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Asymmetric Stochastic Transport Models with $\mathscr{U}_q(\mathfrak{su}(1,1))$ Symmetry

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Abstract By using the algebraic construction outlined in Carinci et al. (arXiv:1407.3367, 2014), we introduce several Markov processes related to the $\mathscr{U}_q(\mathfrak{su}(1,1))$ quantum Lie algebra. These processes serve as asymmetric transport models and their algebraic structure easily allows to deduce duality properties of the systems. The results include: (a) the asymmetric version of the Inclusion Process, which is self-dual; (b) the diffusion limit of this process, which is a natural asymmetric analogue of the and which turns out to have the Symmetric Inclusion Process as a dual process; (c) the asymmetric analogue of the KMP Process, which also turns out to have a symmetric dual process. We give applications of the various duality relations by computing exponential moments of the current.

1 Introduction

1.1 Motivations

Exactly solvable stochastic systems out-of-equilibrium have received considerable attention in recent days [7,9,10,14,19,27,30]. Often in the analysis of these models duality (or self-duality) is a crucial ingredient by which the study of n-point correlations is reduced to the study of n dual particles. For instance, the exact current statistics in the case of the asymmetric exclusion process is obtained by solving the dual particle dynamics via Bethe ansatz [6,21,29].

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The duality property has algebraic roots, as was first noticed by Schütz and Sandow for symmetric exclusion processes [28], which is related to the classical Lie algebra $\mathfrak{su}(2)$. Next this symmetry approach was extended by Schütz [29] to the quantum Lie algebra $\mathscr{U}_q(\mathfrak{su}(2))$ in a representattion of spin $1/2$, thus providing self-duality of the asymmetric exclusion process (see [31] for a recent work on currents in ASEP, using the quantum Lie algebra $\mathscr{U}_q(\mathfrak{su}(2))$). Notice that the connection between Markov processes and quantum spin chains has been studied extensively by Alcaraz et al., see e.g. [1,2]. However, in this literature no connection with duality or self-duality is made.

Recently Markov processes with the $\mathscr{U}_q(\mathfrak{su}(2))$ algebraic structure for higher spin value have been introduced and studied in [13]. This lead to a family of non-integrable asymmetric generalization of the partial exclusion process (see also [25]).

In [16,17] the algebraic approach to duality has been extended by connecting duality functions to the algebra of operators commuting with the generator of the process. In particular for the models of heat conduction studied in [17] the underlying algebraic structure turned out to be $\mathscr{U}(\mathfrak{su}(1,1))$. This class is richer than its fermionic counterpart related to the classical Lie algebra $\mathscr{U}(\mathfrak{su}(2))$ which is at the root of processes of exclusion type. In particular, the classical Lie algebra $\mathscr{U}(\mathfrak{su}(1,1))$ has been shown to be related to a large class of symmetric processes, including: (a) an interacting particle system with attractive interactions (Inclusion Process [17,18]); (b) interacting diffusion processes for heat conduction (Brownian Energy Process [12,17]); (c) redistribution models of KMP-type [11,22]. The dualities and self-dualities of all these processes arise naturally from the symmetries which are built in the construction.

It is the aim of this paper to provide the asymmetric version of these models with (self)-duality property, via the study of the deformed quantum Lie algebra $\mathscr{U}_q(\mathfrak{su}(1,1))$. This provides a new class of bulk-driven non-equilibrium systems with duality, which includes in particular an asymmetric version of the KMP model [22]. The diversity of models related to the classical $\mathscr{U}(\mathfrak{su}(1,1))$ will also appear here in the asymmetric context where we consider the quantum Lie algebra $\mathscr{U}_q(\mathfrak{su}(1,1))$.

1.2 Models and Abbreviations

For the sake of simplicity, we will use the following acronyms in order to describe the class of new processes that arise from our construction.

(a) Discrete representations will provide interacting particle systems in the class of *Inclusion Processes*. For a parameter $k \in \mathbb{R}_+$, the Symmetric Inclusion Process version is denoted by SIP(k), and **ASIP**(q, k) is the corresponding asymmetric version, with asymmetry parameter $q \in (0, 1)$.
(b) Continuous representations give rise to diffusion processes in the class of *Brownian Energy Processes*. For $k \in \mathbb{R}_+$, the Symmetric Brownian Energy Process is denoted by BEP(k), and **ABEP**(σ, k) is the asymmetric version with asymmetry parameter $\sigma > 0$.
(c) By instantaneous thermalization, redistribution models are obtained, where energy or particles are redistributed at Poisson event times. This class includes the thermalized version of ABEP(σ, k), which is denoted by Th-ABEP(σ, k). In the particular case $k = 1/2$ the Th-ABEP(σ, k) is called the Asymmetric KMP (Kipnis-Marchioro-Presutti) model, denoted by **AKMP**(σ), which becomes the KMP model as $\sigma \to 0$. The instantaneous thermalization of the ASIP(q, k) yields the Th-ASIP(q, k) process.

1.3 Markov Processes with Algebraic Structure

In [13] we constructed a generalization of the asymmetric exclusion process, allowing $2j$ particles per site with self-duality properties reminiscent of the self-duality of the standard ASEP found initially by Schütz [29]. This construction followed a general scheme where one starts from the Casimir operator C of the quantum Lie algebra $\mathscr{U}_q(\mathfrak{su}(2))$, and applies a coproduct to obtain an Hamiltonian $H_{i,i+1}$ working on the occupation number variables at sites i and $i+1$. The operator $H = \sum_{i=1}^{L-1} H_{i,i+1}$ then naturally allows a rich class of commuting operators (symmetries), obtained from the n-fold coproduct applied to any generator of the algebra. This operator H is not yet the generator of a Markov process. But H allows a strictly positive ground state, which can also be constructed from the symmetries applied to a trivial ground state. Via a ground state transformation, H can then be turned into a Markov generator L of a jump process where particles hop between nearest neighbor sites and at most $2j$ particles per site are allowed. The symmetries of H directly translate into the symmetries of L, which in turn directly translate into self-duality functions.

This construction is in principle applicable to every quantum Lie algebra with a non-trivial center. However, it is not guaranteed that a Markov generator can be obtained. This depends on the chosen representation of the generators of the algebra, and the choice of the co-product. Recently the construction has been applied to algebras with higher rank, such as $\mathscr{U}_q(\mathfrak{gl}(3))$ [5,23] or $\mathscr{U}_q(\mathfrak{sp}(4))$ [23], yielding two-component asymmetric exclusion process with multiple conserved species of particles.

1.4 Informal Description of Main Results

In [17] we introduced a class of processes with $\mathfrak{su}(1,1)$ symmetry which in fact arise from this construction for the Lie algebra $\mathscr{U}(\mathfrak{su}(1,1))$. In this paper we look for natural asymmetric versions of the processes constructed in [17], and [11]. In particular the natural asymmetric analogue of the KMP process is a target. The main results are the following

(a) *Self-duality of ASIP*(q,k). We proceed via the same construction as in [13] for the algebra $\mathscr{U}_q(\mathfrak{su}(1,1))$ to find the ASIP(q,k) which is the "correct" asymmetric analogue of the SIP(k). The parameter q tunes the asymmetry: $q \to 1$ gives back the SIP(k). This process is then via its construction self-dual with a non-local self-duality function.

(b) *Duality between ABEP*(σ,k) *and* SIP(k). We then show that in the limit $\epsilon \to 0$ where simultaneously the asymmetry is going to zero ($q = 1-\epsilon\sigma$ tends to unity), and the number of particles to infinity $\eta_i = \lfloor \epsilon^{-1} x_i \rfloor$, we obtain a diffusion process ABEP(σ,k) which is reminiscent of the Wright–Fisher diffusion with mutation and a selective drift. As a consequence of self-duality of ASIP(q,k) we show that this diffusion process is dual to the SIP(k), i.e., the dual process is symmetric, and the asymmetry is in the duality function. Notice that this is the first example of duality between a truly asymmetric system (i.e. bulk-driven) and a symmetric system (with zero current).

(c) *Duality of instantaneous thermalization models*. Finally, we then consider instantaneous thermalization of ABEP(σ,k) to obtain an asymmetric energy redistribution model of KMP type. Its dual is the instantaneous thermalization of the SIP(k) which for $k=1/2$ is exactly the dual KMP process.

1.5 Organization of the Paper

The rest of our paper is organized as follows. In Sect. 2 we introduce the process ASIP(q, k). After discussing some limiting cases, we show that this process has reversible profile product measures on $\mathbb{Z}_+$ (but not on $\mathbb{Z}$).

In Sect. 3 we consider the weak asymmetry limit of ASIP(q, k). This leads to the diffusion process ABEP(σ, k), that also has reversible inhomogeneous product measures on the half-line. We prove that ABEP(σ, k) is a genuine non-equilibrium asymmetric system in the sense that it has a non-zero average current. Nevertheless in the last part of Sect. 3 we show that the ABEP(σ, k) can be mapped – via a global change of coordinates – to the BEP(k), which is a symmetric system with zero-current. In Sect. 3.6 this is also explained in the framework of the representation theory of the classical Lie algebra $\mathscr{U}(\mathfrak{su}(1, 1))$.

In Sect. 4 we introduce the instantaneous thermalization limits of both ASIP(q, k) and ABEP(σ, j) which are a particle, resp. energy, redistribution model at Poisson event times. This provides asymmetric redistribution models of KMP type.

In Sect. 5 we introduce the self-duality of the ASIP(q, k) and prove various other duality relations that follow from it. In particular, once the self-duality of ASIP(q, k) is obtained, duality of ABEP(σ, k) with SIP(k) follows from a limiting procedure which is proved in Sect. 5.2. In the limit of an infinite number of particles with weak-asymmetry, the original process scales to ABEP(σ, k), whereas in the dual process the asymmetry disappears because the number of particles is finite. Next the self-duality and duality of thermalized models is derived in Sect. 5.3.

In Sect. 6 we illustrate the use of the duality relations in various computations of exponential moments of currents. Finally, the last section is devoted to the full construction of the ASIP(q, k) from a $\mathscr{U}_q(\mathfrak{su}(1, 1))$ symmetric quantum Hamiltonian and the proof of self-duality from the symmetries of this Hamiltonian.

2 The Asymmetric Inclusion Process ASIP(q, k)

2.1 Basic Notation

We will consider as underlying lattice the finite lattice $\Lambda_L = \{1, \ldots, L\}$ or the periodic lattice $\mathbb{T}_L = \mathbb{Z}/L\mathbb{Z}$. At the sites of Λ_L we allow an arbitrary number of particles. The particle system configuration space is $\Omega_L = \mathbb{N}^{\Lambda_L}$. Elements of Ω_L are denoted by η, ξ and for $\eta \in \Omega_L$, $i \in \Lambda_L$, we denote by $\eta_i \in \mathbb{N}$ the number of particles at site i. For $\eta \in \Omega_L$ and $i, j \in \Lambda_L$ such that $\eta_i > 0$, we denote by $\eta^{i,j}$ the configuration obtained from η by removing one particle from i and putting it at j.

We need some further notation of q-numbers. For $q \in (0, 1)$ and $n \geq 0$ we introduce the q-number

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{2.1}$$

satisfying the property $\lim_{q\to 1}[n]_q = n$. The first q-natural number's ($n \in \mathbb{N}_0$) are thus given by

$$[0]_q = 0, \qquad [1]_q = 1, \qquad [2]_q = q + q^{-1}, \qquad [3]_q = q^2 + 1 + q^{-2}, \quad \ldots$$

We also introduce the q-factorial for $n \in \mathbb{N}_0$

$$[n]_q! := [n]_q \cdot [n-1]_q \cdot \dots \cdot [1]_q \quad \text{for } n \geq 1, \qquad \text{and} \quad [0]_q! := 1$$

and the q-binomial coefficient

$$\binom{n}{m}_q := \frac{[n]_q!}{[m]_q![n-m]_q!} \quad \text{for} \quad n \geq m \tag{2.2}$$

and, for $m \in \mathbb{N}$, $\alpha \in (-1, 1)$

$$\binom{m+\alpha}{m}_q := \frac{[m+\alpha]_q \cdot [m-1+\alpha]_q \cdot \ldots \cdot [1+\alpha]_q}{[m]_q!} \tag{2.3}$$

Further we denote

$$(a; q)_m := (1-a)(1-aq)\cdots\left(1-aq^{m-1}\right). \tag{2.4}$$

2.2 The ASIP(q, k) Process

We introduce the process in finite volume by specifying its generator.

Definition 2.1 (*ASIP(q,k) process*)

1. The ASIP(q, k) with closed boundary conditions is defined as the Markov process on Ω_L with generator defined on functions $f : \Omega_L \to \mathbb{R}$

$$(\mathscr{L}_{(L)}^{ASIP(q,k)} f)(\eta) := \sum_{i=1}^{L-1} (\mathscr{L}_{i,i+1}^{ASIP(q,k)} f)(\eta) \qquad \text{with}$$
$$\begin{aligned}(\mathscr{L}_{i,i+1}^{ASIP(q,k)} f)(\eta) := {} & q^{\eta_i-\eta_{i+1}+(2k-1)}[\eta_i]_q[2k+\eta_{i+1}]_q(f(\eta^{i,i+1}) - f(\eta)) \\ & + q^{\eta_i-\eta_{i+1}-(2k-1)}[2k+\eta_i]_q[\eta_{i+1}]_q(f(\eta^{i+1,i}) - f(\eta))\end{aligned} \tag{2.5}$$

2. The ASIP(q, k) with periodic boundary conditions is defined as the Markov process on $\mathbb{N}^{\mathbb{T}_L}$ with generator

$$(\mathscr{L}_{(\mathbb{T}_L)}^{ASIP(q,k)} f)(\eta) := \sum_{i\in\mathbb{T}_L} (\mathscr{L}_{i,i+1}^{ASIP(q,k)} f)(\eta) \tag{2.6}$$

Since in finite volume we always start with finitely many particles, and the total particle number is conserved, the process is automatically well defined as a finite state space continuous time Markov chain. Later on (see Sect. 6.1) we will consider expectations of the self-duality functions in the infinite volume limit. In this way we can deal with relevant infinite volume expectations without having to solve the full existence problem of the ASIP(q, k) in infinite volume for a generic initial data. This might actually be an hard problem due to the lack of monotonicity.

2.3 Limiting Cases

The ASIP (q, k) degenerates to well known interacting particle systems when its parameters take the limiting values $q \to 1$ and $k \to \infty$ recovering the cases of symmetric evolution or totally asymmetric zero range interaction. Notice in particular that these two limits do not commute.

- Convergence to symmetric processes

(i) $q \to 1, k$ *fixed* The ASIP(q, k) reduces to the SIP(k), i.e. the Symmetric Inclusion Process with parameter k. All the results of the present paper apply also to this symmetric case. In particular, in the limit $q \to 1$, the self-duality functions that will be given in theorem 5.1 below converge to the self-duality functions of the SIP(k) (given in [11]).
(ii) $q \to 1, k \to \infty$ Furthermore, when the Symmetric Inclusion Process is time changed so that time is scaled down by a factor $1/2k$, then in the limit $k \to \infty$ the symmetric inclusion converges weakly in path space to a system of symmetric independent random walkers (moving at rate 1).

- Convergence to totally asymmetric processes

(iii) $k \to \infty, q$ *fixed* If the limit $k \to \infty$ is performed first, then a totally asymmetric system is obtained under proper time rescaling. Indeed, by multiplying the ASIP(q, k) generator by $(1-q^2)q^{4k-1}$ one has

$$(1-q^2)q^{4k-1}\left[\mathscr{L}^{ASIP}_{i,i+1} f\right](\eta) = q^{4k}\,\frac{\left(q^{2\eta_i}-1\right)\left(q^{4k}-q^{-2\eta_{i+1}}\right)}{\left(1-q^2\right)}\left[f\left(\eta^{i,i+1}\right)-f(\eta)\right] + \frac{\left(q^{-2\eta_{i+1}}-1\right)\left(1-q^{2\eta_i+4k}\right)}{\left(q^{-2}-1\right)}\left[f\left(\eta^{i+1,i}\right)-f(\eta)\right]$$

Therefore, considering the family of processes $y^{(k)}(t) := \{y_i^{(k)}(t)\}_{i\in\Lambda_L}$ labeled by $k \geq 0$ and defining

$$y_i^{(k)}(t) := \eta_i\left((1-q^2)q^{4k-1}t\right)$$

one finds that in the limit $k \to \infty$ the process $y^{(k)}(t)$ converges weakly to the Totally Asymmetric Zero Range process $y(t)$ with generator given by:

$$\left(\mathscr{L}^{q-\text{TAZRP}}_{su(1,1)} f\right)(y) = \sum_{i=1}^{L-1} \frac{q^{-2y_{i+1}}-1}{q^{-2}-1}\left[f\left(y^{i+1,i}\right)-f(y)\right], \qquad f:\Omega_L \to \mathbb{R} \tag{2.7}$$

In this system, particles jump to the left only with rates that are monotone increasing functions of the occupation variable of the departure site. Note that the rates are unbounded for $y_{i+1} \to \infty$, nevertheless the process is well defined even in the infinite volume, as it belongs to the class considered in [4]. This is to be compared to the case of the deformed algebra $U_q(\mathfrak{sl}_2)$ [13] whose scaling limit with infinite spin is given by [6]

$$\left(\mathscr{L}^{(q-\text{TAZRP})}_{su(2)} f\right)(y) = \sum_{i=1}^{L-1} \frac{1-q^{2y_i}}{1-q^2}\left[f\left(y^{i,i+1}\right)-f(y)\right], \qquad f:\Omega_L \to \mathbb{R} \tag{2.8}$$

Here particles jump to the right only with rates that are also a monotonous increasing function of the occupation variable of the departure site, however now it is a bounded function approaching 1 in the limit $y_i \to \infty$. In [15] it is proved that the totally asymmetric zero range process (2.8) is in the KPZ universality class. It is an interesting open problem to prove or disprove that the same conclusion holds true for (2.7) [26]. We remark that the rates of (2.7) are (discrete) convex function and this also translates into convexity of the stationary current $j(\rho)$ as a function of the density ρ, whereas for (2.8) we have concave relations.

(iv) $k \to \infty$, $q \to 1$ In the limit $q \to 1$ the zero range process in (2.7) reduces to a system of totally asymmetric independent walkers. This is to be compared to item (ii) where symmetric walkers were found if the two limits were performed in the reversed order.

2.4 Reversible Profile Product Measures

Here we describe the reversible measures of ASIP(q, k).

