x\in[0,1]} \int_0^t |w_{xxt}|^2(x,\tau)\,\mathrm{d}\tau \le C\left(\|w_{xxt}\|_2^2 + \|w_{xxxt}\|_2^2\right),
$$

hence by the regularity part of Theorem 7.3 and Hölder's inequality we obtain

$$
\int_0^t |\theta w_{xxt}-\tilde\theta\tilde w_{xxt}|(x,\tau)\,\mathrm{d}\tau \le C\left(\left(\int_0^t |\theta-\tilde\theta|^2(x,\tau)\,\mathrm{d}\tau\right)^{1/2} + \int_0^t |w_{xxt}-\tilde w_{xxt}|(x,\tau)\,\mathrm{d}\tau\right).
$$

Similarly

$$\int_0^t |w_{xxt}^2 - \tilde{w}_{xxt}^2|(x,\tau)\,\mathrm{d}\tau = \int_0^t (|w_{xxt} - \tilde{w}_{xxt}|\,|w_{xxt} + \tilde{w}_{xxt}|)(x,\tau)\,\mathrm{d}\tau$$
$$\leq C\left(\int_0^t |w_{xxt} - \tilde{w}_{xxt}|^2(x,\tau)\,\mathrm{d}\tau\right)^{1/2}.$$

The dissipation term is estimated as

$$\begin{aligned}
&\int_0^t |\mathcal{D}[m,\chi,w_{xx}] - \mathcal{D}[\tilde{m},\tilde{\chi},\tilde{w}_{xx}]|\,(x,\tau)\,\mathrm{d}\tau \\
&= \int_0^t \left|\int_0^\infty r\gamma(m,\chi,r)|\mathfrak{p}_r[w_{xx}]_t|\,\mathrm{d}r - \int_0^\infty r\gamma(\tilde{m},\tilde{\chi},r)|\mathfrak{p}_r[\tilde{w}_{xx}]_t|\,\mathrm{d}r\right|\,\mathrm{d}\tau \\
&= \int_0^t \left|\int_0^\infty r(\gamma(m,\chi,r) - \gamma(\tilde{m},\chi,r))|\mathfrak{p}_r[w_{xx}]_t|\,\mathrm{d}r + \int_0^\infty r(\gamma(\tilde{m},\chi,r) - \gamma(\tilde{m},\tilde{\chi},r))|\mathfrak{p}_r[w_{xx}]_t|\,\mathrm{d}r\right. \\
&\left. + \int_0^\infty r\gamma(\tilde{m},\tilde{\chi},r)|\mathfrak{p}_r[w_{xx}]_t - \mathfrak{p}_r[\tilde{w}_{xx}]_t|\,\mathrm{d}r\right|\,\mathrm{d}\tau \\
&\leq C\int_0^t (|m-\tilde{m}||w_{xxt}| + |\chi-\tilde{\chi}||w_{xxt}| + |w_{xxt} - \tilde{w}_{xxt}|)\,(x,\tau)\,\mathrm{d}\tau,
\end{aligned}$$

where we have used identity (B.21) for the play, Proposition B.7 and Hypothesis 7.1 (i). Arguing as above, by the regularity part of Theorem 7.3 we then obtain

$$\begin{aligned}
&\int_0^t |\mathcal{D}[m,\chi,w_{xx}] - \mathcal{D}[\tilde{m},\tilde{\chi},\tilde{w}_{xx}]|\,(x,\tau)\,\mathrm{d}\tau \\
&\leq C\left(\left(\int_0^t |m-\tilde{m}|^2(x,\tau)\,\mathrm{d}\tau\right)^{1/2} + \left(\int_0^t |\chi-\tilde{\chi}|^2(x,\tau)\,\mathrm{d}\tau\right)^{1/2} + \int_0^t |w_{xxt} - \tilde{w}_{xxt}|(x,\tau)\,\mathrm{d}\tau\right).
\end{aligned} \tag{7.132}$$

The terms containing K and M are estimated similarly as

$$\begin{aligned}
&\int_0^t |m_t\,K[m,\chi,w_{xx}] - \tilde{m}_t\,K[\tilde{m},\tilde{\chi},\tilde{w}_{xx}]|\,(x,\tau)\,\mathrm{d}\tau \\
&= \frac{1}{2}\int_0^t \left|m_t\int_0^\infty \gamma_m(m,\chi,r)\mathfrak{s}_r^2[w_{xx}]\,\mathrm{d}r - \tilde{m}_t\int_0^\infty \gamma_m(\tilde{m},\tilde{\chi},r)\,\mathfrak{s}_r^2[\tilde{w}_{xx}]\,\mathrm{d}r\right|\,\mathrm{d}\tau \\
&= \frac{1}{2}\int_0^t \left|m_t\int_0^\infty (\gamma_m(m,\chi,r) - \gamma_m(\tilde{m},\chi,r))\mathfrak{s}_r^2[w_{xx}]\,\mathrm{d}r\right. \\
&+ m_t\int_0^\infty (\gamma_m(\tilde{m},\chi,r) - \gamma_m(\tilde{m},\tilde{\chi},r))\mathfrak{s}_r^2[w_{xx}]\,\mathrm{d}r + m_t\int_0^\infty \gamma_m(\tilde{m},\tilde{\chi},r)(\mathfrak{s}_r^2[w_{xx}] - \mathfrak{s}_r^2[\tilde{w}_{xx}])\,\mathrm{d}r \\
&\left. + (m_t - \tilde{m}_t)\int_0^\infty \gamma_m(\tilde{m},\tilde{\chi},r)\mathfrak{s}_r^2[\tilde{w}_{xx}]\,\mathrm{d}r\right|\,\mathrm{d}\tau \\
&\leq C\int_0^t \Big(|m_t|(|m-\tilde{m}| + |\chi-\tilde{\chi}| + |w_{xxt} - \tilde{w}_{xxt}|) + |m_t - \tilde{m}_t|\Big)(x,\tau)\,\mathrm{d}\tau
\end{aligned} \tag{7.133}$$

and

$$
\begin{aligned}
&\int_0^t |\chi_t\, M[m,\chi,w_{xx}] - \tilde\chi_t\, M[\tilde m,\tilde\chi,\tilde w_{xx}]|\,(x,\tau)\,\mathrm{d}\tau \\
&= \frac{1}{2}\int_0^t \left| \chi_t \int_0^\infty \gamma_\chi(m,\chi,r)\mathfrak{s}_r^2[w_{xx}]\,\mathrm{d}r - \tilde\chi_t \int_0^\infty \gamma_\chi(\tilde m,\tilde\chi,r)\,\mathfrak{s}_r^2[\tilde w_{xx}]\,\mathrm{d}r \right| \mathrm{d}\tau \\
&= \frac{1}{2}\int_0^t \Bigg| \chi_t \int_0^\infty (\gamma_\chi(m,\chi,r) - \gamma_\chi(\tilde m,\chi,r))\mathfrak{s}_r^2[w_{xx}]\,\mathrm{d}r + \chi_t \int_0^\infty (\gamma_\chi(\tilde m,\chi,r) - \gamma_\chi(\tilde m,\tilde\chi,r))\mathfrak{s}_r^2[w_{xx}]\,\mathrm{d}r \\
&+ \chi_t \int_0^\infty \gamma_\chi(\tilde m,\tilde\chi,r)(\mathfrak{s}_r^2[w_{xx}] - \mathfrak{s}_r^2[\tilde w_{xx}])\,\mathrm{d}r + (\chi_t - \tilde\chi_t)\int_0^\infty \gamma_\chi(\tilde m,\tilde\chi,r)\mathfrak{s}_r^2[\tilde w_{xx}]\,\mathrm{d}r \Bigg|\,\mathrm{d}\tau \\
&\le C\int_0^t \Big(|\chi_t|(|m-\tilde m| + |\chi-\tilde\chi| + |w_{xxt} - \tilde w_{xxt}|) + |\chi_t - \tilde\chi_t| \Big)(x,\tau)\,\mathrm{d}\tau,
\end{aligned} \tag{7.134}
$$

where we used Hypothesis 7.1 (i) and Proposition B.5. Our aim is now to estimate the terms $|m-\tilde m|$, $|m_t - \tilde m_t|$, $|\chi-\tilde\chi|$, $|\chi_t - \tilde\chi_t|$. Arguing in a similar way as we did in (7.121) we deduce

$$
|\chi-\tilde\chi|(x,t) \le \int_0^t |\chi_t - \tilde\chi_t|(x,\tau)\,\mathrm{d}\tau \le C\int_0^t (|\theta-\tilde\theta| + |m-\tilde m| + |w_{xxt} - \tilde w_{xxt}|)(x,\tau)\,\mathrm{d}\tau, \tag{7.135}
$$

whereas arguing as for (7.122)–(7.125) we obtain

$$
\begin{aligned}
&|m-\tilde m|(x,t) \le \int_0^t |m_t - \tilde m_t|(x,\tau)\,\mathrm{d}\tau \\
&\le C\int_0^t \left(|\chi_t - \tilde\chi_t|(x,\tau) + \int_0^1 (|m-\tilde m||w_{xxt}| + |\chi-\tilde\chi||w_{xxt}| + |w_{xxt} - \tilde w_{xxt}|)(y,\tau)\,\mathrm{d}y \right)\mathrm{d}\tau.
\end{aligned} \tag{7.136}
$$

We observe that, again by the regularity part of Theorem 7.3,

$$
\begin{aligned}
\int_0^t\int_0^1 |m-\tilde m||w_{xxt}|\,\mathrm{d}y\,\mathrm{d}\tau &\le C\int_0^t \left(\int_0^1 |m-\tilde m|^2\,\mathrm{d}y\right)^{1/2}\mathrm{d}\tau, \\
\int_0^t\int_0^1 |\chi-\tilde\chi||w_{xxt}|\,\mathrm{d}y\,\mathrm{d}\tau &\le C\int_0^t \left(\int_0^1 |\chi-\tilde\chi|^2\,\mathrm{d}y\right)^{1/2}\mathrm{d}\tau.
\end{aligned}
$$

Plugging this back into (7.136) together with (7.135) we have

$$
\begin{aligned}
&|m-\tilde m|(x,t) \le \int_0^t |m_t - \tilde m_t|(x,\tau)\,\mathrm{d}\tau \\
&\le C\int_0^t (|\theta-\tilde\theta| + |m-\tilde m| + |w_{xxt} - \tilde w_{xxt}|)(x,\tau)\,\mathrm{d}\tau \\
&+ C\int_0^t \left(\int_0^1 |m-\tilde m|^2(x,\tau)\,\mathrm{d}y\right)^{1/2}\mathrm{d}\tau + C\int_0^t \left(\int_0^1 |\chi-\tilde\chi|^2(x,\tau)\,\mathrm{d}y\right)^{1/2}\mathrm{d}\tau \\
&+ C\left(\int_0^t\int_0^1 |w_{xxt} - \tilde w_{xxt}|^2(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau\right)^{1/2}.
\end{aligned} \tag{7.137}
$$

Then (7.135) and (7.137) imply

$$
\begin{aligned}
&|m-\tilde m|^2(x,t) + |\chi-\tilde\chi|^2(x,t) \\
&\le C\Bigg(\int_0^t (|\theta-\tilde\theta|^2 + |m-\tilde m|^2 + |w_{xxt} - \tilde w_{xxt}|^2)(x,\tau)\,\mathrm{d}\tau \\
&+ \int_0^t\int_0^1 (|m-\tilde m|^2 + |\chi-\tilde\chi|^2 + |w_{xxt} - \tilde w_{xxt}|^2)(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \Bigg).
\end{aligned}
$$

Integrating in space and using Grönwall's lemma (A.2), we obtain from (7.137) that

$$
\begin{aligned}
&\int_0^t |m_t - \tilde m_t|(x,\tau)\,\mathrm{d}\tau \\
&\le C\Bigg(\int_0^t (|\theta-\tilde\theta| + |m-\tilde m| + |w_{xxt}-\tilde w_{xxt}|)(x,\tau)\,\mathrm{d}\tau \\
&+ \bigg(\int_0^t\int_0^1 (|\theta-\tilde\theta|^2 + |w_{xxt}-\tilde w_{xxt}|^2)(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau\bigg)^{1/2}\Bigg),
\end{aligned}
$$

from which by (7.135) also

$$
\begin{aligned}
&\int_0^t |m_t - \tilde m_t|(x,\tau)\,\mathrm{d}\tau + \int_0^t |\chi_t - \tilde\chi_t|(x,\tau)\,\mathrm{d}\tau \\
&\le C\Bigg(\int_0^t (|\theta-\tilde\theta| + |w_{xxt}-\tilde w_{xxt}|)(x,\tau)\,\mathrm{d}\tau + \int_0^t\int_0^\tau |m_t-\tilde m_t|(x,\tau')\,\mathrm{d}\tau'\,\mathrm{d}\tau \\
&+ \bigg(\int_0^t\int_0^1 (|\theta-\tilde\theta|^2 + |w_{xxt}-\tilde w_{xxt}|^2)(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau\bigg)^{1/2}\Bigg).
\end{aligned}
$$

Applying Grönwall's lemma A.2 one more time produces the desired inequality

$$
\begin{aligned}
|m-\tilde m|(x,t) + |\chi-\tilde\chi|(x,t) &\le \int_0^t (|m_t-\tilde m_t| + |\chi_t-\tilde\chi_t|)(x,\tau)\,\mathrm{d}\tau \\
&\le C\Bigg(\int_0^t (|\theta-\tilde\theta| + |w_{xxt}-\tilde w_{xxt}|)(x,\tau)\,\mathrm{d}\tau \\
&\quad + \bigg(\int_0^t\int_0^1 (|\theta-\tilde\theta|^2 + |w_{xxt}-\tilde w_{xxt}|^2)(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau\bigg)^{1/2}\Bigg).
\end{aligned}
\tag{7.138}
$$

Substituting in (7.132) and exploiting the regularity part of Theorem 7.3 we obtain

$$
\begin{aligned}
&\int_0^t \left|\mathcal{D}[m,\chi,w_{xx}] - \mathcal{D}[\tilde m,\tilde\chi,\tilde w_{xx}]\right|(x,\tau)\,\mathrm{d}\tau \\
&\le C\bigg(\int_0^t (|\theta-\tilde\theta|^2 + |w_{xxt}-\tilde w_{xxt}|^2)(x,\tau)\,\mathrm{d}\tau + \int_0^t\int_0^1 (|\theta-\tilde\theta|^2 + |w_{xxt}-\tilde w_{xxt}|^2)(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau\bigg)^{1/2},
\end{aligned}
$$

and from (7.133)–(7.134)

$$
\begin{aligned}
&\int_0^t \left|m_t\,K[m,\chi,w_{xx}] - \tilde m_t\,K[\tilde m,\tilde\chi,\tilde w_{xx}]\right|(x,\tau)\,\mathrm{d}\tau \\
&\le C\Bigg(\int_0^t (|\theta-\tilde\theta| + |w_{xxt}-\tilde w_{xxt}|)(x,\tau)\,\mathrm{d}\tau + \bigg(\int_0^t\int_0^1 (|\theta-\tilde\theta|^2 + |w_{xxt}-\tilde w_{xxt}|^2)(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau\bigg)^{1/2}\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^t \left|\chi_t\,M[m,\chi,w_{xx}] - \tilde\chi_t\,M[\tilde m,\tilde\chi,\tilde w_{xx}]\right|(x,\tau)\,\mathrm{d}\tau \\
&\le C\Bigg(\int_0^t (|\theta-\tilde\theta| + |w_{xxt}-\tilde w_{xxt}|)(x,\tau)\,\mathrm{d}\tau + \bigg(\int_0^t\int_0^1 (|\theta-\tilde\theta|^2 + |w_{xxt}-\tilde w_{xxt}|^2)(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau\bigg)^{1/2}\Bigg).
\end{aligned}
$$

We finally observe that, by Hypothesis 7.1 (vii),

$$
\int_0^t |g(\theta,x,\tau) - g(\tilde\theta,x,\tau)|\,\mathrm{d}\tau \le g_1 \int_0^t |\theta-\tilde\theta|(x,\tau)\,\mathrm{d}\tau.
$$

Coming back to (7.131), applying Hölder's inequality in space we see that all these estimates yield

$$\begin{aligned}&\int_0^1 |\theta-\tilde\theta|^2(x,t)\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left(\int_0^t(\theta_x-\tilde\theta_x)(x,\tau)\,\mathrm{d}\tau\right)^2\mathrm{d}x\\ &\le C\int_0^t\int_0^1\Big(|w_{xxt}-\tilde w_{xxt}|^2+|\theta-\tilde\theta|^2\Big)(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}\tag{7.139}$$

Next we test the difference of the time derivatives of (6.23) for w and $\tilde w$ by $w_t-\tilde w_t$, the difference of (6.22) for u and $\tilde u$ by $w_{xxt}-\tilde w_{xxt}$ and subtract the two equations. We obtain

$$\begin{aligned}&\int_0^1\mu(w_{tt}-\tilde w_{tt})(w_t-\tilde w_t)\,\mathrm{d}x-\int_0^1\alpha(w_{xxtt}-\tilde w_{xxtt})(w_t-\tilde w_t)\,\mathrm{d}x-\int_0^1(u_t-\tilde u_t)(w_{xxt}-\tilde w_{xxt})\,\mathrm{d}x\\ &=-\int_0^1(u_{xxt}-\tilde u_{xxt})(w_t-\tilde w_t)\,\mathrm{d}x-\int_0^1 B(w_{xx}-\tilde w_{xx})(w_{xxt}-\tilde w_{xxt})\,\mathrm{d}x\\ &-\int_0^1(P[m,\chi,w_{xx}]-P[\tilde m,\tilde\chi,\tilde w_{xx}])(w_{xxt}-\tilde w_{xxt})\,\mathrm{d}x-\int_0^1\nu(w_{xxt}-\tilde w_{xxt})^2\,\mathrm{d}x\\ &+\int_0^1\beta(\theta-\tilde\theta)(w_{xxt}-\tilde w_{xxt})\,\mathrm{d}x.\end{aligned}$$

Integrating by parts and rearranging the terms we get, up to constants,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left(|w_t-\tilde w_t|^2+|w_{xt}-\tilde w_{xt}|^2+|w_{xx}-\tilde w_{xx}|^2\right)(x,t)\,\mathrm{d}x+\int_0^1|w_{xxt}-\tilde w_{xxt}|^2\,\mathrm{d}x\\ &=-\int_0^1(P[m,\chi,w_{xx}]-P[\tilde m,\tilde\chi,\tilde w_{xx}])(w_{xxt}-\tilde w_{xxt})\,\mathrm{d}x+\int_0^1(\theta-\tilde\theta)(w_{xxt}-\tilde w_{xxt})\,\mathrm{d}x.\end{aligned}$$

Note that arguing as for (7.65) and using (7.138) gives

$$\begin{aligned}&|P[m,\chi,w_{xx}]-P[\tilde m,\tilde\chi,\tilde w_{xx}]|\le C\left(|m-\tilde m|+|\chi-\tilde\chi|+\int_0^t|w_{xxt}-\tilde w_{xxt}|\,\mathrm{d}\tau\right)\\ &\le C\Bigg(\int_0^t(|\theta-\tilde\theta|+|w_{xxt}-\tilde w_{xxt}|)(x,\tau)\,\mathrm{d}\tau+\left(\int_0^t\int_0^1(|\theta-\tilde\theta|^2+|w_{xxt}-\tilde w_{xxt}|^2)(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau\right)^{1/2}\Bigg),\end{aligned}$$

hence by Young's inequality we finally get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\Big(|w_t-\tilde w_t|^2+|w_{xt}-\tilde w_{xt}|^2+|w_{xx}-\tilde w_{xx}|^2\Big)(x,t)\,\mathrm{d}x+\int_0^1|w_{xxt}-\tilde w_{xxt}|^2(x,t)\,\mathrm{d}x\\ &\le C\left(\int_0^1|\theta-\tilde\theta|^2(x,t)\,\mathrm{d}x+\int_0^t\int_0^1\Big(|w_{xxt}-\tilde w_{xxt}|^2+|\theta-\tilde\theta|^2\Big)(x,t)\,\mathrm{d}x\,\mathrm{d}\tau\right).\end{aligned}\tag{7.140}$$

We now multiply (7.139) by $2C$ and add the result to (7.140) to obtain

$$\begin{aligned}&\int_0^1\Big(|w_{xxt}-\tilde w_{xxt}|^2+C|\theta-\tilde\theta|^2\Big)(x,t)\,\mathrm{d}x\\ &+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left(2C\left(\int_0^t(\theta_x-\tilde\theta_x)(x,\tau)\,\mathrm{d}\tau\right)^2+|w_t-\tilde w_t|^2+|w_{xt}-\tilde w_{xt}|^2+|w_{xx}-\tilde w_{xx}|^2\right)(x,t)\,\mathrm{d}x\\ &\le(C+2C^2)\int_0^t\int_0^1\Big(|w_{xxt}-\tilde w_{xxt}|^2+|\theta-\tilde\theta|^2\Big)(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}$$

Integrating in time $\int_0^{t'}\mathrm{d}t$ for some $t'>t$ and using Grönwall's lemma A.2, we see that $w=\tilde w$, $\theta=\tilde\theta$, and by equation (6.23) and estimate (7.138) also $u=\tilde u$, $m=\tilde m$, $\chi=\tilde\chi$. Hence the proof of Theorem 7.3 is complete.

Part IV

Regularity for double-phase variational problems

Introduction

In this last part we deal with phase transitions from a different point of view, namely, by using the directs methods pertaining to the regularity theory in the field of Calculus of Variations. More precisely, we are going to prove regularity results for a class of integral functionals that belong to the realm of nonuniformly elliptic problems. An example of such functionals is given by those satisfying the so-called *nonstandard growth conditions*. Here the primary model we have in mind is given by the *double-phase functional*

$$\mathcal{J}_p^q(u,\Omega) := \int_\Omega (|Du|^p + a(x)|Du|^q)\ \mathrm{d}x, \tag{IV.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open domain, $n > 2$, and

$$1 < p \le q, \quad 0 \le a(\cdot) \in L^\infty(\Omega).$$

The significant case occurs when $p < q$. The main feature of the functional $\mathcal{J}_p^q$ is the change of ellipticity/growth type occurring on the zero set $\{a(x) = 0\}$. Indeed, while in those points x where $a(x)$ is positive the energy density of $\mathcal{J}_p^q$ exhibits a growth/ellipticity in the gradient which is of order q, on $\{a(x) = 0\}$ the energy density has p-growth in the gradient.

In his seminal works [133–136], V. V. Zhikov was the first to introduce and study functionals whose integrands change their ellipticity rate according to the point, and, in particular, the one in (IV.1). Such functionals provide a useful paradigm for describing the behavior of strongly anisotropic materials whose hardening properties (linked to the exponent ruling the growth of the gradient variable) drastically change with the point. The coefficient $a(\cdot)$ serves to regulate the mixture between two different materials, with p and q hardening, respectively. In this class of functionals $\mathcal{J}_p^q$ appears to be the one exhibiting the most dramatic phase-transition and therefore the most difficult to treat. However in this last part we will be more interested in the theoretical aspects of the problem rather than to its applications, since the functional $\mathcal{J}_p^q$ appears to be very interesting from the point of view of regularity theory. Indeed, while in the standard situation $p = q$ the coefficient $a(\cdot)$ acts in the energy density as a local perturbation of the main elliptic terms, this is not obviously the case when $q > p$, since it is $a(\cdot)$ to dictate the ellipticity rate of the energy density.

As mentioned above, the functional in (IV.1) is the primary example of a functional exhibiting a nonstandard growth. More generally, the Lagrangian F may be requested to satisfy the following growth conditions

$$\alpha\left(|u|^p-1\right)\leq F(x,u)\leq\beta\left(|u|^q+1\right),\qquad 1<p\leq q,$$

for some positive constants α,β. However, here we are also interested in relaxing the lower bound by essentially taking $p=1$, hence allowing *nearly-linear growth conditions* in the gradient. In this case the model functional is given by

$$\mathcal{J}_1^q(u,\Omega):=\int_\Omega\Big(|Du|\log(1+|Du|)+a(x)(1+|Du|^2)^{q/2}\Big)\,\mathrm{d}x$$

with $q>1$ and $a\in L^\infty(\Omega)$. Energy densities with logarithmic growth are not only of interest from a theoretical point of view, since they naturally occur in the context of generalized Newtonian fluids where they serve as models for so-called Prandtl-Eyring fluids. Moreover, this kind of behavior is observed in the theory of plasticity with logarithmic hardening law.

For nonuniformly elliptic energy functionals as above, we are concerned with the study of the regularity of their minimizers among all functions belonging to some given set. The variational problem we have in mind is the so-called *obstacle problem*. Given $\Omega\subset\mathbb{R}^n$, $n>2$, bounded open domain, we consider the variational obstacle problem in the form

$$\min\left\{\int_\Omega F(x,Dw)\,\mathrm{d}x : w\in\mathcal{K}_\psi(\Omega)\right\}. \tag{IV.2}$$

The function $\psi:\ \Omega\to[-\infty,+\infty)$, called *obstacle*, belongs to the Sobolev class $W^{1,p}(\Omega)$ and the set $\mathcal{K}_\psi(\Omega)$ contains the functions above the obstacle that share at least its regularity, that is,

$$\mathcal{K}_\psi(\Omega)=\Big\{w\in u_0+W_0^{1,p}(\Omega):w\geq\psi\text{ a.e. in }\Omega\Big\} \tag{IV.3}$$

where $u_0\in W^{1,p}(\Omega)$ is a fixed boundary value. For the Lagrangian F we assume nonuniform ellipticity conditions as the ones outlined above.

The obstacle problem appeared in the mathematical literature in the work by G. Stampacchia [127] in the special case $\psi=\chi_E$ and related to the capacity of a subset $E\Subset\Omega$. In an earlier independent work, G. Fichera [62] solved the first unilateral problem, the so-called Signorini problem in elastostatics. It consists in finding the elastic equilibrium configuration of an anisotropic nonhomogeneous elastic body, resting on a rigid frictionless surface and subject only to its mass forces.

The study of the regularity theory for obstacle problems is now a classical topic in Partial Differential Equations and Calculus of Variations. It is well known that its solution cannot be of class C^2 independently of how regular the obstacle is: this led to the concept of weak solution and to the theory of variational inequalities, after the fundamental work of J. L. Lions and G. Stampacchia [108]. Thus, in general, the regularity of solutions to the obstacle problems is influenced by the one of the obstacle. For example, for linear obstacle problems, obstacle and solutions have the same regularity (see [20, 28, 87]). This does not apply in the nonlinear setting, hence along the years there have been intense research activities for the regularity of the obstacle problem in this direction.

A first important result by J.H. Michael and W.P. Ziemer [114] establishes Hölder continuity of solutions to the obstacle problem when the obstacle itself is Hölder continuous. H.J. Choe [33] proved that if the gradient of the obstacle is Hölder continuous, the same happens for the gradient of solutions. Other results that deserve to be quoted are [34, 66, 106]. Since then, many regularity results have been obtained in different situations: for instance we quote [41] in the setting of Morrey and Campanato spaces, [10, 118] for gradient continuity for nonlinear obstacle problems, [46] for global results up to the boundary, [18] for the parabolic case, [19] for the porous medium problem. A higher differentiability result in the case of standard growth conditions was recently obtained in [56] (see Theorem 9.3). Here both the integer and the fractional differentiability are established.

Our purpose is to investigate the regularity of solutions to (IV.2)–(IV.3) for nonuniformly elliptic functionals. In the nonstandard setting we quote [26, 43, 45, 119] in the case of a single obstacle problem (see also [27, 44, 117] for Calderón–Zygmund case), and [13] in the case of double obstacle problems. In the nearly-linear setting we refer to the papers [38, 66], dealing with Lipschitz and $C^{1,\alpha}$ regularity, respectively. However, to our knowledge, no result had been given in the direction of higher differentiability prior to the the papers [71–73]. More precisely, these works investigate the extra integer differentiability of solutions to the obstacle problem assuming that the gradient of the obstacle has some differentiability property. We are going to report the results therein.

The program is as follows. In Chapter 8 we report some preliminary results that will be used in the sequel, whereas Chapter 9 contains the main higher differentiability results for the obstacle problem. In particular, in Section 9.1 we report some already known facts abound the standard case. Then we turn our attention to the nonstandard case, and in Section 9.2 we state and prove the two (independent) results from [71, 72]. Finally, Section 9.3 is devoted to the nearly-linear case, and contains the result developed in the paper [73].

CHAPTER 8

Notation and preliminary results

This chapter is devoted to fix the notation and collect some results that will be needed in the next chapter.

Here and in the sequel we shall denote by C or c a generic positive constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies will be suitably emphasized using parentheses or subscripts. The norm we use on $\mathbb{R}^n$ will be the standard euclidean one. In what follows, $B(x,r) = B_r(x) = \{y \in \mathbb{R}^n : \ |y - x| < r\}$ will denote the ball centered at x of radius r. We shall omit the dependence on the center and on the radius when no confusion arises.

8.1 Difference quotient

Our proof is achieved by means of the difference quotient method (which goes back to L. Nirenberg, see [116]), that is quite natural when trying to establish higher differentiability results. Thus we recall some properties of the finite difference operator that will be needed in the following chapter. Let us recall that, for every function $F : \mathbb{R}^n \to \mathbb{R}^N$, the finite difference operator is defined by

$$\tau_{s,h} F(x) := F(x + he_s) - F(x)$$

where $h \in \mathbb{R}$, e_s is the unit vector in the x_s direction and $s \in \{1, \ldots, n\}$.

We start with the description of some elementary properties that can be found, for example, in [81].

Proposition 8.1. *Let F and G be two functions such that $F, G \in W^{1,p}(\Omega)$, with $p \geq 1$, and let us consider the set*

$$\Omega_{|h|} := \{x \in \Omega : dist(x, \partial\Omega) > |h|\} .$$

Then

(d1) $\tau_h F \in W^{1,p}(\Omega)$ and

$$D_i(\tau_h F) = \tau_h(D_i F) .$$

(*d*2) *If at least one of the functions F or G has support contained in $\Omega_{|h|}$, then*

$$\int_\Omega F\,\tau_h G\,\mathrm{d}x = \int_\Omega G\,\tau_{-h} F\,\mathrm{d}x.$$

(*d*3) *We have*

$$\tau_h(FG)(x) = F(x+h)\,\tau_h G(x) + G(x)\,\tau_h F(x).$$

The next result about finite difference operator is a kind of integral version of Lagrange Theorem.

