Università degli Studi di Modena e Reggio Emilia

Scuola di Dottorato in Matematica - XXXIII ciclo in convenzione con l'Università degli Studi di Ferrara e l'Università degli Studi di Parma. Coordinatore: Prof. Cristian Giardinà

On the Kolmogorov equation: regularity theory & applications

Candidato:
Francesca Anceschi

Relatore:
Prof. Sergio Polidoro

26 Febbraio 2021

Abstract

The Kolmogorov equation was firstly introduced in 1934 as a fundamental ingredient of a kinetic model for the study of the density of a system of N particles of gas in the phase space. Kolmogorov pointed out that, although the dimension of the phase space is $2N$ and the diffusion term acts on the velocity variable, whose dimension is N, the differential operator is strongly degenerate. Nevertheless, Kolmogorov exhibited the explicit expression of the fundamental solution of the operator and pointed out that it is a smooth function, in fact proving that the operator is hypoelliptic. Throughout this work, we are mainly concerned with degenerate Kolmogorov equations in divergence form, for which the regularity theory for classical solutions had widely been developed during the years. Chapter 1 of this work is devoted to a survey of results on the classical regularity theory for Kolmogorov operators with constant or continuous coefficients. In Chapter 2 we consider an application of the Kolmogorov equation in finance, where the Black and Scholes theory is applied to the pricing problem for Asian options. The price of the option is computed by solving a Cauchy problem, where the initial data represents the payoff of the option and the associated PDE is a Kolmogorov type equation with local Hölder continuous coefficients. The existence and uniqueness of the fundamental solution of the associated PDO are proved, alongside with a uniqueness result for the solution of the Cauchy problem, through a limiting procedure whose convergence is ensured by Schauder type estimates. Furthermore, in Chapter 3 we consider an application of the Kolmogorov equation to the kinetic theory. Specifically, we introduce a space inhomogeneous kinetic model associated to a nonlinear Kolmogorov-Fokker-Planck operator and we investigate the classical theory for the associated Cauchy problem in Hölder spaces. The second part of my thesis is devoted to the regularity theory for weak solutions to the Kolmogorov equation with measurable coefficients, which is nowadays the main focus of the research community. It has been developed during the last decade, and the most advanced achievement in this framework have been established in the particular case of the Kolmogorov-Fokker-Planck equation. In Chapter 4 we give proof of a geometric statement for the Harnack inequality for weak solutions to the Kolmogorov-Fokker-Planck equation proved by Golse, Imbert, Mouhot and Vasseur in 2017, based on the concepts of Harnack chains and attainable set. As far as we are concerned with the more general Kolmogorov equation in divergence form, Chapter 5 is devoted to the extension of the Moser's iterative procedure (proved by Pascucci and Polidoro in 2004 for the dilation invariant case) to weak solutions to the Kolmogorov equation under minimal integrability assumptions for the lower order coefficients in the non-dilation invariant case.

Sunto

L'equazione di Kolmogorov è stata introdotta nel 1934 come ingrediente fondamentale di un modello cinetico per lo studio della densità di un sistema di N particelle di gas nello spazio delle fasi. Kolmogorov osservò che tale operatore è fortemente degenere in quanto la dimensione dello spazio delle fasi è $2N$, mentre il termine di diffusione agisce sulla variabile velocità di dimensione N. Nonostante ciò, egli fornì l'espressione esplicita della soluzione fondamentale per tale operatore, una funzione differenziabile infinite volte, così dimostrando che l'operatore è ipoellittico. Nella mia tesi mi occupo prevalentemente di equazioni di Kolmogorov degeneri in forma di divergenza, per le quali la teoria della regolarità classica è stata ampiamente sviluppata nel corso degli anni. Nel Capitolo 1 presento i principali risultati di tale teoria per operatori di Kolmogorov a coefficienti costanti o continui. Nel Capitolo 2 considero un'applicazione dell'equazione di Kolmgorov in ambito finanziario, dove la teoria di Black & Scholes si applica al pricing problem per le opzioni Asiatiche. Il prezzo di un'opzione si calcola risolvendo un problema di Cauchy, il cui dato iniziale rappresenta il payoff dell'opzione e la EDP associata è un'equazione di tipo Kolmogorov a coefficienti localmente Hölderiani. Attraverso una procedura di limite, la cui convergenza è assicurata da stime di tipo Schauder, si dimostrano l'esistenza e l'unicità della soluzione per l'ODP associato ed un risultato di unicità per la soluzione del problema di Cauchy. Nel Capitolo 3 considero un'ulteriore applicazione dell'equazione di Kolmogorov alla teoria cinetica. In particolare, introduco un modello cinetico non omogeneo associato ad un operatore non lineare di tipo Kolmogorov-Fokker-Planck e studio la teoria della regolarità classica per il problema di Cauchy associato in spazi Hölderiani. La seconda parte della mia tesi è dedicata alla teoria della regolarità per soluzioni deboli dell'equazione di Kolmogorov a coefficienti misurabili, argomento su cui è prevalentemente concentrata la comunità scientifica oggigiorno. Gli sviluppi più recenti in questa direzione sono stati ottenuti nel caso particolare dell'equazione di Kolmogorov-Fokker-Planck. Nel Capitolo 4 dimostro un enunciato di tipo geometrico per la disuguaglianza di Harnack provata da Golse, Imbert, Mouhot e Vasseur nel 2017 per le soluzioni deboli dell'equazione di Kolmogorov-Fokker-Planck a coefficienti misurabili, basandomi sul concetto di catene di Harnack e insieme ammissibile. Per quanto riguarda invece l'equazione di Kolmogorov in forma di divergenza nella sua forma più generale, il Capitolo 5 è dedicato all'estensione dell'iterazione di Moser (dimostrata da Polidoro e Pascucci nel 2004 nel caso invariante per dilatazioni) alle soluzioni deboli per l'equazione di Kolmogorov sotto ipotesi minimali di integrabilità per i coefficienti di ordine inferiore nel caso non invariante per dilatazioni.

Contents

Introduction

Kolmogorov equations appear in the theory of stochastic processes as linear second order parabolic equations with non-negative characteristic form. Throughout this work, we are mainly concerned with degenerate Kolmogorov equations. In its simplest form, if $(W_t)_{t\geq 0}$ denotes a real Brownian motion, the density $p = p(t, v, y, v_0, y_0)$ of the stochastic process $(Y_t, V_t)_{t\geq 0}$

$$\begin{cases} Y_t = y_0 + \sigma W_t \\ V_t = v_0 + \int_0^t Y_s \, ds \end{cases} \tag{1}$$

is a solution to a strongly degenerate Kolmogorov equation, that is

$$\tfrac{1}{2}\sigma^2 \partial_v^2 p + v \partial_y p = \partial_t p, \qquad t \geq 0, \qquad (v, y) \in \mathbb{R}^2. \tag{2}$$

This equation was firstly introduced by Kolmogorov in 1934, as a fundamental ingredient of a kinetic model for the study of the density of a system of N particles of gas in the phase space. Kolmogorov pointed out that, although the dimension of the phase space is $2N$ and the diffusion term acts on the velocity variable, whose dimension is N, the differential operator is strongly degenerate. Nevertheless, Kolmogorov provided us with the explicit expression of the density $p = p(t, v, y, v_0, y_0)$ of the above equation (see [78])

$$p(t, v, y, v_0, y_0) = \tfrac{\sqrt{3}}{2\pi t^2} \exp\left(-\tfrac{(v-v_0)^2}{t} - 3\tfrac{(v-v_0)(y-y_0-tv_0)}{t^2} - 3\tfrac{(y-y_0-ty_0)^2}{t^3} \right) \qquad t > 0, \tag{3}$$

and pointed out that it is a smooth function despite the strong degeneracy of the equation (2). As it is suggested by the smoothness of the density p, the operator $\mathcal{K}$ associated to equation (2)

$$\mathcal{K} := \tfrac{1}{2}\sigma^2 \partial_v^2 + v \partial_y - \partial_t, \tag{4}$$

is hypoelliptic, in the sense of the following definition, that we state for a general second order differential operator $\mathcal{K}$ acting on an open subset Ω of $\mathbb{R}^N$.

Hypoellipticity. *The operator $\mathcal{K}$ is hypoelliptic if, for every distributional solution $u \in L^1_{\mathrm{loc}}(\Omega)$ to the equation $\mathcal{K}u = f$, we have that*

$$f \in C^\infty(\Omega) \quad \Rightarrow \quad u \in C^\infty(\Omega). \tag{5}$$

Hörmander considered the operator $\mathcal{K}$ in (4) as a prototype for the family of hypoelliptic operators studied in his seminal work [64]. Specifically, the operators considered by Hörmander are of the form

$$\mathcal{K} = \sum_{k=1}^{m} X_k^2 + Y, \tag{6}$$

where X_k are smooth vector fields of the form

$$X_k = \sum_{j=1}^{N+1} b_{j,k}(z)\,\partial_{z_j}, \qquad Y = \sum_{j=1}^{N+1} b_{j,m+1}(z)\,\partial_{z_j} \quad k = 1, \dots, m, \tag{7}$$

with $b_{j,k} \in C^\infty(\Omega)$ for every $j = 1, \dots, N+1$, $k = 1, \dots, m+1$ and Ω is any open subset of $\mathbb{R}^{N+1}$. The main result presented in [64] is a sufficient condition to the hypoellipticity of $\mathcal{K}$. Its statement requires some notation. Given two vector fields Z_1, Z_2, the commutator of Z_1 and Z_2 is the vector field:

$$[Z_1, Z_2] = Z_1\,Z_2 - Z_2\,Z_1. \tag{8}$$

Moreover, we recall that $\mathrm{Lie}(X_1, \dots, X_m, Y)$ is the Lie algebra generated by the vector fields $X_1, \dots, X_m, Y$ and their commutators.

Hörmander's Rank Condition. *Suppose that*

$$\operatorname{rank} \mathrm{Lie}(X_1, \dots, X_m, Y)(z) = N + 1 \qquad \textit{for every} \quad z \in \Omega. \tag{9}$$

Then the operator $\mathcal{K}$ defined in (6) *is hypoelliptic in* Ω*,*

Let us consider again the operator $\mathcal{K}$ defined in (4) with $\sigma = \sqrt{2}$, to simplify the notation. It can be written in the form (6) if we choose

$$X = \partial_v \sim (0,1,0)^T, \qquad Y = v\partial_y + \partial_t \sim (-1,0,v)^T,$$

and the Hörmander's rank condition is satisfied, as

$$[X,Y] = XY - YX = \partial_y \sim (0,0,1)^T.$$

As the regularity properties of Hörmander's operators $\mathcal{K}$ are related to a Lie algebra, it became clear that the natural framework for the regularity theory of Hörmander's operators is the non-euclidean setting of Lie groups, as Folland and Stein pointed out in [48]. Later on, Rothschild and Stein developed a general regularity theory for Hörmander's operators in [112]. We refer to the more recent monograph by Bonfiglioli, Lanconelli and Uguzzoni [19] for a comprehensive treatment of the recent achievements of the theory. As far as we are concerned with the operator $\mathcal{K}$, we show that it is invariant with respect to the non-commutative translation given by the following composition law

$$(t,v,y) \circ (t_0,v_0,y_0) = (t_0 + t, v_0 + v, y_0 + y + tv_0), \qquad (t,v,y), (t_0,v_0,y_0) \in \mathbb{R}^3.$$

Indeed, if $w(t,v,y) = u(t_0 + t, v_0 + v, y_0 + y + tv_0)$ and $g(t,v,y) = f(t_0 + t, v_0 + v, y_0 + y + tv_0)$, then

$$\mathcal{K}u = f \quad \Longleftrightarrow \quad \mathcal{K}w = g \quad \text{for every} \quad (t_0,v_0,y_0) \in \mathbb{R}^3.$$

As we will see in the sequel, in several applications the couple (v,y) denotes the velocity and the position of a particle. For this reason the above operation is also known as *Galilean* change of variable.

Another remarkable property of the operator $\mathcal{K}$ is its dilation invariance. More precisely, the operator $\mathcal{K}$ is invariant with respect to the following family of dilations

$$\delta_r(t,v,y) := (r^2 t, rv, r^3 y), \qquad r > 0,$$

with the following meaning: if we define $w(t,v,y) = u(r^2 t, rv, r^3 y)$ and $g(t,v,y) = f(r^2 t, rv, r^3 y)$ we have that

$$\mathcal{K}u = f \quad \Longleftrightarrow \quad \mathcal{K}w = r^2 g \quad \text{for every} \quad r > 0.$$

As we will see in the sequel, this underlying invariance property plays a fundamental role in the study of the operator $\mathcal{K}$, even though it does not hold true for every Kolmogorov operator (see Chapter 1, Proposition 1.1), as it happens in the family of uniformly parabolic operators. Indeed, we usually consider parabolic dilations $\delta_r(x,t) = (rx, r^2 t)$ also when considering the parabolic Ornstein-Uhlenbeck operator $\mathcal{K} = \Delta - \langle x, \nabla \rangle - \partial_t$.

Throughout this work, we are mainly concerned with degenerate Kolmogorov equations in divergence form, for which the regularity theory for classical solutions had widely been developed during the years, starting from the work by Lanconelli and Polidoro in [84], and it is referred to the case of the Kolmogorov operator

$$\mathcal{K} = \sum_{i,j=1}^{m_0} a_{i,j}(x,t) \partial^2_{x_i x_j} + \sum_{j=1}^{m_0} b_j(x,t) \partial_{x_j} + \langle Bx, D \rangle - \partial_t, \qquad \text{for } (x,t) \in \mathbb{R}^{N+1}$$

with either constant or continuous coefficients $a_{i,j}$'s and b_j's, and $1 \le m_0 < N$. As in the parabolic case, the classical theory for degenerate Kolmogorov operators is developed for suitable spaces of Hölder continuous functions that we introduce in Definition 1.11. This definition relies on the Lie group $\mathbb{G}$ that we define in (1. 12), a non-Euclidean invariant structure for the *constant coefficients* operators of the type $\mathcal{K}$. This non-Euclidean invariant structure was implicitly used for the first time by Garofalo and E. Lanconelli in [51], and then later on properly written and thoroughly studied by Lanconelli and Polidoro in [84]. In this framework, we deal with classic solution to the equation $\mathcal{K}u = f$ under minimal regularity assumptions on u in the following sense.

Classic solution. *A function u is a solution to the equation $\mathcal{K}u = f$ in a domain Ω of $\mathbb{R}^{N+1}$ if there exist the Euclidean derivatives $\partial_{x_i} u, \partial_{x_i, x_j} u \in C(\Omega)$ for $i, j = 1, \dots, m_0$, the Lie derivative $Yu \in C(\Omega)$, and the equation*

$$\sum_{i,j=1}^{m_0} a_{i,j}(z) \partial^2_{x_i x_j} u(x,t) + \sum_{j=1}^{m_0} b_j(z) \partial_{x_j} u(x,t) + Yu(x,t) = f(x,t)$$

is satisfied at any $(x,t) \in \Omega$.

In the following of this work, Chapter 1 is devoted to a survey of results on the classical regularity theory for Kolmogorov operators, which can nowadays be considered complete, starting from the ideas presented in the paper [6]. In Chapter 2 we consider an application of the Kolmogorov equation to finance, and in particular we address the problem of the existence and uniqueness of the fundamental solution for PDEs with local Hölder continuous coefficients associated to Asian options, that we have presented for the first time in the paper [5]. Asian options are a family of *path-dependent options* whose payoff depends on the average of the underlying stock price over a certain time interval. Indeed, in the Black & Scholes framework, the price of the underlying Stock S_t and of the bond B_t are described by the processes

$$S_t = S_0 e^{\mu t + \sigma W_t}, \qquad B_t = B_0 e^{rt}, \qquad 0 \le t \le T,$$

where μ, r, T, and σ are given constants. In particular, in this work we consider continuous Asian Options and the price $(Z_t)_{0 \le t \le T}$ of a path dependent option is considered as a function $Z_t = Z(S_t, A_t, t)$ that depends on the *stock price* S_t, the *time to maturity* t and of an *average* A_t of the stock price

$$A_t = \int_0^t f(S_\tau)\, d\tau, \qquad t \in [0,T].$$

The price of the option is computed by solving the Cauchy problem associated to the process $(S_t, B_t, A_t)_{t\geq 0}$:

$$\begin{cases} \frac{1}{2}\sigma^2(S,A,t)S^2\frac{\partial^2 Z}{\partial S^2} + f(S)\frac{\partial Z}{\partial A} + r(S,A,t)\left(S\frac{\partial Z}{\partial S} - Z\right) + \frac{\partial Z}{\partial t} = 0 & (S,A,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \times]0,T[, \\ Z(S,A,T) = \varphi(S,A) & (S,A) \in \mathbb{R}^+ \times \mathbb{R}^+, \end{cases} \tag{10}$$

where the initial data φ represents the payoff of the option, and depending on the choice of the function $f(S)$ we have a different Kolmogorov type equation with local Hölder continuous coefficients associated to it. Indeed, when $f(S) = \log S$ we recover the case of Geometric Average Asian Options and the Partial Differential Operator (PDO) associated to (10) is the classical Kolmogorov operator with only one commutator:

$$\mathcal{K}u(x,y,t) \;=\; \frac{\partial}{\partial x}\left(a(x,y,t)\frac{\partial u}{\partial x}\right) \,+\, b(x,y,t)\,\frac{\partial u}{\partial x} \,+\, x\frac{\partial u}{\partial y} \,-\, r(x,y,t)u - \,\frac{\partial u}{\partial t} \qquad (x,y,t) \in \mathbb{R}^{2n+1}.$$

On the other hand, by choosing $f(S) = S$ the PDO associated to (10) is a generalization of the constant coefficients operator (i.e. $a = 1$ and $b = 1$) introduced by Yor in his seminal paper [123] for the study of Arithmetic Average Asian Option:

$$\mathscr{L}u(x,y,t) \;:=\; x\,\frac{\partial}{\partial x}\left(a(x,y,t)\,x\,\frac{\partial u}{\partial x}\right) \,+\, b(x,y,t)\,x\,\frac{\partial u}{\partial x} \,+\, x\,\frac{\partial u}{\partial y} - r(x,y,t)u - \,\frac{\partial u}{\partial t} \qquad (x,y,t) \in \mathbb{R}^{2n+1}.$$

In Chapter 2 the existence and uniqueness of the fundamental solution for both the operators $\mathcal{K}$ and $\mathscr{L}$ is proved (see Theorem 2.1 and Theorem 2.12), alongside with a uniqueness result for the solution of the Cauchy problem (10) (see Theorem 2.2 and Theorem 2.13). Our approach is based on a limiting procedure whose convergence is ensured by Schauder types estimates for the Kolmogorov operator $\mathcal{K}$, and on the idea that locally the operator $\mathscr{L}$ behaves as the classical Kolmogorov operator $\mathcal{K}$.

Furthermore, in Chapter 3 we consider an application of the Kolmogorov equation to the kinetic theory. The new results we present here are part of a joint project with Yuzhe Zhu from the ENS of Paris (France), where the author has spent a research period under the supervision of Prof. Cyril Imbert (CNR). In particular, we are interested in the following nonlinear spatial inhomogeneous drift-diffusion equation

$$\begin{cases} \left(\partial_t + v\cdot\nabla_x\right)u(v,x,t) = \rho_u^\beta(x,t)\mathscr{L}u(v,x,t), \\ u(v,x,0) = \varphi(v,x), \end{cases} \tag{11}$$

for an unknown $u(v,x,t) \geq 0$ with $(v,x,t) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^+$, where the constant $\beta \in [0,1]$, and

$$\rho_u(x,t) := \int_{\mathbb{R}^n} u(v,x,t)\,dv, \qquad \text{with} \quad \rho_\varphi(x) = \int_{\mathbb{R}^n} \varphi(v,x)dv.$$

Equation (11) arises in various different research fields, such as plasma physics and polymer dynamics, and it is a fundamental tool for the modeling of the collisional evolution of a system of a large number of particles. If we denote by $\mathscr{L}$ the kinetic Kolmogorov-Fokker-Planck diffusive operator appearing on the right-hand side of the equation (11)

$$\mathscr{L}u := \nabla_v \cdot \left(\nabla_v + v\right)u,$$

the nonlinear diffusive collision term $\rho_u^\beta\mathscr{L}$ models the collision of particles in a certain surrounding bath, where the aggregation of particles induces friction contribution. Moreover, we remark that the linear operator associated to equation (11), given by $\mathscr{L} - \partial_t + v\cdot\nabla_x$ is a particular case of the operator $\mathcal{K}$, in fact obtained by choosing $N = 2n$, $m_0 = n$, $x = (v,x,t)$ and

$$a_{ij}(v,x,t) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}, \qquad \text{and } b_i = 1 \text{ for every } i = 1,\ldots,n.$$

Our aim is to investigate well-posedness results for classical solutions to the Cauchy problem (11). In the framework of Sobolev spaces results of this type have been proved by Imbert and Mouhot in [68], and by Liao, Wang and Yang in [87]. Our results improve that of [68] and [87] because we consider the Cauchy problem (11) in Hölder spaces. Indeed, if the initial data is bounded we prove the existence and uniqueness of a positive weak solution for (11). Moreover, given that the initial data is continuous the global existence and uniqueness are proved, alongside with C^∞ a priori estimates, obtained by adapting an iterative procedure firstly introduced by Imbert and Silvestre in [69] for the Boltzmann equation.

The second part of this work is devoted to the study of the weak regularity theory for solutions to the Kolmogorov equation in divergence form, that is nowadays the main focus of the research community. In particular, we are interested in weak solutions to the Kolmogorov equation $\mathcal{K}u = f$, with measurable coefficients a_{ij}'s and b_{ij}'s:

$$\begin{aligned} \mathcal{K}u(x,t) := \sum_{i,j=1}^{m_0} \partial_{x_i}\left(a_{ij}(x,t)\partial_{x_j}u(x,t)\right) + \sum_{i,j=1}^{N} b_{ij}x_j\partial_{x_i}u(x,t) - \partial_t u(x,t) + \\ + \sum_{i=1}^{m_0} b_i(x,t)\partial_i u(x,t) - \sum_{i=1}^{m_0} \partial_{x_j}\left(a_i(x,t)u(x,t)\right) + c(x,t)u(x,t) = f(x,t), \end{aligned} \tag{12}$$

where $(x,t) \in \mathbb{R}^{N+1}$ and $1 \le m_0 < N$. The most recent developments in this framework have been established in the particular case of the kinetic Kolmogorov-Fokker-Planck equation. This equation belongs to a class of evolution equations arising in the kinetic theory of gases and takes the following form

$$\sum_{j=1}^{n} v_j\partial_{x_j}u + \partial_t u = \sum_{i,j=1}^{n} \partial_{v_i}(a_{ij}\,\partial_{v_j}u + b_i u) + \sum_{i=1}^{n} a_i\partial_{v_i}u + au + f, \qquad (v,x,t) \in \mathbb{R}^{2n+1} \tag{13}$$

where $u = u(v,x,t)$ represents in this case the density of particles with velocity $v = (v_1,\dots,v_n)$ and position $x = (x_1,\dots,x_n)$ at time t. We remark that we recover this case from (12) by choosing $m_0 = n$, $N = 2n$ and $b_{ii} = 1$ for $i = n+1,\dots,2n$ and zero everywhere else. The lefthand side of (13) is the so called total derivative with respect to time in the phase space $\mathbb{R}^{2n+1}$. Whereas, the righthand side is the collision operator, where a_{ij}, a_i and a are functions of (v,x,t). Indeed, this latter equation is the one considered by Golse, Imbert, Mouhot and Vasseur in [58], where the authors prove the Hölder continuity and a Harnack inequality for weak solutions to the kinetic Kolmogorov-Fokker-Planck equation in divergence form (13). The Harnack inequality proved in [58] is the only one available in the framework of weak regularity theory for Kolmogorov equations in divergence form, and it is based on the De Giorgi method. In Chapter 4 we prove a geometric statement for that Harnack inequality, based on the concepts of Harnack chains and attainable set. The results we present here appeared for the first time in the paper [4] by the author, Eleuteri and Polidoro. As far as we are concerned with the more general Kolmogorov equation (12), Chapter 5 is devoted to the extension of the Moser's iterative scheme to weak solutions to $\mathcal{K}u = 0$ in the sense of the following definition.

Weak solution (in the L^2 sense). *Let Ω be an open subset of $\mathbb{R}^{N+1}$. A weak solution to $\mathcal{K}u = 0$ is a function u such that $u, \partial_{x_1}u, \dots, \partial_{x_{m_0}}u, Yu \in L^2_{\text{loc}}(\Omega)$ and*

$$\int_\Omega -\langle ADu, D\varphi\rangle + Yu\,\varphi + \langle b, Du\rangle\varphi + \langle a, D\varphi\rangle u + cu\varphi = 0, \quad \textit{for every } \varphi \in C_0^\infty(\Omega),$$

where the operator $\mathcal{K}$ is written in his compact form, given that

- *the matrix $A(x,t) = (a_{ij}(x,t))_{1\le i,j\le N}$ has real measurable entries defined as the coefficients a_{ij} appearing in (12) for $i,j = 1,\dots,m_0$ and $a_{ij} \equiv 0$ whenever $i > m_0$, or $j > m_0$;*

- *the vectors* $a(x,t) = (a_1(x,t), \dots, a_{m_0}(x,t), 0, \dots, 0)$ *and* $b(x,t) = (b_1(x,t), \dots, b_{m_0}(x,t), 0, \dots, 0)$ *are the coefficients appearing in front of the lower order terms; moreover, the drift term is defined as* $Y = \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} - \partial_t$.

The main advantage of this definition is that it allows us to directly handle the computations involving the drift term Yu. In particular, it allows us to extend the previously known results for the Moser's iterative scheme proved by Polidoro and Pascucci [104], Cinti, Pascucci and Polidoro [33] and Wang and Zhang [119] to weak solutions to (12) with lower order measurable coefficients with positive divergence under minimal integrability assumptions in the non-dilation invariant framework. These results were firstly presented in the paper [7] by the author, Ragusa and Polidoro in 2019. The main difficulty of this case lies in the lower order term. Our study has been inspired by the article of Nazarov and Uralt'seva [99], who prove L^∞_{loc} estimates and Harnack inequalities for uniformly elliptic and parabolic operators in divergence form that are those with $m_0 = N$ according to our notation. As it is known, in order to prove the Moser's iterative scheme we need to combine a Caccioppoli inequality and a Sobolev inequality. Nevertheless, since we are considering degenerate equations, the Caccioppoli inequality gives an *a priori* L^2 estimates for the derivatives $\partial_{x_1} u, \dots, \partial_{x_{m_0}} u$ of the solution u, that are the derivatives with respect to the *non-degeneracy* directions of $\mathcal{K}$. Moreover, the standard Sobolev inequality cannot be used to obtain an improvement of the integrability of the solution as in the non-degenerate case. For this reason we rely on a representation formula for the solution u firstly applied in [104] that allows us to represent a solution u to $\mathcal{K}u = 0$ in terms of the fundamental solution of its principal part operator.

Chapter 1

Classical regularity theory

This chapter is devoted to the study of the classical regularity theory for Kolmogorov operators of the form

$$\begin{aligned}\mathcal{K}u :=& \sum_{i,j=1}^{N} a_{ij}\partial^2_{x_i x_j}u + \sum_{i,j=1}^{N} b_{ij}x_i\partial_{x_j}u - \partial_t u \\ =&\mathrm{Tr}(AD^2u) + \langle Bx, Du\rangle - \partial_t u, \qquad x \in \mathbb{R}^N,\, t \in \mathbb{R},\end{aligned} \tag{1. 1}$$

where $A = (a_{ij})_{i,j=1,\dots,N}$ and $B = (b_{ij})_{i,j=1,\dots,N}$ are matrices with real constant coefficients, A symmetric and non negative. In the following, we present a survey of results for the classical theory appeared for the first time in the paper [6] by the author and Polidoro, and we conclude this introductory chapter with the proof of a Strong Maximum Principle for Kolmogorov operators with continuous coefficients based on the Harnack inequality stated in Theorem 1.26.

As we have already pointed out in the introduction of this work, the simplest Kolmogorov equation (2) has appeared for the first time in 1934, when Kolmogorov considered it to describe the probability density of a system with $2n$ degrees of freedom in his seminal paper [78]. There the author also proved the existence of its fundamental solution (3), and wrote it as the density of the solution to the stochastic differential equation (1). This is also the case when we consider a higher dimension. Specifically, let σ be a $N \times m$ constant matrix, B as in (1. 1), and let $(W_t)_{t\geq 0}$ be a m-dimensional Wiener process. Denote by $(X_t)_{t\geq 0}$ the solution to the following N-dimensional Stochastic Differential Equation (SDE in short)

$$\begin{cases} dX_t = -BX_t\,dt + \sigma\,dW_t \\ X_{t_0} = x_0. \end{cases} \tag{1. 2}$$

Then the *backward Kolmogorov operator* $\mathcal{K}_b$ of $(X_t)_{t\geq 0}$ acts on sufficiently regular functions u as follows

$$\mathcal{K}_b u(y,s) = \partial_s u(y,s) + \sum_{i,j=1}^{N} a_{ij}\partial^2_{y_i y_j}u(y,s) - \sum_{i,j=1}^{N} b_{ij}y_i\partial_{y_j}u(y,s).$$

where

$$A = \tfrac{1}{2}\sigma\sigma^T, \tag{1. 3}$$

and the *forward Kolmogorov operator* $\mathcal{K}_f$ of $(X_t)_{t\geq 0}$ is the adjoint $\mathcal{K}_b^*$ of $\mathcal{K}_b$, that is

$$\mathcal{K}_f v(x,t) = -\partial_t v(x,t) + \sum_{i,j=1}^{N} a_{ij}\partial^2_{x_i x_j}v(x,t) + \sum_{i,j=1}^{N} b_{ij}x_i\partial_{x_j}v(x,t) + \mathrm{tr}(B)v(x,t),$$

for sufficiently regular functions v. Note that $\mathcal{K}_f$ operator agrees with $\mathcal{K}$ in (1. 1) up to a multiplication of the solution by $\exp(t\,\mathrm{tr}(B))$. Also note that, because of (1. 3), it is natural to consider in (1. 1) a symmetric and non negative matrix A. When the matrix A is strictly positive, the solution $(X_t)_{t\geq 0}$ of the SDE (1. 2) has a density $p = p(t-s,x,y)$ which is a solutions of the equations $\mathcal{K}_b p = 0$ and $\mathcal{K}_f p = 0$ in the following sense. For every $(x,t) \in \mathbb{R}^{N+1}$, the function $u(y,s) := p(t-s,x,y)$ is a classical solution to the equation $\mathcal{K}_b u = 0$ in $\mathbb{R}^n \times]-\infty, t[$ and, for every $(y,s) \in \mathbb{R}^{N+1}$, the function $v(x,t) = p(t-s,x,y)$ is a classical solution to $\mathcal{K}_f v = 0$ in $\mathbb{R}^n \times]s, +\infty[$. This is not always the case when A is degenerate. In the sequel we give necessary and sufficient conditions on A and B for the existence of a density p for the stochastic process $(X_t)_{t\geq 0}$. These conditions are also necessary and sufficient for the hypoellipticity of $\mathcal{K}$. In order to state the aforementioned conditions, we introduce some further notation. Following Hörmander (see p. 148 in [64]), we set, for every $t \in \mathbb{R}$,

$$E(t) = \exp(-tB), \qquad C(t) = \int_0^t E(s)\,A\,E^T(s)\,ds. \tag{1. 4}$$

The matrix $C(t)$ is symmetric and non-negative for every $t > 0$, nevertheless it may occur that it is strictly positive. If this is the case, then $C(t)$ is invertible and the fundamental solution $\Gamma(x_0,t_0;x,t)$ of $\mathcal{K}$ is

$$\Gamma(x,t;\xi,\tau) = \Gamma(x - E(t-\tau)\xi, t-\tau), \tag{1. 5}$$

where $\Gamma(x,t) = \Gamma(x,t;0,0)$. Moreover, $\Gamma(x,t) = 0$ for every $t \leq 0$ and

$$\Gamma(x,t) = \frac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det C(t)}} \exp\left(-\frac{1}{4}\langle C^{-1}(t)x, x\rangle - t\,\mathrm{tr}(B)\right), \qquad t > 0. \tag{1. 6}$$

The last notation we need to introduce allows us to write the operator $\mathcal{K}$ in the form (6). To do that, we recall that $\sigma = (\sigma_{jk})_{\substack{j=1,\dots,N\\k=1,\dots,m}}$ is a matrix with constant coefficients, and we set

$$X_k := \frac{1}{\sqrt{2}}\sum_{j=1}^N \sigma_{jk}\partial_{x_j}, \quad k = 1,\dots,m, \qquad Y := \sum_{i,j=1}^N b_{ij}x_i\partial_{x_j} - \partial_t. \tag{1. 7}$$

This allows us to rewrite the operator $\mathcal{K}$ as a *sum of squares*

$$\mathcal{K} = \sum_{j=1}^m X_j^2 + Y,$$

analogous to the form (6) introduced in Chapter , when talking about hypoelliptic operators. The following result holds true.

Proposition 1.1 *Consider an operator $\mathcal{K}$ of the form* (1. 1)*, and let σ be a $N \times m$ constant matrix such that A writes as in* (1. 3)*. Let $X_1,\dots,X_m$, and Y be the vector fields defined in* (1. 7)*. Then the following statements are equivalent*

C1. *(Hörmander's condition):* $\mathrm{rank}\,\mathrm{Lie}(X_1,\dots,X_m,Y)(x,t) = N+1$ *for every* $(x,t) \in \mathbb{R}^{N+1}$*;*

C2. $\ker(A)$ *does not contain non-trivial subspaces which are invariant for B;*

C3. $C(t) > 0$ *for every $t > 0$, where $C(t)$ is defined in* (1. 4)*;*

C4. *(Kalman's rank condition):* $\mathrm{rank}\,\left(\sigma, B\sigma, \dots, B^{N-1}\sigma\right) = N$*;*

C5. *for some basis of* $\mathbb{R}^N$ *the matrices* A *and* B *take the following block form*

$$A = \begin{pmatrix} \overline{A} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix} \tag{1. 8}$$

where $\overline{A}$ *is a symmetric strictly positive* $m_0 \times m_0$ *matrix, with* $m_0 \leq m$*, and*

$$B = \begin{pmatrix} * & * & \dots & * & * \\ B_1 & * & \dots & * & * \\ \mathbb{O} & B_2 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_\kappa & * \end{pmatrix} = \begin{pmatrix} B_{0,0} & B_{0,1} & \dots & B_{0,\kappa-1} & B_{0,\kappa} \\ B_1 & B_{1,1} & \dots & B_{\kappa-1,1} & B_{\kappa,1} \\ \mathbb{O} & B_2 & \dots & B_{\kappa-1,2} & B_{\kappa,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_\kappa & B_{\kappa,\kappa} \end{pmatrix} \tag{1. 9}$$

where every block B_j *is a* $m_j \times m_{j-1}$ *matrix of rank* m_j *with* $j = 1, 2, \dots, \kappa$*. Moreover, the* $m_j s$ *are positive integers such that*

$$m_0 \geq m_1 \geq \dots \geq m_\kappa \geq 1, \quad \text{and} \quad m_0 + m_1 + \dots + m_\kappa = N \tag{1. 10}$$

and the entries of the blocks denoted by $*$ *are arbitrary.*

When the above conditions are satisfied, then $\mathcal{K}$ *is hypoelliptic, its fundamental solution* Γ *defined in* (1. 5) *and* (1. 6)*, is the density of the solution* $(X_t)_{t\geq 0}$ *to* (1. 2)*, and the problem* (1. 11) *is controllable.*

The equivalence between **C1** and **C2** is proved by Hörmander in [64]. The equivalence between **C1**, **C2**, **C3** and **C5** can be found in [84] (see Proposition A.1, and Proposition 2.1). The equivalence between **C3** and **C4** was first pointed out by Lunardi in [89].

Remark 1.2 *The condition* **C4** *arises in control theory and it is related to the following controllability problem. For* $x_0, x_1 \in \mathbb{R}^N$ *and* $t_0, t_1 \in \mathbb{R}$ *with* $t_0 < t_1$*, find a "control"* $\omega \in L^1([t_0, t_1], \mathbb{R}^m)$ *such that*

$$\begin{cases} \dot{x}(t) = -Bx(t) + \sigma\omega(t), \\ x(t_0) = x_0, \quad x(t_1) = x_1, \end{cases} \tag{1. 11}$$

where σ, B *are the same matrices appearing in* (1. 2)*. It is known that a solution to the above control problem exists if, and only if, Kalmann's rank condition holds true (see [125]).*

Remark 1.3 *We discuss the meaning of the matrix* $C(t)$*.*

- *From the SDEs point of view,* $2C(t)$ *is the covariance matrix of the solution* $(X_t)_{t\geq 0}$ *to the SDE* (1. 2)*. In general,* $(X_t)_{t\geq 0}$ *is a Gaussian process and its density* p *is defined on* $\mathbb{R}^N$ *when its covariance matrix is positive definite. If this is not the case, the trajectories of* $(X_t)_{t\geq 0}$ *belong to a proper subspace of* $\mathbb{R}^N$*.*
- *The matrix* $C(t)$ *has a meaning also for the optimal control point of view. Indeed, it is known that*

  $$\langle C(t-t_0)^{-1}\left(x - E(t-t_0)x_0\right), x - E(t-t_0)x_0 \rangle = \inf \int_{t_0}^{t} |\omega(s)|^2 \, ds,$$

  where the infimum is taken in the set of all controls for (1. 11) *(see [86], Theorem 3, p. 180). In particular, when* $(x_0, t_0) = (0, 0)$ *the optimal cost is* $\langle C(t)^{-1}x, x\rangle$*, a quantity that appears in the expression for the fundamental solution* Γ *in* (1. 6)*. As we will see in the sequel, this fact will be used to prove asymptotic bounds for positive solutions to Kolmogorov equations (see* (1. 42) *in Theorem 1.15).*

In view of the above assertions, the equivalence of **C3** *and* **C4** *can be interpreted as follows. A control* $\omega \in L^1([t_0, t_1], \mathbb{R}^m)$ *for the problem* (1. 11) *exists if, and only if, the trajectories of the Stocastic Process* $(X_t)_{t\geq 0}$ *reach every point of* $\mathbb{R}^N$*.*

1.1 Lie Group

In this section we focus on the non-Euclidean invariant structure for Kolmogorov operators of the form (1. 1). This non commutative structure was first used by Garofalo and Lanconelli in [51], then explicitly written and thoroughly studied by Lanconelli and Polidoro in [84]. Here and in the sequel we denote by $\mathbb{K}$, the family of Kolmogorov operators $\mathcal{K}$ satisfying the equivalent conditions of Proposition 1.1. We also assume the basis of $\mathbb{R}^N$ is such that the *constant* matrices A and B have the form (1. 8) and (1. 9), respectively.

We now define a non commutative algebraic structure on $\mathbb{R}^{N+1}$ introduced in [84], that replaces the Euclidean one in the study of Kolmogorov operators.

LIE GROUP. *Consider an operator $\mathcal{K}$ in the form* (1. 1) *and recall the notation* (1. 4)*. Let*

$$\mathbb{G} = (\mathbb{R}^{N+1}, \circ), \qquad (x,t)\circ(\xi,\tau) = (\xi + E(\tau)x,\ t+\tau). \tag{1. 12}$$

Then $\mathbb{G}$ is a group with zero element $(0,0)$*, and inverse*

$$(x,t)^{-1} := (-E(-t)x,\ -t). \tag{1. 13}$$

For a given $\zeta \in \mathbb{R}^{N+1}$, we denote by ℓ_ζ the left traslation defined as

$$\ell_\zeta : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}, \quad \ell_\zeta(z) = \zeta \circ z.$$

Then the operator $\mathcal{K}$ is left invariant with respect to the Lie product $\circ$, that is

$$\mathcal{K}\circ\ell_\zeta = \ell_\zeta\circ\mathcal{K} \qquad \textit{or, equivalently,} \qquad \mathcal{K}\left(u(\zeta\circ z)\right) = \left(\mathcal{K}u\right)\left(\zeta\circ z\right), \tag{1. 14}$$

for every u sufficiently smooth.

We omit the details of the proof of the above statements as they are elementary. We remark that, even though we are interested in hypoelliptic operators $\mathcal{K}$, the definition of the Lie product $\circ$ is well posed wether or not we assume the Hörmander's condition. Also note that

$$(\xi,\tau)^{-1}\circ(x,t) = (x - E(t-\tau)\xi, t-\tau), \qquad (x,t),(\xi,\tau)\in\mathbb{R}^{N+1}, \tag{1. 15}$$

then the meaning of (1. 5) can be interpreted as follows:

$$\Gamma(x,t;\xi,\tau) = \Gamma\big((\xi,\tau)^{-1}\circ(x,t)\big). \tag{1. 16}$$

Among the class of Kolmogorov operators $\mathbb{K}$, the invariant operators with respect to a certain family of dilations $(\delta_r)_{r>0}$ play a central role. We say that $\mathcal{K}\in\mathbb{K}$ is invariant with respect to $(\delta_r)_{r>0}$ if

$$\mathcal{K}\left(u\circ\delta_r\right) = r^2\delta_r\left(\mathcal{K}u\right), \quad \text{for every} \quad r>0, \tag{1. 17}$$

for every function u sufficiently smooth. This property can be read in the expression of the matrix B (see Proposition 2.2 of [84]).

Proposition 1.4 *Let $\mathcal{K}$ be an operator of the family $\mathbb{K}$. Then $\mathcal{K}$ satisfies* (1. 17) *if, and only if, the matrix B as this form*

$$B_0 = \begin{pmatrix} \mathbb{O} & \mathbb{O} & \ldots & \mathbb{O} & \mathbb{O} \\ B_1 & \mathbb{O} & \ldots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & B_2 & \ldots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \ldots & B_\kappa & \mathbb{O} \end{pmatrix}. \tag{1. 18}$$

In this case

$$\delta_r = \mathrm{diag}(r\mathbb{I}_{m_0}, r^3\mathbb{I}_{m_1}, \ldots, r^{2\kappa+1}\mathbb{I}_{m_\kappa}, r^2) \quad \text{for every } r > 0, \tag{1. 19}$$

where $\mathbb{I}_{m_j}$ *denotes the identity matrix in* $\mathbb{R}^{m_j}$. *In the sequel we denote by* $\mathbb{K}_0$ *the family of dilation-invariant operators belonging to* $\mathbb{K}$.

It is useful to denote by $(\delta_{r,0}(r))_{r>0}$ the family of spatial dilations defined as

$$\delta_{r,0} = \mathrm{diag}(r\mathbb{I}_{m_0}, r^3\mathbb{I}_{m_1}, \ldots, r^{2\kappa+1}\mathbb{I}_{m_\kappa}) \quad \text{for every } r > 0. \tag{1. 20}$$

Homogeneous Lie group. *If the matrix* B *has the form* (1. 18)*, we say that the following structure*

$$\mathbb{G}_0 = \left(\mathbb{R}^{N+1}, \circ, (\delta_r)_{r>0}\right) \tag{1. 21}$$

is a homogeneous Lie group. In this case, because $\delta_{r,0}\, E(t)\, \delta_{r,0} = E(r^2 t)$ *is verified when* B *has the form* (1. 18)*, the following distributive property holds*

$$\delta_r(\zeta \circ z) = (\delta_r \zeta) \circ (\delta_r z), \qquad \delta_r(z^{-1}) = (\delta_r z)^{-1}. \tag{1. 22}$$

Remark 1.5 *A measurable function* u *on* $\mathbb{G}_0$ *will be called homogeneous of degree* $\alpha \in \mathbb{R}$ *if*

$$u(\delta_r(z)) = r^\alpha u(z) \qquad \textit{for every } z \in \mathbb{R}^{N+1}.$$

A differential operator X *will be called homogeneous of degree* $\beta \in \mathbb{R}$ *with respect to* $(\delta_r)_{r\geq 0}$ *if*

$$Xu(\delta_r(z)) = r^\beta\, (Xu)\, (\delta_r(z)) \qquad \textit{for every } z \in \mathbb{R}^{N+1},$$

and for every sufficiently smooth function u. *Note that, if* u *is homogeneous of degree* α *and* X *is homogeneous of degree* β*, then* Xu *is homogeneous of degree* $\alpha - \beta$.
As far as we are concerned with the vector fields of the Kolmogorov operators as defined in (1. 7)*, we have that* $X_1, \ldots, X_m$ *are homogeneous of degree* 1 *and* Y *is homogeneous of degree* 2 *with respect to* $(\delta_r)_{r\geq 0}$. *In particular,* $\mathcal{K} = \sum_{j=1}^m X_j + Y$ *is is homogeneous of degree* 2.

Remark 1.6 *The presence of the exponents* $1, 3, \ldots, 2\kappa + 1$ *in the matrix* δ *can be explained as follows. The usual parabolic dilation in the first* m_0 *coordinates of* $\mathbb{R}^N$ *and in time is due to the fact that* $\mathcal{K}$ *is non degenerate with respect to* $x_1, \ldots, x_{m_0}$. *The remaining coordinates appear as we check the Hörmander's condition. For instance, consider the Kolmogorov operator*

$$\mathcal{K} = \partial^2_{x_1} + x_1\partial_{x_2} + x_2\partial_{x_3} - \partial_t = X_1^2 + Y.$$

To satisfy the Hörmander condition we need $\kappa = 2$ *commutators* $\partial_{x_2} = [X_1, Y] = X_1Y - YX_1$ *and* $\partial_{x_3} = [[X_1, Y], Y]$. *Because* Y *needs to be considered as a second order derivative, we have that* ∂_{x_2} *and* ∂_{x_3} *are derivatives of order* 3 *and* 5*, respectively. On the other hand, the matrices* A*,* B *and* $D_0(r)$ *associated to this operator are*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \delta_{r,0}(r) = \begin{pmatrix} r & 0 & 0 \\ 0 & r^3 & 0 \\ 0 & 0 & r^5 \end{pmatrix}.$$

The same argument can be applied to operators that need $\kappa > 2$ *steps to satisfy Hörmander's rank condition.*

The integer numbers

$$Q := m_0 + 3m_1 + \ldots + (2\kappa+1)m_k, \quad \text{and} \quad Q+2 \tag{1. 23}$$

will be named *homogeneous dimension of* $\mathbb{R}^N$ *with respect to* $(\delta_{r,0})_{r>0}$, and *homogeneous dimension of* $\mathbb{R}^{N+1}$ *with respect to* $(\delta_r)_{r>0}$, because we have that

$$\det \delta_{r,0} = r^Q \quad \text{and} \quad \det \delta_r = r^{Q+2} \qquad \text{for every } r>0.$$

We now introduce a homogeneous semi-norm of degree 1 with respect to the family of dilations $(\delta_r)_{r>0}$ and a quasi-distance which is invariant with respect to the group operation $\circ$.