Theorem 2.1 (Reversible measures of ASIP(q, k)) *For all $L \in \mathbb{N}$, $L \geq 2$, the following results hold true:*

1. *the ASIP(q, k) on Λ_L with closed boundary conditions admits a family labeled by α of reversible product measures with marginals given by*

$$\mathbb{P}^{(\alpha)}(\eta_i = n) = \frac{\alpha^n}{Z_i^{(\alpha)}} \binom{n+2k-1}{n}_q \cdot q^{4kin} \qquad n \in \mathbb{N} \tag{2.9}$$

for $i \in \Lambda_L$ and $\alpha \in [0, q^{-(2k+1)})$ (with the convention $\binom{2k-1}{0}_q = 1$). The normalization is

$$Z_i^{(\alpha)} = \sum_{n=0}^{+\infty} \binom{n+2k-1}{n}_q \cdot \alpha^n q^{4kin} = \frac{1}{(\alpha q^{4ki-(2k-1)}; q^2)_{2k}} \tag{2.10}$$

and for this measure

$$\mathbb{E}^{(\alpha)}(\eta_i) = \sum_{l=0}^{2k-1} \frac{1}{q^{-2l}(\alpha q^{4ki-2k+1})^{-1} - 1}. \tag{2.11}$$

(2) *The ASIP(q, k) process on the torus $\mathbb{T}_L$ with periodic boundary condition does not admit homogeneous product measures.*

Proof The proof of item (2) is similar to the proof of Theorem 3.1, item (d) in [13] and we refer the reader to that paper for full details. The main idea is that if we have the rate $c^+(\eta_i, \eta_{i+1})$ (resp. $c^-(\eta_i, \eta_{i-1})$) for a particle to jump from i to $i+1$ (resp. from i to $i-1$) and there exists a homogeneous product measure $\bar{\mu}$ with marginals $\mu(\eta_i)$, then the stationarity condition implies that the function

$$F(\eta_1, \eta_2) = c^+(\eta_1 + 1, \eta_2 - 1) + c^-(\eta_2 + 1, \eta_1 - 1) - c^+(\eta_1, \eta_2) - c^-(\eta_2, \eta_1)$$

is of the form $g(\eta_1) - g(\eta_2)$, which is not the case, as can be seen from the explicit expressions of the rates just as in [13].

To prove item (1) consider the detailed balance relation

$$\mu(\eta) c_q(\eta, \eta^{i,i+1}) = \mu(\eta^{i,i+1}) c_q(\eta^{i,i+1}, \eta) \tag{2.12}$$

where the hopping rates are given by

$$\begin{aligned} c_q(\eta, \eta^{i,i+1}) &= q^{\eta_i - \eta_{i+1} + 2k - 1} [\eta_i]_q [2k + \eta_{i+1}]_q \\ c_q(\eta^{i,i+1}, \eta) &= q^{\eta_i - \eta_{i+1} - 2k - 1} [2k + \eta_i - 1]_q [\eta_{i+1} + 1]_q \end{aligned}$$

and μ denotes a reversible measure. Suppose now that μ is a product measure of the form $\mu = \otimes_{i=1}^L \mu_i$. Then (2.12) holds if and only if

$$\begin{aligned} &\mu_i(\eta_i - 1)\mu_{i+1}(\eta_{i+1} + 1) q^{-2k} [2k + \eta_i - 1]_q [\eta_{i+1} + 1]_q \\ &\quad = \mu_i(\eta_i)\mu_{i+1}(\eta_{i+1}) q^{2k} [\eta_i]_q [2k + \eta_{i+1}]_q \end{aligned} \tag{2.13}$$

which implies that there exists $\alpha \in \mathbb{R}$ so that for all $i \in \Lambda_L$

$$\frac{\mu_i(n)}{\mu_i(n-1)} = \alpha q^{4ki} \frac{[2k+n-1]_q}{[n]_q}. \tag{2.14}$$

Then (2.9) follows from (2.14) after using an induction argument on n. The normalization $Z_i^{(\alpha)}$ is computed by using Corollary 10.2.2 of [3]. We have that

$$Z_i^{(\alpha)} < \infty \quad \text{if and only if} \quad 0 \leq \alpha < q^{-4ki+(2k-1)} \quad \text{for any } i \in \Lambda_L \tag{2.15}$$

As a consequence (since $q < 1$ and $i = 1$ is the worst case) α must belong to the interval $[0, q^{-(2k+1)})$. The expectation (2.11) is obtained by exploiting the identity

$$\mathbb{E}^{(\alpha)}(\eta_i) = \alpha \frac{d}{d\alpha} \ln Z_i^{(\alpha)}.$$

□

The following comments are in order:

(i) *vanishing asymmetry*: in the limit $q \to 1$ the reversible product measure of ASIP(q, k) converges to a product of Negative Binomial distributions with shape parameter $2k$ and success probability α, which are the reversible measures of the SIP(k) [11].

(ii) *monotonicity of the profile*: the average occupation number $\mathbb{E}^{(\alpha)}(\eta_i)$ in formula (2.11) is a decreasing function of i, and $\lim_{i\to\infty} \mathbb{E}^{(\alpha)}(\eta_i) = 0$.

(iii) *infinite volume*: the reversible product measures with marginal (2.9) are also well-defined in the limit $L \to \infty$. One could go further to $[-M, \infty) \cap \mathbb{Z}$ for $\alpha < q^{4kM+2k-1}$ (but not to the full line $\mathbb{Z}$). These infinite volume measure concentrate on configurations with a finite number of particles, and thus are the analogue of the profile measures in the asymmetric exclusion process [24].

3 The Asymmetric Brownian Energy Process ABEP(σ, k)

Here we will take the limit of weak asymmetry $q = 1 - \epsilon\sigma \to 1$ ($\epsilon \to 0$) combined with the number of particles proportional to ϵ^{-1}, going to infinity, and work with rescaled particle numbers $x_i = \lfloor \epsilon \eta_i \rfloor$. Reminiscent of scaling limits in population dynamics, this leads to a diffusion process of Wright–Fisher type [12], with σ-dependent drift term, playing the role of a selective drift in the population dynamics language, or bulk driving term in the non-equilibrium statistical physics language.

3.1 Definition

We define the ABEP(q, k) process via its generator. It has state space $\mathscr{X}_L = (\mathbb{R}_+)^L$, $\mathbb{R}_+ := [0, +\infty)$. Configurations are denoted by $x \in \mathscr{X}_L$, with x_i being interpreted as the energy at site $i \in \Lambda_L$.

Definition 3.1 (*ABEP(σ, k) process*)

1. Let $\sigma > 0$ and $k \geq 0$. The Markov process ABEP(σ, k) on the state space $\mathscr{X}_L$ with closed boundary conditions is defined by the generator working on the core of smooth functions $f : \mathscr{X}_L \to \mathbb{R}$ via

$$\left[\mathscr{L}_{(L)}^{ABEP^{(\sigma,k)}} f\right](x) = \sum_{i=1}^{L-1} \left[\mathscr{L}_{i,i+1}^{ABEP^{(\sigma,k)}} f\right](x) \tag{3.1}$$

with

$$\left[\mathscr{L}_{i,i+1}^{ABEP(\sigma,k)} f\right](x) = \frac{1}{4\sigma^2}\left(1-\mathrm{e}^{-2\sigma x_i}\right)\left(\mathrm{e}^{2\sigma x_{i+1}}-1\right)\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_{i+1}}\right)^2 f(x)$$
$$-\frac{1}{2\sigma}\left\{\left(1-\mathrm{e}^{-2\sigma x_i}\right)\left(\mathrm{e}^{2\sigma x_{i+1}}-1\right)+2k\left(2-\mathrm{e}^{-2\sigma x_i}-\mathrm{e}^{2\sigma x_{i+1}}\right)\right\}$$
$$\times\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_{i+1}}\right) f(x)$$

2. The ABEP(σ, k) with periodic boundary conditions is defined as the Markov process on $\mathbb{R}_+^{\mathbb{T}_L}$ with generator

$$\left[\mathscr{L}_{(\mathbb{T}_L)}^{ABEP(\sigma,k)} f\right](x) := \sum_{i\in\mathbb{T}_L}\left[\mathscr{L}_{i,i+1}^{ABEP(\sigma,k)} f\right](x) \tag{3.2}$$

The ABEP(σ, k) is a genuine asymmetric non-equilibrium system, in the sense that its translation-invariant stationary state may sustain a non-zero current. To see this, let $\mathbb{E}$ denote expectation with respect to the translation invariant measure for the ABEP(σ, k) on $\mathbb{T}_L$. Let $f_i(x) := x_i$, then from (3.1) we have

$$\left[\mathscr{L}^{ABEP(\sigma,k)} f_i\right](x) = \Theta_{i,i+1}(x) - \Theta_{i-1,i}(x) \tag{3.3}$$

with

$$\Theta_{i,i+1}(x) = -\frac{1}{2\sigma}\left\{\left(1-\mathrm{e}^{-2\sigma x_i}\right)\left(\mathrm{e}^{2\sigma x_{i+1}}-1\right)+2k\left(2-\mathrm{e}^{-2\sigma x_i}-\mathrm{e}^{2\sigma x_{i+1}}\right)\right\} \tag{3.4}$$

So we have

$$\frac{d}{dt}\mathbb{E}_x\left[f_i(x(t))\right] = \mathbb{E}_x\left[\Theta_{i,i+1}(x(t))\right] - \mathbb{E}_x\left[\Theta_{i-1,i}(x(t))\right]$$

and then, from the continuity equation we have that, in a translation invariant state, $\mathscr{J}_{i,i+1} := -\mathbb{E}\left[\Theta_{i,i+1}\right]$ is the instantaneous stationary current over the edge $(i, i+1)$. Thus we have the following

Proposition 3.1 (Non-zero current of ABEP(σ, k))

$$\mathscr{J}_{i,i+1} = -\mathbb{E}\left[\Theta_{i,i+1}\right] < 0 \qquad \textit{if} \;\; k > 1/2$$

and

$$\mathscr{J}_{i,i+1} = -\mathbb{E}\left[\Theta_{i,i+1}\right] > 0 \qquad \textit{if} \;\; k = 0.$$

Proof In the case $k > 1/2$, taking expectation of (3.4) we obtain

$$\mathbb{E}\left[\Theta_{i,i+1}\right] = \frac{1}{2\sigma}\left\{(1-4k)+(2k-1)\mathbb{E}\left(\mathrm{e}^{2\sigma x_{i+1}}+\mathrm{e}^{-2\sigma x_i}\right)+\mathbb{E}\left(\mathrm{e}^{2\sigma(x_{i+1}-x_i)}\right)\right\}$$

Since expectation in the translation invariant stationary state of local variables are the same on each site and $\cosh(x) \geq 1$ one obtains

$$\mathbb{E}\left[\Theta_{i,+1}\right] \geq \frac{1}{2\sigma}\left\{(1-4k)+2(2k-1)+\mathbb{E}\left[\mathrm{e}^{2\sigma(x_{i+1}-x_i)}\right]\right\}$$

Furthermore, Jensen inequality and translation invariance implies that

$$\mathbb{E}\left[\Theta_{i,i+1}\right] > \frac{1}{2\sigma}\left\{(1-4k)+2(2k-1)+1\right\} = 0$$

In the case $k = 0$ one has

$$\mathbb{E}\left[\Theta_{i,i+1}\right] = \frac{1}{2\sigma}\,\mathbb{E}\Big[\big(1-\mathrm{e}^{-2\sigma x_i}\big)\big(1-\mathrm{e}^{2\sigma x_{i+1}}\big)\Big] < 0$$

which is negative because the function is negative a.s. □

3.2 Limiting Cases

- Symmetric processes

(i) $\sigma \to \mathbf{0}$, **k fixed:** we recover the Brownian Energy Process with parameter k, BEP(k) (see [11]) whose generator is

$$\mathscr{L}_{i,i+1}^{BEP(k)} = x_i x_{i+1}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^2 - 2k(x_i - x_{i+1})\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right) \tag{3.5}$$

(ii) $\sigma \to \mathbf{0}$, $\mathbf{k} \to \infty$: under the time rescaling $t \to t/2k$, one finds that in the limit $k \to \infty$ the BEP(k) process scales to a symmetric deterministic system evolving with generator

$$\left[\mathscr{L}_{i,i+1}^{DEP} f\right](x) = -(x_i - x_{i+1})\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right) f(x) \tag{3.6}$$

This deterministic system is symmetric in the sense that if the initial condition is given by $(x_i(0), x_{i+1}(0)) = (a, b)$ then the asymptotic solution is given by the fixed point $\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$ where the initial total energy $a + b$ is equally shared among the two sites.

- Wright–Fisher diffusion

(iii) $\sigma \simeq \mathbf{0}$, **k fixed:** the ABEP(σ, k) on the simplex can be read as a Wright Fisher model with mutation and selection, however we have not been able to find in the literature the specific form of selection appearing in (3.1) (see [12] for the analogous result when $\sigma = 0$). For fixed k, to first order in σ one recovers the standard Wright–Fisher model with constant mutation k and constant selection σ, i.e.

$$\mathscr{L}^{WF(\sigma,k)} = x_i x_{i+1}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^2 - (2\sigma x_i x_{i+1} + 2k(x_i - x_{i+1}))\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)$$

- Asymmetric Deterministic System

(iv) $\mathbf{k} \to \infty$, σ **fixed:** if the limit $k \to \infty$ is taken directly on the ABEP(σ, k) then, by time rescaling $t \to t/2k$ one arrives at an asymmetric deterministic system with generator

$$\mathscr{L}_{i,i+1}^{ADEP(\sigma)} = -\frac{1}{2\sigma}\left(2 - \mathrm{e}^{-2\sigma x_i} - \mathrm{e}^{2\sigma x_{i+1}}\right)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right) \tag{3.7}$$

This deterministic system is asymmetric in the sense that if the initial condition is given by $(x_i(0), x_{i+1}(0)) = (a, b)$ then the asymptotic solution is given by the fixed point

$$(A, B) := \left(\frac{1}{2\sigma}\ln\left(\frac{1+\mathrm{e}^{2\sigma(a+b)}}{2}\right), a+b-\frac{1}{2\sigma}\ln\left(\frac{1+\mathrm{e}^{2\sigma(a+b)}}{2}\right)\right)$$

where $A > B$.

(v) $\mathbf{k} \to \infty$, $\sigma \to \mathbf{0}$: in the limit $\sigma \to 0$ (3.7) converges to (3.6) and one recovers again the symmetric equi-distribution between the two sites of DEP process with generator (3.6).

(vi) $\mathbf{k} \to \infty$, $\sigma \to \infty$: in the limit $\sigma \to \infty$ one has the totally asymmetric stationary solution $(a + b, 0)$.

3.3 The ABEP(σ, k) as a Diffusion Limit of ASIP(q, k)

Here we show that the ABEP(σ, k) arises from the ASIP(q, k) in a limit of vanishing asymmetry and infinite particle number.

Theorem 3.1 (Weak asymmetry limit of ASIP(q, k)) *Fix $T > 0$. Let $\{\eta^\epsilon(t) : 0 \leq T\}$ denote the ASIP$(1-\sigma\epsilon, k)$ starting from initial condition $\eta^\epsilon(0)$. Assume that*

$$\lim_{\epsilon \to 0} \epsilon\eta^\epsilon(0) = x \in \mathscr{X}_L \tag{3.8}$$

Then as $\epsilon \to 0$, the process $\{\eta^\epsilon(t) : 0 \leq t \leq T\}$ converges weakly on path space to the ABEP(σ, k) starting from x.

Proof The proof follows the lines of the corresponding results in population dynamics literature, i.e., Taylor expansion of the generator and keeping the relevant orders. Indeed, by the Trotter–Kurtz theorem [24], we have to prove that on the core of the generator of the limiting process, we have convergence of generators. Because the generator is a sum of terms working on two variables, our theorem follows from the computational lemma below. □

Lemma 3.1 *If $\eta^\epsilon \in \Omega_L$ is such that $\epsilon\eta^\epsilon \to x \in \mathscr{X}_L$ then, for every smooth function $F : \mathscr{X}_L \to \mathbb{R}$, and for every $i \in \{1, \ldots, L-1\}$ we have*

$$\lim_{\epsilon \to 0} (\mathscr{L}_{i,i+1}^{ASIP(1-\epsilon\sigma,k)} F_\epsilon)(\eta^\epsilon) = \mathscr{L}_{i,i+1}^{ABEP(\sigma,k)} F(x) \tag{3.9}$$

where $F_\epsilon(\eta) = F(\epsilon\eta)$, $\eta \in \Omega_L$.

Proof Define $x^\epsilon = \epsilon\eta^\epsilon$. Then we have, by the regularity assumptions on F that

$$\begin{aligned} &F_\epsilon((\eta^\epsilon)^{i,i+1}) - F_\epsilon(\eta) \\ &= \epsilon \left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i} \right) F(x^\epsilon) + \epsilon^2 \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}} \right)^2 F(x^\epsilon) + O(\epsilon^3) \end{aligned} \tag{3.10}$$

and similarly

$$\begin{aligned} &F_\epsilon((\eta^\epsilon)^{i+1,i}) - F_\epsilon(\eta) \\ &= -\epsilon \left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i} \right) F(x^\epsilon) + \epsilon^2 \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}} \right)^2 F(x^\epsilon) + O(\epsilon^3) \end{aligned} \tag{3.11}$$

Then using $q = 1 - \epsilon\sigma$, and

$$(1-\epsilon\sigma)^{x_i^\epsilon/\epsilon} = \mathrm{e}^{-\sigma x_i} - 2x_i\sigma^2\mathrm{e}^{-2\sigma x_i}\epsilon + O(\epsilon^2)$$

straightforward computations give

$$\left[\mathscr{L}_{i,i+1}^\epsilon F\right](x^\epsilon) = \left[B_\epsilon(x^\epsilon) \left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i} \right) + D_\epsilon(x^\epsilon) \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}} \right)^2 \right] F(x^\epsilon) + O(\epsilon)$$

with

$$\begin{aligned} B_\epsilon(x) &= \frac{1}{2\sigma} \left\{ (1-\mathrm{e}^{-2\sigma x_i})(\mathrm{e}^{2\sigma x_{i+1}} - 1) + 2k\left(2 - \mathrm{e}^{-2\sigma x_i} - \mathrm{e}^{2\sigma x_{i+1}}\right) \right\} + O(\epsilon) \\ D_\epsilon(x) &= \frac{1}{4\sigma^2} (1-\mathrm{e}^{-2\sigma x_i})(\mathrm{e}^{2\sigma x_{i+1}} - 1) + O(\epsilon) \end{aligned} \tag{3.12}$$

Then we recognize

$$\left[B_\epsilon(x^\epsilon)\left(\frac{\partial}{\partial x_{i+1}}-\frac{\partial}{\partial x_i}\right)+D_\epsilon(x^\epsilon)\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_{i+1}}\right)^2\right]F(x^\epsilon)$$
$$=\left(\mathscr{L}_{i,i+1}^{ABEP(\sigma,k)}F\right)(x^\epsilon)$$

which ends the proof of the lemma by the smoothness of F and because by assumption, $x^\epsilon \to x$. □

The weak asymmetry limit can also be performed on the q-TAZRP. This yields a totally asymmetric deterministic system as described in the following theorem.

Theorem 3.2 (Weak asymmetry limit of q-TAZRP) *Fix $T > 0$. Let $\{y^\epsilon(t) : 0 \le T\}$ denote the q^ϵ-TAZRP, $q_\epsilon := 1 - \sigma\epsilon$, with generator (2.7) and initial condition $y^\epsilon(0)$. Assume that*

$$\lim_{\epsilon\to 0}\epsilon y^\epsilon(0) = y \in \mathscr{X}_L \tag{3.13}$$

Then as $\epsilon \to 0$, the process $\{y^\epsilon(t) : 0 \le t \le T\}$ converges weakly on path space to the Totally Asymmetric Deterministic Energy Process, TADEP(σ) with generator

$$(\mathscr{L}_{i,i+1}^{TADEP}f)(z) = -\left(\frac{1-e^{2\sigma z_{i+1}}}{2\sigma}\right)\left(\frac{\partial}{\partial z_i}-\frac{\partial}{\partial z_{i+1}}\right)f(z), \qquad f:\mathbb{R}_+^L\to\mathbb{R} \tag{3.14}$$

initialized from the configuration y.