Lemma 8.2. *If $0<\rho<R$, $|h|<\frac{R-\rho}{2}$, $1<p<\infty$, and $F, DF \in L^p(B_R)$ then*

$$\int_{B_\rho} |\tau_h F(x)|^p\,\mathrm{d}x \le c(n,p)|h|^p \int_{B_R} |DF(x)|^p\,\mathrm{d}x.$$

Moreover

$$\int_{B_\rho} |F(x+h)|^p\,\mathrm{d}x \le \int_{B_R} |F(x)|^p\,\mathrm{d}x.$$

Now, we recall the fundamental Sobolev embedding property.

Lemma 8.3. *Let $F:\mathbb{R}^n \to \mathbb{R}^N$, $F \in L^p(B_R)$ with $1<p<n$. Suppose that there exist $\rho \in (0,R)$ and $M>0$ such that*

$$\sum_{s=1}^{n} \int_{B_\rho} |\tau_{s,h} F(x)|^p\,\mathrm{d}x \le M^p |h|^p$$

for every h with $|h|<\frac{R-\rho}{2}$. Then $F \in W^{1,p}(B_\rho) \cap L^{\frac{np}{n-p}}(B_\rho)$. Moreover

$$\|DF\|_{L^p(B_\rho)} \le M$$

and

$$\|F\|_{L^{\frac{np}{n-p}}(B_\rho)} \le c\left(M + \|F\|_{L^p(B_R)}\right),$$

with $c = c(n,N,p,\rho,R)$.

8.2 Other auxiliary results

In view of the next chapter, it is convenient to introduce auxiliary functions

$$V_1(\xi) := \frac{\xi}{\left(1^2+|\xi|^2\right)^{\frac{1}{4}}}, \tag{8.1}$$

$$V_p(\xi) := \left(\mu^2+|\xi|^2\right)^{\frac{p-2}{4}} \xi \quad \text{for } p \ge 2, \tag{8.2}$$

defined for all $\xi \in \mathbb{R}^n$ and for $\mu \in [0,1]$. Note that for V_p we allow for the degenerate case, which corresponds to $\mu = 0$. One can easily check that there exist positive constants c_1, c_2 such that

$$c_1\left(|\xi|-1\right) \le |V_1(\xi)|^2, \tag{8.3}$$

$$c_2\,|\xi|^p \le |V_p(\xi)|^2. \tag{8.4}$$

We also recall the following results from [2, Lemma 2.1] and [81, Lemma 8.3], respectively.

Lemma 8.4. *For every $\gamma \in (-1/2, 0)$ and $\mu \geq 0$ we have*

$$(2\gamma+1)|\xi-\eta| \leq \frac{|(\mu^2+|\xi|^2)^\gamma\,\xi - (\mu^2+|\eta|^2)^\gamma\,\eta|}{(\mu^2+|\xi|^2+|\eta|^2)^\gamma} \leq \frac{c(n)}{2\gamma+1}|\xi-\eta|$$

for all $\xi, \eta \in \mathbb{R}^n$, not both zero if $\mu = 0$.

Lemma 8.5. *Let $1 < p < \infty$. There exists a constant $c = c(n,p) > 0$ such that*

$$c^{-1}\Big(\mu^2+|\xi|^2+|\eta|^2\Big)^{\frac{p-2}{2}} \leq \frac{|V_p(\xi)-V_p(\eta)|^2}{|\xi-\eta|^2} \leq c\Big(\mu^2+|\xi|^2+|\eta|^2\Big)^{\frac{p-2}{2}}$$

for any ξ, $\eta \in \mathbb{R}^n$.

Finally, we we state a very well-known iteration lemma (see [81], pp. 191–192 for the proof).

Lemma 8.6. *Let $\Phi : \left[\frac{R}{2}, R\right] \to \mathbb{R}$ be a bounded nonnegative function, where $R > 0$. Assume that for all $\frac{R}{2} \leq r < s \leq R$ we have*

$$\Phi(r) \leq \vartheta\,\Phi(s) + A + \frac{B}{(s-r)^2} + \frac{C}{(s-r)^\gamma}$$

where $\vartheta \in (0,1)$, $A, B, C \geq 0$ and $\gamma > 0$ are constants. Then there exists a constant $c = c(\vartheta, \gamma)$ such that

$$\Phi\left(\frac{R}{2}\right) \leq c\left(A + \frac{B}{R^2} + \frac{C}{R^\gamma}\right).$$

CHAPTER 9

Higher differentiability for the obstacle problem

The obstacle problem is a well known motivating example in the Calculus of Variations. It comes from a classical problem in elasticity theory, namely, finding the equilibrium position of an elastic membrane whose boundary is held fixed, and which is constrained to lie above a given obstacle.

In its classical mathematical formulation, the problem consists in seeking minimizers of the Dirichlet energy functional

$$\mathcal{I}(u) := \int_{\Omega} |\nabla u|^2 \,\mathrm{d}x$$

in some domain $\Omega \subset \mathbb{R}^n$, $n > 2$, where u represents the vertical displacement of the membrane. In addition to satisfying Dirichlet boundary conditions corresponding to the fixed boundary of the membrane, the function u is constrained to be greater than some given obstacle function ψ.

Existence and uniqueness of a solution for the above problem is easy to prove. The same argument works for more general functionals, actually. If we consider

$$\mathcal{J}(u) := \int_{\Omega} F(x, Du) \,\mathrm{d}x$$

such that F satisfies the coercivity inequality $F(x,\xi) \geq \alpha|\xi|^p - \beta$ for some $\alpha > 0$, $\beta \geq 0$ and $1 < p < \infty$, then the constrained minimization problem has at least one solution. Furthermore, if the mapping $\xi \mapsto F(x,\xi)$ is uniformly convex for each x, the solution is also unique. A good reference for such results is e. g. [37].

The regularity problem is less straightforward to address, especially in the nonlinear setting. As already mentioned in the introduction, the regularity of the obstacle influences the regularity of the solution. In this chapter we are interested in the problem of *higher differentiability*, that is, determining whether (and, possibly, how) higher differentiability assumptions on the obstacle transfer to the solutions.

9.1 The case of standard growth conditions

We start our analysis by reporting some already known results about the obstacle problem satisfying standard growth conditions. More precisely, on the Lagrangian F in (IV.2) we assume that there exist an exponent $1 < p < \infty$ and some positive constants $\tilde{\nu}, \tilde{L}, \tilde{\ell}$ such that

$$\frac{1}{\tilde{\ell}}\left(|\xi|^p - \mu^2\right) \le F(x,\xi) \le \tilde{\ell}\left(\mu^2 + |\xi|^p\right) \tag{$\hat{F}1$}$$

$$\langle D_{\xi\xi}F(x,\xi)\lambda, \lambda\rangle \ge \tilde{\nu}\left(\mu^2 + |\xi|^2\right)^{\frac{p-2}{2}}|\lambda|^2 \tag{$\hat{F}2$}$$

$$|D_{\xi\xi}F(x,\xi)| \le \tilde{L}\left(\mu^2 + |\xi|^2\right)^{\frac{p-2}{2}} \tag{$\hat{F}3$}$$

$$|D_{\xi x}F(x,\xi)| \le k(x)\left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}} \tag{$\hat{F}4$}$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$, where $\mu \in [0,1]$ is a parameter that allows to consider both the degenerate and the nondegenerate situation. Here k is a function whose regularity is related to the "extra-differentiability transfer" from the obstacle to the solution, and that will be specified when needed.

The first result states that, in case of functionals with standard growth, the obstacle problem can be reformulated as a problem in the theory of variational inequalities.

Proposition 9.1. *Let the Lagrangian F satisfy $(\hat{F}1) - (\hat{F}4)$. Then $u \in W^{1,p}(\Omega)$ is a solution to the obstacle problem* (IV.2)–(IV.3) *if and only if $u \in \mathcal{K}_\psi(\Omega)$ solves the variational inequality*

$$\int_\Omega \langle \mathcal{A}(x, Du), D(\varphi - u)\rangle \,\mathrm{d}x \ge 0 \qquad \text{for all } \varphi \in \mathcal{K}_\psi(\Omega), \tag{9.1}$$

where we set

$$\mathcal{A}(x,\xi) := D_\xi F(x,\xi)\,.$$

Before proving the above statement, we point out that from conditions $(\hat{F}1) - (\hat{F}4)$ we deduce the existence of positive constants ν, L, ℓ such that in terms of $\mathcal{A}$ the p-ellipticity and p-growth conditions read

$$\langle \mathcal{A}(x,\xi) - \mathcal{A}(x,\eta), \xi - \eta\rangle \ge \nu\,|\xi - \eta|^2(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \tag{$\mathcal{A}1$}$$

$$|\mathcal{A}(x,\xi) - \mathcal{A}(x,\eta)| \le L\,|\xi - \eta|(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \tag{$\mathcal{A}2$}$$

$$|\mathcal{A}(x,\xi)| \le \ell\,(\mu^2 + |\xi|^2)^{\frac{p-1}{2}} \tag{$\mathcal{A}3$}$$

$$|D_x A(x,\xi)| \le k(x)\left(\mu^2 + |Du|^2\right)^{\frac{p-1}{2}} \tag{$\tilde{\mathcal{A}}4$}$$

for a.e. $x \in \Omega$ and every $\xi, \eta \in \mathbb{R}^n$.

Proof. Let us prove the two implications separately.

1. (IV.2)–(IV.3) $\Rightarrow$ (9.1)

   Let $u \in W^{1,p}(\Omega)$ be a solution to (IV.2)–(IV.3) under the assumptions $(\hat{F}1)-(\hat{F}4)$. Let $\varphi \in \mathcal{K}_\psi(\Omega)$. Since the set $\mathcal{K}_\psi(\Omega)$ is convex, also the combination $(1-\varepsilon)u + \varepsilon\varphi = u + \varepsilon(\varphi - u)$, $0 \le \varepsilon \le 1$, belongs to this set. Then, being u a minimizer, it holds

$$0 \le \int_\Omega \Big(F(x, D(u + \varepsilon(\varphi - u))) - F(x, Du) \Big)\, \mathrm{d}x.$$

   Therefore, we have

$$\begin{aligned}
0 &\le \varepsilon \int_\Omega \int_0^1 \langle D_\xi F(x, Du + s\,\varepsilon D(\varphi - u)), D(\varphi - u)\rangle \; \mathrm{d}s\, \mathrm{d}x \\
&= \varepsilon \int_\Omega \langle D_\xi F(x, Du), D(\varphi - u)\rangle \; \mathrm{d}s\, \mathrm{d}x \\
&\quad + \varepsilon \int_\Omega \int_0^1 \langle D_\xi F(x, Du + s\,\varepsilon D(\varphi - u)) - D_\xi F(x, Du), D(\varphi - u)\rangle \; \mathrm{d}s\, \mathrm{d}x \\
&= \varepsilon \int_\Omega \langle D_\xi F(x, Du), D(\varphi - u)\rangle \; \mathrm{d}s\, \mathrm{d}x \\
&\quad + \varepsilon^2 \int_\Omega \int_0^1 s \int_0^1 \langle D_{\xi\xi} F(x, Du + ts\,\varepsilon D(\varphi - u)) D(\varphi - u), D(\varphi - u)\rangle \; \mathrm{d}t\, \mathrm{d}s\, \mathrm{d}x.
\end{aligned}$$

   Dividing both sides of the previous inequality by ε we obtain

$$\begin{aligned}
0 &\le \int_\Omega \langle D_\xi F(x, Du), D(\varphi - u)\rangle \; \mathrm{d}s\, \mathrm{d}x \\
&\quad + \varepsilon \int_\Omega \int_0^1 s \int_0^1 \langle D_{\xi\xi} F(x, Du + ts\,\varepsilon D(\varphi - u)) D(\varphi - u), D(\varphi - u)\rangle \; \mathrm{d}t\, \mathrm{d}s\, \mathrm{d}x \\
&=: I_1 + I_2.
\end{aligned} \tag{9.2}$$

   We focus on the term I_2. Note that by $(F3)$ with $q = p$ and by Hölder's inequality with conjugate exponents $p/(p-2)$, $p/2$

$$\begin{aligned}
I_2 &\le \varepsilon \int_\Omega \int_0^1 s \int_0^1 |D_{\xi\xi} F(x, Du + ts\,\varepsilon D(\varphi - u))| |D(\varphi - u)|^2 \, \mathrm{d}t\, \mathrm{d}s\, \mathrm{d}x \\
&\le \varepsilon \int_\Omega \int_0^1 s \int_0^1 \tilde{L} \left(\mu^2 + |Du + ts\,\varepsilon D(\varphi - u)|^2\right)^{\frac{p-2}{2}} |D(\varphi - u)|^2 \, \mathrm{d}t\, \mathrm{d}s\, \mathrm{d}x \\
&\le \varepsilon \tilde{L} \int_\Omega \left(\mu^2 + |Du|^2 + \varepsilon^2 |D(\varphi - u)|^2\right)^{\frac{p-2}{2}} |D(\varphi - u)|^2 \, \mathrm{d}x \\
&\le \varepsilon \tilde{L} \left(\left(\int_\Omega |Du|^p \, \mathrm{d}x \right)^{\frac{p-2}{2}} \left(\int_\Omega |D(\varphi - u)|^p \, \mathrm{d}x \right)^{\frac{2}{p}} + \int_\Omega (1 + |D(\varphi - u)|^p) \, \mathrm{d}x \right).
\end{aligned}$$

   Since the above integrals are finite, we infer that $\lim_{\varepsilon \to 0} I_2 = 0$. Hence passing to the limit as $\varepsilon \to 0$ in (9.2) yields (9.1).

2. (9.1) $\Rightarrow$ (IV.2)–(IV.3)

   This implication is straightforward, and does not depend on the growth hypotheses. Indeed, defining

$$j(\varepsilon) := \int_\Omega F(x, D(u + \varepsilon(\varphi - u)))\, \mathrm{d}x,$$

condition (9.1) is essentially saying that $j'(0) \geq 0$ for every $\varphi \in \mathcal{K}_\psi(\Omega)$. This means that there exists a sufficiently small neighborhood in which u is a minimum for $\int_\Omega F(x, Dw)\,\mathrm{d}x$.

□

Remark 9.2. We obtain a variational inequality instead of the usual Euler-Lagrange equation since we can only take one-sided variations of the minimizer. However, this happens only in the contact set, that is, the set defined as $\{x \in \Omega : u(x) = \psi(x)\}$. If we are in a region where the minimizer u is strictly above the obstacle, then we are allowed to take double-sided variations for sufficiently small ε. As a result, (9.1) will turn into an equality, as in unconstrained problems. Indeed, this is reasonable. If we take a minimizer u of an obstacle problem in the region of the domain where u is away from the constraint, then it will not be influenced by the obstacle. The obstacle shapes the minimizer only where u is forced to stay in contact with it.

Dealing with nonstandard growth, the equivalence between the two formulations is not guaranteed if the functional F is not regular enough. Actually, already for non constrained problems with non-standard growth conditions, the relation between minima and extremals (i. e. solutions of the corresponding Euler-Lagrange system) is an issue that requires a careful investigation (see e. g. [30, 31]). More precisely, looking at the proof of Proposition 9.1, we see that in case of nonstandard growth conditions the summand labeled I_2 may be unbounded. However, once we have gained more regularity by means of our higher differentiability result, we will show in Remark 9.11 that our hypotheses (in particular the closeness condition between the growth and the ellipticity exponents) still allow us to reformulate the minimization problem as a variational inequality, provided we consider a new class $\mathcal{K}_\psi^{\text{new}}(\Omega)$.

We are now in the position of briefly commenting on the higher differentiability for the obstacle problem with standard growth, which can be considered in the formulation (9.1), $(\mathcal{A}1) - (\tilde{\mathcal{A}}4)$. It is usually observed that the regularity of its solutions is strictly connected to the analysis of the regularity of solutions to partial differential equations of the form

$$\mathrm{div}\mathcal{A}(x, Du) = \mathrm{div}\mathcal{A}(x, D\psi).$$

It is well known that no extra differentiability properties for the solutions of partial differential equations of the type

$$\mathrm{div}\mathcal{A}(x, Du) = \mathrm{div}\,\mathcal{G} \tag{9.3}$$

can be expected even if $\mathcal{G}$ is smooth, unless some assumption is given on the x-dependence of $\mathcal{A}$. On the other hand, recent results concerning the higher differentiability of solutions to (9.3) show that the weak differentiability of the map $\mathcal{A}$, as function of the x-variable, is a sufficient condition (see

[79, 80, 120, 121, 123]). In [79, 120, 121] the higher differentiability of solutions to the equation in (9.3) is obtained assuming a $W^{1,n}$-type regularity on the partial map $x \mapsto \mathcal{A}(x,\xi)$ that is expressed through a pointwise condition on $\mathcal{A}(\cdot,\xi)$. This condition relies on the characterization of the Sobolev spaces due to P. Hajłasz ([83]). More precisely, for Carathéodory functions $\mathcal{A}$ satisfying assumptions $(\mathcal{A}1)-(\mathcal{A}3)$ we reformulate $(\tilde{\mathcal{A}}4)$ assuming that there exists a non negative function κ depending on k and with its same summability such that the following inequality

$$|\mathcal{A}(x,\xi)-\mathcal{A}(y,\xi)| \le (\kappa(x)+\kappa(y))|x-y|(\mu^2+|\xi|^2)^{\frac{p-1}{2}} \qquad (\mathcal{A}4)$$

holds true for a.e. $x, y \in \Omega$ and for every $\xi \in \mathbb{R}^n$.
In case of standard growth conditions, the differentiability of the map is sufficient also in the context of obstacle problems to prove that the differentiability of the gradient of the obstacle transfers to the gradient of the solution with no loss in the order. More precisely, the following higher differentiability result from [56, Theorem 1.1] holds.

Theorem 9.3. *Let $u \in \mathcal{K}_\psi(\Omega)$ be the solution to the obstacle problem* (9.1)*, and suppose that $\mathcal{A}$ satisfies* $(\mathcal{A}1)-(\mathcal{A}4)$ *for $2 \le p < n$ and for a function $\kappa \in L^n_{\mathrm{loc}}(\Omega)$. Then we have*

$$D\psi \in W^{1,p}_{\mathrm{loc}}(\Omega) \quad \Rightarrow \quad (\mu^2+|Du|^2)^{\frac{p-2}{4}}Du \in W^{1,2}_{\mathrm{loc}}(\Omega).$$

Note that, thanks to (8.2) and (8.4), the above theorem essentially states that $Du \in W^{1,p}_{\mathrm{loc}}(\Omega)$. Thus it effectively represents a higher differentiability result.

Remark 9.4. The reader has certainly noticed that Theorem 9.3 does not take into account the case $1 < p < 2$, namely, the case of *sub-quadratic growth conditions*. Already for unconstrained problems it is known that the sub-quadratic growth conditions require specific tools and, in general, the expected regularity of the solution strongly differs from the case $p \ge 2$ (for a detailed explanation of this phenomenon see [8]). A higher differentiability result for the obstacle problem with p-growth, $1 < p < 2$, was recently proved in [77], thus it was not available when we started to investigate the higher differentiability for the obstacle problem with nonstandard growth. For this reason, in Section 9.2 we will limit ourselves to the case $p \ge 2$. The (p,q)-growth case with $1 < p < 2$ will be the object of further studies.

Starting from these facts, our purpose is to investigate the extra integer differentiability of solutions to the obstacle problem for nonuniformly elliptic functionals, assuming that the gradient of the obstacle has some differentiability property. This will be done in the next two sections. More precisely, Section 9.2 is devoted to the nonstandard case with $p \ge 2$, whereas Section 9.3 deals with the nearly-linear one, which essentially corresponds to the choice $p = 1$.

9.2 The case of nonstandard growth conditions

An important contribution to the study of problems in the field of Calculus of Variations involving functionals with nonstandard growth was given by P. Marcellini in his seminal papers [109–112]. He was able to show that the standard growth condition is not actually a necessary condition in the regularity theory for minimizers, and regularity theorems can be obtained also under nonstandard growth assumptions. More precisely, the regularity can only be expected if the difference between the growth exponents p and q is not "too large", that is, the q/p gap ratio cannot differ too much from 1. A huge number of papers have been devoted to the subject since then (see [29, 36, 53–55] for the elliptic case and [15–17, 113] for the parabolic case), and it turned out that the right condition to impose on the two exponents $1 < p < q$ is of the form

$$\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{r} \tag{9.4}$$

with $r > n$.

Our purpose consists in extending the analysis of the regularity to the case of the obstacle problem (IV.2)–(IV.3), investigating the higher differentiability of the solutions. To this aim we require the data of our problem to fulfill the following hypotheses.

Hypothesis 9.5. Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain. For the Lagrangian functional F : $\Omega \times \mathbb{R}^n \to [0, +\infty)$ we shall assume that:

(i) it is a Carathéodory function, that is, $F(\cdot, \xi)$ measurable for every $\xi \in \mathbb{R}^n$ and $F(x, \cdot)$ continuous for a. e. $x \in \Omega$;

(ii) $\xi \mapsto F(x, \xi)$ is a strictly convex C^2 function for a. e. $x \in \Omega$;

(iii) there exists a function $\tilde{f} : \Omega \times [0, +\infty) \to [0, +\infty)$ such that $F(x, \xi) = \tilde{f}(x, |\xi|)$;

(iv) there exist positive constants $\tilde{\nu}, \tilde{L}, \tilde{\ell}$ and exponents $2 \le p < q < \infty$ such that

$$\frac{1}{\tilde{\ell}} \left(|\xi|^p - \mu^2\right) \le F(x, \xi) \le \tilde{\ell} \left(\mu^2 + |\xi|^q\right) \tag{F1}$$

$$\langle D_{\xi\xi} F(x, \xi)\lambda, \lambda\rangle \ge \tilde{\nu} \left(\mu^2 + |\xi|^2\right)^{\frac{p-2}{2}} |\lambda|^2 \tag{F2}$$

$$|D_{\xi\xi} F(x, \xi)| \le \tilde{L} \left(\mu^2 + |\xi|^2\right)^{\frac{q-2}{2}} \tag{F3}$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$. We assume moreover that there exists a nonnegative function $k(x)$ belonging to a suitable Sobolev space such that

$$|D_{\xi x} F(x, \xi)| \le k(x) \left(\mu^2 + |\xi|^2\right)^{\frac{q-1}{2}} \tag{F4}$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$. We denote with $\mu \in [0,1]$ a parameter that will allow us to consider in our analysis both the degenerate and the nondegenerate situation.

Finally, to avoid trivialities, we shall assume that the set of admissible functions $\mathcal{K}_\psi(\Omega)$ is not empty and that u_0 in (IV.3) is such that $F(x, Du_0(x)) \in L^1_{\mathrm{loc}}(\Omega)$. Note that, as long as our regularity results have local nature, we are not requiring further assumptions on the boundary datum u_0.

Remark 9.6. Hypothesis 9.5 (iii), known in the literature as *Uhlenbeck structure*, makes the functional "less anisotropic". For example, in the vectorial case (that is, when the Lagrangian F is such that $F : \Omega \times \mathbb{R}^{nN} \to [0, +\infty)$ for some $N > 1$) it rules out singularities of minima. In the case of systems an important problem is clearly the one of identifying classes of functionals for which everywhere $C^{1,\alpha}$ (or even just continuity) of minimizers occurs, in analogy with the De Giorgi-Nash-Moser theorem for single equations. Indeed, in 1968 E. De Giorgi showed that this result cannot be extended to systems, providing a famous counterexample (see [39]). Hence it is crucial to find conditions for everywhere regularity of minimizers or, in other words, additional structure assumptions on the integrand F under which the singular set is void. This is still an open problem and, up to now, the only known structure preventing the formation of singularities for minimizers is the Uhlenbeck one (see [128]).

However, the above considerations do not affect our problem, since we are dealing with the scalar case. Here we are interested in a secondary effect of the Uhlenbeck structure, which is of fundamental importance in the framework of nonstandard growth conditions, both in the scalar and in the vector case. Indeed, if the functional satisfies the above structure condition, then we are allowed to approximate it from below by means of a family of functionals with p-growth (see Lemma 9.10). This rules out the occurrence of the so-called *Lavrentiev phenomenon*, which nonautonomous functionals typically exhibit when under (p, q)-growth conditions. Indeed, in this framework there might occur an inequality of the type

$$\inf_{u \in u_0 + W^{1,p}_0(\Omega)} \int_\Omega F(x, Du)\,\mathrm{d}x < \inf_{u \in u_0 + W^{1,q}_0(\Omega)} \int_\Omega F(x, Du)\,\mathrm{d}x, \tag{9.5}$$

for a suitable (even smooth) boundary datum u_0. In other words, it is not possible to achieve the minimum of the functional via more regular maps, although these are dense. This is a tautological obstruction to regularity of minima since it prevents minimizers to be in $W^{1,q}$, and several counterexamples in regularity are based on the occurrence of (9.5) (see e. g. [58, 135]). It is interesting and significant to see that F never exhibits the Lavrentiev phenomenon either when $p = q$ or when $F(x, \xi) = F(\xi)$, see again [58].

A useful way to quantify the Lavrentiev phenomenon can be introduced according to [25] by considering the relaxed functional

$$\overline{\mathcal{F}}(u, B) := \inf_{u_j} \left\{ \liminf_j \mathcal{F}(u_j, B) := \int_B F(x, Du_j)\,\mathrm{d}x \,:\, u_j \in W^{1,q}(B),\ u_j \rightharpoonup u \text{ in } W^{1,p}(B) \right\}$$

where $B \Subset \Omega$. Since $F(x, z)$ is convex with respect to z, we have

$$\mathcal{F}(u, B) := \int_B F(x, Du)\,\mathrm{d}x \le \liminf_j \mathcal{F}(u_j, B)$$

whenever $u_j \rightharpoonup u$ and $v_j \in W^{1,p}(B)$. Therefore $\mathcal{F}(u,B) \leq \overline{\mathcal{F}}(u,B)$, and it is possible to define the following nonnegative Lavrentiev gap functional

$$\mathcal{L}(u,B) := \overline{\mathcal{F}}(u,B) - \mathcal{F}(u,B) \geq 0 \quad \text{for all } u \in W^{1,p}(B).$$

The value of the functional $\mathcal{L}(u,B)$ gives a measure of the impossibility of finding a sequence of more regular maps $u_j \in W^{1,q}(B)$ such that $u_j \rightharpoonup u$ in $W^{1,p}(B)$ and $\int_B F(x,Du_j)\,\mathrm{d}x \to \int_B F(x,Du)\,\mathrm{d}x$ (see Lemma 9.15). Hence, in general, approximating the functional from above does not guarantee the convergence to the desired minimizer, but it could result in the convergence to the minimum of the relaxed functional.
This leads to the following developments: usually one assumes that $\mathcal{L}(u,B) \equiv 0$ (and this is the case of Section 9.3) or, alternatively, some structure conditions must be imposed on the integrand in order to perform an approximation from below.

Remark 9.7. Conditions $(F2)-(F3)$ are nothing but a uniform (with respect to x) boundedness condition on the eigenvalues of $D_{\xi\xi}F(x,\xi)$. In other words, we are asking that there exist two functions $\gamma(\xi)$, $\Gamma(\xi) : \mathbb{R}^n \to [0,+\infty)$ such that

$$\gamma(\xi)|\lambda|^2 \leq \langle D_{\xi\xi}F(x,\xi)\lambda,\lambda\rangle \leq \Gamma(\xi)|\lambda|^2,$$

and since in $(F1)$ we asked for p,q-growth conditions then $\gamma(\xi) = \gamma|\xi|^{p-2}$ and $\Gamma(\xi) = \Gamma|\xi|^{q-2}$ for some positive constants γ, Γ. Furthermore, since we want to consider both the degenerate and the nondegenerate situation, we replace ξ by $\sqrt{\mu^2+|\xi|^2}$ in both functions. Then the condition on the eigenvalues becomes

$$\gamma\,(\mu^2+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2 \leq \langle D_{\xi\xi}F(x,\xi)\lambda,\lambda\rangle \leq \Gamma\,(\mu^2+|\xi|^2)^{\frac{q-2}{2}}|\lambda|^2,$$

which is just a different formulation of $(F2)-(F3)$. This leads us to the definition of the *rate of nonuniform ellipticity*, quantified by the ratio

$$\mathcal{R}(\xi,B) := \frac{\sup_{x\in B} \text{of the highest eigenvalue of } D_{\xi\xi}F(x,\xi)}{\inf_{x\in B} \text{of the lowest eigenvalue of } D_{\xi\xi}F(x,\xi)}$$

on any ball $B \subset \Omega$, that in the nonuniformly elliptic case becomes unbounded as $|\xi| \to \infty$. In the case of the double-phase functional (IV.1) it is $\mathcal{R}(\xi,B) \approx 1 + \|a\|_{L^\infty(B)}|z|^{q-p}$ on any ball B intersecting the set $\{a(x)=0\}$.

If Hypothesis 9.5 holds, then the obstacle problem (IV.2)–(IV.3) has a unique solution. This can be proved by using standard existence results like, e.g., [37, Theorem 3.30] together with the fact that the admissible set $\mathcal{K}_\psi(\Omega)$ is closed. In particular, the requirement that the functional $\int_\Omega F(x,Dw)\,\mathrm{d}x$

is finite at some point is essential in order to rule out the possibility that the infimum is infinite. In case of standard growth conditions, this is automatically ensured by the p-growth of the functional.

The main result of this section is the following theorem from [71].

Theorem 9.8. *Let u be the solution to the obstacle problem* (IV.2)–(IV.3). *Suppose that F satisfies Hypothesis 9.5 for exponents $2 \leq p < n$, $p < q$, $r > n$ as in* (9.4) *and for a function $k \in L^r_{\mathrm{loc}}(\Omega)$. Then it holds*

$$D\psi \in W^{1,2q-p}_{\mathrm{loc}}(\Omega) \quad \Rightarrow \quad (\mu^2 + |Du|^2)^{\frac{p-2}{4}} Du \in W^{1,2}_{\mathrm{loc}}(\Omega)\,. \tag{9.6}$$

Contrary to the standard case, here the $W^{1,r}$-type regularity on the partial map $x \mapsto F_\xi(x,\xi)$ no longer allows the differentiability of the gradient of the obstacle to entirely transfer to the gradient of the solution, and there is a loss in the order of differentiability. However the exponent r plays an important role in determining the distance between p and q (which must be "small"), so that this loss turns out to be not "too big".