Definition 1.7 *For every* $z=(x,t) \in \mathbb{R}^{N+1}$ *we set*

$$\|z\| = |t|^{\frac{1}{2}} + |x|, \qquad |x| = \sum_{j=1}^{N} |x_j|^{\frac{1}{q_j}}, \tag{1. 24}$$

where the numbers q_j *are associated to the dilation group* $(\delta_r)_{r>0}$ *as follows*

$$\delta_r = \operatorname{diag}\left(r^{q_1}, \ldots, r^{q_N}, r^2\right).$$

Remark 1.8 *The norm* $\|\cdot\|$ *is homogeneous of degree* 1 *with respect to* $\{\delta_r\}_{r>0}$*, that is*

$$\| \delta_r(x,t) \| = r \, \| (x,t) \| \qquad \textit{for every} \, r>0 \;\; \textit{and} \;\; (x,t) \in \mathbb{R}^{N+1}.$$

Because every norm is equivalent to any other in $\mathbb{R}^{N+1}$*, other definitions have been used in the literature. For instance in [90] it is chosen the following one. For every* $z=(x_1,\ldots,x_N,t) \in \mathbb{R}^{N+1} \setminus \{0\}$ *the norm of* z *is the unique positive solution* r *to the following equation*

$$\frac{x_1^{q_1}}{r^{2q_1}} + \frac{x_2^{q_2}}{r^{2q_2}} + \ldots + \frac{x_N^{q_N}}{r^{2q_N}} + \frac{t^2}{r^4} = 1. \tag{1. 25}$$

Note that, if we choose (1. 25)*, the set* $\left\{z \in \mathbb{R}^{N+1} : \|z\| = r\right\}$ *is a smooth manifold for every positive* r*, which is note the case for* (1. 24)*.*

A further example may be the following norm

$$\| (x,t) \|_1 = |x_1|^{\frac{1}{\alpha_1}} + \ldots + |x_N|^{\frac{1}{\alpha_N}} + |t|^{\frac{1}{2}},$$

where the homogeneity with respect to $\{\delta_r\}_{r>0}$ *can easily be showed. We prefer the norm of Definition 1.7 to* $\|\cdot\|_1$ *because its level sets (spheres) are smooth surfaces.*

Based on Definition 1.7, in the following we introduce a quasi-distance $d : \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \to [0,+\infty[$ (see Definition 1.10 below). This means that:

1. $d(z,w)=0$ if and only if $z=w$ for every $z,w \in \mathbb{R}^{N+1}$;
2. for every compact subset K of $\mathbb{R}^{N+1}$, there exists a positive constant $C_K \geq 1$ such that

$$\begin{aligned} &d(z,w) \leq C_K d(w,z); \\ &d(z,w) \leq C_K \left(d(z,\zeta) + d(\zeta,w)\right), \quad \text{for every } \; z,w,\zeta \in K. \end{aligned} \tag{1. 26}$$

The proof of (1. 26) is given in Lemma 2.1 of [41]. Definition 1.10 is given for general non-homogeneous Lie groups. This requires the notion of *principal part operator* discussed in the next section. We point out that the constant C_K doesn't depend on K in the case of homogeneous groups (see Proposition 2.1 in [90]).

1.2 Principal part operator

This section is devoted to show that dilation invariant operators are the blow-up limit of the operator belonging to $\mathbb{K}$. In order to identify the appropriate dilation, we denote by $\mathcal{K}_0$ the *principal part operator of* $\mathcal{K}$ obtained from (1. 1) by substituting the matrix B with B_0 as defined in (1. 18), that is

$$\mathcal{K}_0 = \operatorname{div}(AD) + \langle B_0 x, D\rangle - \partial_t. \tag{1. 27}$$

Since $\mathcal{K}_0$ is dilation-invariant with respect to $(\delta_r)_{r>0}$, we define $\mathcal{K}_r$ as the *scaled operator* of $\mathcal{K}$ in terms of $(\delta_r)_{r>0}$ as follows

$$\mathcal{K}_r := r^2\, \delta_r\, \circ \mathcal{K} \circ \delta_{1/r} = \operatorname{Tr}(AD^2) + \langle B_r x, D\rangle - \partial_t, \tag{1. 28}$$

where $B_r = \delta_r\, B\delta_{1/r}$ is given by

$$B_r = \begin{pmatrix} r^2 B_{0,0} & r^4 B_{0,1} & \dots & r^{2\kappa} B_{0,\kappa-1} & r^{2\kappa+2} B_{0,\kappa} \\ B_1 & r^2 B_{1,1} & \dots & r^{2\kappa-2} B_{\kappa-1,1} & r^{2\kappa} B_{\kappa,1} \\ \mathbb{O} & B_2 & \dots & r^{2\kappa-4} B_{\kappa-1,2} & r^{2\kappa-2} B_{\kappa,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_\kappa & r^2 B_{\kappa,\kappa} \end{pmatrix}. \tag{1. 29}$$

Clearly $\mathcal{K}_r = \mathcal{K}$ for every $r > 0$ if and only if $B = B_0$, and the principal part $\mathcal{K}_0$ of $\mathcal{K}$ is obtained as the limit of (1. 28) as $r \to 0$.

The invariance structures of the operator $\mathcal{K}$ also reveal themselves in the expression of the fundamental solution Γ. In particular, as noticed above, Γ is translation invariant, as it satisfies the identity (1. 16). As far as we are concerned with the dilation invariance, the fundamental solution Γ_0 of $\mathcal{K}_0$ is a homogeneous function of degree $-Q$ with respect to the dilation $(\delta_r)_{r>0}$, that is

$$\Gamma_0(\delta_r z) = r^{-Q}\Gamma_0(z) \qquad \text{for every} \quad z \in \mathbb{R}^{N+1} \setminus \{0\},\, r > 0, \tag{1. 30}$$

where Q is the spatial homogeneous dimension of $\mathbb{R}^{N+1}$ introduced in (5. 19). Moreover, the expression of Γ_0 writes in terms of $\delta_{r,0}(r)$. Indeed, the matrix $C(t)$ defined in (1. 4) satisfies the following identity

$$C(t) = \delta_{\sqrt{t},0}\; C(1)\; \delta_{\sqrt{t},0} \qquad \text{for every}\, t > 0,$$

and

$$\Gamma_0(x,t) = \frac{C_N}{t^{\frac{Q}{2}}} \exp\left(-\tfrac{1}{4} \langle C^{-1}(1)\, \delta_{1/\sqrt{t},0}\; x, \delta_{1/\sqrt{t},0}\; x\rangle \right),$$

where C_N is the positive constant

$$C_N = (4\pi)^{-\frac{N}{2}} (\det C(1))^{-\frac{1}{2}}.$$

We refer to [84], [80], [82] for the proof of the above statements. Eventually, Theorem 3.1 in [84] provides us with a quantitative comparison between Γ and Γ_0.

Theorem 1.9 *Let $\mathcal{K}$ be an operator of the class $\mathbb{K}$ and let $\mathcal{K}_0$ be its principal part as defined in* (1. 27)*. Then for every $K > 0$ there exists a positive constant $\varepsilon > 0$ such that*

$$(1-\varepsilon)\Gamma_0(z) \le \Gamma(z) \le (1+\varepsilon)\Gamma_0(z) \tag{1. 31}$$

for every $z \in \mathbb{R}^{N+1}$ such that $\Gamma_0(z) \ge K$. Moreover, $\varepsilon = \varepsilon(K) \to 0$ as $K \to +\infty$.

Note that the above result does not hold true in the set $\left\{\Gamma_0 < K\right\}$ (see formula (1.30) in [84]).

We now introduce the quasi-distance d for a generic Lie group $\mathbb{G}$. In the following definition "$\circ$" denotes the traslation associated to $\mathcal{K}$, and the norm $\|\cdot\|$ is the one associated to $\mathcal{K}_0$.

Definition 1.10 *For every $z, w \in \mathbb{R}^{N+1}$, we define a quasi-distance $d(z,w)$ invariant with respect to the translation group $\mathbb{G}_0$ as follows*
$$d(z,w) = \|z^{-1} \circ w\|, \tag{1. 32}$$
and we denote by $B_r(z)$ the $d-$ball of center z and radius r.

Definition 1.11 *Let α be a positive constant, $\alpha \le 1$, and let Ω be an open subset of $\mathbb{R}^{N+1}$. We say a function $f : \Omega \longrightarrow \mathbb{R}$ is Hölder continuous with exponent α in Ω with respect to the groups $\mathbb{G} = (\mathbb{R}^{N+1}, \circ)$ and $(\delta_r)_{r>0}$ (in short: Hölder continuous with exponent α, $f \in C^\alpha(\Omega)$) if there exists a positive constant $k > 0$ such that*
$$|f(z) - f(\zeta)| \le k\, d(z,\zeta)^\alpha \qquad \text{for every } z, \zeta \in \Omega.$$
To every bounded function $f \in C^\alpha(\Omega)$ we associate the norm
$$|f|_{\alpha,\Omega} \;=\; \sup_{\Omega}|f| \;+\; \sup_{\substack{z,\zeta\in\Omega\\ z\neq\zeta}} \frac{|f(z)-f(\zeta)|}{d(z,\zeta)^\alpha}.$$
Moreover, we say a function f is locally Hölder continuous, and we write $f \in C^\alpha_{\text{loc}}(\Omega)$, if $f \in C^\alpha(\Omega')$ for every compact subset Ω' of Ω.

Remark 1.12 *Let Ω be a bounded subset of $\mathbb{R}^{N+1}$. If f is a Hölder continuous function of exponent α in the usual Euclidean sense, then f is Hölder continuous of exponent α. Vice versa, if $f \in C^\alpha(\Omega)$ then f is a $\beta-$Hölder continuous in the Euclidean sense, where $\beta = \frac{\alpha}{2\kappa+1}$ and κ is the constant appearing in* (1. 9).

1.3 Kolmogorov operator with Hölder continuous coefficients

In this section we consider Kolmogorov operator in non-divergence form in $\mathbb{R}^{N+1}$
$$\mathcal{K} = \sum_{i,j=1}^{m_0} a_{ij}(x,t)\partial^2_{x_i x_j} + \sum_{j=1}^{m_0} b_j(x,t)\partial_{x_j} + \langle Bx, D\rangle - \partial_t, \qquad \text{for } (x,t) \in \mathbb{R}^{N+1} \tag{1. 33}$$
with continuous coeficients a_{ij}'s and b_j's. As in the parabolic case, the classical theory for degenerate Kolmogorov operators is developed for spaces of Hölder continuous functions introduced in Definition 1.11. We remark that this definition relies on the Lie group $\mathbb{G}$ in (1. 12), that is an invariant structure for the *constant coefficients* operators. Even though the *non-constant coefficients* operators in (1. 33) are not invariant with respect to $\mathbb{G}$, we will rely on the Lie group invariance of the model operator
$$\Delta_{m_0} + Y = \sum_{j=1}^{m_0} \partial^2_{x_j} + \langle Bx, D\rangle - \partial_t, \tag{1. 34}$$
associated to $\mathcal{K}$. Indeed, this is a standard procedure in the study of uniformly parabolic operators. We next list the standing assumptions of this section:

(H1) $B = (b_{i,j})$ is a $N \times N$ real constant matrix of the type (1. 9), with blocks B_j of rank m_j and $*-$blocks arbitrary;

(H2) $A = (a_{ij}(z))_{i,j=1,\dots,m_0}$ is a symmetric matrix of the form (1. 8), i.e. $a_{ij}(z) = a_{j,i}(z)$ for $i,j = 1,\dots,m_0$, with $1 \le m_0 \le N$. Moreover, it is positive definite in $\mathbb{R}^{m_0}$ and there exist a positive constant λ such that
$$\frac{1}{\lambda}\sum_{i=1}^{m_0}|\xi_i|^2 \le \sum_{i,j=1}^{m_0} a_{ij}(z)\xi_i\xi_j \le \lambda\sum_{i=1}^{m_0}|\xi_i|^2$$
for every $(\xi_1,\dots,\xi_{m_0}) \in \mathbb{R}^{m_0}$ and $z \in \mathbb{R}^{N+1}$;

(H3) there exist $0 < \alpha \le 1$ and $M > 0$ such that

$$|a_{ij}(z) - a_{ij}(\zeta)| \le M\, d(z,\zeta)^{\alpha}, \qquad |b_j(z) - b_j(\zeta)| \le M\, d(z,\zeta)^{\alpha},$$

for every $z, \zeta \in \mathbb{R}^{N+1}$ and for every $i, j = 1, \dots, m_0$.

Note that, if $m_0 = N$, the operator $\mathcal{K}$ is uniformly parabolic and $B = \mathbb{O}$. In particular the model operator (1. 34) is the heat operator and we have $d\big((\xi,\tau),(x,t)\big) = |\xi - x| + |\tau - t|^{1/2}$, so that we are considering the parabolic modulus of continuity.
In the sequel we refer to the assumption **(H3)** by saying that the coefficients a_{ij}'s and b_j's belong to the space C^{α} introduced in Definition 1.11. We next give the definion of classic solution to the equation $\mathcal{K}u = f$ under minimal regularity assumptions on u. A function u is Lie differentiable with respect to the vector field Y defined in (1. 7) at the point $z = (x,t)$ if there exists and is finite

$$Yu(z) := \lim_{s \to 0} \frac{u(\gamma(s)) - u(\gamma(0))}{s}, \qquad \gamma(s) = (E(-s)x, t-s). \tag{1. 35}$$

Note that γ is the integral curve of Y from z. Clearly, if $u \in C^1(\Omega)$, with Ω open subset of $\mathbb{R}^{N+1}$, then $Yu(x,t)$ agrees with $\langle Bx, Du(x,t)\rangle - \partial_t u(x,t)$ considered as a linear combination of the derivatives of u.

Definition 1.13 *A function u is a solution to the equation $\mathcal{K}u = f$ in a domain Ω of $\mathbb{R}^{N+1}$ if there exists the Euclidean derivatives $\partial_{x_i}u, \partial_{x_i x_j}u \in C(\Omega)$ for $i, j = 1, \dots, m_0$, the Lie derivative $Yu \in C(\Omega)$, and the equation*

$$\sum_{i,j=1}^{m_0} a_{ij}(z)\partial^2_{x_i x_j}u(z) + \sum_{j=1}^{m_0} b_j(z)\partial_{x_j}u(z) + Yu(z) = f(z)$$

is satisfied at any point $z = (x,t) \in \Omega$.

The natural functional setting for the study of classical solutions is the space

$$C^{2,\alpha}(\Omega) = \Big\{ u \in C^{\alpha}(\Omega) \ \mid \ \partial_{x_i}u, \partial^2_{x_i x_j}u, Yu \in C^{\alpha}(\Omega), \quad \text{for } i,j = 1,\dots,m_0 \Big\}, \tag{1. 36}$$

where $C^{\alpha}(\Omega)$ is given in Definition 1.11. Moreover, if $u \in C^{2,\alpha}(\Omega)$ then we define the norm

$$|u|_{2+\alpha,\Omega} := |u|_{\alpha,\Omega} + \sum_{i=1}^{m_0} |\partial_{x_i}u|_{\alpha,\Omega} + \sum_{i,j=1}^{m_0} |\partial^2_{x_i x_j}u|_{\alpha,\Omega} + |Yu|_{\alpha,\Omega}. \tag{1. 37}$$

Clearly, the definition of $C^{2,\alpha}_{\mathrm{loc}}(\Omega)$ follows straightforwardly from the definition of $C^{\alpha}_{\mathrm{loc}}(\Omega)$. A definition of the space $C^{k,\alpha}(\Omega)$ for every positive integer k is given and discussed in the work [101] by Pagliarani, Pascucci and Pignotti, where a proof of the Taylor expansion for $C^{k,\alpha}(\Omega)$ functions is given. It is worth noting that the authors of [101] require weaker regularity assumptions for the definition of the space $C^{2,\alpha}$ than the ones considered here in (1. 36).
As in the uniformly elliptic and parabolic case, fundamental results in the classical regularity theory are the Schauder estimates. We recall that Schauder estimates for the dilation invariant Kolmogorov operator (i.e. where the matrix $B = B_0$) with Hölder continuous coefficients were proved by Manfredini in [90] (see Theorem 1.4). Manfredini result was then extended by Di Francesco and Polidoro in [41] to the non-dilation invariant case.

Theorem 1.14 *Let us consider an operator $\mathcal{K}$ of the type* (1. 33) *satisfying assumptions **(H1)**, **(H2)**, **(H3)** with $\alpha < 1$. Let Ω be an open subset of $\mathbb{R}^{N+1}$, $f \in C^{\alpha}_{\mathrm{loc}}(\Omega)$ and let u be a classical solution to $\mathcal{K}u = f$ in Ω. Then for every $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ there exists a positive constant C such that*

$$|u|_{2+\alpha,\Omega'} \le C\Big(\sup_{\Omega''} |u| + |f|_{\alpha,\Omega''} \Big).$$

A more precise estimate taking into account the distance between the point and the boundary of the set Ω can be found in [90] (see Theorem 1.4) for the dilation invariant case. We omit here this precise statement because it requires the introduction of further notation. We also recall that analogous Schauder estimates have been proved by several authors in the framework of semigroup theory, where they consider solutions which are not classical in the sense of Definition 2.6. Among others, we refer to Lunardi [89], Lorenzi [88], Priola [111], Delarue and Menozzi [38].

1.4 Fundamental Solution and Cauchy Problem

The existence of a fundamental solution Γ for the operator $\mathcal{K}$ satisfying the assumptions **(H1)**, **(H2)** and **(H3)** has been proved using the Levi's parametrix method. The first results of this type are due to M. Weber [121], to Il'In [67] and to Sonin [114] who assumed an Euclidean regularity on the coeficients a_{ij}'s and b_j's. Later on, Polidoro applied in [107] the Levi parametrix method for the *dilation inviariant* operator $\mathcal{K}$ (i.e. under the additional assumption that B has the form (1. 18)), then Di Francesco and Pascucci removed this last assumption in [40].
The Levi's parametrix method is a constructive argument to prove existence and bounds of the fundamental solution. For every $\zeta \in \mathbb{R}^{N+1}$, the parametrix $Z(\,\cdot\,,\zeta)$ is the fundamental solution, with pole at ζ, of the following operator

$$\mathcal{K}_\zeta = \sum_{i,j=1}^{m_0} a_{ij}(\zeta)\,\partial^2_{x_i x_j} \;+\; \langle Bx, D\rangle \;-\; \partial_t. \tag{1. 38}$$

The method is based on the fact that, if the coeficients a_{ij}'s are continuous and the coefficiens b_j's are bounded, then Z is a good approximation of the fundamental solution of $\mathcal{K}$, because

$$\mathcal{K}Z(z,\zeta) = \sum_{i,j=1}^{m_0} \left(a_{ij}(z) - a_{ij}(\zeta)\right)\,\partial^2_{x_i x_j} Z(z,\zeta) + \sum_{j=1}^{m_0} b_j(z)\,\partial_{x_j} Z(z,\zeta),$$

at least as z is close to the pole ζ. We look for the fundamental solution Γ as a solution of the following Volterra equation

$$\Gamma(x,t,\xi,\tau) = Z(x,t,\xi,\tau) + \int_\tau^t \int_{\mathbb{R}^N} Z(x,t,y,s) G(y,s,\xi,\tau)\,dy\,ds, \tag{1. 39}$$

where the unknown function G is obtained by a fixed point argument. It turns out that

$$G(z,\zeta) = \sum_{k=1}^{+\infty} (\mathcal{K}Z)_k(z,\zeta), \tag{1. 40}$$

where $(\mathcal{K}Z)_1(z,\zeta) = \mathcal{K}Z(z,\zeta)$ and, for every $k \in \mathbb{N}$,

$$(\mathcal{K}Z)_{k+1}(x,t,\xi,\tau) = \int_\tau^t \int_{\mathbb{R}^N} \mathcal{K}Z(x,t,y,s)(\mathcal{K}Z)_k(y,s,\xi,\tau)\,dy\,ds.$$

Let's point out that Z is explicitly known by formulas (1. 5) and (1. 6), then the equations (1. 39) and (1. 40) give explicit bounds for Γ and for its derivatives (see equations (1. 42) and (1. 53) below). We summarize here the main results of the articles [107] and [40] on the existence and bounds for the fundamental solution.

Theorem 1.15 *Let $\mathcal{K}$ be an operator of the form* (1. 33) *under the assumptions* ***(H1)**, **(H2)**, **(H3)**. Then there exists a fundamental solution $\Gamma(\cdot,\zeta)$ to $\mathcal{K}$ with pole at $\zeta \in \mathbb{R}^{N+1}$ such that:*

1. $\Gamma(\cdot,\zeta) \in L^1_{\text{loc}}(\mathbb{R}^{N+1}) \cap C(\mathbb{R}^{N+1} \setminus \{\zeta\})$*;*

2. for every $\varphi \in C_b(\mathbb{R}^N)$ *the function*

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)\varphi(\xi)d\xi,$$

is a classical solution of the Cauchy problem

$$\begin{cases} \mathcal{K}u = 0, & (x,t) \in \mathbb{R}^N \times \mathbb{R}^+ \\ u(x,0) = \varphi(x) & (x,t) \in \mathbb{R}^N. \end{cases} \tag{1. 41}$$

3. For every $(x,t), (\xi,\tau) \in \mathbb{R}^{N+1}$ *such that* $\tau < t$ *we have that*

$$\int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau)\, d\xi = 1;$$

4. the reproduction property holds for every $(y,s) \in \mathbb{R}^{N+1}$ *with* $\tau < s < t$*:*

$$\Gamma(x,t,\xi,\tau) = \int_{\mathbb{R}^N} \Gamma(x,t,y,s)\,\Gamma(y,s,\xi,\tau)\,dy;$$

5. for every positive T *and for every* $\Lambda > \lambda$*, with* λ *as in* **(H1)***, there exists a positive constant* $c^+ = c^+(\Lambda,\lambda,T)$ *such that*

$$c^-\,\Gamma^-(z,\zeta) \le \Gamma(z,\zeta) \le c^+\,\Gamma^+(z,\zeta) \quad \text{for every } z,\zeta \in \mathbb{R}^{N+1},\, 0 < t-\tau < T, \tag{1. 42}$$

for every $(x,t), (\xi,\tau) \in \mathbb{R}^{N+1}$ *with* $0 < t-\tau < T$*. Here,* Γ^+ *and* Γ^- *are, respectively, the fundamental solutions of the following operators:*

$$\mathcal{K}^+ = \lambda\Delta_{m_0} + \langle Bx, D\rangle - \partial_t \quad \text{and} \quad \mathcal{K}^- = \lambda^{-1}\Delta_{m_0} + \langle Bx, D\rangle - \partial_t.$$

Once the uniqueness of the Cauchy problem is guaranteed, points 3. and 4. of the above theorem will follow from point 2. The lower bound in (1. 42) is proved by using the Harnack inequality presented in Theorem 1.26 and following the technique introduced by Aronson and Serrin [11] for the classic parabolic case. We remark that property 3. of Theorem 1.15 does not hold true unless we require further regularity assumptions on the coefficients a_{ij}'s and b_j's needed to define the formal adjoint $\mathcal{K}^*$ of $\mathcal{K}$.

In view of (2. 14), the fundamental solution is the most natural tool to deal with the Cauchy problem associated to the equation $\mathcal{K}u = f$. For a given positive T we denote by S_T the strip of $\mathbb{R}^{N+1}$ defined as follows

$$S_T = \mathbb{R}^N \times]0,T[,$$

and we look for a classical solution to the Cauchy problem

$$\begin{cases} \mathcal{K}u = f & \text{in } S_T, \\ u(\cdot,0) = \varphi & \text{in } \mathbb{R}^N, \end{cases} \tag{1. 43}$$

with $f \in C(S_T)$ and $\varphi \in C(\mathbb{R}^N)$. Once again in view of (2. 14) it is clear that growth condition on f and φ are required to ensure existence and uniqueness for the solution to (1. 43). The following result is due to Di Francesco and Pascucci in [40].

Theorem 1.16 *Let $\mathcal{K}$ be an operator of the form* (1. 33) *under the assumptions **(H1), (H2), (H3)**. Consider the Cauchy problem* (1. 43) *with $\varphi \in C(\mathbb{R}^N)$ and $f \in C^\alpha(\Omega)$, in the sense of Definition 1.11. Let us suppose for some positive constant C*

$$|f(x,t)| \le C\, e^{C|x|^2} \qquad |\varphi(x)| \le C\; e^{C|x|^2}.$$

for every $x \in \mathbb{R}^N$ and $0 < t < T$. Then there exists $0 < T_0 \le T$ such that the function

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t,\xi,0)\,\varphi(\xi)\,d\xi - \int_0^t \int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau)\,f(\xi,\tau)\,d\xi\,d\tau. \tag{1. 44}$$

is well defined for every $(x,t) \in \mathbb{R}^N \times]0, T_0[$. Moreover, it is a solution to the Cauchy problem (1. 43) *and the initial condition is attained by continuity*

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = \varphi(x_0), \qquad \textit{for every } x_0 \in \mathbb{R}^N.$$

Uniqueness results for the Cauchy problem (1. 43) can be found in [108], [40] and [41]. Later on, Cinti and Polidoro proved in [34] the following result.

Theorem 1.17 *Let $\mathcal{K}$ be an operator of the form* (1. 33) *under the assumptions **(H1), (H2), (H3)**. If u and v are two solutions to the same Cauchy problem* (1. 43) *satisfying the following estimate*

$$\int_0^T \int_{\mathbb{R}^N} (|u(x,t)| + |v(x,t)|)\; e^{-C\left(|x|^2 + \frac{1}{t^\beta}\right)}\, dx\, dt < +\infty \tag{1. 45}$$

with $0 < \beta < 1$, then $u \equiv v$.

We eventually quote the main uniqueness result of [41], that doesn't require any growth assumptions on the solutions u and v.

Theorem 1.18 *Let $\mathcal{K}$ be an operator of the form* (1. 33) *under the assumptions **(H1), (H2), (H3)**. If u and v are two non-negative solutions to the same Cauchy problem* (1. 43)*, with $f = 0$ and $\varphi \ge 0$, then $u \equiv v$.*

1.4.1 The Dirichlet problem

In the sequel Ω will denote a bounded domain of $\mathbb{R}^{N+1}$. For every $f \in C(\Omega)$ and $\varphi \in C(\partial\Omega, \mathbb{R})$, we consider the Dirichlet problem for the operator $\mathcal{K}$ with Hölder continuous coefficients

$$\begin{cases} \mathcal{K}u = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1. 46}$$

This problem has been studied by Manfredini in [90] in the framework of the Potential Theory. In accordance with the usual axiomatic approach, we denote by H_φ^Ω the Perron-Wiener-Brelot-Bauer solution to the Dirichlet problem (1. 46) with $f = 0$. In order to discuss the boundary condition of the problem (1. 46) we say that a point $z_0 \in \partial\Omega$ is $\mathcal{K}-$regular for Ω if

$$\lim_{z\to z_0} H_\varphi^\Omega(z) \qquad \text{for every } \varphi \in C(\partial\Omega). \tag{1. 47}$$

The first result for the existence of a solution to the Dirichlet problem (1. 46) for an operator $\mathcal{K}$ with Hölder continuous coefficiens is proved by Manfredini in [90], Theorem 1.4.

Theorem 1.19 *Let $\mathcal{K}$ be an operator in the form* (1. 33) *satisfying conditions **(H1), (H2), (H3)**, and assume that the matrix B has the form* (1. 18). *Suppose that $f \in C^{\alpha}(\overline{\Omega})$ and $\varphi \in C(\partial\Omega)$. Then there exixts a unique solution $u \in C^{2,\alpha}_{\mathrm{loc}}(\Omega)$ to the Dirichlet problem* (1. 46). *The function u is a classical solution to $\mathcal{K}u = f$ in Ω, and $\lim\limits_{z \to z_0} u(z) = \varphi(z_0)$ for every $\mathcal{K}-$regular point $z_0 \in \partial\Omega$.*

The assumption that the matrix B is of the form (1. 18) has been introduced to simplify the problem and seems to be unnecessary. Indeed, this condition is removed in [41], where a specific family of open sets Ω is considered. The uniqueness of the solution follows straightforwardly from the following *weak maximum principle* that can be found in the proof of Proposition 4.2 of [90].

Theorem 1.20 *Let $\mathcal{K}$ be an operator in the form* (1. 33) *satisfying conditions **(H1), (H2), (H3)**, and assume that the matrix B has the form* (1. 18). *Let Ω be a bounded open set of $\mathbb{R}^{N+1}$, and let u be a continuous function in $\overline{\Omega}$, such that $\partial_{x_j} u, \partial^2_{x_i x_j} u$, for $i, j = 1, \ldots, m_0$ and Yu are continuous in Ω. If moreover*

$$\begin{cases} \mathcal{K}u \geq 0 & \text{in } \Omega, \\ u \leq 0 & \text{on } \partial\Omega, \end{cases}$$

then $u \leq 0$ in Ω.

In order to discuss the boundary regularity of Ω, we recall that the analogous of the Bouligand theorem for operators $\mathcal{K}$ has been proved in [90]. Specifically, a point $z_0 \in \partial\Omega$ is $\mathcal{K}-$regular if there exists a local barrier at z_0, that is there exists a neighborhood V of z_0 and a function $w \in C^{2,\alpha}(V)$ such that

$$w(z_0) = 0, \ w(z) > 0 \text{ for } z \in \overline{\Omega \cap V} \setminus \{z_0\} \qquad \text{and} \qquad \mathcal{K}w \leq 0 \text{ in } \Omega \cap V.$$

Let z_0 be point belonging to $\partial\Omega$. We say that a vector $\nu \in \mathbb{R}^{N+1}$ is an *outer normal to Ω at z_0* if there exists a positive r such that $B(z_1, r|\nu|) \cap \overline{\Omega} = \{z_0\}$. Here $B(z_1, r|\nu|)$ is the Euclidean ball centered at $z_1 = z_0 + r\nu$ and radius $r|\nu|$. Note that this definition does not require any regularity on $\partial\Omega$ and several linearly independent vectors are allowed to be outer normal to Ω at the same point z_0. The following result proved in [90] gives a very simple geometric condition for the boundary regularity of Ω and is in accordance with the Fichera's classification of $\partial\Omega$.

Theorem 1.21 *Let $\mathcal{K}$ be an operator in the form* (1. 33) *satisfying conditions **(H1), (H2), (H3)**. Consider the Dirichlet problem* (1. 46), *and let $z_0 \in \partial\Omega$. Assume that ν is an outer normal to Ω at z_0. Then it holds*

- *if $\langle A(z_0)\nu, \nu\rangle \neq 0$, then there exists a local barrier at z_0;*
- *if $\langle A(z_0)\nu, \nu\rangle = 0$, and $\langle Y(z_0), \nu\rangle > 0$ then there exists a local barrier at z_0;*
- *if $\langle A(z_0)\nu, \nu\rangle = 0$, and $\langle Y(z_0), \nu\rangle < 0$ then z_0 is non regular.*

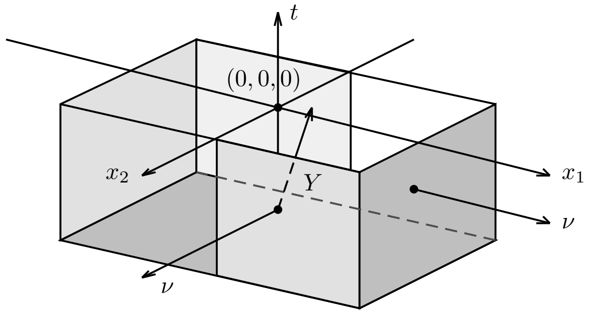

FIG. 1 - REGULAR POINTS FOR $\partial^2_{x_1} + x_1\partial_{x_2} - \partial_t$ ON THE SET $]-1,1[^2\times]-1,0[$.

The following more refined condition extends the Zaremba cone criterium. Let $\bar{U}$ be an open set of $\mathbb{R}^N$ and let $\bar{t} > 0$. We denote by $Z_{\bar{U},\bar{t}}(z_0)$ the following *tusk-shaped* cone

$$Z_{\bar{U},\bar{t}}(z_0) := \left\{ z_0 \circ D_r(\bar{x}, -\bar{t}) \mid \bar{x} \in \bar{U}, 0 \le r \le 1 \right\}.$$

Theorem 1.22 *Let $\mathcal{K}$ be an operator in the form* (1. 33) *satisfying conditions **(H1), (H2), (H3)**, and assume that the matrix B has the form* (1. 18)*. Consider the Dirichlet problem* (1. 46)*, and let $z_0 \in \partial\Omega$. If there exist $\bar{U}$ and $\bar{t}$ such that $Z_{\bar{U},\bar{t}}(z_0) \cap \overline{\Omega} = \{z_0\}$, then there exists a local barrier at z_0.*

Theorems 1.21 and 1.22 have been first proved in [90] assuming that the matrix B has the form (1. 18), this assumption has been removed from Theorem 1.21 in [41]. We also recall the work [85] by Lascialfari and Morbidelli, where a quasilinear problem is considered, and the article [72] by Kogoj for a complete treatment of the potential theory in the study of the Dirichlet problem for a general class of evolution hypoelliptic equations.

Recently, Kogoj, Lanconelli and Tralli prove in [75] a characterization of the $\mathcal{K}-$regular boundary points for constant coefficients operators $\mathcal{K}$ of the form (1. 1). Their main result is stated in terms of a series involving $\mathcal{K}-$potentials of regions contained in $\mathbb{R}^{N+1} \setminus \Omega$, within different level sets of Γ, the fundamental solution of $\mathcal{K}$. Specifically, if F is a compact subset of $\mathbb{R}^{N+1}$, then V_F denotes the $\mathcal{K}-$equilibrium potential of F. That is,

$$V_F(z) = \liminf_{\zeta \to z} W_F(\zeta), \qquad z \in \mathbb{R}^{N+1}, \tag{1. 48}$$

where if $\overline{\mathcal{K}}(\mathbb{R}^{N+1})$ denotes the family of $\mathcal{K}-$super harmonic functions in $\mathbb{R}^{N+1}$

$$W_F := \inf \left\{ v : \, v \in \overline{\mathcal{K}}(\mathbb{R}^{N+1}), v \ge 0 \,\text{in}\, \mathbb{R}^{N+1}, v \ge 1 \,\text{in}\, F \right\}. \tag{1. 49}$$

Moreover, for given $\mu \in]0,1[$, $z_0 \in \partial\Omega$, and for every positive integer k we denote by $\Omega^c_k(z_0)$ the set

$$\Omega^c_k(z_0) := \left\{ z \in \mathbb{R}^{N+1} \setminus \Omega \;\mid\; \left(\tfrac{1}{\mu}\right)^{k \log k} \le \Gamma(z_0; z) \le \left(\tfrac{1}{\mu}\right)^{(k+1)\log(k+1)} \right\}.$$

We then have (Theorem 1.1, [75]).

Theorem 1.23 *Let $\mathcal{K}$ be an hypoelliptic operator in the form* (1. 1)*, let Ω be a bounded open subset of $\mathbb{R}^{N+1}$ and let $z_0 \in \partial\Omega$. Then z_0 is $\mathcal{K}-$regular for $\partial\Omega$ if and only if*

$$\sum_{k=1}^{+\infty} V_{\Omega^c_k(z_0)}(z_0) = +\infty. \tag{1. 50}$$

We remark that this criterion is sharper than the Zaremba cone condition, moreover it provides us with a *necessary* regularity condition. On the other hand, it only applies to constant coefficients operators of the form (1. 1).

1.5 Mean value formulas, Harnack inequalities and Strong Maximum Principle

In the first part of this section we consider *divergence form operators* acting on functions $u = u(x,t) \in C^{2,\alpha}(\Omega)$ as follows

$$\mathcal{K}u = \sum_{i,j=1}^{m_0} \partial_{x_i}\left(a_{ij}(x,t)\partial_{x_j}u\right) + \sum_{j=1}^{m_0} b_j(x,t)\partial_{x_j}u + \langle Bx, Du\rangle - \partial_t u, \tag{1. 51}$$

under the structural assumptions **(H1), (H2), (H3)**. Moreover, we suppose the following additional assumption for the first order derivatives holds true:

(H4) for every $i, j = 1, \dots, m_0$ the derivatives $\partial_{x_i} a_{ij}(x,t)$ $\partial_{x_j} b_j(x,t)$ exist and are bounded Hölder continuous functions of the exponent α in **(H3)**.

The reason to consider classical solutions to divergence form operators is that the adjoint $\mathcal{K}^*$ of $\mathcal{K}$ is well defined and the function $\Gamma^*(x,t,\xi,\tau) = \Gamma(\xi,\tau,x,t)$ built via the parametrix method is the fundamental solution of $\mathcal{K}^*$.

1.5.1 Mean value formula

The mean value formula we present here is based on the Green's identity and on the fundamental solution to $\mathcal{K}$ and is derived in the same way as for the classic parabolic case. In order to give the precise statement we need to introduce some notation. For every $r > 0$ and for every $z_0 \in \mathbb{R}^{N+1}$, we denote by $\Omega_r(z_0)$ the *super-level set of the fundamental solution* Γ of $\mathcal{K}$ defined as

$$\Omega_r(z_0) := \left\{ z \in \mathbb{R}^{N+1} \ \mid \ \Gamma(z_0; z) > \tfrac{1}{r} \right\}. \tag{1. 52}$$

We remark that Γ is constructed via the parametrix method as the sum of a series of functions (see (1. 39) and (1. 40)), then the definition of the set $\Omega_r(z_0)$ is implicit. However the parametrix method provides us with the following local estimate, useful to identify $\Omega_r(z_0)$. For every $\varepsilon > 0$ there exists a positive K such that

$$(1-\varepsilon)\, Z(z_0,\zeta) \ \leq \Gamma(z_0,\zeta) \ \leq \ (1+\varepsilon)\, Z(z_0,\zeta) \tag{1. 53}$$

for every $\zeta \in \mathbb{R}^{N+1}$ with $Z(z_0,\zeta) \geq K$, where Z is the fundamental solution associated to the operator $\mathcal{K}_\zeta$ defined in (1. 38) and its explicit expression is available. Moreover, every super-level set of Z is bounded whenever B has the form 1. 18. This fact and Theorem 1.9 imply that $\Omega_r(z_0)$ is bounded for every sufficiently small positive r.
Mean value formulas for constant coefficients operators in the form (1. 1) have been proved by Kuptsov [80], Garofalo and Lanconelli [51], then by Lanconelli and Polidoro [84]. Later on, Polidoro considers operators $\mathcal{K}$ with Hölder continuous coefficients in [107] and proves mean value formulas for operators $\mathcal{K}$ of this kind under the qualitative assumptions that the coefficients of $\mathcal{K}$ are smooth.

Theorem 1.24 *Let $\mathcal{K}$ be an operator in the form* (1. 33) *satisfying conditions **(H1), (H2)**, assume the coefficients a_{ij} are smooth and that the matrix B has the form* (1. 18). *Let u be a solution to $\mathcal{K}u = 0$ on Ω. Then, for every $z_0 \in \Omega$ such that $\overline{\Omega_r(z_0)} \subset \Omega$, we have*

$$u(z_0) = \frac{1}{r} \int\limits_{\Omega_r(z_0)} M(z_0; z)\, u(z)\, dz.$$

Here

$$M(z_0; z) = \frac{\langle A(z)\, D_x\Gamma(z_0; z)\,,\, D_x\Gamma(z_0; z)\rangle}{\Gamma^2(z_0; z)}. \tag{1. 54}$$

As in Theorem 1.19, the assumption that the matrix B has the form (1. 18) has been introduced to simplify the problem and seems to be unnecessary. We finally remark that mean value formulas analogous to the one stated in Theorem 1.24, where the kernel (1. 54) is replaced by a bounded continuous one, have been proved in [80], [51], [84] and [107]. Lastly, we recall a recent paper by Cupini and Lanconelli [36], where the authors give a general proof of Mean Value formulas for solutions to linear second order PDEs, only based on the local properties of the fundamental solution.

1.5.2 Harnack inequality

The first proofs of Harnack type inequalities for Kolmogorov operators have been derived using mean value formulas, and are due to Kuptsov [80] [81]. This result has been improved by Garofalo and Lanconelli (see Theorem 1.1 in [51]) for some specific constant coefficients operators of the type (1. 1). Their approach follows the ideas introduced for the heat equation by Pini [106] and Hadamard [61] in their seminal works. Later on, Lanconelli and Polidoro proved the Harnack inequality for every operator (1. 1). The statement of this result requires a further notation. For every positive ε we denote

$$K_r(z_0,\varepsilon) := \Omega_r(z_0) \cap \left\{ (x,t) \in \mathbb{R}^{N+1} \mid t \le t_0 - \varepsilon r^{2/Q} \right\}. \tag{1. 55}$$

We recall here Theorem 5.1 in [84].

Theorem 1.25 *Let $\mathcal{K}$ be an operator of the form* (1. 1) *satisfying the equivalent conditions of Proposition 1.1. Then there exist three positive constants $c, r_0 > 0$ and ε, only dependent on $\mathcal{K}$, such that*

$$\sup_{z \in K_r(z_0,\varepsilon)} u(z) \le c u(z_0), \tag{1. 56}$$

for every non negative solution u to $\mathcal{K}u = 0$ in an open subset Ω of $\mathbb{R}^{N+1}$, for every $z_0 \in \Omega$ such that $\overline{\Omega_{2r}(z_0)} \subset \Omega$ and for every $r \in]0, r_0[$.

The same result has been proved in [107] for variable coefficients operators (1. 51) satisfying **(H1)- (H4)**, with B in the form (1. 18). We point out that the geometry of the above Harnack inequality is quite complicated. The natural analogy between the parabolic case and the Kolmogorov case is restored in [84], where the Harnack inequality is written in terms of *cylinders* (see equation (1. 59) below). Here and in the following, we consider the *unit box* $\mathcal{Q}$ defined as

$$\mathcal{Q} =]-1,1[^N \times]-1,0[. \tag{1. 57}$$

Moreover, for given constants $\alpha, \beta, \gamma, \delta$ with $0 < \alpha < \beta < \gamma < 1$ and $0 < \eta < 1$, we set

$$\mathcal{Q}^+ = \delta_{\eta,0} \left( ]-1,1[^N \right) \times]-\alpha, 0[, \quad \mathcal{Q}^- = \delta_{\eta,0} \left( ]-1,1[^N \right) \times]-\gamma, -\beta[. \tag{1. 58}$$

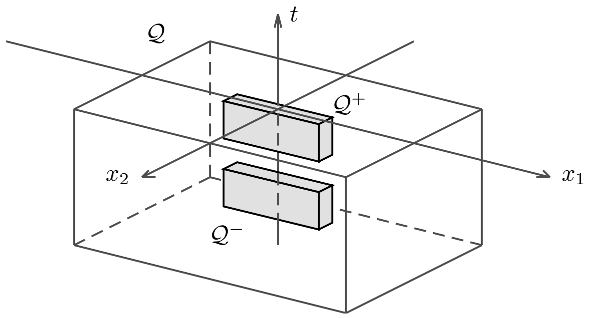

Fig. 2 - Harnack inequality.

Based on the translation and on the dilation respectively defined in (1. 12) and (1. 19), we introduce for every $r > 0$ the cylinders

$$\begin{aligned} \mathcal{Q}_r &:= \delta_r \mathcal{Q} = \{ \delta_r(x,t) \mid (x,t) \in \mathcal{Q} \} \\ \mathcal{Q}_r(x_0,t_0) &:= (x_0,t_0) \circ \mathcal{Q}_r \\ &= \{ (x_0,t_0) \circ \delta_r(x,t) \mid (x,t) \in \mathcal{Q} \} \end{aligned}$$

centered at the origin and at a point $(x_0, t_0) \in \mathbb{R}^{N+1}$, respectively. Analogously, we define

$$\mathcal{Q}_r^+(x_0, t_0) := (x_0, t_0) \circ \delta_r \mathcal{Q}^+, \qquad \mathcal{Q}_r^-(x_0, t_0) := (x_0, t_0) \circ \delta_r \mathcal{Q}^-.$$

Given the above notation, we recall that in Theorem 5.1 of [84] is proved a Harnack inequality analogous to (1. 56), where the sets $\overline{\Omega_{2r}(z_0)}$ and $K_r(z_0, \varepsilon)$ are replaced by cylinders. Specifically, we have

$$\sup_{z \in \mathcal{Q}_r^-(z_0)} u(z) \leq c\, u(z_0), \tag{1. 59}$$

whenever $\overline{\mathcal{Q}_r(z_0)} \subset \Omega$. We next quote the most general Harnack inequality for operators in non-divergence form as defined in (1. 33) proved in [41].

Theorem 1.26 *Let $\mathcal{K}$ be an operator of the form* (1. 33) *satisfying **(H1)-(H3)**. Then there exist positive constants $c, r_0, \alpha, \beta, \gamma$ and δ, only dependent on the parameters of the assumptions **(H1)-(H3)**, such that*

$$\sup_{z \in \mathcal{Q}_r^-(z_0)} u(z) \leq c \inf_{z \in \mathcal{Q}_r^+(z_0)} u(z), \tag{1. 60}$$

for every non negative solution u to $\mathcal{K}u = 0$ in an open subset Ω of $\mathbb{R}^{N+1}$, for every $z_0 \in \Omega$ such that $\overline{\mathcal{Q}_r(z_0)} \subset \Omega$ and for every $r \in]0, r_0[$.

In spite of their local nature, Harnack inequalities are essential tools for the proof of non-local results. Among them, we find the Liouville theorems proved by Kogoj and Lanconelli in [73, 74] and the ones proved by Kogoj, Pinchover and Polidoro in [76]. Moreover, they are also used to derive asymptotic estimates for positive solutions by a repeated application of them. Harnack chains are the tools needed to prove this kind of estimates.

Harnack chain. *We say that a finite sequence $(x_0, t_0), (x_1, t_1), \dots, (x_k, t_k)$ is a* Harnack chain *if there exist positive constants $r_0, r_1, \dots, r_{k-1}$ such that $\mathcal{Q}_{r_j}(x_j, t_j) \subset \Omega$ and $(x_{j+1}, t_{j+1}) \in \mathcal{Q}_{\theta r_j}(x_j, t_j)$ for $j = 0, \dots, k-1$, so that, by the repeated use of the Harnack inequality, we obtain*

$$u(x_k, t_k) \leq c u(x_{k-1}, t_{k-1}) \leq \dots \leq c^k u(x_0, t_0),$$

for every non-negative solution u to $\mathcal{K}u = 0$ in Ω.