Proof The proof is analogous to the proof of Theorem 3.1

3.4 Reversible Measure of the ABEP(σ, k)

Theorem 3.3 (ABEP(σ, k) reversible measures) *For all $L \in \mathbb{N}$, $L \ge 2$, the ABEP(q, k) on $\mathscr{X}_L$ with closed boundary conditions admits a family (labeled by $\gamma > -4\sigma k$) of reversible product measures with marginals given by*

$$\mu_i(x_i) := \frac{1}{\mathscr{Z}_i^{(\gamma)}}(1-e^{-2\sigma x_i})^{(2k-1)}e^{-(4\sigma k i+\gamma)x_i} \qquad x_i \in \mathbb{R}^+ \tag{3.15}$$

for $i \in \Lambda_L$ and

$$\mathscr{Z}_i^{(\gamma)} = \frac{1}{2\sigma}\,\mathrm{Beta}\left(2ki+\frac{\gamma}{2\sigma}, 2k\right) \tag{3.16}$$

where $\mathrm{Beta}(s,t) = \Gamma(s)\Gamma(t)/\Gamma(s+t)$ *is the Beta function.*

Proof The adjoint of the generator of the ABEP(σ, k) is given by

$$\left(\mathscr{L}_{(L)}^{ABEP^{(\sigma,k)}}\right)^* = \sum_{i=1}^{L-1}\left(\mathscr{L}_{i,i+1}^{ABEP}\right)^* \tag{3.17}$$

with

$$\left(\mathscr{L}_{i,i+1}^{ABEP}\right)^* f = \frac{1}{4\sigma^2}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_{i+1}}\right)^2\left(\left(1-e^{-2\sigma x_i}\right)\left(e^{2\sigma x_{i+1}}-1\right)f\right)$$
$$-\frac{1}{2\sigma}\left(\frac{\partial}{\partial x_{i+1}}-\frac{\partial}{\partial x_i}\right)\left(\left\{\left(1-e^{-2\sigma x_i}\right)\left(e^{2\sigma x_{i+1}}-1\right)\right.\right.$$
$$\left.\left.+2k\left[\left(1-e^{-2\sigma x_i}\right)-\left(e^{2\sigma x_{i+1}}-1\right)\right]\right\}f\right)$$

Let μ be a product measure with $\mu(x) = \prod_{i=1}^{L} \mu_i(x_i)$, then in order for μ to be a stationary measure it is sufficient to impose that the conditions

$$\frac{1}{4\sigma^2}\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right)\left(1-\mathrm{e}^{-2\sigma x_i}\right)\left(\mathrm{e}^{2\sigma x_{i+1}}-1\right)\mu(x)$$
$$-\frac{1}{2\sigma}\left\{\left(1-\mathrm{e}^{-2\sigma x_i}\right)\left(\mathrm{e}^{2\sigma x_{i+1}}-1\right)+2k\left[\left(1-\mathrm{e}^{-2\sigma x_i}\right)-\left(\mathrm{e}^{2\sigma x_{i+1}}-1\right)\right]\right\}\mu(x)=0$$

are satisfied for any $i \in \{1, \dots, L-1\}$. This is true if and only if

$$\frac{\mu_i'(x_i)}{\mu_i(x_i)} - 2\sigma\,\frac{2k-\mathrm{e}^{-2\sigma x_i}}{1-\mathrm{e}^{-2\sigma x_i}} + \sigma = \frac{\mu_{i+1}'(x_{i+1})}{\mu_{i+1}(x_{i+1})} + 2\sigma\,\frac{\mathrm{e}^{2\sigma x_{i+1}}-2k}{\mathrm{e}^{2\sigma x_{i+1}}-1} - \sigma \tag{3.18}$$

for any $x_i, x_{i+1} \in \mathbb{R}^+$. The conditions (3.18) are verified if and only if the marginals $\mu_i(x)$ are of the form (3.15) for some $\gamma \in \mathbb{R}$, $\mathscr{Z}_i^{(\gamma)}$ is a normalization constant, and the constraint $\gamma > -4\sigma k$ is imposed in order to assure the integrability of $\mu(\cdot)$ on $\mathscr{X}_L$. Thus we have proved that the product measure with marginal (3.15) are stationary. One can also verify that for any $f : \mathscr{X}_L \to \mathbb{R}$

$$\mathscr{L}^{ABEP} f = \frac{1}{\mu}\left(\mathscr{L}^{ABEP}\right)^*(\mu f)$$

which then implies that the measure is reversible. □

Remark 3.1 In the limit $\sigma \to 0$ the reversible product measure of ABEP(σ, k) converges to a product of Gamma distributions with shape parameter $2k$ and scale parameter $1/\gamma$, which are the reversible homogeneous measures of the BEP(k) [11]. In the case $\sigma \neq 0$ the reversible product measure of ABEP(σ, k) has a decreasing average profile (see Proposition 4.1).

3.5 Transforming the ABEP(σ, k) to BEP(k)

In this subsection we show that the ABEP(σ, k), which is an asymmetric process, can be mapped via a global change of coordinates to the BEP(k) process which is symmetric. Here we focus on the analytical aspects of such σ-dependent mapping. In Sect. 3.6 we will show that this map induces a conjugacy at the level of the underlying $\mathfrak{su}(1,1)$ algebra. This implies that the ABEP(q, k) generator has a classical (i.e. non deformed) $\mathfrak{su}(1,1)$ symmetry. This is remarkable because ABEP(q, k) is a bulk-driven non-equilibrium process with non-zero average current (as it has been shown in Proposition 3.1) and yet its generator is an element of the classical $\mathfrak{su}(1,1)$ algebra.

Definition 3.2 (*Partial energy*) We define the partial energy functions $E_i : \mathscr{X}_L \to \mathbb{R}_+$, $i \in \{1, \dots, L+1\}$

$$E_i(x) := \sum_{\ell=i}^{L} x_\ell, \qquad \text{for } i \in \Lambda_L \qquad \text{and} \qquad E_{L+1}(x) = 0. \tag{3.19}$$

We also define the total energy $E : \mathscr{X}_L \to \mathbb{R}_+$ as

$$E(x) := E_1(x).$$

Definition 3.3 (*The mapping* g) We define the map $g : \mathscr{X}_L \to \mathscr{X}_L$

$$g(x) := (g_i(x))_{i\in\Lambda_L} \qquad \text{with} \quad g_i(x) := \frac{\mathrm{e}^{-2\sigma E_{i+1}(x)} - \mathrm{e}^{-2\sigma E_i(x)}}{2\sigma} \tag{3.20}$$

Notice that g does not have full range, i.e. $g[\mathscr{X}_L] \neq \mathscr{X}_L$. Indeed

$$E(g(x)) = \frac{1}{2\sigma}\left(1 - \mathrm{e}^{-2\sigma E(x)}\right) \leq \frac{1}{2\sigma} \tag{3.21}$$

so that in particular $g[\mathscr{X}_L] \subseteq \{x \in \mathscr{X}_L : E(x) \leq 1/2\sigma\}$. Moreover g is a bijection from $\mathscr{X}_L$ to $g[\mathscr{X}_L]$. Indeed, for $z \in g[\mathscr{X}_L]$ we have

$$(g^{-1}(z))_i = \frac{1}{2\sigma} \ln \left\{ \frac{1 - 2\sigma \sum_{j=i+1}^{L} z_j}{1 - 2\sigma \sum_{j=i}^{L} z_j} \right\} \tag{3.22}$$

Theorem 3.4 (Mapping from ABEP(σ, k) to BEP(k)) *Let $X(t) = (X_i(t))_{i\in\Lambda_L}$ be the ABEP(σ, k) process starting from $X(0) = x$, then the process $Z(t) := (Z_i(t))_{i\in\Lambda_L}$ defined by the change of variable $Z(t) := g(X(t))$ is the BEP(k) with initial condition $Z(0) = g(x)$.*

Proof It is sufficient to prove that, for any $f : \mathscr{X}_L \to \mathbb{R}_+$ smooth, $x \in \mathscr{X}_L$ and g defined above

$$\left[\mathscr{L}_{i,i+1}^{\mathrm{BEP}} f\right](g(x)) = \left[\mathscr{L}_{i,i+1}^{\mathrm{ABEP}}(f \circ g)\right](x) \tag{3.23}$$

for any $i \in \Lambda_L$. Define $F := f \circ g$, then

$$\begin{aligned}
&[\mathscr{L}^{\mathrm{ABEP}}(f \circ g)](x) \\
&= [\mathscr{L}^{\mathrm{ABEP}}(F)](x) = \frac{1}{4\sigma^2}(1 - \mathrm{e}^{-2\sigma x_i})(\mathrm{e}^{2\sigma x_{i+1}} - 1)\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right)^2 F(x) \\
&\quad + \frac{1}{2\sigma}\left\{(1 - \mathrm{e}^{-2\sigma x_i})(\mathrm{e}^{2\sigma x_{i+1}} - 1) + 2k\left(2 - \mathrm{e}^{-2\sigma x_i} - \mathrm{e}^{2\sigma x_{i+1}}\right)\right\}\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right) F(x)
\end{aligned} \tag{3.24}$$

The computation of the Jacobian of g

$$\frac{\partial g_j}{\partial x_i}(x) = \begin{cases} -2\sigma g_j(x) & \text{for } j \leq i-1 \\ \mathrm{e}^{-2\sigma E_j(x)} & \text{for } j = i \\ 0 & \text{for } j \geq i+1 \end{cases} \tag{3.25}$$

implies that

$$\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right) g_j(x) = \begin{cases} 0 & \text{for } j \leq i-1 \\ -\mathrm{e}^{-2\sigma E_{i+1}(x)} & \text{for } j = i \\ \mathrm{e}^{-2\sigma E_{i+1}(x)} & \text{for } j = i+1 \\ 0 & \text{for } j \geq i+2 \end{cases} \tag{3.26}$$

and

$$\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right) F(x) = \mathrm{e}^{-2\sigma E_{i+1}(x)}\left[\left(\frac{\partial}{\partial z_{i+1}} - \frac{\partial}{\partial z_i}\right) f\right](g(x)) \tag{3.27}$$

$$\begin{aligned}
\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right)^2 F(x) &= -2\sigma \mathrm{e}^{-2\sigma E_{i+1}(x)}\left[\left(\frac{\partial}{\partial z_{i+1}} - \frac{\partial}{\partial z_i}\right) f\right](g(x)) \\
&\quad + \mathrm{e}^{-4\sigma E_{i+1}(x)}\left[\left(\frac{\partial}{\partial z_{i+1}} - \frac{\partial}{\partial z_i}\right)^2 f\right](g(x)).
\end{aligned} \tag{3.28}$$

Then, using (3.27) and (3.28), (3.24) can be rewritten as

$$
\begin{aligned}
&[\mathscr{L}_{i,i+1}^{\text{ABEP}}(f\circ g)](x)\\
&=\frac{1}{4\sigma^2}(1-\mathrm{e}^{-2\sigma x_i})(\mathrm{e}^{2\sigma x_{i+1}}-1)\mathrm{e}^{-4\sigma E_{i+1}(x)}\left[\left(\frac{\partial}{\partial z_{i+1}}-\frac{\partial}{\partial z_i}\right)^2 f\right](g(x))\\
&\quad+\left\{2\sigma+\frac{1}{2\sigma}\left((1-\mathrm{e}^{-2\sigma x_i})(\mathrm{e}^{2\sigma x_{i+1}}-1)+2k\left(2-\mathrm{e}^{-2\sigma x_i}-\mathrm{e}^{2\sigma x_{i+1}}\right)\right)\right\}\mathrm{e}^{-2\sigma E_{i+1}(x)}\\
&\quad\times\left[\left(\frac{\partial}{\partial z_{i+1}}-\frac{\partial}{\partial z_i}\right)f\right](g(x))
\end{aligned}
$$

Simplifying, this gives

$$
\begin{aligned}
&[\mathscr{L}_{i,i+1}^{\text{ABEP}}(f\circ g)](x)\\
&=\left\{\frac{\mathrm{e}^{-2\sigma E_{i+1}(x)}-\mathrm{e}^{-2\sigma E_i(x)}}{2\sigma}\cdot\frac{\mathrm{e}^{-2\sigma E_{i+2}(x)}-\mathrm{e}^{-2\sigma E_{i+1}(x)}}{2\sigma}\left[\left(\frac{\partial}{\partial z_{i+1}}-\frac{\partial}{\partial z_i}\right)^2 f\right](g(x))\right.\\
&\quad\left.-\frac{k}{\sigma}\left(\mathrm{e}^{-2\sigma E_i(x)}-2\mathrm{e}^{-2\sigma E_{i+1}(x)}+\mathrm{e}^{-2\sigma E_{i+2}(x)}\right)\left[\left(\frac{\partial}{\partial z_{i+1}}-\frac{\partial}{\partial z_i}\right)f\right](g(x))\right.\\
&=\left[\mathscr{L}_{i,i+1}^{\text{BEP}}f\right](g(x))
\end{aligned}
$$

□

The ABEP(σ,k) has a single conservation law given by the total energy $E(x)=\sum_{i\in\Lambda_L}x_i$. As a consequence there exists an infinite family of invariant measures which is hereafter described.

Proposition 3.2 (Microcanonical measure of ABEP(σ,k)) *The stationary measure of the ABEP(σ,k) process on Λ_L with given total energy E is unique and is given by the inhomogeneous product measure with marginals (3.15) conditioned to a total energy $E(x)=E$. More explicitly*

$$
d\mu^{(E)}(y)=\frac{\prod_{i=1}^L\mu_i(y_i)\mathbf{1}_{\{\sum_{i\in\Lambda_L}y_i=E\}}dy_i}{\int\ldots\int\prod_{i=1}^L\mu_i(y_i)\mathbf{1}_{\{\sum_{i\in\Lambda_L}y_i=E\}}dy_i} \tag{3.29}
$$

Proof We start by observing that the stationary measure of the BEP(k) process on Λ_L with given total energy $\mathscr{E}$ is unique and is given by a product of i.i.d. Gamma random variable $(X_i)_{i\in\Lambda_L}$ with shape parameter $2k$ conditioned to $\sum_{i\in\Lambda_l}X_i=\mathscr{E}$. This is a consequence of duality between BEP(k) and SIP(k) processes [17]. Furthermore, an explicit computation shows that the reversible measure of ABEP(σ,k) conditioned to energy E are transformed by the mapping g (see Definition 3.3) to the stationary measure of the BEP(k) with energy $\mathscr{E}$ given by

$$
\mathscr{E}=\frac{1}{2\sigma}\left(1-\mathrm{e}^{-2\sigma E}\right).
$$

The uniqueness for ABEP(σ,k) follows from the uniqueness for BEP(σ,k) and the fact that g is a bijection from $\mathscr{X}_L$ to $g[\mathscr{X}_L]$.

3.6 The Algebraic Structure of ABEP(σ, k)

First we recall from [17] that the BEP(k) generator can be written in the form

$$\mathscr{L}^{BEP(k)} = \sum_{i=1}^{L-1} \left(K_i^+ K_{i+1}^- + K_i^- K_{i+1}^+ - 2K_i^o K_{i+1}^o + 2k^2\right) \tag{3.30}$$

where

$$\begin{aligned} K_i^+ &= z_i \\ K_i^- &= z_i \frac{\partial^2}{\partial z_i^2} + 2k \frac{\partial}{\partial z_i} \\ K_i^o &= z_i \frac{\partial}{\partial z_i} + k \end{aligned} \tag{3.31}$$

is a representation of the classical $\mathfrak{su}(1, 1)$ algebra. We show here that the ABEP(σ, k) has the same algebraic structure. This is proved by using a representation of $\mathfrak{su}(1, 1)$ that is conjugated to (3.31) and is given by

$$\tilde{K}_i^a = C_g \circ K_i^a \circ C_{g^{-1}} \qquad \text{with} \quad a \in \{+, -, o\} \tag{3.32}$$

where g is the function of Definition 3.3 and

$$(C_{g^{-1}} f)(x) = (f \circ g^{-1})(x)$$

$$(C_g f)(x) = (f \circ g)(x).$$

Explicitly one has

$$(\tilde{K}_i^a f)(x) = \left(K_i^a f \circ g^{-1}\right)(g(x)) \qquad \text{with} \quad a \in \{+, -, o\} \tag{3.33}$$

Theorem 3.5 (Algebraic structure of ABEP(σ, k)) *The generator of the ABEP*(σ, k) *process is written as*

$$\mathscr{L}^{ABEP(\sigma,k)} = \sum_{i=1}^{L-1} \left(\tilde{K}_i^+ \tilde{K}_{i+1}^- + \tilde{K}_i^- \tilde{K}_{i+1}^+ - 2\tilde{K}_i^o \tilde{K}_{i+1}^o + 2k^2\right) \tag{3.34}$$

where the operators $\tilde{K}_i^a$ *with* $a \in \{+, -, o\}$ *are defined in* (3.32) *and provide a representation of the* $\mathfrak{su}(1, 1)$ *Lie algebra.*

Proof The proof is a consequence of the following two results:

$$\mathscr{L}^{ABEP(\sigma,k)} = C_g \circ \mathscr{L}^{BEP(k)} \circ C_{g^{-1}} \tag{3.35}$$

and the operators $\tilde{K}_i^a$ with $a \in \{+, -, o\}$ satisfy the commutation relations of the $\mathfrak{su}(1, 1)$ algebra. The first property is an immediate consequence of Theorem 3.4, as Eq. (3.35) is simply a rewriting of Eq. (3.23) by using the definition of C_g and $C_{g^{-1}}$. The second property can be obtained by the following elementary Lemma, which implies that the commutation relations of the $\tilde{K}_i^a$ operators with $a \in \{+, -, o\}$ are the same of the K_i^a operators with $a \in \{+, -, o\}$. □

Lemma 3.2 *Consider an operator A working on function $f : \mathscr{X}_L \to \mathbb{R}$ and let $g : \mathscr{X}_L \to X \subset \mathscr{X}_L$ be a bijection. Then defining*

$$\tilde{A} = C_g \circ A \circ C_{g^{-1}}$$

we have that $A \to \tilde{A}$ is an algebra homomorphism.

Proof We need to verify that

$$\widetilde{A+B} = \tilde{A} + \tilde{B} \qquad \text{and} \qquad \widetilde{AB} = \tilde{A}\tilde{B}$$

The first is trivial, the second is proved as follows

$$\widetilde{AB} = C_g \circ AB \circ C_{g^{-1}} = \left(C_g \circ A \circ C_{g^{-1}}\right) \circ \left(C_g \circ B \circ C_{g^{-1}}\right) = \tilde{A}\tilde{B}$$

As a consequence

$$\widetilde{[A, B]} = [\tilde{A}, \tilde{B}].$$

□

4 The Asymmetric KMP Process, AKMP(σ)

4.1 Instantaneous Thermalizations

The procedure of instantaneous thermalization has been introduced in [17]. We consider a generator of the form

$$\mathscr{L} = \sum_{i=1}^{L-1} \mathscr{L}_{i,i+1} \tag{4.1}$$

where $\mathscr{L}_{i,i+1}$ is such that, for any initial condition (x_i, x_{i+1}), the corresponding process converges to a unique stationary distribution $\mu_{(x_i,x_{i+1})}$.

Definition 4.1 (*Instantaneous thermalized process*) The instantaneous thermalization of the process with generator $\mathscr{L}$ in (4.1) is defined to be the process with generator

$$\mathscr{A} = \sum_{i=1}^{L-1} \mathscr{A}_{i,i+1}$$

where

$$\begin{aligned} \mathscr{A}_{i,i+1} f &= \lim_{t\to\infty} (\mathrm{e}^{t\mathscr{L}_{i,i+1}} f - f) \\ &= \int [f(x_1, \ldots, x_{i-1}, y_i, y_{i+1}, x_{i+2}, \ldots, x_L) \\ &\quad - f(x_1, \ldots, x_L)] d\mu_{(x_i,x_{i+1})}(y_i, y_{i+1}) \end{aligned} \tag{4.2}$$

In words, in the process with generator $\mathscr{A}$ each edge $(i, i+1)$ is updated at rate one, and after update its variables are replaced by a sample of the stationary distribution of the process with generator $\mathscr{L}_{i,i+1}$ starting from (x_i, x_{i+1}).