Remark 9.9. Inequality (9.4) yields the sharp estimate

$$\frac{(2q-p)r}{r-2} < \frac{np}{n-2}\,, \tag{9.7}$$

which will be frequently used in the sequel. Indeed, setting $m := \frac{r}{r-2}$, $2^* := \frac{2n}{n-2}$, (9.4) entails

$$2q - p < 2p\left(1 + \frac{1}{n} - \frac{1}{r}\right) - p = p\left(1 + \frac{1}{m} - \frac{2}{2^*}\right)$$

so that

$$\frac{(2q-p)r}{r-2} = (2q-p)m < pm\left(1 + \frac{1}{m} - \frac{2}{2^*}\right).$$

It turns out that

$$pm\left(1 + \frac{1}{m} - \frac{2}{2^*}\right) \leq \frac{np}{n-2} = \frac{2^*}{2}p \;\Leftrightarrow\; 1 + \frac{1}{m} - \frac{2}{2^*} \leq \frac{2^*}{2m} \;\Leftrightarrow\; 1 + \frac{1}{m} - \frac{2}{2^*} - \frac{2^*}{2m} \leq 0.$$

Note now that $\frac{2^*}{2m} > 1$ since $r > n$, hence we can write

$$\frac{2^*}{2m} = 1 + \varepsilon$$

for some $\varepsilon > 0$. Therefore

$$1 + \frac{1}{m} - \frac{2}{2^*} - \frac{2^*}{2m} = 1 + \frac{2}{2^*}(1+\varepsilon) - \frac{2}{2^*} - 1 - \varepsilon = \left(\frac{2}{2^*} - 1\right)\varepsilon \leq 0,$$

which is what we wanted to prove.

9.2.1 Proof of the main result

We are now going to prove Theorem 9.8. In the first part of the proof we construct an approximating minimization problem satisfying standard growth conditions. This problems will then be suitable for the application of Theorem 9.3, whose conclusion will become the starting point for deriving a priori estimates for the solution to the approximating problems. Then in the second part we conclude by showing that the a priori estimate is preserved in passing to the limit.

Step 1: Approximation and derivation of estimates

We start our proof by constructing a suitable approximating minimization problem. The main tool is the following lemma, which can be found in [35, Lemma 4.1].

Lemma 9.10. *Let $F : \Omega \times \mathbb{R}^n \to [0,+\infty)$ be a Carathéodory function, convex with respect to ξ, such that $F(x,\xi) = \tilde{f}(x,|\xi|)$ for some $\tilde{f} : \Omega \times [0,+\infty) \to [0,+\infty)$, and which satisfies assumptions $(F1) - (F4)$. Then there exists a sequence (F_ε) of Carathéodory functions $F_\varepsilon : \Omega \times \mathbb{R}^n \to [0,+\infty)$, convex with respect to ξ and monotonically convergent to F, such that*

(I) for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^n$ and for every $\varepsilon_1 < \varepsilon_2$, it holds

$$F_{\varepsilon_1}(x,\xi) \le F_{\varepsilon_2}(x,\xi) \le F(x,\xi),$$

(II) for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$, we have

$$\langle D_{\xi\xi}F_\varepsilon(x,\xi)\lambda, \lambda\rangle \ge \bar{\nu}\,(\mu^2 + |\xi|^2)^{\frac{p-2}{2}} |\lambda|^2$$

with $\overline{\nu}$ depending only on p and $\tilde{\nu}$,

(III) for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$, there exist K_0, K_1 independent of ε and $\overline{K}_1$ depending on ε such that

$$K_0(|\xi|^p - \mu^2) \le F_\varepsilon(x,\xi) \le K_1(\mu^2 + |\xi|)^q,$$
$$F_\varepsilon(x,\xi) \le \overline{K}_1(\varepsilon)\,(\mu^2 + |\xi|)^p,$$

(IV) for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$, there exists a constant $C(\varepsilon) > 0$ such that

$$|D_{\xi x}F_\varepsilon(x,\xi)| \le k(x)\,(\mu^2 + |\xi|^2)^{\frac{q-1}{2}},$$
$$|D_{\xi x}F_\varepsilon(x,\xi)| \le C(\varepsilon)\,k(x)\,(\mu^2 + |\xi|^2)^{\frac{p-1}{2}}.$$

In addition, it holds $F_\varepsilon(x,\xi) = \tilde{f}_\varepsilon(x,|\xi|)$ for every $\xi \in \mathbb{R}^n$.

In other words, the Lagrangian F can be approximated by means of a sequence of Carathéodory functionals F_ε monotonically convergent to F, and satisfying nonstandard growth conditions with constants independent of ε and standard growth conditions with constants all depending on ε.

Let us consider the sequence of functionals $F_\varepsilon(x,\xi)$ obtained applying Lemma 9.10 to the integrand $F(x,\xi)$ of problem (IV.2)–(IV.3), and fix a ball $B_R \Subset \Omega$. Let $u_\varepsilon \in u + W_0^{1,p}(B_R)$ be the solution to the obstacle problem

$$\min\left\{\int_\Omega F_\varepsilon(x,Dz) : z \in \mathcal{K}_\psi(\Omega)\right\}. \tag{9.8}$$

Setting

$$\mathcal{A}_\varepsilon(x,\xi) = D_\xi F_\varepsilon(x,\xi),$$

by Proposition 9.1 we infer that u_ε solves the variational inequality

$$\int_\Omega \langle \mathcal{A}_\varepsilon(x,Du_\varepsilon), D(\varphi - u_\varepsilon)\rangle \,\mathrm{d}x \geq 0 \qquad \text{for all } \varphi \in \mathcal{K}_\psi(\Omega). \tag{9.9}$$

Moreover the Carathéodory function $\mathcal{A}_\varepsilon$ is such that for a.e. $x, y \in \Omega$ and every $\xi, \eta \in \mathbb{R}^n$ the following conditions hold

$$\langle \mathcal{A}_\varepsilon(x,\xi) - \mathcal{A}_\varepsilon(x,\eta), \xi - \eta\rangle \geq \nu\,|\xi-\eta|^2(\mu^2+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}} \tag{$\mathcal{A}_\varepsilon 1$}$$

$$|\mathcal{A}_\varepsilon(x,\xi) - \mathcal{A}_\varepsilon(x,\eta)| \leq 2L\,|\xi-\eta|(\mu^2+|\xi|^2+|\eta|^2)^{\frac{q-2}{2}} \tag{$\mathcal{A}_\varepsilon 2$}$$

$$|\mathcal{A}_\varepsilon(x,\xi)| \leq 2\ell\,(\mu^2+|\xi|^2)^{\frac{q-1}{2}} \tag{$\mathcal{A}_\varepsilon 3$}$$

$$|\mathcal{A}_\varepsilon(x,\xi) - \mathcal{A}_\varepsilon(y,\xi)| \leq (\kappa(x)+\kappa(y))\,|x-y|(\mu^2+|\xi|^2)^{\frac{q-1}{2}} \tag{$\mathcal{A}_\varepsilon 4$}$$

with constants independent of ε, and also

$$|\mathcal{A}_\varepsilon(x,\xi) - \mathcal{A}_\varepsilon(x,\eta)| \leq \tilde{L}_\varepsilon\,|\xi-\eta|(\mu^2+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}} \tag{$H_\varepsilon 1$}$$

$$|\mathcal{A}_\varepsilon(x,\xi)| \leq \tilde{\ell}_\varepsilon\,(\mu^2+|\xi|^2)^{\frac{p-1}{2}} \tag{$H_\varepsilon 2$}$$

$$|\mathcal{A}_\varepsilon(x,\xi) - \mathcal{A}_\varepsilon(y,\xi)| \leq C(\varepsilon)\,(\kappa(x)+\kappa(y))\,|x-y|(\mu^2+|\xi|^2)^{\frac{p-1}{2}} \tag{$H_\varepsilon 3$}$$

with constants all depending on ε.

Before we start with the derivation of estimates, observe that, since $2q-p>p$, we have

$$D\psi \in W^{1,2q-p}_{\mathrm{loc}}(\Omega) \;\Rightarrow\; D\psi \in W^{1,p}_{\mathrm{loc}}(\Omega).$$

Thus, by virtue of $(\mathcal{A}_\varepsilon 1)$, $(H_\varepsilon 1)-(H_\varepsilon 3)$ and since $r>n$, all the hypotheses of Theorem 9.3 are satisfied. We then get, according to the notation introduced in (8.2),

$$V_p(Du_\varepsilon) = (\mu^2+|Du_\varepsilon|^2)^{\frac{p-2}{4}}Du_\varepsilon \in W^{1,2}_{\mathrm{loc}}(\Omega)$$

from which we deduce, applying Lemma 8.3 with $p = 2$, $F = V_p(Du_\varepsilon)$ together with inequality (8.4),

$$Du_\varepsilon \in L^{\frac{np}{n-2}}_{\mathrm{loc}}(\Omega).$$

Thus the integral

$$\int_{B_R} (1 + |Du_\varepsilon(x)|)^{\frac{np}{n-2}}\, \mathrm{d}x,$$

which will frequently appear in the sequel, is finite. In particular by the Sobolev embedding we obtain $u_\varepsilon \in W^{1,\frac{np}{n-2}}_{\mathrm{loc}}(\Omega)$.

Now let us fix radii $0 < \frac{R}{8} < \rho < s < t < t' < \frac{R}{4}$ and a cut off function $\eta \in C_0^\infty(B_t)$ such that $0 \le \eta \le 1$, $\eta = 1$ on B_s, $|\nabla\eta| \le \frac{C}{t-s}$. Consider the function

$$\varphi(x) = u_\varepsilon(x) - \frac{\lambda}{h^2}\, \tau_{-h}\big(\eta^2(x)\, \tau_h(u_\varepsilon - \psi)(x)\big)$$

with τ_h defined in Subsection 8.1. Observe that, for $\lambda > 0$ sufficiently small, φ is an admissible test function in the variational inequality (9.9), that is, $\varphi \in \mathcal{K}_\psi(\Omega)$. Indeed

$$\begin{aligned}
\varphi(x) - \psi(x) &= (u_\varepsilon - \psi)(x) - \frac{\lambda}{h^2}\, \tau_{-h}\big(\eta^2(x)\, \tau_h(u_\varepsilon - \psi)(x)\big) \\
&= (u_\varepsilon - \psi)(x) + \frac{\lambda}{h^2}\eta^2(x)\Big((u_\varepsilon - \psi)(x+h) - (u_\varepsilon - \psi)(x)\Big) \\
&\quad - \frac{\lambda}{h^2}\eta^2(x-h)\Big((u_\varepsilon - \psi)(x) - (u_\varepsilon - \psi)(x-h)\Big) \\
&= (u_\varepsilon - \psi)(x)\left(1 - \frac{\lambda}{h^2}\eta^2(x) - \frac{\lambda}{h^2}\eta^2(x-h)\right) \\
&\quad + \frac{\lambda}{h^2}\Big(\eta^2(x)(u_\varepsilon - \psi)(x+h) + \eta^2(x-h)(u_\varepsilon - \psi)(x-h)\Big) \ge 0\,,
\end{aligned}$$

provided $0 < \lambda < \frac{h^2}{2}$. Hence, using φ as test function in (9.9) we get, with the aid of Proposition 8.1,

$$\int_\Omega \langle \tau_h(\mathcal{A}_\varepsilon(x, Du_\varepsilon)), D(\eta^2 \tau_h(u_\varepsilon - \psi))\rangle\, \mathrm{d}x \le 0.$$

Carrying out the computations, we get

$$\begin{aligned}
&\int_\Omega \Big\langle \mathcal{A}_\varepsilon(x+h, Du_\varepsilon(x+h)) - \mathcal{A}_\varepsilon(x, Du_\varepsilon(x)), D(\eta^2\tau_h(u_\varepsilon - \psi))\Big\rangle\, \mathrm{d}x \\
&= \int_\Omega \Big\langle \mathcal{A}_\varepsilon(x+h, Du_\varepsilon(x+h)) - \mathcal{A}_\varepsilon(x, Du_\varepsilon(x)), \eta^2\tau_h(Du_\varepsilon - D\psi)\Big\rangle\, \mathrm{d}x \\
&+ \int_\Omega \Big\langle \mathcal{A}_\varepsilon(x+h, Du_\varepsilon(x+h)) - \mathcal{A}(x, Du_\varepsilon(x)), 2\eta\, D\eta\, \tau_h(u_\varepsilon - \psi)\Big\rangle\, \mathrm{d}x \le 0.
\end{aligned}$$

We can rewrite the previous inequality as

$$\begin{aligned}
&\int_\Omega \Big\langle \mathcal{A}_\varepsilon(x+h, Du_\varepsilon(x+h)) - \mathcal{A}_\varepsilon(x+h, Du_\varepsilon(x)), \eta^2 \tau_h Du_\varepsilon \Big\rangle \,\mathrm{d}x \\
&- \int_\Omega \Big\langle \mathcal{A}_\varepsilon(x+h, Du_\varepsilon(x+h)) - \mathcal{A}_\varepsilon(x+h, Du_\varepsilon(x)), \eta^2 \tau_h D\psi \Big\rangle \,\mathrm{d}x \\
&+ \int_\Omega \Big\langle \mathcal{A}_\varepsilon(x+h, Du_\varepsilon(x+h)) - \mathcal{A}_\varepsilon(x+h, Du_\varepsilon(x)), 2\eta\, D\eta\, \tau_h (u_\varepsilon - \psi) \Big\rangle \,\mathrm{d}x \\
&+ \int_\Omega \Big\langle \mathcal{A}_\varepsilon(x+h, Du_\varepsilon(x)) - \mathcal{A}_\varepsilon(x, Du_\varepsilon(x)), \eta^2 \tau_h Du_\varepsilon \Big\rangle \,\mathrm{d}x \\
&- \int_\Omega \Big\langle \mathcal{A}_\varepsilon(x+h, Du_\varepsilon(x)) - \mathcal{A}_\varepsilon(x, Du_\varepsilon(x)), \eta^2 \tau_h D\psi \Big\rangle \,\mathrm{d}x \\
&+ \int_\Omega \Big\langle \mathcal{A}_\varepsilon(x+h, Du_\varepsilon(x)) - \mathcal{A}_\varepsilon(x, Du_\varepsilon(x)), 2\eta\, D\eta\, \tau_h (u_\varepsilon - \psi) \Big\rangle \,\mathrm{d}x \\
&=: I + II + III + IV + V + VI \le 0,
\end{aligned}$$

which yields

$$I \le |II| + |III| + |IV| + |V| + |VI|. \tag{9.10}$$

We now estimate each of the six summands. The ellipticity assumption expressed by $(\mathcal{A}_\varepsilon 1)$ implies

$$I \ge \nu \int_\Omega \eta^2 |\tau_h Du_\varepsilon|^2 (\mu^2 + |Du_\varepsilon(x+h)|^2 + |Du_\varepsilon(x)|^2)^{\frac{p-2}{2}} \,\mathrm{d}x. \tag{9.11}$$

By virtue of assumption $(\mathcal{A}_\varepsilon 2)$ and by Young's and Hölder's inequalities, we get

$$\begin{aligned}
|II| &\le 2L \int_\Omega \eta^2 |\tau_h Du_\varepsilon| (\mu^2 + |Du_\varepsilon(x+h)|^2 + |Du_\varepsilon(x)|^2)^{\frac{q-2}{2}} \,|\tau_h D\psi| \,\mathrm{d}x \\
&\le \vartheta \int_\Omega \eta^2 |\tau_h Du_\varepsilon|^2 (\mu^2 + |Du_\varepsilon(x+h)|^2 + |Du_\varepsilon(x)|^2)^{\frac{p-2}{2}} \,\mathrm{d}x \\
&+ C_\vartheta(L) \int_\Omega \eta^2 |\tau_h D\psi|^2 (1 + |Du_\varepsilon(x+h)|^2 + |Du_\varepsilon(x)|^2)^{\frac{2q-p-2}{2}} \,\mathrm{d}x \\
&\le \vartheta \int_\Omega \eta^2 |\tau_h Du_\varepsilon|^2 (\mu^2 + |Du_\varepsilon(x+h)|^2 + |Du_\varepsilon(x)|^2)^{\frac{p-2}{2}} \,\mathrm{d}x \\
&+ C_\vartheta(L) \left(\int_{B_t} |\tau_h D\psi|^{2q-p} \,\mathrm{d}x \right)^{\frac{2}{2q-p}} \left(\int_{B_{t'}} (1 + |Du_\varepsilon(x)|)^{2q-p} \,\mathrm{d}x \right)^{\frac{2q-p-2}{2q-p}}
\end{aligned}$$

where we used also the properties of η, and with a constant ϑ that will be specified later. Note that (9.7) implies

$$2q - p < \frac{(2q-p)r}{r-2} < \frac{np}{n-2}\,. \tag{9.12}$$

Hence, being $D\psi \in L^{2q-p}_{\mathrm{loc}}(\Omega)$ by hypothesis, we can use Lemma 8.2 together with Hölder's inequality to get

$$\begin{aligned}
|II| &\le \vartheta \int_\Omega \eta^2 |\tau_h Du_\varepsilon|^2 (\mu^2 + |Du_\varepsilon(x+h)|^2 + |Du_\varepsilon(x)|^2)^{\frac{p-2}{2}} \,\mathrm{d}x \\
&+ C_\theta(n,p,q,L,R)\, |h|^2 \left(\int_{B_{t'}} |D^2\psi|^{2q-p} \,\mathrm{d}x \right)^{\frac{2}{2q-p}} \left(\int_{B_{t'}} (1 + |Du_\varepsilon(x)|)^{\frac{np}{n-2}} \,\mathrm{d}x \right)^{\frac{(2q-p-2)(n-2)}{np}},
\end{aligned}$$

and by Young's inequality with conjugate exponents $\frac{p}{2+2p-2q}$ and $\frac{p}{2q-p-2}$ (which are bigger than 1 thanks to (9.4)) also

$$
\begin{aligned}
|II| &\leq \vartheta \int_\Omega \eta^2 |\tau_h Du_\varepsilon|^2 (\mu^2 + |Du_\varepsilon(x+h)|^2 + |Du_\varepsilon(x)|^2)^{\frac{p-2}{2}}\, dx \\
&+ C_{\theta,\Theta}(L,R)\, |h|^2 \left(\int_{B_{t'}} |D^2\psi|^{2q-p}\, dx \right)^{\frac{p}{(2q-p)(2+p-q)}} \\
&+ \Theta\, |h|^2 \left(\int_{B_{t'}} (1+|Du_\varepsilon(x)|)^{\frac{np}{n-2}}\, dx \right)^{\frac{n-2}{n}}
\end{aligned}
\tag{9.13}
$$

for some constant $\Theta = \Theta(n,p,q,r)$, $0 < \Theta < 1$, that will be determined later. Arguing analogously, we get

$$
\begin{aligned}
|III| &\leq 4L \int_\Omega |\tau_h Du_\varepsilon|\, |D\eta|\, \eta\, (\mu^2 + |Du_\varepsilon(x+h)|^2 + |Du_\varepsilon(x)|^2)^{\frac{q-2}{2}}\, |\tau_h(u_\varepsilon - \psi)|\, dx \\
&\leq \vartheta \int_\Omega \eta^2 |\tau_h Du_\varepsilon|^2\, (\mu^2 + |Du_\varepsilon(x+h)|^2 + |Du_\varepsilon(x)|^2)^{\frac{p-2}{2}}\, dx \\
&+ C_\vartheta(L) \int_\Omega |D\eta|^2 (1 + |Du_\varepsilon(x+h)|^2 + |Du_\varepsilon(x)|^2)^{\frac{2q-p-2}{2}} |\tau_h(u_\varepsilon - \psi)|^2\, dx \\
&\leq \vartheta \int_\Omega \eta^2 |\tau_h Du_\varepsilon|^2\, (\mu^2 + |Du_\varepsilon(x+h)|^2 + |Du_\varepsilon(x)|^2)^{\frac{p-2}{2}}\, dx \\
&+ \frac{C_\vartheta(L)}{(t-s)^2} \left(\int_{B_{t'}} (1+|Du_\varepsilon(x)|)^{2q-p}\, dx \right)^{\frac{2q-p-2}{2q-p}} \left(\int_{B_t} |\tau_h(u_\varepsilon - \psi)|^{2q-p}\, dx \right)^{\frac{2}{2q-p}}
\end{aligned}
$$

where we used the fact that $|D\eta| \leq \frac{C}{t-s}$. Since we already noticed that $u_\varepsilon \in W^{1,\frac{np}{n-2}}_{\mathrm{loc}}(\Omega) \hookrightarrow W^{1,2q-p}_{\mathrm{loc}}(\Omega)$, we deduce $D(u_\varepsilon - \psi) \in L^{2q-p}_{\mathrm{loc}}(\Omega)$. Hence we may use the first estimate of Lemma 8.2 to control the last integral on the right-hand side, obtaining

$$
\begin{aligned}
|III| &\leq \vartheta \int_\Omega \eta^2 |\tau_h Du_\varepsilon|^2\, (\mu^2 + |Du_\varepsilon(x+h)|^2 + |Du_\varepsilon(x)|^2)^{\frac{p-2}{2}}\, dx \\
&+ |h|^2 \frac{C_\vartheta(n,p,q,L)}{(t-s)^2} \left(\int_{B_{t'}} (1+|Du_\varepsilon(x)|)^{2q-p}\, dx \right)^{\frac{2q-p-2}{2q-p}} \left(\int_{B_{t'}} |D(u_\varepsilon - \psi)|^{2q-p}\, dx \right)^{\frac{2}{2q-p}} \\
&\leq \vartheta \int_\Omega \eta^2 |\tau_h Du_\varepsilon|^2\, (\mu^2 + |Du_\varepsilon(x+h)|^2 + |Du_\varepsilon(x)|^2)^{\frac{p-2}{2}}\, dx \\
&+ |h|^2\, \frac{C_\vartheta(n,p,q,L)}{(t-s)^2} \int_{B_{t'}} (1+|Du_\varepsilon(x)|)^{2q-p}\, dx \\
&+ |h|^2\, \frac{C_\vartheta(n,p,q,L)}{(t-s)^2} \int_{B_{R/2}} |D\psi(x)|^{2q-p}\, dx,
\end{aligned}
\tag{9.14}
$$

where we used also Young's inequality. Since (9.12) holds, we can use the interpolation inequality

$$
\|Dv\|_{L^{2q-p}(\Omega')} \leq \|Dv\|^{\delta_0}_{L^p(\Omega')}\, \|Dv\|^{1-\delta_0}_{L^{\frac{np}{n-2}}(\Omega')} \qquad \forall\, \Omega' \Subset \Omega\,,
\tag{9.15}
$$

where $0 < \delta_0 < 1$ is defined through the condition

$$
\frac{1}{2q-p} = \frac{\delta_0}{p} + \frac{(1-\delta_0)(n-2)}{np}
$$

which implies

$$\delta_0 = \frac{p(n-1) - q(n-2)}{2q-p}\,, \qquad\qquad 1-\delta_0 = \frac{n(q-p)}{2q-p}\,.$$

We get

$$\begin{aligned} &\int_{B_{t'}} (1+|Du_\varepsilon(x)|)^{2q-p}\,\mathrm{d}x \\ &\le \left(\int_{B_{t'}} (1+|Du_\varepsilon(x)|)^p\,\mathrm{d}x\right)^{\frac{\delta_0(2q-p)}{p}} \left(\int_{B_{t'}} (1+|Du_\varepsilon(x)|)^{\frac{np}{n-2}}\,\mathrm{d}x\right)^{\frac{(q-p)(n-2)}{p}}. \end{aligned} \tag{9.16}$$

Inserting (9.16) in (9.14) and using Young's inequality which conjugate exponents $\frac{p}{p-n(q-p)}$ and $\frac{p}{n(q-p)}$ (which are bigger than 1 thanks to (9.4)), we conclude that

$$\begin{aligned} |III| &\le \vartheta \int_\Omega \eta^2 |\tau_h Du_\varepsilon|^2 \left(\mu^2 + |Du_\varepsilon(x+h)|^2 + |Du_\varepsilon(x)|^2\right)^{\frac{p-2}{2}}\mathrm{d}x \\ &+ |h|^2 \frac{C_{\vartheta,\Theta}(L)}{(t-s)^{\frac{2p}{p-n(q-p)}}} \left(\int_{B_{R/2}} (1+|Du_\varepsilon(x)|)^p\,\mathrm{d}x\right)^{\frac{\delta_0(2q-p)}{p-n(q-p)}} \\ &+ \Theta\,|h|^2 \left(\int_{B_{t'}} (1+|Du_\varepsilon(x)|)^{\frac{np}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}} + |h|^2 \frac{C_\vartheta(n,p,q,L)}{(t-s)^2} \int_{B_{R/2}} |D\psi(x)|^{2q-p}\,\mathrm{d}x \end{aligned} \tag{9.17}$$

with Θ as above. In order to estimate the integral IV, we use assumption $(\mathcal{A}_\varepsilon 4)$, Young's and Hölder's inequalities as follows

$$\begin{aligned} |IV| &\le |h| \int_\Omega \eta^2 (\kappa(x+h)+\kappa(x))(\mu^2+|Du_\varepsilon(x)|^2)^{\frac{q-1}{2}}\,|\tau_h Du_\varepsilon|\,\mathrm{d}x \\ &\le \vartheta \int_\Omega \eta^2 |\tau_h Du_\varepsilon|^2 \left(\mu^2 + |Du_\varepsilon(x)|^2 + |Du_\varepsilon(x+h)|^2\right)^{\frac{p-2}{2}}\mathrm{d}x \\ &+ C_\vartheta\,|h|^2 \int_{B_t} (\kappa(x+h)+\kappa(x))^2 (1+|Du_\varepsilon(x)|^2)^{\frac{2q-p}{2}}\,\mathrm{d}x \\ &\le \vartheta \int_\Omega \eta^2 |\tau_h Du_\varepsilon|^2 \left(\mu^2 + |Du_\varepsilon(x)|^2 + |Du_\varepsilon(x+h)|^2\right)^{\frac{p-2}{2}}\mathrm{d}x \\ &+ C_\vartheta\,|h|^2 \left(\int_{B_t} (\kappa(x+h)+\kappa(x))^r\,\mathrm{d}x\right)^{\frac{2}{r}} \left(\int_{B_t} (1+|Du_\varepsilon(x)|)^{\frac{r(2q-p)}{r-2}}\,\mathrm{d}x\right)^{\frac{r-2}{r}}, \end{aligned} \tag{9.18}$$

where we also used that $\operatorname{supp}\eta \subset B_t \subset B_R$. Since (9.7) holds, we can use the interpolation inequality

$$\|Dv\|_{L^{\frac{(2q-p)r}{r-2}}(\Omega')} \le \|Dv\|^{\delta}_{L^p(\Omega')}\,\|Dv\|^{1-\delta}_{L^{\frac{np}{n-2}}(\Omega')} \qquad \forall\,\Omega' \Subset \Omega\,,$$

where $0<\delta<1$ is defined through the condition

$$\frac{r-2}{(2q-p)r} = \frac{\delta}{p} + \frac{(1-\delta)(n-2)}{np}$$

which implies

$$\delta = \frac{p(nr-n-r) - q(nr-2r)}{(2q-p)r}\,, \qquad\qquad 1-\delta = \frac{n(qr-pr+p)}{(2q-p)r}\,.$$

We get

$$
\begin{aligned}
&\left(\int_{B_t}(1+|Du_\varepsilon(x)|)^{\frac{(2q-p)r}{r-2}}\,\mathrm{d}x\right)^{\frac{r-2}{r}}\\
&\le\left(\int_{B_t}(1+|Du_\varepsilon(x)|)^{p}\,\mathrm{d}x\right)^{\frac{\delta(2q-p)}{p}}\left(\int_{B_t}(1+|Du_\varepsilon(x)|)^{\frac{np}{n-2}}\,\mathrm{d}x\right)^{\frac{(qr-pr+p)(n-2)}{pr}}.
\end{aligned}
\tag{9.19}
$$

Inserting (9.19) in (9.18), using the second inequality of Lemma 8.2 and Young's inequality with conjugate exponents $\frac{pr}{pr-n(qr-pr+p)}$ and $\frac{pr}{n(qr-pr+p)}$ (which are bigger than 1 thanks to (9.4))

$$
\begin{aligned}
|IV|&\le\vartheta\int_\Omega\eta^2|\tau_h Du_\varepsilon|^2\left(\mu^2+|Du_\varepsilon(x)|^2+|Du_\varepsilon(x+h)|^2\right)^{\frac{p-2}{2}}\,\mathrm{d}x\\
&+C_{\vartheta,\Theta}\,|h|^2\left(\int_{B_{R/2}}\kappa^r(x)\,\mathrm{d}x\right)^{\frac{2p}{pr-n(qr-pr+p)}}\left(\int_{B_{R/2}}(1+|Du_\varepsilon(x)|)^p\,\mathrm{d}x\right)^{\frac{\delta(2q-p)r}{pr-n(qr-pr+p)}}\\
&+\Theta\,|h|^2\left(\int_{B_t}(1+|Du_\varepsilon(x)|)^{\frac{np}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}}
\end{aligned}
\tag{9.20}
$$

with Θ as above. Assumption $(\mathcal{A}_\varepsilon 4)$ also entails

$$
\begin{aligned}
|V|&\le|h|\int_\Omega\eta^2(\kappa(x+h)+\kappa(x))(\mu^2+|Du_\varepsilon(x)|^2)^{\frac{q-1}{2}}\,|\tau_h D\psi|\,\mathrm{d}x\\
&\le|h|\left(\int_{B_t}(\kappa(x+h)+\kappa(x))^r\,\mathrm{d}x\right)^{\frac{1}{r}}\\
&\cdot\left(\int_{B_t}(1+|Du_\varepsilon(x)|)^{\frac{(q-1)(2q-p)r}{(2q-p)(r-1)-r}}\,\mathrm{d}x\right)^{\frac{(2q-p)(r-1)-r}{(2q-p)r}}\left(\int_{B_t}|\tau_h D\psi|^{2q-p}\,\mathrm{d}x\right)^{\frac{1}{2q-p}}\\
&\le C(n,p,q)\,|h|^2\left(\int_{B_{R/2}}\kappa^r(x)\,\mathrm{d}x\right)^{\frac{1}{r}}\\
&\cdot\left(\int_{B_t}(1+|Du_\varepsilon(x)|)^{\frac{(q-1)(2q-p)r}{(2q-p)(r-1)-r}}\,\mathrm{d}x\right)^{\frac{(2q-p)(r-1)-r}{(2q-p)r}}\left(\int_{B_{R/2}}|D^2\psi|^{2q-p}\,\mathrm{d}x\right)^{\frac{1}{2q-p}}
\end{aligned}
$$

where we used Hölder's inequality, the properties of η and Lemma 8.2 by virtue of the fact that $D^2\psi\in L^{2q-p}_{\mathrm{loc}}(\Omega)$ by hypothesis. On the other hand, our assumptions on p,q,r and (9.7) imply

$$
\frac{(q-1)(2q-p)r}{(2q-p)(r-1)-r}<\frac{(2q-p)r}{r-2}<\frac{np}{n-2}\,. \tag{9.21}
$$