In particular, a first application of this tool can be found in the proof of Proposition 1.32 in the following subsection, where Harnack chains are used to prove a geometric version of Theorem 1.26. Further applications can be found in the papers by Polidoro [105], Di Francesco and Polidoro [41], Boscain and Polidoro [21] and Cibelli and Polidoro [28] to obtain asymptotic estimates for the fundamental solution. We also recall the work by Cinti, Nyström and Polidoro [31, 32] where a boundary Harnack inequality is proved.

1.5.3 Strong Maximum Principle

The most general statement of the strong maximum principle for *subsolutions* to Kolmogorov equations is proved by Amano in [3]. It extends the Bony's *maximum propagation principle* [20] to a wide family of possibly degenerate operators with coefficients $a_{ij} \in C^1$, among which we find the ones in the form (1. 33). To our knowledge, a proof of the strong maximum principle for operators of the form (1. 33) with continuous coefficients a_{ij}'s is not available in literature, even though it is expected to be true. For this reason, in the following we derive from Theorem 1.26 a strong maximum principle for *solutions* to $\mathcal{K}u = 0$, assuming that the coefficients a_{ij}'s are Hölder continuous.

In order to state the strong maximum principle, we introduce the notion of $\mathcal{K}$-*admissible curve* and that of $\mathcal{K}$-*admissible set*. Recall that to every operator $\mathcal{K}$ in the form (1. 33) we associate the model operator (1. 34), which can be written in the Hörmander form

$$\sum_{j=1}^{m_0} X_j^2 + Y, \quad \text{with} \quad X_j = \partial_{x_j} \quad \text{for} \quad j = 1, \dots, m_0.$$

Definition 1.27 *Let $\mathcal{K}$ be an operator of the form* (1. 33)*, satisfying assumptions **(H1)-(H3)**. We say that a curve $\gamma : [0,T] \to \mathbb{R}^{N+1}$ is* $\mathcal{K}$-admissible *if is absolutely continuous and*

$$\dot{\gamma}(s) = \sum_{k=1}^{m_0} \omega_k(s) X_k(\gamma(s)) + Y(\gamma(s))$$

for almost every $s \in [0,T]$ and with $\omega_1, \omega_2, \dots, \omega_{m_0} \in L^1[0,T]$.

Definition 1.28 *Let Ω be any open subset of $\mathbb{R}^{N+1}$, and let $\mathcal{K}$ be an operator of the form* (1. 33)*, satisfying assumptions **(H1)-(H3)**. For every point $(x_0,t_0) \in \Omega$ we denote by $\mathscr{A}_{(v_0,x_0,t_0)}(\Omega)$ the* attainable set *defined as*

$$\mathscr{A}_{(v_0,x_0,t_0)}(\Omega) = \left\{ \begin{array}{c} (x,t) \in \Omega \mid \text{ there exists an } \mathcal{K} - \text{admissible curve} \\ \gamma : [0,T] \to \Omega \text{ such that } \gamma(0) = (x_0,t_0) \text{ and } \gamma(T) = (x,t) \end{array} \right\}.$$

Whenever there is no ambiguity on the choice of the set Ω we denote $\mathscr{A}_{(v_0,x_0,t_0)} = \mathscr{A}_{(v_0,x_0,t_0)}(\Omega)$.

We are now in position to state the strong maximum principle.

Theorem 1.29 *Let Ω be any open subset of $\mathbb{R}^{N+1}$, and let $\mathcal{K}$ be an operator of the form* (1. 33)*, satisfying assumptions **(H1)-(H3)**. Let $u \geq 0$ be a solution to $\mathcal{K}u = 0$ in Ω. If $u(x_0,t_0) = 0$ for some point $(x_0,t_0) \in \Omega$, then $u(x,t) = 0$ for every $(x,t) \in \overline{\mathscr{A}_{(v_0,x_0,t_0)}}$.*

We remark that the attainable set $\mathscr{A}_{(v_0,x_0,t_0)}$ strongly depends on the domain Ω. For instance, when Ω agrees with the unit box $\mathcal{Q} =]-1,1[^2 \times]-1,0[$ we have

$$\mathscr{A}_{(0,0,0)} = \Big\{ (x_1,x_2,t) \in \mathcal{Q} \mid |x_1| \leq |t| \Big\}. \tag{1. 61}$$

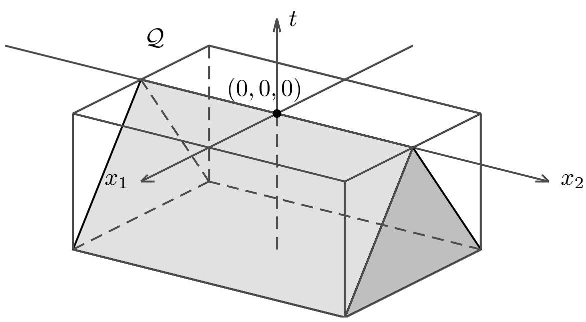

Fig. 3 - $\mathscr{A}_{(0,0,0)}(\mathcal{Q})$ with $\mathcal{Q} =]-1,1[^2 \times]-1,0[$.

For the proof of this fact we refer to [30], Proposition 4.5, p.353. Moreover, the statement of Theorem 1.29 is optimal. Indeed, in Proposition 4.5 of [30] it is also shown that there exists a non-negative solution u to $\mathcal{K}u = 0$ in $\mathcal{Q}$ such that $u(x,t) = 0$ for every $(x,t) \in \overline{\mathscr{A}_{(0,0)}}$, and $u(x,t) > 0$ for every $(x,t) \in \mathcal{Q} \backslash \overline{\mathscr{A}_{(0,0)}}$.

In order to prove Theorem 1.29, we first need to prove the following intermediate result.

Theorem 1.30 *Let $\mathcal{K}$ be an operator of the form* (1. 33) *satisfying **(H1)-(H3)**, and let Ω be an open subset of $\mathbb{R}^{N+1}$. For every $z_0 \in \Omega$, and for any compact set $K \subseteq \operatorname{int}\big(\mathscr{A}_{(v_0,x_0,t_0)}\big)$, there exists a positive constant C_K, only dependent on Ω, z_0, K and on the operator $\mathcal{K}$, such that*

$$\sup_{z \in K} u(z) \ \le \ C_K \, u(z_0),$$

for every non negative solution u to $\mathcal{K}u = 0$ in Ω.

We then obtain, as a direct consequence, the proof of the Strong Maximum Principle stated in Theorem 1.29. In order to achieve this program, we introduce a further notation and we recall a lemma, whose proof can be found in Lemma 2.2 of [21]. Given β, η as in the definition of $\mathcal{Q}^-$ and for every $z \in \mathbb{R}^{N+1}$, $r > 0$ we set

$$\begin{aligned} &\widetilde{\mathcal{Q}} :=]-1,1[^{N+1} \qquad \widetilde{\mathcal{Q}}_r(x_0,t_0) := (x_0,t_0) \circ \delta_r \widetilde{\mathcal{Q}}; \\ &K^- = \delta_{\eta,0}\left( ]-1,1[^N \right) \times \left\{ -\tfrac{\beta+\gamma}{2} \right\} \qquad K_r^-(x_0,t_0) := (x_0,t_0) \circ \delta_r K^-. \end{aligned}$$

Lemma 1.31 *Let $\gamma : [0,T] \to \mathbb{R}^{N+1}$ be an $\mathcal{K}-$admissible path and let a, b be two constants s.t. $0 \le a < b \le T$. Then there exists a positive constant h, only depending on $\mathcal{K}$, such that*

$$\int_a^b |\omega(\tau)|^2 \delta\tau \le h \qquad \Longrightarrow \qquad \gamma(b) \in K_r^-(\gamma(a)), \ \text{ with } \ r = \sqrt{2\frac{b-a}{\beta+\gamma}}.$$

Note that $K_r^-(z)$ is a subset of $\mathcal{Q}_r^-(z)$, then Lemma 1.31 implies $\mathcal{Q}_r^-(\gamma(a))$ is an open neighborhood of $\gamma(b)$. Our first result of this section is a local version of Theorem 1.26, whose proof only relies on the Harnack chains and on Lemma 1.31.

Proposition 1.32 *Let z_0 be a point of Ω, an open subset of $\mathbb{R}^{N+1}$. For every $z \in \operatorname{int}\big(\mathscr{A}_{z_0}\big)$ there exist an open neighborhood U_z of z and a positive constant C_z such that*

$$\sup_{U_z} u \ \le \ c_z \, u(z_0)$$

for every non-negative solution u to $\mathcal{K}u = 0$ in an open subset Ω of $\mathbb{R}^{N+1}$.

Proof. Let z be any point of $\operatorname{int}\big(\mathscr{A}_{z_0}\big)$. We plan to prove our claim by constructing a finite Harnack chain connecting z to z_0. Because of the very definition of $\mathscr{A}_{z_0}$, there exists a $\mathcal{K}-$admissible curve $\gamma : [0,T] \to \Omega$ steering z_0 to z. Our Harnack chain will be a finite subset of $\gamma([0,T])$. As $\widetilde{\mathcal{Q}}_r(x_0,t_0)$ is an open neighborhood of (x_0,t_0), for every $s \in [0,T]$ we can set

$$r(s) \ := \ \sup \left\{ r > 0 \ : \ \widetilde{\mathcal{Q}}_r(\gamma(s)) \subseteq \Omega \right\}.$$

We remark that the function $r(s)$ is continuous, then it is well defined the positive number

$$r_0 \ := \ \min_{s \in [0,T]} r(s). \tag{1. 62}$$

Moreover $\mathcal{Q}_r(\gamma(s)) \subset \widetilde{\mathcal{Q}}_r(\gamma(s))$, then

$$\mathcal{Q}_r(\gamma(s)) \subseteq \Omega \quad \text{for every } s \in [0,T] \quad \text{and } r \in]0, r_0]. \tag{1. 63}$$

On the other hand, we notice that the following function is (uniformly) continuous in $[0,T]$

$$I(s) \ := \ \int_0^s |\omega(\tau)|^2 dt, \tag{1. 64}$$

then there exists a positive η_0 such that $\eta_0 \le \beta r_0$ and that

$$\int_a^b |\omega(\tau)|^2 dt \le h \qquad \text{for every } a,b \in [0,T], \quad \text{such that } 0 < a-b \le \eta_0, \tag{1. 65}$$

where h is the constant appearing in Lemma 1.31.
We are now ready to construct our Harnack chain. Let k be the unique positive integer such that $(k-1)\eta_0 < T$, and $k\eta_0 \ge T$. We define $\{s_j\}_{j\in\{0,1,\dots,k\}} \in [0,T]$ as follows: $s_j = j\eta_0$ for $j = 0,1,\dots,k-1$, and $s_k = T$. As noticed before, the equation (4. 17) allows us to apply Lemma 1.31. We then obtain

$$\gamma(s_{j+1}) \in \mathcal{Q}^-_{r_0}(\gamma(s_j)) \quad\ j=0,\dots,k-2, \qquad \gamma(s_k) \in \mathcal{Q}^-_{r_1}(\gamma(s_{k-1})), \tag{1. 66}$$

for some $r_1 \in]0, r_0]$. We next show that $\left(\gamma(s_j)\right)_{j=0,1,\dots,k}$ is a Harnack chain and we conclude the proof. We proceed by induction. For every $j = 1,\dots,k-2$ we have that $\gamma(s_{j+1}) \in \mathcal{Q}^-_{r_0}(\gamma(s_j))$. From (1. 63) we know that $\mathcal{Q}_{r_0}(\gamma(s_j)) \subseteq \Omega$, then we apply Theorem 1.26 and we find

$$u(\gamma(s_{j+1})) \le \sup_{\mathcal{Q}^-_{r_0}(\gamma(s_j))} u \le c \inf_{\mathcal{Q}^+_{r_0}(\gamma(s_j))} u \le cu(\gamma(s_j).$$

As a consequence, we obtain

$$u(\gamma(s_{k-1})) \le cu(\gamma(s_{k-2})) \le M^2 u(\gamma(s_{k-3})) \le \ldots \le c^{k-1}u(\gamma(0)).$$

We eventually apply Theorem 1.26 to the set $\mathcal{Q}_{r_1}(\gamma(s_{k-1})) \subseteq \Omega$ and we obtain

$$\sup_{U_z} u \ \le \ c^k\, u(z_0),$$

where $U_z = \mathcal{Q}^-_{r_1}(\gamma(s_{k-1}))$. As we noticed above, $\mathcal{Q}^-_{r_1}(\gamma(s_{k-1}))$ is an open neighborhood of $\gamma(T)$. This concludes the proof. □

Proof of Theorem 1.30. Let K be any compact subset of $\operatorname{int}(\mathscr{A}_{z_0})$. For every $z \in K$ we consider the open set U_z appearing in the statement of Proposition 1.32. Clearly we have

$$K \subseteq \bigcup_{z\in K} U_z.$$

Because of its compactness, there exists a finite covering of K

$$K \subseteq \bigcup_{j=1,\dots,m_K} U_{z_j},$$

and Proposition 1.32 yields

$$\sup_{U_{z_j}} u \le C_{z_j}\, u(z_0) \qquad j = 1,\dots,m_K.$$

This concludes the proof of Theorem 1.30, if we choose

$$C_K \ = \max_{j=1,\dots,m_K} C_{z_j}.$$

□

Proof of Theorem 1.29. If u is a non-negative solution to $\mathcal{K}u = 0$ in Ω and K is a compact subset of $\operatorname{int}(\mathscr{A}_{z_0})$, then $\sup_K u \le C_K u(z_0)$. If moreover $u(z_0) = 0$, we have $u(z) = 0$ for every $z \in K$ and, thus, for every $z \in \mathscr{A}_{z_0}$. The conclusion of the proof then follows from the continuity of u. □

Chapter 2

Existence of a Fundamental Solution of PDEs associated to Asian Options

Asian options belong to the family of *path-dependent options* whose payoff depends on the average of the underlying stock price over a certain time interval. In the Black & Scholes framework, the price of the underlying Stock S_t and of the bond B_t are described by the processes

$$S_t = S_0 e^{\mu t + \sigma W_t}, \qquad B_t = B_0 e^{rt}, \qquad 0 \le t \le T,$$

where μ, r, T, and σ are given constants. If the price observations are considered as a set of regularly spaced time points we refer to a discrete Asian Option, otherwise when we consider a continuum of price observations and its average it is computed by means of an integral we have a continuous Asian Option. In particular, in this work we consider continuous Asian Options. In the Black & Scholes setting, the price $(Z_t)_{0\le t\le T}$ of a path dependent option is considered as a function $Z_t = Z(S_t, A_t, t)$ that depends on the *stock price* S_t, the *time to maturity* t and on an averageA_t of the stock price

$$A_t = \int_0^t f(S_\tau)\, d\tau, \qquad t \in [0, T]. \tag{2. 1}$$

From a financial point of view, Asian Options have several advantages. Indeed, they are less expensive than Plain Vanilla Options thanks to the averaging mechanism which allows to reduce their volatility as well. Secondly, they reduce the risk of market manipulation of the underlying instrument at maturity (see [115]). In this sense, Asian Options are suitable to fulfill some of the needs of corporate treasures. We refer to the Black & Scholes [17] and to Merton [95] articles for the seminal works of this theory, and to the books by Björk [16], Hull [65] and Pascucci [102] for a comprehensive treatment of the recent development of this subject. The most common techniques to price path-dependent derivatives are:

- **the Monte Carlo simulations**, relying on the Feynman–Kac formula

$$Z(S, A, t) = \mathbb{E}^{\mathbb{Q}} \left[e^{-r(T-t)} \varphi(S_T, A_T) \,|\, (S_t, A_t) = (S, A) \right], \tag{2. 2}$$

  where $\mathbb{Q}$ is a measure such that the process $e^{-rt} Z_t$ is a martingale under $\mathbb{Q}$, and the fast Fourier transform (see for example [56], [12], [50]). In [56], the authors derived an analytical expression for the Laplace transform in maturity for the continuous call option case when the asset price follows a geometric Brownian motion. However, as pointed out by [49], [44] and [50] the analytical method of [56] can lead to numerical problems for short maturities or small volatilities. These problems are consequences of the slowly decaying oscillatory nature of the integrand for such parameter values (see [50]).

- **The PDE approach**, which has the aim to solve numerically the Cauchy problem associated with the no-arbitrage PDE. Related works following this line are those of [29], [15], [116]. In [37], the author applies a method on conditioning on the geometric mean price. In this case an approximation of Arithmetic Asian option prices is available. In [39], the author derives an accurate approximation formulae for Asian-rate Call options in the Black & Scholes model by a matched asymptotic expansion. In this work we rely on the results proved in [29], where the authors prove via probabilistic techniques the existence of the fundamental solution Γ for the operator $\mathscr{L}$ with smooth coefficients a and b. Moreover, we recall the existence and local regularity results proved by Lanconelli, Pascucci and Polidoro [83], under the assumption that the coefficients a and b belong to some space of Hölder continuous functions.

The results we present here firstly appeared in the paper [5] and follow an analytical approach based on PDEs, since it has several advantages compared to the Monte Carlo approach. As it is stressed by [49], PDE based approaches provide an analytical approximation of the solution in closed-form gives evidence of the explicit dependency of the results on the underlying parameters. Secondly, they produce better and faster sensitivities than Monte Carlo methods.

In order to explain our main results, we introduce some notation. From now on, we consider the stochastic differential equation of the process $(S_t, B_t, A_t)_{t\geq 0}$

$$\begin{cases} dS_t \;=\; \mu(S_t, A_t, t)\, S_t dt \;+\; \sigma(S_t, A_t, t)\, S_t dW_t, \\ dB_t \;=\; r(S_t, A_t, t)\, B_t\, dt \\ dA_t \;=\; f(S_t)\, dt, \end{cases} \tag{2. 3}$$

where $t \in]0, T[$, μ, r and σ depend on S_t, A_t and t. Then we construct the replicating portfolio $(Z_t)_{0\leq t\leq T}$ for the option, we consider it as a function $Z_t = Z(S_t, A_t, t)$ and we apply Itô's formula. Thus, we obtain the following Cauchy problem

$$\begin{cases} \frac{1}{2}\sigma^2(S,A,t)S^2\frac{\partial^2 Z}{\partial S^2} + f(S)\frac{\partial Z}{\partial A} + r(S,A,t)\left(S\frac{\partial Z}{\partial S} - Z\right) + \frac{\partial Z}{\partial t} = 0 & (S,A,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \times]0,T[, \\ Z(S,A,T) = \varphi(S,A) & (S,A) \in \mathbb{R}^+ \times \mathbb{R}^+, \end{cases} \tag{2. 4}$$

where φ is the *payoff* of the Asian Option. Depending on the choice of the function f, either equal to S or to the $\log(S)$, we deal with different kinds of options: Arithmetic Average Asian Options and Geometric Average Asian Options, respectively. As we shall see in the following Section 2.1 and Section 2.2, respectively devoted to Geometric Average Asian Options and Arithmetic Average Asian Options, we can associate to the pricing problem (2. 4) a second order partial differential operator of Kolmogorov type. Thus, our aim is to prove the existence and uniqueness of the fundamental solution for those operators. We improve previous results in that we provide a closed form expression for the solution of the Cauchy problem (2. 4) under weak regularity assumptions on the coefficients of the associated differential operator. In Section 2.3 we present our method, which is based on a limiting procedure, whose convergence relies on some barrier arguments and uniform a priori estimates recently discovered. Moreover, we emphasize that our approach improves the previously known results in that it allows us to consider differential operators with *locally Hölder continuous* coefficients, which is a milder assumption than the usual ones. We will be more specific in the following, as we introduce the required notation. Lastly, our approach can be also applied to more general problems than the one described above. For instance, in a further investigation we will consider the pricing problem for an Option on a basket containing n assets $S_t = (S_t^1, \ldots, S_t^n)$ whose dynamic is

$$dS_t^j \;=\; S_t^j\, \mu_j(S_t, A_t, t) \;+\; S_t^j \sum_{k=1}^{n} \sigma_{jk}(S_t, A_t, t)\, dW_t^k, \qquad j = 1, \ldots, n, \tag{2. 5}$$

where $(W_t^1, ..., W_t^n)_{t\geq 0}$ is a n-dimensional Wiener process and $(A_t)_{t\geq 0}$ is an average of the assets.

2.1 Geometric Average Asian Options

First of all, we address the case of Geometric Average Asian Options, that we recover by choosing $f(S) = \log(S)$ in the formula (2. 1). Through a simple change of variable $v(x, y, t) := Z(e^x, y, T - t)$ we transform the PDE (2. 4), with its final condition, into the following Cauchy problem

$$\begin{cases} \frac{1}{2}\sigma^2(x,y,t)\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) + x\frac{\partial v}{\partial y} + r(x,y,t)\left(\frac{\partial v}{\partial x} - v\right) = \frac{\partial v}{\partial t} \\ v(x,y,0) = \widetilde{\varphi}(x,y), \end{cases} \tag{2. 6}$$

where $\widetilde{\varphi}(x, y) := \varphi(e^x, y)$. Note that, if we assume that $\frac{\partial \sigma}{\partial x}$ is a continuous function, then the differential operator in (2. 6) can be written in its *divergence form*. Precisely, for every sufficiently smooth function u, we have that the PDE in (2. 6) writes as $\mathcal{K}u = 0$, with

$$\mathcal{K}u(x,y,t) = \frac{\partial}{\partial x}\left(a(x,y,t)\frac{\partial u}{\partial x}\right) + b(x,y,t)\,\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} - r(x,y,t)u - \frac{\partial u}{\partial t}. \tag{2. 7}$$

Here $a(x, y, t) = \frac{1}{2}\sigma^2(x, y, t)$ and $b(x, y, t) = r(x, y, t) - \frac{1}{2}\sigma^2(x, y, t) - \sigma(x, y, t)\frac{\partial\sigma(x,y,t)}{\partial x}$. The reason to write $\mathcal{K}$ in this form is that we need apply some results that have been proved only for divergence form operators. We also introduce its formal adjoint $\mathcal{K}^*$, acting on smooth functions $w = w(x, y, t)$ as

$$\mathcal{K}^*w(x,y,t) = \frac{\partial}{\partial x}\left(a(x,y,t)\frac{\partial w}{\partial x}\right) - \frac{\partial}{\partial x}\left(b(x,y,t)\,w\right) - x\frac{\partial w}{\partial y} - r(x,y,t)w + \frac{\partial w}{\partial t}. \tag{2. 8}$$

The operator $\mathcal{K}$ belongs to the class of Kolmogorov operators (1. 1) and (1. 33). Thus, here we simply recall the basic facts necessary for our proof and we refer to Chapter 1 for a survey of results regarding the classical regularity theory for the operator $\mathcal{K}$.

The Cauchy problem (2. 6) has been studied over the years, and the fundamental solution Γ_K associated to the operator $\mathcal{K}$ provides us with a representation formula for its solution (see Theorem 1.15, where we summarize the main results on the existence and bounds for the fundamental solution we have at our disposal). In particular, if $\widetilde{\varphi}$ is a bounded continuous function, then

$$u(x,y,t) = \int\limits_{\mathbb{R}^+\times\mathbb{R}} \Gamma_K(x,y,t;\xi,\eta,0)\,\widetilde{\varphi}(\xi,\eta)\,d\xi\,d\eta, \tag{2. 9}$$

is a classical solution to (2. 6). Kolmogorov wrote in [78] the explicit expression of the fundamental solution Γ_K for the operator $\mathcal{K}$ with constant coefficients σ and r. In this case the function $u(x, y, t) := e^{rt}v\left(x + \left(\frac{1}{2}\sigma^2 - r\right)t, y, t\right)$ is a solution to the Cauchy problem

$$\begin{cases} \mathcal{K}_\lambda u = 0, \\ u(x,y,0) = \widetilde{\varphi}(x,y), \end{cases}$$

where $\lambda = \frac{1}{2}\sigma^2$ and

$$\mathcal{K}_\lambda := \lambda\,\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y} - \frac{\partial u}{\partial t}, \tag{2. 10}$$

Moreover, the fundamental solution Γ_K^λ of the operator $\mathcal{K}_\lambda$ is

$$\Gamma_K^\lambda(x,y,t,\xi,\eta,\tau) = \begin{cases} \frac{\sqrt{3}}{2\lambda\pi(t-\tau)^2}\exp\left(-\frac{\left|x-\xi\right|^2}{4\lambda(t-\tau)} - 3\frac{\left|y-\eta+(t-\tau)\frac{x+\xi}{2}\right|^2}{\lambda(t-\tau)^3}\right) & t > \tau, \\ 0 & t \leq \tau. \end{cases} \tag{2. 11}$$

Thus, we obtain a closed form for the price of the Geometric Average Asian Option in the case of constant volatility σ and interest rate r.

The Levy parametrix method provied us with a fundamental solution for operators in the form $\mathcal{K}$ with Hölder continuous coefficients. This method has been used by several authors [107, 40, 41, 79] and requires a uniform Hölder continuity of the coefficients of $\mathcal{K}$. The definition of the Hölder space C^α_K is given in Definition 1.11, nevertheless for the sake of completeness it will be recall in Section 2.1.2. We will see that a function f belongs to the space C^α_K, with $0 < \alpha \le 1$, if there exists a positive constant M such that

$$|f(x,y,t) - f(\xi,\eta,\tau))| \le M \left(|x-\xi| + \left| y - \eta + (t-\tau)\tfrac{x+\xi}{2} \right|^{\frac{1}{3}} + |t-\tau|^{\frac{1}{2}} \right)^{\alpha}, \tag{2. 12}$$

for every $(x,y,t), (\xi,\eta,\tau) \in \mathbb{R}^3$. On one hand, the intrinsic Hölder space C^α_K associated to $\mathcal{K}$ complies with the fundamental solution Γ^λ written in (2. 11). On the other hand, intrinsic Hölder regularity can be a rather restrictive property, as it has already been pointed out by Pascucci and Pesce in the Example 1.3 of [103]. In particular, Pascucci and Pesce show that, whenever a function $f = f(y)$ only depends on y and belongs to C^α_K, is necessarily constant. As we will see in the sequel, we only require a *local* Hölder regularity of the coefficients of the operator $\mathcal{K}$. This allows us to consider a wider family of continuous functions. More precisely, we consider the following assumption on the coefficients a and b:

($\mathbf{H}_K$) $a, b, r, \frac{\partial a}{\partial x}, \frac{\partial b}{\partial x} \in C^\alpha_{\text{loc}}(\mathbb{R}^3)$. Moreover, there exist two positive constants λ, Λ such that

$$\lambda \le a(x,y,t) \le \Lambda, \quad |b(x,y,t)|, |r(x,y,t)|, \left|\tfrac{\partial a}{\partial x}(x,y,t)\right|, \left|\tfrac{\partial b}{\partial x}(x,y,t)\right| \le \Lambda,$$

for every $(x,y,t) \in \mathbb{R}^3$.

In the above display, C^α_{loc} denotes the usual space of Hölder continuous functions. Proposition 2.5 states that a function f belongs to the space of *locally Hölder continuous* functions $C^\alpha_{K,\text{loc}}$ if, and only if, it belongs to the space C^β_{loc} for some positive β. We are now in position to state our main results concerning the operator $\mathcal{K}$.

Theorem 2.1 *Let us consider the operator $\mathcal{K}$ under the assumption ($\boldsymbol{H}_K$). Then there exists a unique fundamental solution Γ_K of $\mathcal{K}$ in the sense of Definition 2.8. Moreover, the following properties hold:*

1. *Support of Γ_K: for every (x,y,t), $(\xi,\eta,\tau) \in \mathbb{R}^3$ with $t \le \tau$*

$$\Gamma_K(x,y,t;\xi,\eta,\tau) = 0;$$

2. *Reproduction property: for every $(x,y,t), (x_0,y_0,t_0) \in \mathbb{R}^3$ and $\tau \in \mathbb{R}$ with $t_0 < \tau < t$*

$$\Gamma_K(x,y,t;x_0,y_0,t_0) = \int_{\mathbb{R}^2} \Gamma_K(x,y,t;\xi,\eta,\tau)\, \Gamma_K(\xi,\eta,\tau;x_0,y_0,t_0)\, d\xi\, d\eta;$$

3. *Integral of Γ_K: the following bound holds true*

$$e^{-\Lambda(t-\tau)} \le \int_{\mathbb{R}^2} \Gamma_K(x,y,t;\xi,\eta,\tau)\, d\xi\, d\eta \le e^{\Lambda(t-\tau)};$$

4. *Bounds for Γ_K: let $I =]T_0, T_1[$ be a bounded interval, then there exist four positive constants λ^+, λ^-, C^+, C^- such that for every $(x,y,t), (\xi,\eta,\tau) \in \mathbb{R}^3$ with $T_0 < t < T < T_1$*

$$C^- \, \Gamma_K^{\lambda^-}(x,y,t;\xi,\eta,\tau) \le \Gamma_K(x,y,t;\xi,\eta,\tau) \le C^+ \, \Gamma_K^{\lambda^+}(x,y,t;\xi,\eta,\tau).$$

The constants λ^+, λ^-, C^+, C^- depend only on $\mathcal{K}$ and $T_1 - T_0$. $\Gamma_K^{\lambda^-}$ and $\Gamma_K^{\lambda^+}$ respectively denote the fundamental solution of $\mathcal{K}_{\lambda^-}$ and $\mathcal{K}_{\lambda^+}$, defined in (2. 11) *and* (2. 10) *respectively.*

Moreover, the function $\Gamma_K{}^*(\xi, \eta, \tau; x, y, t) = \Gamma_K(x, y, t; \xi, \eta, \tau)$ *is the fundamental solution of the adjoint operator* $\mathcal{K}^*$ *with pole at* (ξ, η, τ) *and satisfies all of the previous properties accordingly.*

We remark that in the Black & Scholes setting it is natural to consider the Cauchy problem (2. 4) with an unbounded initial condition φ that grows linearly. After the change of variable $v(x, y, t) := Z(e^x, y, T - t)$, it corresponds to an *exponential growth* for the Cauchy data $\widetilde{\varphi}$. As we will see in Remark 2.21, the formula (2. 9) supports initial data satisfying the following condition

$$|\widetilde{\varphi}(x, y)| \leq M \exp(C|(x, y)|^\alpha), \qquad (x, y) \in \mathbb{R}^2, \tag{2. 13}$$

for some positive constants M, C and α, with $\alpha < 2$. Note that if we consider $\alpha = 2$ then the solution to the Cauchy problem (2. 4) is defined in a suitably small interval of time. Moreover, the following uniqueness result holds.

Theorem 2.2 *Let us consider the operator* $\mathcal{K}$ *under the assumption* ***(H***$_K$***)***. *Let* u_1 *and* u_2 *be classical solutions to*

$$\begin{cases} \mathcal{K}u = 0, & (x, y, t) \in \mathbb{R}^2 \times]t_0, T] \\ u(x_0, y_0, t_0) = \varphi(x_0, y_0) & (x_0, y_0) \in \mathbb{R}^2. \end{cases} \tag{2. 14}$$

in the sense of Definition 2.6, and

$$|u_1(x, y, t)| + |u_2(x, y, t)| \leq M \exp\left(C(x^2 + y^2)\right),$$

for some positive consants M *and* C*, then* $u_1 = u_2$ *in* $\mathbb{R}^2 \times]t_0, T]$.

2.1.1 Geometric setting and fundamental solution for $\mathcal{K}$

In this section we recall some basic notions, already introduce and thoroughly presented in Chapter 1, regarding the geometric setting suitable for the study of the Kolmogorov operator $\mathcal{K}$, and some well known results concerning its fundamental solution Γ_K. Firstly, let us consider the operator $\mathcal{K}_1$ defined in (2. 10) with $\lambda = 1$:

$$\mathcal{K}_1 := \frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y} - \frac{\partial u}{\partial t}. \tag{2. 15}$$

Even tough it is a strongly degenerate operator, it is *hypoelliptic* in the following sense. Let u be a distributional solution of $\mathcal{K}_1 u = f$ in $\Omega \subset \mathbb{R}^3$, then

$$u \in C^\infty(\Omega) \quad \text{whenever} \quad f \in C^\infty(\Omega). \tag{2. 16}$$

Hörmander introduced in his seminal paper [64] a simple sufficient condition to check the hypoellipticity of any second order linear differential operator defined on some open set $\Omega \subset \mathbb{R}^{N+1}$ that can be written as a sum of squares of smooth vector fields $X_0, X_1, \ldots, X_m$, as follows

$$\sum_{i=1}^m X_i^2 + X_0. \tag{2. 17}$$

The celebrated hypoellipticity result due to Hörmander reads as follows.

Theorem (Hörmander hypoellipticity condition). *Let us consider the operator* (2. 17). *If* $\mathrm{Lie}\{X_0, X_1, \ldots, X_m\}(x, y, t) = \mathbb{R}^{N+1}$ *at every* $(x, y, t) \in \Omega$*, then* $\sum_{i=1}^m X_i^2 + X_0$ *is hypoelliptic.*

We recall that the notation $\mathrm{Lie}\{X_0, X_1, \ldots, X_m\}(x, t)$ denotes the vector space generated by the vector fields $\{X_0, X_1, \ldots, X_m\}$ and their commutator. The *commutator* of two vector fields W and Z acting on $u \in C^\infty(\Omega)$ is defined as $[W, Z]u := WZu - ZWu$.

As far as we are concerned with the operator $\mathcal{K}_1$ defined in (2. 15), we can write it as follows

$$\mathcal{K}_1 \,=\, X^2 \,+\, Y, \tag{2. 18}$$

where

$$X = \frac{\partial}{\partial x} \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad Y = x\frac{\partial}{\partial y} - \frac{\partial}{\partial t} \sim \begin{pmatrix} 0 \\ x \\ -1 \end{pmatrix} \quad \text{and} \quad [X,Y] = \frac{\partial}{\partial y} \sim \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{2. 19}$$

Hence, the vector fields X, Y and $[X,Y]$ form a basis of $\mathbb{R}^3$ at every point $(x,y,t) \in \mathbb{R}^3$, so that K_1 satisfies the Hörmander's rank condition.

The commutators are strongly related to a non-Euclidean invariant structure for the Kolmogorov operator, as was firstly pointed out by Garofalo and Lanconelli in [51]. Later on, a non commutative algebraic structure was introduced by Lanconelli and Polidoro in [84] to replace the Euclidean one in the study of Kolmogorov operators (2. 15).

Lie group. *Consider an operator $\mathcal{K}_1$ in the form* (2. 15)*. Then* $\mathbb{G} = (\mathbb{R}^3, \bullet)$*,*

$$(x_0, y_0, t_0) \bullet (x, y, t) = (x_0 + x, y_0 + y - tx_0, t_0 + t). \tag{2. 20}$$

is a group with zero element $(0,0,0)$*, and inverse* $(x,y,t)^{-1} := (-x, -y - tx, -t)$*.*

Indeed, if we set $v(x,y,t) = u(x_0 + x, y_0 + y - tx_0, t_0 + t)$, then

$$\mathcal{K}_1 v = 0 \quad \text{if, and only if,} \quad \mathcal{K}_1 u = 0.$$

Moreover, the operator $\mathcal{K}_1$ is invariant with respect to the following family of dilations of $\mathbb{R}^3$

$$\delta_r(x,y,t) = (rx, r^3 y, r^2 t) \qquad \text{for every } r > 0, \tag{2. 21}$$

in the sense that if we set $v(x,y,t) = u(rx, r^3 y, r^2 t)$, then

$$\mathcal{K}_1 v = 0 \quad \text{if, and only if,} \quad \mathcal{K}_1 u = 0.$$

We now introduce a quasi-distance invariant with respect to the group operation "$\bullet$".

Definition 2.3 *For every $z = (x,y,t), \zeta = (\xi, \eta, \tau) \in \mathbb{R}^3$, we define a quasi-distance $d_K(z,\zeta)$ invariant with respect to the translation group $\mathbb{G}$ as follows*

$$d_K(z,\zeta) = \Big|x - \xi\Big| \,+\, \Big|y - \eta + (t-\tau)\tfrac{x+\xi}{2}\Big|^{\frac{1}{3}} \,+\, \Big|t - \tau\Big|^{\frac{1}{2}}.$$

Here we recall the meaning of quasi-distance $d_K : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, +\infty[$:

1. $d_K(z,w) = 0$ if and only if $z = w$ for every $z, w \in \mathbb{R}^3$;
2. $d_K(z,w) = d_K(w,z)$;
3. for every $z, w, \zeta \in \mathbb{R}^3$ there exists a constant $C > 0$ such that (see Lemma 2.1 of [40])

$$d_K(z,w) \le C\left(d_K(z,\zeta) + d_K(\zeta,w)\right).$$

Moreover, we remark that the quasi-distance d_K is homogeneous of degree 1 with respect to the family of dilations $\{\delta_r\}_{r>0}$ in the sense that for every $z, \zeta \in \mathbb{R}^3$

$$d_K\left(\delta_r(z), \delta_r(\zeta)\right) \,=\, r\left(d_K(z,\zeta)\right) \qquad \text{for every } r > 0.$$

We are now in position to define the space of Hölder continuous functions C^α_K.

Definition 2.4 *Let α be a positive constant, $\alpha \leq 1$, and let Ω be an open subset of $\mathbb{R}^3$. We say that a function $f : \Omega \longrightarrow \mathbb{R}$ is Hölder continuous with exponent α in Ω with respect to the group $\mathbb{G} = (\mathbb{R}^3, \bullet, \delta_r)$ (in short: Hölder continuous with exponent α, $f \in C^\alpha_K(\Omega)$) if there exists a positive constant $C > 0$ such that*

$$|f(z) - f(\zeta)| \leq C\, d_K(z,\zeta)^\alpha \qquad \text{for every } z, \zeta \in \Omega.$$

To every bounded function $f \in C^\alpha_K(\Omega)$ we associate the norm

$$|f|_{\alpha,\Omega} \;=\; \sup_\Omega |f| \;+\; \sup_{\substack{z,\zeta\in\Omega \\ z\neq\zeta}} \frac{|f(z) - f(\zeta)|}{d_K(z,\zeta)^\alpha}.$$

Moreover, we say a function f is locally Hölder continuous, and we write $f \in C^\alpha_{K,\mathrm{loc}}(\Omega)$, if $f \in C^\alpha_K(\Omega')$ for every compact subset Ω' of Ω.

We recall the following result, due to Manfredini (see p. 833 in [90]), where the space C^α_K is compared with the usual Euclidean Hölder space C^α.

Proposition 2.5 *Let Ω be a bounded subset of $\mathbb{R}^3$. If $f \in C^\alpha(\Omega)$ in the usual Euclidean sense, then $f \in C^\alpha_K(\Omega)$ in the sense of Definition 1.11. Vice versa, if $f \in C^\alpha_K(\Omega)$, then $f \in C^\beta(\Omega)$ in the Euclidean sense with $\beta = \frac{\alpha}{3}$.*

We remark that the local Hölder regularity assumption we assume on the coefficients of the operator $\mathcal{K}$ is less restrictive than the global Hölder regularity, as pointed out by Pascucci and Pesce (see Example 1.3, [103]). Indeed, for every $y, \eta, t, \tau \in \mathbb{R}$ with $t \neq \tau$, let us consider the following couple of points in $\mathbb{R}^3$

$$z = \left(\frac{\eta - y}{t-\tau}, y, t\right) \qquad \text{and} \qquad \zeta = \left(\frac{\eta - y}{t-\tau}, \eta, \tau\right), \tag{2. 22}$$

then we have $d(z,\zeta) = |t-\tau|^{\frac{1}{2}}$. Since y and η are arbitrary real numbers, we see that points in $\mathbb{R}^3$ that are far from each other in the Euclidean sense can be very close with respect to the distance d. It follows that, if a function $f(x,y,t) = f(y)$ depends only on y and it belongs to $C^\alpha_K(\mathbb{R}^3)$ for some positive α, then it must be constant. In fact, for z, ζ as defined in (2. 22), we have

$$|f(y) - f(\eta)| \;=\; |f(z) - f(\zeta)| \;\leq\; C|t-\tau|^\alpha$$

for some positive constants C, α and for any $y, \eta \in \mathbb{R}$ and $t \neq \tau$.

2.1.2 Hölder continuous coefficients

The results we present in this chapter regards classical solutions to the equation $\mathcal{K}u = f$. In this section we recall the definition of Lie derivative, classical solution for the equation $\mathcal{K}u = f$ and fundamental solution. We first recall the notion of Lie derivative Yu of a function u with respect to the vector field Y defined in (2. 19):

$$Yu(x,y,t) := \lim_{s\to 0} \frac{u(\gamma(s)) - u(\gamma(0))}{s}, \qquad \gamma(s) = (x, y + sx, t - s). \tag{2. 23}$$

Note that γ is the integral curve of Y, i.e. $\dot\gamma(s) = Y(\gamma(s))$. Clearly, if $u \in C^1(\Omega)$, with Ω open subset of $\mathbb{R}^3$, then $Yu(x,y,t)$ agrees with $x\partial_y u(x,y,t) - \partial_t u(x,y,t)$ considered as a linear combination of the derivatives of u.

Definition 2.6 *A function u is a solution to the equation $\mathcal{K}u = f$ in a domain Ω of $\mathbb{R}^3$ if the derivatives $\partial_x u, \partial_x^2 u$ and the Lie derivative Yu exist as continuous functions in Ω, and the equation*

$$\frac{\partial}{\partial x}\left(a(x,y,t)\frac{\partial u}{\partial x}\right) + b(x,y,t)\,\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} - r(x,y,t)u - \frac{\partial u}{\partial t} = f(x,y,t)$$

is satisfied at any point $(x,y,t) \in \Omega$. Moreover, we say that u is a classical supersolution *to $\mathcal{K}u = 0$ if $f \leq 0$ in Ω, and we write $\mathcal{K}u \leq 0$. We say that u is a* classical subsolution *if $-u$ is a classical supersolution.*

Fundamental tools in the classical regularity theory for Partial Differential Equations are the Schauder estimates. In particular, we recall the result proved by Manfredini in [90] (see Theorem 1.4) for classical solutions to $\mathcal{K}u = f$, where the natural functional setting is

$$C_K^{2+\alpha}(\Omega) = \left\{u \in C_K^{\alpha}(\Omega) \ \mid \ \partial_x u, \partial_x^2 u, Yu \in C_K^{\alpha}(\Omega)\right\},$$

and $C_K^{\alpha}(\Omega)$ is given in Definition 2.4. Moreover, if $u \in C_K^{2+\alpha}(\Omega)$ then we define the norm

$$|u|_{2+\alpha,\Omega} := |u|_{\alpha,\Omega} \ + \ |\partial_x u|_{\alpha,\Omega} \ + \ |\partial_x^2 u|_{\alpha,\Omega} \ + \ |Yu|_{\alpha,\Omega}.$$

Clearly, the definition of $C_{K,\mathrm{loc}}^{2+\alpha}(\Omega)$ follows straightforwardly from the definition of $C_{K,\mathrm{loc}}^{\alpha}(\Omega)$.

Theorem 2.7 *Let us consider an operator $\mathcal{K}$ of the type* (2. 7) *satisfying assumptions **($H_{\mathcal{K}}$)** with $0 < \alpha \leq 1$. Let Ω be an open subset of $\mathbb{R}^3$, $f \in C_{K,\mathrm{loc}}^{\alpha}(\Omega)$ and let u be a classical solution to $\mathcal{K}u = f$ in Ω. Then for every $\Omega^{'} \subset\subset \Omega^{''} \subset\subset \Omega$ there exists a positive constant C such that*

$$|u|_{2+\alpha,\Omega'} \leq C\Big(\ \sup_{\Omega''} |u| \ + \ |f|_{\alpha,\Omega''}\Big).$$

We refer to the survey paper [6] for a more recent bibliography on this subject, and we recall that for further information the topic of this section is extensively treated in Chapter 1 of this work. We also recall the notion of fundamental solution.

Definition 2.8 *We say a function $\Gamma_K : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is a fundamental solution for $\mathcal{K}$ if*

1. *for every $(x_0, y_0, t_0) \in \mathbb{R}^3$ the function $x \mapsto \Gamma_K(x,y,t;x_0,y_0,t_0)$:*
   (a) belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^3)$;

   (b) is a classical solution of $\mathcal{K}u = 0$ in $\mathbb{R}^3 \setminus \{(x_0,y_0,t_0))\}$ in the sense of Definition 2.6;
2. *for every bounded function $\varphi \in C(\mathbb{R}^2)$, we have that*

$$u(x,y,t) = \int_{\mathbb{R}^2} \Gamma_K(x,y,t;\xi,\eta,t_0)\,\varphi(\xi,\eta)\,d\xi\,d\eta,$$

   is a classical solution of the Cauchy problem (2. 14)*;*
3. *The function $\Gamma_K^*(x,y,t;x_0,y_0,t_0) := \Gamma_K(x_0,y_0,t_0;x,y,t)$ satisfies 1. and 2. with $\mathcal{K}$ replaced by its adjoint operator $\mathcal{K}^*$ as defined in* (2. 8)*.*

The existence of a fundamental solution Γ_K for the operator $\mathcal{K}$ has widely been investigated over the years, and as we have already pointed out in the Introduction of this paper the Levy parametrix method provides us with a fundamental solution for the operator $\mathcal{K}$ under global Hölder regularity assumptions for the coefficients. Among the first results of this type we recall [121], [67] and [114]. We summarize here the main results of the articles [107], [40] and [83] on the existence and bounds for the fundamental solution under the following assumption for the coefficients of the operator $\mathcal{K}$:

$a, \frac{\partial a}{\partial x}, b, r \in C^{\alpha}_K(\mathbb{R}^3)$ and there exist two positive constants λ, Λ such that

$$\lambda \leq a(x,y,t) \leq \Lambda \quad \left|\tfrac{\partial a}{\partial x}\right|, |b(x,y,t)|, |r(x,y,t)| \leq \Lambda \quad \text{for every} \quad (x,y,t) \in \mathbb{R}^3. \tag{2. 24}$$

For more references on this subject we refer to the survey paper [6].