Notice that, by definition, if a measure is stationary for the process with generator $\mathscr{L}_{i,i+1}$ then it is also stationary for the process with generator $\mathscr{A}_{i,i+1}$.

An example of thermalized processes is the Th-BEP(k) process, where the local redistribution rule is

$$(x, y) \to (B(x+y), (1-B)(x+y)) \tag{4.3}$$

with B a Beta($2k$, $2k$) distributed random variable [12]. In particular for $k = 1/2$ this gives the KMP process [22] that has a uniform redistribution rule on [0, 1]. Among discrete models we mention the Th-SIP(k) process where the redistribution rule is

$$(n, m) \to (R, n+m-R) \tag{4.4}$$

where R is Beta-Binomial($n+m$, $2k$, $2k$). For $k = 1/2$ this corresponds to discrete uniform distributions on $\{0, 1, \dots, n+m\}$. Other examples are described in [12]. In the following we introduce the asymmetric version of these redistribution models.

4.2 Thermalized Asymmetric Inclusion Process Th-ASIP(q, k)

The instantaneous thermalization limit of the Asymmetric Inclusion Process is obtained as follows. Imagine on each bond $(i, i+1)$ to run the ASIP(q, k) dynamics for an infinite amount of time. Then the total number of particles on the bond will be redistributed according to the stationary measure on that bond, conditioned to conservation of the total number of particles of the bond. We consider the independent random variables $(M_1, \dots, M_L)$ distributed according to the stationary measure of the ASIP(q, k) at equilibrium. Thus M_i and M_{i+1} are distributed according to

$$p_i^{(\alpha)}(\eta_i) := \mathbb{P}^{(\alpha)}(M_i = \eta_i) = \frac{\alpha^{\eta_i}}{Z_i^{(\alpha)}} \binom{\eta_i + 2k - 1}{\eta_i}_q \cdot q^{4ki\eta_i} \qquad \eta_i \in \mathbb{N} \tag{4.5}$$

and

$$p_{i+1}^{(\alpha)}(\eta_{i+1}) := \mathbb{P}^{(\alpha)}(M_{i+1} = \eta_{i+1}) = \frac{\alpha^{\eta_{i+1}}}{Z_{i+1}^{(\alpha)}} \binom{\eta_{i+1} + 2k - 1}{\eta_{i+1}}_q \cdot q^{4k(i+1)\eta_{i+1}} \qquad \eta_{i+1} \in \mathbb{N} \tag{4.6}$$

for some $\alpha \in [0, q^{-(2k+1)})$.

Hence the distribution of M_i, given that the sum is fixed to $M_i + M_{i+1} = n+m$ has the following probability mass function:

$$\begin{aligned} \nu_{q,k}^{ASIP}(r \mid n+m) &:= \mathbb{P}(M_i = r \mid M_i + M_{i+1} = n+m) \\ &= \frac{p_i^{(\alpha)}(r) p_{i+1}^{(\alpha)}(n+m-r)}{\sum_{l=0}^{n+m} p_i^{(\alpha)}(l) p_{i+1}^{(\alpha)}(n+m-l)} \\ &= \widetilde{\mathscr{C}}_{q,k}(n+m)\, q^{-4kr} \binom{r+2k-1}{r}_q \cdot \binom{2k+n+m-r-1}{n+m-r}_q \end{aligned} \tag{4.7}$$

where $r \in \mathbb{N}$ and $\widetilde{\mathscr{C}}_{q,k}(n+m)$ is a normalization constant.

Definition 4.2 (*Th-ASIP*(q, k) *process*) The Th-ASIP(q, k) process on Λ_L is defined as the thermalized discrete process with state space Ω_L and local redistribution rule

$$(n, m) \to (R_q, n+m-R_q) \tag{4.8}$$

where R_q has a q-deformed Beta-Binomial$(n+m, 2k, 2k)$ distribution with mass function (4.7). The generator of this process is given by

$$\mathscr{L}_{th}^{ASIP(q,k)} f(\eta) = \sum_{i=1}^{L-1} \sum_{r=0}^{\eta_i+\eta_{i+1}} \big[f(\eta_1, \ldots, \eta_{i-1}, r, \eta_i + \eta_{i+1} - r, \eta_{i+2}, \ldots, \eta_L) - f(\eta) \big] \, \nu_{q,k}^{ASIP}(r \mid \eta_i + \eta_{i+1}) \tag{4.9}$$

4.3 Thermalized Asymmetric Brownian Energy Process Th-ABEP(σ, k)

We define the instantaneous thermalization limit of the Asymmetric Brownian Energy Process as follows. On each bond we run the ABEP(σ, k) for an infinite time. Then the energies on the bond will be redistributed according to the stationary measure on that bond, conditioned to the conservation of the total energy of the bond. If we take two independent random variables X_i and X_{i+1} with distributions as in (3.15), i.e.

$$\mu_i(x_i) := \frac{1}{\mathscr{Z}_i^{(\gamma)}} (1 - \mathrm{e}^{-2\sigma x_i})^{(2k-1)} \mathrm{e}^{-(4\sigma k i + \gamma) x_i} \qquad x_i \in \mathbb{R}^+ \tag{4.10}$$

$$\mu_{i+1}(x_{i+1}) := \frac{1}{\mathscr{Z}_{i+1}^{(\gamma)}} (1 - \mathrm{e}^{-2\sigma x_{i+1}})^{(2k-1)} \mathrm{e}^{-(4\sigma k (i+1) + \gamma) x_{i+1}} \qquad x_{i+1} \in \mathbb{R}^+ \tag{4.11}$$

then the distribution of X_i, given the sum fixed to $X_i + X_{i+1} = E$, has density

$$\begin{aligned} p(x_i | X_i + X_{i+1} = E) &= \frac{\mu_i(x_i)\mu_{i+1}(E - x_i)}{\int_0^E \mu_i(x)\mu_{i+1}(E-x)\, dx} \\ &= \mathscr{C}_{\sigma,k}(E)\, \mathrm{e}^{4\sigma k x_i} \left[\left(1 - \mathrm{e}^{-2\sigma x_i}\right) \left(1 - \mathrm{e}^{-2\sigma (E - x_i)}\right) \right]^{2k-1} \end{aligned}$$

where $\mathscr{C}_{\sigma,k}(E)$ is a normalization constant. Equivalently, let $W_i := X_i/E$, then W_i is a random variable taking values on $[0, 1]$. Conditioned to $X_i + X_{i+1} = E$, its density is given by

$$\nu_{\sigma,k}(w|E) = \widehat{\mathscr{C}_{\sigma,k}}(E)\, \mathrm{e}^{2\sigma E w} \left\{ \left(\mathrm{e}^{2\sigma E w} - 1\right) \left(1 - \mathrm{e}^{-2\sigma E (1-w)}\right) \right\}^{2k-1} \tag{4.12}$$

with

$$\widehat{\mathscr{C}_{\sigma,k}}(E) := \int_0^1 \mathrm{e}^{2\sigma E w} \left\{ \left(\mathrm{e}^{2\sigma E w} - 1\right) \left(1 - \mathrm{e}^{-2\sigma E (1-w)}\right) \right\}^{2k-1} dw \tag{4.13}$$

Definition 4.3 (*Thermalized ABEP*(σ, k)) The Th-ABEP(σ, k) process on Λ_L is defined as the thermalized process with state space $\mathscr{X}_L$ and local redistribution rule

$$(x, y) \to (B_\sigma(x + y), (1 - B_\sigma)(x + y)) \tag{4.14}$$

where B_σ has a distribution with density function $\nu_{\sigma,k}(\cdot|x+y)$ in (4.12). Thus the generator of Th-ABEP(σ, k) is given by

$$\mathscr{L}_{th}^{ABEP(\sigma,k)} f(x) = \sum_{i=1}^{L-1} \int_0^1 \big[f(x_1, \ldots, w(x_i + x_{i+1}), (1-w)(x_i + x_{i+1}), \ldots, x_L) - f(x) \big] \, \nu_{\sigma,k}(w | x_i + x_{i+1})\, dw \tag{4.15}$$

In the limit $\sigma \to 0$, the conditional density $\nu_{0^+,k}(\cdot|E)$ does not depend on E, and for any $E \geq 0$ we recover the Beta$(2k, 2k)$ distribution with density

$$\nu_{0^+,k}(w|E) = \frac{1}{\mathrm{Beta}(2k, 2k)} [w(1-w)]^{2k-1}. \tag{4.16}$$

Then the generator $\mathscr{L}_{th}^{ABEP(0^+,k)}$ coincides with the generator of the thermalized Brownian Energy Process Th-BEP(k) defined in Eq. (5.13) of [11].

The redistribution rule with the random variable B_σ in Definition 4.3 is truly asymmetric, meaning that—on average—the energy is moved to the left.

Proposition 4.1 *Let B_σ be the random variable on* $[0, 1]$ *distributed with density* (4.12)*, then $\mathbb{E}[B_\sigma] \geq \frac{1}{2}$. As a consequence B_σ and $1 - B_\sigma$ are not equal in distribution and for $(X_1, \ldots, X_L)$ distributed according to the reversible product measure μ of ABEP(σ, k) defined in* (3.15)*, we have that the energy profile is decreasing, i.e.*

$$\mathbb{E}_\mu[X_i] \geq \mathbb{E}_\mu[X_{i+1}], \qquad \forall\, i \in \{1, \ldots, L-1\}. \tag{4.17}$$

Proof Let $X = (X_1, X_2)$ be a two-dimensional random vector taking values in $\mathscr{X}_2$ distributed according to the microcanonical measure $\mu^{(E)}$ of ABEP(σ, k) with fixed total energy $E \geq 0$, defined in (3.29). Then, from Definition 4.3,

$$(X_1, X_2) \overset{d}{=} (EB_\sigma, E(1 - B_\sigma)) \qquad \text{with} \qquad B_\sigma \sim \nu_{\sigma,k}(\cdot|E) \tag{4.18}$$

Then, as already remarked in the proof of Proposition 3.2, $Z := g(X)$ with $g(\cdot)$ as in Definition 3.3 is a two-dimensional random variable taking values in $g[\mathscr{X}_2] \subset \mathscr{X}_2$ and distributed according to the microcanonical measure of BEP(k) with fixed total energy $\mathscr{E} = \frac{1}{2\sigma}(1 - \mathrm{e}^{-2\sigma E})$. It follows from (4.3) that

$$g(X) \overset{d}{=} (\mathscr{E}B, \mathscr{E}(1 - B)) \qquad \text{with} \qquad B \sim \mathrm{Beta}(2k, 2k). \tag{4.19}$$

Then, by (3.22) we have

$$(1 - B_\sigma)E = (g^{-1}(Z))_2 = \frac{1}{2\sigma} \ln \left\{ \frac{1}{1 - 2\sigma(1 - B)\mathscr{E}} \right\} \tag{4.20}$$

and therefore

$$B_\sigma = 1 + \frac{1}{2\sigma E} \ln \left(1 - B(1 - \mathrm{e}^{-2\sigma E}) \right) \tag{4.21}$$

Put $2\sigma E = 1$ without loss of generality, for simplicity. Then to prove that $\mathbb{E}[B_\sigma] > 1/2$ we have to prove that

$$\mathbb{E}\left(1 + \ln\left(1 - B\left(1 - \mathrm{e}^{-1}\right)\right)\right) \geq \frac{1}{2}$$

Defining $a = 1 - \mathrm{e}^{-1}$ we then have to prove that

$$\mathbb{E}(-\ln(1 - aB)) \leq \frac{1}{2} \tag{4.22}$$

It is useful to write

$$-\ln(1 - aB) = \sum_{n=1}^{\infty} \frac{a^n B^n}{n}$$

and remark that for a Beta(α, α) distributed B one has

$$\mathbb{E}(B^n) = \prod_{r=0}^{n-1} \frac{\alpha + r}{2\alpha + r}.$$

So we have to prove that

$$\psi(\alpha, a) := \sum_{n=1}^{\infty} \frac{a^n}{n} \prod_{r=0}^{n-1} \frac{\alpha + r}{2\alpha + r} < 1/2$$

First consider the limit $\alpha \to \infty$ then we find

$$\lim_{\alpha\to\infty} \phi(\alpha, a) = \sum_{n=1}^{\infty} \frac{a^n}{2^n n} = -\ln\left(1 - \frac{1}{2}\left(1 - \mathrm{e}^{-1}\right)\right) = -\ln\left(\frac{1}{2} + \frac{\mathrm{e}^{-1}}{2}\right) \approx 0.379 < 1/2$$

Next remark when $\alpha = 0$ the B is distributed like $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ which gives

$$\mathbb{E}(-\ln(1 - aB)) = -\frac{1}{2}\ln(\mathrm{e}^{-1}) = \frac{1}{2}$$

Now we prove that ψ is monotonically decreasing in α. To see this notice that

$$\frac{d}{d\alpha} \frac{\alpha + r}{2\alpha + r} = \frac{-r}{(2\alpha + r)^2} < 0$$

So the derivative

$$\frac{d}{d\alpha}\psi(\alpha, a) = \sum_{n=1}^{\infty} \sum_{r'=0}^{n-1} \frac{a^n}{n} \left(\prod_{r=0, r\neq r'}^{n-1} \frac{\alpha + r}{2\alpha + r} \right) \frac{-r'}{(2\alpha + r)^2} < 0$$

Therefore $\psi(\alpha, a)$ is monotonically decreasing in α and $\psi(\alpha, a) \le \frac{1}{2}$. Thus the claim $\mathbb{E}[B_\sigma] > 1/2$ is proved.

Now let $X = (X_1, X_2)$ be a two-dimensional r.v. distributed according to the profile measure μ defined in (3.15) with $L = 2$ and with abuse of notation let $\nu_{\sigma,k}[B_\sigma | E] = \mathbb{E}[B_\sigma]$. Then we can write $X = (E\, B_\sigma, E(1 - B_\sigma))$ where now E is a random variable. We have

$$\begin{aligned}\mathbb{E}_\mu[X_2] &= \mathbb{E}_\mu\left[\mathbb{E}_\mu[X_2 |\, E]\right] = \mathbb{E}_\mu\left[\mathbb{E}_\mu[E(1 - B_\sigma)|\, E]\right] = \mathbb{E}_\mu\left[E\ \nu_{\sigma,k}[(1 - B_\sigma)|\, E]\right] \\ &\le \mathbb{E}_\mu\left[E\ \nu_{\sigma,k}[B_\sigma |\, E]\right] = \mathbb{E}_\mu\left[\mathbb{E}_\mu[X_1 |\, E]\right] = \mathbb{E}_\mu[X_1]\end{aligned} \tag{4.23}$$

The proof can be easily generalized to the case $L \ge 2$, yielding (4.17). □

For $k = 1/2$ and $\sigma \to 0$ the Th-ABEP(σ, k) is exactly the KMP process [22]. For $k = 1/2$ and $\sigma > 0$

$$\nu_{\sigma,1/2}(w|E) = \frac{2\sigma E}{\mathrm{e}^{2\sigma E} - 1}\mathrm{e}^{2\sigma E w}, \qquad w \in [0, 1] \tag{4.24}$$

The Th-ABEP$(\sigma, \frac{1}{2})$ can therefore be considered as the natural asymmetric analogue of the KMP process. This justifies the following definition.

Definition 4.4 (*AKMP*(σ) *process*) We define the Asymmetric KMP with asymmetry parameter $\sigma \in \mathbb{R}_+$ on Λ_L as the process with generator given by:

$$\mathscr{L}^{AKMP(\sigma)} f(x) = \sum_{i=1}^{L-1} \Bigg\{ \frac{2\sigma(x_i + x_{i+1})}{e^{2\sigma(x_i+x_{i+1})} - 1} \cdot \int_0^1 \big[f(x_1, \ldots, w(x_i + x_{i+1}), (1-w)(x_i + x_{i+1}), \ldots, x_L) - f(x) \big] e^{2\sigma w(x_i+x_{i+1})} \, dw \Bigg\}$$

5 Duality Relations

In this section we derive various duality properties of the processes introduced in the previous sections. We start by recalling the definition of *duality*.

Definition 5.1 Let $\{X_t\}_{t\ge0}$, $\{\widehat{X}_t\}_{t\ge0}$ be two Markov processes with state spaces Ω and $\widehat{\Omega}$ and $D : \Omega \times \widehat{\Omega} \to \mathbb{R}$ a bounded measurable function. The processes $\{X_t\}_{t\ge0}$, $\{\widehat{X}_t\}_{t\ge0}$ are said to be dual with respect to D if

$$\mathbb{E}_x\big[D(X_t, \widehat{x})\big] = \widehat{\mathbb{E}}_{\widehat{x}}\big[D(x, \widehat{X}_t)\big] \tag{5.1}$$

for all $x \in \Omega, \widehat{x} \in \hat{\Omega}$ and $t > 0$. In (5.1) $\mathbb{E}_x$ is the expectation with respect to the law of the $\{X_t\}_{t\ge0}$ process started at x, while $\widehat{\mathbb{E}}_{\widehat{x}}$ denotes expectation with respect to the law of the $\{\widehat{X}_t\}_{t\ge0}$ process initialized at $\widehat{x}$.

5.1 Self-Duality of ASIP(q, k)

The basic duality relation is the self-duality of ASIP(q, k). This self-duality property is derived from a symmetry of the underlying Hamiltonian which is a sum of co-products of the Casimir operator. In [13] this construction was achieved for the algebra $\mathscr{U}_q(\mathfrak{su}(2))$, and from the Hamiltonian a Markov generator was constructed via a positive ground state. Here the construction and consequent symmetries is analogous, but for the algebra $\mathscr{U}_q(\mathfrak{su}(1,1))$. For the proof of the following Theorem we refer to Appendix, where we implement the steps of [13] for the algebra $\mathscr{U}_q(\mathfrak{su}(1,1))$.

Theorem 5.1 (Self-duality of the finite ASIP(q, k)) *The ASIP*(q, k) *on* Λ_L *with closed boundary conditions is self-dual with the following self-duality function*

$$D_{(L)}(\eta, \xi) = \prod_{i=1}^{L} \frac{\binom{\eta_i}{\xi_i}_q}{\binom{\xi_i+2k-1}{\xi_i}_q} \cdot q^{(\eta_i-\xi_i)\left[2\sum_{m=1}^{i-1}\xi_m+\xi_i\right]-4ki\xi_i} \cdot \mathbf{1}_{\xi_i\le\eta_i} \tag{5.2}$$

or, equivalently,

$$D_{(L)}(\eta, \xi) = \prod_{i=1}^{L} \frac{(q^{2(\eta_i-\xi_i+1)}; q^2)_{\xi_i}}{(q^{4k}; q^2)_{\xi_i}} \cdot q^{(\xi_i-4ki+2N_{i+1}(\eta))\xi_i} \cdot \mathbf{1}_{\xi_i\le\eta_i} \tag{5.3}$$

with $(a; q)_m$ *as defined in* (2.4) *and*

$$N_i(\eta) := \sum_{k=i}^{L} \eta_k. \tag{5.4}$$

Remark 5.1 For $n \in \mathbb{N}$, let $\xi^{(\ell_1,\dots,\ell_n)}$ be the configurations with n particles located at sites $\ell_1, \dots, \ell_n$.