Hence from Hölder's inequality we obtain

$$
\begin{aligned}
|V|\le C(n,p,q)\,|h|^2&\left(\int_{B_{R/2}}\kappa^r(x)\,\mathrm{d}x\right)^{\frac{1}{r}}\left(\int_{B_t}(1+|Du_\varepsilon(x)|)^{\frac{np}{n-2}}\,\mathrm{d}x\right)^{\frac{(q-1)(n-2)}{np}}\\
&\cdot\left(\int_{B_{R/2}}|D^2\psi|^{2q-p}\,\mathrm{d}x\right)^{\frac{1}{2q-p}},
\end{aligned}
$$

and by Young's inequality with conjugate exponents $\frac{p}{q-1}$ and $\frac{p}{1+p-q}$ (which are bigger than 1 thanks to (9.4)) also

$$
\begin{aligned}
|V| &\leq C_\Theta(R)\,|h|^2 \left(\int_{B_{R/2}} \kappa^r(x)\,\mathrm{d}x\right)^{\frac{p}{(1+p-q)r}} \left(\int_{B_{R/2}} |D^2\psi|^{2q-p}\,\mathrm{d}x\right)^{\frac{p}{(1+p-q)(2q-p)}} \\
&\quad + \Theta\,|h|^2 \left(\int_{B_t} (1+|Du_\varepsilon(x)|)^{\frac{np}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}}
\end{aligned}
\tag{9.22}
$$

with Θ as above. Finally, using assumption $(\mathcal{A}_\varepsilon 4)$ again and the properties of $D\eta$ we get

$$
\begin{aligned}
|VI| &\leq 2|h| \int_\Omega \eta\,|D\eta|\,(\kappa(x+h)+\kappa(x))\,(\mu^2+|Du_\varepsilon(x)|^2)^{\frac{q-1}{2}}\,|\tau_h(u_\varepsilon-\psi)|\,\mathrm{d}x \\
&\leq |h|\,\frac{C}{(t-s)^2} \left(\int_{B_t} |\tau_h u_\varepsilon|^{\frac{qr}{r-1}}\,\mathrm{d}x\right)^{\frac{r-1}{qr}} \left(\int_{B_t} (\kappa(x+h)+\kappa(x))^r\,\mathrm{d}x\right)^{\frac{1}{r}} \\
&\quad \cdot \left(\int_{B_t} (1+|Du_\varepsilon(x)|)^{\frac{qr}{r-1}}\,\mathrm{d}x\right)^{\frac{(r-1)(q-1)}{qr}} \\
&\quad + |h|\,\frac{C}{(t-s)^2} \left(\int_{B_t} |\tau_h \psi|^{2q-p}\,\mathrm{d}x\right)^{\frac{1}{2q-p}} \left(\int_{B_t} (\kappa(x+h)+\kappa(x))^r\,\mathrm{d}x\right)^{\frac{1}{r}} \\
&\quad \cdot \left(\int_{B_t} (1+|Du_\varepsilon(x)|)^{\frac{(q-1)(2q-p)r}{(2q-p)(r-1)-r}}\,\mathrm{d}x\right)^{\frac{(2q-p)(r-1)-r}{(2q-p)r}} \\
&=: VI_a + VI_b,
\end{aligned}
$$

where we used also Hölder's inequality. Let us focus on the term VI_a. Note that (9.7) implies

$$
\frac{qr}{r-1} < \frac{(2q-p)r}{r-1} < \frac{(2q-p)r}{r-2} < \frac{np}{n-2}\,. \tag{9.23}
$$

Since we already noticed that $u_\varepsilon \in W^{1,\frac{np}{n-2}}_{\mathrm{loc}}(\Omega)$, we can apply Lemma 8.2 obtaining

$$
VI_a \leq |h|^2\,\frac{C(n,q,r)}{(t-s)^2} \left(\int_{B_{t'}} (1+|Du_\varepsilon(x)|)^{\frac{qr}{r-1}}\,\mathrm{d}x\right)^{\frac{r-1}{r}} \left(\int_{B_{R/2}} \kappa^r(x)\,\mathrm{d}x\right)^{\frac{1}{r}}. \tag{9.24}
$$

Hence, owing again to (9.23), we can use the interpolation inequality

$$
\|Dv\|_{L^{\frac{qr}{r-1}}(\Omega')} \leq \|Dv\|^{\delta'}_{L^p(\Omega')}\,\|Dv\|^{1-\delta'}_{L^{\frac{np}{n-2}}(\Omega')} \qquad \forall\,\Omega' \Subset \Omega\,,
$$

where $0<\delta'<1$ is defined through the condition

$$
\frac{r-1}{qr} = \frac{\delta'}{p} + \frac{(1-\delta')(n-2)}{np}
$$

which implies

$$
\delta' = \frac{p(nr-n)-q(nr-2r)}{2qr}\,, \qquad\qquad 1-\delta' = \frac{n(qr-pr+p)}{2qr}\,.
$$

We get

$$
\begin{aligned}
&\left(\int_{B_{t'}} (1+|Du_\varepsilon(x)|)^{\frac{qr}{r-1}}\,\mathrm{d}x\right)^{\frac{r-1}{r}} \\
&\leq \left(\int_{B_{t'}} (1+|Du_\varepsilon(x)|)^p\,\mathrm{d}x\right)^{\frac{\delta' q}{p}} \left(\int_{B_{t'}} (1+|Du_\varepsilon(x)|)^{\frac{np}{n-2}}\,\mathrm{d}x\right)^{\frac{(n-2)(qr-pr+p)}{2pr}}.
\end{aligned}
\tag{9.25}
$$

Plugging (9.25) into (9.24) and using Young's inequality with conjugate exponents $\frac{2pr}{2pr-n(qr-pr+p)}$ and $\frac{2pr}{n(qr-pr+p)}$ (which are bigger that 1 thanks to (9.4)) yields

$$
\begin{aligned}
VI_a \le |h|^2 \, & \frac{C_\Theta}{(t-s)^{\frac{4pr}{2pr-n(qr-pr+p)}}} \left(\int_{B_{R/2}} (1+|Du_\varepsilon(x)|)^p \, \mathrm{d}x \right)^{\frac{\delta' 2qr}{2pr-n(qr-pr+p)}} \\
& \cdot \left(\int_{B_{R/2}} \kappa^r(x) \, \mathrm{d}x \right)^{\frac{2p}{2pr-n(qr-pr+p)}} + \Theta \, |h|^2 \left(\int_{B_t} (1+|Du_\varepsilon(x)|)^{\frac{np}{n-2}} \, \mathrm{d}x \right)^{\frac{n-2}{n}}
\end{aligned}
$$

with Θ as above. Finally, concerning the term VI_b, since $D\psi \in L^{2q-p}_{\mathrm{loc}}(\Omega)$ we can apply Lemma 8.2 and then argue as we did to obtain (9.22), getting

$$
\begin{aligned}
VI_b \le |h|^2 \, & \frac{C_\Theta(R)}{(t-s)^{\frac{2p}{1+p-q}}} \left(\int_{B_{R/2}} |D\psi(x)|^{2q-p} \, \mathrm{d}x \right)^{\frac{p}{(1+p-q)(2q-p)}} \left(\int_{B_{R/2}} \kappa^r(x) \, \mathrm{d}x \right)^{\frac{p}{(1+p-q)r}} \\
& + \Theta \, |h|^2 \left(\int_{B_t} (1+|Du_\varepsilon(x)|)^{\frac{np}{n-2}} \, \mathrm{d}x \right)^{\frac{n-2}{n}}
\end{aligned}
$$

with Θ as above. Summarizing the above computations we come to

$$
\begin{aligned}
|VI| \le |h|^2 \, & \frac{C_\Theta}{(t-s)^{\frac{4pr}{2pr-n(qr-pr+p)}}} \left(\int_{B_{R/2}} (1+|Du_\varepsilon(x)|)^p \, \mathrm{d}x \right)^{\frac{\delta' 2qr}{2pr-n(qr-pr+p)}} \\
& \cdot \left(\int_{B_{R/2}} \kappa^r(x) \, \mathrm{d}x \right)^{\frac{2p}{2pr-n(qr-pr+p)}} \\
& + |h|^2 \, \frac{C_\Theta(R)}{(t-s)^{\frac{2p}{1+p-q}}} \left(\int_{B_{R/2}} |D\psi(x)|^{2q-p} \, \mathrm{d}x \right)^{\frac{p}{(1+p-q)(2q-p)}} \left(\int_{B_{R/2}} \kappa^r(x) \, \mathrm{d}x \right)^{\frac{p}{(1+p-q)r}} \\
& + \Theta \, |h|^2 \left(\int_{B_{t'}} (1+|Du_\varepsilon(x)|)^{\frac{np}{n-2}} \, \mathrm{d}x \right)^{\frac{n-2}{n}} .
\end{aligned}
\tag{9.26}
$$

Inserting estimates (9.11), (9.13), (9.17), (9.20), (9.22) and (9.26) in (9.10), we infer

$$
\begin{aligned}
&\nu \int_\Omega \eta^2 |\tau_h Du_\varepsilon|^2 (\mu^2 + |Du_\varepsilon(x+h)|^2 + |Du_\varepsilon(x)|^2)^{\frac{p-2}{2}} \,\mathrm{d}x \\
&\le 3\vartheta \int_\Omega \eta^2 |\tau_h Du_\varepsilon|^2 (\mu^2 + |Du_\varepsilon(x+h)|^2 + |Du_\varepsilon(x)|^2)^{\frac{p-2}{2}} \,\mathrm{d}x \\
&+ 2\Theta\, |h|^2 \left(\int_{B_t} (1 + |Du_\varepsilon(x)|)^{\frac{np}{n-2}} \,\mathrm{d}x \right)^{\frac{n-2}{n}} + 3\Theta\, |h|^2 \left(\int_{B_{t'}} (1 + |Du_\varepsilon(x)|)^{\frac{np}{n-2}} \,\mathrm{d}x \right)^{\frac{n-2}{n}} \\
&+ C_{\theta,\Theta}(L,R)\, |h|^2 \left(\int_{B_{t'}} |D^2\psi|^{2q-p} \,\mathrm{d}x \right)^{\frac{p}{(2q-p)(2+p-q)}} \\
&+ |h|^2 \, \frac{C_{\vartheta,\Theta}(L)}{(t-s)^{\frac{2p}{p-n(q-p)}}} \left(\int_{B_{R/2}} (1 + |Du_\varepsilon(x)|)^p \,\mathrm{d}x \right)^{\frac{\delta_0(2q-p)}{p-n(q-p)}} + |h|^2 \, \frac{C_\vartheta(n,p,q,L)}{(t-s)^2} \int_{B_{R/2}} |D\psi(x)|^{2q-p} \,\mathrm{d}x \\
&+ C_{\vartheta,\Theta}\, |h|^2 \left(\int_{B_{R/2}} \kappa^r(x) \,\mathrm{d}x \right)^{\frac{2p}{pr-n(qr-pr+p)}} \left(\int_{B_{R/2}} (1 + |Du_\varepsilon(x)|)^p \,\mathrm{d}x \right)^{\frac{\delta(2q-p)r}{pr-n(qr-pr+p)}} \\
&+ C_\Theta(R)\, |h|^2 \left(\int_{B_{R/2}} \kappa^r(x) \,\mathrm{d}x \right)^{\frac{p}{(1+p-q)r}} \left(\int_{B_{R/2}} |D^2\psi|^{2q-p} \,\mathrm{d}x \right)^{\frac{p}{(1+p-q)(2q-p)}} \\
&+ |h|^2 \, \frac{C_\Theta}{(t-s)^{\frac{4pr}{2pr-n(qr-pr+p)}}} \left(\int_{B_{R/2}} (1 + |Du_\varepsilon(x)|)^p \,\mathrm{d}x \right)^{\frac{\delta' 2qr}{2pr-n(qr-pr+p)}} \left(\int_{B_{R/2}} \kappa^r(x) \,\mathrm{d}x \right)^{\frac{2p}{2pr-n(qr-pr+p)}} \\
&=: 3\vartheta \int_\Omega \eta^2 |\tau_h Du_\varepsilon|^2 (\mu^2 + |Du_\varepsilon(x+h)|^2 + |Du_\varepsilon(x)|^2)^{\frac{p-2}{2}} \,\mathrm{d}x \\
&+ 2\Theta\, |h|^2 \left(\int_{B_t} (1 + |Du_\varepsilon(x)|)^{\frac{np}{n-2}} \,\mathrm{d}x \right)^{\frac{n-2}{n}} + 3\Theta\, |h|^2 \left(\int_{B_{t'}} (1 + |Du_\varepsilon(x)|)^{\frac{np}{n-2}} \,\mathrm{d}x \right)^{\frac{n-2}{n}} \\
&+ |h|^2 A + |h|^2 \frac{B}{(t-s)^2} + |h|^2 \frac{C_1}{(t-s)^{\gamma_1}} + |h|^2 \frac{C_2}{(t-s)^{\gamma_2}} + |h|^2 \frac{C_3}{(t-s)^{\gamma_3}} \,.
\end{aligned}
$$

Choosing $\vartheta = \frac{\nu}{6}$, we can reabsorb the first integral on the right-hand side of previous estimate by the left-hand side thus getting

$$
\begin{aligned}
&\frac{\nu}{2} \int_\Omega \eta^2 |\tau_h Du_\varepsilon|^2 (\mu^2 + |Du_\varepsilon(x+h)|^2 + |Du_\varepsilon(x)|^2)^{\frac{p-2}{2}} \,\mathrm{d}x \\
&\le 2\Theta\, |h|^2 \left(\int_{B_t} (1 + |Du_\varepsilon(x)|)^{\frac{np}{n-2}} \,\mathrm{d}x \right)^{\frac{n-2}{n}} + 3\Theta\, |h|^2 \left(\int_{B_{t'}} (1 + |Du_\varepsilon(x)|)^{\frac{np}{n-2}} \,\mathrm{d}x \right)^{\frac{n-2}{n}} \\
&+ |h|^2 A + |h|^2 \frac{B}{(t-s)^2} + |h|^2 \frac{C_1}{(t-s)^{\gamma_1}} + |h|^2 \frac{C_2}{(t-s)^{\gamma_2}} + |h|^2 \frac{C_3}{(t-s)^{\gamma_3}} \,.
\end{aligned}
\tag{9.27}
$$

Using Lemma 8.5 on the left-hand side of the previous estimate and recalling that $\eta \equiv 1$ on B_s we get

$$
\begin{aligned}
&\frac{\nu}{2c} \int_{B_s} |\tau_h V_p(Du_\varepsilon)|^2 \,\mathrm{d}x \\
&\le \Bigg(2\Theta \left(\int_{B_t} (1 + |Du_\varepsilon(x)|)^{\frac{np}{n-2}} \,\mathrm{d}x \right)^{\frac{n-2}{n}} + 3\Theta \left(\int_{B_{t'}} (1 + |Du_\varepsilon(x)|)^{\frac{np}{n-2}} \,\mathrm{d}x \right)^{\frac{n-2}{n}} \\
&\quad + A + \frac{B}{(t-s)^2} + \frac{C_1}{(t-s)^{\gamma_1}} + \frac{C_2}{(t-s)^{\gamma_2}} + \frac{C_3}{(t-s)^{\gamma_3}} \Bigg) |h|^2 \,,
\end{aligned}
$$

and from Lemma 8.3 and inequality (8.4) also

$$\begin{aligned}
&\int_{B_s} |Du_\varepsilon(x)|^{\frac{np}{n-2}}\,\mathrm{d}x \\
&\le 2\Theta \left(\int_{B_t} (1+|Du_\varepsilon(x)|)^{\frac{np}{n-2}}\,\mathrm{d}x \right)^{\frac{n-2}{n}} + 3\Theta \left(\int_{B_{t'}} (1+|Du_\varepsilon(x)|)^{\frac{np}{n-2}}\,\mathrm{d}x \right)^{\frac{n-2}{n}} \\
&\quad + A + \frac{B}{(t-s)^2} + \frac{C_1}{(t-s)^{\gamma_1}} + \frac{C_2}{(t-s)^{\gamma_2}} + \frac{C_3}{(t-s)^{\gamma_3}} \,.
\end{aligned}$$

Setting

$$\phi(r) = \left(\int_{B_r} |Du_\varepsilon(x)|^{\frac{np}{n-2}}\,\mathrm{d}x \right)^{\frac{n-2}{n}}$$

and possibly redefining the constants, we can rewrite the previous inequality as

$$\phi(s) \le 2\Theta\,\phi(t) + 3\Theta\,\phi(t') + A + \frac{B}{(t-s)^2} + \frac{C_1}{(t-s)^{\gamma_1}} + \frac{C_2}{(t-s)^{\gamma_2}} + \frac{C_3}{(t-s)^{\gamma_3}} \,.$$

Notice that we can choose Θ small enough to satisfy $\Theta < 1/3 < 1/2$. Thus we can apply Lemma 8.6 obtaining

$$\phi(\rho) \le c \left(2\Theta\,\phi(t') + A + \frac{B}{(t'-\rho)^2} + \frac{C_1}{(t'-\rho)^{\gamma_1}} + \frac{C_2}{(t'-\rho)^{\gamma_2}} + \frac{C_3}{(t'-\rho)^{\gamma_3}} \right)$$

for some $c = c\,(\Theta, \gamma_1, \gamma_2, \gamma_3)$. But then, applying Lemma 8.6 one more time we get

$$\phi\left(\frac{R}{8}\right) \le \tilde{c} \left(A + \frac{B}{R^2} + \frac{C_1}{R^{\gamma_1}} + \frac{C_2}{R^{\gamma_2}} + \frac{C_3}{R^{\gamma_3}} \right)$$

with $\tilde{c} = \tilde{c}\,(\Theta, \gamma_1, \gamma_2, \gamma_3)$. Then, using again Lemma 8.3 with $p = 2$, $F = V_p(Du_\varepsilon)$ on the left-hand side of (9.27), by the arbitrariness of the ball $B_{R/8}$ and our regularity hypothesis on the obstacle we infer the a priori estimate

$$\int_{B_{R/8}} |D\left(V_p(Du_\varepsilon(x))\right)|^2\,\mathrm{d}x \le C \left(\int_{B_{R/2}} (1+|Du_\varepsilon(x)|)^p\,\mathrm{d}x \right)^{\beta} \tag{9.28}$$

with $C = C(n,p,q,r,\nu,L,R)$ and

$$\beta = \max\left\{ \frac{\delta_0(2q-p)}{p-n(q-p)}\,,\ \frac{\delta(2q-p)r}{pr-n(qr-pr+p)}\,,\ \frac{\delta' 2qr}{2pr-n(qr-pr+p)} \right\}.$$

Step 2: Passage to the limit

Now we conclude the proof by passing to the limit in the approximating problem. The limit procedure is standard (see e. g. [35]). By the growth conditions at (III) of Lemma 9.10 and the minimality of u_ε for F_ε we get

$$\begin{aligned}
\int_{B_R} |Du_\varepsilon(x)|^p\,\mathrm{d}x &\le C(K_0) \int_{B_R} F_\varepsilon(x, Du_\varepsilon(x))\,\mathrm{d}x \\
&\le C(K_0) \int_{B_R} F_\varepsilon(x, Du(x))\,\mathrm{d}x \\
&\le C(K_0) \int_{B_R} F(x, Du(x))\,\mathrm{d}x,
\end{aligned}$$

where in the last inequality we used also (I) of Lemma 9.10. Since $F(x, Du_0(x)) \in L^1_{\mathrm{loc}}(\Omega)$ by assumption, by the minimality of u for F we deduce also $F(x, Du) \in L^1_{\mathrm{loc}}(\Omega)$. Then, up to subsequences, there exists $\bar{u} \in u + W^{1,p}_0(B_R)$ such that

$$u_\varepsilon \to \bar{u} \qquad \text{weakly in } u + W^{1,p}_0(B_R).$$

Our next step is to show that $\bar{u}$ is a solution to our obstacle problem over the ball B_R, that is, it minimizes the functional $\int_\Omega F(x, Dw)\,\mathrm{d}x$ in $u + W^{1,p}_0(B_R)$. To this aim fix ε_0 and observe that, by (I) in Lemma 9.10, it holds

$$\int_{B_R} F_{\varepsilon_0}(x, Du_\varepsilon(x))\,\mathrm{d}x \leq \int_{B_R} F_\varepsilon(x, Du_\varepsilon(x))\,\mathrm{d}x \leq \int_{B_R} F(x, Du_\varepsilon(x))\,\mathrm{d}x \tag{9.29}$$

for $\varepsilon > \varepsilon_0$. By the lower semicontinuity of the functional $v \mapsto \int_{B_R} F_{\varepsilon_0}(x, Dv(x))\,\mathrm{d}x$ on $W^{1,p}$, estimate (9.29) and again the minimality of u_ε it follows

$$\int_{B_R} F_{\varepsilon_0}(x, D\bar{u})\,\mathrm{d}x \leq \liminf_{\varepsilon \to 0} \int_{B_R} F_{\varepsilon_0}(x, Du_\varepsilon)\,\mathrm{d}x \leq \int_{B_R} F(x, Du)\,\mathrm{d}x.$$

Again by (I) in Lemma 9.10, we can use the Monotone Convergence Theorem on the left-hand side of the previous estimate to deduce

$$\int_{B_R} F(x, D\bar{u})\,\mathrm{d}x = \lim_{\varepsilon_0 \to 0} \int_{B_R} F_{\varepsilon_0}(x, D\bar{u})\,\mathrm{d}x \leq \int_{B_R} F(x, Du)\,\mathrm{d}x.$$

Note also that, since $u_\varepsilon \in \mathcal{K}_\psi(\Omega)$ for every ε and $\mathcal{K}_\psi(\Omega)$ is a closed set, we have $\bar{u} \in \mathcal{K}_\psi(\Omega)$. We have then proved that the limit function $\bar{u}$ is a solution to the minimization problem

$$\min \left\{ \int_\Omega F(x, Dw)\,\mathrm{d}x : w \in u + W^{1,p}_0(B_R),\, w \in \mathcal{K}_\psi(\Omega) \right\}.$$

It only remains to show that the minimizer has the regularity stated in Theorem 9.8. First of all, note that the strict convexity of F yields $\bar{u} = u$. From estimate (9.28) and by compact embedding we infer

$$\begin{aligned} V_p(Du_\varepsilon) &\to v \qquad \text{weakly in } W^{1,2}_{\mathrm{loc}}(\Omega), \\ V_p(Du_\varepsilon) &\to v \qquad \text{strongly in } L^2_{\mathrm{loc}}(\Omega), \end{aligned}$$

from which we deduce, together with inequality (8.2),

$$Du_\varepsilon \to \bar{w} \qquad \text{strongly in } L^p_{\mathrm{loc}}(\Omega).$$

We thus have the strong convergence

$$u_\varepsilon \to \bar{u} = u \qquad \text{strongly in } u + W^{1,p}_0(B_R).$$

Hence we can pass to the limit in estimate (9.28) and conclude that $V_p(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega)$, which is the claim of Theorem 9.8.

Remark 9.11. As already mentioned, now that we have gained some extra regularity for the solution of the obstacle problem (IV.2), we can reformulate it as a variational inequality in the spirit of Proposition 9.1, provided we consider a new class $\mathcal{K}_\psi^{\mathrm{new}}(\Omega)$.
Let $u \in \mathcal{K}_\psi(\Omega)$ be a solution to (IV.2) under Hypothesis 9.5. Let $\varphi \in \mathcal{K}_\psi(\Omega)$. Looking at the proof of Proposition 9.1, we see that the growth conditions came into play only when we proved that I_2 was finite. Now, by $(F3)$,

$$\begin{aligned} I_2 &\le \varepsilon \int_\Omega \int_0^1 s \int_0^1 |D_{\xi\xi}F(x, Du + ts\,\varepsilon D(\varphi - u))||D(\varphi-u)|^2 \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}x \\ &\le \varepsilon \int_\Omega \int_0^1 s \int_0^1 \tilde{L}\left(\mu^2 + |Du + ts\,\varepsilon D(\varphi-u)|^2\right)^{\frac{q-2}{2}} |D(\varphi-u)|^2 \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}x \\ &\le \varepsilon \tilde{L} \int_\Omega \left(\mu^2 + |Du|^2 + \varepsilon^2 |D(\varphi-u)|^2\right)^{\frac{q-2}{2}} |D(\varphi-u)|^2 \,\mathrm{d}x. \end{aligned}$$

Note that the sharp inequality (9.7) coming from condition (9.4) entails

$$q < \frac{(2q-p)r}{r-2} < \frac{np}{n-2} < p^*\,,$$

hence by Theorem 9.8 and the Sobolev embedding we have that

$$Du \in L^{p^*}_{\mathrm{loc}}(\Omega) \hookrightarrow L^q_{\mathrm{loc}}(\Omega).$$

Thus, arguing as in the proof of Proposition 9.1 and letting $\varepsilon \to 0$ we see that

$$\int_\Omega \langle \mathcal{A}(x, Du), D(\varphi-u)\rangle \,\mathrm{d}x \ge 0 \qquad \text{for all } \varphi \in \mathcal{K}_\psi^{\mathrm{new}}(\Omega),$$

where

$$\mathcal{K}_\psi^{\mathrm{new}}(\Omega) := \left\{ w \in u_0 + W_0^{1,q}(\Omega) : w \ge \psi \text{ a.e. in } \Omega \right\}$$

with $u_0 \in W^{1,q}(\Omega)$ fixed boundary value.

9.2.2 A second higher differentiability result

In this section we report and briefly comment the achievements from [72], where we proved that the higher differentiability result remains true also under the hypothesis $D\psi \in W^{1,r}_{\mathrm{loc}}(\Omega)$. The statement is the following.

Theorem 9.12. *Let $u \in \mathcal{K}_\psi(\Omega)$ be the solution to the obstacle problem* (IV.2)*. Suppose that F satisfies Hypothesis 9.5 for exponents $2 \le p < n$, $p < q$, $r > n$ as in* (9.4) *and for a function $\kappa \in L^r_{\mathrm{loc}}(\Omega)$. Then it holds*

$$D\psi \in W^{1,r}_{\mathrm{loc}}(\Omega) \quad \Rightarrow \quad (\mu^2 + |Du|^2)^{\frac{p-2}{4}} Du \in W^{1,2}_{\mathrm{loc}}(\Omega)\,. \tag{9.30}$$

We are not going to give the proof of this result, since it exploits the same techniques we have already seen in the previous section.

It is worth noting that if $q < n$ then Theorem 9.12 ends up to be a consequence of Theorem 9.8, since in this case $2q - p < r$. Indeed (9.4) is equivalent to

$$r > \frac{1}{1 + \frac{1}{n} - \frac{q}{p}} .$$

If we are able to show that

$$2q - p < \frac{1}{1 + \frac{1}{n} - \frac{q}{p}} ,$$

then our result follows. We have

$$2q - p < \frac{1}{1 + \frac{1}{n} - \frac{q}{p}} \;\Leftrightarrow\; 2q - p < \frac{np}{np + p - nq} \;\Leftrightarrow\; pq + p(q-p) < np + nq(q-p) + n(q-p)^2$$

which holds true as long as $p(q-p) < nq(q-p)$ and $qp < np$, since we are assuming $q < n$.

However, if nothing is assumed on the relationship between q and n, the previous turns out to be a completely independent result.

9.3 The case of nearly-linear growth conditions

There are few results in the literature dealing with the case of nearly-linear growth conditions. We mention the classical papers [67, 82] and the more recent [122] in the case of equations of functionals, and [38, 66] in the case of obstacle problems.

The aim of this section is the study of the higher differentiability properties of the gradient of functions $u \in W^{1,1}(\Omega)$ which are solutions to variational obstacle problems of the form (IV.2) in the setting of nearly-linear growth conditions. The obstacle ψ belongs to the Sobolev space $W^{1,1}(\Omega)$, and the class $\mathcal{K}_\psi(\Omega)$ is defined as

$$\mathcal{K}_\psi(\Omega) := \left\{ w \in W^{1,1}(\Omega) : w \geq \psi \text{ a.e. in } \Omega \right\} . \tag{9.31}$$

We require the data of our problem to fulfill the following hypotheses.

Hypothesis 9.13. Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain. For the Lagrangian functional $F : \Omega \times \mathbb{R}^n \to [0, +\infty)$ we shall assume that:

(i) it is a Carathéodory function, that is, $F(\cdot, \xi)$ measurable for every $\xi \in \mathbb{R}^n$ and $F(x, \cdot)$ continuous for a. e. $x \in \Omega$;

(ii) $\xi \mapsto F(x, \xi)$ is a strictly convex C^2 function for a. e. $x \in \Omega$;

(iii) there exist positive constants $\tilde{\nu}, \tilde{L}, \tilde{\ell}$ and an exponent $1 < q < \infty$ such that

$$\frac{1}{\tilde{\ell}}\,\bar{F}(|\xi|) \leq F(x,\xi) \leq \tilde{\ell}\,(1+|\xi|^q) \tag{$F_{\mathrm{nl}}1$}$$

$$\langle D_{\xi\xi}F(x,\xi)\lambda,\lambda\rangle \geq \tilde{\nu}\,(1+|\xi|^2)^{-\frac{1}{2}}|\lambda|^2 \tag{$F_{\mathrm{nl}}2$}$$

$$|D_{\xi\xi}F(x,\xi)| \leq \tilde{L}\,(1+|\xi|^2)^{\frac{q-2}{2}} \tag{$F_{\mathrm{nl}}3$}$$

$$|D_{\xi x}F(x,\xi)| \leq k(x)(1+|\xi|^2)^{\frac{q-1}{2}} \tag{$F_{\mathrm{nl}}4$}$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$. The two functions $k : \Omega \to [0,+\infty)$ and $\bar{F} : [0,+\infty) \to [0,+\infty)$ satisfy respectively $k \in L^r_{\mathrm{loc}}(\Omega)$ with $r > n$ and

$$\lim_{t\to+\infty} \frac{\bar{F}(t)}{t} = +\infty. \tag{9.32}$$

Finally, to avoid trivialities, we shall assume that the set of admissible functions $\mathcal{K}_\psi(\Omega)$ is not empty and that a solution u to (IV.2), (9.31) is such that $F(x,Du) \in L^1_{\mathrm{loc}}(\Omega)$.