Theorem 2.9 *Let $\mathcal{K}$ be an operator of the form* (2. 7) *under the assumption* (2. 24)*. Then there exists a fundamental solution Γ_K in the sense of Definition 2.8. Moreover, for every $(x_0,y_0,t_0) \in \mathbb{R}^3$, Γ_K belongs to $C^{2+\alpha}_{\text{loc}}(\mathbb{R}^3 \setminus \{(x_0,y_0,t_0)\})$ and the following properties hold:*

1. *Support of Γ_K: for every $(x,y,t),(\xi,\eta,\tau) \in \mathbb{R}^3$ with $t \leq \tau$*
$$\Gamma_K(x,y,t;\xi,\eta,\tau) = 0;$$
2. *Reproduction property: for every $(x,y,t),(x_0,y_0,t_0) \in \mathbb{R}^3$ and $\tau \in \mathbb{R}$ with $t_0 < \tau < t$:*
$$\Gamma_K(x,y,t;x_0,y_0,t_0) = \int_{\mathbb{R}^2} \Gamma_K(x,y,t;\xi,\eta,\tau)\,\Gamma_K(\xi,\eta,\tau;x_0,y_0,t_0)\,d\xi\,d\eta;$$
3. *Integral of Γ_K: the following bound holds true*
$$e^{-\Lambda(t-\tau)} \;\leq\; \int_{\mathbb{R}^2} \Gamma_K(x,y,t;\xi,\eta,\tau)\,d\xi\,d\eta \;\leq\; e^{\Lambda(t-\tau)}; \tag{2. 25}$$
4. *let $I =]T_0,T_1[$ be a bounded interval, then there exist four positive constants λ^+, λ^-, C^+, C^- such that for every $(x,y,t),(\xi,\eta,\tau) \in \mathbb{R}^3$ with $T_0 < t < \tau < T_1$*
$$C^-\,\Gamma_K^{\lambda^-}(x,y,t;\xi,\eta,\tau) \;\leq\; \Gamma_K(x,y,t;\xi,\eta,\tau) \;\leq\; C^+\,\Gamma_K^{\lambda^+}(x,y,t;\xi,\eta,\tau). \tag{2. 26}$$
The constants λ^+, λ^-, C^+, C^- depend only on $\mathcal{K}$ and $T_1 - T_0$. $\Gamma_K^{\lambda^-}$ and $\Gamma_K^{\lambda^+}$ respectively denote the fundamental solution of $\mathcal{K}_{\lambda^-}$ and $\mathcal{K}_{\lambda^+}$, defined in (2. 11) *and* (2. 10) *respectively.*

Furthermore, for every $(x_0,y_0,t_0) \in \mathbb{R}^3$ also $\Gamma_K{}^*$ belongs to $C^{2+\alpha}_{\text{loc}}(\mathbb{R}^3 \setminus \{(x_0,y_0,t_0)\})$.

The properties *1.* and *2.* of the above statement have been proved in [107] and [40]. The inequalities (2. 25) follow from the comparison principle for classical solutions, as the functions $e^{-\Lambda(t-\tau)}$ and $e^{\Lambda(t-\tau)}$ are respectively subsolution and supersolution to the Cauchy problem (2. 14) with inital datum $\varphi(x,y) = 1$. We remark that the constants λ^+, λ^-, C^+, C^- appearing in the bounds (2. 26) proved in [107, 40] also depend on the $C^\alpha(\mathbb{R}^N \times I)$ norm of the coefficients $a, \frac{\partial a}{\partial x}, b, r$. We rely here on the bounds proved in [83], where the dependence on the regularity of the coefficients is removed thanks to the Harnack inequality proved by Golse, Imbert, Mouhot and Vasseur in [58]. We conclude this section with the following Gaussian bound for Γ_K.

Corollary 2.10 *Let $(x,y,t) \in \mathbb{R}^3$, with $t > t_0$. Then there exist two positive constants $\overline{C}$, only depending on the operator $\mathcal{K}$, and R_0, also depending on (x,y,t), on t_0, such that*

$$\Gamma_K\,(x,y,t;\xi,\eta,\tau) \leq \overline{C} e^{-\overline{C}\,\frac{\xi^2+\eta^2}{t-\tau}},$$

for every $(\xi,\eta) \in \mathbb{R}^2$ such that $\|(\xi,\eta)\| \geq R_0$ and for every $\tau \in \mathbb{R}$ with $t_0 < \tau < t$.

The proof of this result directly follows from the upper bound (2. 26) combined with the explicit expression of the fundamental solution (2. 11). We refer to the Lemma 3.1 of [108] for the proof, that will be omitted here.

2.2 Arithmetic Average Asian Options

Now, we address the case of Arithmetic Average Asian Options, that we recover by choosing $f(S) = S$. Through the change of variable $v(x, y, T - t) := Z(x, y, t)$ we transform the Cauchy problem (2. 4) into the following

$$\begin{cases} \frac{1}{2}\sigma^2(x,y,t)x^2\frac{\partial^2 v}{\partial x^2} + r(x,y,t)\left(x\frac{\partial v}{\partial x} - v\right) + x\frac{\partial v}{\partial y}(x,y,t) = \frac{\partial v}{\partial t} \\ v(x,y,0) = \varphi(x,y). \end{cases}$$

As we did in the previous section for the case of Geometric Average Asian Options, we write the operator appearing in the above PDE in its divergence form

$$\begin{cases} x\,\frac{\partial}{\partial x}(a(x,y,t)\,x\,\frac{\partial v}{\partial x}) + b(x,y,t)\,x\,\frac{\partial v}{\partial x} - r(x,y,t)v + x\,\frac{\partial v}{\partial y} = \frac{\partial v}{\partial t}, \\ v(x,y,0) = \varphi(x,y), \end{cases} \tag{2. 27}$$

where $a(x,y,t) = \frac{1}{2}\sigma^2(x,y,t)$ and $b(x,y,t) = r(x,y,t) - 2xa(x,y,t) - x^2\partial_x a(x,y,t)$. Note that the coefficient x in front of the derivative $\frac{\partial}{\partial x}$ introduces new difficulties. We denote by $\mathscr{L}$ the differential operator in (2. 27), acting on sufficiently smooth functions $u = u(x,y,t)$ as

$$\mathscr{L}u(x,y,t) := x\,\frac{\partial}{\partial x}\left(a(x,y,t)\,x\,\frac{\partial u}{\partial x}\right) + b(x,y,t)\,x\,\frac{\partial u}{\partial x} + x\,\frac{\partial u}{\partial y} - r(x,y,t)u - \frac{\partial u}{\partial t}, \tag{2. 28}$$

and its formal adjoint $\mathscr{L}^*$, acting on differentiable functions $w = w(x,y,t)$ as follows

$$\mathscr{L}^* w(x,y,t) := x\,\frac{\partial w}{\partial x}\left(a(x,y,t)\,x\,\frac{\partial w}{\partial x}\right) - \frac{\partial}{\partial x}\left(x\,b(x,y,t)\,w\right) - x\,\frac{\partial}{\partial y}w - r(x,y,t)w + \frac{\partial}{\partial t}w. \tag{2. 29}$$

The simplest form of the operator $\mathscr{L}$ is associated to the stochastic process $(S_t, A_t)_{t\geq 0}$

$$S_t = S_0 e^{\mu t + \sigma W_t}, \qquad A_t = \int_0^t \exp(\mu s + \sigma W_s)ds, \tag{2. 30}$$

where W_t is a real valued Brownian motion starting from 0. Indeed, when $\mu = 0$ and σ is a positive constant we can consider the following model operator

$$\mathscr{L}_\lambda := \lambda\,x^2\partial_x^2 + x\partial_x + x\partial_y - \partial_t, \tag{2. 31}$$

where $\lambda = \frac{1}{2}\sigma^2$. As it is pointed out by Yor in [123], thanks to the scaling invariance properties of the Brownian motion we can restrict ourselves to the case where $\sigma = \sqrt{2}$, for which he proves the existence of the transition density of the associated process $(S_t, A_t)_{t\geq 0}$, which reads as follows

$$p(w,y,t) = \frac{e^{\frac{\pi^2}{2t}}}{\pi\sqrt{2\pi t}}\,\exp\left(-\frac{1+e^{2w}}{2y}\right)\,\frac{e^w}{y^2}\,\Theta\left(\frac{e^w}{y}, t\right),$$

where

$$\Theta(z,t) = \int_0^\infty e^{-\frac{\xi^2}{2t}}\,e^{-z\cosh(\xi)}\,\sinh(\xi)\,\sin\left(\frac{\pi\xi}{t}\right)\,d\xi.$$

Thus, the explicit expression of the fundamental solution Γ_L^1 associated with the operator

$$\mathscr{L}_1 := x^2\partial_x^2 + x\partial_x + x\partial_y - \partial_t, \tag{2. 32}$$

reads as follows

$$\Gamma_L^1(x,y,t;x_0,y_0,t_0) = \frac{1}{2xx_0}\, p\left(\frac{1}{2}\log\left(\frac{x_0}{x}\right), \frac{y-y_0}{2x}, \frac{t-t_0}{2}\right). \tag{2. 33}$$

Thus, the pricing problem for the simplest case of Arithmetic Average Asian Options can be solved by the argument outlined in the previous subsection when considering (2. 6), but in this case the differential operator $\mathcal{K}$ needs to be replaced by $\mathscr{L}_\lambda$. As we can see from the explicit expression (2. 33) of the fundamental solution Γ_L^λ of $\mathscr{L}_\lambda$, and as several authors point out (for instance, see [2, 44, 49, 50, 113]), the explicit representation of the Asian option prices given by Geman and Yor in [56] is hardly numerically treatable, in particular when pricing Asian Options with short maturities or small volatilities. We quote [124, 56] for an exhaustive presentation of the topic, other related works are due to Matsumoto, Geman and Yor [94, 56, 93], Carr and Schröder [26], Bally and Kohatsu-Higa [13].

As we have already pointed out at the beginning of the Introduction, in this work we consider the operator $\mathscr{L}$ with variable coefficients. This allows one to deal with more general market models, but the mathematical theory for this kind of operator $\mathscr{L}$ is nowadays still incomplete. Indeed, the unique result available on the existence of a fundamental solution for $\mathscr{L}$ has been proved by Cibelli, Polidoro and Rossi in [29] and requires the C^∞ smoothness of the coefficients a and b. Moreover, only the case $r = 0$ is considered in [29]. Our research weakens the regularity requirements on the coefficients in that only the *local* Hölder continuity is needed to produce classical solutions to the pricing problem. The class of Hölder continuous functions C_L^α related to the operator $\mathscr{L}$ is strongly linked to the definition of the space C_K^α related to the operator $\mathcal{K}$, as we will see in the sequel of this article. Moreover, in the following we prove that *locally* the two definitions coincide (see Proposition 2.16). This allows us to consider a wider family of continuous functions, since the local Hölder condition is really easy to check and less restrictive that the *global* Hölder continuity assumption, required for instance by the parametrix method, that is an alternative approach to produce a fundamental solution. We are now ready to state the precise assumption for the coefficients a and b of the operator $\mathscr{L}$.

($\mathbf{H}_L$) $a, b, \frac{\partial a}{\partial x}, \frac{\partial b}{\partial x} \in C^\alpha_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$. Moreover, there exist two positive constants λ, Λ such that

$$\lambda \le a(x,y,t) \le \Lambda, \quad |b(x,y,t)|, \left|\tfrac{\partial}{\partial x}\left(x\, a(x,y,t)\right)\right|, \left|\tfrac{\partial}{\partial x}\left(x\, b(x,y,t)\right)\right| \le \Lambda,$$

for every $(x,y,t) \in \mathbb{R}^+ \times \mathbb{R}^2$.

Remark 2.11 *As said above, we only consider the operator $\mathscr{L}$ without the zero order term r, as we rely on the results proved in [29], where this condition was assumed. However a simple change of function allows us to consider any continuous function $r = r(t)$ only depending on t. Indeed, if u is a solution to $\mathscr{L}u = 0$, where the term r doesn't appear in $\mathscr{L}$, then the function*

$$v(x,y,t) = e^{R(t)}u(x,y,t), \qquad R(t) = \int_{t_0}^t r(s)ds,$$

solves the equation $\mathscr{L}v(x,y,t) - r(t)v(x,y,t) = 0$.

We are now in position to state our main results regarding the operator $\mathscr{L}$.

Theorem 2.12 *Let us consider the operator $\mathscr{L}$ under the assumption ($\boldsymbol{H}_L$). Then there exists a unique fundamental solution Γ_L of $\mathscr{L}$ in the sense of Definition 2.18. Moreover, the following properties hold:*

1. *Support of Γ_L: for every (x,y,t), $(\xi,\eta,\tau) \in \mathbb{R}^+ \times \mathbb{R}^2$ with $t \le \tau$ or $y \ge \eta$*

$$\Gamma_L(x,y,t;\xi,\eta,\tau) = 0;$$

2. *Reproduction property: for every* (x, y, t), $(x_0, y_0, t_0) \in \mathbb{R}^+ \times \mathbb{R}^2$ *and* $\tau \in \mathbb{R}$ *with* $t_0 < \tau < t$

$$\Gamma_L(x, y, t; x_0, y_0, t_0) = \int\limits_{\mathbb{R}^+ \times \mathbb{R}} \Gamma_L(x, y, t; \xi, \eta, \tau)\, \Gamma_L(\xi, \eta, \tau; x_0, y_0, t_0)\, d\xi\, d\eta;$$

3. *Integral of* Γ_L*: there exists a positive constant* $\overline{C}$ *depending on* $t - \tau$ *and such that* $\overline{C} \to 1$ *as* $t \to \tau$ *for which*

$$\int\limits_{\mathbb{R}^+ \times \mathbb{R}} \Gamma_L(x, y, t; \xi, \eta, \tau)\, d\xi\, d\eta \;=\; 1, \qquad \int\limits_{\mathbb{R}^+ \times \mathbb{R}} \Gamma_L(x, y, t; \xi, \eta, \tau)\, dx\, dy \;=\; \overline{C};$$

4. *Bounds for* Γ_L*: for every* $\varepsilon \in]0, 1[$*, and* $T > 0$ *there exist two positive constants* $c_\varepsilon^-, C_\varepsilon^+$ *depending on* ε*, on* T *and on the operator* $\mathscr{L}$*, and two positive constants* C^-, c^+*, only depending on the operator* $\mathscr{L}$*, such that*

$$\frac{c_\varepsilon^-}{x_0^2(t - t_0)^2} \exp\left(-C^- \psi(x, y + x_0\varepsilon(t - t_0), t - \varepsilon(t - t_0); x_0, y_0, t_0)\right) \leq$$
$$\leq \Gamma_L(x, y, t; x_0, y_0, t_0) \leq \frac{C_\varepsilon^+}{x_0^2(t - t_0)^2} \exp\left(-c^+ \psi(x, y - x_0\varepsilon, t + \varepsilon; x_0, y_0, t_0)\right)$$

for every $(x, y, t) \in \mathbb{R}^+ \times]-\infty, y_0 - x_0\varepsilon(t - t_0)[\times]t_0, T]$*, where* ψ *is the value function for the optimal control problem* (2. 39)*.*

Moreover, the function ${\Gamma_L}^*(\xi, \eta, \tau; x, y, t) = \Gamma_L(x, y, t; \xi, \eta, \tau)$ *is the fundamental solution of the adjoint operator* $\mathscr{L}^*$ *with pole at* (ξ, η, τ) *and satisfies all of the previous properties accordingly.*

We note that the upper and lower bounds in the above point *4.* can be written in terms of the function (2. 33) as stated in Corollary 2.20 below. As in the case of Geometric Average Asian Options, we consider the Cauchy problem (2. 4) with an initial condition φ that grows linearly. However, in the case of Arithmetic Average Asian Options the change of variable $v(x, y, t) = Z(e^x, y, T - t)$ doesn't simplify the proof of our results. Therfore we don't apply it and we keep the linear growth as the natural assumption on the function φ. We will see in Remark 2.24 that the formula (2. 42) supports initial data satisfying this condition. As far as we are concerned with the uniqueness of the solution to the Cauchy problem for operators of the form (2. 28), we have the following result.

Theorem 2.13 *Let us consider the operator* $\mathscr{L}$ *under the assumption* ***(H***$_L$***)****. Let* u_1 *and* u_2 *be classical solutions to*

$$\begin{cases} \mathscr{L}u = 0, & (x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times]t_0, T] \\ u(x_0, y_0, t_0) = \varphi(x_0, y_0) & (x_0, y_0) \in \mathbb{R}^+ \times \mathbb{R}. \end{cases}$$

in the sense of Definition 2.17, and

$$|u_1(x, y, t)| + |u_2(x, y, t)| \;\leq\; M \exp\left(C(\log(x^2 + y^2 + 1) - \log(x)) + 1\right)^2,$$

for some positive consants M *and* C*, then* $u_1 = u_2$ *in* $\mathbb{R}^+ \times \mathbb{R} \times]t_0, T]$*.*

2.2.1 The operator $\mathscr{L}_1$

In this section we collect known facts on the operator $\mathscr{L}$. Let us consider the operator $\mathscr{L}_1$ introduced in (2. 31) as the prototype operator for the operator $\mathscr{L}$. As we have already pointed out in the Introduction of this paper, the function Γ_L^1 defined in (2. 33) is the fundamental solution Γ_L^1 of $\mathscr{L}_1$. Its expression agrees with that of the density of the process $(W_t, A_t)_{t \geq 0}$ in (2. 30), first considered by Yor in [123].

As far as we are concerned with the invariance properties of $\mathscr{L}_1$, Monti and Pascucci observe in [96] that it is invariant with respect to the group operation on $\mathbb{R}^+ \times \mathbb{R}^2$:

$$(\xi, \eta, \tau) \circ (x, y, t) = (\xi\, x, \eta + \xi\, y, \tau + t). \tag{2. 34}$$

Indeed, if we set $v(x, y, t) = u(\xi\, x, \eta + \xi\, y, \tau + t)$, then $\mathscr{L}_1 v = 0$ if, and only if, $\mathscr{L}_1 u = 0$. We also remark that

$$\mathbb{L} := (\mathbb{R}^+ \times \mathbb{R}^2, \circ)$$

is a Lie group, where the identity $\mathbb{I}_{\mathbb{G}}$ and the inverse of the element (x, y, t) are defined as

$$\mathbb{I}_{\mathbb{L}} = (1, 0, 0), \qquad (x, y, t)^{-1} = \left(\tfrac{1}{x}, -\tfrac{y}{x}, -t\right).$$

Let us notice that the translation defined in (2. 34) reflects the mixed nature of the stochastic process $(S_t, A_t)_{t\geq 0}$ defined in (2. 30). Indeed its first component S_t is log-normal, then is related to a multiplicative group, while the component A_t is defined as the integral of S_t, then is related to an additive group. In particular, the null element of the group is $(1, 0, 0)$, the left-translation $(r, 0, 0) \circ (x, y, t)$ acts as a dilation with respect to (x, y), while the left-translation $(1, \eta, t) \circ (x, y, t)$ acts as an Euclidean translation with respect to (y, t)

$$(r, 0, 0) \circ (x, y, t) = (rx, ry, t), \qquad (1, \eta, t) \circ (x, y, t) = (x, \eta + y, \tau + t).$$

As far as we are concerned with the regularizing properties of the operator $\mathscr{L}_1$, one can easily prove it is hypoelliptic in the sense of (2. 16). Indeed, we can write the vector fields associated to $\mathscr{L}_1$ as follows

$$X = x\partial_x \sim \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}, \quad Y = x\partial_y - \partial_t \sim \begin{pmatrix} 0 \\ x \\ -1 \end{pmatrix}, \quad \text{and} \quad [X, Y] = x\partial_y \sim \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix}. \tag{2. 35}$$

Thus, $\text{Lie}\{X, Y, [X, Y]\}(x, y, t) = \mathbb{R}^+ \times \mathbb{R}^2$ for every $(x, y, t) \in \mathbb{R}^+ \times \mathbb{R}^2$, hence $\mathscr{L}_1$ satisfies the Hörmander's hypoellipticity condition.

2.2.2 The optimal control problem for $\mathscr{L}_1$

We now introduce the function ψ appearing in the formula (2. 45). Let us consider the vector fields X and Y defined in (2. 35) associated to the operator $\mathscr{L}_1$. We consider the following optimal control problem. For any *end point* $(x_0, y_0, t_0) \in \mathbb{R}^+ \times \mathbb{R}^2$ and *starting point* $(x_1, y_1, t_1) \in \mathbb{R}^+ \times \mathbb{R}^2$, with $t_1 > t_0$:

$$\psi(x_1, y_1, t_1; x_0, y_0, t_0) := \inf_{\omega \in L^1([0,T])} \int_0^T \omega^2(\tau)\, d\tau \qquad \text{subject to constraint} \tag{2. 36}$$

$$\begin{cases} \dot{x}(s) = \omega(s)\, x(s) \\ \dot{y}(s) = x(s) \qquad\qquad 0 \leq s \leq T \\ \dot{t}(s) = -1 \end{cases}$$

$$(x, y, t)(0) = (x_1, y_1, t_1), \qquad (x, y, t)(T) = (x_0, y_0, t_0).$$

The constraint $\dot{t}(s) = -1$ implies that admissible paths satisfy $t(s) = t_1 - s$, hence $T = t_1 - t_0$. For this reason, in the sequel we drop the time variable, and we set $T := t_1 - t_0$. Moreover, as $x(s) > 0$ for every s, the second equation yields $y_1 < y_0$. The knowledge of the explicit expression of the function ψ is particularly important, and we summarize here some quantitative informations about it in terms of the following function

$$g(r) = \begin{cases} \frac{\sinh(\sqrt{r})}{\sqrt{(r)}}, & r > 0, \\ 1 & r = 0, \\ \frac{\sinh(\sqrt{-r})}{\sqrt{(-r)}}, & -\pi^2 < r < 0. \end{cases}$$

Indeed, for every $(x_1, y_1, t_1), (x_0, y_0, t_0) \in \mathbb{R}^+ \times \mathbb{R}^2$, with $t_0 < t_1$ and $y_0 > y$, we have

$$\begin{cases} \psi(x_1, y_1, t_1; x_0, y_0, t_0) = E(t_1 - t_0) + \frac{4(x_1+x_0)}{y_0-y_1} - 4\sqrt{E + \frac{4x_1x_0}{(y_0-y_1)^2}}, \quad \text{if } E \geq -\frac{\pi^2}{(t_1-t_0)^2} \; ; \\ \\ \psi(x_1, y_1, t_1; x_0, y_0, t_0) = E(t_1 - t_0) + \frac{4(x_1+x_0)}{y_0-y_1} + 4\sqrt{E + \frac{4x_1x_0}{(y_0-y_1)^2}}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } -\frac{4\pi^2}{(t_1-t_0)^2} < E < -\frac{\pi^2}{(t_1-t_0)^2} \; , \end{cases} \tag{2. 37}$$

where

$$E = \frac{4}{(t_1 - t_0)^2}\, g^{-1}\left(\frac{y_0 - y_1}{(t_1 - t_0)\sqrt{x_1 x_0}}\right).$$

For further informations we refer to [29], Section 4, where also the solution of the control problem (2. 36) is provided. Moreover, we recall that one of the results of [29] are the following bounds for the fundamental solution Γ_L^1 constructed by Geman and Yor in [56]:

$$\frac{c_\varepsilon^-}{x_0^2(t - t_0)^2} \exp\left(-C^- \psi(x, y + x_0\varepsilon(t - t_0), t - \varepsilon(t - t_0); x_0, y_0, t_0)\right) \leq \tag{2. 38}$$
$$\leq \Gamma_L^1(x, y, t; x_0, y_0, t_0) \leq \frac{C_\varepsilon^+}{x_0^2(t - t_0)^2} \exp\left(-c^+ \psi(x, y - x_0\varepsilon, t + \varepsilon; x_0, y_0, t_0)\right),$$

where ψ is the cost function of the optimal control problem (2. 39).

As the vector fields $X = x\partial_x$ and $Y = x\partial_y - \partial_t$ are invariant with respect to the left translation "$\circ$" in (2. 34), it turns out that also the solution ψ to the optimal control problem (2. 36) is invariant with respect to $\mathbb{L} = (\mathbb{R}^+ \times \mathbb{R}^2, \circ)$. In particular we have

$$\psi(x_1, y_1, t_1; x_0, y_0, t_0) = \psi((x_0, y_0, t_0)^{-1} \circ (x_1, y_1, t_1); 1, 0, 0).$$

Hence, from now on we fix the final condition $(x_0, y_0, t_0) = (1, 0, 0)$ in the optimal control problem (2. 36), and then use the invariance property to solve it with a general initial condition, and we use the simplified notation $\psi(x, y, t) = \psi(x, y, t; 1, 0, 0)$. For all of the above reasons, the optimal control problem (2. 36) now reads as follows for a general starting point $(x, y, t) \in \mathbb{R}^+ \times \mathbb{R}^- \times \mathbb{R}^+$:

$$\psi(x, y, t) := \inf_{\omega \in L^1([0,t])} \int_0^t \omega^2(\tau)\, d\tau \qquad \text{subject to constraint} \tag{2. 39}$$
$$\begin{cases} \dot{x}(s) = \omega(s)\, x(s), & x(0) = x, \quad x(t) = 1, \\ \dot{y}(s) = x(s), & y(0) = y, \quad y(t) = 0. \end{cases}$$

2.2.3 The space C_L^α

In order to define the Hölder spaces relevant to the operator $\mathscr{L}$ we note that the operators $\mathscr{L}$ and $\mathcal{K}$ agree in every compact set of $\mathbb{R}^+ \times \mathbb{R}^2$. We then borrow the regularity theory developed for the opeator

$\mathcal{K}$, and described in the above subsection, to obtain analogous results for the operator $\mathscr{L}$. This point of view was adopted in the work [29] to obtain an invariant Harnack inequality for $\mathscr{L}$.

Consider a function $f = f(x,y,t)$ defined in $\mathbb{R}^+ \times \mathbb{R}^2$, and let (ξ,η,τ) be a point in $\mathbb{R}^+ \times \mathbb{R}^2$. In accordance with the operation (2. 34), we define the function

$$\widetilde{f}(x,y,t) := f(\xi\, x, \eta + \xi\, y, \tau + t).$$

We note that

$$f(x,y,t) = \widetilde{f}\left(\tfrac{x}{\xi}, \tfrac{y-\eta}{\xi}, t-\tau\right),$$

and we apply the Definition 1.11 to $\widetilde{f}(x,y,t)$ in a neighborhood of $(1,0,0)$. We find

$$\begin{aligned} |f(x,y,t) - f(\xi,\eta,\tau)| &= \left|\widetilde{f}\left(\tfrac{x}{\xi}, \tfrac{y-\eta}{\xi}, t-\tau\right) - \widetilde{f}(1,0,0)\right| \leq \\ & C\left(\left|\tfrac{x-\xi}{\xi}\right| + \left|\tfrac{y-\eta}{\xi} + (t-\tau)\tfrac{x+\xi}{2\xi}\right|^{1/3} + |t-\tau|^{1/2}\right)^{\alpha}. \end{aligned} \tag{2. 40}$$

Let us remark that the operators $\mathscr{L}$ and $\mathcal{K}$ are comparable only when the points x and ξ are close each other. Indeed, if we exchange the role of the points (x,y,t) and (ξ,η,τ), we find the inequality

$$|f(\xi,\eta,\tau) - f(x,y,t)| \leq C\left(\left|\tfrac{\xi-x}{x}\right| + \left|\tfrac{\eta-y}{x} + (\tau-t)\tfrac{\xi+x}{2x}\right|^{1/3} + |t-\tau|^{1/2}\right)^{\alpha},$$

which doesn't agree with (2. 40), unless ξ and x have similar size. For this reason, we give the following definition of quasi-distance and Hölder continuous function with respect to the operation "$\circ$", which is equivalent to (2. 40) when $\frac{x}{\xi}$ is close to 1.

Definition 2.14 *For every $z = (x,y,t), \zeta = (\xi,\eta,\tau) \in \mathbb{R}^+ \times \mathbb{R}^2$, we define a symmetric quasi-distance $d_L(z,\zeta)$ invariant with respect to the translation group $\mathbb{L}$ as follows*

$$d_L(z,\zeta) = \left|\frac{x-\xi}{\sqrt{x\xi}}\right| + \left|\frac{y-\eta+(t-\tau)\frac{x+\xi}{2}}{\sqrt{x\xi}}\right|^{1/3} + |t-\tau|^{1/2}.$$

Definition 2.15 *Let α be a positive constant, $\alpha \leq 1$, and let Ω be an open subset of $\mathbb{R}^+ \times \mathbb{R}^2$. We say a function $f : \Omega \longrightarrow \mathbb{R}$ is Hölder continuous with exponent α in Ω with respect to the group $\mathbb{L} = (\mathbb{R}^+ \times \mathbb{R}^2, \circ)$ (in short: Hölder continuous with exponent α, $f \in C^{\alpha}_{L}(\Omega)$) if there exists a positive constant $C > 0$ such that*

$$|f(x,y,t) - f(\xi,\eta,\tau)| \leq C d_L(z,\zeta)^{\alpha}, \tag{2. 41}$$

for every $(x,y,t),(\xi,\eta,\tau) \in \Omega$. Moreover, we say a function f is locally Hölder continuous, and we write $f \in C^{\alpha}_{L,\mathrm{loc}}(\Omega)$, if $f \in C^{\alpha}_{L}(\Omega')$ for every compact subset Ω' of Ω.

As the definitions $C^{\alpha}_{L}(\Omega')$ and $C^{\alpha}_{K}(\Omega')$ agree in every compact subset Ω' of $\mathbb{R}^+ \times \mathbb{R}^2$, the following statement is an immediate consequence of Proposition 2.5. For this reason, we omit its proof, which is immediate.

Proposition 2.16 *Let Ω' be a compact subset of $\mathbb{R}^+ \times \mathbb{R}^2$. If $f \in C^{\alpha}(\Omega)$ in the usual Euclidean sense, then $f \in C^{\alpha}_{L}(\Omega')$ in the sense of Definition 2.15. Vice versa, if $f \in C^{\alpha}_{L}(\Omega')$, then $f \in C^{\beta}(\Omega')$ in the Euclidean sense with $\beta = \frac{\alpha}{3}$.*

Definition 2.17 *A function u is a solution to the equation $\mathscr{L}u = f$ in a domain Ω of $\mathbb{R}^+ \times \mathbb{R}^2$ if the derivatives $x\partial_x u, x^2\partial_x^2 u$, and the Lie derivative Yu exist as continuous functions in Ω, and the equation*

$$x\,\frac{\partial u}{\partial x}\left(a(x,y,t)\,x\,\frac{\partial u}{\partial x}\right) + b(x,y,t)\,x\,\frac{\partial u}{\partial x} + x\,\frac{\partial u}{\partial y} - r(x,y,t)u - \frac{\partial u}{\partial t}, = f(x,y,t)$$

is satisfied at any point $(x,y,t) \in \Omega$.

Moreover, we define the natural functional setting for $\mathscr{L}u = f$ as follows

$$C_L^{2+\alpha}(\Omega) = \left\{u \in C_L^{\alpha}(\Omega) \;\middle|\; x\partial_x u, x^2\partial_x^2 u, Yu \in C_L^{\alpha}(\Omega)\right\},$$

where $C_L^{\alpha}(\Omega)$ is given in Definition 2.15. Clearly, the definition of $C_{L,\mathrm{loc}}^{2+\alpha}(\Omega)$ follows straightforwardly from the definition of $C_{L,\mathrm{loc}}^{\alpha}(\Omega)$.

2.2.4 The fundamental solution Γ_L

We now focus on the fundamental solution Γ_L for the operator $\mathscr{L}$.

Definition 2.18 *A function $\Gamma_L : \left(\mathbb{R}^+ \times \mathbb{R}^2\right) \times \left(\mathbb{R}^+ \times \mathbb{R}^2\right) \to \mathbb{R}$ is a fundamental solution for $\mathscr{L}$ if*

1. *for every $(x_0,y_0,t_0) \in \mathbb{R}^+ \times \mathbb{R}^2$ the function $x \mapsto \Gamma_L(x,y,t;x_0,y_0,t_0)$:*
   (a) *belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2) \cap C^\infty(\mathbb{R}^+ \times \mathbb{R}^2 \setminus \{(x_0,y_0,t_0)\})\}$;*

   (b) *is a classical solution of $\mathscr{L}u = 0$ in $\mathbb{R}^+ \times \mathbb{R}^2 \setminus \{(x_0,y_0,t_0)\}\}$ in the sense of Definition 2.17;*
2. *for every bounded function $\varphi \in C(\mathbb{R}^2)$, we have that*

$$u(x,y,t) = \int_{\mathbb{R}^2} \Gamma_L(x,y,t;\xi,\eta,0)\,\varphi(\xi,\eta)\,d\xi\,d\eta, \tag{2. 42}$$

   is a classical solution of the Cauchy problem

$$\begin{cases} \mathscr{L}u = 0, & (x,y,t) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \\ u(x,y,0) = \varphi(x,y) & (x,y) \in \mathbb{R}^2. \end{cases} \tag{2. 43}$$

3. *The function $\Gamma_L^*(x,y,t;x_0,y_0,t_0) := \Gamma_L(x_0,y_0,t_0;x,y,t)$ satisfies 1. and 2. with $\mathscr{L}$ replaced by its adjoint operator $\mathscr{L}^*$ as defined in* (2. 29).

Under the following assumption (2. 44), which is stronger than $\mathbf{(H}_L\mathbf{)}$, Cibelli, Polidoro and Rossi prove the existence of the fundamental solution Γ_L for $\mathscr{L}$ and bounds analogous to (2. 38) by applying methods coming from the stochastic theory (see Proposition 3.7 and Theorem 1.3 of [29], respectively). We summarize here the main results of the paper [29], under the following assumption for the coefficients a and b:

a,$b \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$. Moreover, there exist two positive constants λ, Λ such that

$$\lambda \le a(x,y,t) \le \Lambda, \quad \left|\tfrac{\partial}{\partial x}\left(x\,a(x,y,t)\right)\right|, \left|\tfrac{\partial}{\partial x}\left(x\,a(x,y,t)\right)\right| \le \Lambda, \tag{2. 44}$$

for every $(x,y,t) \in \mathbb{R}^+ \times \mathbb{R}^2$.

Theorem 2.19 *Let $\mathscr{L}$ be an operator of the form* (2. 28) *under the assumption* (2. 44). *Then there exists a fundamental solution Γ_L in the sense of Definition 2.18. Moreover, the following properties hold:*

1. *Support of* Γ_L*: for every* (x,y,t), $(\xi,\eta,\tau) \in \mathbb{R}^+ \times \mathbb{R} \times]0,T]$ *with* $t \le \tau$ *and* $y \ge \eta$

$$\Gamma_L(x,y,t;\xi,\eta,\tau) = 0;$$

2. *Reproduction property: for every* (x,y,t), (x_0,y_0,t_0), $(\xi,\eta,\tau) \in \mathbb{R}^+ \times \mathbb{R} \times]0,T]$ *with* $T \le t > \tau > t_0 > 0$

$$\Gamma_L(x,y,t;x_0,y_0,t_0) = \int_{\mathbb{R}^+ \times \mathbb{R}} \Gamma_L(x,y,t;\xi,\eta,\tau)\, \Gamma_L(\xi,\eta,\tau;x_0,y_0,t_0)\, d\xi\, d\eta;$$

3. *Integral of* Γ_L*: there exists a positive constant* $\overline{C}$ *depending on* $t-\tau$ *and such that* $\overline{C} \to 1$ *as* $t \to \tau$ *for which*

$$\int_{\mathbb{R}^+ \times \mathbb{R}} \Gamma_L(x,y,t;\xi,\eta,\tau)\, d\xi\, d\eta \;=\; 1, \qquad \int_{\mathbb{R}^+ \times \mathbb{R}} \Gamma_L(x,y,t;\xi,\eta,\tau)\, dx\, dy \;=\; \overline{C};$$

4. *Bounds for* Γ_L*: for every* $\varepsilon \in]0,1[$, *and* $T > 0$ *there exist two positive constants* $c_\varepsilon^-, C_\varepsilon^+$ *depending on* ε, *on* T *and on the operator* $\mathscr{L}$, *and two positive constants* C^-, c^+, *only depending on the operator* $\mathscr{L}$, *such that*

$$\frac{c_\varepsilon^-}{x_0^2(t-t_0)^2} \exp\left(-C^- \psi(x, y + x_0\varepsilon(t-t_0), t - \varepsilon(t-t_0); x_0, y_0, t_0)\right) \le \tag{2. 45}$$
$$\le \Gamma_L(x,y,t;x_0,y_0,t_0) \le \frac{C_\varepsilon^+}{x_0^2(t-t_0)^2} \exp\left(-c^+ \psi(x, y - x_0\varepsilon, t+\varepsilon; x_0, y_0, t_0)\right)$$

for every $(x,y,t) \in \mathbb{R}^+ \times]-\infty, y_0 - x_0\varepsilon(t-t_0)[\times]t_0,T]$, *where* ψ *is the value function for the optimal control problem* (2. 39).

We remark that the bounds obtained in (2. 45) for the operator $\mathscr{L}$ by [29] are analogous to the bounds (2. 26) obtained for the fundamental solution of the Kolmogorov operator $\mathcal{K}$. Let us consider the fundamental solutions ${\Gamma_L}^{\pm}$ of the operators

$$\mathscr{L}^{\pm} u = \lambda^{\pm} x^2 \frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} - \frac{\partial u}{\partial t}, \quad (x,y,t) \in R^+ \times \mathbb{R} \times \mathbb{R}^+. \tag{2. 46}$$

By applying the bounds (2. 45) to Γ_L and to ${\Gamma_L}^{\pm}$, we obtain the following corollary to the previous Theorem 2.19 (see Proposition 1.5 of [29]).

Corollary 2.20 *For every* $\varepsilon \in]0,1[$, *there exist the fundamental solutions* $\Gamma^{\pm}$ *of the operators* (2. 46), *and positive constants* $k^{\pm}$ *such that*

$$\begin{aligned} & k^- {\Gamma_L}^-(x, y + x_0\varepsilon(t-t_0+1), t - \varepsilon(t-t_0+1); x_0,y_0,t_0) \le \\ & \quad \le \Gamma_L(x,y,t;x_0,y_0,t_0) \le \\ & \qquad \le k^+ {\Gamma_L}^+ \left(x, y - x_0\frac{\varepsilon}{1-\varepsilon}(t-t_0+1), t + \frac{\varepsilon}{1-\varepsilon}(t-t_0+1); x_0, y_0, t_0\right) \end{aligned}$$

for every $(x,y,t),(x_0,y_0,t_0) \in \mathbb{R}^+ \times \mathbb{R} \times]0,T]$, *with* $y + x_0\varepsilon(t-t_0+1) < y_0$ *and* $t > t_0 + \varepsilon/(1-\varepsilon)$.

2.3 Existence and uniqueness of the fundamental solution Γ_K

This section is devoted to the proof of our main results on the existence of the fundamental solution for the operator $\mathcal{K}$. Our approach relies on the local regularity properties of the solutions and on the bounds for the fundamental solution. Let us consider first the operator $\mathcal{K}$. We build a sequence of operators $(\mathcal{K}_n)_{n\in\mathbb{N}}$ satisfying the hypotheses of Theorem 2.9, then a fundamental solution Γ_K^n exists for every $n \in \mathbb{N}$. Moreover, the sequence $(\Gamma_K^n)_{n\in\mathbb{N}}$ is equibounded by (2. 26), and locally equicontinuous, thanks to the Schauder estimates of Theorem 2.7. The existence of Γ_K then follows from the Ascoli-Arzelà's theorem and a diagonal argument. The proof of the existence of a fundamental solution Γ_L for $\mathscr{L}$ is analogous, relies on Theorem 2.19, and it is presented in Section 2.4.

Existence of Γ_K

Proof of Theorem 2.1 *(Existence of the fundamental solution).* We construct a sequence of operators $(\mathcal{K}_n)_{n\in\mathbb{N}}$ satisfying the hypotheses of Theorem 2.9. In particular, we need the coefficients a_n, b_n, r_n to be uniformly Hölder continuous and satisfying the assumption (2. 24). For this reason, we introduce a cut-off function $\chi_n \in C_0^\infty(\mathbb{R}^3)$ such that $0 \le \chi_n(x, y, t) \le 1$, and

$$\chi_n(x,y,t) = 1 \quad \text{for} \quad x^2+y^2 \le n^2, \qquad \chi_n(x,y,t) = 0 \quad \text{for} \quad x^2+y^2 \ge (n+1)^2.$$

For every $n \in \mathbb{N}$ we set

$$\begin{aligned} a_n(x,y,t) &:= \chi_n(x,y,t)a(x,y,t) + (1-\chi_n(x,y,t))\lambda, \\ b_n(x,y,t) &:= \chi_n(x,y,t)b(x,y,t), \qquad r_n(x,y,t) := \chi_n(x,y,t)r(x,y,t). \end{aligned}$$

Then we apply Theorem 2.9 to the operator $\mathcal{K}_n$ for every $n \in \mathbb{N}$. Thus, there exists a sequence of equibounded fundamental solutions $(\Gamma_K^n)_{n\in\mathbb{N}}$, in the sense that each of them satisfies (2. 26).

We define a sequence of open subsets $(\Omega_p)_{p\in\mathbb{N}}$ of $\mathbb{R}^6$ such that $\Omega_p \subset\subset \Omega_{p+1}$ for every $p \in \mathbb{N}$ and $\bigcup_{p=1}^{+\infty} \Omega_p = \{(x,y,t;\xi,\eta,\tau) \in \mathbb{R}^6 \mid (x,y,t) \neq (\xi,\eta,\tau)\}$:

$$\Omega_p := \left\{ \begin{matrix} (x,y,t;\xi,\eta,\tau) \in \mathbb{R}^6 \mid x^2+y^2+t^2 < p^2, \xi^2+\eta^2+\tau^2 < p^2, \\ (x-\xi)^2+(y-\eta)^2+(t-\tau)^2 > \left(\frac{1}{p}\right)^2 \end{matrix} \right\}.$$

We note that the sequence $(\Gamma_K^n)_{n\ge2}$ is equicontinuous in Ω_1 thanks to Theorem 3. 41. Moreover, by Theorem 2.9 and Theorem 3. 41, we also have that

$$\left(\frac{\partial\Gamma_K^n}{\partial x}\right)_{n\ge2}, \quad \left(\frac{\partial\Gamma_K^n}{\partial \xi}\right)_{n\ge2}, \quad \left(\frac{\partial^2\Gamma_K^n}{\partial x^2}\right)_{n\ge2}, \quad \left(\frac{\partial^2\Gamma_K^n}{\partial \xi^2}\right)_{n\ge2}, \quad (Y\Gamma_K^n)_{n\ge2}, \quad \text{and} \quad \left(Y^*_{(\xi,\eta,\tau)}\Gamma_K^n\right)_{n\ge2}$$

are bounded sequences in $C^\alpha(\Omega_1)$. Here Y is the Lie derivative defined in (2. 23) and $Y^*_{(\xi,\eta,\tau)}$ is its adjoint, computed with respect to the variable (ξ, η, τ). Thus, there exists a subsequence $(\Gamma_K^{1,m})_{m\in\mathbb{N}}$ that converges uniformly to some function Γ_1 that satisfies (2. 26). Moreover, $\Gamma_1 \in C^{2+\alpha}(\Omega_1)$ and, for every $(x_0, y_0, t_0) \in \mathbb{R}^3$ such that $x_0^2 + y_0^2 + t_0^2 < 1$ the function $u(x, y, t) := \Gamma_1(x, y, t; x_0, y_0, t_0)$ is a classical solution to $\mathcal{K}u = 0$ in the set $\{(x,y,t) \in \mathbb{R}^3 \mid (x,y,t;x_0,y_0,t_0) \in \Omega_1\}$, and the function $v(\xi,\eta,\tau) := \Gamma_1(x_0,y_0,t_0;\xi,\eta,\tau)$ is a classical solution to $\mathcal{K}^*v = 0$ in the set $\{(\xi,\eta,\tau) \in \mathbb{R}^3 \mid (x_0,y_0,t_0;\xi,\eta,\tau) \in \Omega_1\}$.

We next apply the same argument to the sequence $(\Gamma_K^{1,m})_{m\in\mathbb{N}}$ on the set Ω_2, and obtain a subsequence $(\Gamma_K^{2,m})_{m\in\mathbb{N}}$ that converges in $C^{2+\alpha}(\Omega_2)$ to some function Γ_2, that belongs to $C^{2+\alpha}(\Omega_2)$ and satisfies the bounds (2. 26), and the following condition. For every $(x_0, y_0, t_0) \in \mathbb{R}^3$ such that $x_0^2 + y_0^2 + t_0^2 < 4$ the function $u(x, y, t) := \Gamma_2(x, y, t; x_0, y_0, t_0)$ is a classical solution to $\mathcal{K}u = 0$ in the set $\{(x,y,t) \in \mathbb{R}^3 \mid (x,y,t;x_0,y_0,t_0) \in \Omega_2\}$, and the function $v(\xi,\eta,\tau) := \Gamma_2(x_0,y_0,t_0;\xi,\eta,\tau)$ is a classical solution to

$\mathcal{K}^* v = 0$ in the set $\{(\xi, \eta, \tau) \in \mathbb{R}^3 \mid (x_0, y_0, t_0; \xi, \eta, \tau) \in \Omega_2\}$. We remark that, since Γ_2 is the limit of a subsequence of $(\Gamma_K^{1,m})_{m\in\mathbb{N}}$, it must coincide with Γ_1 in Ω_1.

We next proceed by induction. Let us assume that the sequence $(\Gamma_K^{q-1,m})_{m\in\mathbb{N}}$ on the set Ω_q has been defined for some $q \in \mathbb{N}$. We extract from it a subsequence $(\Gamma_K^{q,m})_{m\in\mathbb{N}}$ converging in $C^{2+\alpha}(\Omega_q)$ to some function Γ_q, satisfying (2. 26). Moreover, $(x, y, t) \mapsto \Gamma_q(x, y, t; x_0, y_0, t_0)$ is a classical solution to $\mathcal{K}u = 0$ and $(\xi, \eta, \tau) \mapsto \Gamma_q(x_0, y_0, t_0; \xi, \eta, \tau)$ is a classical solution to $\mathcal{K}^* v = 0$. Moreover, it agrees with Γ_{q-1} on the set Ω_{q-1}.

Next, we define a function Γ_K in the following way: for every $(x, y, t), (\xi, \eta, \tau) \in \mathbb{R}^3$ with $(x, y, t) \neq (\xi, \eta, \tau)$ we choose $q \in \mathbb{N}$ such that $(x, y, t; \xi, \eta, \tau) \in \Omega_q$ and we set $\Gamma_K(x, y, t; \xi, \eta, \tau) := \Gamma_q(x, y, t; \xi, \eta, \tau)$. This definition is well-posed, since if $(x, y, t) \in \Omega_p$, then $\Gamma_p(x, y, t; \xi, \eta, \tau) = \Gamma_q(x, y, t; \xi, \eta, \tau)$.