Then for the configuration $\xi^{(\ell)}$ with one particle at site ℓ

$$D(\eta, \xi^{(\ell)}) = \frac{q^{-(4k\ell+1)}}{q^{2k} - q^{-2k}} \cdot \left(q^{2N_\ell(\eta)} - q^{2N_{\ell+1}(\eta)}\right) \tag{5.5}$$

and, more generally, for the configuration $\xi^{(\ell_1,\dots,\ell_n)}$ with n particles at sites $\ell_1, \dots, \ell_n$ with $\ell_i \neq \ell_j$

$$D(\eta, \xi^{(\ell_1,\dots,\ell_n)}) = \frac{q^{-4k\sum_{m=1}^n \ell_m - n^2}}{(q^{2k} - q^{-2k})^n} \cdot \prod_{m=1}^n \left(q^{2N_{\ell_m}(\eta)} - q^{2N_{\ell_m+1}(\eta)}\right)$$

The duality relation with duality function (5.3) makes sense in the limit $L \to \infty$. Indeed, if $N_i(\eta) = \infty$ for some i, then $\lim_{L\to\infty} D_{(L)}(\eta, \xi) = 0$ for all ξ with $\xi_i \neq 0$. If the initial configuration $\eta \in \Omega_\infty$ has a finite number of particles at the right of the origin, then from the duality relation, we deduce that it remains like this for all later times $t > 0$, which implies that $N_\ell(\eta_t) < \infty$ for all $t \geq 0$. Conversely, if η is such that $N_0(\eta) = \infty$, then $N_0(\eta_t) = \infty$ for all later times because, from the duality relation, $\mathbb{E}_\xi\left[D(\eta, \xi_t)\right] = 0$ for all $t > 0$. To extract some non-trivial informations from the duality relation in the infinite volume case, a suitable renormalization is required (see Sect. 6.1).

5.2 Duality Between ABEP(σ, k) and SIP(k)

We remind the reader that in the limit of zero asymmetry $q \to 1$ the ASIP(q, k) converges to the SIP(k). Therefore from the self-duality of ASIP(q, k), and the fact that the ABEP(σ, k) arises as a limit of ASIP(q, k) with $q \to 1$, a duality between ABEP(σ, k) and SIP(k) follows.

Theorem 5.2 (Duality ABEP(σ, k) and SIP(k)) *The ABEP(σ, k) on Λ_L with closed boundary conditions is dual to the SIP(k) on Λ_L with closed boundary conditions, with the following self-duality function*

$$D^\sigma_{(L)}(x, \xi) = \prod_{i\in\Lambda_L} \frac{\Gamma(2k)}{\Gamma(2k+\xi_i)} \left(\frac{e^{-2\sigma E_{i+1}(x)} - e^{-2\sigma E_i(x)}}{2\sigma}\right)^{\xi_i} \tag{5.6}$$

with $E_i(\cdot)$ the partial energy function defined in Definition 3.2.

Proof The duality function in (5.6) is related to the duality function between BEP(k) and SIP(k), $D^0_{(L)}(x, \eta)$ (see e.g. Sect. 4.1 of [11]) by the following relation

$$D^\sigma_{(L)}(x, \xi) = D^0_{(L)}(g(x), \eta) \tag{5.7}$$

where $g(\cdot)$ is the map defined in (3.3). Thus, omitting the subscript (L) in the following, from (3.35) we have

$$\begin{aligned}\left[\mathscr{L}^{\mathrm{ABEP}(\sigma,k)} D^\sigma(\cdot, \eta)\right](x) &= \left[\mathscr{L}^{\mathrm{ABEP}(\sigma,k)} \left(D^0(\cdot, \eta) \circ g\right)\right](x) \\ &= \left[\mathscr{L}^{\mathrm{BEP}(k)} D^0(\cdot, \eta)\right](g(x))\end{aligned}$$

$$= \left[\mathscr{L}^{\mathrm{SIP}(k)} D^0(g(x), \cdot) \right](\eta)$$
$$= \left[\mathscr{L}^{\mathrm{SIP}(k)} D^\sigma(x, \cdot) \right](\eta) \tag{5.8}$$

and this proves the Theorem. □

Remark 5.2 In the limit as $\sigma \to 0$ one recovers the duality $D^0_{(L)}(\cdot, \cdot)$ between BEP(k) and SIP(k). However it is remarkable here that for finite σ there is duality between a bulk driven asymmetric process, the ABEP(σ, k), and an equilibrium symmetric process, the SIP(k). Indeed, the asymmetry is hidden in the duality function. This is somewhat reminiscent of the dualities between systems with reservoirs and absorbing systems [11], where also the source of non-equilibrium, namely the different parameters of the reservoirs has been moved to the duality function.

The following proposition explains how $D^\sigma_{(L)}(x, \xi)$ arises as the limit of ASIP(q, k) self-duality function for $q = 1 - N^{-1}\sigma$, $N \to \infty$.

Proposition 5.1 *For any fixed $L \geq 2$ we have*

$$\lim_{N\to\infty} \left(\frac{\sigma}{N}\right)^{|\xi|} D^{ASIP(1-\sigma/N,k)}_{(L)}(\lfloor Nx \rfloor, \xi) = D^{ABEP(\sigma,k)}_{(L)}(x, \xi) \tag{5.9}$$

where $D^{ASIP(q,k)}_{(L)}(\eta, \xi)$ denotes the self-duality function of ASIP(q, k) defined in (5.3) and $D^{ABEP(\sigma,k)}_{(L)}(x, \xi)$ denotes the duality function defined in (5.6).

Proof Let

$$N := |\eta| := \sum_{i=1}^{L} \eta_i, \qquad q = 1 - \frac{\sigma}{N}, \qquad x := N^{-1}\eta, \tag{5.10}$$

then

$$D^{\mathrm{ASIP}(q,k)}_{(L)}(\eta, \xi) = \prod_{i=1}^{L} \frac{[\eta_i]_q [\eta_i - 1]_q \dots [\eta_i - \xi_i + 1]_q}{[2k + \xi_i - 1]_q [2k + \xi_i - 2]_q \dots [2k]_q}$$
$$\times q^{(\eta_i - \xi_i)\left[2\sum_{m=1}^{i-1} \xi_m + \xi_i\right] - 4ki\xi_i} \cdot \mathbf{1}_{\xi_i \leq \eta_i} \tag{5.11}$$

Now, for any m

$$[\eta_i - m]_{1-\frac{\sigma}{N}} = [Nx_i - m]_{1-\frac{\sigma}{N}}$$
$$= \frac{N}{2\sigma}\left[\mathrm{e}^{\sigma x_i} - \mathrm{e}^{-\sigma x_i} + O(N^{-1})\right]$$
$$= \frac{N}{\sigma} \sinh(\sigma x_i) + O(1) \tag{5.12}$$

hence

$$\prod_{m=0}^{\xi_i - 1} [Nx_i - m]_{1-\frac{\sigma}{N}} = \left(\frac{N}{\sigma} \sinh(\sigma x_i) + O(1)\right)^{\xi_i} \tag{5.13}$$

On the other hand

$$[2k+m]_{1-\frac{\sigma}{N}} = 2k+m+O(N^{-1}) \quad \text{thus} \quad \prod_{m=0}^{\xi_i-1}[2k+m]_{1-\frac{\sigma}{N}} = \frac{\Gamma(2k+\xi_i)}{\Gamma(2k)} + O(N^{-1}) \tag{5.14}$$

finally, let $f_i(\xi) := 2\sum_{m=1}^{i-1}\xi_m + \xi_i$ and $g_i(\xi) := -\xi_i\left[2\sum_{m=1}^{i-1}\xi_m + \xi_i\right] - 4ki\xi_i$ we have

$$q^{\eta_i f_i(\xi)} = \left(1-\frac{\sigma}{N}\right)^{Nx_i f_i(\xi)} = \mathrm{e}^{-\sigma x_i f_i(\xi)} + O(N^{-1}), \quad \text{and}$$

$$q^{g(\xi)} = \left(1-\frac{\sigma}{N}\right)^{g(\xi)} = 1 + O(N^{-1}) \tag{5.15}$$

then (5.9) immediately follows. □

5.3 Duality for the Instantaneous Thermalizations

In this section we will prove that the self-duality of ASIP(q,k) and the duality between ABEP(σ,k) and SIP(k) imply duality properties also for the thermalized models.

Proposition 5.2 *If a process $\{\eta(t) : t \geq 0\}$ with generator $\mathscr{L} = \sum_{i=1}^{L-1}\mathscr{L}_{i,i+1}$ is dual to a process $\{\xi(t) : t \geq 0\}$ with generator $\widehat{\mathscr{L}} = \sum_{i=1}^{L-1}\widehat{\mathscr{L}}_{i,i+1}$ with duality function $D(\cdot,\cdot)$ in such a way that for all i*

$$\left[\mathscr{L}_{i,i+1}D(\cdot,\xi)\right](\eta) = [\widehat{\mathscr{L}}_{i,i+1}D(\eta,\cdot)](\xi)$$

then, if the instantaneous thermalization processes of η_t, resp. ξ_t both exist, they are each other's dual with the same duality function $D(\cdot,\cdot)$.

Proof Let $\mathscr{A}$, resp. $\widehat{\mathscr{A}}$ be the generators of the instantaneous thermalization of η_t, resp. ξ_t, then, from (4.2) we know that

$$\mathscr{A} = \sum_{i\in\Lambda_L}\mathscr{A}_{i,i+1}, \qquad \mathscr{A}_{i,i+1} = \lim_{t\to\infty}\left(\mathrm{e}^{t\mathscr{L}_{i,i+1}} - I\right)$$

and

$$\widehat{\mathscr{A}} = \sum_{i\in\Lambda_L}\widehat{\mathscr{A}}_{i,i+1}, \qquad \widehat{\mathscr{A}}_{i,i+1} = \lim_{t\to\infty}\left(\mathrm{e}^{t\widehat{\mathscr{L}}_{i,i+1}} - I\right)$$

where I denotes identity and where the exponential $\mathrm{e}^{t\mathscr{L}_{i,i+1}}$ is the semigroup generated by $\mathscr{L}_{i,i+1}$ in the sense of the Hille Yosida theorem. Hence we immediately obtain that

$$\left[\left(\mathrm{e}^{t\mathscr{L}_{i,i+1}} - I\right)D(\cdot,\xi)\right](\eta) = \left[\left(\mathrm{e}^{t\widehat{\mathscr{L}}_{i,i+1}} - I\right)D(\eta,\cdot)\right](\xi)$$

which proves the result. □

As a consequence of this Proposition we obtain duality between the thermalized ABEP(q,k) and the thermalized SIP(k) as well as self-duality of the thermalized ASIP(q,k).

Theorem 5.3 *(a) The Th-ASIP(q,k) with generator (4.9) is self-dual with self-duality function given by (5.2).*

(b) The Th-ABEP(σ, k) with generator (4.15) is dual, with duality function (5.6) to the Th-SIP(k) in Λ_L whose generator is given by

$$\mathscr{L}_{th}^{SIP(k)} f(\xi) = \sum_{i=1}^{L-1} \sum_{r=0}^{\xi_i+\xi_{i+1}} \big[f(\xi_1, \ldots, \xi_{i-1}, r, \xi_i + \xi_{i+1} - r, \xi_{i+2}, \ldots, \xi_L) - f(\xi) \big] \, \nu_k^{SIP}(r \,|\, \xi_i + \xi_{i+1}) \tag{5.16}$$

where $\nu_k^{SIP}(r \,|\, n+m)$ is the probability density of a Beta-Binomial distribution of parameters $(n+m, 2k, 2k)$.

Remark 5.3 For $k = 1/2$ (5.16) gives the KMP-dual, i.e., the Asymmetric KMP has the same dual as the symmetric KMP, but of course with different σ-dependent duality function given by

$$D_{(L)}^{\mathrm{AKMP}(\sigma)}(x, \xi) = \prod_{i \in \Lambda_L} \frac{1}{\xi_i!} \left(\frac{\mathrm{e}^{-2\sigma E_{i+1}(x)} - \mathrm{e}^{-2\sigma E_i(x)}}{2\sigma} \right)^{\xi_i} \tag{5.17}$$

6 Applications to Exponential Moments of Currents

The definition of the ASIP(q, k) process on the infinite lattice requires extra conditions on the initial data. Indeed, when the total number of particles is infinite, there is the possibility of the appearance of singularities, since a single site can accommodate an unbounded number of particles. By self-duality we can however make sense of expectations of duality functions in the infinite volume limit. This is the aim of the next section.

6.1 Infinite Volume Limit for ASIP(q, k)

In this section we approximate an infinite-volume configuration by a finite-volume configuration and we appropriately renormalize the self-duality function to avoid divergence in the thermodynamical limit.

Definition 6.1 (*Good infinite-volume configuration*)

(a) We say that $\eta \in \mathbb{N}^{\mathbb{Z}}$ is a *"good infinite-volume configuration"* for ASIP(q, k) iff for $\eta^{(L)} \in \mathbb{N}^{\mathbb{Z}}$, $L \in \mathbb{N}$, the restriction of η on $[-L, L]$, i.e.

$$\eta_i^{(L)} = \begin{cases} \eta_i & \text{for} \quad i \in [-L, L] \\ 0 & \text{otherwise} \end{cases} \tag{6.1}$$

the limit

$$\lim_{L \to \infty} \prod_{i \in \mathbb{Z}} q^{-2\xi_i N_{i+1}(\eta^{(L)})} \, \mathbb{E}_\xi \left[D(\eta^{(L)}, \xi(t)) \right] \tag{6.2}$$

exists and is finite for all $t \geq 0$ and for any $\xi \in \mathbb{N}^{\mathbb{Z}}$ finite (i.e. such that $\sum_{i \in \mathbb{Z}} \xi_i < \infty$).

(b) Let μ be a probability measure on $\mathbb{N}^{\mathbb{Z}}$, then we say that it is a *"good infinite-volume measure"* for ASIP(q, k) iff it concentrates on good infinite-volume configurations.

Proposition 6.1 *(1) If $\eta \in \mathbb{N}^{\mathbb{Z}}$ is a "good infinite-volume configuration" for ASIP(q, k) and $\xi^{(\ell_1, \ldots, \ell_n)}$ is the configurations with n particles located at sites $\ell_1, \ldots, \ell_n \in \mathbb{Z}$, then the limit*

$$\lim_{L\to\infty}\prod_{m=1}^{n} q^{-2N_{\ell_m+1}(\eta^{(L)})}\ \mathbb{E}_{\eta^{(L)}}\left[D(\eta(t),\xi^{(\ell_1,\dots,\ell_n)})\right] \tag{6.3}$$

is well-defined for all $t \geq 0$ *and is equal to*

$$\lim_{L\to\infty}\prod_{m=1}^{n} q^{-2N_{\ell_m+1}(\eta^{(L)})}\ \mathbb{E}_{\xi^{(\ell_1,\dots,\ell_n)}}\left[D(\eta^{(L)},\xi(t))\right] \tag{6.4}$$

(2) If $\eta \in \mathbb{N}^{\mathbb{Z}}$ *is bounded, i.e.* $\sup_{i\in\mathbb{Z}}\eta_i < \infty$*, then it is a "good infinite-volume configuration".*

(3) Let us denote by $\mathscr{N}_\lambda(t)$ *a Poisson process of rate* $\lambda > 0$*, and by* $\mathbf{E}[\cdot]$ *the expectation w.r. to its probability law. If* μ *is a probability measure on* $\mathbb{N}^{\mathbb{Z}}$ *such that for any* $\lambda > 0$ *the expectation*

$$\mathbb{E}_\mu\left[\mathbf{E}\left[e^{\sum_{i=1}^{\mathscr{N}_\lambda(t)}\eta_{\ell+i}}\right]\right] \tag{6.5}$$

is finite for all $t \geq 0$ *and for any* $\ell \in \mathbb{Z}$*, then* μ *is a "good infinite-volume measure".*

Proof (1) If $\eta \in \mathbb{N}^{\mathbb{Z}}$ is a good infinite volume configuration, then the duality relation with duality function (5.3) makes sense after the following renormalization:

$$\begin{aligned}
&\mathbb{E}_{\eta^{(L)}}\left[D(\eta(t),\xi^{(\ell_1,\dots,\ell_n)})\right]\prod_{m=1}^{n} q^{-2N_{\ell_m+1}(\eta^{(L)})}\\
&\quad= \mathbb{E}_{\xi^{(\ell_1,\dots,\ell_n)}}\left[D(\eta^{(L)},\xi(t))\right]\prod_{m=1}^{n} q^{-2N_{\ell_m+1}(\eta^{(L)})}
\end{aligned} \tag{6.6}$$

then the first statement of the Theorem follows after taking the limit as $L \to \infty$ of (6.6).

(2) Let ξ be a finite configuration in $\mathbb{N}^{\mathbb{Z}}$. We prove that for any bounded $\eta \in \mathbb{N}^{\mathbb{Z}}$ the family of functions

$$\mathscr{S}_L(t) := \prod_{i\in\mathbb{Z}} q^{-2\xi_i N_{i+1}(\eta^{(L)})}\ \mathbb{E}_\xi\left[D(\eta^{(L)},\xi(t))\right], \qquad L \in \mathbb{N} \tag{6.7}$$

is uniformly bounded. Without loss of generality we can suppose that $\xi = \xi^{(\ell_1,\dots,\ell_n)}$, for some $\{\ell_1,\dots,\ell_n\} \subset \mathbb{Z}$, $n \in \mathbb{N}$. Moreover we denote by $(\ell_1(t),\dots,\ell_n(t))$ the positions of the n ASIP(q,k) walkers starting at time $t = 0$ from $(\ell_1,\dots,\ell_n)$. We then have $\xi(t) = \xi^{(\ell_1(t),\dots,\ell_n(t))}$, and

$$\begin{aligned}
\mathscr{S}_L(t) &= \prod_{m=1}^{n} q^{-2N_{\ell_m+1}(\eta^{(L)})}\ \mathbb{E}_{\xi^{(\ell_1,\dots,\ell_n)}}\left[D(\eta^{(L)},\xi(t))\right]\\
&= \mathbb{E}_{\xi^{(\ell_1,\dots,\ell_n)}}\Bigg[\prod_{i=1}^{L}\frac{(q^{2(\eta_i^{(L)}-\xi_i(t)+1)};q^2)_{\xi_i(t)}}{(q^{4k};q^2)_{\xi_i(t)}}\cdot q^{\xi_i^2(t)}\cdot \mathbf{1}_{\xi_i(t)\le\eta_i^{(L)}}\\
&\qquad\times\prod_{m=1}^{n} q^{-4k\ell_m(t)+2[N_{\ell_m(t)+1}(\eta^{(L)})-N_{\ell_m+1}(\eta^{(L)})]}\Bigg].
\end{aligned}$$

As a consequence, since

$$\left(q^{2(\eta-\xi+1)};q^2\right)_\xi\cdot q^{\xi^2}\cdot\mathbf{1}_{\xi\le\eta} \le 1 \tag{6.8}$$

and

$$\sup_{\ell \le n} \frac{1}{(q^{4k}; q^2)_\xi} \le c \tag{6.9}$$

for some $c > 0$, we have that there exists $C > 0$ such that

$$\left|\mathscr{S}_L(t)\right| \le C \, \mathbb{E}_{\xi(\ell_1,\dots,\ell_n)} \left[\prod_{m=1}^{n} q^{-4k\ell_m(t)+2[N_{\ell_m(t)+1}(\eta^{(L)})-N_{\ell_m+1}(\eta^{(L)})]} \right] \tag{6.10}$$

for all $L \in \mathbb{N}$, $t \ge 0$. Then, from the Cauchy–Schwarz inequality, in order to find an upper bound for (6.10), it is sufficient to find an upper bound for

$$s_{L,m}(t) := \mathbb{E}_{\xi(\ell_1,\dots,\ell_n)} \left[q^{\kappa\{-4k\ell_m(t)+2[N_{\ell_m(t)+1}(\eta^{(L)})-N_{\ell_m+1}(\eta^{(L)})]\}} \right]$$

for any fixed $m \in \{1, \dots, n\}$ and $\kappa \in \mathbb{N}$. Now, let $M := \sup_{i \in \mathbb{Z}} \eta_i < \infty$, then

$$\left|N_{\ell_m(t)+1}(\eta^{(L)}) - N_{\ell_m+1}(\eta^{(L)})\right| \le M |\ell_m(t) - \ell_m|$$

hence there exists $C', \omega > 0$ such that

$$\left|s_{L,m}(t)\right| \le C' \, \mathbb{E}_{\xi(\ell_1,\dots,\ell_n)} \left[\mathrm{e}^{\omega|\ell_m(t)-\ell_m|} \right] \tag{6.11}$$

for any $L \in \mathbb{N}$, $t \ge 0$. Since $\xi(t)$ has a finite number of particles, for each $m \in \{1, \dots, n\}$ the process $|\ell_m(t) - \ell_m|$ is stochastically dominated by a Poisson process $\mathscr{N}(t)$ with parameter

$$\lambda := \max_{0 \le \eta, \eta' \le n} \left\{ q^{\eta-\eta'+(2k-1)} [\eta]_q [2k+\eta']_q \right\} \vee \max_{0 \le \eta, \eta' \le n} \left\{ q^{\eta-\eta'-(2k-1)} [2k+\eta]_q [\eta']_q \right\} \tag{6.12}$$

then the right hand side of (6.11) is less or equal than

$$\mathbf{E}\left[\mathrm{e}^{\omega \mathscr{N}(t)} \right] = \mathrm{e}^{-\lambda t} \sum_{i=0}^{\infty} \mathrm{e}^{\omega i} \frac{(\lambda t)^i}{i!} < \infty. \tag{6.13}$$

This proves that $\mathscr{S}_L(t)$ is uniformly bounded.