Remark 9.14. The growth assumption (F_{nl}) together with (9.32) guarantees the coercivity of the functional F, so that we can still apply [37, Theorem 3.30] to deduce the existence of a solution to the obstacle problem (IV.2), (9.31).

Note that functionals with nearly-linear growth have features in common with the ones satisfying nonstandard growth since, by virtue of (9.32), we have that

$$c\,|\xi| \leq F(x,\xi) \leq \tilde{\ell}\,(1+|\xi|^q)\,.$$

It is well-known that in this setting the Lavrentiev phenomenon may occur (see Remark 9.7). We therefore assume to be in the situation where irregularity phenomena can be ruled out, in the following sense. According to [38], for any $B \Subset \Omega$ we consider the set

$$\mathcal{K}^*_\psi(\Omega) := W^{1,1}_{\mathrm{loc}}(\Omega) \cap \big\{w \in W^{1,q}(B) : w \geq \psi \text{ a.e. in } B\big\}\,.$$

Then, in the spirit of Remark 9.7, we define

$$\mathcal{F}(u,\Omega) := \int_\Omega F(x,Du)\,\mathrm{d}x$$

and introduce the relaxed functional

$$\overline{\mathcal{F}}(u,B) := \inf_{\mathcal{C}(u)} \{\liminf_j \mathcal{F}(u_j,B)\}, \tag{9.33}$$

where

$$\mathcal{C}(u) := \big\{\{u_j\}_{j\in\mathbb{N}} \subset \mathcal{K}^*_\psi(\Omega) \,:\, u_j \rightharpoonup u \text{ weakly in } W^{1,1}(B)\big\}\,. \tag{9.34}$$

Then we consider the Lavrentiev gap functional

$$\mathcal{L}(u,B) := \overline{\mathcal{F}}(u,B) - \mathcal{F}(u,B).$$

We will assume that in our setting it holds $\mathcal{L}(u,B) = 0$ for all $B \Subset \Omega$. This leads to the following nice characterization that will be useful in the passage to the limit procedure.

Lemma 9.15. *For any fixed ball $B \Subset \Omega$, let $u \in \mathcal{K}_\psi(\Omega)$ be such that $\mathcal{F}(u,B) < \infty$. Then $\mathcal{L}(u,B) = 0$ if and only if there exists a sequence $\{u_j\}_{j\in N} \subset W^{1,q}(B)$ such that $u_j \geq \psi$ a. e. in B, $u_j \rightharpoonup u$ weakly in $W^{1,1}(B)$ and $\mathcal{F}(u_j,B) \to \mathcal{F}(u,B)$.*

Proof. Following [1], we define the *set of good sequences for u in B* as

$$\mathcal{G}(u,B) := \left\{ \{u_j\}_{j\in\mathbb{N}} \in \mathcal{C}(u) \, : \, \overline{\mathcal{F}}(u,B) = \lim_{j\to\infty} \mathcal{F}(u_j,B) \right\},$$

where $\mathcal{C}(u)$ is defined in (9.34). It is clear that $\mathcal{G}(u,B)$ is nonempty. Hence, that $\mathcal{L}(u,B) = 0$ implies the existence of the desired sequence follows immediately from the definition of $\overline{\mathcal{F}}(u,B)$ by choosing any good sequence for u in B.

Conversely, in view of Ioffe's Theorem (see [86]), the functional $\mathcal{F}$ is lower semicontinuous and thus $\mathcal{F}(u,B) \leq \overline{\mathcal{F}}(u,B)$ for all $u \in \mathcal{K}_\psi(\Omega)$. On the other hand, since we are assuming $\mathcal{F}(u_j,B) \to \mathcal{F}(u,B)$, by the definition of $\overline{\mathcal{F}}(u,B)$ in (9.33) we obtain the converse inequality. Thus $\mathcal{L}(u,B) = 0$. □

In analogy with (9.4), we shall assume also that

$$q < 1 + \frac{1}{n} - \frac{1}{r}\,. \tag{9.35}$$

Remark 9.16. Inequality (9.35) implies that q is such that

$$1 < q < 1 + \frac{1}{2} = \frac{3}{2}.$$

The main result of this section is the following theorem from [73].

Theorem 9.17. *Let u be a solution to the obstacle problem* (IV.2), (9.31) *such that $\mathcal{L}(u,B_R) = 0$ for all $B_R \Subset \Omega$. Suppose that F satisfies Hypothesis 9.13 for exponents q, r as in* (9.35)*. Then*

$$D\psi \in W^{1,r}_{\mathrm{loc}}(\Omega) \quad \Rightarrow \quad (1+|Du|^2)^{-1/4} Du \in W^{1,2}_{\mathrm{loc}}(\Omega).$$

9.3.1 Proof of the main result

We are now going to prove Theorem 9.17. The first part of the proof is devoted to the construction of an approximating minimization problem, whose solution satisfies a suitable a priori estimate. In the second step we conclude showing that the estimate is preserved in passing to the limit, and this yields the desired regularity result.

Step 1: Approximation and derivation of estimates

Let us fix a ball $B_R \Subset \Omega$. We first use Proposition 9.5 to get the existence of a sequence $\{\tilde{u}_j\}_{j\in\mathbb{N}} \subset W^{1,q}(B_R)$ such that

$$\begin{gathered} \tilde{u}_j \geq \psi \;\text{ a.e. in } B_R, \quad \tilde{u}_j \rightharpoonup u \;\text{ weakly in } W^{1,1}(B_R), \\ \int_{B_R} F(x, D\tilde{u}_j)\,\mathrm{d}x \to \int_{B_R} F(x, Du)\,\mathrm{d}x. \end{gathered} \tag{9.36}$$

Then, following [38], we consider the sequence of functionals $F_j(x,\xi)$ defined by

$$F_j(x,\xi) := F(x,\xi) + \frac{\varepsilon_j}{q}\left(1+|\xi|^2\right)^{\frac{q}{2}}, \tag{9.37}$$

where

$$\varepsilon_j := \left(1 + j + \|D\tilde{u}_j\|^{2q}_{L^q(B_R)}\right)^{-1}. \tag{9.38}$$

By Direct Methods and convexity, we get that for any $j \in \mathbb{N}$ there exists a unique solution $u_j \in \tilde{u}_j + W^{1,q}_0(B_R)$ to the obstacle problem

$$\min\left\{\int_\Omega F_j(x, Dw)\,\mathrm{d}x : w \in \mathcal{K}_\psi(\Omega)\right\}.$$

Since $F_j(x,\xi)$ satisfies standard growth conditions, setting

$$\mathcal{A}_j(x,\xi) = D_\xi F_j(x,\xi)$$

from Proposition 9.1 we get that u_j is a solution to the variational inequality

$$\int_\Omega \langle \mathcal{A}_j(x, Du_j), D(\varphi - u_j)\rangle\,\mathrm{d}x \geq 0 \qquad \forall\,\varphi \in \mathcal{K}_\psi(\Omega)\,. \tag{9.39}$$

Moreover, from $(F_{\mathrm{nl}}1) - (F_{\mathrm{nl}}4)$ it follows that there exist positive constants ν, L, ℓ and a function $\kappa \in L^r_{\mathrm{loc}}(\Omega)$ such that for a.e. $x, y \in \Omega$ and every $\xi, \eta \in \mathbb{R}^n$ the following conditions hold

$$\langle \mathcal{A}_j(x,\xi) - \mathcal{A}_j(x,\eta), \xi - \eta\rangle \geq \nu\,|\xi-\eta|^2(1+|\xi|^2+|\eta|^2)^{-\frac{1}{2}} \tag{$\mathcal{A}_j1$}$$

$$|\mathcal{A}_j(x,\xi) - \mathcal{A}_j(x,\eta)| \leq L\,|\xi-\eta|(1+|\xi|^2+|\eta|^2)^{\frac{q-2}{2}} \tag{$\mathcal{A}_j2$}$$

$$|\mathcal{A}_j(x,\xi)| \leq \ell\,(1+|\xi|^2)^{\frac{q-1}{2}} \tag{$\mathcal{A}_j3$}$$

$$|\mathcal{A}_j(x,\xi) - \mathcal{A}_j(y,\xi)| \leq (\kappa(x)+\kappa(y))\,|x-y|(1+|\xi|^2)^{\frac{q-1}{2}}. \tag{$\mathcal{A}_j4$}$$

Before we start with the derivation of estimates, note that it holds, according to the notation introduced in (8.1),

$$|V_1(Du_j)|^2 = \frac{|Du_j|^2}{\sqrt{1+|Du_j|^2}} \leq |Du_j|$$

Thus, since we are considering $u_j \in W^{1,q}(B_R)$ with $q > 1$, we find that $V_1(Du_j) \in L^2(B_R)$. Applying Lemma 8.3 with $p = 2$, $F = V_1(Du_j)$ together with inequality (8.3) we obtain

$$Du_j \in L^{\frac{n}{n-2}}(B_R).$$

Thus the integral

$$\int_{B_R} (1 + |Du_j(x)|)^{\frac{n}{n-2}} \, \mathrm{d}x,$$

which will frequently appear in the sequel, is finite.

Let us now fix radii $0 < \frac{R}{8} < \rho < s < t < t' < \frac{R}{4}$ and a cut off function $\eta \in C_0^\infty(B_t)$ such that $0 \le \eta \le 1$, $\eta = 1$ on B_s, $|D\eta| \le \frac{C}{t-s}$. Consider the function

$$\varphi(x) = u_j(x) - \frac{\lambda}{h^2}\, \tau_{-h}\big(\eta^2(x)\, \tau_h(u_j - \psi)(x)\big)$$

with τ_h defined in Subsection 8.1. Notice that, when $\lambda > 0$ is sufficiently small, φ is an admissible test function in the variational inequality (9.39), that is, $\varphi \in \mathcal{K}_\psi(\Omega)$. Indeed

$$\begin{aligned}
\varphi(x) - \psi(x) &= (u_j - \psi)(x) - \frac{\lambda}{h^2}\, \tau_{-h}\big(\eta^2(x)\, \tau_h(u_j - \psi)(x)\big) \\
&= (u_j - \psi)(x) + \frac{\lambda}{h^2}\eta^2(x)\Big((u_j - \psi)(x+h) - (u_j - \psi)(x)\Big) \\
&\quad - \frac{\lambda}{h^2}\eta^2(x-h)\Big((u_j - \psi)(x) - (u_j - \psi)(x-h)\Big) \\
&= (u_j - \psi)(x)\left(1 - \frac{\lambda}{h^2}\eta^2(x) - \frac{\lambda}{h^2}\eta^2(x-h)\right) \\
&\quad + \frac{\lambda}{h^2}\Big(\eta^2(x)(u_j - \psi)(x+h) + \eta^2(x-h)(u_j - \psi)(x-h)\Big) \ge 0\,,
\end{aligned}$$

provided $0 < \lambda < \frac{h^2}{2}$. Hence, using φ as test function in (9.39) we get, with the aid of Proposition 8.1,

$$\int_\Omega \langle \tau_h(\mathcal{A}_j(x, Du_j)), D(\eta^2\, \tau_h(u_j - \psi))\rangle \, \mathrm{d}x \le 0\,.$$

Performing the calculations we obtain

$$\begin{aligned}
0 &\ge \int_\Omega \Big\langle \mathcal{A}_j(x+h, Du_j(x+h)) - \mathcal{A}_j(x, Du_j(x)), D(\eta^2 \tau_h(u_j - \psi))\Big\rangle \, \mathrm{d}x \\
&= \int_\Omega \Big\langle \mathcal{A}_j(x+h, Du_j(x+h)) - \mathcal{A}_j(x, Du_j(x)), \eta^2 \tau_h(Du_j - D\psi)\Big\rangle \, \mathrm{d}x \\
&\quad + \int_\Omega \Big\langle \mathcal{A}_j(x+h, Du_j(x+h)) - \mathcal{A}_j(x, Du_j(x)), 2\,\eta\, D\eta \tau_h(u_j - \psi)\Big\rangle \, \mathrm{d}x.
\end{aligned}$$

The previous inequality can be rewritten as follows

$$\begin{aligned}
&\int_\Omega \Big\langle \mathcal{A}_j(x+h, Du_j(x+h)) - \mathcal{A}_j(x+h, Du_j(x)), \eta^2 \tau_h Du_j \Big\rangle \, \mathrm{d}x \\
&- \int_\Omega \Big\langle \mathcal{A}_j(x+h, Du_j(x+h)) - \mathcal{A}_j(x+h, Du_j(x)), \eta^2 \tau_h D\psi \Big\rangle \, \mathrm{d}x \\
&+ \int_\Omega \Big\langle \mathcal{A}_j(x+h, Du_j(x+h)) - \mathcal{A}_j(x+h, Du_j(x)), 2\eta \, D\eta \tau_h (u_j - \psi) \Big\rangle \, \mathrm{d}x \\
&+ \int_\Omega \Big\langle \mathcal{A}_j(x+h, Du_j(x)) - \mathcal{A}_j(x, Du_j(x)), \eta^2 \tau_h Du_j \Big\rangle \, \mathrm{d}x \\
&- \int_\Omega \Big\langle \mathcal{A}_j(x+h, Du_j(x)) - \mathcal{A}_j(x, Du_j(x)), \eta^2 \tau_h D\psi \Big\rangle \, \mathrm{d}x \\
&+ \int_\Omega \Big\langle \mathcal{A}_j(x+h, Du_j(x)) - \mathcal{A}_j(x, Du_j(x)), 2\eta \, D\eta \tau_h (u_j - \psi) \Big\rangle \, \mathrm{d}x \\
&=: I + II + III + IV + V + VI \le 0 \,,
\end{aligned}$$

which entails

$$I \le |II| + |III| + |IV| + |V| + |VI| \,. \tag{9.40}$$

By virtue of $(\mathcal{A}_j 1)$ we get

$$I \ge \nu \int_\Omega \eta^2 |\tau_h Du_j|^2 (1 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{-\frac{1}{2}} \, \mathrm{d}x \,. \tag{9.41}$$

Assumption $(\mathcal{A}_j 2)$ together with Young's inequality implies

$$\begin{aligned}
|II| &\le L \int_\Omega \eta^2 |\tau_h Du_j| (1 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{q-2}{2}} \, |\tau_h D\psi| \, \mathrm{d}x \\
&\le \vartheta \int_\Omega \eta^2 |\tau_h Du_j|^2 (1 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{-\frac{1}{2}} \, \mathrm{d}x \\
&\quad + C_\vartheta(L) \int_\Omega \eta^2 |\tau_h D\psi|^2 (1 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{2q-3}{2}} \, \mathrm{d}x.
\end{aligned}$$

Concerning the second summand, it holds

$$\begin{aligned}
(1 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{2q-3}{2}} &= \frac{(1 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{2q-2}{2}}}{(1 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{1}{2}}} \\
&\le (1 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{2q-2}{2}},
\end{aligned} \tag{9.42}$$

thus employing Hölder's inequality and using the properties of η we obtain

$$\begin{aligned}
|II| &\le \vartheta \int_\Omega \eta^2 |\tau_h Du_j|^2 (1 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{-\frac{1}{2}} \, \mathrm{d}x \\
&\quad + C_\vartheta(L) \left(\int_{B_t} |\tau_h D\psi|^r \, \mathrm{d}x \right)^{\frac{2}{r}} \left(\int_{B_{t'}} (1 + |Du_j(x)|)^{\frac{(2q-2)r}{r-2}} \, \mathrm{d}x \right)^{\frac{r-2}{r}} .
\end{aligned}$$

Remark 9.16 yields

$$\frac{(2q-2)r}{r-2} < \frac{r}{r-2} < \frac{n}{n-2} \,.$$

Hence, being $D^2\psi \in L^r_{\mathrm{loc}}(\Omega)$, we can use Lemma 8.2 together with Hölder's inequality to get

$$|II| \le \vartheta \int_\Omega \eta^2 |\tau_h Du_j|^2 (1+|Du_j(x+h)|^2+|Du_j(x)|^2)^{-\frac12}\,\mathrm{d}x$$
$$+ C_\vartheta(n,q,r,L,R)\,|h|^2 \left(\int_{B_{t'}} |D^2\psi|^r\,\mathrm{d}x\right)^{\frac{2}{r}} \left(\int_{B_{t'}} (1+|Du_j(x)|)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{(n-2)(2q-2)}{n}},$$

and by Young's inequality with conjugate exponents $\frac{1}{2q-2}$ and $\frac{1}{3-2q}$ (which are bigger than 1, see Remark 9.16) also

$$\begin{aligned}|II| &\le \vartheta \int_\Omega \eta^2 |\tau_h Du_j|^2 (1+|Du_j(x+h)|^2+|Du_j(x)|^2)^{-\frac12}\,\mathrm{d}x \\ &\quad + C_{\vartheta,\Theta}(n,q,r,L,R)\,|h|^2 \left(\int_{B_{R/2}} |D^2\psi|^r\,\mathrm{d}x\right)^{\frac{2}{(3-2q)r}} \\ &\quad + \Theta\,|h|^2 \left(\int_{B_{t'}} (1+|Du_j(x)|)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}}\end{aligned} \tag{9.43}$$

for some constant $\Theta = \Theta(n,q,r)$, $0<\Theta<1$, that we will specify later. Similarly, we also get

$$\begin{aligned}|III| &\le 4L \int_\Omega |\tau_h Du_j|\,|D\eta|\,\eta\,(1+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{q-2}{2}}\,|\tau_h(u_j-\psi)|\,\mathrm{d}x \\ &\le \vartheta \int_\Omega \eta^2 |\tau_h Du_j|^2\,(1+|Du_j(x+h)|^2+|Du_j(x)|^2)^{-\frac12}\,\mathrm{d}x \\ &\quad + C_\vartheta(L) \int_\Omega |D\eta|^2 (1+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{2q-3}{2}} |\tau_h u_j|^2\,\mathrm{d}x \\ &\quad + C_\vartheta(L) \int_\Omega |D\eta|^2 (1+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{2q-3}{2}} |\tau_h \psi|^2\,\mathrm{d}x\,.\end{aligned}$$

By the fact that $|D\eta| \le \frac{C}{t-s}$ and Hölder's inequality we deduce

$$\begin{aligned}|III| &\le \vartheta \int_\Omega \eta^2 |\tau_h Du_j|^2\,(1+|Du_j(x+h)|^2+|Du_j(x)|^2)^{-\frac12}\,\mathrm{d}x \\ &\quad + \frac{C_\vartheta(L)}{(t-s)^2} \left(\int_{B_{t'}} (1+|Du_j(x)|)^{2q-1}\,\mathrm{d}x\right)^{\frac{2q-3}{2q-1}} \left(\int_{B_t} |\tau_h u_j|^{2q-1}\,\mathrm{d}x\right)^{\frac{2}{2q-1}} \\ &\quad + \frac{C_\vartheta(L)}{(t-s)^2} \left(\int_{B_{t'}} (1+|Du_j(x)|)^{\frac{(2q-3)r}{r-2}}\,\mathrm{d}x\right)^{\frac{r-2}{r}} \left(\int_{B_t} |\tau_h \psi|^r\,\mathrm{d}x\right)^{\frac{2}{r}} \\ &\le \vartheta \int_\Omega \eta^2 |\tau_h Du_j|^2\,(1+|Du_j(x+h)|^2+|Du_j(x)|^2)^{-\frac12}\,\mathrm{d}x \\ &\quad + \frac{C_\vartheta(n,q,L)}{(t-s)^2} \int_{B_{t'}} (1+|Du_j(x)|)^{2q-1}\,\mathrm{d}x \\ &\quad + \frac{C_\vartheta(n,r,L)}{(t-s)^2} \left(\int_{B_{t'}} (1+|Du_j(x)|)^{\frac{(2q-3)r}{r-2}}\,\mathrm{d}x\right)^{\frac{r-2}{r}} \left(\int_{B_R} |D\psi(x)|^r\,\mathrm{d}x\right)^{\frac{2}{r}}.\end{aligned} \tag{9.44}$$

Here we used Lemma 8.2 as long as $D\psi \in L^r_{\mathrm{loc}}(\Omega)$. Now, assumption (9.35) entails

$$2q-1 < 1+\frac{2}{n}-\frac{2}{r} < 1+\frac{2}{n} < \frac{n}{n-2}\,.$$

By means of the interpolation inequality

$$\|Dv\|_{L^{2q-1}(\Omega')} \leq \|Dv\|_{L^1(\Omega')}^{\delta_0} \|Dv\|_{L^{\frac{n}{n-2}}(\Omega')}^{1-\delta_0} \quad \forall \, \Omega' \Subset \Omega \,,$$

with $0 < \delta_0 < 1$ defined as

$$\frac{1}{2q-1} = \delta_0 + \frac{(1-\delta_0)(n-2)}{n}$$

that is

$$\delta_0 = \frac{n-1-q(n-2)}{2q-1}\,, \qquad 1-\delta_0 = \frac{n(q-1)}{2q-1}\,,$$

we obtain

$$\begin{aligned}
&\int_{B_{t'}} (1+|Du_j(x)|)^{2q-1}\,\mathrm{d}x \\
&\leq \left(\int_{B_{t'}} (1+|Du_j(x)|)\,\mathrm{d}x\right)^{\delta_0(2q-1)} \left(\int_{B_{t'}} (1+|Du_j(x)|)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{(q-1)(n-2)}{}} .
\end{aligned} \tag{9.45}$$

Inserting (9.45) in (9.44) and using Young's inequality with conjugate exponents $\frac{1}{n(q-1)}$ and $\frac{1}{1-n(q-1)}$ (which are bigger than 1 thanks to (9.35)), with the aid of (9.42) and the subsequent computations we conclude that

$$\begin{aligned}
|III| &\leq \vartheta \int_\Omega \eta^2 |\tau_h Du_j|^2 \,(1+|Du_j(x+h)|^2+|Du_j(x)|^2)^{-\frac{1}{2}}\,\mathrm{d}x \\
&\quad + |h|^2 \,\frac{C_{\vartheta,\Theta}(n,q,L)}{(t-s)^{\frac{2}{1-n(q-1)}}} \left(\int_{B_R} (1+|Du_j(x)|)\,\mathrm{d}x\right)^{\frac{\delta_0(2q-1)}{1-n(q-1)}} \\
&\quad + |h|^2 \,\frac{C_{\vartheta,\Theta}(n,q,r,L,R)}{(t-s)^{\frac{2}{3-2q}}} \left(\int_{B_R} |D\psi(x)|^r\,\mathrm{d}x\right)^{\frac{2}{(3-2q)r}} \\
&\quad + \Theta\,|h|^2 \left(\int_{B_{t'}} (1+|Du_j(x)|)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}}
\end{aligned} \tag{9.46}$$

with Θ as above. In order to estimate the integral IV, we use assumption $(\mathcal{A}_j4)$ followed by Young's and Hölder's inequalities to obtain

$$\begin{aligned}
|IV| &\leq |h| \int_\Omega \eta^2 (\kappa(x+h)+\kappa(x))(1+|Du_j(x)|^2)^{\frac{q-1}{2}}\,|\tau_h Du_j|\,\mathrm{d}x \\
&\leq \vartheta \int_\Omega \eta^2 |\tau_h Du_j|^2 \,\left(1+|Du_j(x)|^2+|Du_j(x+h)|^2\right)^{-\frac{1}{2}}\,\mathrm{d}x \\
&\quad + C_\vartheta |h|^2 \int_{B_t} (\kappa(x+h)+\kappa(x))^2 (1+|Du_j(x)|^2)^{\frac{2q-1}{2}}\,\mathrm{d}x \\
&\leq \vartheta \int_\Omega \eta^2 |\tau_h Du_j|^2 \,\left(1+|Du_j(x)|^2+|Du_j(x+h)|^2\right)^{-\frac{1}{2}}\,\mathrm{d}x \\
&\quad + C_\vartheta |h|^2 \left(\int_{B_t} (\kappa(x+h)+\kappa(x))^r\,\mathrm{d}x\right)^{\frac{2}{r}} \left(\int_{B_t} (1+|Du_j(x)|)^{\frac{(2q-1)r}{r-2}}\,\mathrm{d}x\right)^{\frac{r-2}{r}} ,
\end{aligned} \tag{9.47}$$

where we also used the fact that $\mathrm{supp}\eta \subset B_t$. On the other hand, assumption (9.35) implies

$$\frac{(2q-1)r}{r-2} < \frac{n}{n-2} \tag{9.48}$$

in a sharp way, as long as $r > n$. The proof of this fact can be carried out as in Remark 9.9. By means of the interpolation inequality

$$\|Dv\|_{L^{\frac{(2q-1)r}{r-2}}(\Omega')} \leq \|Dv\|^{\delta}_{L^1(\Omega')} \|Dv\|^{1-\delta}_{L^{\frac{n}{n-2}}(\Omega')} \qquad \forall\, \Omega' \Subset \Omega\,,$$

with $0 < \delta < 1$ defined as

$$\frac{r-2}{(2q-1)r} = \delta + \frac{(1-\delta)(n-2)}{n}$$

that is

$$\delta = \frac{(nr-n-r)-q(nr-2r)}{(2q-1)r}\,, \qquad\qquad 1-\delta = \frac{n(qr-r+1)}{(2q-1)r}\,,$$

we obtain

$$\begin{aligned}
&\left(\int_{B_t} (1+|Du_j(x)|)^{\frac{(2q-1)r}{r-2}}\,\mathrm{d}x\right)^{\frac{r-2}{r}} \\
&\leq \left(\int_{B_t} (1+|Du_j(x)|)\,\mathrm{d}x\right)^{\delta(2q-1)} \left(\int_{B_t} (1+|Du_j(x)|)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{(qr-r+1)(n-2)}{r}}.
\end{aligned} \tag{9.49}$$

Inserting (9.49) in (9.47), using the second inequality of Lemma 8.2 and Young's inequality with conjugate exponents $\frac{r}{n(qr-r+1)}$ and $\frac{r}{r-n(qr-r+1)}$ (which are bigger than 1 thanks to (9.35)), we obtain

$$\begin{aligned}
|IV| &\leq \vartheta \int_\Omega \eta^2 |\tau_h Du_j|^2 \left(1+|Du_j(x)|^2+|Du_j(x+h)|^2\right)^{-\frac{1}{2}}\,\mathrm{d}x \\
&\quad + C_\vartheta |h|^2 \left(\int_{B_R} \kappa^r(x)\,\mathrm{d}x\right)^{\frac{2}{r}} \left(\int_{B_R} (1+|Du_j(x)|)\,\mathrm{d}x\right)^{\delta(2q-1)} \\
&\quad\ \cdot \left(\int_{B_t} (1+|Du_j(x)|)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{(qr-r+1)(n-2)}{r}} \\
&\leq \vartheta \int_\Omega \eta^2 |\tau_h Du_j|^2 \left(1+|Du_j(x)|^2+|Du_j(x+h)|^2\right)^{-\frac{1}{2}}\,\mathrm{d}x \\
&\quad + C_{\vartheta,\Theta}\, |h|^2 \left(\int_{B_R} \kappa^r(x)\,\mathrm{d}x\right)^{\frac{2}{r-n(qr-r+1)}} \left(\int_{B_R} (1+|Du_j(x)|)\,\mathrm{d}x\right)^{\frac{\delta(2q-1)r}{r-n(qr-r+1)}} \\
&\quad + \Theta\, |h|^2 \left(\int_{B_t} (1+|Du_j(x)|)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}}
\end{aligned} \tag{9.50}$$

with Θ as above. Assumption $(\mathcal{A}_j 4)$ also yields

$$\begin{aligned}
|V| &\leq |h| \int_\Omega \eta^2 (\kappa(x+h)+\kappa(x))(1+|Du_j(x)|^2)^{\frac{q-1}{2}}\, |\tau_h D\psi|\,\mathrm{d}x \\
&\leq |h| \left(\int_{B_t} (\kappa(x+h)+\kappa(x))^r\,\mathrm{d}x\right)^{\frac{1}{r}} \left(\int_{B_t} (1+|Du_j(x)|)^{\frac{(q-1)r}{r-2}}\,\mathrm{d}x\right)^{\frac{r-2}{r}} \left(\int_{B_t} |\tau_h D\psi|^r\,\mathrm{d}x\right)^{\frac{1}{r}} \\
&\leq C(n,r)\, |h|^2 \left(\int_{B_R} \kappa^r(x)\,\mathrm{d}x\right)^{\frac{1}{r}} \left(\int_{B_t} (1+|Du_j(x)|)^{\frac{(q-1)r}{r-2}}\,\mathrm{d}x\right)^{\frac{r-2}{r}} \left(\int_{B_R} |D^2\psi|^r\,\mathrm{d}x\right)^{\frac{1}{r}}
\end{aligned}$$

where we used Hölder's inequality, the properties of η and Lemma 8.2 by virtue of the fact that $D^2\psi \in L^r_{\mathrm{loc}}(\Omega)$. On the other hand, Remark 9.16 yields

$$\frac{(q-1)r}{r-2} < \frac{r}{r-2} < \frac{n}{n-2}\,. \tag{9.51}$$

Hence we can apply Hölder's inequality again to obtain

$$|V| \le C(n,q,r,R)\,|h|^2 \left(\int_{B_R} \kappa^r(x)\,\mathrm{d}x\right)^{\frac{1}{r}} \left(\int_{B_t} (1+|Du_j(x)|)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{(n-2)(q-1)}{n}} \left(\int_{B_R} |D^2\psi|^r\,\mathrm{d}x\right)^{\frac{1}{r}},$$

and by Young's inequality with conjugate exponents $\frac{1}{q-1}$ and $\frac{1}{2-q}$ (which are bigger than 1, see Remark 9.16) also

$$\begin{aligned} |V| &\le C_\Theta(n,q,r,R)\,|h|^2 \left(\int_{B_R} \kappa^r(x)\,\mathrm{d}x\right)^{\frac{1}{(2-q)r}} \left(\int_{B_R} |D^2\psi|^r\,\mathrm{d}x\right)^{\frac{1}{(2-q)r}} \\ &\quad + \Theta\,|h|^2 \left(\int_{B_t} (1+|Du_j(x)|)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}} \end{aligned} \tag{9.52}$$

with Θ as above. Finally, assumption $(\mathcal{A}_j 4)$ and the properties of $D\eta$ yield

$$\begin{aligned} |VI| &\le 2|h| \int_\Omega \eta\,|D\eta|(\kappa(x+h)+\kappa(x))(1+|Du_j(x)|^2)^{\frac{q-1}{2}}\,|\tau_h(u_j-\psi)|\,\mathrm{d}x \\ &\le 2|h| \int_\Omega \eta\,|D\eta|(\kappa(x+h)+\kappa(x))(1+|Du_j(x)|^2)^{\frac{q-1}{2}}\,|\tau_h u_j|\,\mathrm{d}x \\ &\quad + 2|h| \int_\Omega \eta\,|D\eta|(\kappa(x+h)+\kappa(x))(1+|Du_j(x)|^2)^{\frac{q-1}{2}}\,|\tau_h \psi|\,\mathrm{d}x \\ &\le |h|\,\frac{C}{(t-s)^2} \left(\int_{B_t} |\tau_h u_j|^{\frac{qr}{r-1}}\,\mathrm{d}x\right)^{\frac{r-1}{qr}} \left(\int_{B_t} (\kappa(x+h)+\kappa(x))^r\,\mathrm{d}x\right)^{\frac{1}{r}} \\ &\quad \cdot \left(\int_{B_t} (1+|Du_j(x)|)^{\frac{qr}{r-1}}\,\mathrm{d}x\right)^{\frac{(q-1)(r-1)}{qr}} \\ &\quad + |h|\,\frac{C}{(t-s)^2} \left(\int_{B_t} |\tau_h \psi|^r\,\mathrm{d}x\right)^{\frac{1}{r}} \left(\int_{B_t} (\kappa(x+h)+\kappa(x))^r\,\mathrm{d}x\right)^{\frac{1}{r}} \\ &\quad \cdot \left(\int_{B_t} (1+|Du_j(x)|)^{\frac{(q-1)r}{r-2}}\,\mathrm{d}x\right)^{\frac{r-2}{r}} \end{aligned} \tag{9.53}$$

where we used also Hölder's inequality. Moreover, since $D\psi \in L^r_{\mathrm{loc}}(\Omega)$ we can employ Lemma 8.2 to obtain

$$\begin{aligned} |VI| &\le |h|^2\,\frac{C(n,q,r)}{(t-s)^2} \left(\int_{B_{t'}} (1+|Du_j(x)|)^{\frac{qr}{r-1}}\,\mathrm{d}x\right)^{\frac{r-1}{r}} \left(\int_{B_R} \kappa^r(x)\,\mathrm{d}x\right)^{\frac{1}{r}} \\ &\quad + |h|^2\,\frac{C(n,r)}{(t-s)^2} \left(\int_{B_R} |D\psi(x)|^r\,\mathrm{d}x\right)^{\frac{1}{r}} \left(\int_{B_R} \kappa^r(x)\,\mathrm{d}x\right)^{\frac{1}{r}} \left(\int_{B_t} (1+|Du_j(x)|)^{\frac{(q-1)r}{r-2}}\,\mathrm{d}x\right)^{\frac{r-2}{r}}. \end{aligned}$$