We next check that Γ_K has the properties listed in the statement of the Theorem 2.1. As every $\Gamma_K^n(x, y, t; x_0, y_0, t_0) = 0$ whenever $t \leq t_0$, also $\Gamma_K(x, y, t; x_0, y_0, t_0) = 0$ for every $t \leq t_0$. For the same reason, it satisfies (2. 26). Moreover, for every $(x_0, y_0, t_0) \in \mathbb{R}^3$, $(x, y, t) \mapsto \Gamma_K(x, y, t; x_0, y_0, t_0) \in L^1_{\mathrm{loc}}(\mathbb{R}^3) \cap C^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{(x_0, y_0, t_0)\})$, and is a classical solution to $\mathcal{K}u = 0$ in $\mathbb{R}^3 \setminus \{(x_0, y_0, t_0)\}$. Analogously, $(\xi, \eta, \tau) \mapsto \Gamma_K(x_0, y_0, t_0; \xi, \eta, \tau) \in L^1_{\mathrm{loc}}(\mathbb{R}^3) \cap C^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{(x_0, y_0, t_0)\})$ and is a classical solution to $\mathcal{K}^* v = 0$ in $\mathbb{R}^3 \setminus \{(x_0, y_0, t_0)\}$. This proves the point *1.* of the Definition 2.8 and the point *1.* of Theorem 2.1. We remark that points *3.* and *4.* of Theorem 2.1 follow immediately from the construction of the fundamental solution Γ_K and the pointwise convergence.

As far as we are concerned with the reproduction property *2.* of Theorem 2.1, we use the upper bound in (2. 26), which yields

$$\Gamma_K^n(x, y, t; \xi, \eta, \tau)\Gamma_K^n(\xi, \eta, \tau; x_0, y_0, t_0) \leq C^+\, \Gamma_K^{\lambda^+}(x, y, t; \xi, \eta, \tau) C^+\, \Gamma_K^{\lambda^+}(\xi, \eta, \tau; x_0, y_0, t_0),$$

and the reproduction property

$$\int_{\mathbb{R}^2} \Gamma_K^{\lambda^+}(x, y, t; \xi, \eta, \tau)\Gamma_K^{\lambda^+}(\xi, \eta, \tau; x_0, y_0, t_0) d\xi\, d\eta\, d\tau = \Gamma_K^{\lambda^+}(x, y, t; x_0, y_0, t_0) < +\infty,$$

which allows us to use the Lebesgue convergence theorem. Thus the property *2.* of Theorem 2.1 holds true.

To proceed with the proof of Theorem 2.1 we have to verify that, for every $\varphi \in C_b(\mathbb{R}^2)$, the function

$$u(x, y, t) = \int_{\mathbb{R}^2} \Gamma_K(x, y, t; \xi, \eta, t_0)\, \varphi(\xi, \eta)\, d\xi\, d\eta \tag{2. 47}$$

is a classical solution to the Cauchy problem

$$\begin{cases} \mathcal{K}u = 0, & (x, y, t) \in \mathbb{R}^2 \times]t_0, \infty[; \\ u(x, y, t_0) = \varphi(x, y) & (x, y) \in \mathbb{R}^2. \end{cases} \tag{2. 48}$$

By the usual standard argument, we differentiate under the integral sign

$$\mathcal{K}u(x, y, t) = \int_{\mathbb{R}^2} \mathcal{K}\Gamma_K(x, y, t; \xi, \eta, t_0)\, \varphi(\xi, \eta)\, d\xi\, d\eta = 0.$$

Thus, we are left with the proof that $u(x, y, t) \to \varphi(x_0, y_0)$ as $(x, y, t) \to (x_0, y_0, t_0)$. We first note that

$$u(x, y, t) - \varphi(x_0, y_0) = \underbrace{\boxed{\int_{\mathbb{R}^2} \Gamma_K(x, y, t; \xi, \eta, t_0)\, (\varphi(\xi, \eta) - \varphi(x_0, y_0))\, d\xi\, d\eta}}_{A} +$$

$$+ \underbrace{\boxed{\varphi(x_0, y_0)\Big(\int_{\mathbb{R}^2} \Gamma_K(x, y, t; \xi, \eta, t_0) d\xi\, d\eta - 1\Big).}}_{B}$$

The term B plainly vanishes, as t goes to t_0, because of the bound (2. 25). The integral A can be handled considering that (2. 26) holds true, and that the expression of $\Gamma_K^{\lambda^+}$ is (2. 11). Specifically, we apply the change of variable

$$\overline{x} = \frac{1}{2\sqrt{\lambda^+(t-t_0)}}(x-\xi) \qquad \overline{y} = \frac{\sqrt{3}}{\sqrt{\lambda^+(t-t_0)^3}}\left(y-\eta+(t-t_0)\frac{x+\xi}{2}\right),$$

and we obtain the following bound

$$|A| \leq \frac{1}{\pi}\int_{\mathbb{R}^2} e^{-\left(\overline{x}^2+\overline{y}^2\right)}|\widetilde{\varphi}(\overline{x},\overline{y}) - \varphi(x_0,y_0)|\, d\overline{x}\, d\overline{y}, \tag{2. 49}$$

where

$$\widetilde{\varphi}(\overline{x},\overline{y}) := \varphi\left(x - 2\sqrt{\lambda^+(t-t_0)}\,\overline{x},\ y - \sqrt{\frac{\lambda^+(t-t_0)^3}{3}}\,\overline{y} + (t-t_0)\frac{x-\sqrt{\lambda^+(t-t_0)}\,\overline{x}}{\sqrt{3}}\right).$$

Note that, for every fixed $(\overline{x},\overline{y})$, the above expression converges to $\varphi(x_0,y_0)$ as $(x,y,t)\to(x_0,y_0,t_0)$. Moreover $\widetilde{\varphi}-\varphi(x_0,y_0)$ is bounded as a function of $(\overline{x},\overline{y})$, then the Lebesgue theorem implies that A vanishes as $(x,y,t)\to(x_0,y_0,t_0)$. Thus $u(x,y,t)\to\varphi(x_0,y_0)$ as $(x,y,t)\to(x_0,y_0,t_0)$, and the proof of the point *1.* of Definition 2.8 accomplilshed.
The proof that ${\Gamma_K}^*(\xi,\eta,\tau;x,y,t) = \Gamma_K(x,y,t;\xi,\eta,\tau)$ is the fundamental solution of the adjoint operator $\mathcal{K}^*$ and satisfies the properties of Theorem 2.1 follows accordingly. □

Remark 2.21 *The growth condition* (2. 13) *can be used instead of the boundedness assumption on the initial data φ. Indeed, it ensures, alongside with the upper bound* (2. 26) *for the fundamental solution Γ_K, that the integral* (2. 47) *is convergent for every $(x,y,t)\in\mathbb{R}^2\times]t_0,+\infty[$, that it can be differentiated twice with respect to x and once in the direction of the vector field $x\partial_y-\partial_t$, so that u a solution to $\mathscr{L}u=0$. Moreover, the condition* (2. 13) *and the inequality* (2. 49) *yield that the expression $|A|$ vanishes, as $(x,y,t)\to(x_0,y_0,t_0)$.*

Uniqueness and comparison principle for the operator $\mathcal{K}$

Now, we recall a technical result, an a priori estimate for the derivatives of the fundamental solution $\Gamma_K(x,y,t;\xi,\eta,\tau)$ necessary to conclude the proof of Proposition 2.23. In order to state our result, we introduce for every $R>1$ the set

$$\widetilde{Q}_R := \left\{R \leq \sqrt{\xi^2+\eta^2} \leq R+1\right\} \subset \mathbb{R}^2. \tag{2. 50}$$

Lemma 2.22 *Let **(H_K)** hold, and let Γ_K be the fundamental solution for the operator $\mathcal{K}$ in the sense of Definition 2.8. Let Q_R be the cylinder defined in* (5) *, there exists a constant C such that*

$$\frac{\lambda}{2}\int_{t_0}^{t}\int_{\widetilde{Q}_R}\left|\frac{\partial\Gamma_K}{\partial\xi}(x,y,t;\xi,\eta,\tau)\right|^2 d\xi d\eta d\tau \leq$$

$$\leq C_3\int_{t_0}^{t}\int_{\widetilde{Q}_R}\Gamma_K(x,y,t;\xi,\eta,\tau)^2 d\xi d\eta d\tau + \frac{1}{2}\int_{\widetilde{Q}_R}\Gamma_K(x,y,t;\xi,\eta,t_0)^2 d\xi d\eta.$$

where C_3 is a positive constant only depending on λ, Λ, C_0 and C_1.

The proof of this a priori estimate, also known as Caccioppoli inequality, is presented at the end of this section and is based on the representation formula for solutions to the equation $\mathcal{K}u=0$. For further applications of this technique see for instance [104], [33] and [7].

As a first step in the proof of our uniqueness result, we state and prove a comparison principle for the operator $\mathcal{K}$.

Proposition 2.23 *Let us consider the operator $\mathcal{K}$ under the assumption* **(*H*$_K$)**. *Let u be a classical solution to*

$$\begin{cases} \mathcal{K}u \geq 0, & (x,y,t) \in \mathbb{R}^2 \times]t_0, T]; \\ u(x,y,t_0) \leq 0 & (x,y) \in \mathbb{R}^2. \end{cases} \tag{2. 51}$$

in the sense of Definition 2.6. If moreover

$$|u(x,y,t)| \leq M e^{C(x^2+y^2)},$$

for some positive consants M and C, then $u \leq 0$ in $\mathbb{R}^2 \times]t_0, T]$.

Proof. We fix a positive constant $\bar{t}$ such that $\bar{t} \leq 1$, and we prove that, if we choose $\bar{t}$ small enough, we have $u = 0$ in $\mathbb{R}^2 \times]t_0, t_0 + \bar{t}\,]$. We then iterate our argument on the strip $\mathbb{R}^2 \times]t_0 + \bar{t}, t_0 + 2\bar{t}\,]$, then on $\mathbb{R}^2 \times]t_0 + 2\bar{t}, t_0 + 3\bar{t}\,]$. As the choice of $\bar{t}$ only depends on the operator $\mathcal{K}$ and on the constant C in our assumption $|u(x,y,t)| \leq M e^{C(x^2+y^2)}$, after a finite number of steps we cover the whole set $\mathbb{R}^2 \times]t_0, T]$.

Fix $(x,y,s) \in \mathbb{R}^2 \times]t_0, t_0 + \bar{t}\,]$ and, denote by $|(x,y)|$ the Euclidean norm of (x,y). For every $R > |(x,y)|$, we let h_R be a $C^\infty(\mathbb{R}^2)$ smooth function, such that $0 \leq h_R \leq 1$, $h_R(\xi,\eta) = 1$ whenever $|(\xi,\eta)| \leq R$, and $h_R(\xi,\eta) = 0$ for every $(\xi,\eta) \in \mathbb{R}^2$ with $|(\xi,\eta)| \geq R+1$. We also assume that its first and second order derivatives are bounded uniformly with respect to R.

We next recall the Green identity

$$v\mathcal{K}u - u\mathcal{K}^* v = \tfrac{\partial}{\partial x}\left(va\tfrac{\partial u}{\partial x} - ua\tfrac{\partial v}{\partial x} + buv\right) + x\tfrac{\partial}{\partial y}(uv) - \tfrac{\partial}{\partial t}(uv).$$

We then choose a constant $\delta \in]0, s - t_0[$ and we apply the divergence theorem with $v_R(\xi,\eta,\tau) = h_R(\xi,\eta)\Gamma_K(x,y,s;\xi,\eta,\tau)$, to the cylinder

$$Q_{R,\delta} := \left\{(\xi,\eta,\tau) \in \mathbb{R}^2 \times]t_0, t_0 + \bar{t}\,] \mid |(\xi,\eta)| \leq R+2, \tau \leq s - \delta\right\}.$$

As v_R, $\frac{\partial v_R}{\partial x}$, and $\frac{\partial v_R}{\partial y}$ vanish at the lateral part of the boundary of $Q_{R,\delta}$, we find

$$\int_{Q_{R,\delta}} (v_R\mathcal{K}u - u\mathcal{K}^* v_R)(\xi,\eta,\tau) d\xi\, d\eta\, d\tau = -\int_{\mathbb{R}^2} (uv_R)(\xi,\eta,s-\delta) d\xi\, d\eta + \int_{\mathbb{R}^2} (uv_R)(\xi,\eta,0) d\xi\, d\eta. \tag{2. 52}$$

Because of the properties of the fundamental solution we have

$$u(x,y,s) = \lim_{\delta \to 0} \int_{\mathbb{R}^2} \Gamma_K(x,y,s,\xi,\eta,s-\delta) h_R(\xi,\eta) u(\xi,\eta,s-\delta) d\xi\, d\eta.$$

Moreover, by our assumption, we have $v_R\mathcal{K}u \geq 0$ in $\mathbb{R}^2 \times]t_0, t_1[$, and $(uv_R)(\cdot,\cdot,0) \leq 0$. Hence (2. 52) gives

$$\begin{aligned} u(x,y,s) &= \int_{\mathbb{R}^2} (uv_R)(\xi,\eta,0) d\xi\, d\eta - \int_{\mathbb{R}^2 \times]t_0, s[} (v_R\mathcal{K}u - u\mathcal{K}^* v_R)(\xi,\eta,\tau) d\xi\, d\eta\, d\tau \\ &\leq \int_{\mathbb{R}^2 \times]t_0, s[} u(\xi,\eta,\tau)\mathcal{K}^* v_R(\xi,\eta,\tau) d\xi\, d\eta\, d\tau. \end{aligned}$$

We are left with the proof that the right hand side of the above inequality vanishes as $R \to +\infty$. From now on, we only sketch the proof since it suffices to proceed as in the proof of Theorem 1.6 of [40]. Since $\mathcal{K}^*\Gamma_K(x,y,s;\xi,\eta,\tau) = 0$, we deduce

$$\begin{aligned} u(x,y,s) \leq & 2\Lambda \int_{t_0}^{s} \left(\int_{\widetilde{Q}_R} |u(\xi,\eta,\tau)| \left| \frac{\partial \Gamma_K}{\partial \xi}(x,y,s;\xi,\eta,\tau) \right| \left| \frac{\partial h_R}{\partial \xi}(\xi,\eta,\tau) \right| d\xi\, d\eta \right) d\tau + \\ & + \int_{t_0}^{s} \left(\int_{\widetilde{Q}_R} |u(\xi,\eta,\tau)|\, |\Gamma_K(x,y,s;\xi,\eta,\tau)|\, |\mathcal{K}^* h_R(\xi,\eta,\tau)| d\xi\, d\eta \right) d\tau, \end{aligned}$$

where $\widetilde{Q}_R$ is defined in (2. 50). We recall that first and second order derivatives of the function h_R are bounded because of its definition, more precisely we have that

$$\begin{aligned} &|Y(h_R(x,y,t)| \le C_0 R, \qquad \left|\frac{\partial h_R(x,y,t)}{\partial x}\right| \le C_1, \quad \text{and} \\ &|K^* h_R(x,y,t)| \le \Lambda(2+C_1) + C_0 R =: C_2(1+R), \end{aligned} \tag{2. 53}$$

where C_0, C_1 and C_2 are positive constants. Thus, by applying the Hölder inequality and the estimates (2. 53), we get the following inequality

$$\begin{aligned} u(x,y,s) \le 2\Lambda C_1 \Big(\int_{t_0}^{s} \int_{\widetilde{Q}_R} |u(\xi,\eta,\tau)|^2 d\xi d\eta d\tau \Big)^{\frac12} \Big(\int_{t_0}^{s} \int_{\widetilde{Q}_R} \left| \frac{\partial \Gamma_K}{\partial \xi}(x,y,s;\xi,\eta,\tau)\right|^2 d\xi d\eta d\tau \Big)^{\frac12} + \\ C_2(1+R) \Big(\int_{t_0}^{s} \int_{\widetilde{Q}_R} |u(\xi,\eta,\tau)|^2 d\xi d\eta d\tau \Big)^{\frac12} \Big(\int_{t_0}^{s} \int_{\widetilde{Q}_R} \Gamma_K(x,y,s;\xi,\eta,\tau)^2 d\xi d\eta d\tau \Big)^{\frac12}. \end{aligned}$$

Then, Lemma 2.22 yields

$$\begin{aligned} u(x,y,s) \le (2\Lambda C_1 C_3 + C_2(1+R)) \Big(\int_{t_0}^{s} \int_{\widetilde{Q}_R} |u(\xi,\eta,\tau)|^2 d\xi d\eta d\tau \Big)^{\frac12} \cdot \\ \Big(\int_{t_0}^{s} \int_{\widetilde{Q}_R} \Gamma_K(x,y,s;\xi,\eta,\tau)^2 d\xi d\eta d\tau + \int_{\widetilde{Q}_R} \Gamma_K(x,y,s;\xi,\eta,t_0)^2 d\xi d\eta \Big)^{\frac12}. \end{aligned}$$

By our assumption $|u(\xi,\eta,\tau)| \le M e^{C(\xi^2+\eta^2)}$, we have that

$$\Big(\int_{t_0}^{s} \int_{\widetilde{Q}_R} |u(\xi,\eta,\tau)|^2 d\xi d\eta d\tau \Big)^{\frac12} \le 2\sqrt{\pi \bar{t}} R\, M e^{C(R+1)^2}.$$

Moreover, the Corollary 5.8 gives

$$\Big(\int_{t_0}^{s} \int_{\widetilde{Q}_R} \Gamma_K(x,y,s;\xi,\eta,\tau)^2 d\xi d\eta d\tau + \int_{\widetilde{Q}_R} \Gamma_K(x,y,s;\xi,\eta,t_0)^2 d\xi d\eta \Big)^{\frac12} \le 2\sqrt{\pi(1+\bar{t})R\overline{C}} e^{-\overline{C}\frac{(R-1)^2}{\bar{t}}},$$

provided that $R-1$ is greater than the constant R_0 appearing in its statement. Finally, recalling that $0 < \bar{t} \le 1$, we conclude that

$$u(x,y,s) \le C_4(1+R)^2 e^{C(R+1)^2} e^{-\overline{C}\frac{(R-1)^2}{\bar{t}}},$$

where C_4 is a positive constant depending on the operator $\mathcal{K}$. In order to conclude our proof, it sufficies to choose $\bar{t} < \frac{\overline{C}}{C}$. The concludion the follows by letting $R \to +\infty$. Hence $u(x,y,s) \le 0$. The thesis follows by repeating the previous argument finitely many times, as the choice of $\bar{t}$ does not depend on (x,y,s). □

Proof of Theorem 2.2. This uniqueness result plainly follows from Proposition 2.23 firstly applied to $u = u_1 - u_2$, and then to $u = u_2 - u_1$. □

PROOF OF THEOREM 2.1 *(Uniqueness of the fundamental solution).* Suppose that Γ_1 and Γ_2 are two fundamental solutions for the operator $\mathcal{K}$. For every $\varphi \in C_b^\infty(\mathbb{R}^2)$ we define

$$u_1(x,y,t) = \int_{\mathbb{R}^2} \Gamma_1(x,y,t;\xi,\eta,t_0)\,\varphi(\xi,\eta)\,d\xi\,d\eta, \quad u_2(x,y,t) = \int_{\mathbb{R}^2} \Gamma_2(x,y,t;\xi,\eta,t_0)\,\varphi(\xi,\eta)\,d\xi\,d\eta$$

and we note that are bounded classical solutions to the same Cauchy problem (2. 51). Then $u_1 = u_2$ by Theorem 2.2. Since φ is arbitrarily chosen we have that $\Gamma_1 = \Gamma_2$. $\square$

PROOF OF LEMMA 2.22. For every $R > 0$, let us consider the following cylinder

$$Q_R := \big\{(\xi,\eta,\tau) \in \mathbb{R}^2 \times]t_0, t] \mid |(\xi,\eta)| \le R+2, \tau \le t\big\}, \tag{2. 54}$$

which is a slight modification of the cylinder $Q_{R,\delta}$ previously introduced in the proof of Proposition 2.23. Let us consider the fundamental solution Γ_K associated with the operator $\mathcal{K}$. By definition, Γ_K satisfies the equation $\mathcal{K}u = 0$. Thus, by multiplying the equation by a certain test function $\varphi(x,y,t) \in C_0^\infty(\mathbb{R}^3)$, integrating on the cylinder Q_R and then proceeding by parts, we get the following equality

$$\begin{aligned} 0 = &- \int_{\mathcal{Q}_R} \Big\langle a \frac{\partial \Gamma_K}{\partial \xi}(x,y,t;\xi,\eta,\tau), \frac{\partial \varphi}{\partial \xi}\Big\rangle\, d\xi\, d\eta\, d\tau + \int_{\mathcal{Q}_R} b\, \frac{\partial \Gamma_K}{\partial \xi}(x,y,t;\xi,\eta,\tau)\,\varphi\, d\xi\, d\eta\, d\tau + \\ &+ \int_{\mathcal{Q}_R} Y\Gamma_K(x,y,t;\xi,\eta,\tau)\varphi\, d\xi\, d\eta\, d\tau - \int_{\mathcal{Q}_R} r\Gamma_K(x,y,t;\xi,\eta,\tau)\,\varphi\, d\xi\, d\eta\, d\tau. \end{aligned}$$

This equality is also known as *weak formulation* of the equation $\mathcal{K}u = 0$, and for the sake of clarity from now on we set $\Gamma_K = \Gamma_K(x,y,t;\xi,\eta,\tau)$. In particular, as a test function we can consider $\varphi(\xi,\eta,\tau) := [h_{R+1}(\xi,\eta) - h_{R-1}(\xi,\eta)]^2\,\Gamma_K$, where h_R is the same smooth function introduced in the proof of Proposition 2.23, we get

$$0 \le \varphi \le 1, \quad \varphi = \begin{cases} 0 & \text{for} \quad |(\xi,\eta)| \le R-1, \\ (1-h_{R-1})^2\,\Gamma_K & \text{for} \quad R-1 < |(\xi,\eta)| \le R \\ \Gamma_K & \text{for} \quad R < |(\xi,\eta)| \le R+1 \\ h_{R+1}^2\,\Gamma_K & \text{for} \quad R+1 < |(\xi,\eta)| < R+2 \\ 0 & \text{for} \quad |(\xi,\eta)| \ge R+2. \end{cases}$$

Since $\partial_\xi \varphi = (h_{R+1} - h_R)\,\partial_\xi \Gamma_K + 2\,(h_{R+1} - h_R)\,\Gamma_K \partial_\xi\,(h_{R+1} - h_R)$, assumption $(\mathbf{H}_K)$ holds true and the first and second order derivatives of the function h_R are bounded as in (2. 53), we get the following inequality

$$\begin{aligned} \lambda \int_{t_0}^{t} \int_{\widetilde{Q}_R} \left|\frac{\partial \Gamma_K}{\partial \xi}\right|^2 \le\; & \boxed{2\lambda \int_{\mathcal{Q}_R} \left|\Gamma_K\,(h_{R+1} - h_{R-1})\,\frac{\partial \Gamma_K}{\partial \xi}\,\frac{\partial}{\partial \xi}\,(h_{R+1} - h_{R-1})\right|}_A + \\ & + \boxed{\frac{1}{2}\int_{\mathcal{Q}_R} Y\,(\Gamma_K{}^2)\,(h_{R+1}(\xi,\eta) - h_{R-1}(\xi,\eta))^2}_B \\ & + \boxed{\Lambda \int_{\mathcal{Q}_R} \left|\Gamma_K\,\frac{\partial \Gamma_K}{\partial \xi}\,(h_{R+1} - h_{R-1})^2\right|}_C + \Lambda \int_{\mathcal{Q}_R} \Gamma_K{}^2\,(h_{R+1} - h_{R-1})^2, \end{aligned} \tag{2. 55}$$

where the set $\widetilde{Q}_R$ has previously been defined in (2. 50). Now, we can estimate terms A and C by Young's inequality. As far as we are concerned with term B, we begin considering the following identity:

$$(h_{R+1} - h_R)^2\,Y(\Gamma_K{}^2) \;=\; Y\left(\Gamma_K{}^2\,(h_{R+1} - h_R)^2\right) \;-\; \Gamma_K{}^2\,Y\left((h_{R+1} - h_R)^2\right).$$

Thus, we can rewrite term B as the sum of two terms, and by applying the divergence theorem to B_1 (${\Gamma_K}^2 \left(h_{R+1} - h_R\right)^2$ is null on the lateral boundary of $\widetilde{Q}_R$), we get

$$\begin{aligned}
&\underbrace{\boxed{\frac{1}{2}\int_{\mathcal{Q}_R} Y\left({\Gamma_K}^2\right)\left(h_{R+1}(\xi,\eta) - h_R(\xi,\eta)\right)^2}}_{B} \leq \\
&\leq \underbrace{\boxed{\frac{1}{2}\int_{\mathcal{Q}_R} Y\left({\Gamma_K}^2 \left(h_{R+1} - h_R\right)^2\right)}}_{B_1} + \underbrace{\boxed{\int_{\mathcal{Q}_R} {\Gamma_K}^2\, Y\left(\left(h_{R+1} - h_R\right)^2\right)}}_{B_2} \\
&\leq \frac{1}{2}\int_{\widetilde{Q}_R} \Gamma_K(x,y,t;\xi,\eta,t_0)^2 d\xi d\eta + 2C_0 \int_{t_0}^{t}\int_{\widetilde{Q}_R} {\Gamma_K}^2 d\xi d\eta d\tau.
\end{aligned}$$

By choosing $\varepsilon = \frac{\lambda}{2(4\lambda C_1 - \Lambda)}$ we get

$$\frac{\lambda}{2}\int_{t_0}^{t}\int_{\widetilde{Q}_R}\left|\frac{\partial \Gamma_K}{\partial \xi}\right|^2 d\xi d\eta d\tau \leq C_3 \int_{t_0}^{t}\int_{\widetilde{Q}_R} {\Gamma_K}^2 d\xi d\eta d\tau + \frac{1}{2}\int_{\widetilde{Q}_R} {\Gamma_K}^2(x,y,t;\xi,\eta,t_0) d\xi d\eta,$$

where $C_3 = C_3(\lambda, \Lambda, C_0, C_1)$ is a positive constant. □

2.4 Existence and uniqueness of the fundamental solution Γ_L

This section is devoted to the proof of our main results concerning the existence of the fundamental solution Γ_L for the operator $\mathscr{L}$. The proof relies on Theorem 2.19, and is analogous to the proof of the existence of the fundamental solution Γ_K for the operator $\mathcal{K}$ presented in Section 2.3.

Existence of Γ_L

Proof of Theorem 2.12 *(Existence of the fundamental solution).* The proof of this theorem is analogous to the proof of Theorem 2.1. In this case, we construct a sequence of operators $(\mathscr{L}_n)_{n\in\mathbb{N}}$ satisfying the assumptions of Theorem 2.19. In particular, we need the coefficients a_n, b_n to be smooth and satisfying a suitable version of the condition (2. 44). For this reason, we introduce a non-negative function $\rho \in C_0^\infty(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \rho = 1, \qquad B_0 := \operatorname{supp}\rho \subset \left\{(x,y,t) \in \mathbb{R}^3 \mid x^2 + y^2 + t^2 < \tfrac{1}{4}\right\},$$

and then proceed with a standard mollifying procedure. In order to take into consideration the fact that the domain of the coefficients a and b is $\mathbb{R}^+ \times \mathbb{R}^2$, for every $(x,y,t) \in \mathbb{R}^+ \times \mathbb{R}^2$ and for every $n \in \mathbb{N}$ we set

$$\begin{aligned}
a_n(x,y,t) &:= \int_{B_0} a\left(x - \tfrac{x\xi}{n}, y - \tfrac{\eta}{n}, t - \tfrac{\tau}{n}\right)\rho(\xi,\eta,\tau)\, d\xi\, d\eta\, d\tau, \\
b_n(x,y,t) &:= \int_{B_0} b\left(x - \tfrac{x\xi}{n}, y - \tfrac{\eta}{n}, t - \tfrac{\tau}{n}\right)\rho(\xi,\eta,\tau)\, d\xi\, d\eta\, d\tau.
\end{aligned}$$

Note that $\left(x - \frac{x\xi}{n}, y - \frac{\eta}{n}, t - \frac{\tau}{n}\right) \in B(x,y,t)$ for every $(\xi,\eta,\tau) \in B_0$ and for every $n \in \mathbb{N}$, where

$$B(x,y,t) := \left[\tfrac{1}{2}x, \tfrac{3}{2}x\right] \times \left[y - \tfrac{1}{2}, y + \tfrac{1}{2}\right] \times \left[t - \tfrac{1}{2}, t + \tfrac{1}{2}\right].$$

Then for every $n \in \mathbb{N}$ the coefficients a_n are smooth and satisfy the following version of $(\mathbf{H}_L)$

$$|a_n(x,y,t)| \le \sup_{B(x,y,t)} |a| \le \Lambda, \quad \left|\frac{\partial a_n}{\partial x}(x,y,t)\right| \le \sup_{B(x,y,t)} \left|\frac{\partial a}{\partial x}\right| \le \frac{2\Lambda}{x}, \quad a_n(x,y,t) \ge \inf_{B(x,y,t)} a \ge \lambda.$$

The same statement holds true for the coefficients b_n, with $n \in \mathbb{N}$. Then we apply Theorem 2.19 to the operator $\mathscr{L}_n$ for every $n \in \mathbb{N}$. Thus, there exists a sequence of equibounded fundamental solutions $(\Gamma_L^n)_{n\in\mathbb{N}}$, in the sense that each of them satisfies (2. 45).

Then we apply the same diagonal argument as in the proof of Theorem 2.1, but with a different choice for the open sets $(\Omega_p)_{p\in\mathbb{N}}$ of $(\mathbb{R}^+ \times \mathbb{R}^2)^2$. Indeed, we define

$$\Omega_p := \left\{ \begin{matrix} (x,y,t;\xi,\eta,\tau) \in (\mathbb{R}^+ \times \mathbb{R}^2)^2 \mid x^2+y^2+t^2 \le p^2, \quad \xi^2+\eta^2+\tau^2 \le p^2 \\ (x-\xi)^2+(y-\eta)^2+(t-\tau)^2 \ge \frac{1}{2p}, \quad x > \frac{1}{p}, \quad \xi > \frac{1}{p} \end{matrix} \right\},$$

such that $\bigcup_{p=1}^{+\infty} \Omega_p = \left\{(x,y,t;\xi,\eta,\tau) \in (\mathbb{R}^+ \times \mathbb{R}^2)^2 \mid (x,y,t) \neq (\xi,\eta,\tau)\right\}$ and $\Omega_p \subset\subset \Omega_{p+1}$ for every $p \in \mathbb{N}$. Thus, we define a function Γ_L in the following way: for every $(x,y,t),(\xi,\eta,\tau) \in \mathbb{R}^+ \times \mathbb{R}^2$ with $(x,y,t) \neq (\xi,\eta,\tau)$ we choose $q \in \mathbb{N}$ such that $(x,y,t;\xi,\eta,\tau) \in \Omega_q$ and we set $\Gamma_L(x,y,t;\xi,\eta,\tau) := \Gamma_q(x,y,t;\xi,\eta,\tau)$. This definition is well-posed, since if $(x,y,t) \in \Omega_p$, then $\Gamma_p(x,y,t;\xi,\eta,\tau) = \Gamma_q(x,y,t;\xi,\eta,\tau)$.

We next check that Γ_L has the properties listed in the statement of the Theorem 2.12. As every $\Gamma_L^n(x,y,t;x_0,y_0,t_0) = 0$ whenever $t \le t_0$ or $y \ge y_0$, also $\Gamma_L(x,y,t;x_0,y_0,t_0) = 0$ whenever $t \le t_0$ or $y \ge y_0$. For the same reason, it satisfies (2. 45). Moreover, for every $(x_0,y_0,t_0) \in \mathbb{R}^+ \times \mathbb{R}^2$, $(x,y,t) \mapsto \Gamma_L(x,y,t;x_0,y_0,t_0) \in L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^2) \cap C^{2+\alpha}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^2 \setminus \{(x_0,y_0,t_0)\})$, and is a classical solution to $\mathscr{L}u = 0$ in $\mathbb{R}^+ \times \mathbb{R}^2 \setminus \{(x_0,y_0,t_0)\}$. Analogously, $(\xi,\eta,\tau) \mapsto \Gamma_L(x_0,y_0,t_0;\xi,\eta,\tau) \in L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^2) \cap C^{2+\alpha}_{\text{loc}}(\mathbb{R}^3 \setminus \{(x_0,y_0,t_0)\})$ and is a classical solution to $\mathscr{L}^* v = 0$ in $\mathbb{R}^+ \times \mathbb{R}^2 \setminus \{(x_0,y_0,t_0)\}$. This proves the point *1.* of the Definition 2.18 and the point *1.* of Theorem 2.12. We remark that points *3.* and *4.* of Theorem 2.12 follow immediately from the construction of the fundamental solution Γ_L and the pointwise convergence. As far as we are concerned with the reproduction property *2.* of Theorem 2.12, we proceed as in the proof of Theorem 2.1 thanks to Corollary 2.20.

To proceed with the proof of Theorem 2.12 we have to verify that for every $\varphi \in C_b(\mathbb{R}^2)$ the function

$$u(x,y,t) = \int_{\mathbb{R}^2} \Gamma_L(x,y,t;\xi,\eta,t_0)\,\varphi(\xi,\eta)\,d\xi\,d\eta$$

is a classical solution to the Cauchy problem

$$\begin{cases} \mathscr{L}u(x,y,t) = 0, & (x,y,t) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+; \\ u(x,y,t_0) = \varphi(x,y) & (x,y) \in \mathbb{R}^+ \times \mathbb{R}. \end{cases}$$

By a very standard argument we differentiate under the integral sign and we find

$$\mathscr{L}u(x,y,t) = \int_{\mathbb{R}^+ \times \mathbb{R}} \mathscr{L}\Gamma_L(x,y,t;\xi,\eta,t_0)\,\varphi(\xi,\eta)\,d\xi\,d\eta = 0.$$

Thus, to conclude the proof we have to verify that for any $(x_0,y_0) \in \mathbb{R}^+ \times \mathbb{R}$ we have

$$\lim_{(x,y,t)\to(x_0,y_0,0)} u(x,y,t) = \varphi(x_0,y_0). \tag{2. 56}$$

The proof of this fact is based on the use of "barriers", and on Theorems 6.1 and 6.3 of [90]. The following argument relies on the fact that the operator $\mathscr{L}$ behaves as the operator $\mathcal{K}$ in every compact set of $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$. Let us consider the sequence of functions

$$u_n(x,y,t) = \int_{\mathbb{R}^2} \Gamma_L^n(x,y,t;\xi,\eta,t_0)\,\varphi(\xi,\eta)\,d\xi\,d\eta$$

and note that $u(x,y,t) = \lim_{n\to\infty} u_n(x,y,t)$. Since Γ_L^n is the fundamental solution of $\mathscr{L}_n$, we have that

$$\lim_{(x,y,t)\to(x_0,y_0,0)} u_n(x,y,t) = \varphi(x_0,y_0) \quad \text{for every}\, n \in \mathbb{N}. \tag{2. 57}$$

Let us introduce the cylinder

$$Q := \left]\tfrac{1}{2}x_0, \tfrac{3}{2}x_0\right[\times]y_0 - 1, y_0 + 1[\times]0, T[$$

centered at $(x_0, y_0, 0)$. As the sequence $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded, it is possible to construct two barrier functions u^+ and u^-, a super and a sub solution respectively, such that

$$u^-(x,y,t) \le u_n(x,y,t) \le u^+(x,y,t) \quad \text{for every}\, (x,y,t) \in Q$$

and for every $n \in \mathbb{N}$, and such that

$$\lim_{(x,y,t)\to(x_0,y_0,0)} u^-(x,y,t) = \varphi(x_0,y_0), \qquad \lim_{(x,y,t)\to(x_0,y_0,0)} u^+(x,y,t) = \varphi(x_0,y_0).$$

The claim (2. 56) directly follows. □

Remark 2.24 *The linear growth of the initial condition in the Cauchy problem* (2. 4) *is allowed in the formula* (2. 42)*. Indeed, the Corollary 2.20 holds for the operator $\mathscr{L}$ satisfying the assumption* $(\mathbf{H}_L)$*, and it is known that the Geman-Yor process* (2. 30) *has finite first order moments.*

Uniqueness and comparison principle for the operator $\mathscr{L}$

Following the steps of the proof of Theorem 2.2 for the uniqueness of the solution for the Cauchy problem associated to the operator $\mathcal{K}$, we need to prove an intermediate result for the operator $\mathscr{L}$, also known as comparison principle. In particular, we can apply the general result due to Aronson and Besala proved in [10], that in the case of the operator $\mathscr{L}$ reads as follows.

THEOREM B *(Aronson - Besala). Let us consider for $T > 0$ the open set $\Omega = \mathbb{R}^+ \times \mathbb{R}\times]0, T]$ and let $\mathscr{L}$ be the differential operator defined in* (2. 28) *under the assumption* $(\boldsymbol{H}_L)$*. If u is a classical solution of $\mathscr{L}u \le 0$ in Ω such that*

$$u(x,y,0) \ge 0, \quad \textit{for}\, (x,y) \in \mathbb{R}^+ \times \mathbb{R} \quad \textit{and} \quad u(0,y,t) \ge 0 \quad \textit{for}\, (y,t) \in \mathbb{R}\times]0,T]$$

and for some positive constant M and k

$$u(x,y,t) \ge -M \,\exp\left\{k \,\log\left(x^2 + y^2 + 1\right) + 1\right\}^2$$

in Ω, then $u(x,y,t) \ge 0$ in $\overline{\Omega}$.

We remark that the above results would be enough to ensure the uniqueness of the solution for the Cauchy problem associated to the operator $\mathscr{L}$ in the form (2. 28). Nevertheless, when considering the operator $\mathscr{L}$ with locally Hölder continuous coefficients and satisfying the assumption $(\mathbf{H}_L)$ as in our case, we can improve the previous result by requiring the solution u to have a positive sign only on the boundary related to the initial data

$$u(x,y,0) \ge 0, \quad \text{for}\, (x,y) \in \mathbb{R}^+ \times \mathbb{R},$$

and getting rid of the sign assumption on the part of the boundary $\{0\} \times \mathbb{R}\times]0,T]$.

Theorem 2.25 *Let us consider for $T>0$ the open set $\Omega = \mathbb{R}^+ \times \mathbb{R} \times]0,T]$ and let $\mathscr{L}$ be the differential operator defined in* (2. 28) *under the assumption* $(\boldsymbol{H}_L)$. *If u is a classical solution of $\mathscr{L}u \le 0$ in Ω such that*

$$u(x,y,0) \ge 0, \quad \textit{for}\,(x,y) \in \mathbb{R}^+ \times \mathbb{R}, \tag{2. 58}$$

and for some positive constant M and k

$$u(x,y,t) \ge -M \exp\left(C(\log(x^2+y^2+1) - \log(x)) + 1\right)^2, \tag{2. 59}$$

for every $(x,y,t) \in \mathbb{R}^+ \times \mathbb{R} \times]0,T]$. Then $u \ge 0$ in $\mathbb{R}^+ \times \mathbb{R} \times [0,T]$.

PROOF Let us consider, for a given $\beta > 0$, the auxiliary function

$$v(x,y,t) = \exp\left(2e^{\beta t}\, C(\log(x^2+y^2+1) - \log(x)) + 1\right)^2.$$

It is easily verified that if $t \in]0, 1/\beta]$ we have

$$\mathscr{L}v(x,y,t) \le e^{\beta t} v \left(C(\log(x^2+y^2+1) - \log(x)) + 1\right)^2 (E - 2\beta),$$

where E is a positive constant only depending on the constants C and λ, Λ appearing in $(\mathbf{H}_L)$. Thus, if we set $\beta = E$ it follows that $\mathscr{L}v < 0$ in $\mathbb{R}^+ \times \mathbb{R} \times]0, 1/\beta]$.

In the following, we let $\widetilde{\psi}(x,y) := \log(x^2+y^2+1) - \log(x)$ and we note that, for every $K > \log(2)$, we have

$$\left\{(x,y) \in \mathbb{R}^+ \times \mathbb{R} \mid \widetilde{\psi}(x,y) < K\right\} = \left\{(x,y) \in \mathbb{R}^2 \mid (x - x_K)^2 + y^2 < r_K^2\right\},$$

where $x_K := \frac{e^K}{2}$ and $r_K = \sqrt{x_K^2 - 1}$. If $\beta > 0$ is as above, we consider, for arbitrary $K > \log(2)$ and $M > 0$, the function

$$w(x,y,t) = u(x,y,t) + Me^{-(CK+1)^2} v(x,y,t),$$

It is clear that $\mathscr{L}w < 0$ in $\mathbb{R}^+ \times \mathbb{R} \times]0, 1/\beta]$, and that $w(x,y,0) \ge 0$ for $(x,y) \in \mathbb{R} \times \mathbb{R}^+$, by (2. 58). Moreover, because of (3.14), we have that

$$w(x,y,t) \ge 0 \qquad \text{for}\,(x,y,t) \in \left\{\mathbb{R}^+ \times \mathbb{R} \times [0,T] \mid \widetilde{\psi}(x,y) = K\right\}.$$

From the weak minimum principle it follows that $w(x,y,t) \ge 0$ for every $(x,y,t) \in \mathbb{R}^+ \times \mathbb{R} \times [0, 1/\beta]$ such that $\widetilde{\psi}(x,y) \le K$.

Now, if (x,y,t) is any point in $\mathbb{R}^+ \times \mathbb{R} \times]0, 1/\beta]$, we choose K such that $\widetilde{\psi}(x,y) \le K$, and by the above argument it follows that $w(x,y,t) \ge 0$. The case $t > 1/\beta$ straightly follows by repeating the above argument. □

PROOF OF THEOREM 2.13. This uniqueness result plainly follows from Proposition 2.25 firstly applied to $u = u_1 - u_2$, and then to $u = u_2 - u_1$. □

PROOF OF THEOREM 2.12 *(Uniqueness of the fundamental solution).* Suppose that Γ_1 and Γ_2 are two fundamental solutions for the operator $\mathscr{L}$. For every $\varphi \in C_b^\infty(\mathbb{R}^2)$ we define

$$u_1(x,y,t) = \int_{\mathbb{R}^2} \Gamma_1(x,y,t;\xi,\eta,t_0)\, \varphi(\xi,\eta)\, d\xi\, d\eta, \quad u_2(x,y,t) = \int_{\mathbb{R}^2} \Gamma_2(x,y,t;\xi,\eta,t_0)\, \varphi(\xi,\eta)\, d\xi\, d\eta$$

and we note that are bounded classical solutions to the same Cauchy problem. Then $u_1 = u_2$ by Theorem 2.13. Since φ is arbitrarily chosen we have that $\Gamma_1 = \Gamma_2$. □

Chapter 3

On a nonlinear kinetic Kolmogorov-Fokker-Planck model: well-posedness results in Hölder spaces and diffusion asymptotics

This chapter is devoted to the study of an application of the Kolmogorov equation to the kinetic theory. The new results we present here are part of a joint project with Yuzhe Zhu from the ENS of Paris (France), where the author has spent a research period under the supervision of Prof. Cyril Imbert (CNR). In particular, we are interested in proving well-posedness results and diffusion asymptotics in Hölder spaces for positive solutions $u = u(v,x,t) \geq 0$ to the following Cauchy problem

$$\begin{cases} (\partial_t + v \cdot \nabla_x)\, u(v,x,t) = \rho_u^\beta(x,t)\, \nabla_v \cdot (\nabla_v + v)\, u(v,x,t), & (v,x,t) \in \mathbb{R}^n \times \mathbb{T}^n \times (0,T] \\ u(v,x,0) = \varphi(v,x) & (v,x) \in \mathbb{R}^n \times \mathbb{T}^n, \end{cases} \tag{3. 1}$$

where T is a positive constant, $\beta \in [0,1]$ and the nonlinear term ρ_u is defined as follows

$$\rho_u(x,t) := \int_{\mathbb{R}^n} u(v,x,t)\, dv, \qquad \text{with} \quad \rho_\varphi(x) = \int_{\mathbb{R}^n} \varphi(v,x)\, dv. \tag{3. 2}$$

From now on, we denote by $\mathscr{L}$ the stationary kinetic Kolmogorov-Fokker-Planck diffusive operator

$$\mathscr{L}u(v,x,t) := \nabla_v \cdot (\nabla_v + v)\, u(v,x,t) = \nabla_v \cdot (\nabla_v u(v,x,t) + v u(v,x,t)) \tag{3. 3}$$

appearing on the right-hand side of equation (3. 1). This operator only acts on the velocity variable and ceases to be dissipative on its unique steady state μ, that is the following global Maxwellian

$$\mu(v) := (2\pi)^{-\frac{n}{2}} e^{-\frac{|v|^2}{2}} \qquad v \in \mathbb{R}^n. \tag{3. 4}$$

For this reason is natural to assume Gaussian bounds for the initial data φ of this type

$$\varphi(v,x) \leq \Lambda\mu(v) \quad \text{or} \quad \lambda\mu(v) \leq \varphi(v,x) \leq \Lambda\mu(v) \quad \text{for } (v,x) \in \mathbb{R}^n \times \mathbb{T}^n.$$

The nonlinear drift-diffusion equation appearing in (3. 1) arises in various different research fields, such as plasma physics and polymer dynamics, and it is a fundamental tool for the modeling of the collisional evolution of a system of a large number of particles. From the perspective of a stochastic process $\{(X_t, V_t) : t \geq 0\}$

$$\begin{cases} dX_t = V_t dt, \\ dV_t = V_t\, dt + \sqrt{2\rho_u^\beta(X_t,t)}\; dW_t, \end{cases}$$

driven by a Brownian motion $\{W_t\}_{t\geq 0}$, the function $\mu^{-1}u$ is the evolving density of the law of the process $\{(X_t, V_t) : t \geq 0\}$, where u is a solution to the equation in (3. 1) and μ is the global Maxwellian introduced in (3. 4). For further information we refer to the following works [27] and [117], respectively by Chado and Villani. Moreover, if we recall the definition of the stationary kinetic Kolmogorov-Fokker-Planck diffusive operator $\mathscr{L}$ introduced in (3. 3), the nonlinear diffusive collision term $\rho_u^\beta \mathscr{L}$ models the collision of particles in a certain surrounding bath, where the aggregation of particles induces friction contribution. Specifically, on one hand the diffusion coefficient ρ_u^β describes that the friction effect in the collision interaction is positively correlated to the mass of particles occupying the position x at time t. On the other hand, the drift-diffusion operator $\mathscr{L}$, that only acts on the velocity variable and ceases to be dissipative on its unique steady state μ, ensures that its null space is spanned by the global Maxwellian μ and the local conservation law of mass is satisfied.