(3) Suppose that the probability measure μ satisfies (6.5). Then, in order to prove that it is a "good" measure, it is sufficient to show that

$$\lim_{L \to \infty} \mathbb{E}_\mu \left[\prod_{i \in \mathbb{Z}} q^{-2\xi_i N_{i+1}(\eta^{(L)})} \, \mathbb{E}_\xi \left[D(\eta^{(L)}, \xi(t)) \right] \right] < \infty \tag{6.14}$$

By exploiting the same arguments used in the proof of item 2), we claim that, in order to prove (6.14) it is sufficient to show that for each fixed $m = 1, \dots, n$, $\kappa > 0$, the function

$$\Theta_{L,m}(t) := \mathbb{E}_\mu \left[\mathbb{E}_{\xi(\ell_1,\dots,\ell_n)} \left[q^{\kappa\{-4k\ell_m(t)+2[N_{\ell_m(t)+1}(\eta^{(L)})-N_{\ell_m+1}(\eta^{(L)})]\}} \right] \right] \tag{6.15}$$

is uniformly bounded. We have that

$$\begin{aligned}
&\Theta_{L,m}(t) \\
&= \mathbb{E}_\mu \left[\mathbb{E}_{\xi(\ell_1,\dots,\ell_n)} \left[q^{-4\kappa k \ell_m(t)} \left(q^{-2\kappa \sum_{i=\ell_m+1}^{\ell_m(t)} \eta_i^{(L)}} \mathbf{1}_{\ell_m < \ell_m(t)} + q^{2\kappa \sum_{i=\ell_m(t)+1}^{\ell_m} \eta_i^{(L)}} \mathbf{1}_{\ell_m(t) < \ell_m} \right) \right] \right] \\
&\le \mathbb{E}_\mu \left[\mathbb{E}_{\xi(\ell_1,\dots,\ell_n)} \left[q^{-4\kappa k \ell_m(t)} \left(q^{-2\kappa \sum_{i=1}^{\ell_m(t)-\ell_m} \eta_{i+\ell_m}^{(L)}} \mathbf{1}_{\ell_m < \ell_m(t)} + 1 \right) \right] \right].
\end{aligned}$$

Then the result follows as in proof of item 2) from the fact that the process $\ell_m(t) - \ell_m$ is stochastically dominated by a Poisson process of rate λ (6.12), and from the hypothesis (6.5).

Later on, if we write expectations in the infinite volume we always refer to the limiting procedure described above. Namely, for a "good infinite-volume configuration" $\eta \in \mathbb{N}^{\mathbb{Z}}$, with an abuse of notation we will write

$$\prod_{m=1}^{n} q^{-2N_{\ell_m+1}(\eta)} \, \mathbb{E}_\eta \left[D(\eta(t), \xi^{(\ell_1,\dots,\ell_n)}) \right]$$
$$:= \lim_{L\to\infty} \prod_{m=1}^{n} q^{-2N_{\ell_m+1}(\eta^{(L)})} \, \mathbb{E}_{\eta^{(L)}} \left[D(\eta(t), \xi^{(\ell_1,\dots,\ell_n)}) \right] \tag{6.16}$$

and

$$\prod_{m=1}^{n} q^{-2N_{\ell_m+1}(\eta)} \, \mathbb{E}_{\xi^{(\ell_1,\dots,\ell_n)}} \left[D(\eta, \xi(t)) \right]$$
$$:= \lim_{L\to\infty} \prod_{m=1}^{n} q^{-2N_{\ell_m+1}(\eta^{(L)})} \, \mathbb{E}_{\xi^{(\ell_1,\dots,\ell_n)}} \left[D(\eta^{(L)}, \xi(t)) \right] \tag{6.17}$$

6.2 q-Exponential Moment of the Current of ASIP(q, k)

We start by defining the current for the ASIP(q, k) process on $\mathbb{Z}$.

Definition 6.2 (*Current*) Let $\{\eta(t),\ \ t \geq 0\}$ be a càdlàg trajectory on the infinite-volume configuration space $\mathbb{N}^{\mathbb{Z}}$, then the total integrated current $J_i(t)$ in the time interval $[0, t]$ is defined as the net number of particles crossing the bond $(i-1, i)$ in the right direction. Namely, let $(t_i)_{i\in\mathbb{N}}$ be the sequence of the process jump times. Then

$$J_i(t) = \sum_{k: t_k \in [0,t]} \left(\mathbf{1}_{\{\eta(t_k)=\eta(t_k^-)^{i-1,i}\}} - \mathbf{1}_{\{\eta(t_k)=\eta(t_k^-)^{i,i-1}\}} \right) \tag{6.18}$$

Lemma 6.1 (Current) *The total integrated current of a càdlàg trajectory* $(\eta(s))_{0\leq s\leq t}$ *with* $\eta(0) = \eta$ *is given by*

$$J_i(t) = N_i(\eta(t)) - N_i(\eta) := \lim_{L\to\infty} \left(N_i(\eta^{(L)}(t)) - N_i(\eta^{(L)}) \right) \tag{6.19}$$

where $N_i(\eta)$ *is defined in* (5.4) *and* $\eta^{(L)}$ *is defined in* (6.1). *Moreover*

$$\lim_{i\to-\infty} J_i(t) = 0 \tag{6.20}$$

Proof (6.19) immediately follows from the definition of $J_i(t)$, whereas (6.20) follows from the conservation of the total number of particles. □

Proposition 6.2 (Current q-exponential moment via a dual walker) *Let* $\eta \in \mathbb{N}^{\mathbb{Z}}$ *a good infinite-volume configuration in the sense of Definition 6.1, then the first* q*-exponential moment of the current when the process is started from* η *at time* $t = 0$ *is given by*

$$\mathbb{E}_\eta \left[q^{2J_i(t)} \right] = q^{2(N(\eta)-N_i(\eta))} - \sum_{n=-\infty}^{i-1} q^{4kn} \, \mathbf{E}_n \left[q^{-4km(t)} \left(1 - q^{-2\eta_{m(t)}}\right) q^{2(N_{m(t)}(\eta)-N_i(\eta))} \right] \tag{6.21}$$

where $m(t)$ denotes a continuous time asymmetric random walker on $\mathbb{Z}$ jumping left at rate $q^{-2k}[2k]_q$ and jumping right at rate $q^{2k}[2k]_q$ and $\mathbf{E}_i$ denotes the expectation with respect to the law of $m(t)$ started at site $i \in \mathbb{Z}$ at time $t = 0$. Furthermore $N(\eta) - N_i(\eta) = \sum_{n<i} \eta_n$ and the first term on the right hand side of (6.21) is zero when there are infinitely many particles to the left of $i \in \mathbb{Z}$ in the configuration η.

Proof To prove (6.21) we consider the configuration $\xi^{(i)} \in \mathbb{N}^{\mathbb{Z}}$ with a single dual particle at site i. Since the ASIP(q, k) is self-dual the dynamics of the single dual particle is given an asymmetric random walk $m(t)$ on $\mathbb{Z}$ whose rates are computed from the process definition and coincides with those in the statement of the Proposition. From (6.16), (6.17) and item (1) of Proposition 6.1 we have that

$$q^{-2N_i(\eta)}\, \mathbb{E}_\eta \left[D(\eta(t), \xi^{(i)}) \right] = \frac{q^{-(4ki+1)}}{q^{2k} - q^{-2k}}\, q^{-2N_i(\eta)}\, \mathbb{E}_\eta \left[q^{2N_i(\eta(t))} - q^{2N_{i+1}(\eta(t))} \right]$$

is equal to

$$\begin{aligned} &q^{-2N_i(\eta)}\, \mathbb{E}_{\xi^{(i)}} \left[D(\eta, \xi^{(m(t))}) \right] \\ &= q^{-2N_i(\eta)} \frac{q^{-1}}{q^{2k} - q^{-2k}}\, \mathbf{E}_i \left[q^{-4km(t)} (q^{2N_{m(t)}(\eta)} - q^{2N_{m(t)+1}(\eta)}) \right] \end{aligned}$$

Then from (6.19) we get

$$\begin{aligned} \mathbb{E}_\eta \left[q^{2J_i(t)} \right] &= q^{-2\eta_i}\, \mathbb{E}_\eta \left[q^{2J_{i+1}(t)} \right] \\ &\quad + q^{4ki}\, \mathbf{E}_i \left[q^{-4km(t)} \left(q^{2(N_{m(t)}(\eta) - N_i(\eta))} - q^{2(N_{m(t)+1}(\eta) - N_i(\eta))} \right) \right] \end{aligned} \tag{6.22}$$

By iterating the relation in (6.22), for any $n \ge 0$ we get

$$\begin{aligned} \mathbb{E}_\eta \left[q^{2J_{i+1}(t)} \right] &= q^{2(N_{i-n}(\eta) - N_{i+1}(\eta))}\, \mathbb{E}_\eta \left[q^{2J_{i-n}(t)} \right] \\ &\quad - \sum_{j=0}^{n} q^{2(N_{i-j}(\eta) - N_{i+1}(\eta))} q^{4k(i-j)} \\ &\quad \times \mathbf{E}_{i-j} \left[q^{-4km(t)} \left(q^{2(N_{m(t)}(\eta) - N_{i-j}(\eta))} - q^{2(N_{m(t)+1}(\eta) - N_{i-j}(\eta))} \right) \right]. \end{aligned} \tag{6.23}$$

By taking the limit $n \to \infty$ we get

$$\begin{aligned} \mathbb{E}_\eta \left[q^{2J_{i+1}(t)} \right] &= \lim_{n\to\infty} q^{2(N_{i-n}(\eta) - N_{i+1}(\eta))}\, \mathbb{E}_\eta \left[q^{2J_{i-n}(t)} \right] \\ &\quad - \sum_{j=0}^{\infty} q^{-2N_{i+1}(\eta)} q^{4k(i-j)}\, \mathbf{E}_{i-j} \left[q^{-4km(t)} \left(q^{2N_{m(t)}(\eta)} - q^{2N_{m(t)+1}(\eta)} \right) \right] \end{aligned}$$

and using (6.20) we obtain (6.21). □

We continue with a lemma that is useful in the following (for the proof see e.g. [20]).

Lemma 6.2 *Let $x(t)$ be the random walk on $\mathbb{Z}$ jumping to the right with rate $a \ge 0$ and to the left with rate $b \ge 0$, let $\alpha \in \mathbb{R}$, and $A \subseteq \mathbb{R}$ then*

$$\lim_{t\to\infty} \frac{1}{t} \ln \mathbf{E}_0 \left[\alpha^{x(t)} \,\middle|\, x(t) \in A \right] = \sup_{x\in A} \{ x \ln \alpha - \mathscr{I}(x) \} - \inf_{x\in A} \mathscr{I}(x) \tag{6.24}$$

with

$$\mathscr{I}(x) = (a+b) - \sqrt{x^2+4ab} + x\ln\left(\frac{x+\sqrt{x^2+4ab}}{2a}\right) \tag{6.25}$$

Remark 6.1 Let $m(t)$ be the random walk defined in Proposition 6.2, then (6.24) holds with

$$\mathscr{I}(x) = [4k]_q - \sqrt{x^2+(2[2k]_q)^2} + x\ln\left\{\frac{1}{2[2k]_q q^{2k}}\left[x+\sqrt{x^2+(2[2k]_q)^2}\right]\right\} \tag{6.26}$$

We denote by $\mathbb{E}^{\otimes\mu}$ the expectation of the ASIP(q,k) process on $\mathbb{Z}$ initialized with the homogeneous product measure on $\mathbb{N}^{\mathbb{Z}}$ with marginals μ at time 0, i.e.

$$\mathbb{E}^{\otimes\mu}[f(\eta(t))] = \sum_{\eta}(\otimes_{i\in\mathbb{Z}}\mu(\eta_i))\,\mathbb{E}_\eta[f(\eta(t))].$$

Proposition 6.3 (q-moment for product initial condition) *Consider an homogeneous product probability measure μ on $\mathbb{N}$. Then, for the infinite volume ASIP(q,k), we have*

$$\mathbb{E}^{\otimes\mu}\left[q^{2J_i(t)}\right] = \mathbf{E}_0\left[\left(\frac{q^{-4k}}{\lambda_q}\right)^{m(t)}\mathbf{1}_{m(t)\le 0}\right] + \mathbf{E}_0\left[q^{-4km(t)}\left(\lambda_{1/q}^{m(t)} - \lambda_{1/q} + \lambda_q^{-1}\right)\mathbf{1}_{m(t)\ge 1}\right] \tag{6.27}$$

where $\lambda_y := \sum_{n=0}^{\infty} y^n\mu(n)$ and $m(t)$ is the random walk defined in Proposition 6.2. In particular we have

$$\lim_{t\to\infty}\frac{1}{t}\ln \mathrm{e}^{\otimes\mu}[q^{2J_i(t)}] = \sup_{x\ge 0}\left\{x\ln M_q - \mathscr{I}(x)\right\} - \inf_{x\ge 0}\mathscr{I}(x) \tag{6.28}$$

with $M_q := q^{-4k}\lambda_{1/q}$ and $\mathscr{I}(x)$ given by (6.26).

Proof It is easy to check that an homogeneous product measure μ verifies the condition (6.5) in Proposition 6.1, thus it is a good infinite-volume probability measure in the sense of Definition 6.1. For this reason we can apply Proposition 6.2, and from (6.21) we have

$$\begin{aligned}\mathbb{E}^{\otimes\mu}\left[q^{2J_i(t)}\right] &= \int \otimes\mu(d\eta)\,\mathbb{E}_\eta\left[q^{2J_i(t)}\right] \\ &= \int \otimes\mu(d\eta) q^{2(N(\eta)-N_i(\eta))} \\ &\quad + \sum_{n=-\infty}^{i-1} q^{4kn}\int\otimes\mu(d\eta)\mathbf{E}_n\left[q^{-4km(t)}\left(q^{-2\eta_{m(t)}}-1\right)q^{2(N_{m(t)}(\eta)-N_i(\eta))}\right]\end{aligned} \tag{6.29}$$

Since

$$\int\otimes\mu(d\eta)q^{2(N_m(\eta)-N_i(\eta))} = \lambda_q^{i-m}\,\mathbf{1}_{\{m\le i\}} + \lambda_{1/q}^{m-i}\,\mathbf{1}_{\{m>i\}} \tag{6.30}$$

then, in particular, $\int\otimes\mu(d\eta)q^{2(N(\eta)-N_i(\eta))} = 0$ since $\lambda_q < 1$, where we recall the interpretation of $N(\eta) - N_i(\eta)$ from Proposition 6.2. Hence

$$
\begin{aligned}
\mathbb{E}^{\otimes\mu}\left[q^{2J_i(t)}\right] &= \sum_{n=-\infty}^{i-1} q^{4kn} \sum_{m\in\mathbb{Z}} \mathbf{P}_n\left(m(t)=m\right) q^{-4km} \\
&\quad \times \int \otimes\mu(d\eta)\left[q^{2(N_{m+1}(\eta)-N_i(\eta))} - q^{2(N_m(\eta)-N_i(\eta))}\right] \\
&= \left(\lambda_q^{-1}-1\right) A(t) + \left(\lambda_{1/q}-1\right) B(t)
\end{aligned} \tag{6.31}
$$

with

$$
A(t) := \sum_{n\le i-1} q^{4kn} \sum_{m\le i} \mathbf{P}_n\left(m(t)=m\right) q^{-4km} \lambda_q^{i-m} \tag{6.32}
$$

and

$$
B(t) := \sum_{n\le i-1} q^{4kn} \sum_{m\ge i+1} \mathbf{P}_n\left(m(t)=m\right) q^{-4km} \lambda_{1/q}^{m-i} \tag{6.33}
$$

Now, let $\alpha := q^{-4k}\lambda_q^{-1}$, then

$$
\begin{aligned}
A(t) &= \sum_{n\le i-1} q^{4kn}\lambda_q^{i} \sum_{m\le i} \mathbf{P}_n\left(m(t)=m\right) \alpha^{m} \\
&= \sum_{j\ge 1} \lambda_q^{j} \sum_{\bar m\le j} \mathbf{P}_0\left(m(t)=\bar m\right) \alpha^{\bar m} \\
&= \sum_{\bar m\le 0} \alpha^{\bar m}\mathbf{P}_0\left(m(t)=\bar m\right) \sum_{j\ge 1}\lambda_q^{j} + \sum_{\bar m\ge 1} \alpha^{\bar m}\mathbf{P}_0\left(m(t)=\bar m\right) \sum_{j\ge \bar m}\lambda_q^{j} \\
&= \frac{1}{1-\lambda_q}\left\{\lambda_q\,\mathbf{E}_0\left[\alpha^{m(t)}\,\mathbf{1}_{m(t)\le 0}\right] + \mathbf{E}_0\left[q^{-4km(t)}\,\mathbf{1}_{m(t)\ge 1}\right]\right\}
\end{aligned} \tag{6.34}
$$

Analogously one can prove that

$$
B(t) = \frac{1}{\lambda_{1/q}-1}\left\{\mathbf{E}_0\left[\beta^{m(t)}\,\mathbf{1}_{m(t)\ge 2}\right] - \lambda_{1/q}\mathbf{E}_0\left[q^{-4km(t)}\,\mathbf{1}_{m(t)\ge 2}\right]\right\} \tag{6.35}
$$

with $\beta = q^{-4k}\lambda_{1/q}$ then (6.27) follows by combining (6.31), (6.34) and (6.35).