On the other hand, the fact that $q>1$ and the sharp inequality (9.48) imply

$$\frac{qr}{r-1} < \frac{(2q-1)r}{r-2} < \frac{n}{n-2}\,.$$

By means of the interpolation inequality

$$\|Dv\|_{L^{\frac{qr}{r-1}}(\Omega')} \le \|Dv\|^{\delta'}_{L^1(\Omega')}\,\|Dv\|^{1-\delta'}_{L^{\frac{n}{n-2}}(\Omega')} \qquad \forall\,\Omega' \Subset \Omega\,,$$

with $0<\delta'<1$ defined as

$$\frac{r-1}{qr} = \delta' + \frac{(1-\delta')(n-2)}{n}$$

that is

$$\delta' = \frac{(nr-n) - q(nr-2r)}{2qr}\,, \qquad\qquad 1-\delta' = \frac{n(qr-r+1)}{2qr}\,,$$

we obtain

$$\begin{aligned}
&\left(\int_{B_{t'}} (1+|Du_j(x)|)^{\frac{qr}{r-1}}\,\mathrm{d}x\right)^{\frac{r-1}{r}} \\
&\leq \left(\int_{B_{t'}} (1+|Du_j(x)|)\,\mathrm{d}x\right)^{\delta' q} \left(\int_{B_{t'}} (1+|Du_j(x)|)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{(n-2)(qr-r+1)}{2r}}\,.
\end{aligned} \tag{9.54}$$

Inserting (9.54) into (9.53) and using Young's inequality with conjugate exponents $\frac{2r}{n(qr-r+1)}$ and $\frac{2r}{2r-n(qr-r+1)}$ (which are bigger that 1 thanks to (9.35)), with the aid of (9.51) and the subsequent computations we finally obtain

$$\begin{aligned}
|VI| \leq{}& |h|^2 \frac{C_\Theta(n,q,r)}{(t-s)^{\frac{4r}{2r-n(qr-r+1)}}} \left(\int_{B_R}(1+|Du_j(x)|)\,\mathrm{d}x\right)^{\frac{\delta' 2qr}{2r-n(qr-r+1)}} \left(\int_{B_R}\kappa^r(x)\,\mathrm{d}x\right)^{\frac{2}{2r-n(qr-r+1)}} \\
&+ |h|^2 \frac{C_\Theta(n,q,r,R)}{(t-s)^{\frac{2}{2-q}}} \left(\int_{B_R}\kappa^r(x)\,\mathrm{d}x\right)^{\frac{1}{(2-q)r}} \left(\int_{B_R}|D\psi(x)|^r\,\mathrm{d}x\right)^{\frac{1}{(2-q)r}} \\
&+ \Theta\,|h|^2 \left(\int_{B_{t'}} (1+|Du_j(x)|)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}}
\end{aligned} \tag{9.55}$$

with Θ as above. Inserting estimates (9.41), (9.43), (9.46), (9.50), (9.52) and (9.55) in (9.40), we infer

$$
\begin{aligned}
&\nu \int_\Omega \eta^2 |\tau_h Du_j|^2 (1+|Du_j(x+h)|^2+|Du_j(x)|^2)^{-\frac{1}{2}}\,\mathrm{d}x \\
&\le 3\vartheta \int_\Omega \eta^2 |\tau_h Du_j|^2 (1+|Du_j(x+h)|^2+|Du_j(x)|^2)^{-\frac{1}{2}}\,\mathrm{d}x \\
&+ 2\Theta\,|h|^2 \left(\int_{B_t} (1+|Du_j(x)|)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}} + 3\Theta\,|h|^2 \left(\int_{B_{t'}} (1+|Du_j(x)|)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}} \\
&+ C_{\vartheta,\Theta}(n,q,r,L,R)\,|h|^2 \left(\int_{B_R} |D^2\psi|^r\,\mathrm{d}x\right)^{\frac{2}{(3-2q)r}} \\
&+ |h|^2 \,\frac{C_{\vartheta,\Theta}(n,q,L)}{(t-s)^{\frac{2}{1-n(q-1)}}} \left(\int_{B_R} (1+|Du_j(x)|)\,\mathrm{d}x\right)^{\frac{\delta_0(2q-1)}{1-n(q-1)}} \\
&+ |h|^2 \frac{C_{\vartheta,\Theta}(n,q,r,L,R)}{(t-s)^{\frac{2}{3-2q}}} \left(\int_{B_R} |D\psi(x)|^r\,\mathrm{d}x\right)^{\frac{2}{(3-2q)r}} \\
&+ C_{\vartheta,\Theta}\,|h|^2 \left(\int_{B_R} \kappa^r(x)\,\mathrm{d}x\right)^{\frac{2}{r-n(qr-r+1)}} \left(\int_{B_R} (1+|Du_j(x)|)\,\mathrm{d}x\right)^{\frac{\delta(2q-1)r}{r-n(qr-r+1)}} \\
&+ C_\Theta(n,q,r,R)\,|h|^2 \left(\int_{B_R} \kappa^r(x)\,\mathrm{d}x\right)^{\frac{1}{(2-q)r}} \left(\int_{B_R} |D^2\psi|^r\,\mathrm{d}x\right)^{\frac{1}{(2-q)r}} \\
&+ |h|^2 \,\frac{C_\Theta(n,q,r)}{(t-s)^{\frac{4r}{2r-n(qr-r+1)}}} \left(\int_{B_R} (1+|Du_j(x)|)\,\mathrm{d}x\right)^{\frac{\delta' 2qr}{2r-n(qr-r+1)}} \left(\int_{B_R} \kappa^r(x)\,\mathrm{d}x\right)^{\frac{2}{2r-n(qr-r+1)}} \\
&+ |h|^2 \frac{C_\Theta(n,q,r,R)}{(t-s)^{\frac{2}{2-q}}} \left(\int_{B_R} \kappa^r(x)\,\mathrm{d}x\right)^{\frac{1}{(2-q)r}} \left(\int_{B_R} |D\psi(x)|^r\,\mathrm{d}x\right)^{\frac{1}{(2-q)r}} \\
&=: 3\vartheta \int_\Omega \eta^2 |\tau_h Du|^2 (1+|Du_j(x+h)|^2+|Du_j(x)|^2)^{-\frac{1}{2}}\,\mathrm{d}x + 2\Theta\,|h|^2 \left(\int_{B_t} (1+|Du_j(x)|)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}} \\
&+ 3\Theta\,|h|^2 \left(\int_{B_{t'}} (1+|Du_j(x)|)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}} \\
&+ |h|^2 A + |h|^2 \frac{C_1}{(t-s)^{\gamma_1}} + |h|^2 \frac{C_2}{(t-s)^{\gamma_2}} + |h|^2 \frac{C_3}{(t-s)^{\gamma_3}} + |h|^2 \frac{C_4}{(t-s)^{\gamma_4}}\,.
\end{aligned}
$$

Choosing $\vartheta = \frac{\nu}{6}$, we can reabsorb the first integral on the right-hand side of the previous estimate by the left-hand side, thus obtaining

$$
\begin{aligned}
&\nu \int_\Omega \eta^2 |\tau_h Du_j|^2 (1+|Du_j(x+h)|^2+|Du_j(x)|^2)^{-\frac{1}{2}}\,\mathrm{d}x \\
&\le 2\Theta\,|h|^2 \left(\int_{B_t} (1+|Du_j(x)|)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}} + 3\Theta\,|h|^2 \left(\int_{B_{t'}} (1+|Du_j(x)|)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}} \\
&+ |h|^2 A + |h|^2 \frac{C_1}{(t-s)^{\gamma_1}} + |h|^2 \frac{C_2}{(t-s)^{\gamma_2}} + |h|^2 \frac{C_3}{(t-s)^{\gamma_3}} + |h|^2 \frac{C_4}{(t-s)^{\gamma_4}}\,.
\end{aligned}
$$

Using Lemma 8.4 with $\gamma = -1/4$ we have

$$\begin{aligned}
&\frac{|\tau_h Du_j|^2}{(1+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{1}{2}}}\\
&= \left(\frac{|Du_j(x+h)-Du_j(x)|}{(1+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{1}{4}}}\right)^2\\
&\geq c(n)|(1+|Du_j(x+h)|^2)^{-\frac{1}{4}}Du_j(x+h)-(1+|Du_j(x)|^2)^{-\frac{1}{4}}Du_j(x)|^2\\
&= c(n)|\tau_h V_1(Du_j)|^2,
\end{aligned}$$

where V_1 was defined in (8.1). Therefore, inserting this last estimate in the previous one and recalling that $\eta \equiv 1$ on B_s, we get

$$\begin{aligned}
&\int_{B_s} |\tau_h V_1(Du_j)|^2 \,\mathrm{d}x\\
&\leq \Bigg(2\Theta\left(\int_{B_t}(1+|Du_j(x)|)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}} + 3\Theta\left(\int_{B_{t'}}(1+|Du_j(x)|)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}}\\
&+ A + \frac{C_1}{(t-s)^{\gamma_1}} + \frac{C_2}{(t-s)^{\gamma_2}} + \frac{C_3}{(t-s)^{\gamma_3}} + \frac{C_4}{(t-s)^{\gamma_4}}\Bigg)|h|^2,
\end{aligned} \tag{9.56}$$

where we absorbed ν in the constants on the right-hand side. We now apply Lemma 8.3 with $F = V_1(Du)$ and $p = 2$, obtaining

$$\begin{aligned}
&\left(\int_{B_s}|V_1(Du_j)|^{\frac{2n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}}\\
&\leq 2\Theta\left(\int_{B_t}(1+|Du_j(x)|)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}} + 3\Theta\left(\int_{B_{t'}}(1+|Du_j(x)|)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}}\\
&+ A + \frac{C_1}{(t-s)^{\gamma_1}} + \frac{C_2}{(t-s)^{\gamma_2}} + \frac{C_3}{(t-s)^{\gamma_3}} + \frac{C_4}{(t-s)^{\gamma_4}}\,.
\end{aligned} \tag{9.57}$$

Hence, setting

$$\phi(r) = \left(\int_{B_r}|Du_j(x)|^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}}$$

and possibly redefining the constants, exploiting (8.3) we can rewrite inequality (9.57) as

$$\phi(s) \leq 2\Theta\,\phi(t) + 3\Theta\,\phi(t') + A + \frac{C_1}{(t-s)^{\gamma_1}} + \frac{C_2}{(t-s)^{\gamma_2}} + \frac{C_3}{(t-s)^{\gamma_3}} + \frac{C_4}{(t-s)^{\gamma_4}}\,.$$

Note that we can choose the constant Θ small enough to satisfy $\Theta < 1/3 < 1/2$. Thus we can apply Lemma 8.6 with $B = 0$ obtaining

$$\phi(\rho) \leq c\left(2\Theta\,\phi(t') + A + \frac{C_1}{(t'-\rho)^{\gamma_1}} + \frac{C_2}{(t'-\rho)^{\gamma_2}} + \frac{C_3}{(t'-\rho)^{\gamma_3}} + \frac{C_4}{(t'-\rho)^{\gamma_4}}\right)$$

for some $c = c\,(\Theta, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$. But then, employing Lemma 8.6 one more time we get

$$\phi\left(\frac{R}{4}\right) \leq \tilde{c}\left(A + \frac{C_1}{R^{\gamma_1}} + \frac{C_2}{R^{\gamma_2}} + \frac{C_3}{R^{\gamma_3}} + \frac{C_4}{R^{\gamma_4}}\right)$$

with $\tilde{c} = \tilde{c}(\Theta, \gamma_1, \gamma_2, \gamma_3, \gamma_4) = \tilde{c}(n, q, r)$. Recalling the definition of ϕ and using Lemma 8.3, from inequality (9.56) and our hypotheses on the data we infer the a priori estimate

$$\int_{B_{R/8}} |D\left(V_1(Du_j(x))\right)|^2 \,\mathrm{d}x \le C \left(\int_{B_{R/2}} (1 + |Du_j(x)|) \,\mathrm{d}x \right)^{\sigma} \tag{9.58}$$

with $C = C(\nu, n, q, r, L, R)$ and

$$\sigma := \max\left\{ \frac{\delta(2q-1)r}{r - n(qr - r + 1)}, \frac{\delta_0(2q-1)}{1 - n(q-1)}, \frac{\delta' 2qr}{2r - n(qr - r + 1)} \right\}.$$

Step 2: Passage to the limit. We conclude the proof by passing to the limit in the approximating problem. For simplicity, here we denote by $\mathrm{o}(j)$ a quantity such that $\lim_{j\to\infty} \mathrm{o}(j) = 0$. By the growth condition ($F_{\mathrm{nl}}1$), the minimality of u_j for F_j and the definition of F_j in (9.37) we have

$$\begin{aligned} \frac{1}{\tilde{\ell}} \int_{B_R} \bar{F}(|Du_j(x)|) \,\mathrm{d}x &\le \int_{B_R} F_j(x, D\tilde{u}_j(x)) \,\mathrm{d}x \\ &= \int_{B_R} F(x, D\tilde{u}_j(x)) \,\mathrm{d}x + \frac{\varepsilon_j}{q} \int_{B_R} \left(1 + |D\tilde{u}_j(x)|^2\right)^{\frac{q}{2}} \,\mathrm{d}x \end{aligned} \tag{9.59}$$

where $\tilde{u}_j$ is as in (9.36). The definition of ε_j in (9.38) entails

$$\frac{\varepsilon_j}{q} \int_{B_R} \left(1 + |D\tilde{u}_j(x)|^2\right)^{\frac{q}{2}} \,\mathrm{d}x = \frac{1}{q\left(1 + j + \|D\tilde{u}_j\|^{2q}_{L^q(B_R)}\right)} \int_{B_R} \left(1 + |D\tilde{u}_j(x)|^2\right)^{\frac{q}{2}} \,\mathrm{d}x \xrightarrow[j\to\infty]{} 0\,,$$

hence we obtain, using also (9.36),

$$\frac{1}{\tilde{\ell}} \int_{B_R} \bar{F}(|Du_j(x)|) \,\mathrm{d}x \le \int_{B_R} F(x, D\tilde{u}_j(x)) \,\mathrm{d}x + \mathrm{o}(j) = \int_{B_R} F(x, Du(x)) \,\mathrm{d}x + \mathrm{o}(j)\,. \tag{9.60}$$

The fact that we are assuming $F(x, Du) \in L^1_{\mathrm{loc}}(\Omega)$ implies that the sequence $\{\bar{F}(|Du_j|)\}$ is bounded in $L^1(B_R)$. Now, (9.32) and the Dunford-Pettis criterion imply that there exists $\bar{u} \in u + W^{1,1}_0(B_R)$ such that

$$u_j \rightharpoonup \bar{u} \quad \text{weakly in } W^{1,1}_0(B_R)\,.$$

Our next step is to show that $\bar{u}$ is a solution to our obstacle problem over the ball B_R, that is, it minimizes the functional $\int_\Omega F(x, Dw) \,\mathrm{d}x$ in $u + W^{1,1}_0(B_R)$. To this aim note that (9.37) and (9.59)–(9.60) entail

$$\int_{B_R} F(x, Du_j(x)) \,\mathrm{d}x \le \int_{B_R} F_j(x, Du_j(x)) \,\mathrm{d}x \le \int_{B_R} F(x, Du(x)) \,\mathrm{d}x + \mathrm{o}(j).$$

Hence by the lower semicontinuity of the map $v \mapsto \int_{B_R} F(x, Dv(x)) \,\mathrm{d}x$ we get

$$\int_{B_R} F(x, D\bar{u}(x)) \,\mathrm{d}x \le \liminf_{j\to\infty} \int_{B_R} F(x, Du_j(x)) \,\mathrm{d}x \le \int_{B_R} F(x, Du(x)) \,\mathrm{d}x.$$

Moreover, since $u_j \in \mathcal{K}_\psi(\Omega)$ for every $j \in \mathbb{N}$ and $\mathcal{K}_\psi(\Omega)$ is a closed set, we have $\bar{u} \in \mathcal{K}_\psi(\Omega)$. We have then proved that the limit function $\bar{u} \in u + W^{1,1}_0(B_R)$ is a solution to the minimization problem

$$\min\left\{ \int_\Omega F(x, Dw) \,\mathrm{d}x : w \in u + W^{1,1}_0(B_R),\ w \in \mathcal{K}_\psi(\Omega) \right\}.$$

Note also that the strict convexity of F yields $\bar{u} = u$. It only remains to show that the minimizer has the regularity stated in Theorem 9.17. From estimate (9.58) and by compact embedding we infer

$$V_1(Du_j) \rightharpoonup v \quad \text{weakly in } W^{1,2}_{\text{loc}}(\Omega),$$
$$V_1(Du_j) \to v \quad \text{strongly in } L^2_{\text{loc}}(\Omega),$$

from which we deduce, together with inequality (8.3),

$$Du_j \to \bar{w} \quad \text{strongly in } L^1_{\text{loc}}(\Omega).$$

We thus have the strong convergence

$$u_j \to \bar{u} = u \quad \text{strongly in } u + W^{1,1}_0(B_R).$$

Hence we can pass to the limit in estimate (9.58), which allows us to conclude that $V_1(Du) \in W^{1,2}_{\text{loc}}(\Omega)$. Recalling the definition of V_1 in (8.1), we see that this is exactly the claim of Theorem 9.17.

Final remark

The higher differentiability results contained in this last part of the thesis have already been used to prove further regularity results. In particular, Theorem 9.8 is employed in [32], where the authors prove the local Lipschitz continuity of solutions to a quite large class of variational inequalities whose principal part satisfies nonstandard growth conditions. This was the first result about Lipschitz regularity for the obstacle problem with p, q-growth. In the same spirit, Theorem 9.17 has been recently used in [11] to prove Lipschitz continuity results for functionals with nearly-linear growth conditions.

Appendix

APPENDIX A

Auxiliary tools

This Appendix contains several results and concepts which are recalled and used in this thesis. They are all presented without proof, but we quote in each case some references where further details may be found. Here Ω will be a nonempty open bounded connected subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, with Lipschitz boundary.

A.1 Some useful results

A.1.1 Grönwall's lemma

The result stated below is well-known and fundamental when dealing with differential equations. Its proof can be found for example in [59].

Lemma A.1 (Grönwall's Lemma – differential form)**.** *Let η be a nonnegative, absolutely continuous function on $[0,T]$, which satisfies for a. e. t the differential inequality*

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t),$$

where ϕ, ψ are nonnegative, summable functions on $[0,T]$. Then

$$\eta(t) \leq \mathrm{e}^{\int_0^t \phi(s)\,\mathrm{d}s}\left(\eta(0) + \int_0^t \psi(s)\,\mathrm{d}s\right)$$

for all $0 \leq t \leq T$.

Lemma A.2 (Grönwall's Lemma – integral form)**.** *Let ξ be a summable function on $[0,T]$ which satisfies for a. e. t the integral inequality*

$$\xi(t) \leq \beta(t) + \int_0^t \alpha(s)\xi(s)\,\mathrm{d}s,$$

where α, β are summable functions on $[0,T]$ and, in addition, α is nonnegative. Then

$$\xi(t) \leq \beta(t) + \int_0^t \alpha(s)\beta(s)\,\mathrm{e}^{\int_s^t \alpha(r)\,\mathrm{d}r}\,\mathrm{d}s$$

for all $0 \le t \le T$. *If, in addition,* β *is nondecreasing, then*

$$\xi(t) \le \beta(t)\, \mathrm{e}^{\int_0^t \alpha(s)\,\mathrm{d}s}$$

for all $0 \le t \le T$.

A.1.2 Anisotropic embedding theorems

In evolution problems, one deals with functions depending on a space variable $x \in \Omega$ and time $t \in \omega$, where $\omega \subset \mathbb{R}$ is an open interval corresponding to the time of the process. It is thus natural to expect that the partial derivatives of u with respect to space and time are integrable with different exponents. For $1 \le p, q \le \infty$ we introduce the norms

$$|u|_{p,q,\Omega,\omega} := \begin{cases} \left(\displaystyle\int_\omega |u(\cdot,t)|^p_{q,\Omega}\,\mathrm{d}t \right)^{1/p} & \text{if } 1 \le p < \infty, \\ \displaystyle\sup_{t\in\omega} \mathrm{ess}\, |u(\cdot,t)|_{q,\Omega} & \text{if } p = \infty, \end{cases} \tag{A.1}$$

where $|\cdot|_{q,\Omega}$ denotes the usual norm of $L^q(\Omega)$, and the spaces

$$L^p(\omega; L^q(\Omega)) := \left\{ u \in L^1(\Omega \times \omega) : |u|_{p,q,\Omega,\omega} < \infty \right\}.$$

We further define

$$\begin{aligned} W^{1,(p_0,q_0);(p_1,q_1)}(\omega;\Omega) := \Bigg\{ & u \in L^1(\Omega\times\omega) : \frac{\partial u}{\partial t} \in L^{p_0}(\omega; L^{q_0}(\Omega)), \\ & \frac{\partial u}{\partial x_i} \in L^{p_1}(\omega; L^{q_1}(\Omega)) \text{ for } i = 1,\dots,N \Bigg\}. \end{aligned}$$

Here we report two anisotropic embedding theorems established in [12].

Theorem A.3. *If* $q_2 \ge \max\{q_0, q_1\}$, $p_2 \ge \max\{p_0, p_1\}$ *and*

$$\left(1 - \frac{1}{p_0} + \frac{1}{p_2}\right)\left(\frac{1}{N} - \frac{1}{q_1} + \frac{1}{q_2}\right) > \left(\frac{1}{p_1} - \frac{1}{p_2}\right)\left(\frac{1}{q_0} - \frac{1}{q_2}\right), \tag{A.2}$$

then $W^{1,(p_0,q_0);(p_1,q_1)}(\omega;\Omega)$ *is compactly embedded in* $L^{p_2}(\omega; L^{q_2}(\Omega))$. *If moreover* (A.2) *holds for* $q_2 = \infty$ *with the convention* $1/\infty = 0$, *that is,*

$$\left(1 - \frac{1}{p_0} + \frac{1}{p_2}\right)\left(\frac{1}{N} - \frac{1}{q_1}\right) > \frac{1}{q_0}\left(\frac{1}{p_1} - \frac{1}{p_2}\right),$$

then $W^{1,(p_0,q_0);(p_1,q_1)}(\omega;\Omega)$ *is compactly embedded in* $L^{p_2}(\omega; C(\bar{\Omega}))$. *If* (A.2) *holds for* $p_2 = q_2 = \infty$, *that is,*

$$\frac{p_0'}{p_1 q_0} + \frac{1}{q_1} < \frac{1}{N}, \tag{A.3}$$

then $W^{1,(p_0,q_0);(p_1,q_1)}(\omega;\Omega)$ *is compactly embedded in* $C(\bar{\Omega} \times \bar{\omega})$.

Note that the order of integration in (A.1) cannot be reversed. For $p \geq q$ we have that $L^q(\Omega; L^p(\omega))$ is embedded into $L^p(\omega; L^q(\Omega))$, but the opposite inclusion does not hold. On the other hand, denoting

$$W^{1,(q_0,p_0);(q_1,p_1)}(\Omega;\omega) := \Big\{ u \in L^1(\Omega \times \omega) : \frac{\partial u}{\partial t} \in L^{q_0}(\Omega; L^{p_0}(\omega)), \frac{\partial u}{\partial x_i} \in L^{q_1}(\Omega; L^{p_1}(\omega)) \text{ for } i = 1, \dots, N \Big\},$$

it is possible to repeat the computations leading to Theorem A.3 with reversed order of integration, to check that conditions (A.2) and (A.3) remain valid for the compact embedding of $W^{1,(q_0,p_0);(q_1,p_1)}(\Omega;\omega)$ into $L^{q_2}(\Omega; L^{p_2}(\omega))$ and $C(\bar{\Omega} \times \bar{\omega})$, respectively. Let us report one important particular case which frequently occurs in applications.

Theorem A.4. *If $q_2 \geq \max\{q_0, q_1\}$ and*

$$\frac{1}{p_0'}\left(\frac{1}{N} - \frac{1}{q_1} + \frac{1}{q_2}\right) > \frac{1}{p_1}\left(\frac{1}{q_0} - \frac{1}{q_2}\right),$$

then the space $W^{1,(q_0,p_0);(q_1,p_1)}(\Omega;\omega)$ is compactly embedded in $L^{q_2}(\Omega; C(\bar{\omega}))$.

From Theorem A.4 we immediately obtain the following result which we use in the first part of this thesis.

Corollary A.5. *The space $W^{1,2}(0,T; L^2(\Omega)) \cap L^\infty(0,T; W^{1,2}(\Omega))$ is compactly embedded in the space $L^2(\Omega; C([0,T]))$ endowed with the norm*

$$\|w\|_{2,\infty} = \left(\int_\Omega \max_{t \in [0,T]} |w(x,t)|^2 \, \mathrm{d}x \right)^{1/2} \quad \text{for } w \in L^2(\Omega; C([0,T])). \tag{A.4}$$

We report also the following compactness result from [107, Theorem 5.1].

Theorem A.6. *Let B_0, B and B_1 be three Banach spaces with $B_0 \subset B \subset B_1$, B_0 and B_1 reflexive. Suppose that the embedding of B_0 into B is compact. For $1 < p_0, p_1 < \infty$ let*

$$W := \Big\{ w \in L^{p_0}(0,T; B_0) : w_t \in L^{p_1}(0,T; B_1) \Big\},$$

where $T > 0$ is finite. Then the embedding of W into $L^{p_0}(0,T; B)$ is compact.

The following statement about Nemytskiĭ (or superposition) operators is frequently used in the passage to the limit procedure. Its proof can be found in [69].

Theorem A.7. *Let $p_1, p_2, \dots, p_m$ and r be real numbers, $p_i \geq 1$ ($i = 1, \dots, m$), $r \geq 1$. Let $h = h(\xi)$ be a function defined for $\xi \in \mathbb{R}^m$, and let $h \in C(\mathbb{R}^m)$. Denote by $H(u_1, \dots, u_m)$ the Nemytskiĭ operator determined by h.*

(i) Then, for an arbitrary m-tuple of functions $u_i \in L^{p_i}(\Omega)$ $(i = 1, \dots, m)$,

$$H(u_1, \dots, u_m) \in L^r(\Omega)$$

holds if and only if the following condition is satisfied: (a) *A function $g \in L^r(\Omega)$ and a number $c \geq 0$ exist such that for a. e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^m$*

$$|h(\xi_1, \dots, \xi_m)| \leq g(x) + c \sum_{i=1}^{m} |\xi_i|^{p_i/r}.$$

(ii) If condition (a) *is satisfied, then the Nemytskiĭ operator H is continuous from $L^{p_1}(\Omega) \times L^{p_2}(\Omega) \times \cdots \times L^{p_m}(\Omega)$ into $L^r(\Omega)$.*

A.1.3 Traces on the boundary

Trace operators arise from the fact that one needs to assign "boundary values" along $\partial\Omega$ to a function $w \in W^{1,p}(\Omega)$, assuming that $\partial\Omega$ is regular enough. The problem is that a typical function in this space is not in general continuous and, even worse, is only defined a. e. in Ω. Since $\partial\Omega$ has n-dimensional Lebesgue measure zero, there is no direct meaning we can give to the expression "w restricted to $\partial\Omega$". The notion of *trace operator* solves this problem.

We say that a function $\bar{u}$ belongs to $L^p(\partial\Omega)$ if the boundary norm

$$|\bar{u}|_{p,\partial\Omega} := \begin{cases} \left(\int_{\partial\Omega} |\bar{u}(x)|^p \, \mathrm{d}S(x) \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \sup\operatorname{ess}_{x \in \partial\Omega} |\bar{u}(x)| & \text{if } p = \infty \end{cases}$$

is finite. One important property of functions in $W^{1,p}(\Omega)$ for domains Ω with Lipschitzian boundary is that the trace of u on $\partial\Omega$ is well defined, as stated by the following theorem whose proof can be found e. g. in [115].