As far as we are concerned with well-posedness results for the Cauchy problem (3. 1), in the framework of Sobolev spaces results of this type have been proved by Imbert and Mouhot in [68], where the authors also prove higher order Schauder estimates for solutions to the equation (3. 1), and by Liao, Wang and Yang in [87]. Our results improve that of [68] and [87] because we study the Cauchy problem (3. 1) in Hölder spaces C^α (see Definition 3.7). Indeed, given that the initial data φ is bounded from above $\varphi \leq \Lambda\mu$ by the global Maxwellian introduced in (3. 4) and for every $T > 0$ there exists a unique positive solution to (3. 1) that is smooth for every $t > 0$.

Theorem 3.1 *Let us consider the Cauchy problem* (3. 1)*, with $\beta \in [0, 1]$. Let $\varphi \in C(\mathbb{R}^n \times \mathbb{T}^n)$ and let $0 < \lambda \leq \Lambda$ be two positive constants such that the following bounds for the initial data φ and the nonlinear term ρ_u defined in* (3. 2) *hold true:*

$$0 \leq \varphi(v, x) \leq \Lambda\mu(v) \;\text{ in } \mathbb{R}^n \times \mathbb{T}^n \qquad \textit{and} \qquad \rho_\varphi(x) \geq \lambda \;\text{ in } \mathbb{T}^n. \tag{3. 5}$$

If φ is continuous in $\mathbb{R}^n \times \mathbb{T}^n$, for every $T > 0$ there exists a unique positive solution u to (3. 1) *in $\mathbb{R}^n \times \mathbb{T}^n \times [0, T]$ such that for any $k \in \mathbb{N}$ and $\tau \in (0, T)$, we have*

$$\|u\|_{W^{k,\infty}(\mathbb{R}^n\times\mathbb{T}^n\times[\tau,T))} \leq C_{k,\tau}, \tag{3. 6}$$

for some constant $C_{k,\tau} > 0$ only depending on α, β, λ, Λ, n, τ, T and k.

On one hand, the proof of the well-posedness result is presented in Section 3.3, see Proposition 3.19 and Proposition 3.20. In particular, the proof of Proposition 3.19 relies on the Schauder-Fixed point theorem, as well as a mass-spreading result based on the Harnack inequality (3. 17) and a barrier function method inspired by the works [62] and [63] for the Landau equation and the Boltzmann equation (see Proposition 3.14, Lemma 3.15 and Lemma 3.16).

On the other hand, the C^∞ a priori estimates (3. 6) are obtained through an iterative procedure that was firstly introduced by Imbert and Silvestre in [69] for the Boltzmann equation that we adapt here to the Cauchy problem (3. 1) in Section 3.4.

Lastly, since the spatial inhomogeneous nonlinear equation (3. 1) locally behaves as the classical Kolmogorov equation $\mathcal{K}u = f$, where the linear operator $\mathcal{K}$ is defined as

$$(\partial_t + v \cdot \nabla_x)\, u(v, x, t) = \operatorname{Tr}\left(\overline{A} D_v^2 u(v, x, t)\right) + \underline{b} \cdot \nabla_v u(v, x, t) + cu(v, x, t), \tag{3. 7}$$

in Section 3.1 we recall some basic facts regarding this equation and the associated Cauchy problem in the more general domain $\mathbb{R}^n \times \mathbb{R}^n \times [0, T]$. Moreover, we recall the statement of the Harnack inequality proved by Golse, Imbert, Mouhot and Vasseur in [58] for the linear equation (3. 7), that is the fundamental tool for the construction of the Harnack chain argument. Finally, in Theorem 3.12 we prove Global Schauder estimates for solutions to (3. 18). We remark that all of these results can be restricted to the domain $\mathbb{T}^n \times \mathbb{R}^n \times [0,T]$ when required.

In the future, our aim is to study, with the help of the estimate (3. 6), the exponential stability of the global equilibria for the equation (3. 1) following the entropic hypocoercivity method developed by Villani in [118] to study the spacial inhomogeneous kinetic equation in a H^1-framework and through a macro-micro scheme decomposition (see [46], [43] and [66]). Moreover, we plan to study the asymptotic stability as $\varepsilon \to 0$ of the following scaled equations

$$\begin{cases} (\varepsilon\partial_t + v\cdot\nabla_x)\, u_\varepsilon(v,x,t) = \frac{1}{\varepsilon}\rho^{\beta}_{u_\varepsilon}(x,t)\,\mathscr{L}u_\varepsilon(v,x,t), \\ u_\varepsilon(v,x,0) = \varphi_\varepsilon(v,x), \end{cases} \tag{3. 8}$$

obtained by applying the parabolic scaling $t \mapsto \varepsilon^2 t$, $x \mapsto \varepsilon x$ to (3. 1) and where $\varepsilon \in (0,1)$ denotes the ratio of the mean free path (microscopic scale) to the typical macroscopic length. In particular, our aim is to prove that the scaled equations (3. 8) lead to the fast porous medium flow equation under the diffusive limit.

3.1 The classical Kolmogorov equation

This section contains a survey of results regarding the Kolmogorov equation (3. 7), that is the linear counterpart of the nonlinear spacial inhomogeneous equation (3. 1), and the associated Cauchy problem in the more general domain $\mathbb{R}^n\times\mathbb{R}^n\times(0,T)$. Moreover, we recall the statement of the Harnack inequality proved by Golse, Imbert, Mouhot and Vasseur in [58] for the linear equation (3. 7), that is the fundamental tool for the construction of the Harnack chain argument appearing in the proof of Theorem 3.1. Finally, in Theorem 3.12 we prove Global Schauder estimates for solutions to (3. 18). We remark that all of these results can be restricted to the domain $\mathbb{T}^n\times\mathbb{R}^n\times(0,T)$ when required. Let us consider the partial differential operator associated to the Kolmogorov equation (3. 7):

$$\mathcal{K}u(v,x,t) := \partial_t u(v,x,t) + v\cdot\nabla_x u(v,x,t) - \mathrm{Tr}\left(\overline{A}D^2_v u(v,x,t)\right) - \underline{b}\cdot\nabla_v u(v,x,t) - cu(v,x,t). \tag{3. 9}$$

Indeed, it is a classical Kolmogorov operator of the type (1. 33) in $\mathbb{R}^{2n+1}$ and can be recovered from (1. 33) by choosing $\underline{b}=0$, $c=0$,

$$A = \begin{pmatrix} -\overline{A} & \mathbb{O}_n \\ \mathbb{O}_n & \mathbb{O}_n \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} \mathbb{O}_n & \mathbb{O}_n \\ \mathbb{I}_n & \mathbb{O}_n \end{pmatrix}.$$

Since the Kolmogorov operator $\mathcal{K}$ is the linear counterpart of the nonlinear operator introduced in (3. 1) and the two of them locally agree in every compact subset of $\mathbb{R}^{2n+1}$, we then borrow the geometrical setting and the regularity theory developed for the Kolmogorov operator (3. 7) to study the nonlinear spacial inhomogeneous operator introduced in (3. 1). Thus, in this section we recall some notation and known results concerning the geometrical structure underlying the operator (3. 7). For a comprehensive treatment of this subject we refer to Chapter 1. First of all, equations of the form (3. 7) are left translation invariant with respect to the Lie product "$\circ$" introduced in (1. 12), that in dimension $\mathbb{R}^{2n+1}$ reads as follows

$$(v_1,x_1,t_1)\circ(v_2,x_2,t_2) = (v_1+v_2, x_1+x_2+t_2v_1, t_1+t_2), \tag{3. 10}$$

where $(v_1,x_1,t_1),(v_2,x_2,t_2)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$. Since the couple (v,x) denotes the position and the velocity of a particle, the above group operation is also known as *Galilean* transformation. One can also define the inverse element z^{-1} for a certain $z=(v,x,t)\in\mathbb{R}^{2n+1}$ as follows

$$z^{-1} := (-v, -x+tv, -t).$$

Since the matrix B is of the form (1. 18), by Proposition 1.4 we have that equation (3. 7) is left translation invariant with respect to the following family of dilations

$$\delta_r(v,x,t) := (rv, r^3x, r^2t), \qquad \text{for every } r>0. \tag{3. 11}$$

As already noticed in Chapter 1, the scaling $\{\delta_r\}_{r>0}$ and the Galilean traslation "$\circ$" naturally require a new definition for cylinders. Given $z_0 \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and $r>0$, we respectively define the unit cylinder and the scaled unit cylinder as follows

$$Q_1 := Q_1(0) = [-1,1]^n \times [-1,1]^n \times (-1,0] \qquad \text{and} \qquad Q_r := Q_r(0) = \delta_r(Q_1). \tag{3. 12}$$

Finally, we define the cylinder $Q_r(z_0) := \{z_0 \circ z : \, z \in Q_r\}$ with radius r and center at z_0. In particular, we have that

$$Q_r(z_0) = \left\{ (v,x,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} : \, |v-v_0| < r, \; |x-x_0-(t-t_0)v_0| < r^3, \, t_0 - r^3 < t \leq t_0 \right\}.$$

Therefore, it may also be convenient to work with a notion of distance, Hölder norm, kinetic degree and kinetic differential homogeneous operator with respect to the scaling $\{\delta_r\}_{r>0}$ and left-invariant with respect to the translation "$\circ$".

Definition 3.2 *Given $z_1 = (v_1,x_1,t_1), z_2 = (v_2,x_2,t_2)$ in $\mathbb{R}^{2n+1}$, we introduce the following distance*

$$d_l(z_1,z_2) := \min_{w \in \mathbb{R}^n} \left\{ \max\left(|v_1 - w|, |v_2 - w|, |x_1 - x_2 - (t_1-t_2)w|^{\frac{1}{3}}, |t_1 - t_2|^{\frac{1}{2}} \right) \right\}.$$

It may also be convenient to introduce the notion of length of a vector $z = (v,x,t) \in \mathbb{R}^{2n+1}$ as

$$\|z\| := \min_{w \in \mathbb{R}^n} \left\{ \max\left(|v-w|, |w|, |x-tw|^{\frac{1}{3}}, |t|^{\frac{1}{2}} \right) \right\}.$$

Remark 3.3 *The distance d_l is left invariant with respect to the Lie group action in the sense that $d_l(z \circ z_1, z \circ z_2) = d_l(z_1,z_2)$ for any $z, z_1, z_2 \in \mathbb{R}^{2n+1}$. Moreover, it is homogeneous of degree 1 with respect to $\{\delta_r\}_{r>0}$. Indeed, $d_l(\delta_r(z_1), \delta_r(z_2)) = r d_l(z_1,z_2)$.*

Remark 3.4 *As it is pointed out in Proposition 2.2 in [70] (we consider here $s=1$), d_l is indeed a distance in the sense that it satisfies the triangle inequality. Nevertheless, there are other equivalent formulations for the distance d_l, such as the one considered in Definition 1.10 of Chapter 1. We remark that this latter formulation is not a proper distance, in the sense that the triangle inequality fails, but instead holds true the quasi-trangle inequality (1. 26). Nevertheless Definition 1.10, and other alternative formulations such as the one of Remark 1.8, give us a good estimate of the distance d_l. For this reason, they are used whenever it is convenient.*

Remark 3.5 *Technically, $\|\cdot\|$ is a homogeneous semi-norm of degree 1 with respect to the family of dilations $\{\delta_r\}_{r>0}$ defined in (3. 11) and, as we point out in Definition 1.7 and Remark 1.8, there are several convenient equivalent expressions for it. Moreover, it does satisfy the traingle inequality with respect to the group action "$\circ$":*

$$\|z_1 \circ z_2\| \leq \|z_1\| + \|z_2\|. \tag{3. 13}$$

It may be convenient to define a modified notion of degree for a polynomial p in $\mathbb{R}[v,x,t]$ that matches the scaling of the equation. For this reason, it is called *kinetic degree.*

Definition 3.6 *Given a monomial m of the form*

$$m(v,x,t) = cv_1^{\alpha_{n+1}} \dots v_n^{\alpha_{2n}} x_1^{\alpha_1} \dots x_n^{\alpha_n} t^{\alpha_0} \quad \textit{with } c \neq 0,$$

we define its kinetic degree as

$$deg_k m = 2a_0 + 3\sum_{j=1}^{n} \alpha_j + \sum_{j=n+1}^{2n} \alpha_j.$$

A polynomial p in $\mathbb{R}[v,x,t]$ is always a finite sum of monomials. In general, we define the kinetic degree of $p = \sum m_j$, and we write $deg_k p$, as the maximum of $deg_k m_j$ for all its monomial terms m_j.

This definition of kinetic degree is justified by the fact that our notion of degree needs to be consistent with the family of dilations $\{\delta_r\}_{r>0}$. Let us consider a monomial m. Its degree is computed counting 2 times the exponent for the variable t, 3 times the exponent for the variables x_i and 1 time the exponent for the variables v_i. With this definition in mind, one can easily check that

$$m(rv, r^3 x, r^2 t) = r^{\deg_k m} m(v,x,t).$$

We remark that the kinetic degree of the zero polynomial is not properly defined in the definition above. It is appropriate to make it equal to $-\infty$, or -1. The fact that the kinetic degree of the zero polynomial is negative is relevant for the definition of the C_l^0 norm given in (3. 14).

In the same spirit, we need to introduce a properly scaled version of Hölder spaces. There are various proper definitions of Hölder spaces that may be feasible for our case. The one we consider here is due to Imbert and Silvestre and it was firstly introduced in [70]. We remark that when $\alpha \in (0,1)$ the folllwing definition is equivalent to the one considered by Manfredini in [90] and in Definition 1.11.

Definition 3.7 *Let Ω be an open subset of $\mathbb{R}^{2n+1}$. For any $\alpha \in (0,+\infty)$, a function $u : \Omega \longrightarrow \mathbb{R}$ is α-Hölder continuous at a point $z_0 \in \mathbb{R}^{2n+1}$ if there exists a polynomial $p \in \mathbb{R}[v,x,t]$ such that $deg_k p < \alpha$ and for any $z \in \Omega$*

$$|u(z) - p(z)| \leq C\, d_l(z,z_0)^\alpha \qquad \text{for every } z, \zeta \in \Omega.$$

When this property holds at every point z_0 in the domain Ω, with a uniform constant C, we say $u \in C_l^\alpha(\Omega)$. The semi-norm $[u]_{C_l^\alpha(\Omega)}$ is the smallest value of the constant C such that the above inequality holds for every $z, z_0 \in \Omega$. With this definition in mind, we have

$$[u]_{C_l^0(\Omega)} = \|u\|_{C_l^0(\Omega)} = \|u\|_{L^\infty(\Omega)}. \tag{3. 14}$$

Thus, we can define the C_l^α-norm of a function u to be

$$\|u\|_{C_l^\alpha(\Omega)} = \|u\|_{L^\infty(\Omega)} + [u]_{C_l^\alpha(\Omega)}. \tag{3. 15}$$

Remark 3.8 *When $\alpha \in (0,1)$ this definition coincides with Definition 1.11. Moreover, in Section 2.4 of [68] the authors prove that when $\beta = (2+\alpha) \in (2,3)$ the polynomial p realizing the infimum in the $C_l^\beta-$seminorm is the Taylor expansion of kinetic degree 2 defined as:*

$$\begin{aligned} T_{z_0}[u](v,x,t) := u(z_0) + (t-t_0)[\partial_t + v_0 \cdot \nabla_x]u(z_0) \\ + (v-v_0)\cdot\nabla_v u(z_0) + \tfrac{1}{2}(v-v_0)^T \cdot D_v^2 u(z_0)\cdot(v-v_0), \end{aligned} \tag{3. 16}$$

where the linear part in x does not appear since it is of kinetic degree 3. We also recall the work by Pagliarani, Pascucci and Pignotti [101], where the authors prove the Taylor expansion for $C^{k,\alpha}(\Omega)$ functions. It is worth noticing that the authors require in [101] a weaker regularity assumption for the definition of the space $C^{2+\alpha}$ than the one considered in [70], [68] and [90].

The usual interpolation estimates for Hölder spaces hold (see Proposition 2.10, [70]).

Proposition 3.9 *Given $0 \leq \alpha_1 < \alpha_2 < \alpha_3$ so that $\alpha_2 = \theta\alpha_1 + (1-\theta)\alpha_3$, we have for all function $u \in C_l^{\alpha_3}(Q_r(z_0))$*

$$[u]_{C_l^{\alpha_2}(Q_r(z_0))} \leq C\left([u]^{\theta}_{C_l^{\alpha_1}(Q_r(z_0))}[u]^{1-\theta}_{C_l^{\alpha_3}(Q_r(z_0))} + r^{\alpha_1-\alpha_2}[u]_{C_l^{\alpha_1}(Q_r(z_0))}\right),$$

for some constant C depending on α_1, α_3 and on dimension only.

Lastly, it may be convenient to define the kinetic degree of a differential operator. We say that the kinetic degree of $\partial_t + v\cdot\nabla_x$ is 2, the kinetic degree of ∂_{x_i} is 3 and the kinetic degree of ∂_{v_i} is 1. Eventually, we recall the following lemma proved by Imbert and Silvestre in [70] (see Lemma 2.7), that relates the definition of Hölder spaces C_l^{α} with the operators lastly introduced.

Proposition 3.10 *Let $D = \partial_t + v\cdot\nabla_x$, $D = \partial_{x_i}$ or $D = \partial_{v_i}$. Let u be a C_l^{α} function in a cilinder Q and let $deg_k D = \kappa$, with $\kappa < \alpha$. Then $Df \in C_l^{\alpha-\kappa}$ and*

$$[Du]_{C_l^{\alpha-\kappa}(Q)} \leq C[u]_{C_l^{\alpha}(Q)}.$$

3.1.1 Harnack inequality and Global Schauder estimates

Let us consider the Kolmogorov-Fokker-Planck equation (3. 7) under the following structural assumption for the matrix A and the vector $\underline{b}$.

(H) $\underline{b} = (b_i(v,x,t))_{i=1}^n$ is a vector and $\overline{A} = (a_{ij}(v,x,t))_{i,j=1,\dots,n}$ is a positive definite symmetric matrix in $\mathbb{R}^n$ and there exist two positive constants λ, Λ such that

$$\lambda\sum_{i=1}^n |\xi_i|^2 \leq \sum_{i,j=1}^n a_{ij}(z)\xi_i\xi_j \leq \Lambda\sum_{i=1}^n|\xi_i|^2$$

for every $\xi \in \mathbb{R}^n$ and $z \in \mathbb{R}^{2n+1}$.

The study of the regularity theory for this linear equation has widely been developed during the last decade and in particular, Golse, Imbert, Mouhot and Vasseur proved the following Harnack inequality for weak solutions (in the sense introduced in (4. 4) of Section 4) to the equation (3. 7).

Harnack inequality (Theorem 2 [58]). *There exist three constants $M > 1, R > 0, \Delta > 0$, with $0 < R^2 < \Delta < \Delta + R^2 < 1$, such that*

$$\sup_{Q^-} u \leq M(\inf_{Q^+} u + \|f\|_{L^\infty(Q_1)})$$

for every non-negative weak solution u to the equation (3. 7) on Q_1, with $f \in L^\infty(Q_1)$ and Q_1 is the unit box introduced in (3. 12). The constants M, R and Δ only depend on the dimension n and on the ellipticity constants λ and Λ. Moreover Q^+, Q^- are defined as follows

$$Q^+ = Q_R \text{ with } 0 < R^2 < \Delta < \Delta + R^2 < 1, \qquad Q^- = Q_R(0,0,-\Delta).$$

As Golse, Imbert, Mouhot and Vasseur notice in Remark 4 in [58], "using the transformation (3. 10), we get a Harnack inequality for cylinders centered at an arbitrary point (v_0, x_0, t_0)". We hereby recall the precise meaning of this assertion and we improve it by also using the dilation (3. 11). We refer to Theorem 4.6 of Chapter 4 for the proof of this statement.

Invariant Harnack inequality. *Let (v_0, x_0, t_0) be any point of $\mathbb{R}^{2n+1}$ and let r be a positive number. There exist three constants $M > 1, R > 0, \Delta > 0$, with $0 < R^2 < \Delta < \Delta + R^2 < 1$, such that*

$$\sup_{Q_r^-(v_0,x_0,t_0)} u \leq M\big(\inf_{Q_r^+(v_0,x_0,t_0)} u + \|f\|_{L^\infty(Q_r(v_0,x_0,t_0))}\big) \tag{3. 17}$$

for every non-negative weak solution u to the equation (3. 7) on $Q_r(v_0, x_0, t_0)$, with $f \in L^\infty(Q_r(v_0, x_0, t_0))$. The constants M, R and Δ only depend on the dimension n and on the ellipticity constant λ. Moreover $Q_r^+(v_0, x_0, t_0),^- Q_r(v_0, x_0, t_0)$ are defined as follows

$$Q_r^+(v_0,x_0,t_0) = (v_0,x_0,t_0) \circ d_r Q^+, \qquad Q_r^-(v_0,x_0,t_0) = (v_0,x_0,t_0) \circ d_r Q^-.$$

Let us consider the Cauchy problem associated to the linear Kolmogorov-Fokker-Planck equation (3. 7) under the structural assumption **(H)**:

$$\begin{cases} (\partial_t + v \cdot \nabla_x)\, u = \mathrm{Tr}\left(\overline{A} D_v^2 u\right) + \underline{b} \cdot \nabla_v u + cu + f & \text{in } \mathbb{R}^n \times \mathbb{R}^n \times (0,T), \\ u(v,x,0) = \varphi(v,x) & \text{in } \mathbb{R}^n \times \mathbb{R}^n, \\ \lim\limits_{|(v,x,t)| \to \infty} u(v,x,t) = 0. \end{cases} \tag{3. 18}$$

There exists a vast literature on the existence and uniqueness of the solution to (3. 18) related to the more general class of ultraparabolic equations (1. 33), for which the kinetic Fokker-Planck operator $\mathcal{K}$ introduced in (3. 9) is a particular case. In particular, Theorem 1.15, Theorem 1.16 and Theorem 1.17 of Chapter 1 are the most complete results we have at our disposal in this field. For the sake of completeness, we hereby recall the following well-posedness result for the Cauchy problem (3. 18), that can be seen as a particular case of the more general Theorem 1.15. Moreover, for the proof of an analogous statement we refer to Chapter 2, Theorem 2.1 and Theorem 2.2.

Proposition 3.11 *Let $T > 0$, $\alpha \in (0,1)$ and $a_{ij}, b_i, c \in C^\alpha\left(\mathbb{R}^{2n} \times [0,T)\right)$ with $1 \leq i,j \leq n$. Then for any $f \in C^\alpha\left(\mathbb{R}^{2n} \times [0,T)\right)$ and $\varphi \in C\left(\mathbb{R}^{2n}\right)$ such that*

$$|f(v,x,t)| \leq C\, e^{C|(v,x)|^2} \quad |\varphi(v,x)| \leq C\; e^{C|(v,x)|^2} \quad \textit{for every } (v,x) \in \mathbb{R}^{2n} \textit{ and } 0 < t < T,$$

there exists a unique solution u to (3. 18). In particular, if $\varphi \in C_l^{2+\alpha}\left(\mathbb{R}^{2n}\right)$, then $u \in C_l^{2+\alpha}\left(\mathbb{R}^{2n} \times [0,T)\right)$.

As far as we are concerned with Schauder estimates for the Cauchy problem (3. 18), optimal results have been obtained by many authors in the framework of semigroup theory. In Theorem 1.2 and Theorem 8.2 of [89], Lunardi proves an optimal Hölder regularity result for the solution u for the Cauchy problem (3. 18), under the assumption that the initial data φ has Hölder continuous derivatives $\partial_{x_i}\varphi$ and $\partial_{x_i x_j}\varphi$, $i,j = 1,\dots,m_0$. It is also assumed that the matrix $\{a_{ij}\}_{i,j}$ is elliptic and that the coefficients a_{ij} are Hölder continuous function of the space variable x that converges as $|x|$ goes to $+\infty$. Lorenzi improves in [88] the results by Lunardi in that the coefficients a_{ij} are not assumed to be bounded functions. In the other hand, in [88] the coefficients a_{ij} have Hölder continuous derivatives up to third order and the Lie algebra related to the constant coefficient operator has step 2. Priola in [111] considers operators with unbounded coefficients a_i, $i = 1,\dots,m_0$. We also recall the work [100] by Nyström, Pascucci and Polidoro, where the authors prove Schauder estimates for the obstacle problem associated to (3. 7). Moreover, Manfredini proves in [90] global Schauder estimates for the Dirichlet problem associated to (3. 7) in the dilation invariant case. Later on, Di Francesco and Polidoro [40] extend these results to the non dilation invariant case. For further information see Chapter 1, Theorem 3. 41. However in all of these works, either the choice of the Hölder spaces was different from the ones introduced here in Definition 3.7, or the assumptions on the coefficients were stronger. For this reason, for interior Schauder estimates we refer to the paper [68] by Imbert and Mouhot, where the authors prove Schauder estimates and localized Schauder estimates for classical solutions to $\mathcal{K}u = f$.

SCAHUDER ESTIMATES (THEOREM 1.1 [68]) *Let us consider $\mathcal{K}u = f$ under the structural assumption **(H)**. Given $\alpha \in (0,1)$, $a_{ij}, b_i, c \in C_l^\alpha\left(\mathbb{R}^{2n} \times \mathbb{R}\right)$ with $1 \le i,j \le n$ and a function $f \in C_l^\alpha\left(\mathbb{R}^{2n} \times \mathbb{R}\right)$, any classical solution u in the sense of Definition 2.6 to $\mathcal{K}u = f$ satisfies*

$$\|u\|_{C^{2+\alpha}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R})} \le C\left([f]_{C^\alpha(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R})} + \|u\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R})}\right) \tag{3. 19}$$

where the constant C depends on the dimension n, the ellipticity constants λ, Λ, the exponent α, the $\|\cdot\|_{C^\alpha}$ norm of the coefficients a_{ij}, b_i for $i = 1, \dots, n$ and c.

LOCAL SCAHUDER ESTIMATES (THEOREM 3.9 [68]) *Let us consider $\mathcal{K}u = f$ under the structural assumption **(H)**. Given $\alpha \in (0,1)$, $a_{ij}, b_i, c \in C_l^\alpha\left(\mathbb{R}^{2n} \times \mathbb{R}\right)$ with $1 \le i,j \le n$ and a function $f \in C_l^\alpha\left(\mathbb{R}^{2n} \times \mathbb{R}\right)$, any classical solution $u \in C_l^{2+\alpha}\left(\mathbb{R}^{2n} \times \mathbb{R}\right)$ in the sense of Definition 2.6 to $\mathcal{K}u = f$ satisfies for every $z_0 \in \mathbb{R}^{2n} \times \mathbb{R}$*

$$\|u\|_{C^{2+\alpha}(Q_1(z_0))} \le C\left([f]_{C^\alpha(Q_2(z_0))} + \|u\|_{L^\infty(Q_2(z_0))}\right) \tag{3. 20}$$

where the constant C depends on the dimension n, the ellipticity constants λ, Λ, the exponent α, the $\|\cdot\|_{C^\alpha}$ norm of the coefficients a_{ij}, b_i for $i = 1, \dots, n$ and c.

Our aim is to prove global Schauder estimates for the Cauchy problem (3. 18) starting from these interior results and taking into account the estimate of the C_l^α norm of the solution around the initial time. For this reason, for $0 \le k \le 2$ and $\alpha \in (0,1)$ we introduce the following weighted norms

$$\|u\|_{k+\alpha}^{(\omega)} := \sum_{0\le j\le k} [u]_j^{(\omega)} + [D^k u]_\alpha^{(\omega)} \tag{3. 21}$$

for a certain weight $\omega \in \mathbb{R}$, where the semi-norms appearing on the right-hand side are defined as follows:

$$[u]_j^{(\omega)} := \sup_{(v,x,t)\in\mathbb{R}^{2n}\times(0,T)} t^{\frac{j}{2}+\omega}\left|D^j u(v,x,t)\right|, \tag{3. 22}$$

$$[u]_{j+\alpha}^{(\omega)} := \sup_{\substack{(v_1,x_1,t_1),(v_2,x_2,t_2)\\ \in\mathbb{R}^{2n}\times(0,T)}} t^{\frac{j+\alpha}{2}+\omega}\frac{\left|D^j f(t_1,x_1,v_1) - D^j f(t_2,x_2,v_2)\right|}{d_l\left((t_1,x_1,v_1),(t_2,x_2,v_2)\right)^\alpha},$$

where $D^0 = 1$, $D^1 = \partial_v$ and D^2 is taken over $\partial_t + v\cdot\partial_x$ and ∇_v^2. We are now in position to state our result regarding global Schauder estimates for classical solution to the Cauchy problem (3. 18).

Theorem 3.12 *Let $T \in (0,\infty]$, $\alpha, \omega \in (0,1)$ and a_{ij}, b_i, $f \in C^\alpha\left(\mathbb{R}^{2n} \times [0,T)\right)$ with $1 \le i,j \le n$. If u is a positive classical solution of the Cauchy problem* (3. 18) *(in the sense of Definition 2.6) with initial data $\varphi = 0$, then*

$$[u]_{2+\alpha}^{(\omega)} \le C[f]_\alpha^{(1-\omega)}. \tag{3. 23}$$

Moreover, provided that $\varphi \in C_l^{2+\alpha}\left(\mathbb{R}^{2n}\right)$ the following estimate holds true

$$\|u\|_{C_l^{2+\alpha}(\mathbb{R}^{2n}\times[0,T))} \le C\left(\|u\|_{L^\infty(\mathbb{R}^{2n}\times[0,T))} + \|\varphi\|_{C_l^{2+\alpha}(\mathbb{R}^{2n})} + \|f\|_{C^\alpha(\mathbb{R}^{2n}\times[0,T))}\right). \tag{3. 24}$$

In both cases, the constant C depends on the dimension n, the ellipticity constants λ, Λ, the exponent α, the $\|\cdot\|_{C^\alpha}$ norm of the coefficients a_{ij}, b_i for $i = 1, \dots, n$ and c.

Before going through the proof of Theorem 3.12 we need to introduce an intermediate result, that is a maximum principle on $\mathbb{R}^{2n} \times [0,T]$ for subsolutions to $\mathcal{K}u = 0$. The proof of this result is inspired by the proof Lemma A.2 of [25]. We remark that the classical case of a bounded domain can be found in Theorem 1.29 and Theorem 4.2 of this work.

Lemma 3.13 *Let us consider the Kolmogorov equation $\mathcal{K}u = 0$ under the structural assumption **(H)** on the parabolic cylinder $\Omega_T := \Omega \times (0,T]$, where Ω is a general unbounded domain of $\mathbb{R}^{2n}$. Let a_{ij}, $b_i, c \in C^0(\Omega_T)$ with $1 \le i,j \le n$ and*

$$\Big|\sum_{i=1}^{n} b_i \xi_i\Big| \le \Lambda \left(1+|v|^2\right)^{\frac{1}{2}} |\xi| \quad \text{for every } \xi \in \mathbb{R}^n,\ (v,x,t) \in \Omega_T \quad \textit{and} \quad |c| \le \Lambda,$$

*where Λ is the ellipticity constant appearing in **(H)**. Let u be a bounded positive subsolution, that is $\mathcal{K}u \le 0$ in Ω_T, then*

$$\sup_{\Omega_T} u \le \sup_{\partial_K \Omega_T} u$$

where the hypoelliptic boundary is defined as $\partial_K \Omega_T := \left([0,T] \times \overline{\Omega}\right) \setminus ((0,T] \times \Omega)$.

Proof. As we have already pointed out, when the domain Ω is bounded the proof of this result can be found in Theorem 1.29 and Theorem 4.2 and also in Proposition A.1 of [25] since we are considering the particular case of the kinetic Kolmogorov-Fokker-Planck equation. For this reason, let us consider a general unbounded domain Ω. Given two positive constant C_1 and C_2 we introduce two auxiliary functions

$$\psi_1(v,t) := e^{C_1 t}\left(1+|v|^2\right) \quad \text{and} \quad \psi_2(t,x) := e^{C_2 t}\left(1+|x|^2\right).$$

Since u is bounded, for every $\varepsilon_1, \varepsilon_2 > 0$ there exist $R(\varepsilon_1), R(\varepsilon_2) > 0$ independent of C_1 and C_2 such that

$$u(v,x,t) - \varepsilon_1 \psi_1(v,t) - \varepsilon_2 \psi_2(x,t) \le \sup_{(v,x,t) \in \partial_K \Omega_T} u(v,x,t) \quad \text{in } \Omega_T \cap \{|v| \ge R(\varepsilon_1) \text{ or } |x| \ge R(\varepsilon_2)\}.$$

By applying the operator $\mathcal{K}$ defined in (3. 9) to the function ψ_1 and by choosing $C_1 = 4\Lambda$ we obtain:

$$\mathcal{K}\psi_1(v,t) = e^{C_1 t}\left((C_1 - c)(1+|v|^2) - \operatorname{Tr}\left(\overline{A} D_v^2 (1+|v|^2)\right) - 2\underline{b} \cdot v\right) \ge (C_1 - 4\Lambda)\left(1+|v|^2\right) = 0 \quad \text{in } \Omega_T.$$

Moreover, for any $R_1 > R(\varepsilon_1)$ we can choose the constant C_2 in such a way that

$$\mathcal{K}\psi_2(x,t) = e^{C_2 t}\left((C_2 - c)(1+|x|^2) + 2v \cdot x\right) \ge (C_2 - \Lambda - 1)(1+|x|^2) - |v|^2 \ge 0 \quad \text{in } \Omega_T \cap \{|v| < R_1\}.$$

Therefore, for any $R_2 > R(\varepsilon_2)$ the function

$$u(v,x,t) - \varepsilon_1 \psi_1(v,t) - \varepsilon_2 \psi_2(x,t)$$

is a subsolution to the general Kolmogorov equation $\mathcal{K}u = f$ in the bounded domain

$$(0,T] \times (\Omega \cap (B_{R_1} \times B_{R_2})),$$

with data f smaller than $\sup_{\partial_K \Omega_T} u$ on the boundary portion contained in the set $\{|v| = R_1, \text{ or } |x| = R_2\}$. Then, the maximum principle for bounded domains yields

$$u - \varepsilon_1 \varphi_1 - \varepsilon_2 \varphi_2 \le \sup_{\partial_K \Omega_T} u \quad \text{in } (0,T] \times (\Omega \cap (B_{R_1} \times B_{R_2})).$$

Sending $R_2 \to \infty$ and $\varepsilon_2 \to 0$, then taking $R_1 \to \infty$, $\varepsilon_1 \to 0$ we get the conclusion. $\square$

Proof of Theorem 3.12. In view of the interior Schauder estimate (3. 19), it suffices to deal with the estimates around the initial time. Without loss of generality, we assume $T \leq 1$. Let $z_0 = (t_0, x_0, v_0) \in \mathbb{R}^{2n} \times (0, T)$. We define the radius r as

$$r = \frac{1}{2} t_0^{\frac{1}{2}}.$$

By applying the local interior Schauder estimates (3. 20), we get by rescaling the following inequality on the cylinder $Q_r(z_0)$:

$$t_0^{\frac{2+\alpha}{2}} [u]_{C_l^{2+\alpha}(Q_r(z_0))} \leq C \left(\|u\|_{L^\infty(Q_{2r}(z_0))} + t_0^{\frac{2+\alpha}{2}} [f]_{C_l^{\alpha}(Q_{2r}(z_0))} \right).$$

Since z_0 is arbitrary and taking into consideration the definition of weighted norm introduced in (3. 22), for any $\omega \in (0,1)$ such that $[u]_0^{(-\omega)} < \infty$ the previous inequality leads to the following one:

$$[u]_{2+\alpha}^{(\omega)} \leq C \left([u]_0^{(-\omega)} + [f]_{\alpha}^{(1-\omega)} \right), \tag{3. 25}$$

where C is the constant appearing in (3. 20). Now, let us consider the function

$$u_1 := \frac{1}{\omega} [f]_0^{(1-\omega)} t^{\omega} - u$$

For every $\omega \in (0,1)$, let us consider the operator $\mathcal{K}$ applied to $\widetilde{u}$:

$$\mathcal{K} u_1 = \mathcal{K} \left(\frac{1}{\omega} [f]_0^{(1-\omega)} t^{\omega} \right) - \mathcal{K} u = t^{\omega - 1} [f]_0^{(1-\omega)} - f \geq 0$$

by definition of weighted norm (3. 22). Moreover, we have that

$$u_1 = \frac{1}{\omega} [f]_0^{(1-\omega)} t^{\omega} - u = 0 \quad \text{on } \{t = 0\}.$$

Thus, we can apply Lemma 3.13 to $\widetilde{u}$. The same reasoning applies when considering

$$u_2 := \frac{1}{\omega} [f]_0^{(1-\omega)} t^{-\omega} + u.$$

Combining the results we obtain for both u_1 and u_2 we obtain the following estimate

$$[u]_0^{(-\omega)} \leq C [f]_0^{(1-\omega)},$$

where C is a universal constant. By combining this last inequality with (3. 25) we conclude the proof of (3. 23). As far as we are concerned with the proof of (3. 24), it is obtained through a direct application of Lemma 3.13. □

We remark that all of these statements can be restricted to the domain $\mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}$, where the periodicity assumption on the x variable is introduced.

3.2 Self-generating lower mass bound

For convenience, let us rewrite the nonlinear equation appearing in the Cauchy problem (3. 1) in terms of the function $h := \mu^{-1} u$ so that

$$(\partial_t + v \cdot \nabla_x) h = \rho_{\mu h}^{\beta} (\nabla_v - v) \cdot \nabla_v h \qquad \text{in } \mathbb{R}^n \times \mathbb{T}^n \times [0, T]. \tag{3. 26}$$

Without any loss of generality (see Lemma 3.17), throughout this subsection we assume that the solution h is valued in $[0,\Lambda]$, and thus it is a bounded positive solution. We remark that under our assumptions the nonlinear term on the righthand side of (3. 26) is bounded:

$$\rho_{\mu h}(x,t) = \int_{\mathbb{R}^n} \mu(v)h(v,x,t)dv \leq \Lambda \int_{\mathbb{R}^n} \mu(v)dv \leq C\Lambda.$$

Now, our aim is to prove the following lower mass bound spreading result for solutions h to the nonlinear equation (3. 26).

Proposition 3.14 *Let h be a classical solution of* (3. 26) *in* $\mathbb{R}^n \times \mathbb{T}^n \times [0,T]$ *valued in* $[0,\Lambda]$ *such that for some* $(v_0,x_0) \in \mathbb{R}^n \times \mathbb{T}^n$*,* $\theta > 0$ *and* $r > 0$ *we have*

$$h(v,x,0) \geq \theta \qquad \text{when } |v-v_0| < r \text{ and } |x-x_0| < r.$$

Then for every fixed $\overline{T} \in (0,T)$ *there exists a (large) constant* $C_* > 0$ *depending on* $\overline{T}$*,* T*,* θ*,* r *and* v_0 *such that for any* $(v,x,t) \in \mathbb{R}^n \times \mathbb{T}^n \times [\overline{T},T]$*,*

$$h(v,x,t) \geq C_*^{-1} e^{-C_*|v|^2}. \tag{3. 27}$$

The proof of this proposition is made of two lemmas and relies on the mixing structure of the maximum principle and the transport operator, but not on the structure of the local mass conservation. In particular, Lemma 3.15 allows us to extend the lower bound from a neighborhood of a given point in $\mathbb{R}^n \times \mathbb{T}^n$ further on in time and to prove it we use a barrier function argument in the same spirit as [63]. Lemma 3.16 allows us to spread the lower bound to all velocities. Essentially, it can be seen as the lower bound estimate of the fundamental solution and its proof is based on the ideas of [62]. The spreading of the lower bound in space is given by selecting a proper velocity to transport the mass, which is guaranteed by Lemma 3.15. By applying these two lemmas repeatedly, we are able to spread the lower mass bound of the solution at any finite time.

We are now in position to state Lemma 3.15, that is responsible for the propagation of the lower bounds forward in time. Indeed, it is used both to preserve a mass core near (v_0,x_0) for short times (which corresponds to the choice of $\tau = 1$ in the statement of the lemma), and to push lower bounds to different locations in x via free transport.

Lemma 3.15 *Let h be a classical solution of* (3. 26) *in* $\mathbb{R}^n \times \mathbb{T}^n \times [0,T]$ *valued in* $[0,\Lambda]$ *such that for some* $(v_0,x_0) \in \mathbb{R}^n \times \mathbb{T}^n$ *and* $\tau,\theta,r > 0$

$$h(v,x,0) \geq \theta \qquad \text{when } |v-v_0| < \frac{r}{\tau} \text{ and } |x-x_0| < r.$$

Then there exist a constant $C_0 > 0$ *depending on* n*,* Λ *and* β *such that*

$$h(v,x,t) \geq \frac{\theta}{8} \quad \text{when } |v-v_0| < \frac{r}{2\tau},\ |x-x_0-tv| < \frac{r}{2} \text{ and} \tag{3. 28}$$

$$t \leq \min\left\{T, \tau, C_0\left(1+\left|\tfrac{\tau}{r}\right|^2\right)^{-1}\left(1+|v_0|^2\right)^{-1}\right\}. \tag{3. 29}$$

Proof. For a constants $\underline{C} > 0$ to be determined, let us consider the following barrier function

$$\underline{h}(v,x,t) := -\underline{C}t + \frac{\theta}{2}\left(1 - \frac{|x-x_0-tv|^2}{r^2} - \frac{\tau^2|v-v_0|^2}{r^2}\right).$$

For any $t \in [0, \min\{T, \tau\}]$ such that $\tau^2|v - v_0|^2 + |x - x_0 - tv|^2 < \frac{r^2}{2}$ a direct computation yields

$$|\rho_{\mu h}^{\beta}(\nabla_v - v) \cdot \nabla_v \underline{h}| \leq C\Lambda^{\beta}\left(|\Delta_v \underline{h}| + |v \cdot \nabla_v \underline{h}|\right) \leq C\Lambda^{\beta}\theta \left(1 + \left|\frac{\tau}{r}\right|^2\right)\left(1 + |v_0|^2\right),$$

where C is a universal constant depending on n. We want to show that $\underline{h}$ is a subsolution to (3. 26), at least in the set where it is positive, that is:

$$\Omega_{\underline{h}} = \{(v, x, t) \in \mathbb{R}^n \times \mathbb{T}^n \times [0, \min\{T, \tau\}] : \underline{h}(v, x, t) > 0\}.$$

Indeed, given the construction of the function $\underline{h}$ we have that

$$\partial_t \underline{h} + v \cdot \nabla_x \underline{h} \leq -\underline{C}.$$

Thus, we obtain that $\underline{h}$ is indeed subsolution to (3. 26):

$$\partial_t \underline{h} + v \cdot \nabla_x \underline{h} - \rho_{\mu h}^{\beta}(\nabla_v - v) \cdot \nabla_v \underline{h} \leq -\underline{C} + C\Lambda^{\beta}\theta \left(1 + \left|\frac{\tau}{r}\right|^2\right)\left(1 + |v_0|^2\right) < 0 \tag{3. 30}$$

where the rightmost inequality is ensured by a proper choice of the constant $\underline{C}$, that needs to be

$$\underline{C} > C\Lambda^{\beta}\theta \left(1 + \left|\frac{\tau}{r}\right|^2\right)\left(1 + |v_0|^2\right).$$

Now it suffices to establish that $h > \underline{h}$ in the set Ω_h. We remark that for $t = 0$ we have

$$\underline{h}(v, x, 0) = \frac{\theta}{2}\left(1 - \frac{|x - x_0|^2}{r^2} - \frac{\tau^2|v - v_0|^2}{r^2}\right) \leq \frac{\theta}{2} < \theta \leq h(v, x, 0)$$

where the rightmost inequality holds true given the definition of the function $\underline{h}$ and the assumption of the lemma. In particular, there exists a (small) universal constant C_0, that incorporates $C\Lambda^{\beta}$ and thus depends on n, Λ and β, such that

$$\underline{C} = \frac{1}{8C_0}\theta \left(1 + \left|\frac{\tau}{r}\right|^2\right)\left(1 + |v_0|^2\right)$$

and $\underline{h}(v, x, t) \geq \frac{\theta}{8}$ in the set

$$\left\{t \leq C_0 \left(1 + \left|\frac{\tau}{r}\right|^2\right)^{-1}\left(1 + |v_0|^2\right)^{-1}, \quad |x - x_0 - tv|^2 + \tau^2|v - v_0|^2 < \frac{r^2}{2}\right\}.$$

We conclude the proof by applying Lemma 3.13 to the function $g = \underline{h} - h$ in the region $\Omega_{\underline{h}}$. □

The spreading of the lower bound to all velocities relies on the construction of a Harnack chain through the iterative application of (3. 17) at the cost of shrinking the x-domain where the lower bound hols. This is possible because locally the nonlinear equation appearing in (3. 1) coincides with the linear kinetic Kolmogorov-Fokker-Planck equation appearing in (3. 18) for which the Harnack inequality (3. 17) was proved by Golse, Imbert, Mouhot and Vasseur in [58].

Lemma 3.16 *Let h be a classical solution of* (3. 26) *in* $\mathbb{R}^n \times \mathbb{T}^n \times [0, T]$ *valued in* $[0, \Lambda]$. *Let* $\overline{\theta} > 0$, $R \in (0, 1]$ *and* $T_0 \in (0, \min\{1, T\}]$ *such that for every* $t \in [0, T_0]$

$$h(t, x, v) \geq \overline{\theta} \qquad \text{when } |x - x_0 - tv_0| < R \text{ and } |v - v_0| < R. \tag{3. 31}$$

for some $(v_0, x_0) \in \mathbb{T}^n \times \mathbb{R}^n$. *Then for every fixed* $\underline{t} \in (0, T_0)$ *there exist a constant* $C_1 > 0$ *depending on* T_0, $\overline{\theta}$, R *and* v_0 *such that*

$$h(v, x, t) \geq C_1^{-1} e^{-C_1|v|^4} \qquad \text{when } |x - x_0 - tv_0| < \frac{R}{2}, \text{ and } t \in [\underline{t}, T_0]. \tag{3. 32}$$

Proof. Let us consider a point $(v_0, x_0) \in \mathbb{R}^n \times \mathbb{T}^n$ such that (3. 31) holds true for every $t \in [0, T_0]$. Let us fix a certain $\underline{t} \in (0, T_0)$. Our aim is to construct a sequence $\{z_i := (t_i, x_i, v_i)\}_i$ of $N+1$ points to reach a certain point $(\overline{v}, \overline{x}, \overline{t})$ in the set

$$\left\{(v, x, t) \in \mathbb{R}^n \times \mathbb{T}^n \times [0, T_0] : \ v \in \mathbb{R}^n, \, |x - x_0 - tv_0| < \tfrac{R}{2}, \, t \in [\underline{t}, T_0]\right\}$$

starting from a certain point $z_1 = (v_1, x_1, t_1)$ belonging to the region where h is positive by the assumption (3. 31), that is

$$\Omega_+ := \{(v, x, t) \in \mathbb{R}^n \times \mathbb{T}^n \times [0, T_0] : \ |v - v_0| < R, \, |x - x_0 - tv_0| < R, \, t \leq \underline{t}\},$$

where $(v_0, x_0) \in \mathbb{T}^n \times \mathbb{R}^n$ and given that $v_1 = v_0$. In particular, x does not exit this region and we remark that the nonlocal nature of the nonlinear term $\rho_f = \rho_{\mu h}$, alongside with the assumption (3. 31) implies the non degeneracy of the diffusion in velocity, so that the positivity of the solution h is propagated over $v \in \mathbb{R}^n$ in a localized space region. To be more specific, we define a sequence of points $z_i := (v_i, x_i, t_i)$ for $i \in \{1, 2, ..., N+1\}$, with $N \in \mathbb{N}$, such that

$$z_{N+1} = (\overline{v}, \overline{x}, \overline{t}), \quad \text{and} \quad z_i = z_{i+1} \circ \delta_r \left(-\tau_2 \frac{\overline{v} - v_0}{|\overline{v} - v_0|}, 0, -\tau_1\right). \tag{3. 33}$$

Thus, our aim is now to determine the coordinates of the starting point $z_1 = (v_1, x_1, t_1) \in \Omega_+$ given that $v_1 = v_0$, alongside with the constants $N, r, \tau_1, \tau_2 > 0$ in such a way that $z_{N+1} = (\overline{v}, \overline{x}, \overline{t})$.