In order to prove (6.28) we use the fact that $m(t)$ has a Skellam distribution with parameters $([2k]_q q^{2k}t, [2k]_q q^{-2k}t)$, i.e. $m(t)$ is the difference of two independent Poisson random variables with those parameters. This implies that

$$
\mathbf{E}_0\left[\left(\frac{q^{-4k}}{\lambda_q}\right)^{m(t)}\mathbf{1}_{m(t)\le 0}\right] = \mathbf{E}_0\left[\lambda_q^{m(t)}\mathbf{1}_{m(t)\ge 0}\right].
$$

Then we can rewrite (6.27) as

$$
\begin{aligned}
\mathbb{E}^{\otimes\mu}\left[q^{2J_i(t)}\right] &= \mathbf{E}_0\left[\lambda_q^{m(t)}\,\mathbf{1}_{m(t)\ge 1}\right] + \mathbf{P}_0\left(m(t)=0\right) \\
&\quad + \left(\lambda_q^{-1}-\lambda_{1/q}\right)\mathbf{E}_0\left[q^{-4km(t)}\mathbf{1}_{m(t)\ge 1}\right] + \mathbf{E}_0\left[M_q^{m(t)}\mathbf{1}_{m(t)\ge 1}\right] \\
&= \mathbf{E}_0\left[M_q^{m(t)}\mathbf{1}_{m(t)\ge 0}\right]\left(1+\mathscr{E}_1(t)+\mathscr{E}_2(t)+\mathscr{E}_3(t)+\mathscr{E}_4(t)\right)
\end{aligned} \tag{6.36}
$$

with

$$
\mathscr{E}_1(t) := \frac{\mathbf{E}_0\left[M_q^{m(t)}\mathbf{1}_{m(t)\ge 1}\right]}{\mathbf{E}_0\left[M_q^{m(t)}\mathbf{1}_{m(t)\ge 0}\right]}, \qquad \mathscr{E}_2(t) := \frac{\mathbf{P}_0\left(m(t)=0\right)}{\mathbf{E}_0\left[M_q^{m(t)}\mathbf{1}_{m(t)\ge 0}\right]}
$$

and

$$\mathscr{E}_3(t) := \frac{\mathbf{E}_0\left[\lambda_q^{m(t)}\mathbf{1}_{m(t)\geq 1}\right]}{\mathbf{E}_0\left[M_q^{m(t)}\mathbf{1}_{m(t)\geq 0}\right]}, \qquad \mathscr{E}_4(t) := \frac{\left(\lambda_q^{-1} - \lambda_{1/q}\right)\mathbf{E}_0\left[q^{-4km(t)}\mathbf{1}_{m(t)\geq 1}\right]}{\mathbf{E}_0\left[M_q^{m(t)}\mathbf{1}_{m(t)\geq 0}\right]} \tag{6.37}$$

To identify the leading term in (6.36) it remains to prove that, for each $i = 1, 2, 3$ there exists $c_i > 0$ such that

$$\sup_{t\geq 0} |\mathscr{E}_i(t)| \leq c_i \tag{6.38}$$

This would imply, making use of Lemma 6.2, the result in (6.28). The bound in (6.38) is immediate for $i = 1, 2, 3$. To prove it for $i = 4$ it is sufficient to show that there exists $c > 0$ such that

$$\lambda_q^{-1}\mathbf{E}_0\left[q^{-4km(t)}\mathbf{1}_{m(t)\geq 1}\right] \leq c\,\mathbf{E}_0\left[\left(q^{-4k}\lambda_{1/q}\right)^{m(t)}\mathbf{1}_{m(t)\geq 1}\right]. \tag{6.39}$$

This follows since there exists $m_* \geq 1$ such that for any $m \geq m_*$ $\lambda_q^{-1} \leq \lambda_{1/q}^m$ and then

$$\begin{aligned}
\lambda_q^{-1}\mathbf{E}_0\left[q^{-4km(t)}\mathbf{1}_{m(t)\geq 1}\right] &\leq \lambda_q^{-1}\mathbf{E}_0\left[q^{-4km(t)}\mathbf{1}_{1\leq m(t)<m_*}\right] + \mathbf{E}_0\left[q^{-4km(t)}\lambda_{1/q}^{m(t)}\mathbf{1}_{m(t)\geq m_*}\right] \\
&\leq \lambda_q^{-1}\mathbf{E}_0\left[q^{-4km(t)}\mathbf{1}_{1\leq m(t)}\right] + \mathbf{E}_0\left[q^{-4km(t)}\lambda_{1/q}^{m(t)}\mathbf{1}_{m(t)\geq 1}\right] \\
&\leq \left(1+\lambda_q^{-1}\right)\mathbf{E}_0\left[\left(q^{-4k}\lambda_{1/q}\right)^{m(t)}\mathbf{1}_{m(t)\geq 1}\right].
\end{aligned} \tag{6.40}$$

This concludes the proof. □

6.3 Infinite Volume Limit for ABEP(σ, k)

Definition 6.3 (*Good infinite-volume configuration*)

(a) We say that $x \in \mathbb{R}_+^{\mathbb{Z}}$ is a *"good infinite-volume configuration"* for ABEP(σ, k) iff for $x^{(L)} \in \mathbb{R}_+^{\mathbb{Z}}$, $L \in \mathbb{N}$, the restriction of x to $[-L, L]$, i.e.

$$x_i^{(L)} = \begin{cases} x_i & \text{for} \quad i \in [-L, L] \\ 0 & \text{otherwise} \end{cases} \tag{6.41}$$

the limit

$$\lim_{L\to\infty} \prod_{i\in\mathbb{Z}} \mathrm{e}^{2\sigma\xi_i E_{i+1}(x^{(L)})}\, \mathbb{E}_\xi\left[D^\sigma(x^{(L)}, \xi(t))\right] \tag{6.42}$$

exists and is finite for all $t \geq 0$ and for any $\xi \in \mathbb{N}^{\mathbb{Z}}$ finite (i.e. such that $\sum_{i\in\mathbb{Z}} \xi_i < \infty$).

(b) Let μ be a probability measure on $\mathbb{R}_+^{\mathbb{Z}}$, then we say that it is a *"good infinite-volume measure"* for ABEP(σ, k) iff it concentrates on good infinite-volume configurations.

Proposition 6.4 *(1) If $x \in \mathbb{R}_+^{\mathbb{Z}}$ is a "good infinite-volume configuration" for ABEP(σ, k) and $\xi^{(\ell_1,\ldots,\ell_n)}$ is the configurations with n particles located at sites $\ell_1, \ldots, \ell_n \in \mathbb{Z}$, then the limit*

$$\lim_{L\to\infty} \prod_{m=1}^{n} \mathrm{e}^{2\sigma E_{\ell_m+1}(x^{(L)})}\, \mathbb{E}_{x^{(L)}}\left[D^\sigma(x(t), \xi^{(\ell_1,\ldots,\ell_n)})\right] \tag{6.43}$$

is well-defined for all $t \geq 0$ and is equal to

$$\lim_{L\to\infty} \prod_{m=1}^{n} \mathrm{e}^{2\sigma E_{\ell_m+1}(x^{(L)})}\, \mathbb{E}_{\xi^{(\ell_1,\ldots,\ell_n)}} \left[D^{\sigma}(x^{(L)}, \xi(t)) \right] \tag{6.44}$$

(2) If $x \in \mathbb{R}_+^{\mathbb{Z}}$ is bounded, i.e. $\sup_{i\in\mathbb{Z}} x_i < \infty$*, then it is a "good infinite-volume configuration" for ABEP(σ, k).*

(3) Let us denote by $\mathscr{N}_\lambda(t)$ a Poisson process of rate $\lambda > 0$, and by $\mathbf{E}[\cdot]$ the expectation w.r.t. to its probability law. If μ is a probability measure on $\mathbb{R}_+^{\mathbb{Z}}$ such that for any $\lambda > 0$ the expectation

$$\mathbb{E}_{\mu}\left[\mathbf{E}\left[\mathrm{e}^{\sum_{i=1}^{\mathscr{N}_\lambda(t)} x_{\ell+i}}\right]\right] \tag{6.45}$$

is finite for all $t \geq 0$ and for any $\ell \in \mathbb{Z}$, then μ is a "good infinite-volume measure" for ABEP(σ, k).

Proof The proof is analogous to the proof of Proposition 6.1. □

Later on for a "good" infinite-volume configuration $x \in \mathbb{R}_+^{\mathbb{Z}}$ we will write

$$\prod_{i\in\mathbb{Z}} \mathrm{e}^{2\sigma\xi_i E_{i+1}(x)}\, \mathbb{E}_{\xi}\left[D^{\sigma}(x, \xi(t))\right] := \lim_{L\to\infty} \prod_{i\in\mathbb{Z}} \mathrm{e}^{2\sigma\xi_i E_{i+1}(x^{(L)})}\, \mathbb{E}_{\xi}\left[D^{\sigma}(x^{(L)}, \xi(t))\right] \tag{6.46}$$

and

$$\begin{aligned}
&\prod_{m=1}^{n} \mathrm{e}^{2\sigma E_{\ell_m+1}(x)}\, \mathbb{E}_{x}\left[D^{\sigma}(x(t), \xi^{(\ell_1,\ldots,\ell_n)})\right] \\
&\quad := \lim_{L\to\infty} \prod_{m=1}^{n} \mathrm{e}^{2\sigma E_{\ell_m+1}(x^{(L)})}\, \mathbb{E}_{x^{(L)}}\left[D^{\sigma}(x(t), \xi^{(\ell_1,\ldots,\ell_n)})\right]
\end{aligned} \tag{6.47}$$

6.4 $\mathbf{e^{-\sigma}}$-Exponential Moment of the Current of ABEP(σ, k)

We start by defining the current for the ABEP(σ, k) process on $\mathbb{Z}$.

Definition 6.4 (*Current*) Let $\{x(t),\ t \geq 0\}$ be a càdlàg trajectory on the infinite-volume configuration space $\mathbb{R}_+^{\mathbb{Z}}$, then the total integrated current $J_i(t)$ in the time interval $[0, t]$ is defined as total energy crossing the bond $(i-1, i)$ in the right direction.

$$J_i(t) = E_i(x(t)) - E_i(x(0)) := \lim_{L\to\infty} \left(E_i(x^{(L)}(t)) - E_i(x^{(L)}) \right) \tag{6.48}$$

where $E_i(x)$ is defined in (3.2) and $x^{(L)}$ as in (6.41).

Lemma 6.3 (Current) *We have* $\lim_{i\to-\infty} J_i(t) = 0$.

Proof It immediately follows from the conservation of the total energy. □

Proposition 6.5 (Current exponential moment via a dual walker) *The first exponential moment of $J_i(t)$ when the process is started from a "good infinite-volume initial configuration" $x \in \mathbb{R}_+^{\mathbb{Z}}$ at time $t = 0$ is given by*

$$\mathbb{E}_x\left[\mathrm{e}^{-2\sigma J_i(x(t))}\right] = \mathrm{e}^{-4kt}\sum_{n\in\mathbb{Z}}\mathrm{e}^{-2\sigma(E_n(x)-E_i(x))}\,I_{|n-i|}(4kt) \tag{6.49}$$

where $I_n(t)$ is the modified Bessel function.

Proof Let $\xi^{(\ell)} \in \mathbb{R}_+^{\mathbb{Z}}$ be the configuration with a single particle at site ℓ. Since the ABEP(σ, k) is dual to the SIP($2k$) the dynamics of the single dual particle is given by a continuous time symmetric random walker $\ell(t)$ on $\mathbb{Z}$ jumping at rate $2k$. Since x is a good configuration we have that the normalized expectation

$$\mathrm{e}^{2\sigma E_i(x)}\,\mathbb{E}_x\left[D(x(t),\xi^{(\ell)})\right] = \frac{1}{4k\sigma}\,\mathrm{e}^{2\sigma E_i(x)}\,\mathbb{E}_x\left[\mathrm{e}^{-2\sigma E_{\ell+1}(x(t))} - \mathrm{e}^{-2\sigma E_\ell(x(t))}\right]$$

and, from the duality relation (5.5) this is also equal to:

$$\mathrm{e}^{2\sigma E_i(x)}\,\mathbb{E}_{\xi^{(\ell)}}\left[D(x,\xi^{(\ell(t))})\right] = \frac{1}{4k\sigma}\,\mathrm{e}^{2\sigma E_i(x)}\,\mathbf{E}_\ell\left[\mathrm{e}^{-2\sigma E_{\ell(t)+1}(x)} - \mathrm{e}^{-2\sigma E_{\ell(t)}(x)}\right]$$

where $\mathbf{E}_\ell$ denotes the expectation with respect to the law of $\ell(t)$ started at site $\ell \in \mathbb{Z}$ at time $t = 0$. As a consequence, for any $\ell \in \mathbb{Z}$

$$\begin{aligned}\mathrm{e}^{2\sigma E_i(x)}\,\mathbb{E}_x\left[\mathrm{e}^{-2\sigma E_{\ell+1}(x(t))}\right] &= \mathrm{e}^{2\sigma E_i(x)}\,\mathbb{E}_x\left[\mathrm{e}^{-2\sigma E_\ell(x(t))}\right]\\ &\quad + \mathrm{e}^{2\sigma E_i(x)}\,\mathbf{E}_\ell\left[\mathrm{e}^{-2\sigma E_{\ell(t)+1}(x)} - \mathrm{e}^{-2\sigma E_{\ell(t)}(x)}\right]\end{aligned} \tag{6.50}$$

from which it follows

$$\begin{aligned}\mathrm{e}^{2\sigma E_i(x)}\,\mathbb{E}_x\left[\mathrm{e}^{-2\sigma E_i(x(t))}\right] &= \mathrm{e}^{2\sigma E_i(x)}\sum_{\ell\le i-1}\mathbf{E}_\ell\left[\mathrm{e}^{-2\sigma E_{\ell(t)+1}(x)} - \mathrm{e}^{-2\sigma E_{\ell(t)}(x)}\right]\\ &= \mathrm{e}^{2\sigma E_i(x)}\sum_{\ell\le i-1}\mathbf{E}_0\left[\mathrm{e}^{-2\sigma E_{\ell(t)+\ell+1}(x)} - \mathrm{e}^{-2\sigma E_{\ell(t)+\ell}(x)}\right]\\ &= \mathrm{e}^{2\sigma E_i(x)}\sum_{m\le i}\mathbf{E}_0\left[\mathrm{e}^{-2\sigma E_{\ell(t)+m}(x)}\right] - \sum_{\ell\le i-1}\mathbf{E}_0\left[\mathrm{e}^{-2\sigma E_{\ell(t)+\ell}(x)}\right]\\ &= \mathrm{e}^{2\sigma E_i(x)}\,\mathbf{E}_0\left[\mathrm{e}^{-2\sigma E_{\ell(t)+i}(x)}\right]\\ &= \mathrm{e}^{2\sigma E_i(x)}\,\mathbf{E}_i\left[\mathrm{e}^{-2\sigma E_{\ell(t)}(x)}\right].\end{aligned} \tag{6.51}$$

Thus we have arrived to

$$\mathbb{E}_x\left[\mathrm{e}^{-2\sigma J_i(t)}\right] = \mathbf{E}_i\left[\mathrm{e}^{-2\sigma\left(E_{\ell(t)}(x)-E_i(x)\right)}\right] \tag{6.52}$$

and the result (6.49) follows since

$$\mathbf{E}_i(f(\ell(t))) = \sum_{n\in\mathbb{Z}} f(n)\cdot\mathbf{P}_i(\ell(t)=n)$$

with

$$\begin{aligned}\mathbf{P}_i(\ell(t)=n) &= \mathbb{P}(\ell(t)=n \mid \ell(0)=i)\\ &= \mathrm{e}^{-4kt}I_{|n-i|}(4kt)\end{aligned} \tag{6.53}$$

where $I_n(x)$ is the modified Bessel function. □

Remark 6.2 Let $\ell(t)$ be a continuous time symmetric random walk on $\mathbb{Z}$ jumping at rate $2k$, then (6.24) holds with

$$\mathscr{I}(x) = 4k - \sqrt{x^2 + (4k)^2} + x \ln \left\{ \frac{1}{4k} \left[x + \sqrt{x^2 + (4k)^2} \right] \right\} \tag{6.54}$$

We denote by $\mathbb{E}^{\otimes \mu}$ the expectation of the ABEP(σ, k) process on $\mathbb{Z}$ initialized with the omogeneous product measure on $\mathbb{R}^{\mathbb{Z}}$ with marginals μ at time 0, i.e.

$$\mathbb{E}^{\otimes \mu}[f(x(t))] = \int (\otimes_{i \in \mathbb{Z}} \mu(dx_i)) \; \mathbb{E}_x[f(x(t))] \tag{6.55}$$

Proposition 6.6 (Exponential moment for product initial condition) *Consider a probability measure μ on $\mathbb{R}^+$. Then, for the infinite volume ABEP(σ, k), we have*

$$\mathbb{E}^{\otimes \mu}\left[\mathrm{e}^{-2\sigma J_i(t)}\right] = \mathbf{P}_0\left[\ell(t) = 0\right] + \mathbf{E}_0\left[\left(\lambda_+^{\ell(t)} + \lambda_-^{\ell(t)}\right) \mathbf{1}_{\ell(t) \geq 1}\right] \tag{6.56}$$

where $\lambda_\pm := \int \mu(dy) \mathrm{e}^{\pm 2\sigma y}$ and $\ell(t)$ is the random walk defined in Remark 6.2. In particular we have

$$\lim_{t \to \infty} \frac{1}{t} \ln \mathrm{e}^{\otimes \mu}[\mathrm{e}^{-2\sigma J_i(t)}] = \sup_{x \geq 0} \{x \ln \lambda_+ - \mathscr{I}(x)\} - \inf_{x \geq 0} \mathscr{I}(x) \tag{6.57}$$

with $\mathscr{I}(x)$ given by (6.54).

Proof It is easy to check that an homogeneous product measure μ verifies the condition (6.45) in Proposition 6.1, thus it is a good infinite-volume probability measure for ABEP(σ, k) in the sense of Definition 6.3. Thus we can apply Proposition 6.5, in particular from (6.52) we have

$$\begin{aligned} \mathbb{E}^{\otimes \mu}\left[\mathrm{e}^{-2\sigma J_i(t)}\right] &= \int \otimes \mu(dx) \, \mathbb{E}_x \left[\mathrm{e}^{-2\sigma J_i(t)}\right] \\ &= \int \otimes \mu(dx) \mathbf{E}_i \left[\mathrm{e}^{-2\sigma \left(E_{\ell(t)}(x) - E_i(x)\right)}\right] \\ &= \sum_{n \in \mathbb{Z}} \mathbf{P}_i\left(\ell(t) = n\right) \int \otimes \mu(dx) \mathrm{e}^{-2\sigma (E_n(x) - E_i(x))}. \end{aligned}$$

Since

$$\int \otimes \mu(d\eta) \mathrm{e}^{-2\sigma (E_x(\eta) - E_i(\eta))} = \lambda_-^{i-n} \, \mathbf{1}_{\{n \leq i\}} + \lambda_+^{n-i} \, \mathbf{1}_{\{n > i\}} \tag{6.58}$$

it follows that

$$\begin{aligned} \mathbb{E}^{\otimes \mu}\left[\mathrm{e}^{-2\sigma J_i(t)}\right] &= \sum_{n \leq i} \mathbf{P}_i\left(\ell(t) = n\right) \lambda_-^{i-n} + \sum_{n \geq i+1} \mathbf{P}_i\left(\ell(t) = n\right) \lambda_+^{n-i} \\ &= \mathbf{E}_i \left[\lambda_-^{i-\ell(t)} \mathbf{1}_{\ell(t) \leq i} + \lambda_+^{\ell(t)-i} \mathbf{1}_{\ell(t) \geq i+1}\right] \\ &= \mathbf{E}_0 \left[\lambda_-^{-\ell(t)} \mathbf{1}_{\ell(t) \leq 0} + \lambda_+^{\ell(t)} \mathbf{1}_{\ell(t) \geq 1}\right] \\ &= \mathbf{E}_0 \left[\lambda_-^{\ell(t)} \mathbf{1}_{\ell(t) \geq 0} + \lambda_+^{\ell(t)} \mathbf{1}_{\ell(t) \geq 1}\right] \end{aligned} \tag{6.59}$$

where the last identity follows from the symmetry of $\ell(t)$. Then (6.56) is proved.