Theorem A.8 (Trace Theorem)**.** *Let $1 \leq p < \infty$. Then there exists a linear continuous mapping $T_p : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that for every $u \in C^1(\bar{\Omega})$ and every $x \in \partial\Omega$ we have $T_p u(x) = u(x)$.*

The trace operator satisfies also the following compactness result.

Corollary A.9. *Let*

$$q \geq \frac{Np - p}{N - p}.$$

Then the trace operator T_p is continuous from $W^{1,p}(\Omega)$ to $L^q(\partial\Omega)$. If moreover the inequality is strict, then the trace operator is compact.

A.2 Some useful inequalities

We recall here some classical inequalities that we often employ throughout the thesis. To simplify the presentation we use the notation $|v|_r$ for the $L^r(\Omega)$-norm of a function $v \in L^r(\Omega)$ for $r \in [0,\infty]$, whereas the norm of a function $v \in W^{1,r}(\Omega)$ will be denoted by $|v|_{1;r}$.
For the proofs see e. g. [12, 78, 103, 115].

Hölder's inequality

Let p, $q \in [1,\infty]$ with $\frac{1}{p}+\frac{1}{q}=1$. Then the inequality

$$|vw|_1 \le |v|_p |w|_q \tag{A.5}$$

holds for every $v \in L^p(\Omega)$, $w \in L^q(\Omega)$.

Remark A.10. In the third part of the thesis we make use of a discrete version of Hölder's inequality (A.5) by setting

$$|\mathbf{v}|_p := \left(\frac{1}{n}\sum_{k=1}^{n} |v_k|^p\right)^{1/p} \qquad \text{if } 1 \le p < \infty \tag{A.6}$$

$$|\mathbf{v}|_\infty := \max_{k=1,\dots,n} |v_k| \qquad \text{if } p = \infty \tag{A.7}$$

for a vector $\mathbf{v} = (v_1,\dots,v_n)$.

Gagliardo-Nirenberg inequality

There exists a constant $C > 0$ such that for every $v \in W^{1,p}(\Omega)$ the inequality

$$|v|_q \le C\left(|v|_s + |v|_s^{1-\varrho}|\nabla v|_p^{\varrho}\right) \tag{A.8}$$

or, equivalently,

$$|v|_q \le C|v|_s^{1-\varrho}|v|_{1;p}^{\varrho}, \tag{A.9}$$

holds for every $1 \le s < q$, $1/q > 1/p - 1/N$, where

$$\varrho = \frac{\frac{1}{s}-\frac{1}{q}}{\frac{1}{s}+\frac{1}{N}-\frac{1}{p}} \in (0{,}1).$$

Remark A.11. In the third part of the thesis we make use of a discrete version of Gagliardo-Nirenberg inequality (A.8). Let $\mathbf{v} = (v_0, v_1, \dots, v_n)$ be a vector, and let $\mathbf{D}\mathbf{v} = (n(v_1 - v_0), \dots, n(v_n - v_{n-1}))$ be the associated vector of difference quotients of $\mathbf{v}$. For norms defined as in (A.6) and (A.7), the discrete counterpart of (A.8) reads

$$|\mathbf{v}|_q \le C\left(|\mathbf{v}|_s + |\mathbf{v}|_s^{1-\varrho}|\mathbf{D}\mathbf{v}|_p^{\varrho}\right), \qquad \varrho = \frac{\frac{1}{s}-\frac{1}{q}}{\frac{1}{s}+1-\frac{1}{p}} \in (0{,}1).$$

It can be derived from (A.8) when $N = 1$ by defining v as equidistant piecewise linear interpolations of v_k.

Poincaré's inequality

Let $b \in L^\infty(\partial\Omega)$ be such that $b(x) \geq 0$ a. e. and $\int_{\partial\Omega} b(x)\,\mathrm{d}s(x) > 0$. Then there exists a constant $C > 0$ such that the inequality

$$|v|_{1;2}^2 \leq C \left(\int_\Omega |\nabla v|^2 \,\mathrm{d}x + \int_{\partial\Omega} b(x)|v|^2 \,\mathrm{d}s(x) \right) \tag{A.10}$$

holds for every $v \in W^{1,2}(\Omega)$.

Korn's inequality

This inequality essentially states that, in the case of zero Dirichlet boundary condition, the L^2-norms of ∇ and ∇_s are equivalent. More precisely it holds

$$\int_\Omega |\nabla_s w|^2(x)\,\mathrm{d}x \geq c\|w\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)} \tag{A.11}$$

for every $w \in W_0^{1,2}(\Omega;\mathbb{R}^3)$, with a constant $c > 0$ independent of w.

APPENDIX B

Hysteresis operators

Hysteresis occurs in several phenomena in physics, engineering, chemistry, biology and economics. Even if this phenomenon has been known and studied since the end of the eighteenth century, it was only more or less fifty years ago that, dealing with plasticity, a small group of Russian mathematicians introduced the concept of hysteresis operator and started a systematic investigation of its properties. The pioneers in this new field were certainly M. A. Krasnosel'skiĭ and A. V. Pokrovskiĭ with their important monograph [88]. From that moment many scientists have contributed to the mathematical study of hysteresis: see M. Brokate & J. Sprekels [24], P. Krejčí [91] and A. Visintin [130].

According to this formalism, the state of the system is characterized by two scalar variables u and w, that play the role of independent and dependent variable, and that are also called input and output. The construction of the hysteresis relation $u \mapsto w$ is made by choosing a suitable hysteresis operator, with the aim of describing *rate independent memory effects*.

(i) *Memory effects*: this means that the output $w(t)$ is not determined by the value of the input $u(t)$ at the same instant, but it depends also on the previous evolution of u.

(ii) *Rate independence*: this means that the couple $(u(t), w(t))$ is invariant with respect to any increasing time homeomorphism. In other words, at any time t, $w(t)$ depends only on the range of the restriction $u : [0, t] \to \mathbb{R}$ and on the order in which values have been attained. So there is no dependence on the derivatives of u, which may even fail to exist. Note that this condition is essential for giving a graphic representation of hysteresis in the (u, w)-plane by means of the so-called hysteresis loop.

The two variables u and w are assumed to depend continuously on time. Since at any instant the state is described by the value of the couple (u, w), it is then possible to define the operator

$$\begin{array}{rcl} \mathcal{F} : D(\mathcal{F}) \subset C^0([0,T]) \times \mathbb{R} & \longrightarrow & C^0([0,T]) \\ (u, u(0)) & \mapsto & w \end{array}$$

where $D(\mathcal{F})$ stands for the domain of $\mathcal{F}$.

In the next sections we are going to give a precise characterization of the operator $\mathcal{F}$, reporting the definitions and properties of the hysteresis operators that are needed in the first three parts of the thesis.

B.1 Hysteresis in elastoplasticity I

In the second part of this thesis we model elastoplasticity by means of a constitutive operator $\mathcal{P}$, which represents the elastoplastic part of the stress tensor σ. Following [105], we assume that a convex subset $0 \in Z \subset \mathbb{R}^{3\times3}_{\mathrm{sym}}$ with nonempty interior representing the admissible plastic stress domain is given in the space $\mathbb{R}^{3\times3}_{\mathrm{sym}}$ of symmetric tensors, and that the constitutive relation between the strain tensor ε and the stress tensor σ involves two fourth order tensors $\mathbf{A}_h$ (the kinematic hardening tensor) and $\mathbf{A}_e$ (the elasticity tensor) satisfying Hypothesis 5.1 (i). We define the constitutive operator $\mathcal{P}$ by the formula

$$\mathcal{P}[\varepsilon] = \mathbf{A}_h\varepsilon + \sigma^p, \tag{B.1}$$

where σ^p is the solution of the variational inequality

$$\sigma^p \in Z, \quad \big(\dot\varepsilon - \mathbf{A}_e^{-1}\dot\sigma^p\big) : \big(\sigma^p - z\big) \geq 0 \ \text{ a. e.} \ \ \forall z \in Z, \quad \sigma^p(0) = Q_Z(\varepsilon(0)) \tag{B.2}$$

for a given $\varepsilon \in W^{1,1}(0,T;\mathbb{R}^{3\times3}_{\mathrm{sym}})$, where $Q_Z : \mathbb{R}^{3\times3}_{\mathrm{sym}} \to Z$ is the orthogonal projection onto Z.

We list some properties of the variational problem (B.2) whose proof can be found in [91, Chapter I].

Proposition B.1. *For every $\varepsilon \in W^{1,1}(0,T;\mathbb{R}^{3\times3}_{\mathrm{sym}})$ there exists a unique $\sigma^p \in W^{1,1}(0,T;\mathbb{R}^{3\times3}_{\mathrm{sym}})$ satisfying the variational inequality* (B.2)*. The solution mapping*

$$W^{1,1}(0,T;\mathbb{R}^{3\times3}_{\mathrm{sym}}) \to W^{1,1}(0,T;\mathbb{R}^{3\times3}_{\mathrm{sym}}) : \varepsilon \mapsto \sigma^p$$

is strongly continuous and the operator $\mathcal{P}$ has the following additional properties.

(i) The operator $\mathcal{P}$ can be extended to a continuous operator in the space $C([0,T];\mathbb{R}^{3\times3}_{\mathrm{sym}})$ in the sense that if $\{\varepsilon_m; m \in \mathbb{N}\}$ is a sequence in $C([0,T];\mathbb{R}^{3\times3}_{\mathrm{sym}})$, then

$$\lim_{m\to\infty} \max_{t\in[0,T]} |\varepsilon_m(t) - \varepsilon(t)| = 0 \implies \lim_{m\to\infty} \max_{t\in[0,T]} |\mathcal{P}[\varepsilon_m](t) - \mathcal{P}[\varepsilon](t)| = 0.$$

(ii) For two inputs $\varepsilon_1, \varepsilon_2 \in W^{1,1}(0,T;\mathbb{R}^{3\times3}_{\mathrm{sym}})$ we denote $\sigma_i = \mathcal{P}[\varepsilon_i]$, $i = 1{,}2$. Then

$$(\sigma_1 - \sigma_2) : (\dot\varepsilon_1 - \dot\varepsilon_2) \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{A}_h(\varepsilon_1 - \varepsilon_2) : (\varepsilon_1 - \varepsilon_2) + \mathbf{A}_e^{-1}(\sigma_1^p - \sigma_2^p) : (\sigma_1^p - \sigma_2^p)\Big) \ \textit{a. e.,} \tag{B.3}$$

$$|\sigma_1(t) - \sigma_2(t)| \leq C\left(|\varepsilon_1(0) - \varepsilon_2(0)| + \int_0^t |\dot\varepsilon_1 - \dot\varepsilon_2|(\tau)\,\mathrm{d}\tau\right) \quad \forall t \in [0,T] \tag{B.4}$$

with a constant C depending only on $\mathbf{A}_h$ and $\mathbf{A}_e$.

(iii) *For inputs* $\varepsilon \in L^2(\Omega; W^{1,1}(0,T;\mathbb{R}^{3\times 3}_{\mathrm{sym}}))$ *we obtain from* (B.4)*, similarly to [40, Formula (6.25)], the inequality*

$$|\nabla\sigma(x,t)| \le C\left(|\nabla\varepsilon(x,0)| + \int_0^t |\nabla\varepsilon_t(x,\tau)|\,\mathrm{d}\tau\right) \quad \textit{a. e..} \tag{B.5}$$

It can also be proved that for all $\varepsilon \in W^{1,1}(0,T;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ it holds

$$|\mathcal{P}[\varepsilon]_t| \le |\varepsilon_t| \quad \text{a. e.} \tag{B.6}$$

The energy potential $U_{\mathcal{P}}$ and the dissipation operator $D_{\mathcal{P}}$ associated with $\mathcal{P}$ are defined by the formula

$$U_{\mathcal{P}}[\varepsilon] = \frac{1}{2}\mathbf{A}_h\varepsilon : \varepsilon + \frac{1}{2}\mathbf{A}_e^{-1}\sigma^p : \sigma^p, \qquad D_{\mathcal{P}}[\varepsilon] = \varepsilon - \mathbf{A}_e^{-1}\sigma^p. \tag{B.7}$$

Let M_{Z^*} denote the Minkowski functional of the polar set Z^* to Z. The energy identity

$$\mathcal{P}[\varepsilon] : \varepsilon_t - U_{\mathcal{P}}[\varepsilon]_t = \|D_{\mathcal{P}}[\varepsilon]_t\|_* \ \text{ a. e.,} \tag{B.8}$$

where $\|\cdot\|_* = M_{Z^*}(\cdot)$ is a seminorm in $\mathbb{R}^{3\times 3}_{\mathrm{sym}}$, and the pointwise inequalities

$$U_{\mathcal{P}}[\varepsilon] \ge \frac{A^\flat}{2}|\varepsilon|^2, \qquad \|D_{\mathcal{P}}[\varepsilon]_t\|_* \le C|\varepsilon_t| \tag{B.9}$$

hold for all inputs $\varepsilon \in W^{1,1}(0,T;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$.

B.2 The scalar stop and play operators

Given a parameter $r > 0$, the *scalar stop operator* with threshold r is defined as the mapping which with a function $u \in W^{1,1}(0,T)$ and with an initial condition $s^{r,0} \in [-r,r]$ associates the solution $s^r \in W^{1,1}(0,T)$ of the differential inclusion

$$\dot{s}^r(t) + \partial I\left(\frac{1}{r}s^r(t)\right) \ni \dot{u}(t) \quad \text{a. e.,}\ s^r(0) = s^{r,0}, \tag{B.10}$$

where

$$I(y) = I_{[-r,r]}(y) = \begin{cases} 0 & \text{if } y \in [-r,r], \\ +\infty & \text{otherwise,} \end{cases} \tag{B.11}$$

is the indicator function of the interval $[-r,r]$ and

$$\partial I(y) = \partial I_{[-r,r]}(y) = \begin{cases} (-\infty,\,0] & \text{if } y = -r, \\ \{0\} & \text{if } -r < y < r, \\ [0,+\infty) & \text{if } y = r, \end{cases} \tag{B.12}$$

is its subdifferential. Let us recall the meaning of the differential inclusion:

$$\gamma \in \partial I_{[-r,r]}(y) \qquad \Longleftrightarrow \qquad y \in [-r,r], \quad \gamma(y-z) \ge 0 \ \ \forall\, z \in [-r,r]. \tag{B.13}$$

Thus (B.10) can be equivalently rewritten in the form of a variational inequality

$$\begin{cases} |s^r(t)| \leq r & \forall\, t \in [0,T], \\ (\dot{u}(t) - \dot{s}^r(t))(s^r(t) - z) \geq 0 & \text{a. e. for all } z \in [-r,r], \\ s^r(0) = s^{r,0}. \end{cases} \tag{B.14}$$

It is well known, see [24, 88, 91, 130], that for each $u \in W^{1,1}(0,T)$ and each $s^{r,0} \in [-r,r]$, the solution $s^r \in W^{1,1}(0,T)$ of (B.10) or (B.14) exists and is unique. We denote the stop operator, that is, the solution mapping of (B.10) or (B.14), by $\mathfrak{s}_r : W^{1,1}(0,T) \times [-r,r] \to W^{1,1}(0,T)$, and write

$$s^r(t) = \mathfrak{s}_r[u, s^{r,0}](t). \tag{B.15}$$

Remark B.2. From (B.12) (or, equivalently, (B.13)) it follows

$$\partial I_{[-1,1]}(y)\, y_t = 0\,. \tag{B.16}$$

The above identity implies that, in the case of relaxed phase-dynamics of the form

$$\gamma \in \partial I_{[-1,1]}(y) + y_t$$

(or, equivalently, in the case of the stop operator (B.10)), we obtain $\gamma\, y_t = y_t^2 \geq 0$.

We now turn our attention to the *scalar play operator*. For given memory parameter $r > 0$, input function $u \in W^{1,1}(0,T)$ and initial condition $\xi^{r,0} \in [u(0) - r, u(0) + r]$, we define the function $\xi^r(t)$ as the solution of the variational inequality

$$\begin{cases} |u(t) - \xi^r(t)| \leq r & \forall\, t \in [0,T], \\ \dot{\xi}^r(t)(u(t) - \xi^r(t) - z) \geq 0 & \text{a. e. for all } z \in [-r,r], \\ \xi^r(0) = \xi^{r,0}. \end{cases} \tag{B.17}$$

This is indeed a scalar version of (B.2) with Z replaced by the interval $[-r,r]$, ε replaced by u and σ^p replaced by $u - \xi^r$.

The mapping $\mathfrak{p}_r : W^{1,1}(0,T) \times [-r,r] \to W^{1,1}(0,T)$ which with each $u \in W^{1,1}(0,T)$ associates the solution

$$\xi^r(t) = \mathfrak{p}_r[u, \xi^{r,0}](t)$$

in $W^{1,1}(0,T)$ of (B.17) is called the *play*. This concept goes back to [88].

Remark B.3. Comparing (B.14) and (B.17) we see that, provided we choose $\xi^{r,0} = u(0) - s^{r,0}$, the play operator is related to the stop by the simple formula

$$\xi^r(t) = u(t) - s^r(t). \tag{B.18}$$

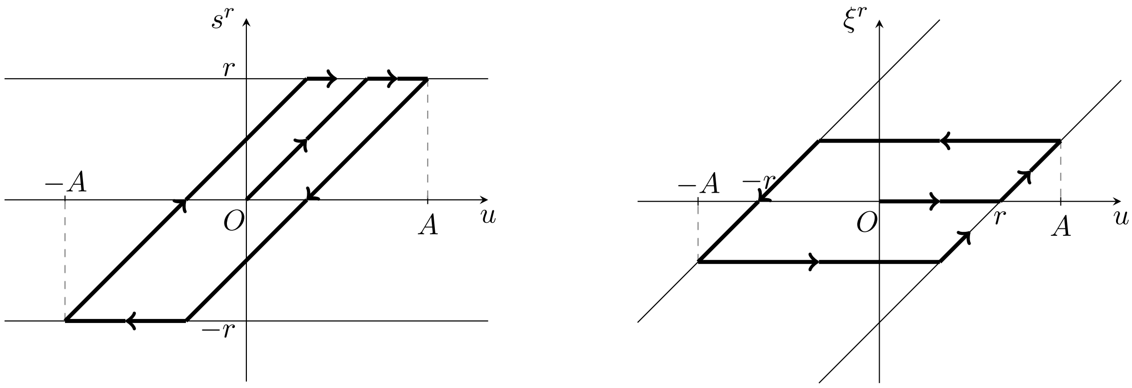

Figure B.1. Input-output diagram for the stop (on the left) and play (on the right) in the case $u(t) = A \sin \omega t$ for $A > r > 0$.

Remark B.4. The initially unperturbed state ("virginal state" in the terminology of [129]) is characterized by the choice

$$s^r(0) = Q_r(u(0)) := \max\{-r, \min\{u(0), r\}\}$$
$$\xi^r(0) = P_r(u(0)) := \min\{u(0) + r, \max\{0, u(0) - r\}\} = u(0) - Q_r(u(0))$$

of the initial conditions in (B.14) and (B.17), respectively. This is be the case of Section B.4, where we use the simplified notation

$$\mathfrak{s}_r[u] := \mathfrak{s}_r[u, Q_r(u(0))], \qquad \mathfrak{p}_r[u] := \mathfrak{p}_r[u, P_r(u(0))].$$

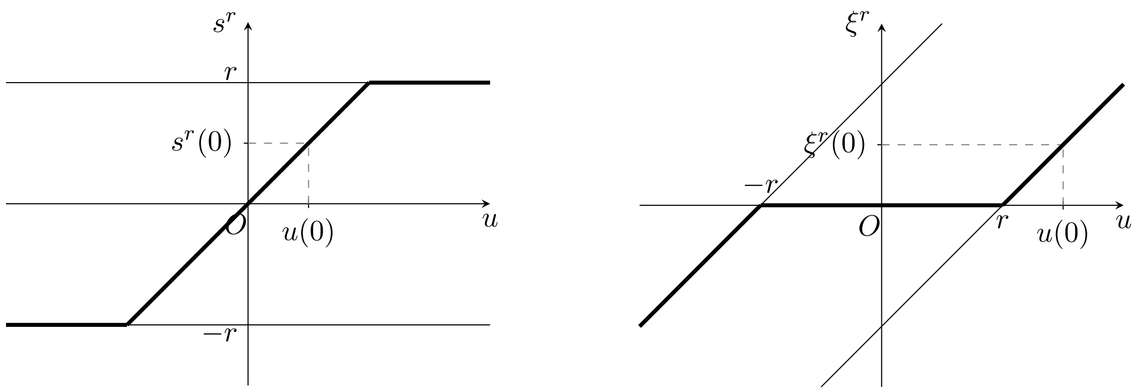

Figure B.2. Canonical initial conditions for the stop (on the left) and play (on the right).

We give a brief survey of basic properties of these two operators that are needed in the thesis. The proofs can be found in [91].

Proposition B.5 (Lipschitz continuity of the stop)**.** *For each $r > 0$, the mapping $s_r : W^{1,1}(0,T) \times [-r,r] \to W^{1,1}(0,T)$ is Lipschitz continuous and admits a Lipschitz continuous extension to $s_r : C[0,T] \times [-r,r] \to C[0,T]$ with Lipschitz constant 2.*

Also the dependence of $s^r(t)$ on r, which can be interpreted as a memory variable and represents the *memory depth*, is Lipschitz continuous in the following sense.

Proposition B.6. *For all $u \in C[0,T]$ and all $t \in [0,T]$ we have the implication*

$$|s^{r_1,0} - s^{r_2,0}| \leq |r_1 - r_2| \quad \forall r_1, r_2 > 0 \;\Rightarrow\; |s^{r_1}(t) - s^{r_2}(t)| \leq |r_1 - r_2| \quad \forall r_1, r_2 > 0.$$

A result like Proposition B.5 holds also for the play operator.

Proposition B.7 (Lipschitz continuity of the play)**.** *For each $r > 0$, the mapping $\mathfrak{p}_r : W^{1,1}(0,T) \to W^{1,1}(0,T)$ is Lipschitz continuous and admits a Lipschitz continuous extension to $\mathfrak{p}_r : C[0,T] \to C[0,T]$.*

It can also be proved that the variational inequalities (B.14) and (B.17) can be rewritten equivalently in "energetic form", that is, the energy balance equations

$$\dot{\xi}^r(t)\, u(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}(\xi^r)^2(t)\right) + |r\dot{\xi}^r(t)| \tag{B.19}$$

$$s^r(t)\, \dot{u}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}(s^r)^2(t)\right) + |r\dot{\xi}^r(t)| \tag{B.20}$$

hold a. e. in $(0,T)$. With this notation, $\dot{\xi}^r(t)\, u(t)$ and $s^r(t)\, \dot{u}(t)$ represent the power supplied to the system: part of it is used for the increase of the corresponding potentials $\frac{1}{2}(\xi^r)^2(t)$ and $\frac{1}{2}(s^r)^2(t)$, and the rest $|r\dot{\xi}^r(t)|$ is dissipated. Furthermore, directly from the definitions (B.14) and (B.17) (compare also with Remark B.2), we can infer that the identities

$$\dot{s}^r(t)\, \dot{u}(t) = (\dot{s}^r(t))^2, \qquad \dot{\xi}^r(t)\, \dot{u}(t) = (\dot{\xi}^r(t))^2 \tag{B.21}$$

hold a. e. for every $u \in W^{1,1}(0,T)$.

The play and the stop can be extended to space and time dependent inputs in the following way. For each function $u : \Omega \times (0,T) \to \mathbb{R}$ such that for a. e. $x \in \Omega$ it holds $u(x,\cdot) \in W^{1,1}(0,T)$ we define

$$\mathfrak{s}_r[u](x,t) = \mathfrak{s}_r[u(x,\cdot)](t), \qquad \mathfrak{p}_r[u](x,t) = \mathfrak{p}_r[u(x,\cdot)](t)$$

for all $t \in [0,T]$, where for the sake of simplicity we used the same symbol both for the operators and their extensions. Note that they are applied at each point $x \in \Omega$ independently: the output depends on $u(x,\cdot)|_{[0,t]}$, but not on $u(y,\cdot)|_{[0,t]}$ for any $y \neq x$. Hence this model can represent memory effects, but not space interaction.

By virtue of Propositions B.5 and B.7, the stop and the play are both Lipschitz continuous in $L^p(\Omega; C[0,T])$ for all $1 \leq p \leq \infty$.

The play and the stop are the main building blocks of more complex hysteresis operators. In the next sections we are going to introduce the Preisach operator and the Prandtl-Ishlinskiĭ operator.

B.3 Hysteresis in capillarity phenomena: the Preisach operator

In order to define the Preisach hysteresis operator, we need to consider the whole continuous family of variational inequalities (B.17) parameterized by $r > 0$. We then introduce the *configuration space*

$$\Lambda = \left\{\lambda \in W^{1,\infty}(0,\infty) : |\lambda'(r)| \le 1 \text{ a.e.}\right\}$$

of *memory configurations* λ, and its subspaces

$$\Lambda_K = \{\lambda \in \Lambda : \lambda(r) = 0 \text{ for } r \ge K\}.$$

By fixing $K > 0$ and an initial state $\lambda_{-1} \in \Lambda_K$, the initial condition is chosen in the form

$$\xi^r(0) = \max\{u(0) - r, \min\{\lambda_{-1}(r), u(0) + r\}\}.$$

We have for all $r > 0$ the initial bound

$$\xi^r(0) \le \max\{|u(0)|, K\}. \tag{B.22}$$

The proof of the following statement can be found in [91].

Proposition B.8. *Let $\lambda_{-1} \in \Lambda_K$ be given, and let $\{\mathfrak{p}_r : r > 0\}$ be the family of play operators. Then for every $u \in C[0,T]$ and every $t \in [0,T]$ we have*

(i) $\mathfrak{p}_r[u,\lambda](t) = 0$ *for* $r \ge K^*(t) := \max\{K, \max_{\tau\in[0,t]} |u(\tau)|\}$*;*

(ii) the function $r \mapsto \mathfrak{p}_r[u,\lambda](t)$ *belongs to* $\Lambda_{K^*(t)}$.

The pressure-saturation operator $\mathcal{G}$ appearing in the second part of this thesis is considered as a sum

$$\mathcal{G}[p] = f(p) + \mathcal{G}_0[p], \tag{B.23}$$

where f is a monotone function satisfying Hypothesis 5.1 (vi) and $\mathcal{G}_0$ is a Preisach operator that we describe here.

Note that the original Preisach construction in [124] was based on averaging over a two-parameter system of the so-called *delayed relay operators*, see also [130]. It was shown in [90] that the Preisach definition of a hysteresis relationship $w(t) = \mathcal{F}[u](t)$ can be equivalently expressed by the following integral formula. Given a nonnegative function $\psi \in L^\infty(\Omega; L^1((0,\infty)\times\mathbb{R}))$ (the *Preisach density*), we define the Preisach operator $\mathcal{G}_0 : L^p(\Omega; C[0,T] \to L^p(\Omega; C[0,T]))$ for $(x,t) \in \Omega \times [0,T]$ as the integral

$$\mathcal{G}_0[p](x,t) = \int_0^\infty \int_0^{\mathfrak{p}_r[p](x,t)} \psi(x,r,v)\,\mathrm{d}v\,\mathrm{d}r, \tag{B.24}$$

where $\mathfrak{p}_r$ is the play operator introduced in the previous Section B.2.

Remark B.9. Definition (B.24) is meaningful (that is, the integral is finite) since $p_r[p](t) = 0$ for r sufficiently large, see Proposition B.8.

For our purposes, we adopt the following hypotheses on the Preisach density.

Hypothesis B.10. There exists a function $\psi^* \in L^1(0,\infty)$ such that for a. e. $x \in \Omega$ and a. e. $v \in \mathbb{R}$ we have $0 \le \psi(x,r,v) \le \psi^*(r)$, $0 \le r\psi(x,r,v) \le \psi^*(r)$ and we set

$$C_\psi^* = \int_0^\infty \psi^*(r)\,\mathrm{d}r.$$

Remark B.11. We require the above hypotheses to hold for mathematical purposes. Nevertheless they are reasonable, since in applications the Preisach density decays exponentially.

The following statement is an easy consequence of Proposition B.7.

Proposition B.12 (Lipschitz continuity of the Preisach operator)**.** *The mapping*

$$\mathcal{G}_0 : L^p(\Omega; C[0,T]) \to L^p(\Omega; C[0,T])$$

is Lipschitz continuous for every $1 \le p \le \infty$ *with Lipschitz constant* C_ψ^*.