Let us consider a certain point $\tilde{z} := (\tilde{v}, \tilde{x}, \tilde{t}) \in Q_1$, where $\mathcal{Q}_1$ is the unit box defined in (3. 12) and let us consider the point:

$$z_{i+1} \circ \delta_r(\tilde{z}) = \left(v_{i+1} + r\tilde{v}, \ x_{i+1} + r^3\tilde{x} + r^2\tilde{t}v_{i+1}, \ t_{i+1} + r^2\tilde{t}\right) \qquad \text{for } i = 1, \ldots, N.$$

If for every $\tilde{z} \in Q_1$ the point $z_{i+1} \circ \delta_r(\tilde{z})$ satisfies the following estimates

$$Nr\tau_2 \leq |\overline{v} - v_0|, \ \left|x_{i+1} + r^3\tilde{x} + r^2\tilde{t}v_{i+1} - x_0 - \left(t_{i+1} + r^2\tilde{t}\right) v_0\right| < R, \ t_{i+1} + r^2\tilde{t} \in [0, T_0], \tag{3. 34}$$

then for every $1 \leq i \leq N$ the function

$$h_{i+1}(\tilde{z}) := h\left(z_{i+1} \circ \delta_r(\tilde{z})\right)$$

verifies the equation

$$\left(\partial_{\tilde{t}} + \tilde{v} \cdot \nabla_{\tilde{x}}\right) h_{i+1} = \rho^{\beta}_{\mu h_{i+1}} \nabla_{\tilde{v}} \cdot \left(\nabla_{\tilde{v}} - r(v_{i+1} + r\tilde{v}) \cdot \nabla_{\tilde{v}}\right) h_{i+1} \quad \text{in } \mathcal{Q}_1,$$

where the coefficients satisfy the following bounds

$$c\theta^{\beta} R^{n\beta} \leq \rho^{\beta}_{\mu h_{i+1}} \leq C\Lambda \quad \text{and} \quad |r(v_i + r\tilde{v})| \leq r(1 + |v_0| + |\overline{v} - v_0|) \leq 1,$$

where c and C are universal constant and provided that

$$r \leq (1 + |v_0| + |\overline{v} - v_0|)^{-1}.$$

Applying the Harnack inequality (3. 17) to the function h_{i+1} we get that there exist constants $c_0, \tau_1, \overline{\tau} \in (0, 1)$, depending only on the constants $\overline{\theta}$ and R stated in the assumptions of the lemma, such that for any $\tau_2 \in [0, \overline{\tau}]$ and $1 \leq i \leq N$ we have:

$$h(v_{i+1}, x_{i+1}, t_{i+1}) = h_{i+1}(0, 0, 0) \geq c_0 h_{i+1} \left(-\tau_2 \frac{\overline{v} - v_0}{|\overline{v} - v_0|}, \, 0, \, -\tau_1\right) = c_0 h(v_i, x_i, t_i). \tag{3. 35}$$

Now we have to determine the coordinates of the starting point $z_1 := (v_1, x_1, t_1)$ of the Harnack chain and the constants N, r, τ_2, since τ_1 is determined by the application of the Harnack inequality to obtain (3. 35). For a certain constant $\mathcal{M} > 0$, we set

$$t_1 := \max\left\{\frac{\underline{t}}{2}, \bar{t} - \frac{R}{8}(1 + |v_0| + |\overline{v} - v_0|)^{-1}\right\} \quad \text{and} \quad r := \frac{R}{\mathcal{M}}(1 + |v_0| + |\overline{v} - v_0|)^{-2}.$$

Recalling that $T, R \in (0, 1]$, by choosing $\mathcal{M} \geq \frac{2}{\underline{t}} + \frac{\tau_1}{\overline{\tau}}\left(8 + \frac{2}{\underline{t}}\right)$, we have

$$r^2 \leq \frac{\underline{t}}{2} \quad \text{and} \quad \tau_2 := \frac{r\tau_1|\overline{v} - v_0|}{\overline{t} - t_1} \leq \overline{\tau}$$

This also ensures the first condition of (3. 34). To determine the parameter $\mathcal{M} > 0$, we point out that there exists some constant $\overline{C}$ depending only on universal constants, $\underline{t}$, $\overline{\theta}$, R and v_0, such that $\mathcal{M} \leq \overline{C}$. Moreover, since $v_1 = v_0$ by assumption we have that in order to satisfy the third condition of (3. 34) we choose:

$$N := \frac{\overline{t} - t_1}{r^2\tau_1} \in \mathbb{N}^+.$$

From the iterative definition of a certain point z_i, with $1 \leq i \leq N + 1$, it follows that

$$t_i = t_1 + (i-1)r^2\tau_1, \qquad v_i = v_0 + (i-1)r\tau_2\frac{v - v_0}{|v - v_0|}, \qquad x_i = x - r^2\tau_1\sum_{j=i}^{N} v_{j+1}. \tag{3. 36}$$

Therefore, given our definition of the parameters r and N, τ_2 we have that

$$|x_{i+1} - x_1 - (t_{i+1} - t_1)v_0| = \frac{i(i+1)}{2}r\tau_2 \leq N^2r^3\tau_1\tau_2 = (t - t_1)|v - v_0| \leq \frac{R}{8}.$$

Thus, for any $\overline{x} \in B_{\frac{R}{2}}(x_0 + tv_0)$ there exists some $x_1 \in B_{\frac{5R}{8}}(x_0 + t_1v_0)$ such that $x_{N+1} = \overline{x}$. In this setting, for any $1 \leq i \leq N$, we also have

$$\begin{aligned}
&\left|x_{i+1} + r^3\tilde{x} + r^2\tilde{t}v_{i+1} - x_0 - \left(t_{i+1} + r^2\tilde{t}\right)v_0\right| \\
&\qquad \leq |x_{i+1} - x_1 - (t_{i+1} - t_1)v_0| + |x_1 - x_0 - t_1v_0| + r^2|r\tilde{x} + \tilde{t}v_{i+1} - \tilde{t}v_0| \\
&\qquad \leq \frac{R}{8} + \frac{5R}{8} + r^2(1 + |v - v_0|) < \frac{3R}{4} + \frac{R^2}{\mathcal{M}^2} < R.
\end{aligned}$$

Thus, the condition (3. 34) ensuring inequality (3. 35) is satisfied for $1 \leq i \leq N$, which yields

$$h(t, x, v) \geq c_0^N h(v_0, x_1, t_1) \geq \theta e^{-N\log\frac{1}{c_0}}.$$

Recalling that $c_0 \in (0, 1)$ appears in (3. 35) and

$$N \leq \frac{T\overline{C}^2\left(1 + |v_0| + |v - v_0|\right)^4}{\tau_1 R^2},$$

for a positive universal constant $\overline{C}$ we obtain the desired result. $\square$

Proof of Proposition 3.14 The proof is split into four steps, and it is inspired by the proof of Theorem 1.2 of the paper [63] by Henderson, Snelson and Tarfulea for the Boltzmann equation.

Step 1: sustaining mass for a small time.
By assumption we have that for a given $(v_0, x_0) \in \mathbb{R}^n \times \mathbb{T}^n$, $\theta, r > 0$ we have

$$h(v, x, 0) \geq \theta \mathbb{1}_{\{|v-v_0|<r,\, |x-x_0|<r\}}.$$

Thus, it is always possible to apply Lemma 3.15 with $\tau = 1$ on the smaller neighborhood of radius $r/2$. Then there exists a universal constant $C_0 > 0$ such that, for every

$$0 \leq t \leq t^* := \min\left\{\frac{1}{2}, T, C_0 \left(1 + \left|\tfrac{\tau}{r}\right|^2\right)^{-1} \left(1 + |v_0|^2\right)^{-1}\right\}$$

we have that

$$h(v, x, t) \geq \frac{\theta}{8} \mathbb{1}_{\left\{|v-v_0|<\frac{r}{4},\, |x-x_0-tv|<\frac{r}{4}\right\}}.$$

We remark that for the choice of t^* we have deliberately chosen to restrict ourselves at $t \leq \frac{1}{2}$, even if Lemma 3.15 would have allowed us to consider values up to $\tau = 1$. In particular,

$$t \leq \tfrac{1}{2} \quad \text{implies} \quad |x - x_0 - tv| < \tfrac{3}{8} r \quad \text{if } |x - x_0 - tv_0| < \frac{r}{4}.$$

Step 2: spreading mass to all v (localized in x) for small times.
Let us apply Lemma 3.16 by considering $T_0 = t^*$, $\overline{\theta} = \theta/8$ and $R = r/4$. Thus,

$$h(v, x, t) \geq C_1^{-1} e^{-C_1|v|^4} \mathbb{1}_{\left\{|x-x_0-tv_0|<\frac{r}{8}\right\}} \qquad \text{for } 0 < t \leq t^*, \ v \in \mathbb{R}^n \tag{3. 37}$$

where C_1 depends on the choice of the radius r, θ and $|v_0|$. We remark that, in order to spread the mass to all v we shrink the domain in the x direction.

Step 3: spreading mass in x for small times.
Let us fix $x_1 \in \mathbb{T}^n$ and a time t_1 such that

$$0 < t_1 \leq \min\left\{t^*, \frac{r}{32|v_0|}\right\}.$$

We observe that the triangle inequality implies that at time $t_1/2$ the estimate (3. 37) holds for $|x - x_0| < r/32$, and thus there exist a certain parameter $\theta_0 > 0$ (it always exists given our construction) and a certain point

$$v_1 = \frac{2(x_1 - x_0)}{t_1}$$

such that, if $r_0 = r/32$,

$$h\left(v, x, \frac{t_1}{2}\right) \geq \theta_0 \mathbb{1}_{\left\{|v-v_1|<\frac{2r_0}{t_1},\, |x-x_0|<r_0\right\}}$$

Our aim is now to apply Lemma 3.15 with $v_0 = v_1$, $R = r_0$ and $\tau = t_1/2$ applied to $h(v, x, t_1/2 + t)$ to propagate this lower bound along trajectories of the type $x \sim x_0 + tv_1$ up to $t = t_1/2$. For this reason we also require t_1 to satisfy the assumption (3. 28) on the time, that is

$$t_1 \left(1 + \left|\tfrac{t_1}{2r_0}\right|^2\right) < C_{01} \left(1 + |v_0|^2\right)^{-1}. \tag{3. 38}$$

If t_1 satisfies this inequality, then Lemma 3.15 implies

$$h(v, x, t) \geq \frac{\theta_0}{8} \mathbb{1}_{\left\{|v-v_1|<\frac{r_0}{2}, |x-x_0-tv_1|<\frac{r_0}{2}\right\}} \qquad \text{for} \quad \frac{t_1}{2} < t < t_1.$$

Now, we are in position to apply Lemma 3.16 with $T_0 = t_1$, $\overline{T} = t_1/2$, $R = r_0/2$ and $v_0 = v_1$

$$h(v, x_1, t_1) \geq C_{11}^{-1} e^{-C_{11}|v|^4} \mathbb{1}_{\left\{|x-x_0-t_1v_1|<\frac{r_0}{4}\right\}}, \tag{3. 39}$$

where $C_{11} > 0$ depends on constants θ, r, $|v_0|$ and $|x_1 - x_0|$.

Step 4: extending the lower bound for moderate times.
We observe that Step 3 holds true if and only if the time t_1 satisfies assumption (3. 38). If this is not the case, we chose $\tilde{t}_1$ sufficiently small depending on r and $|x_1 - x_0|$ in such a way that the inequality (3. 38) is satisfied. Proceeding as above, with $\tilde{t}_1$ replacing t_1, through Lemma 3.16 we obtain a lower bound at $t = \tilde{t}_1$, x near x_1 and v close to zero

$$h(v, x_1, \tilde{t}_1) \geq \theta_1 \mathbb{1}_{\left\{|x-x_1|<\frac{r_0}{4}, |v|<\frac{r_0}{4}\right\}},$$

for some constant $\theta_1 > 0$ with the same dependence as C_1. Next, we propagate this estimate forward in time by applying Lemma 3.15 with $v_0 = 0$, $r = r_0/4$ and $\tau = 1$ to $h(\cdot, \cdot, \bar{t}_0 + \cdot)$ (with $\tau = 1$, $v_0 = 0$), we see that, for any $t \in \left[\bar{t}_0, \min\left\{T_0, \bar{t}_0 + T_1\right\}\right]$ with $T_1 := c_0\left(\frac{r_0}{16}\right)^2$,

$$h(v, x, t) \geq \frac{\theta_1}{8} \mathbb{1}_{\left\{|v|<\frac{r_0}{8}, |x-\overline{x}|<\frac{r_0}{8}\right\}} \quad \tilde{t}_1 \leq t \leq \min\left\{\tilde{t}_1 + T^*, T\right\}$$

where T^* is given by the time condition (3. 28) appearing in Lemma 3.15, that is

$$T^* = C_{02}\frac{r_0}{16}.$$

As long as $t_1 \leq \min\{\tilde{t}_1 + T^*, T\}$ this lower bound extends up to the time t_1 and by applying Lemma 3.16 to $h(v, x, \tilde{t}_1 + t)$ we obtain the following lower bound

$$h(v, x_1, t_1) \geq C_{12}^{-1} e^{-C_{12}|v|^4},$$

with C_1 depending on θ, r, v_0, t_1 and $|x - x_0|$. Since T_0 and T^* depend only on universal constants, r and v_0, by applying the above arguments finitely many times we obtain the desired result.

Step 4: improving the exponential tail.
Up to now we have proved there exists a constant $\underline{c} > 0$ depending only on universal constants T_0, T, θ, r and $|v_0|$ such that

$$h(v, x, t) \geq \underline{c} = C_{13}^{-1} e^{-C_{13}|v|^4} \quad \text{for } (v, x, t) \in \mathbb{R}^n \times \mathbb{T}^n \times [\overline{T}, T],$$

where $\overline{T} \in (0, T]$. Let us now consider the following barrier function

$$\underline{h}(v, x, t) := \underline{c} e^{-C(t-\overline{T})^{-1}|v|^2} \quad \text{in } B_1(0)^c \times \mathbb{T}^n \times [\overline{T}, T],$$

where the constant $C > 1$ is to be determined. A direct computation yields

$$(\partial_t + v \cdot \nabla_x) \underline{h} - \rho_{\mu h}^{\beta} (\nabla_v - v) \cdot \nabla_v \underline{h} \leq \frac{-C_0}{(t - \overline{T})^2} \left(4C - 1 - 4nC(t - \overline{T})\right) \underline{h} \quad \text{in } B_1(0)^c \times \mathbb{T}^n \times (\overline{T}, T].$$

In particular, we have that

$$(\partial_t + v \cdot \nabla_x) \underline{h} - \rho_{\mu h}^{\beta} (\nabla_v - v) \cdot \nabla_v \underline{h} \leq 0 \qquad \text{in } B_1(0)^c \times \mathbb{T}^n \times (\overline{T}, T],$$

given C sufficiently large (depending only on n and T). Besides, by definition we have that

$$h \geq \underline{h} \qquad \text{in } .$$

on the boundary of the set $B_1(0)^c \times \mathbb{T}^n \times [\underline{T}, T]$, that is wherever $t = \overline{T}$ or $|v| = 1$ Thus, Lemma 3.13 implies that

$$h \geq \underline{h} \qquad \text{in } B_1(0)^c \times \mathbb{T}^n \times [\underline{;} T].$$

This allows us to obtain the Gaussian lower bound (3. 27) for any $(v, x, t) \in \mathbb{R}^n \times \mathbb{T}^n \times [\underline{T}, T]$. $\square$

3.3 Existence and uniqueness of the solution

This section is devoted to the proof of the existence and uniqueness of the solution for the Cauchy problem (3. 1). In order to do this, let us rewrite it in terms of the unknown function $g := \mu^{-\frac{1}{2}} u$ with $g_{\text{in}} := \mu^{-\frac{1}{2}}\varphi$ as follows,

$$\begin{cases} (\partial_t + v \cdot \nabla_x)\, g = \mathcal{R}[g]\,\mathcal{U}[g], \\ g(v,x,0) = g_{\text{in}}(v,x), \end{cases} \tag{3. 40}$$

where the terms on the righthand side are defined as follows:

$$\mathcal{R}[g] := \left(\int_{\mathbb{R}^d} g\mu^{\frac{1}{2}}\,\mathrm{d}v\right)^{\beta} \qquad \text{and} \qquad \mathcal{U}[g] := \mu^{-\frac{1}{2}}\nabla_v\left(\mu\nabla_v\left(\mu^{-\frac{1}{2}}g\right)\right) = \Delta_v g + \left(\frac{d}{2} - \frac{|v|^2}{4}\right) g.$$

By such substitution, in contrast with the original equation, we get rid of the first-order term in v and the operator $\mathcal{U}$ becomes self-adjoint in $L^2_{x,v}$. Although the coefficient of zero order term is still unbounded, as firstly proposed by Imbert and Mouhot in [68] we can overcome this difficulty by considering it as a bounded source term, since the Gaussian bounds for the function g propagate in times as it is stated by the following lemma proved by Imbert and Mouhot in [68].

Lemma 3.17 *Let us consider a classical solution g to the Cauchy problem* (3. 40) *in $L^\infty([0,T], H^2(\mathbb{R}^n \times \mathbb{T}^n))$ such that*

$$C_1\sqrt{\mu} \le g(v,x,0) \le C_2\sqrt{\mu} \qquad \text{in } \mathbb{R}^n \times \mathbb{T}^n,$$

then for almost every $t \in [0,T]$ we have

$$C_1\sqrt{\mu(v)} \le g(v,x,t) \le C_2\sqrt{\mu(v)} \qquad \text{for every } (v,x) \in \mathbb{R}^n \times \mathbb{T}^n,$$

We remark that the starting points of this kind of spatially inhomogeneous kinetic equations with a quasilinear diffusive structure in velocity are the works [58] and [4], where the authors develop the kinetic counterpart of the De Giorgi-Nash-Moser theory for classical elliptic equations, and [68], where the Schauder theory is analyzed. We summarize here some basic apriori estimates for the solution g to the Cauchy problem (3. 40) that firstly appeared in Proposition 4.4 of [68] and Corollary 4.6 of [126] respectively. Moreover, we remark that throughout this subsection we set

$$\Omega := \mathbb{R}^n \times \mathbb{T}^n \times (0,T] \qquad \text{with} \quad T \in \mathbb{R}_+.$$

Lemma 3.18 *Let g be a solution to* (3. 40) *in Ω satisfying*

$$0 \le g \le \Lambda\mu^{\frac{1}{2}} \quad \text{in } \Omega \quad \text{and} \quad \mathcal{R}[g] \ge \lambda \quad \text{in } [0,T] \times \mathbb{T}^d.$$

Then, the following two statements hold.

(i) Let $\overline{T} \in (0,T)$ and $\theta \in \left(0, \frac{1}{2}\right)$. There exists some universal constant $\alpha \in (0,1)$ and a positive constant $C = C(\overline{T}, \theta, n, \alpha, \lambda, \Lambda)$ such that for any $Q_{2r}(z_0) \subset \mathbb{R}^n \times \mathbb{T}^n \times [\overline{T}, T]$ and for

$$\|g\|_{C_l^{2+\alpha}(Q_r(z_0))} \le C\mu^\theta(v_0). \tag{3. 41}$$

(ii) Let $\theta \in \left(0, \frac{1}{2}\right)$ and $g_{\text{in}} \in C_l^{\alpha_0}\left(\mathbb{R}^n \times \mathbb{T}^n\right)$ with (universal) $\alpha_0 \in (0,1)$. There exists a universal constant $\alpha \in (0,1)$ and a positive constant $C = C(\theta, n, \alpha, \lambda, \Lambda)$ such that for any $v_0 \in \mathbb{R}^n$, we have

$$\|g\|_{C_l^{\alpha}(B_1(v_0)\times\mathbb{T}^n\times[0,T])} \le C\left(1 + \|g_{\text{in}}\|_{C_l^{\alpha_0}(\mathbb{R}^n\times\mathbb{T}^n)}\right)\mu^\theta(v_0).$$

We remark that the boundedness assumption $\lambda\mu^{\frac{1}{2}} \leq \gamma_{\rm in} \leq \Lambda\mu^{\frac{1}{2}}$ for the initial data ψ of the Cauchy problem (3. 40) reflects on the assumptions of Theorem 3.1 for the initial data φ, in the sense that, given the definition of $g_{\rm in} = \varphi\mu^{-\frac{1}{2}}$, it becomes

$$\rho_\varphi \geq \lambda \text{ in } \mathbb{T}^n \quad \text{and} \quad \varphi \leq \Lambda\mu \text{ in } \mathbb{R}^n \times \mathbb{T}^n,$$

where $\mu = (2\pi)^{-\frac{n}{2}} e^{-|v|^2/2}$ is the global Maxwellian introduced in (3. 4). Moreover, we also remark that Proposition 3.14 holds true (up to a universal constant) also for solutions g to the Cauchy problem (3. 40), since $h = \mu^{\frac{1}{2}} g$. We are now in position to proceed with the proof of Theorem 3.1. In fact, the proof of the existence of the solution for the Cauchy problem (3. 1) is given by Proposition 3.19 for the Cauchy problem (3. 40) and in the same way the proof of the uniqueness of the solution is given by Proposition 3.20.

Proposition 3.19 *Let $0 \leq g_{\rm in} \leq \Lambda\mu^{\frac{1}{2}}$ in $\mathbb{R}^n \times \mathbb{T}^n$. Then, there exists a positive weak solution $g \in C^2(\Omega)$ to (3. 40) in the sense that, for any $\psi \in C_c^\infty(\mathbb{R}^n \times \mathbb{T}^n \times [0,T))$,*

$$\int_{\mathbb{R}^n\times\mathbb{T}^n} g_{\rm in}\, \psi\big|_{t=0}\, dv\, dx = \int_\Omega \left\{ -g\left(\partial_t + v\cdot\nabla_x\right)\psi + \mathcal{R}[g]\nabla_v g\cdot\nabla_v\varphi - \mathcal{R}[g]\left(\frac{n}{2} - \frac{|v|^2}{4}\right) g\psi \right\} dv\, dx\, dt. \tag{3. 42}$$

Furthermore, if $g_{\rm in}$ is continuous in $\mathbb{R}^n \times \mathbb{T}^n$, then g is a classical solution to (3. 40).

Proof. We may assume that $g_{\rm in}$ is not identically zero, i.e. for some point $(v_0, x_0) \in \mathbb{R}^n \times \mathbb{T}^n$ and some constants $\theta, r > 0$ we have

$$g_{\rm in} \geq \theta \mathbb{1}_{\{|v-v_0|<r,\, |x-x_0|<r\}}.$$

By Proposition 3.14, for any solution g to (3. 40) and for any $\overline{T} \in (0,T)$, there exists a $C_* > 0$ depending only on universal constants, $\overline{T}$, T, θ, r and v_0 such that

$$\mathcal{R}[g](x,t) \geq C_* \quad \text{in } \mathbb{T}^n \times [\overline{T}, T]. \tag{3. 43}$$

Step 1. We first approximate the initial data $g_{\rm in}$ by

$$g_{\rm in}^\varepsilon := g_{\rm in} * \varrho_\varepsilon + \varepsilon\mu^{\frac{1}{2}},$$

where $\varrho_1 \in C_c^\infty(B_1 \times B_1)$ is a nonnegative bump function such that for $\varepsilon \in (0,1]$

$$\int_{\mathbb{R}^{2n}} \varrho_1 = 1 \quad \text{and} \quad \varrho_\varepsilon(v,x) := \frac{1}{\varepsilon^{2n}}\varrho_1\left(\frac{v}{\varepsilon}, \frac{x}{\varepsilon}\right) \quad \text{with} \quad (v,x) \in \mathbb{R}^n \times \mathbb{T}^n.$$

Then, we have

$$\varepsilon\mu^{\frac{1}{2}} \leq g_{\rm in}^\varepsilon \leq (1+\Lambda)\mu^{\frac{1}{2}} \quad \text{in} \quad \mathbb{R}^n \times \mathbb{T}^n. \tag{3. 44}$$

Let us fix $\varepsilon \in (0,1]$. In order to establish the existence of classical solution to (3. 40) associated with the initial data $g_{\rm in}^\varepsilon$, we are going to find a fixed point of the mapping $F : w \mapsto g$ defined by solving the following Cauchy problem

$$\begin{cases} \left(\partial_t + v\cdot\nabla_x\right) g = \mathcal{R}[w]\,\mathcal{U}[g] & \text{in } \Omega, \\ g(0,\cdot,\cdot) = g_{\rm in}^\varepsilon & \text{in } \mathbb{R}^n \times \mathbb{T}^n, \end{cases} \tag{3. 45}$$

on the closed convex subset $\mathcal{K}$ of the Banach space $C^\gamma(\overline{\Omega})$,

$$\mathcal{K} := \left\{ w \in C_l^\gamma(\overline{\Omega}) : \|w\|_{C_l^\gamma(\Omega)} \leq \mathcal{N},\ \ \varepsilon\mu^{\frac{1}{2}} \leq w \leq (1+\Lambda)\mu^{\frac{1}{2}} \ \text{ in } \overline{\Omega} \right\},$$

where the constants $\gamma \in (0,1)$ and $\mathcal{N} > 0$ are to be determined. We remark that since $\mathcal{R}[w] \geq \varepsilon$ and (3. 44) hold true, by Lemma 3.17 we have that

$$\varepsilon\mu^{\frac{1}{2}} \leq g \leq (1+\Lambda)\mu^{\frac{1}{2}} \quad \text{in} \quad \overline{\Omega}. \tag{3. 46}$$

In particular, the following estimate for the lower order term holds true

$$\left|\mathcal{R}[w]\,(\frac{n}{2} - \frac{|v|^2}{4})g\right| \leq C \quad \text{for any } w \in \mathcal{K} \tag{3. 47}$$

where C is a universal constant depending on $n, \alpha, \lambda, \Lambda$. Thus, the global Hölder estimate Lemma 3.12 (ii) implies that there exist some constants $\gamma \in (0,1)$ and $\mathcal{N} > 0$ depending only on universal constants and ε such that $\|g\|_{C_l^{2\gamma}(\Omega)} \leq \mathcal{N}$. It then follows from Proposition 3.11 and interior Schauder estimates (3. 41) that the mapping

$$F : \mathcal{K} \to \mathcal{K} \cap C_l^{2\gamma}(\overline{\Omega}) \cap C_l^{2+2\gamma}(\Omega)$$

is well-defined. Besides, by Arzelà–Ascoli theorem we know that $F(\mathcal{K})$ is precompact in $C_l^{\gamma}(\overline{\Omega})$.

As far as the continuity of F is concerned, we take a sequence $\{w_n\}$ converging to w_∞ in $C_l^{\gamma}(\overline{\Omega})$. Since $\{F(w_n)\}$ is precompact in $C_l^{\gamma}(\overline{\Omega})$, there exists a converging subsequence whose limit is $g_\infty \in C_l^{\gamma}(\overline{\Omega})$ which satisfies

$$g_\infty(v,x,0) = g_{\text{in}}^{\varepsilon}(v,x) \qquad (v,x) \in \mathbb{R}^n \times \mathbb{T}^n.$$

In view of the interior Schauder estimate (3. 41), $\{F(w_n)\}$ is precompact in $C^2(K)$ for any compact subset $K \subset \Omega$ and $g_\infty \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Let us consider again the Cauchy problem (3. 45) with the couple (w,g) substituted from $(w_n, F(w_n))$. Thus, sending $n \to \infty$ we see that the equation (3. 45) also holds for the couple of limits $(w,g) = (w_\infty, g_\infty)$. Then, by applying Lemma 3.13 (the maximum principle) to

$$\begin{cases} (\partial_t + v\cdot\nabla_x)\left(\mu^{-\frac{1}{2}}(g_\infty - F(w_\infty))\right) = \mathcal{R}[w_\infty]\,(\nabla_v - v)\cdot\nabla_v\left(\mu^{-\frac{1}{2}}\,(g_\infty - F(w_\infty))\right) & \text{in } \Omega, \\ (g_\infty - F(w_\infty))(0,\cdot,\cdot) = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{T}^n, \end{cases}$$

we obtain $g_\infty = F(w_\infty)$. Then for every $\varepsilon \in (0,1]$ we are allowed to apply the Schauder fixed point theorem (see for instance Corollary 11.2 of [57]) to get $g^\varepsilon \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $F(g^\varepsilon) = g^\varepsilon$, which is a classical solution to (3. 40) associated to the initial data $g_{\text{in}}^{\varepsilon}$.

Step 2. Passage to the limit.
Recalling the lower bound (3. 43) on the coefficient and the interior Schauder estimate (3. 41), we point out that for any $\underline{T} \in (0,T)$, $\{g^\varepsilon\}$ is uniformly bounded in $C_l^{2+\alpha_*}(\mathbb{R}^n \times \mathbb{T}^n \times [\underline{T},T])$, for some constant $\alpha_* \in (0,1)$ with the same dependence as C_*. Hence, g^ε converges uniformly to g in $C_l^2(\mathbb{R}^n \times \mathbb{T}^n \times [\underline{T},T])$, up to a subsequence. Let us consider the equation satisfied by g^ε in the weak formulation, that is for every $\psi \in C_c^\infty(\overline{\Omega})$

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_{\mathbb{T}^n} [g^\varepsilon(v,x,T)\psi(v,x,T) - g_{\text{in}}^{\varepsilon}(v,x)\psi(v,x,0)]\,dv\,dx \\ &\quad = \int_\Omega \left\{ g^\varepsilon\,(\partial_t + v\cdot\nabla_x)\,\psi - \mathcal{R}[g^\varepsilon]\nabla_v g^\varepsilon\cdot\nabla_v\psi + \mathcal{R}[g^\varepsilon]\left(\frac{n}{2} - \frac{|v|^2}{4}\right) g^\varepsilon\psi \right\} dv\,dx\,dt. \end{aligned} \tag{3. 48}$$

Now, we derive from (3. 48) a Caccioppoli type inequality, also known as *energy estimate*, by choosing as a test function $\psi = g^\varepsilon$ and applying Young's inequality we get:

$$\begin{aligned}
&\int_\Omega \mathcal{R}\left[g^\varepsilon\right] |\nabla_v g^\varepsilon|^2 \, dv\, dx\, dt \\
&\quad \leq \int_\Omega \mathcal{R}\left[g^\varepsilon\right] \left(\frac{n}{2} - \frac{|v|^2}{4}\right) |g^\varepsilon|^2 \, dv\, dx\, dt + \int_{\mathbb{R}^n}\int_{\mathbb{T}^n} |g^\varepsilon_{\text{in}}(v,x)|^2 \, dv\, dx - \int_{\mathbb{R}^n}\int_{\mathbb{T}^n} |g^\varepsilon|^2 \, dv\, dx \\
&\quad \leq \underbrace{\boxed{\int_\Omega \mathcal{R}\left[g^\varepsilon\right] \left(\frac{n}{2} - \frac{|v|^2}{4}\right) |g^\varepsilon|^2 \, dv\, dx\, dt}}_{A} + \underbrace{\boxed{\int_{\mathbb{R}^n}\int_{\mathbb{T}^n} |g^\varepsilon_{\text{in}}(v,x)|^2 \, dv\, dx + \int_{\mathbb{R}^n}\int_{\mathbb{T}^n} |g^\varepsilon|^2 \, dv\, dx}}_{B}
\end{aligned}$$

As far as we are concerned with term A, the estimate directly follows from the estimate (3. 47) alongside with the Gaussian bounds (3. 46):

$$\int_\Omega \mathcal{R}\left[g^\varepsilon\right] \left(\frac{n}{2} - \frac{|v|^2}{4}\right) |g^\varepsilon|^2 \, dv\, dx\, dt \leq C \int_\Omega |g^\varepsilon| \, dv\, dx\, dt \leq C \int_\Omega \mu^{\frac{1}{2}} (1+\Lambda) \, dv\, dx\, dt \leq C(1+\Lambda),$$

where C is a universal constant depending on n, Λ, λ. Finally, term B is made of two integrals, each of which is bounded from above by the Gaussian upper bound (3. 46). For instance, let us consider the first term

$$\int_{\mathbb{R}^n}\int_{\mathbb{T}^n} |g^\varepsilon_{\text{in}}(v,x)|^2 \, dv\, dx \leq \int_{\mathbb{R}^n}\int_{\mathbb{T}^n} (1+\Lambda)^2 \mu \, dv\, dx \leq C(1+\Lambda)^2,$$

where C is a positive constant only depending on n. The estimate of the other two terms follow analogously. Thus, the following estimate holds true:

$$\int_\Omega \mathcal{R}\left[g^\varepsilon\right] |\nabla_v g^\varepsilon|^2 \leq C(1+\Lambda)^2,$$

where C is a universal constants depending on $n, \alpha, \lambda, \Lambda$. Therefore, $\mathcal{R}[g^\varepsilon]\nabla_v g^\varepsilon$ converges weakly in $L^2(\Omega)$ to $\mathcal{R}[g]\nabla_v g$ (up to a subsequence). Besides, since $\{\mu^{-\frac{1}{2}} g^\varepsilon\}$ is uniformly bounded thanks to (3. 46), we have that the sequences

$$g^\varepsilon \quad \text{and} \quad \mathcal{R}[g^\varepsilon] \left(\frac{n}{2} - \frac{|v|^2}{4}\right) g^\varepsilon$$

weakly converge in $L^2(\Omega)$ to the functions

$$g \quad \text{and} \quad \mathcal{R}[g] \left(\frac{n}{2} - \frac{|v|^2}{4}\right) g,$$

respectively (up to a subsequence). Then, for any $\psi \in C^\infty_c\left(\mathbb{R}^n \times \mathbb{T}^n \times [0,T)\right)$, sending $\varepsilon \to 0$ in (3. 48) gives (3. 42). This completes the proof. $\square$

Proposition 3.20 *Let the constants $\alpha, \lambda, \Lambda, T > 0$ and g_1, g_2 be two distinct positive solutions to* (3. 40) *in $\mathbb{R}^n \times \mathbb{T}^n \times [0,T]$ with the same initial data $g_{\text{in}} \in C^\alpha\left(\mathbb{T}^n \times \mathbb{R}^n\right)$ such that*

$$R[g_{\text{in}}] \geq \lambda \ \text{ in } \mathbb{T}^n \ \text{ in } \mathbb{T}^n \times \mathbb{R}^n \quad \text{and} \quad 0 \leq g_{\text{in}} \leq \Lambda \mu^{\frac{1}{2}}.$$

Then, the uniqueness holds: $g_1 = g_2$.

Proof. Let g_1 and g_2 be two different solutions to the Cauchy problem (3. 40). The difference of $g_1 - g_2$ satisfies the following Cauchy problem:

$$\begin{cases} (\partial_t + v \cdot \nabla_x)(g_1 - g_2) = (R[g_1] - R[g_2])\, U[g_1] + R[g_2] U[g_1 - g_2] & \text{in } [0,T) \times \mathbb{R}^n \times \mathbb{T}^n \\ g_1(v,x,0) - g_2(v,x,0) = 0 & \text{in } \mathbb{R}^n \times \mathbb{T}^n. \end{cases}$$

By integrating the above equation against $(g_1 - g_2)$ in $\mathbb{R}^n \times \mathbb{T}^n$, and applying integration by parts we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|g_1 - g_2\|^2_{L^2(\mathbb{R}^n\times\mathbb{T}^n)} \\ &\quad = \int\limits_{\mathbb{R}^n\times\mathbb{T}^n} (R[g_1] - R[g_2])\, U[g_1](g_1 - g_2)\, dx\, dv - \int\limits_{\mathbb{R}^n\times\mathbb{T}^n} R[g_2] \left(\nabla_v \left(\frac{g_1 - g_2}{\sqrt{\mu}}\right)\right)^2 \mu\, dx\, dv. \end{aligned}$$

Applying Lemma 3.18 (ii), the elementary inequality $|a^\beta - 1| \le |a - 1|$ $(a \in \mathbb{R}^+)$ and Hölder's inequality we obtain the following estimate

$$\begin{aligned} \frac{d}{dt}\|g_1 - g_2\|^2_{L^2(\mathbb{R}^n\times\mathbb{T}^n)} &\lesssim \|\mu^{-\theta} U[g_1]\|_{L^\infty(\mathbb{T}^n\times\mathbb{R}^n)} \|g_1 - g_2\|^2_{L^2(\mathbb{R}^n\times\mathbb{T}^n)} \\ &\lesssim \left(1 + \|\mu^{-\theta}\Delta_v g_1\|_{L^\infty(\mathbb{R}^n\times\mathbb{T}^n)}\right) \|g_1 - g_2\|^2_{L^2(\mathbb{R}^n\times\mathbb{T}^n)}, \end{aligned} \tag{3. 49}$$

where $\theta \in (0, \frac{1}{4}]$ is arbitrarily chosen. We point out that, from now on $a \lesssim b$ is the shorthand notation for $a \le Cb$, where C is a certain positive constant only depending on n, λ, Λ and α.

Fix $z_0 = (v_0, x_0, t_0) \in \mathbb{R}^n \times \mathbb{T}^n \times (0,1)$ and $2r = t_0^{\frac{1}{2}}$. Then we apply interior Schauder estimates (3. 41) in Q_2 and then rescale back to $\mathcal{Q}_{2r}(z_0)$. Moreover, by observing that

$$(\partial_t + v \cdot \nabla_x)(g_1 - g_1(z_0)) = R[g_1]\Delta_v (g_1 - g_1(z_0)) + R[g_1]\left(\frac{n}{2} - \frac{|v|^2}{4}\right) g_1 \quad \text{in } Q_{2r}(z_0),$$

since $r \in (0,1)$ and $t \ge \frac{t_0}{2}$ we have

$$\begin{aligned} \|\nabla_v^2 g_1\|_{L^\infty(Q_r(z_0))} &\lesssim r^{-2}\|g_1 - g_1(z_0)\|_{L^\infty(Q_{2r}(z_0))} + r^\alpha \left[R[g_1]\left(\frac{n}{2} - \frac{|v|^2}{4}\right) g_1\right]_{C^\alpha(Q_{2r}(z_0))} \\ &\lesssim r^{-2+\alpha}[g_1]_{C^\alpha(Q_{2r}(z_0))} \\ &\lesssim |t|^{-1+\frac{\alpha}{2}} [g_1]^{\frac{1}{2}}_{C^\alpha(Q_{2r}(z_0))} \|g_1\|^{\frac{1}{2}}_{L^\infty(Q_{2r}(z_0))}. \end{aligned}$$

By pointing out that $(v_0, x_0) \in \mathbb{R}^n \times \mathbb{T}^n$ is arbitrarily chosen and applying point (i) of Lemma 3.18, it follows that for any $t \in (0,1)$,

$$\|\mu^{-\theta}\nabla_v^2 g_1(t)\|_{L^\infty(\mathbb{R}^n\times\mathbb{T}^n)} \lesssim |t|^{-1+\frac{\alpha}{2}},$$

with any fixed $\theta \in (0, \frac{1}{8}]$. By applying this estimate and Grönwall's inequality to (3. 49), we get

$$\|(g_1 - g_2)(t)\|^2_{L^2(\mathbb{R}^n\times\mathbb{T}^n)} \lesssim \|(g_1 - g_2)(0)\|^2_{L^2(\mathbb{R}^n\times\mathbb{T}^n)} \exp\left(\int_0^t \left(1 + |s|^{-1+\frac{\alpha}{2}}\right) ds\right) = 0$$

for any $t \in (0,1)$. As for $t \ge 1$, the uniqueness follows directly from (3. 49), point (ii) of Lemma 3.18 and Grönwall's inequality. $\square$

3.4 C^∞ a priori estimates

This section is devoted to the proof C^∞ a priori estimates of Theorem 3.1. Since we want to bootstrap the regularity estimates starting from Lemma 3.18 (*ii*), we are going to apply Schauder estimates (3. 41) to derivatives and increments of the solution g for the equation (3. 40) iteratively. Before proceeding with the proof, we need to adapt some technical lemmas about increments and Hölder norms presented for the first time by Imbert and Silvestre in [69] for the Boltzmann equation. First of all let us write, for some small increment $y \in \mathbb{R}^n$

$$\Delta_y g(z) = g(z \circ (0, y, 0)) - g(z).$$

Roughly speaking, we can say that the Schauder estimate presented in Lemma 3.18 (*ii*) allows us to get only $\frac{2}{3}$ derivatives in x at the first application. In order to get a full derivative, we apply this estimate to increments $\Delta_y g$ as defined above. The following two lemmas allow us to transfer a regularity estimate for an incremental quotient into a higher order differentiation and it is inspired by Lemma 8.1 and Lemma 8.4 proved in [69] by Imbert and Silvestre. In spite of the apparent simplicity of its statement, its proof is rather involved and the first step of the proof is inspired by Lemma 5.6 in [24]. From now on we employ the short hand notation $Q = Q_R(z_0)$ with $R \in (0, 1)$ and $Q_{\text{int}} = Q_{\frac{R}{2}}(z_0)$.

Lemma 3.21 *Let $\alpha > 0$. Given a cylinder $Q = Q_R(z_0)$ with $R \in (0, 1)$ and a bounded continuous function g defined in Q, we consider for any $y \in B_{R^3/2}$ the following function defined in $Q_{int} = Q_{R/2}(z_0)$:*

$$\Delta_y g(z) = g(z \circ (0, y, 0) - g(z).$$

We assume there exists a $N > 0$ such that for every $y \in B_{R^3/2}$

$$\|\Delta_y g\|_{C^0(Q_{int})} \le N, \qquad [\Delta_y g]_{C^{2+\alpha}(Q_{int})} \le N \|(0, y, 0)\|^2. \tag{3. 50}$$

Then for some $\eta = \eta(\alpha) > 0$ we have

$$\|\Delta_y g\|_{C^\eta(Q_{int})} \le N \|(0, y, 0)\|^3. \tag{3. 51}$$

Proof. Let p_z denote the polynomial expansion of g at z of kinetic degree strictly smaller than $2 + \alpha$. The assumptions stated in (3. 50) translate into the following: for every $z \in Q$ and ξ such that $(z \circ \xi) \in Q$

$$|\Delta_y g(z \circ \xi) - \delta_y p_z(\xi)| \lesssim N \|(0, y, 0)\|^2 \|\xi\|^{2+\alpha}, \tag{3. 52}$$

where $\delta_y p_z$ is the polynomial expansion of $\Delta_y g$ at the point z and $\delta_y p_z = \delta_{(0,y,0)} p_z$. Since $\alpha \in (0, 1)$, we aim at proving that for every $z \in Q_{int}$, $z \in Q_{R/2}$ and ξ such that $(z \circ \xi) \in Q_{int}$

$$|\Delta_y g(z \circ \xi) - \delta_y p_z(\xi)| \lesssim N \|(0, y, 0)\|^3 \|\xi\|^\eta, \quad \text{for some } \eta = \eta(\alpha). \tag{3. 53}$$

STEP 1. We claim that for every $z \in Q_{int}$ and for every $k \in \mathbb{N}$ such that $(z \circ (2^k (0, y, 0))) \in Q$, we have

$$|\Delta_y g(z) - 2^{-k} \Delta_{2^k (0,y,0)} g(z)| \lesssim N \|(0, y, 0)\|^{4+\alpha} 2^{k\left(\frac{1+\alpha}{3}\right)}. \tag{3. 54}$$

In order to get such an estimate, let us recall that

$$\Delta_{2y} g(z) = g(z \circ (0, 2y, 0)) - g(z) = \Delta_y g(z) + \Delta_y g(z \circ (0, y, 0)).$$

and by applying estimate (3. 52), we get

$$\begin{aligned} |\Delta_{2y} g(z) - 2\Delta_y g(z)| &= |\Delta_y g(z) + \Delta_y g(z \circ (0, y, 0)) - 2\Delta_y g(z)| \\ &= |\Delta_y g(z \circ (0, y, 0)) - \delta_y p_z(0, y, 0) + \delta_y p_z(0, y, 0) - \Delta_y g(z)| \\ &\lesssim N \|(0, y, 0)\|^{4+\alpha} + |\delta_y p_z(0, y, 0) - \Delta_y g(z)| \end{aligned} \tag{3. 55}$$

Since the polynomial p_z is of degree strictly less than $2+\alpha$, we have for $\xi = (\xi_v, \xi_x, \xi_t) \in \mathbb{R}^{2n+1}$ that

$$\delta_y p_z(\xi) = \Delta_y g(z) + (\partial_t + v \cdot \nabla_x)\Delta_y g(z)\xi_t + D_v \Delta_y g(z) \cdot \xi_v + \tfrac{1}{2} D_v^2 \Delta_y g(z)\xi_v \cdot \xi_v. \tag{3. 56}$$

In particular, we remark that when evaluating the previous expression at the point $\xi = (0, y, 0)$ we get the following

$$\delta_y p_z(\xi) = \Delta_y g(z).$$

Thus, we can conclude from (3. 55)

$$|\Delta_{2y} g(z) - 2\Delta_y g(z)| \lesssim N\|(0, y, 0)\|^{4+\alpha} + |\delta_y p_z(0, y, 0) - \Delta_y g(z)|,$$

or equivalently

$$|\Delta_y g(z) - 2^{-1}\Delta_{2y} g(z)| \lesssim 2^{-1} N\|(0, y, 0)\|^{4+\alpha}. \tag{3. 57}$$

Then we proceed by induction on k. The initial step $k = 0$ is given by (3. 57). Let us suppose (3. 54) holds true for $k-1$ and we prove it's true for k:

$$\begin{aligned}
|\Delta_y g(z) - 2^{-k}\Delta_{2^k y} g(z)| &\lesssim N \sum_{j=1}^{k} 2^{-j} \|2^{j-1}(0, y, 0)\|^{4+\alpha} \\
&\lesssim N\|(0, y, 0)\|^{4+\alpha} \sum_{j=1}^{k} 2^{-j+(j-1)\frac{4+\alpha}{3}} \\
&\lesssim N\|(0, y, 0)\|^{4+\alpha} 2^{k\frac{4+\alpha}{3}-1}.
\end{aligned}$$

This achieves the proof of the claim (3. 54).
STEP 2. We claim that for every $z \in Q_{int}$ and for every $(0, y, 0) \in Q_{R/2}$, we have

$$|\Delta_y g(z)| \lesssim N\|(0, y, 0)\|^3. \tag{3. 58}$$

Indeed, taking into consideration (3. 54) we can write

$$\begin{aligned}
|\Delta_y g(z)| &\lesssim 2^{-k}|\Delta_{2^k(0,y,0)} g(z)| + N\|(0, y, 0)\|^{4+\alpha} 2^{k\left(\frac{1+\alpha}{3}\right)} \\
&\lesssim \|\Delta_{2^k(0,y,0)} g\|_{C^0} \|(0, y, 0)\|^3 + N\|(0, y, 0)\|^3 2^{k\left(\frac{1+\alpha}{3}\right)}.
\end{aligned}$$

□

Lemma 3.22 *Given $y \in B_{R^3/2}$ with $R \le 1$ and $\alpha \in]0, 1]$ and some cylinder $Q = Q_R(z_0)$, let $g \in C^{2+\alpha}(Q)$. Then $\Delta_y g$ lies in $C_l^\alpha(Q_{int})$ with $Q_{int} = Q_{R/2}(z_0)$ and*

$$\|\Delta_y g\|_{C_l^\alpha(Q_{int})} \le C[g]_{C_l^{2+\alpha}(Q)} \|(0, y, 0)\|^2$$

for some constant C only depending on the dimension.