In order to prove (6.57) we rewrite (6.56) as

$$\mathbb{E}^{\otimes\mu}\left[\mathrm{e}^{-2\sigma J_i(t)}\right] = \mathbf{E}_0\left[\lambda_+^{\ell(t)}\mathbf{1}_{\ell(t)\geq 0}\right](1+\mathscr{E}_1(t)+\mathscr{E}_2(t)) \tag{6.60}$$

with

$$\mathscr{E}_1(t) := \frac{\mathbf{E}_0\left[\left(\lambda_+^{\ell(t)}+\lambda_-^{\ell(t)}\right)\mathbf{1}_{\ell(t)\geq 1}\right]}{\mathbf{E}_0\left[\lambda_+^{\ell(t)}\mathbf{1}_{\ell(t)\geq 0}\right]}, \qquad \mathscr{E}_2(t) := \frac{\mathbf{P}_0\left(x(t)=0\right)}{\mathbf{E}_0\left[\lambda_+^{\ell(t)}\mathbf{1}_{\ell(t)\geq 0}\right]}$$

where for $i=1,2$ there exists $c_i > 0$ such that

$$\sup_{t\geq 0}|\mathscr{E}_i(t)| \leq c_i \tag{6.61}$$

This and the result of Remark 6.2 conclude the proof of (6.57). □

Appendix: Algebraic Construction of ASIP(q,k) and Proof of the Self-Duality

In this section we give the a sketch of the proof of Theorem 5.1. Because it follows closely the lines of [13] we will only indicate the places where there are significant differences with the proof in [13]. The main steps are

1. *Central element* Start from a central element C in a Lie algebra (in our case the Casimir element of $\mathscr{U}_q(\mathfrak{su}(1,1))$).
2. *Coproduct, Hamiltonian and ground state* Apply a coproduct to turn C into a two-site Hamiltonian $H_{i,i+1}$, and into a L-site Hamiltonian via $H_{(L)} = \sum_{i=1}^{L-1} H_{i,i+1}$. This Hamiltonian has by construction symmetries and the zero state $|0\rangle\otimes\ldots\otimes|0\rangle$ as a ground state. By acting with a suitable symmetry $S^+_{(L)}$, we obtain a strictly positive ground state.
3. *Markov generator* Turn the Hamiltonian into a Markov generator using the positive ground state.
4. *Self-duality* A self-duality function is obtained by acting with a symmetry obtained from $S^+_{(L)}$ on the cheap self-duality function $d(\eta,\xi) = \frac{1}{\mu(\eta)}\delta_{\eta,\xi}$ with $\mu(\eta)$ a reversible measure.

The Quantum Lie Algebra $\mathscr{U}_q(\mathfrak{su}(1,1))$, Casimir Element and Representation

For $q\in(0,1)$ we consider the algebra with generators K^+, K^-, K^0 satisfying the commutation relations

$$[K^+,K^-] = -[2K^0]_q, \qquad [K^0,K^\pm] = \pm K^\pm\,, \tag{7.1}$$

where $[\cdot,\cdot]$ denotes the commutator, i.e. $[A,B]=AB-BA$, and

$$[2K^0]_q := \frac{q^{2K^0}-q^{-2K^0}}{q-q^{-1}}. \tag{7.2}$$

This is the quantum Lie algebra $\mathscr{U}_q(\mathfrak{su}(1,1))$ (for more details see e.g. [8]), that in the limit $q\to 1$ reduces to the Lie algebra $\mathfrak{su}(1,1)$. The Casimir element is

$$C = [K^0]_q[K^0-1]_q - K^+K^- \tag{7.3}$$

For our construction we chose the following unitary representation of the quantum Lie algebra $\mathscr{U}_q(\mathfrak{su}(1,1))$ is given by

$$\begin{cases} K^+|n\rangle &= \sqrt{[n+2k]_q[n+1]_q}\ |n+1\rangle \\ K^-|n\rangle &= \sqrt{[n]_q[n+2k-1]_q}\ |n-1\rangle \\ K^0|n\rangle &= (n+k)\ |n\rangle. \end{cases} \tag{7.4}$$

$k \in \mathbb{N}$. Here the collection of column vectors $|n\rangle$, with $n \in \mathbb{N}$, denote the standard orthonormal basis with respect to the Euclidean scalar product, i.e. $|n\rangle = (0,\dots,0,1,0,\dots,0)^T$ with the element 1 in the nth position and with the symbol ^{T} denoting transposition. Here and in the following, with abuse of notation, we use the same symbol for a linear operator and the matrix associated to it in a given basis. In the representation (7.4) the ladder operators K^+ and K^- are one the adjoint of the other, namely

$$(K^+)^* = K^- \tag{7.5}$$

and the Casimir element is given by the diagonal matrix

$$C|n\rangle = [k]_q[k-1]_q|n\rangle.$$

The choice of the representation (7.4) is mainly motivated by the fact that, after ground state transformation the Hamiltonian in Definition 6.5 below becomes the generator of the SIP in the limit $q \to 1$.

Co-product

A co-product for the quantum Lie algebra $\mathscr{U}_q(\mathfrak{su}(1,1))$ is defined as the map Δ : $\mathscr{U}_q(\mathfrak{su}(1,1)) \to \mathscr{U}_q(\mathfrak{su}(1,1)) \otimes \mathscr{U}_q(\mathfrak{su}(1,1))$

$$\begin{aligned} \Delta(K^\pm) &= K^\pm \otimes q^{-K^0} + q^{K^0} \otimes K^\pm\ , \\ \Delta(K^0) &= K^0 \otimes 1 + 1 \otimes K^0. \end{aligned} \tag{7.6}$$

The co-product is an isomorphism for the quantum Lie algebra $\mathscr{U}_q(\mathfrak{su}(1,1))$, i.e.

$$[\Delta(K^+),\Delta(K^-)] = -[2\Delta(K^0)]_q, \qquad [\Delta(K^0),\Delta(K^\pm)] = \pm\Delta(K^\pm). \tag{7.7}$$

The Quantum Hamiltonian

A natural candidate for the quantum Hamiltonian operator is obtained by applying the co-product to the Casimir operator C in (7.3). Using the co-product definition (7.6), simple algebraic manipulations yield the following definition.

Definition 6.5 (*Quantum Hamiltonian*) For every $L \in \mathbb{N}$, $L \geq 2$, we consider the operator $H_{(L)}$ defined by

$$H_{(L)} := \sum_{i=1}^{L-1} H_{(L)}^{i,i+1} = \sum_{i=1}^{L-1} \left(h_{(L)}^{i,i+1} + c_{(L)} \right)\ , \tag{7.8}$$

where

$$c_{(L)} = \frac{\left(q^{2k}-q^{-2k}\right)\left(q^{2k-1}-q^{-(2k-1)}\right)}{(q-q^{-1})^2}\underbrace{1\otimes\cdots\otimes 1}_{L\text{ times}} \tag{7.9}$$

and

$$h_{(L)}^{i,i+1} := \underbrace{1 \otimes \cdots \otimes 1}_{(i-1)\text{ times}} \otimes \Delta(C_i) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(L-i-1)\text{ times}} \tag{7.10}$$

and, from (7.3) and (7.6),

$$\Delta(C_i) = \Delta(K_i^+)\Delta(K_i^-) - \Delta([K_i^0]_q)\Delta([K_i^0 - 1]_q) \tag{7.11}$$

Explicitely

$$\begin{aligned}\Delta(C_i) &= q^{K_i^0}\left\{K_i^+ \otimes K_{i+1}^- + K_i^- \otimes K_{i+1}^+\right\}q^{-K_{i+1}^0} \\ &+ K_i^+K_i^- \otimes q^{-2K_{i+1}^0} + q^{2K_i^0} \otimes K_{i+1}^+K_{i+1}^- \\ &- \frac{1}{(q-q^{-1})^2}\left\{q^{-1}q^{2K_i^0} \otimes q^{2K_{i+1}^0} + qq^{-2K_i^0} \otimes q^{-2K_{i+1}^0} - (q+q^{-1})\right\}\end{aligned} \tag{7.12}$$

Remark 6.3 1. The diagonal operator $c_{(L)}$ in (7.9) has been added so that the ground state $|0\rangle_{(L)} := \otimes_{i=1}^{L}|0\rangle_i$ is a right eigenvector with eigenvalue zero, i.e. $H_{(L)}|0\rangle_{(L)} = 0$ as it is immediately seen using (7.4).
2. In the representation (7.4) the operator $H_{(L)}$ is self-adjoint. This follows essentially from the fact that, in this representation, $(K^+)^* = K^-$ and K^0 is self-adjoint.

Symmetries of the Hamiltonian

The symmetries It is easy to construct symmetries for the operator $H_{(L)}$ by using the property that the co-product is an isomorphism for the $\mathscr{U}_q(\mathfrak{su}(1,1))$ algebra. These are the basic symmetries. From them, using a q-deformed exponential we construct a non-trivial symmetry generating both the positive ground state as well as the self-duality functions. We state the main result, and refer for the proof to [13] which can be followed literally, up to a few changes of sign.

Theorem 6.1 (Symmetries of $H_{(L)}$) *Recalling* (7.6)*, we define the operators*

$$\begin{aligned}K_{(L)}^{\pm} &:= \Delta^{L-1}(K_1^{\pm}) = \sum_{i=1}^{L} q^{K_1^0} \otimes \cdots \otimes q^{K_{i-1}^0} \otimes K_i^{\pm} \otimes q^{-K_{i+1}^0} \otimes \ldots \otimes q^{-K_L^0}\ , \\ K_{(L)}^{0} &:= \Delta^{L-1}(K_1^{0}) = \sum_{i=1}^{L} \underbrace{1 \otimes \cdots \otimes 1}_{(i-1)\ times} \otimes K_i^0 \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(L-i)\ times}.\end{aligned} \tag{7.13}$$

They are symmetries of the Hamiltonian (7.8)*, i.e.*

$$[H_{(L)}, K_{(L)}^{\pm}] = [H_{(L)}, K_{(L)}^{0}] = 0. \tag{7.14}$$

Construction of ASIP(q, k) from the Quantum Hamiltonian via a Positive Ground State

In order to construct the suitable (i.e., positive) ground state of the quantum Hamiltonian, we act with a non-trivial symmetry on the trivial groundstate. This symmetry is defined via a q-exponential of $\Delta^{(L-1)}(q^{K_0}K^+)$, where the q-exponential is defined via

$$\exp_{q^2}(x) = \sum_{n\geq 0} \frac{x^n}{[n]_q!}\, q^{-n(n-1)/2} \tag{7.15}$$

From that ground state is then constructed a Markov process.

Theorem 6.2 *1. The operator*

$$S^+_{(L)} := \exp_{q^2}(\Delta^{(L-1)}(q^{K_0}K^+)) \tag{7.16}$$

is a symmetry of $H_{(L)}$. Its matrix elements are given by

$$\langle \eta_1, ..., \eta_L | S^+_{(L)} | \xi_1, ..., \xi_L \rangle = \prod_{i=1}^{L} \sqrt{\binom{\eta_i}{\xi_i}_q \binom{\eta_i + 2k - 1}{\xi_i + 2k - 1}_q} \cdot \mathbf{1}_{\eta_i \geq \xi_i} \, q^{(\eta_i - \xi_i)\left[1 + k + \xi_i + 2\sum_{m=1}^{i-1}(\xi_m + k)\right]} \tag{7.17}$$

2. *As a consequence*

$$|g\rangle = S^+_{(L)}|0, \ldots, 0\rangle = \sum_{\ell_1, \ell_2, ..., \ell_L \geq 0} \prod_{i=1}^{L} \sqrt{\binom{2k + \ell_i - 1}{\ell_i}_q} \cdot q^{\ell_i(1-k+2ki)} \, |\ell_1, ..., \ell_L\rangle$$

is a ground state of $H_{(L)}$.

3. *The operator*

$$\mathscr{L}^{(L)} = G^{-1}_{(L)} H_{(L)} G_{(L)} \tag{7.18}$$

where

$$G_{(L)}|\eta_1, \ldots, \eta_L\rangle = |\eta_1, \ldots, \eta_L\rangle\langle \eta_1, \ldots, \eta_L | S^+ | 0, \ldots, 0\rangle \tag{7.19}$$

is a Markov generator explicitly given by

$$\begin{aligned}(\mathscr{L}^{(L)} f)(\eta) &= \sum_{i=1}^{L-1} (\mathscr{L}_{i,i+1} f)(\eta) \qquad \textit{with} \\ (\mathscr{L}_{i,i+1} f)(\eta) &= q^{\eta_i - \eta_{i+1} + (2k-1)} [\eta_i]_q [2k + \eta_{i+1}]_q (f(\eta^{i,i+1}) - f(\eta)) \\ &\quad + q^{\eta_i - \eta_{i+1} - (2k-1)} [2k + \eta_i]_q [\eta_{i+1}]_q (f(\eta^{i+1,i}) - f(\eta))\end{aligned} \tag{7.20}$$

4.

$$G^{-1}_{(L)} S^+_{(L)} G^{-1}_{(L)} \tag{7.21}$$

is a self-duality function for the process generated by $\mathscr{L}^{(L)}$. It is exactly the self-duality function of (5.2).

Proof As in [13], using pseudo factorization, and acting with $S^+_{(L)}$ on the trivial duality function coming from the reversible measure $\mu(\eta)$ given by $\mu(\eta) = \langle \eta | G^2_{(L)} | \eta \rangle$. □

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References

1. Alcaraz, F.C., Rittenberg, V.: Reaction-diffusion processes as physical realizations of Hecke algebras. Phys. Lett. B **314**, 377–380 (1993)
2. Alcaraz, F.C., Droz, M., Henkel, M., Rittenberg, V.: Reaction-diffusion processes, critical dynamics, and quantum chains. Ann. Phys. **230**, 250–302 (1994)
3. Andrews, G.E., Askey, R., Roy, R.: Special Functions. Encyclopedia of Mathematics and Its Applications, vol. 71. Cambridge University Press, Cambridge (1999)
4. Balázs, M., Rassoul-Agha, F., Seppäläinen, T., Sethuraman, S.: Existence of the zero range process and a deposition model with superlinear growth rates. Ann. Prob. **35**, 1201–1249 (2007)
5. Belitsky, V., Schütz, G.M., Self-Duality for the Two-Component Asymmetric Simple Exclusion Process, preprint arXiv:1504.05096 (2015)
6. Borodin, A., Corwin, I., Sasamoto, T.: From duality to determinants for q-TASEP and ASEP. Ann. Probab. **42**, 2341–2382 (2014)
7. Borodin, A., Petrov, L.: Integrable probability: from representation theory to Macdonald processes. Probab. Surv. **11**, 1–58 (2014)
8. Burban, I.M., Klimyk, A.U.: Representations of the quantum algebra $U_q(su(1, 1))$. J. Phys. A **26**, 2139–2151 (1993)
9. Corwin, I.: The Kardar-Parisi-Zhang equation and universality class. Random Matrices **1**, 1130001 (2012)
10. Corwin, I., Petrov, L., Stochastic Higher Spin Vertex Models on the Line, preprint arXiv:1502.07374 (2015)
11. Carinci, G., Giardinà, C., Giberti, C., Redig, F.: Duality for stochastic models of transport. J. Stat. Phys. **152**, 657–697 (2013)
12. Carinci, G., Giardinà, C., Giberti, C., Redig, F.: Dualities in population genetics: a fresh look with new dualities. Stoch. Process. Appl. **125**, 941–969 (2015)
13. Carinci, G., Giardinà, C., Redig, F., Sasamoto, T., A Generalized Asymmetric Exclusion Process with $U_q(\mathfrak{sl}_2)$ Stochastic Duality, preprint arXiv:1407.3367 (2014)
14. Derrida, B., Non-equilibrium steady states: fluctuations and large deviations of the density and of the current. J. Stat. Mech., P07023 (2007)
15. Ferrari, P., Veto, B., Tracy-Widom Asymptotics for q-TASEP, preprint arXiv:1310.2515 (2013)
16. Giardinà, C., Kurchan, J., Redig, F.: Duality and exact correlations for a model of heat conduction. J. Math. Phys. **48**, 033301 (2007)
17. Giardinà, C., Kurchan, J., Redig, F., Vafayi, K.: Duality and hidden symmetries in interacting particle systems. J. Stat. Phys. **135**, 25–55 (2009)
18. Giardinà, C., Redig, F., Vafayi, K.: Correlation inequalities for interacting particle systems with duality. J. Stat. Phys. **141**, 242–263 (2010)
19. Golinelli, O., Mallick, K.: The asymmetric simple exclusion process: an integrable model for non-equilibrium statistical mechanics. J. Phys. A **39**, 12679 (2006)
20. Harris, R.J., Rákos, A., Schütz, G.M.: Breakdown of Gallavotti-Cohen symmetry for stochastic dynamics. Europhys. Lett. **75**(2), 227–233 (2006)
21. Imamura, T., Sasamoto, T.: Current moments of 1D ASEP by duality. J. Stat. Phys. **142**, 919–930 (2011)
22. Kipnis, C., Marchioro, C., Presutti, E.: Heat flow in an exactly solvable model. J. Stat. Phys. **27**, 65–74 (1982)
23. Kuan, J., Stochastic Duality of ASEP with Two Particle Types via Symmetry of Quantum Groups of Rank Two, preprint arXiv:1504.07173 (2015)
24. Liggett, T.M.: Interacting Particle Systems. Springer, New York (1985)
25. Matsui, C.: Multi-state asymmetric simple exclusion processes. J. Stat. Phys. **158**, 158–191 (2015)
26. Povolotsky, A.M.: Bethe ansatz solution of zero-range process with nonuniform stationary state. Phys. Rev. E **69**, 061109 (2004)
27. Quastel, J., Spohn, H., The One-Dimensional KPZ Equation and Its Universality Class, arXiv:1503.06185
28. Schütz, G., Sandow, S.: Non-Abelian symmetries of stochastic processes: derivation of correlation functions for random-vertex models and disordered-interacting-particle systems. Phys. Rev. E **49**, 2726 (1994)
29. Schütz, G.M.: Duality relations for asymmetric exclusion processes. J. Stat. Phys. **86**, 1265–1287 (1997)
30. Schütz, G.M.: Exactly Solvable Models for Many-Body Systems Far from Equilibrium. In: Domb, C., Lebowitz, J.L. (eds.) Phase Transitions and Critical Phenomena, vol. 19. Academic Press, London (2000)
31. Schütz, G.M., Duality Relations for the ASEP Conditioned on a Low Current, preprint, arXiv:1508.03158 (2015)