From (B.19) and (B.24) we immediately deduce the Preisach energy identity

$$\mathcal{G}_0[p]_t\, p - U_0[p]_t = |D_0[p]_t| \quad \text{a. e.} \tag{B.25}$$

provided we define the Preisach potential U_0 and the dissipation operator D_0 by the integrals

$$U_0[p](x,t) = \int_0^\infty \int_0^{\mathfrak{p}_r[p](x,t)} v\psi(x,r,v)\,\mathrm{d}v\,\mathrm{d}r, \qquad D_0[p](x,t) = \int_0^\infty \int_0^{\mathfrak{p}_r[p](x,t)} r\psi(x,r,v)\,\mathrm{d}v\,\mathrm{d}r. \tag{B.26}$$

Then, considering the whole operator $\mathcal{G} = f + \mathcal{G}_0$, the following counterpart of (B.25)

$$\mathcal{G}[p]_t\, p = U_\mathcal{G}[p]_t + |D_\mathcal{G}[p]_t| \tag{B.27}$$

holds with the choice

$$U_\mathcal{G}[p] = pf(p) - \int_0^p f(z)\,\mathrm{d}z + U_0[p], \qquad D_\mathcal{G}[p] = D_0[p].$$

We denote by

$$\Phi(p) := \int_0^p f(z)\,\mathrm{d}z, \quad V(p) := pf(p) - \Phi(p) = \int_0^p f'(z)z\,\mathrm{d}z, \tag{B.28}$$

so that with this notation we can split the potential $U_\mathcal{G}$ in hysteretic and nonhysteretic part, namely,

$$U_\mathcal{G}[p] = V(p) + U_0[p]. \tag{B.29}$$

From Hypothesis B.10 and identity (B.21) for the play we also obtain

$$0 < U_0[p] \le C_\psi^*(1 + |p|), \qquad |D_0[p]_t| \le C|p_t|. \tag{B.30}$$

We similarly get, using (B.22),

$$|U_0[p](x,0)| = \left| \int_0^\infty \int_0^{\mathfrak{p}_r[p^0](x)} v\psi(x,r,v)\,\mathrm{d}v\,\mathrm{d}r \right| \le C_\psi^* \max\{|p^0(x)|, K\}\,. \tag{B.31}$$

Remark B.13. The Preisach potential is continuous from $L^2(\Omega; C[0,T])$ to $L^1(\Omega; C[0,T])$. Indeed, defining

$$W(\xi, r) = \int_0^\xi v\,\psi(x,r,v)\,\mathrm{d}v,$$

we see that Hypothesis B.10 yields

$$|W(\xi_2,r) - W(\xi_1,r)| \le \left(\left|\int_{\xi_1}^{\xi_2} |v|\,\mathrm{d}v\right|\right) \psi^*(r) \le \frac{1}{2}\left||\xi_2|^2 - |\xi_1|^2\right| \psi^*(r) \le \frac{1}{2}\left(|\xi_2| + |\xi_1|\right)|\xi_2 - \xi_1|\,\psi^*(r).$$

Thus, again from Hypothesis B.10 and from Proposition B.7,

$$\begin{aligned}
|U_0[p_2] - U_0[p_1]|(x,t) &= \int_0^\infty |W(\mathfrak{p}_r[p_2](x,t), r) - W(\mathfrak{p}_r[p_1](x,t), r)|\,\mathrm{d}r \\
&\le \frac{1}{2}\max_{\tau\in[0,t]} |p_2 - p_1|(x,\tau) \int_0^\infty \left(|p_2(x,t)| + |p_1(x,t)| + 2r\right)\psi^*(r)\,\mathrm{d}r \\
&\le \frac{C_\psi^*}{2}\max_{\tau\in[0,t]} |p_2 - p_1|(x,\tau)\left(|p_2(x,t)| + |p_1(x,t)| + 2\right).
\end{aligned}$$

The Preisach operator admits also a family of "nonlinear" energies. As a consequence of (B.19), we have for a. e. t the inequality

$$\mathfrak{p}_r[p]_t(p - \mathfrak{p}_r[p]) \ge 0,$$

thus

$$\mathfrak{p}_r[p]_t(h(p) - h(\mathfrak{p}_r[p])) \ge 0$$

for every nondecreasing function $h : \mathbb{R} \to \mathbb{R}$. Hence a counterpart of (B.25) in the form

$$\mathcal{G}_0[p]_t\, h(p) - U_h[p]_t \ge 0 \quad \text{a. e.} \tag{B.32}$$

holds with a modified potential

$$U_h[p](x,t) = \int_0^\infty \int_0^{\mathfrak{p}_r[p](x,t)} h(v)\psi(x,r,v)\,\mathrm{d}v\,\mathrm{d}r. \tag{B.33}$$

This is related to the fact that for every absolutely continuous nondecreasing function $\hat{h} : \mathbb{R} \to \mathbb{R}$, the mapping $\mathcal{G}_{\hat{h}} := \mathcal{G}_0 \circ \hat{h}$ is also a Preisach operator, see [93].

B.4 Hysteresis in elastoplasticity II: the Prandtl-Ishlinskiĭ operator

In the third part of this thesis we model elastoplasticity by means of a constitutive operator P_0, the Prandtl-Ishlinskiĭ operator, which represents the elastoplastic part of the stress tensor σ. This model is constructed as a linear combination of stops (B.10) or (B.14) with all possible yield points

$r > 0$, and with initial conditions $s^{r,0}$ chosen as in Remark B.4. Given a nonnegative function $\tilde{\gamma} : [0,T] \times (0,\infty) \to [0,\infty)$ such that $\tilde{\gamma}(t,\cdot) \in L^1(0,\infty)$ for all $t \in [0,T]$, we define the Prandtl-Ishlinskiĭ operator by the integral

$$P_0[\varepsilon](t) = \int_0^\infty \tilde{\gamma}(t,r)\,\mathfrak{s}_r[\varepsilon](t)\,\mathrm{d}r\,.$$

Identity (B.20) enables us to establish the energy balance for the Prandtl-Ishlinskiĭ operator. Indeed, if we define the Prantdl-Ishlinskiĭ potential

$$V[\varepsilon](t) = \frac{1}{2}\int_0^\infty \tilde{\gamma}(t,r)\,\mathfrak{s}_r^2[\varepsilon](t)\,\mathrm{d}r$$

and the dissipation operator

$$D[\varepsilon](t) = \int_0^\infty r\tilde{\gamma}(t,r)|\mathfrak{p}_r[\varepsilon]_t(t)|\,\mathrm{d}r\,,$$

we can write the Prandtl-Ishlinskiĭ energy balance in the form

$$\dot{\varepsilon}(t)P_0[\varepsilon](t) = \frac{\mathrm{d}}{\mathrm{d}t}V[\varepsilon](t) + D[\varepsilon](t)\,. \tag{B.34}$$

As a consequence of identity (B.21) for the play we obtain the estimate

$$D[\varepsilon](t) \leq |\dot{\varepsilon}(t)|\int_0^\infty r\tilde{\gamma}(t,r)\,\mathrm{d}r\,. \tag{B.35}$$

By imposing suitable boundedness hypotheses for $\tilde{\gamma}(t,r)$ (and this is the case of Part III), we can estimate the dissipation from above in terms of the input velocity.

Bibliography

[1] E. Acerbi, G. Bouchitté, and I. Fonseca. Relaxation of convex functionals: the gap problem. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, 20(3):359–390, 2003.

[2] E. Acerbi and N. Fusco. Regularity for minimizers of non-quadratic functionals: the case $1 < p < 2$. *Journal of Mathematical Analysis and Applications*, 140(1):115–135, 1989.

[3] H. Al Baba, N.V. Chemetov, Š. Nečasová, and B. Muha. Strong solutions in L^2 framework for fluid-rigid body interaction problem. Mixed case. *Topological Methods in Nonlinear Analysis*, 52(1):337–350, 2018.

[4] B. Albers. Modeling the hysteretic behavior of the capillary pressure in partially saturated porous media: a review. *Acta Mechanica*, 225(8):2163–2189, 2014.

[5] B. Albers and P. Krejčí. Unsaturated porous media flow with thermomechanical interaction. *Mathematical Methods in the Applied Sciences*, 39(9):2220–2238, 2016.

[6] L. Ambrosio. Lecture notes on elliptic partial differential equations. *Unpublished lecture notes. Scuola Normale Superiore di Pisa*, 30, 2015.

[7] F. Bagagiolo. On the controllability of the semilinear heat equation with hysteresis. *Physica B: Condensed Matter*, 407(9):1401–1403, 2012.

[8] A. Kh. Balci, L. Diening, and M. Weimar. Higher order Calderón-Zygmund estimates for the p-laplace equation. *Journal of Differential Equations*, 268(2):590–635, 2020.

[9] V. Barbu. Controllability of parabolic and Navier-Stokes equations. *Scientiae Mathematicae Japonicae*, 56(1):143–211, 2002.

[10] P. Baroni. Lorentz estimates for obstacle parabolic problems. *Nonlinear Analysis: Theory, Methods & Applications*, 96:167–188, 2014.

[11] G. Bertazzoni and S. Riccò. Lipschitz regularity results for a class of obstacle problems with nearly linear growth. *Journal of Elliptic and Parabolic Equations*, 6(2):883–918, 2020.

[12] O.V. Besov, V.P. Il'in, and S.M. Nikol'skii. *Integral Representations of Functions and Embedding Theorems*. Scripta Series in Mathematics. Halsted Press (John Wiley & Sons), New York-Toronto, Ont.-London, 1978 (Vol. 1), 1979 (Vol. 2).

[13] M. Bildhauer, M. Fuchs, and G. Mingione. A priori gradient bounds and local $C^{1,\alpha}$-estimates for (double) obstacle problems under non-standard growth conditions. *Zeitschrift für Analysis und ihre Anwendungen*, 20(4):959–985, 2001.

[14] M.A. Biot. General theory of three-dimensional consolidation. *Journal of Applied Physics*, 12(2):155–164, 1941.

[15] V. Bögelein, F. Duzaar, and P. Marcellini. Parabolic systems with p, q-growth: a variational approach. *Archive for Rational Mechanics and Analysis*, 210(1):219–267, 2013.

[16] V. Bögelein, F. Duzaar, and P. Marcellini. Existence of evolutionary variational solutions via the calculus of variations. *Journal of Differential Equations*, 256(12):3912–3942, 2014.

[17] V. Bogelein, F. Duzaar, and P. Marcellini. A time dependent variational approach to image restoration. *SIAM Journal on Imaging Sciences*, 8(2):968–1006, 2015.

[18] V. Bögelein, F. Duzaar, and G. Mingione. Degenerate problems with irregular obstacles. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2011(650):107–160, 2011.

[19] V. Bögelein, T. Lukkari, and C. Scheven. The obstacle problem for the porous medium equation. *Mathematische Annalen*, 363(1-2):455–499, 2015.

[20] H. Brézis, D. Kinderlehrer, and H. Lewy. The smoothness of solutions to nonlinear variational inequalities. *Indiana University Mathematics Journal*, 23(9):831–844, 1974.

[21] M. Brokate. *Optimale Steuerung von gewöhnlichen Differentialgleichungen mit Nichtlinearitäten vom Hysteresis-Typ*. Verlag Peter D. Lang, Frankfurt am Main-New York, 1987.

[22] M. Brokate. ODE control problems including the Preisach hysteresis operator: Necessary optimality conditions. In G. Feichtinger, editor, *Dynamic economic models and optimal control*, pages 51–68. North-Holland, Amsterdam, 1992.

[23] M. Brokate, K. Dressler, and P. Krejčí. Rainflow counting and energy dissipation for hysteresis models in elastoplasticity. *European journal of mechanics A/Solids*, 15(4):705–737, 1996.

[24] M. Brokate and J. Sprekels. *Hysteresis and Phase Transitions*, volume 121 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1996.

[25] G. Buttazzo and V.J. Mizel. Interpretation of the Lavrentiev phenomenon by relaxation. *Journal of Functional Analysis*, 110(2):434–460, 1992.

[26] S.S. Byun, Y. Cho, and J. Ok. Global gradient estimates for nonlinear obstacle problems with nonstandard growth. *Forum Mathematicum*, 28(4):729–747, 2016.

[27] S.S. Byun, Y. Cho, and L. Wang. Calderón–Zygmund theory for nonlinear elliptic problems with irregular obstacles. *Journal of Functional Analysis*, 263(10):3117–3143, 2012.

[28] L.A. Caffareli and D. Kinderlehrer. Potential methods in variational inequalities. *Journal d'Analyse Mathématique*, 37(1):285–295, 1980.

[29] M. Carozza, J. Kristensen, and A. Passarelli di Napoli. Higher differentiability of minimizers of convex variational integrals. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, 28(3):395–411, 2011.

[30] M. Carozza, J. Kristensen, and A. Passarelli di Napoli. Regularity of minimizers of autonomous convex variational integrals. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, XIII(5):1065–1089, 2014.

[31] M. Carozza, J. Kristensen, and A. Passarelli di Napoli. On the validity of the Euler-Lagrange system. *Communications on Pure & Applied Analysis*, 14(1):51–62, 2015.

[32] M. Caselli, M. Eleuteri, and A. Passarelli di Napoli. Regularity results for a class of obstacle problems with p,q-growth conditions. *arXiv preprint arXiv:1907.08527*, 2019.

[33] H.J. Choe. A regularity theory for a general class of quasilinear elliptic partial differential equations and obstacle problems. *Archive for Rational Mechanics and Analysis*, 114(4):383–394, 1991.

[34] H.J. Choe and J.L. Lewis. On the obstacle problem for quasilinear elliptic equations of p-Laplacian type. *SIAM Journal on Mathematical Analysis*, 22(3):623–638, 1991.

[35] G. Cupini, M. Guidorzi, and E. Mascolo. Regularity of minimizers of vectorial integrals with p,q-growth. *Nonlinear Analysis: Theory, Methods & Applications*, 54(4):591–616, 2003.

[36] G. Cupini, P. Marcellini, and E. Mascolo. Existence and regularity for elliptic equations under p,q-growth. *Advances in Differential Equations*, 19(7-8):693–724, 2014.

[37] B. Dacorogna. *Direct methods in the Calculus of Variations*, volume 78. Springer Science & Business Media, 2007.

[38] C. De Filippis and G. Mingione. On the regularity of minima of non-autonomous functionals. *The Journal of Geometric Analysis*, 30:1584–1626, 2020.

[39] E. De Giorgi. Un esempio di estremali discontinue per un problema variazionale di tipo ellittico. *Bollettino dell'Unione Matematica Italiana*, 4:135–137, 1968.

[40] B. Detmann, P. Krejčí, and E. Rocca. Solvability of an unsaturated porous media flow problem with thermomechanical interaction. *SIAM Journal on Mathematical Analysis*, 48(6):4175–4201, 2016.

[41] M. Eleuteri. Regularity results for a class of obstacle problems. *Applications of Mathematics*, 52(2):137–170, 2007.

[42] M. Eleuteri, C. Gavioli, and J. Kopfová. Fatigue and phase transition in an oscillating elasto-plastic beam. *Mathematical Modelling of Natural Phenomena*, 15, 2020. Art. No. 41.

[43] M. Eleuteri and J. Habermann. Regularity results for a class of obstacle problems under nonstandard growth conditions. *Journal of Mathematical Analysis and Applications*, 344(2):1120–1142, 2008.

[44] M. Eleuteri and J. Habermann. Calderón–Zygmund type estimates for a class of obstacle problems with $p(x)$-growth. *Journal of mathematical analysis and applications*, 372(1):140–161, 2010.

[45] M. Eleuteri and J. Habermann. A Hölder continuity result for a class of obstacle problems under non standard growth conditions. *Mathematische Nachrichten*, 284(11-12):1404–1434, 2011.

[46] M. Eleuteri, P. Harjulehto, and T. Lukkari. Global regularity and stability of solutions to obstacle problems with nonstandard growth. *Revista Matemática Complutense*, 26(1):147–181, 2013.

[47] M. Eleuteri and J. Kopfová. On a new model for fatigue and phase transition in an oscillating elastoplastic plate. *Journal of Differential Equations*, 265(5):1839–1874, 2018.

[48] M. Eleuteri, J. Kopfová, and P. Krejčí. A thermodynamic model for material fatigue under cyclic loading. *Physica B: Condensed Matter*, 407(9):1415–1416, 2012.

[49] M. Eleuteri, J. Kopfová, and P. Krejčí. Fatigue accumulation in an oscillating plate. *Discrete & Continuous Dynamical Systems-S*, 6(4):909–923, 2013.

[50] M. Eleuteri, J. Kopfová, and P. Krejčí. Non-isothermal cyclic fatigue in an oscillating elasto-plastic beam. *Communications on Pure & Applied Analysis*, 12(6):2973–2996, 2013.

[51] M. Eleuteri, J. Kopfová, and P. Krejčí. Fatigue accumulation in a thermo-visco-elastoplastic plate. *Discrete & Continuous Dynamical Systems-B*, 19(7):2091–2109, 2014.

[52] M. Eleuteri, J. Kopfová, and P. Krejčí. A new phase field model for material fatigue in an oscillating elastoplastic beam. *Discrete & Continuous Dynamical Systems-A*, 35(6):2465–2495, 2015.

[53] M. Eleuteri, P. Marcellini, and E. Mascolo. Lipschitz continuity for energy integrals with variable exponents. *Rendiconti Lincei-Matematica e Applicazioni*, 27(1):61–88, 2016.

[54] M. Eleuteri, P. Marcellini, and E. Mascolo. Lipschitz estimates for systems with ellipticity conditions at infinity. *Annali di Matematica Pura ed Applicata*, 195(5):1575–1603, 2016.

[55] M. Eleuteri, P. Marcellini, and E. Mascolo. Regularity for scalar integrals without structure conditions. *Advances in Calculus of Variations*, 13(3):279–300, 2020.

[56] M. Eleuteri and A. Passarelli di Napoli. Higher differentiability for solutions to a class of obstacle problems. *Calculus of Variations and Partial Differential Equations*, 57(5), 2018. Art. ID 115.

[57] O.Yu. Emanuilov. Controllability of parabolic equations. *Sbornik: Mathematics*, 186(6):879–900, 1995.

[58] L. Esposito, F. Leonetti, and G. Mingione. Sharp regularity for functionals with (p,q) growth. *Journal of Differential Equations*, 204(1):5–55, 2004.

[59] L.C. Evans. *Partial Differential Equations*. Graduate Studies in Mathematics. American Mathematical Society, 2010.

[60] E. Feireisl, D. Hilhorst, H. Petzeltová, and P. Takáč. Mathematical analysis of variable density flows in porous media. *Journal of Evolution Equations*, 16(1):1–19, 2016.

[61] E. Feireisl, J. Mikyška, H. Petzeltová, and P. Takáč. On the motion of chemically reacting fluids through porous medium. In *Springer Proceedings in Mathematics & Statistics*, volume 209, pages 139–152. Springer, 2017.

[62] G. Fichera. Problemi elastostatici con vincoli unilaterali: il problema di Signorini con ambigue condizioni al contorno. In *Atti della Accademia Nazionale dei Lincei. Memorie. Classe di scienze fisiche, matematiche e naturali. Ser. 8, sez. 1*, pages 91–140. Accademia Nazionale dei Lincei, 1964.

[63] A. Flatten. *Lokale und nicht-lokale Modellierung und Simulation thermomechanischer Lokalisierung mit Schädigung für metallische Werkstoffe unter Hochgeschwindigkeitsbeanspruchungen.* PhD thesis, BAM Bundesanstalt für Materialforschung und -prüfung, Berlin, Germany, 2008.

[64] D. Flynn. *Modelling the flow of water through multiphase porous media with the Preisach model.* PhD thesis, Department of Civil & Environmental Engineering and Department of Applied Mathematics, University College Cork, Cork, Ireland, 2008.

[65] D. Flynn, H. McNamara, J.P. O'Kane, and A.V. Pokrovskiĭ. Application of the Preisach model in soil-moisture hysteresis. In G. Bertotti and I.D. Mayergoyz, editors, *The Science of Hysteresis*, pages 689–744. Academic Press, 2006.

[66] M. Fuchs and G. Mingione. Full $C^{1,\alpha}$-regularity for free and constrained local minimizers of elliptic variational integrals with nearly linear growth. *Manuscripta Mathematica*, 102(2):227–250, 2000.

[67] M. Fuchs and G. Seregin. A regularity theory for variational integrals with $L \ln L$-growth. *Calculus of Variations and Partial Differential Equations*, 6(2):171–187, 1998.

[68] A.V. Fursikov and O.Yu. Imanuvilov (Emanuilov). On controllability of certain systems simulating a fluid flow. In *The IMA Volumes in Mathematics and its Applications*, volume 68, pages 149–184. Springer-Verlag, New York, 1994.

[69] S. Fučik and A. Kufner. *Nonlinear Differential Equations.* Elsevier, 2014.

[70] C. Gavioli. On the null-controllability of the semilinear heat equation with hysteresis. Master's thesis, Università degli Studi di Trento, Italy, 2017.

[71] C. Gavioli. Higher differentiability of solutions to a class of obstacle problems under non-standard growth conditions. *Forum Mathematicum*, 31(6):1501–1516, 2019.

[72] C. Gavioli. A priori estimates for solutions to a class of obstacle problems under p, q-growth conditions. *Journal of Elliptic and Parabolic Equations*, 5(2):325–347, 2019.

[73] C. Gavioli. Higher differentiability for a class of obstacle problems with nearly linear growth conditions. *Rendiconti Lincei-Matematica e Applicazioni*, 2020. To appear.

[74] C. Gavioli and P. Krejčí. On the null controllability of the heat equation with hysteresis in phase transition modeling. In *Extended Abstracts Spring 2018*, pages 63–71. Springer, 2019.

[75] C. Gavioli and P. Krejčí. Control and controllability of PDEs with hysteresis. *Applied Mathematics & Optimization*, 2020.

[76] C. Gavioli and P. Krejčí. On a viscoelastoplastic porous medium problem with nonlinear interaction. *SIAM Journal on Mathematical Analysis*, 2020. To appear.

[77] A. Gentile. Higher differentiability results for solutions to a class of non-autonomous obstacle problems with sub-quadratic growth conditions. *arXiv preprint arXiv:2007.04064*, 2020.

[78] D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983.

[79] R. Giova. Higher differentiability for n-harmonic systems with Sobolev coefficients. *Journal of Differential Equations*, 259(11):5667–5687, 2015.

[80] R. Giova and A. Passarelli di Napoli. Regularity results for a priori bounded minimizers of non-autonomous functionals with discontinuous coefficients. *Advances in Calculus of Variations*, 12(1):85–110, 2019.

[81] E. Giusti. *Direct Methods in the Calculus of Variations*. World Scientific, 2003.

[82] L. Greco, T. Iwaniec, and C. Sbordone. Variational integrals of nearly linear growth. *Differential and Integral Equations*, 10(4):687–716, 1997.

[83] P. Hajłasz. Sobolev spaces on an arbitrary metric space. *Potential Analysis*, 5(4):403–415, 1996.

[84] R. Haverkamp, P. Reggiani, P.J. Ross, and J.-Y. Parlange. Soil water hysteresis prediction model based on theory and geometric scaling. *Washington DC American Geophysical Union Geophysical Monograph Series*, 129:213–246, 2002.

[85] A. Hundertmark-Zaušková, M. Lukáčová-Medviďová, and Š. Nečasová. On the existence of weak solution to the coupled fluid-structure interaction problem for non-newtonian shear-dependent fluid. *Journal of the Mathematical Society of Japan*, 68(1):193–243, 2016.

[86] A.D. Ioffe. On lower semicontinuity of integral functionals I. *SIAM Journal on Control and Optimization*, 15(4):521–538, 1977.

[87] D. Kinderlehrer and G. Stampacchia. *An Introduction to Variational Inequalities and their Applications*, volume 31. SIAM, 1980.

[88] M.A. Krasnosel'skiĭ and A.V. Pokrovskiĭ. *Systems with Hysteresis*. Springer, Heidelberg, 1989. (Russian edition: Nauka, Moscow, 1983).

[89] P. Krejčí. Hysteresis operators – a new approach to evolution differential inequalities. *Commentationes Mathematicae Universitatis Carolinae*, 30(3):525–536, 1989.

[90] P. Krejčí. On Maxwell equations with the Preisach hysteresis operator: the one-dimensional time-periodic case. *Aplikace Matematiky*, 34(5):364–374, 1989.

[91] P. Krejčí. *Hysteresis, Convexity and Dissipation in Hyperbolic Equations.* GAKUTO International series. Gakkōtoscho, Tokyo, 1996.

[92] P. Krejčí. Elastoplastic reaction of a container to water freezing. *Mathematica Bohemica*, 135(4):423–441, 2010.

[93] P. Krejčí. The Preisach hysteresis model: Error bounds for numerical identification and inversion. *Discrete & Continuous Dynamical Systems-Series S*, 6(1):101–119, 2013.

[94] P. Krejčí. Boundedness of solutions to a degenerate diffusion equation. In P. Colli, A. Favini, E. Rocca, G. Schimperna, and J. Sprekels, editors, *Solvability, Regularity, and Optimal Control of Boundary Value Problems for PDEs: In Honour of Prof. Gianni Gilardi*, volume 22 of *Springer INdAM Series*, pages 305–326. Springer, Cham, 2017.

[95] P. Krejčí and L. Panizzi. Regularity and uniqueness in quasilinear parabolic systems. *Applications of Mathematics*, 56(4):341–370, 2011.

[96] P. Krejčí and E. Rocca. Well-posedness of an extended model for water-ice phase transitions. *Discrete & Continuous Dynamical Systems-S*, 6(2):439–460, 2013.

[97] P. Krejčí, E. Rocca, and J. Sprekels. A bottle in a freezer. *SIAM Journal on Mathematical Analysis*, 41(5):1851–1873, 2009.

[98] P. Krejčí, E. Rocca, and J. Sprekels. Liquid-solid phase transitions in a deformable container. In *Continuous Media with Microstructure*, pages 285–300. Springer, 2010.

[99] P. Krejčí, E. Rocca, and J. Sprekels. Phase separation in a gravity field. *Discrete & Continuous Dynamical Systems-S*, 4(2):391–407, 2011.

[100] P. Krejčí, E. Rocca, and J. Sprekels. Unsaturated deformable porous media flow with thermal phase transition. *Mathematical Models and Methods in Applied Sciences*, 27(14):2675–2710, 2017.

[101] P. Krejčí and J. Sprekels. Elastic–ideally plastic beams and Prandtl–Ishlinskiĭ hysteresis operators. *Mathematical Methods in the Applied Sciences*, 30(18):2371–2393, 2007.

[102] O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N. Ural'tseva. *Linear and Quasilinear Equations of Parabolic Type*. Nauka, Moscow, 1967.

[103] O.A. Ladyzhenskaya and N.N. Ural'tseva. *Linear and Quasilinear Equations of Elliptic Type*. Nauka, Moscow, 1973.

[104] L.D. Landau and E.M. Lifschitz. *Mechanics of Continuous Media*. GITTL, Moscow, 1953.

[105] J. Lemaitre and J.L. Chaboche. *Mechanics of Solid Materials*. Cambridge University Press, Cambridge, 1990.

[106] P. Lindqvist. Regularity for the gradient of the solution to a nonlinear obstacle problem with degenerate ellipticity. *Nonlinear Analysis*, 12(11):1245–1255, 1988.

[107] J.L. Lions. *Quelques méthodes de résolution des problemes aux limites non linéaires*. Dunod, 1969.

[108] J.L. Lions and G. Stampacchia. Variational inequalities. *Communications on pure and applied mathematics*, 20(3):493–519, 1967.

[109] P. Marcellini. Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions. *Archive for Rational Mechanics and Analysis*, 105(3):267–284, 1989.

[110] P. Marcellini. Regularity and existence of solutions of elliptic equations with p, q-growth conditions. *Journal of Differential Equations*, 90(1):1–30, 1991.

[111] P. Marcellini. Regularity for elliptic equations with general growth conditions. *Journal of Differential Equations*, 105(2):296–333, 1993.

[112] P. Marcellini. Regularity for some scalar variational problems under general growth conditions. *Journal of Optimization Theory and Applications*, 90(1):161–181, 1996.

[113] P. Marcellini. A variational approach to parabolic equations under general and p, q-growth conditions. *Nonlinear Analysis*, 194:111456, 2020.

[114] J.H. Michael and W.P. Ziemer. Interior regularity for solutions to obstacle problems. *Nonlinear Analysis: Theory, Methods & Applications*, 10(12):1427–1448, 1986.

[115] J. Nečas and I. Hlaváček. *Mathematical Theory of Elastic and Elasto-Plastic Bodies: An Introduction*, volume 3 of *Studies in Applied Mechanics*. Elsevier Scientific Publishing Co., Amsterdam, 1981.

[116] L. Nirenberg. Remarks on strongly elliptic partial differential equations. *Communications on Pure and Applied Mathematics*, 8(4):648–674, 1955.

[117] J. Ok. Calderón–Zygmund estimates for a class of obstacle problems with nonstandard growth. *Nonlinear Differential Equations and Applications NoDEA*, 23(4), 2016. Art. No. 50.

[118] J. Ok. Gradient continuity for nonlinear obstacle problems. *Mediterranean Journal of Mathematics*, 1(14):1–24, 2016.

[119] J. Ok. Regularity results for a class of obstacle problems with nonstandard growth. *Journal of Mathematical Analysis and Applications*, 444(2):957–979, 2016.

[120] A. Passarelli di Napoli. Higher differentiability of minimizers of variational integrals with Sobolev coefficients. *Advances in Calculus of Variations*, 7(1):59–89, 2014.

[121] A. Passarelli di Napoli. Higher differentiability of solutions of elliptic systems with Sobolev coefficients: the case $p = n = 2$. *Potential Analysis*, 41:715–735, 2014.

[122] A. Passarelli di Napoli. Existence and regularity results for a class of equations with logarithmic growth. *Nonlinear Analysis*, 125:290–309, 2015.

[123] A. Passarelli di Napoli. Regularity results for non-autonomous variational integrals with discontinuous coefficients. *Rendiconti Lincei-Matematica e Applicazioni*, 26(4):475–497, 2015.

[124] F. Preisach. Über die magnetische Nachwirkung. *Zeitschrift für physik*, 94(5-6):277–302, 1935.

[125] R.E. Showalter. Diffusion in deforming porous media. *Dynamis of Continuous Discrete and Impulsive Systems Series A*, 10:661–678, 2003.

[126] R.E. Showalter and U. Stefanelli. Diffusion in poro-plastic media. *Mathematical Methods in the Applied Sciences*, 27(18):2131–2151, 2004.

[127] G. Stampacchia. Formes bilinéaires coercitives sur les ensembles convexes. *Comptes Rendus Hebdomadaires Des Seances De L'Academie Des Sciences*, 258(18):4413–4416, 1964.

[128] K. Uhlenbeck. Regularity for a class of non-linear elliptic systems. *Acta Mathematica*, 138:219–240, 1977.

[129] A. Visintin. On the Preisach model for hysteresis. *Nonlinear Analysis: Theory, Methods & Applications*, 8(9):977–996, 1984.

[130] A. Visintin. *Differential Models of Hysteresis*, volume 111 of *Applied Mathematical Sciences.* Springer-Verlag, Berlin-Heidelberg, 1994.

[131] A. Visintin. *Models of Phase Transitions*, volume 28 of *Progress in Nonlinear Differential Equations and Their Applications*. Birkhäuser, Boston, 1996.

[132] A. Visintin. Introduction to Stefan-type problems. *Handbook of differential equations: evolutionary equations*, 4:377–484, 2008.

[133] V.V. Zhikov. Questions of convergence, duality, and averaging for functionals of the calculus of variations. *Mathematics of the USSR-Izvestiya*, 23(2):243–276, 1984.

[134] V.V. Zhikov. Averaging of functionals of the calculus of variations and elasticity theory. *Mathematics of the USSR-Izvestiya*, 29(1):33–66, 1987.

[135] V.V. Zhikov. On Lavrentiev's phenomenon. *Russian Journal of Mathematical Physics*, 3:249–269, 1995.

[136] V.V. Zhikov. On some variational problems. *Russian Journal of Mathematical Physics*, 5(1):105–116, 1997.