PROOF. We remark that the assumption of this lemma implies that the assumptions of Lemma 3.22 hold true with $N = 2\|g\|_{C_l^\alpha(Q)}$. Applying Lemma 3.22 yields the desired result. □

Proof of Theorem 3.1 (C^∞ a priori estimates). This argument was firstly introduced by Imbert and Silvestre in [69] for the inhomogeneous Boltzmann equation without cut-off and we adapt it here to our nonlinear case (3. 40). From now on, a differential operator $D = \partial_v^{k_v}\partial_x^{k_x}\partial_t^{k_t}$ with $k = (k_v, k_x, k_t) \in \mathbb{N}^{2n+1}$ is intended in the the classical way

$$D = \partial_{v_1}^{k_v^1}\dots\partial_{v_n}^{k_v^n}\partial_{x_1}^{k_x^1}\dots\partial_{x_n}^{k_x^n}\partial_t^{k_t}$$

if $k_v = (k_v^1, \dots, k_v^n)$ and $k_x = (k_x^1, \dots, k_x^n)$. Moreover, we recall that the order of the multi- index $k \in \mathbb{N}^{2n+1}$ is $|k| = k_v^1 + \dots + k_v^n + k_x^1 + \dots + k_x^n + k_t$. In this section, when we refer to the order of D, we literally refer to the classical order of differentiation.

By an iterative process, we will establish the following family of inequalities. For every differential operator $D = \partial_t^{k_t}\partial_x^{k_x}\partial_v^{k_v}$ with $k = (k_v, k_x, k_t) \in \mathbb{N}^{2n+1}$, there exists some constant $\alpha_k \in (0,1)$ so that for every $\tau > 0$ depending on $|k|$ such that for any $Q = Q_R(z_0) \subset \mathbb{R}^n \times \mathbb{T}^n \times [\tau, \infty)$

$$\|Df\|_{C_l^{2+\alpha_k}(\mathcal{Q}_{int})} \le C_k \mu^\nu(v_0). \tag{3. 59}$$

where $\tau > 0$, $z_0 = (t_0, v_0, x_0)$ and the constant C_k depends on k_v, k_x, k_t, τ. The value of α_k we obtain in the iteration depends also on k and tends to be smaller as the order of differentiation increases. A posteriori, we obtain a C^∞ estimate for g, so the particular values of α after each iteration doesn't matter. For simplicity, we will omit the domain in estimates below, since the estimates can be always localized around the center z_0.

Step 0. This is a preliminary step where we consider the case where $Dg = g$, corresponding to the choice of parameters $k_t = 0$, $|k_x| = 0$ and $|k_v| = 0$. With this choice of parameters, inequality (3. 59) follows straightforwardly from the application of Lemma 3.18.

Step 1. In this step, we prove inequality (3. 59) for differential operators of the type $D^k = D_x^{k_x}$, which corresponds to the choice of parameters $k = (0, k_x, 0)$. We proceed by induction on $d = |k_x|$. First of all, we remark that it is not possible to apply directly Proposition 3.10 in order to get inequality (3. 59) for this case, because the kinetic degree of ∂_{x_i} is equal to 3. For this reason, it is convenient to make the inductive statement in terms of increments. We are going to prove by induction on $d \in \mathbb{N}$, $d \ge 1$ that there exists $\alpha_d \in (0,1)$ such that for any $\tau > 0$ there exists a constant $C_d > 0$ such that for every $k_x \in \mathbb{N}^n$ with $|k_x| = d$, $\nu \in (0, \frac{1}{2})$ and $y \in B_{\frac{R^3}{4}}$,

$$\|\Delta_y \partial_x^{k_x} g\|_{C_l^{2+\alpha_d}} \le C_d |y| \mu^\nu(v_0). \tag{3. 60}$$

Passing to the limit as $y \to 0$ completes the step.

The case $d = 0$ is provided by STEP 0. Moreover, note that inequality (3. 60) holds trivially for $d = 0$, since there is no $|k_x| \le -1$. In order to proceed by induction, we suppose that (3. 60) holds for any $k_x \in \mathbb{N}^n$ with $|k_x| \le d-1$. Let $|k_x| = d$ and $h = \Delta_y D_x^{k_x} g$. By the inductive hypothesis (3. 60) combined with Lemma 3.22, we have that

$$\|h\|_{C_l^{\alpha_{d-1}}} \le C[D_x^{k_x} g]_{C_l^{2+\alpha_{d-1}}} \|(0,y,0)\|^2 = C_d |y|^{\frac{2}{3}} \mu^\nu(v_0). \tag{3. 61}$$

Our aim is to enhance the exponent $\frac{2}{3}$ on the right hand side all the way to one. In order to get this result, we adapt Lemma 9.1 proved in [69] to our nonlinear case (for the proof see the end of this section).

Lemma 3.23 *Let $h = \Delta_y D_x^{k_x} g$ (as above), and assume that* (3. 60) *holds true with $|k_x| \le d-1$. If there exists $\overline{\alpha} \in]0, \min(\alpha_0, \alpha_{d-1})]$ such that*

$$\|\Delta_y \partial_x^{k_x} g\|_{C_l^{\overline{\alpha}}(Q)} \le C[D_x^{k_x} g]_{C_l^{2+\alpha_d}(Q)} \|(0,y,0)\|^2 = C_d |y|^{\frac{2}{3}} \mu^\nu(v_0).$$

when $|k_x| = d$*, then with* $\overline{\alpha}' = \frac{2}{3}\overline{\alpha}$ *we have*

$$\|h\|_{C_l^{2+\overline{\alpha}'}(Q')} \leq C_d \|(0,y,0)\|^2 \mu^{\nu'}(v_0) = C_d |y|^{\frac{2}{3}} \mu^{\nu'}(v_0).$$

where $Q' \subset Q \cap (\mathbb{R}^n \times \mathbb{T}^n \times [2\tau, \infty))$.

Thus, by recalling (3. 61) and applying Lemma 3.23 to the function $h = \Delta_y D_x^{k_x} g$ with $|k_x| = d$ we get

$$\|h\|_{C_l^{2+\alpha_d}(Q')} \leq C_d \|(0,y,0)\|^2 \mu^{\nu'}(v_0), \tag{3. 62}$$

where $\alpha_d = \frac{2}{3}\alpha_{d-1}$. Note that the time shift τ was updated to 2τ. This is because the application of Lemma 3.18 in the proof of Lemma 3.23 requires a gap in time. Because we obtain estimates for every value of $\tau > 0$ (with a constant depending on τ), the difference between τ and 2τ is not relevant for the final estimates. In view of this observation, we will omit from now on the domain dependence in the estimates below as a way of decluttering the expressions and focusing on the Hölder exponents. Combining (3. 62) with Lemma 3.21 we finish the proof of (3. 60) for $|k_x| = d$.

Step 2. In this step, we prove inequality (3. 59) for differential operators of the type $D^k = \partial_t^{k_t} D_x^{k_x}$, which correspond to the choice of parameters $k = (0, k_x, k_t)$. Thus, our aim is to control the norm $\|\partial_t^{k_t} D_x^{k_x} g\|_{C_l^{2+\alpha}}$ for some small $\alpha > 0$. We proceed through a bi-dimensional induction on $(m, d) = (k_t, |k_x|)$ such that for any $\tau > 0$ and for any $\nu \in (0, \frac{1}{2})$ there exists a constant $C_{m,d}$ such that

$$\|\partial_t^{k_t} D_x^{k_x} g\|_{C_l^{2+\alpha_{m,d}}} \leq C_{m,d}\, \mu^{\nu}(v_0) \tag{3. 63}$$

Equivalently it can be thought as induction in m and d, where $k_t = m$ and $|k_x| = d$, where the inductive step in m is proved by induction in d. The case $m = 0$ is treated in STEP 1. Let $m, d \in \mathbb{N}$, $m > 0$ and let us assume (3. 63) holds true whenever $k_t \leq m-1$ and $|k_x| \leq d+1$, and also for $k_t = m$ and $|k_x| < d$. Our aim is to prove by induction that (3. 63) also holds for $k_t = m$ and $|k_x| = d$.

Let $k_x \in \mathbb{N}^n$ be any multi-index with $|k_x| = d$. Using the inductive hypothesis (3. 63) with with $k_t = m-1$ we get

$$\|\partial_t^{m-1} D_x^{k_x} g\|_{C_l^{2+\alpha_{m-1,d}}} \leq C_{m-1,d}\, \mu^{\nu}(v_0). \tag{3. 64}$$

Since the kinetic degree of the operator $\partial_t + v \cdot \nabla_x$ is 2, by combining (3. 64) with Lemma 3.10 we get the following bound

$$\|(\partial_t + v \cdot \nabla_x) \partial_t^{m-1} D_x^{k_x} g\|_{C_l^{\alpha_{m-1,d}}}, \leq C_{m,d}\, \mu^{\nu}(v_0). \tag{3. 65}$$

Applying again the inductive assumption (3. 63) with $k_t = m-1$ and $\tilde{k}_x = d+1$ we get the following

$$\|(v \cdot \nabla_x) \partial_t^{m-1} D_x^{\tilde{k}_x} g\|_{C_l^{2+\alpha_{m1,d+1}}} \leq C \|\partial_t^{m-1} \nabla_x D_x^{\tilde{k}_x} g\|_{C_l^{2+\alpha_{m-1,d+1}}} \leq C_{m-1,d+1}\, \mu^{\nu'}(v_0) \tag{3. 66}$$

where $\nu' \in (0, \frac{1}{2})$. Therefore, combining (3. 65) and (3. 66) we get the following inequality for some $\overline{\alpha} > 0$ and some constant $\overline{C}$ depending both on m and d

$$\|\partial_t^n D_x^{k_x} g\|_{C_l^{\overline{\alpha}}} \leq \overline{C}\, \mu^{\nu''}(v_0), \tag{3. 67}$$

where $\nu'' = \min\{\nu, \nu'\}$. Our next objective is to turn estimate (3. 67) into

$$\|\partial_t^m D_x^{k_x} g\|_{C_l^{2+\alpha_{m,d}}} \leq C \mu^{\overline{\nu}}(v_0). \tag{3. 68}$$

In order to do that, we differentiate equation (3. 40) and compute the following equation for $h := \partial_t^m D_x^{k_x} g$

$$(\partial_t + v \cdot \nabla_x) h = R[h] U[g] + R[g] U[h] + \sum_{\substack{i \leq (m, k_x, 0) \\ i \neq (m, k_x, 0)}} R[\hat{D}_i g] U[D_i g],$$

where $\partial_t^n D_x^{k_x} = \hat{D}_i \circ D_i$ and $i \in \mathbb{N}^{2n+1}$ is a multi-index as in the proof of Lemma 3.23 below. The first two terms $R[g]U[h]$ and $R[h]U[g]$ are bounded by the definition of the function g, the intermediate estimate (3. 67) and Lemma 3.18. By induction assumption (3. 63) and Lemma 3.18 each term in the reminder (the summation on the right hand side) can be controlled in C^α.

Step 3. In the third and last step, we establish inequality (3. 59) for every differential operator $D^k = \partial_t^{k_t} D_x^{k_x} D_v^{k_v}$, which correspond to the choice of parameters $k = (k_v, k_x, k_t)$ with $k \in \mathbb{N}^{2n+1}$, and for every $\tau > 0$. More explicitly we will prove, via a bidimensional induction argument similar to the one in STEP 2 where $k_v = m$ and $k_t + |k_x| = d$, that for any $\tau > 0$ there exists a $C_{m,d}$ such that for any $\nu \in (0, \frac{1}{2})$

$$\|\partial_t^{k_t} D_x^{k_x} D_v^{k_v} g\|_{C_l^{2+\alpha_{m,d}}} \leq C_{m,d}\, \mu(v_0)^\nu. \tag{3. 69}$$

The case $m = 0$ is treated in STEP 2. Let $m, d \in \mathbb{N}$, $m > 0$ and let us assume (3. 69) holds true whenever $k_v \leq m-1$ and $k_t + |k_x| \leq d$. Let $m \geq 1$ and $k \in \mathbb{N}^{2n+1}$ be any multi-index with $|k_v| = m-1$ and $k_t + |k_x| = d$. Applying the inductive hypothesis (3. 69) with these parameters we get

$$\|\partial_t^{k_t} D_x^{k_x} D_v^{n-1} g\|_{C_l^{2+\alpha_{m-1,d}}} \leq C_{m-1,d}\, \mu(v_0)^\nu. \tag{3. 70}$$

Since the kinetic degree of ∇_v is 1, we can apply Lemma 3.10 and get the following bound

$$\|\nabla_v \partial_t^{k_t} D_x^{k_x} D_v^{n-1} g\|_{C_l^{\overline{\alpha}}} \leq C_{m,d}\, \mu(v_0)^\nu, \tag{3. 71}$$

where $\overline{\alpha} = 1 + \alpha_{m-1,d}$. Thus, we can compute an equation for $h = \nabla_v \partial_t^{k_t} D_x^{k_x} D_v^{n-1} g$ and proceed like in STEP2. This concludes the proof. □

Proof of Lemma 3.23. The key element to the proof of this lemma is to differentiate (3. 40) with respect to $\Delta_y D_x^{k_x}$. Then we apply Lemma 3.18 together with the estimates we have for each incremental quotient given by the inductive assumption (3. 60). Indeed, by a direct computation we can show $h = \Delta_y D_x^{k_x} g$ verifies the following equation

$$\begin{aligned}(\partial_t + v \cdot \nabla_x)h =& R[g]U[h] + R[h]U[g] + R[\Delta_y g]U[D_x^{k_x} g] + R[D_x^{k_x} g]U[\Delta_y g] + \\ &+ \sum_{\substack{|i|<d \\ i \leq k_x}} \left\{ R[\Delta_y \hat{D}_i g]U[D_i g] + R[\hat{D}_i g]U[\Delta_y D_i g] \right\},\end{aligned} \tag{3. 72}$$

where $i \in \mathbb{N}^n$ is a multi-index such that $i \leq k_x$ (i.e. every component of i is lower or equal than the correspondent component of k_x) and $D_x^{k_x} = \hat{D}_i \circ D_i$. The first two terms $R[g]U[h]$ and $R[h]U[g]$ are bounded by the definition of the function g and Lemma 3.18. Since the index i runs over $|i| < d$, the inductive hypothesis (3. 60) tells us that $\Delta_y g$, $D_x^{k_x} g$, $D_i g$, $\widehat{D}_i g$, $\Delta_y D_i g$ and $\Delta_y \widehat{D}_i g$ are bounded in $C_l^{2+\alpha_d}$ by $C_d |y| \mu^\nu(v_0)$ except for the two extremal cases:

$$\Delta_y D^i g \quad \text{for } i = k_x \quad \text{and} \quad \Delta_y \widehat{D}^i g \quad \text{for } i = (0,0,0).$$

These cases covered by the assumptions of this lemma and Lemma 3.18 (i). □

Chapter 4

The weak regularity theory

The second part of my thesis is devoted to the regularity theory for weak solutions to the Kolmogorov equation with measurable coefficients, which is nowadays the main focus of the research community. The weak regularity theory has been developed during the last decade, and it is still evolving. Here, we briefly recall some of the main results on this subject for different kinds of Kolmogorov operators:

- **Kolmogorov operator with measurable coefficients in divergence form:** the Moser's iterative scheme was firstly proved by Polidoro and Pascucci for the dilation invariant case [104], then later on extended by Cinti, Pascucci and Polidoro [33], and Wand and Zang [119]. As far as we are concerned with Hölder regularity for weak solutions, Wang and Zang prove it alongside with a weak Poincaré inequality in [120] and [119], respectively for the dilation invariant and the non-dilation invariant case. We remark that all of this results consider the definition of weak solution introduce by Pascucci and Polidoro in [104], that we report here in Definition 5.1.
- **Kolmogorov operator with measurable coefficients in non-divergence form:** the only result available is due to Abedin and Tralli [1], who proved a Harnack inequality for this type of operators with additional Cordes-Landis assumption on the coefficients a_{ij}.
- **Kolmogorov operator with VMO coefficients a_{ij}:** these operators have been studied in [22] by Bramanti, Cerutti and Manfredini, [91] by Manfredini and Polidoro, and in [109], [110] by Polidoro and Ragusa.
- **Nonlocal Kolmogorov type operators $(\mathcal{K})^s$:** this kind of nonlocal operators and their stationary counterparts have been introduced in the recent paper [52] by Garofalo and Tralli. In particular, Hardy-Littlewood- Sobolev inequalities, Poincaré-type inequalities, and nonlocal isoperimetric inequalities are proved in [54], [53], and [55] respectively.

The aim of this work is to contribute to the study of weak regularity theory for Kolmogorov equations with measurable coefficients in divergence form. The most recent developments in this framework have been established in the particular case of the kinetic Kolmogorov-Fokker-Planck equation:

$$\partial_t u(v,x,t) + v \cdot \nabla_x u(v,x,t) = \mathrm{div}_v(A(v,x,t)\nabla_v u(v,x,t)) + b(v,x,t) \cdot \nabla_v u(v,x,t) + f(v,x,t), \quad (4.\ 1)$$

where $(v,x,t) \in \mathbb{R}^{2n+1}$. This equation belongs to a class of evolution equations arising in the kinetic theory of gases and $u = u(v,x,t)$ represents in this case the density of particles with velocity $v = (v_1,\ldots,v_n)$ and position $x = (x_1,\ldots,x_n)$ at time t. Indeed, this latter equation is the one considered by Golse, Imbert, Mouhot and Vasseur in [58], where the authors prove the Hölder continuity and a Harnack inequality for weak solutions to the equation (4. 1) with measurable coefficients in divergence form. The

Harnack inequality proved in [58] is the only one available in the framework of weak regularity theory for Kolmogorov equations in divergence form, and it is based on the De Giorgi method. This chapter is devoted to the proof of a geometric statement for this Harnack inequality, based on the concepts of Harnack chains and attainable set.

As far as we are concerned with the more general Kolmogorov equation in divergence form

$$\begin{aligned}\mathcal{K}u(x,t) := \sum_{i,j=1}^{m_0} \partial_{x_i}\left(a_{ij}(x,t)\partial_{x_j}u(x,t)\right) + \sum_{i,j=1}^{N} b_{ij}x_j\partial_{x_i}u(x,t) - \partial_t u(x,t) + \\ + \sum_{i=1}^{m_0} b_i(x,t)\partial_i u(x,t) - \sum_{i=1}^{m_0} \partial_{x_j}\left(a_i(x,t)u(x,t)\right) + c(x,t)u(x,t) = 0\end{aligned} \tag{4. 2}$$

where $(x,t) \in \mathbb{R}^{N+1}$ and $1 \le m_0 \le N$, Chapter 5 is devoted to the extension of the Moser's iterative procedure for weak solutions to the equation $\mathcal{K}u = 0$ under minimal assumptions on the integrability of the lower order coefficients $a_1, \ldots, a_{m_0}, b_1, \ldots, b_{m_0}, c$, where we consider the definition of weak solution proposed by Pascucci and Polidoro in [104] (see Definition 5.1). As we shall see in the forthcoming section, the main advantage of this definition is that it allows us to directly handle the computations involving the drift term. The first result we have at our disposal in this setting is the proof of the Moser's iterative scheme for the dilation invariant Kolmogorov equation in divergence form with measurable coefficients and bounded lower order terms (see [104]). This results was firstly extended to the non dilation invariant case by Cinti, Pascucci and Polidoro in [33]. Then later on, Wang and Zang consider in [119] non dilation invariant Kolmogorov operators with lower order coefficients belonging to some L^q space, with $q > Q + 2$ (where Q is the homogeneous dimension defined in (5. 19)). The results we present here improve these previously known results extending them to the non-dilation invariant case with lower order measurable coefficients with positive divergence, a result that was firstly presented in the paper [7] in 2019 and that has been inspired by the article of Nazarov and Uralt'seva [99], who prove L^∞_{loc} estimates and Harnack inequalities for uniformly elliptic and parabolic operators in divergence form.

4.1 A geometric statement of the Harnack inequality

In this chapter we prove a geometric version of the Harnack inequality proved in [58] for weak solutions to the equation (4. 1), whose statement is recalled in Theorem 4.5 below. As a corollary, we obtain a strong maximum principle. More precisely, we consider second order partial differential equations of Kolmogorov-Fokker-Planck type of the form

$$\begin{aligned}\partial_t u(v,x,t) + \sum_{j=1}^{n} v_j \partial_{x_j} u(v,x,t) = \sum_{j,k=1}^{n} \partial_{v_j}\left(a_{jk}(v,x,t)\partial_{v_k}u(v,x,t)\right) \\ + \sum_{j=1}^{n} b_j(v,x,t)\partial_{v_j}u(v,x,t) + f(v,x,t), \qquad (v,x,t) \in \Omega,\end{aligned} \tag{4. 3}$$

where Ω is an open subset of $\mathbb{R}^{2n+1}$, $f \in L^\infty(\Omega)$, $b = (b_1, \ldots, b_n)$ is a vector of $\mathbb{R}^n$ with bounded measurable coefficients and $A = (a_{jk})_{j,k=1,\ldots,n}$ is a symmetric matrix with real measurable entries. Moreover, there exist two positive constants λ, Λ such that

$$\lambda|\xi|^2 \le \langle A(v,x,t)\xi, \xi\rangle \le \Lambda|\xi|^2, \qquad \forall (v,x,t) \in \Omega, \quad \forall \xi \in \mathbb{R}^n.$$

As the coefficients of the matrix A and of the vector b are measurable, we need to consider *weak solutions* to the equation (4. 3) in the following sense.

Weak solution. *Consider any open subset* Ω *of* $\mathbb{R}^{2n+1}$. *A weak solution to* (4. 3) *is a function* $u \in L^2_{loc}(\Omega)$ *such that* $\partial_{v_1} u, \dots, \partial_{v_n} u$ *and the directional derivative* $\partial_t u + \langle v, \nabla_x u \rangle$ *belong to* $L^2_{loc}(\Omega)$, *and moreover*

$$\int_\Omega (\partial_t u + \langle v, \nabla_x u\rangle - \langle b, \nabla_v u\rangle)\varphi dv\, dx\, dt + \int_\Omega \langle A\nabla_v u, \nabla_v \varphi\rangle dv\, dx\, dt = \int_\Omega f\varphi dv\, dx\, dt, \tag{4. 4}$$

for every test function $\varphi \in C_0^\infty(\Omega)$.

From now on, equation (4. 3) will be understood in the weak sense and will be written in the short form $\mathcal{K}u = f$, where $\mathcal{K}$ is the operator associated to (4. 3) and defined as follows

$$\mathcal{K}u =: \partial_t u + \langle v, \nabla_x u\rangle - \mathrm{div}_v(A\nabla_v u) - \langle b, \nabla_v u\rangle, \qquad (v,x,t) \in \Omega. \tag{4. 5}$$

Let us consider the unit box of $\mathbb{R}^{2n+1}$:

$$Q \ =]-1,1[^n \times]-1,1[^n \times]-1,0[, \tag{4. 6}$$

there the Harnack inequality proved in [58] reads as the usual parabolic Harnack inequality: there exist two *small* boxes Q^+ and Q^- contained in Q (see Fig. 1), with Q^+ located *above* Q^- with respect to the time variable, and a positive constant M, such that

$$\sup_{Q^-} u \le M \left(\inf_{Q^+} u + \|f\|_{L^\infty(Q)} \right)$$

for every non-negative weak solution u of $\mathcal{K}u = f$ in Q, with $f \in L^\infty(Q)$.

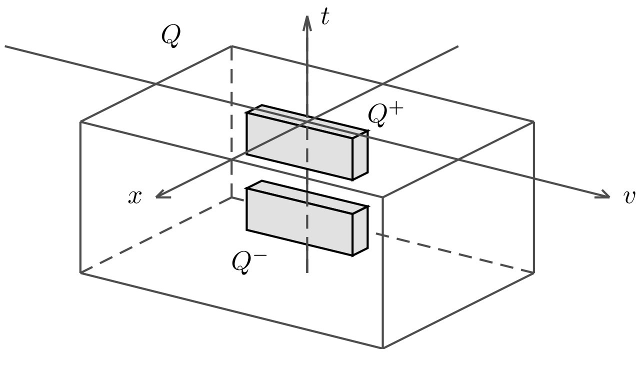

Fig. 1 - Harnack inequality.

We recall that, in the classical statement of the Harnack inequality for uniformly parabolic operators with measurable coefficients, the size of the boxes Q^+ and Q^-, and the gap between the lower basis of Q^+ and upper basis of Q^- can be arbitrarily chosen (see Theorem 1, p. 102 of [97]). On the contrary, in the statement of the Harnack inequality for the operator $\mathcal{K}$ given in [58], neither the size of the boxes Q^+ and Q^-, nor their *position* in Q is characterized. Actually, as we shall see in the sequel, it is known that the Harnack inequality does not hold for any choice of the boxes Q^+ and Q^-. This fact was previously noticed by Cinti, Nyström and Polidoro in [30], where classical solutions of

$$\mathcal{K}_0 u = \partial_t u + \langle v, \nabla_x u\rangle - \frac{1}{2}\mathrm{div}_v\left(\nabla_v u\right) = 0 \tag{4. 7}$$

are considered, and by Kogoj and Polidoro in [77]. We give here a sufficient condition for the validity of the Harnack inequality. For its precise statement we refer to the notion of *attainable set* $\mathscr{A}_{(v_0,x_0,t_0)}$ given in Definition 4.4 below. In the sequel $\mathrm{int}\big(\mathscr{A}_{(v_0,x_0,t_0)}\big)$ denotes the interior of $\mathscr{A}_{(v_0,x_0,t_0)}$.

Theorem 4.1 *Let Ω be an open subset of $\mathbb{R}^{2n+1}$ and let $f \in L^\infty(\Omega)$. For every $(v_0,x_0,t_0) \in \Omega$, and for any compact set $K \subseteq \mathrm{int}\big(\mathscr{A}_{(v_0,x_0,t_0)}\big)$, there exists a positive constant C_K, only dependent on Ω, (v_0,x_0,t_0), K and on the operator $\mathcal{K}$, such that*

$$\sup_K u \le C_K \left(u(v_0,x_0,t_0) + \|f\|_{L^\infty(\Omega)} \right),$$

for every non-negative weak solution to $\mathcal{K}u = f$.

We note that any weak solution u of $\mathcal{K}u = f$ is Hölder continuous (see [120, 119] for the equation $\mathcal{K}u = 0$, and Theorem 2 in [58] for $\mathcal{K}u = f$ with $f \in L^\infty$), then $u(v_0,x_0,t_0)$ is well defined. As we shall see in Definition 4.4, the attainable set $\mathscr{A}_{(v_0,x_0,t_0)}$ depends on the geometry of Ω and it can be easily described. For instance, when it agrees with the unit box Q in (4. 6) we have that

$$\mathscr{A}_{(0,0,0)} = \Big\{ (v,x,t) \in Q \mid |x_j| \le |t|, j = 1,\dots,n \Big\}. \tag{4. 8}$$

The proof of this fact can be seen in [30], Proposition 4.5, p.353 (see Fig. 2).

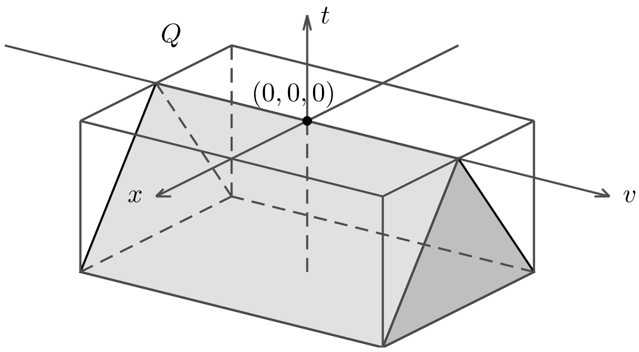

Fig. 2 - $\mathscr{A}_{(0,0,0)}(Q)$.

A direct consequence of our main result inequality is the following strong maximum principle.

Theorem 4.2 *Let Ω be an open subset of $\mathbb{R}^{2n+1}$, and let u be a non-negative solution to $\mathcal{K}u = 0$. If $u(v_0,x_0,t_0) = 0$ for some $(v_0,x_0,t_0) \in \Omega$, then $u(v,x,t) = 0$ for every $(v,x,t) \in \overline{\mathscr{A}_{(v_0,x_0,t_0)}}$.*

Concerning $\overline{\mathscr{A}_{(v_0,x_0,t_0)}}$, we remark that the closure has to be intended with respect to the topology of Ω. Note that the Theorem 4.2 extends to weak solution to $\mathcal{K}u = 0$ the well known Bony's strong maximum principle [20] for classical solutions of degenerate hypoelliptic PDEs with smooth coefficients. We also recall the work of Amano [3], where differential operators with continuous coefficients are considered. Moreover, Theorem 4.2 is somehow optimal. Indeed, in Proposition 4.5 of [30] it is shown that there exists a non-negative solution u to $\mathcal{K}_0 u = 0$ in Q such that $u(v,x,t) = 0$ for every $(v,x,t) \in \overline{\mathscr{A}_{(0,0,0)}}$, and $u(v,x,t) > 0$ for every $(v,x,t) \in Q \backslash \overline{\mathscr{A}_{(0,0,0)}}$.

We remark that our method also applies to the operators considered in the paper [1] *"Harnack inequality for a class of Kolmogorov–Fokker–Planck equations in non-divergence form"*. Thus our Theorems 4.1 and 4.2 hold true also for Kolmogorov operators in non-divergence form with continuous coefficients.

This chapter is organized as follows. Section 4.2 contains some preliminary results and known facts about the regularity properties of the operator $\mathcal{K}_0$, a short discussion on the controllability problem related to $\mathcal{K}_0$ and the definition of Attainable set. In Section 4.3 we recall the Harnack inequality given in [58] and we prove a dilation-invariant version of it. In Section 4.4 we prove our main results.

4.2 Controllability problem for $\mathcal{K}_0$. Definition of attainable set.

In this section, we recall some known facts on the equation (4. 5) and on its prototype (4. 7), that will play an important role in our study. First of all, we recall that the operator $\mathcal{K}$ introduced in (4. 7) belongs to the more general class of constant coefficient operators (1. 1) considered in Chapter 1. In particular, according to the notation of Chapter 1, the operator $\mathcal{K}_0$ can be written as follows:

$$\mathcal{K}_0 u := \sum_{i,j=1}^{2n} \widetilde{a}_{i,j} \partial^2_{y_i y_j} u + \sum_{i,j=1}^{2n} \widetilde{b}_{i,j}\, y_j \partial_{y_i} u + \partial_t u, \qquad (y,t) \in \mathbb{R}^{2n+1}, \tag{4. 9}$$

where $y = (v,x)$, $\widetilde{A} = (\widetilde{a}_{i,j}(y,t))_{i,j=1,\dots,N}$ is a symmetric non-negative matrix with real measurable entries and $\widetilde{B} = (\widetilde{b}_{i,j})_{i,j=1,\dots,N}$ is a constant matrix. We can choose, as it is not restrictive, a basis of $\mathbb{R}^{2n}$ such that $\widetilde{A}$ and $\widetilde{B}$ take the following form (see Proposition 1.1):

$$\widetilde{A} = \begin{pmatrix} \mathbb{I}_n & \mathbb{O}_n \\ \mathbb{O}_n & \mathbb{O}_n \end{pmatrix}, \qquad \text{and} \qquad \widetilde{B} = \begin{pmatrix} \mathbb{O}_n & \mathbb{O}_n \\ \mathbb{I}_n & \mathbb{O}_n \end{pmatrix}.$$

Clearly, as $m_0 = n < 2n$ the operator $\mathcal{K}_0$ is strongly degenerate, and thus by Proposition 1.1 its regularity properties depend on its first order part

$$Y = \langle \widetilde{B} y, D \rangle + \partial_t \sim (0, \dots, 0, v_1, \dots, v_n, 1)^T \tag{4. 10}$$

where $D = (\partial_{y_1}, \dots \partial_{y_{2n}})$ denotes the full gradient of $\mathbb{R}^{2n}$. Moreover, it is known that every operator $\mathcal{K}_0$ is invariant with respect to the non-Euclidean translation introduced in (1. 12), that for every $(v_0, x_0, t_0), (v, x, t) \in \mathbb{R}^{2n+1}$ reads as follows:

$$(v_0, x_0, t_0) \circ (v, x, t) := (v_0 + v, x_0 + x + t v_0, t_0 + t). \tag{4. 11}$$

If u is a solution of the equation $\mathcal{K}_0 u = f$ in some open set $\Omega \subset \mathbb{R}^{2n+1}$, then the function $w(v,x,t) := u\left((v_0, x_0, t_0) \circ (v, x, t)\right)$ is solution to $\mathcal{K}_0 w = g$, where $g(v,x,t) := f\left((v_0, x_0, t_0) \circ (v, x, t)\right)$ in the set $\left\{(v,x,t) \in \mathbb{R}^{2n+1} \mid (v_0, x_0, t_0) \circ (v, x, t) \in \Omega\right\}$. It is known that $\mathbb{R}^{2n+1}$ with the operation "$\circ$" is a non commutative group, with identity $(0,0,0)$. The inverse of the element (v,x,t) is

$$(v,x,t)^{-1} := (-v, -x + tv, -t). \tag{4. 12}$$

Moreover, by Proposition 1.4 the operator $\mathcal{K}_0$ is homogeneous of degree two with respect to the family of the following dilatations:

$$\delta_r(v,x,t) := (rv, r^3 x, r^2 t), \qquad \text{for every } r > 0. \tag{4. 13}$$

In this case the following *distributive property* of the dilation holds

$$\left(d_r(v_0, x_0, t_0)\right) \circ \left(d_r(v,x,t)\right) = d_r\left((v_0, x_0, t_0) \circ (v, x, t)\right),$$

for every $(v_0, x_0, t_0), (v, x, t) \in \mathbb{R}^{2n+1}$ and for every $r > 0$. We refer to Chapter 1 and the reference therein for further information on the subject.

We now introduce some basic notions of Control Theory in order to describe the set where the Harnack inequality holds for non-negative solutions of $\mathcal{K}u = f$. As noticed above, the link between the Regularity Theory for linear PDEs and the Control Theory is not surprising, as the hypoellipticity of $\mathcal{K}$ is equivalent to the Kalman's controllability condition by Proposition 1.1. The first concept we need to introduce is the notion of $\mathcal{K}$-ADMISSIBLE CURVE, the second one is that of ATTAINABLE SET. For the precise statement of these definitions, we first consider the operator $\mathcal{K}_0$ introduced in (4. 7) and we recall that it can be written in the following *sum of squares* form:

$$\mathcal{K}_0 = \sum_{j=1}^{m_0} X_j^2 + Y,$$

where Y is defined in (4. 10) and

$$X_j = \partial_{v_j} \sim e_j = (0, \ldots, 0, 1, 0, \ldots, 0)^T \qquad \text{for } j = 1, \ldots, n. \tag{4. 14}$$

$\mathcal{K}_0$-ADMISSIBLE CURVE AND ATTAINABLE SET. *We say that a curve* $\gamma : [0, T] \to \mathbb{R}^{2n}$ *is* $\mathcal{K}$*-admissible if:*

- *it is absolutely continuous;*
- $\dot\gamma(s) = \sum_{j=1}^{n} \omega_j(s) X_j(\gamma(s)) + Y(\gamma(s))$ *a.e. in* $[0, T]$*, with* $\omega_1, \ldots, \omega_n \in L^1[0, T]$.

Moreover we say that γ *steers* (v_0, x_0, t_0) *to* (v, x, t)*, for* $t_0 > t$*, if* $\gamma(0) = (v_0, x_0, t_0)$ *and* $\gamma(T) = (v, x, t)$*. Note that* $t(s) = t_0 - s$*, then* $T = t_0 - t$ *and* $t_0 > t$*. We denote by* $\mathscr{A}_{(v_0,x_0,t_0)}(\Omega)$ *the following set:*

$$\mathscr{A}_{(v_0,x_0,t_0)}(\Omega) = \left\{ \begin{array}{r} (y,t) \in \Omega \mid \text{ there exists a } \mathcal{K}_0 - \text{admissible curve } \gamma : [0,T] \to \Omega \\ \text{such that } \gamma(0) = (v_0, x_0, t_0) \text{ and } \gamma(T) = (v, x, t). \end{array} \right\}.$$

We will refer to $\mathscr{A}_{(v_0,x_0,t_0)}(\Omega)$ *as attainable set.*

In particular, if we recall the definitions (4. 10) and (4. 14) for the vector fields Y and X_j, respectively, and denote the curve

$$\gamma(s) = (v(s), x(s), t(s)), \quad s \in [0, T],$$

the $\mathcal{K}_0$-admissible curves can be easily described. Indeed, the controllability problem

$$\dot\gamma(s) = \sum_{j=1}^{n} \omega_j(s) X_j(\gamma(s)) + Y(\gamma(s)), \quad \gamma(0) = (v, x, t), \ \gamma(T) = (\eta, \xi, \tau),$$

becomes

$$\dot v(s) = \omega(s), \quad \dot x(s) = v(s), \quad \dot t(s) = -1, \tag{4. 15}$$

and its solution is

$$v(s) = v_0 + \int_0^s \omega(\tau) d\tau, \quad x(s) = x_0 + \int_0^s v(\tau) d\tau, \quad t(s) = t_0 - s.$$

We remark that by Proposition 1.1 our assumption on the matrix $\widetilde{B}$ is equivalent to the Kalman's controllability condition **C4**. Thus, it is guaranteed that for every $(v, x, t) \in \mathbb{R}^{2n+1}$, with $t < t_0$, there is at least a *control* $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in (L^1[0, T])^n$ such that the solution to (4. 15) satisfies $(v(T), x(T), t(T)) = (v, x, t)$.

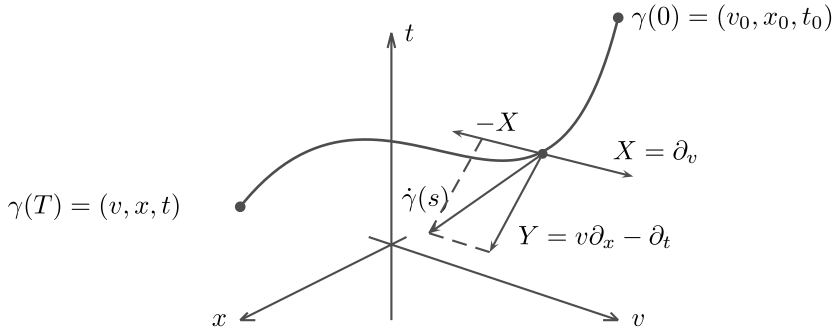

Fig. 3 - A $\mathcal{K}-$admissible curve steering (v_0, x_0, t_0) to (v, x, t).

In the sequel we will use the following notation.

Definition 4.3 *A curve* $\gamma = (v, x, t) : [0, T] \to \mathbb{R}^{2n+1}$ *is said to be* $\mathcal{K}$-ADMISSIBLE *if it is absolutely continuous, and solves the equation* (4. 15) *for almost every* $s \in [0, T]$*, with* $\omega_1, \omega_2, \dots, \omega_n \in L^1[0, T]$*. Moreover we say that* γ *steers* (v_0, x_0, t_0) *to* (v, x, t)*, with* $t_0 > t$*, if* $\gamma(0) = (v_0, x_0, t_0)$ *and* $\gamma(T) = (v, x, t)$*.*

Definition 4.4 *Let* Ω *be any open subset of* $\mathbb{R}^{2n+1}$*, and let* $(v_0, x_0, t_0) \in \Omega$*. We denote by* $\mathscr{A}_{(v_0,x_0,t_0)}(\Omega)$ *the following set:*

$$\mathscr{A}_{(v_0,x_0,t_0)}(\Omega) = \left\{ \begin{matrix} (v, x, t) \in \Omega \mid & \text{there exists a } \mathcal{K}-\text{admissible curve } \gamma : [0, T] \to \Omega \\ & \text{such that } \gamma(0) = (v_0, x_0, t_0) \text{ and } \gamma(T) = (v, x, t). \end{matrix} \right\}.$$

We will refer to $\mathscr{A}_{(v_0,x_0,t_0)}(\Omega)$ *as attainable set. We shall use the notation* $\mathscr{A}_{(v_0,x_0,t_0)} = \mathscr{A}_{(v_0,x_0,t_0)}(\Omega)$ *whenever there is no ambiguity on the choice of the set* Ω *.*

4.3 Harnack inequalities

This Section is devoted to Harnack inequalities for the equation $\mathcal{K}u = f$ introduced in (4. 3). In particular, we recall the Harnack inequality proved by Golse, Imbert, Mouhot and Vasseur (see Theorem 4 in [58]) and then we prove some preliminary results useful for the proof of our Theorem 4.1. In particular, let us introduce some preliminary notation. Let $Q =]-1, 1[^{2n} \times]-1, 0[$ be the *unit box* introduced in (4. 6). Based on the dilation (4. 11) and on the Galilean translation (4. 13), for every positive r and for every (v_0, x_0, t_0) we define the sets

$$Q_r := d_r Q = \big\{ d_r(v, x, t) \mid (v, x, t) \in Q \big\},$$
$$Q_r(v_0, x_0, t_0) := (v_0, x_0, t_0) \circ d_r Q = \big\{ (v_0, x_0, t_0) \circ d_r(v, x, t) \mid (v, x, t) \in Q \big\}.$$

A direct computation shows that

$$\begin{aligned} Q_r \ &=]-r, r[^n \times]-r^3, r^3[^n \times]-r^2, 0[, \\ Q_r(v_0, x_0, t_0) &= \{ (v, x, t) \in \mathbb{R}^{2n+1} \mid |(v - v_0)_j| < r, \\ &\qquad |(x - x_0 - (t - t_0) v_0)_j| < r^3, j = 1, \dots, n, t_0 - r^2 < t < t_0 \}. \end{aligned}$$

With the above notation, the following result holds.

Theorem 4.5 (Theorem 4 in [58]) *There exist three constants $M > 1, R > 0, \Delta > 0$, with $0 < R^2 < \Delta < \Delta + R^2 < 1$, such that*

$$\sup_{Q^-} u \leq M \left(\inf_{Q^+} u + \|f\|_{L^\infty(Q)} \right)$$

for every non-negative weak solution u to the equation $\mathcal{K}u = f$ on Q, with $f \in L^\infty(Q)$. The constants M, R and Δ only depend on the dimension n and on the ellipticity constants λ and Λ. Moreover Q^+, Q^- are defined as follows

$$Q^+ = Q_R \text{ with } 0 < R^2 < \Delta < \Delta + R^2 < 1, \qquad Q^- = Q_R(0, 0, -\Delta).$$

As the authors notice in Remark 4 of [58], "using the transformation (4. 11), we get a Harnack inequality for cylinders centered at an arbitrary point (v_0, x_0, t_0)". We next give a precise meaning to this assertion and we improve it by also considering the dilation (4. 13).

Theorem 4.6 *Let (v_0, x_0, t_0) be any point of $\mathbb{R}^{2n+1}$ and let r be a positive number. There exist three constants $M > 1, R > 0, \Delta > 0$, with $0 < R^2 < \Delta < \Delta + R^2 < 1$, such that*

$$\sup_{Q_r^-(v_0,x_0,t_0)} u \leq M \left(\inf_{Q_r^+(v_0,x_0,t_0)} u + \|f\|_{L^\infty(Q_r(v_0,x_0,t_0))} \right)$$

for every non-negative weak solution u to the equation $\mathcal{K}u = f$ on $Q_r(v_0, x_0, t_0)$, with $f \in L^\infty(Q_r(v_0, x_0, t_0))$. The constants M, R and Δ only depend on the dimension n and on the ellipticity constants λ and Λ. Moreover $Q_r^+(v_0, x_0, t_0), Q_r^-(v_0, x_0, t_0)$ are defined as follows

$$Q_r^+(v_0, x_0, t_0) = (v_0, x_0, t_0) \circ d_r Q^+, \qquad Q_r^-(v_0, x_0, t_0) = (v_0, x_0, t_0) \circ d_r Q^-.$$

Proof. We rely on the invariance of the operator $\mathcal{K}_0$ with respect to the group (4. 13). If u is a non-negative solution to $\mathcal{K}u = f$ in $Q_r(v_0, x_0, t_0)$, then the function

$$\widetilde{u}(v, x, t) := u \left(d_{1/r}\big((v_0, x_0, t_0)^{-1} \circ (v, x, t)\big) \right)$$

is a solution in the unit box Q to the following equation

$$\widetilde{\mathcal{K}}\widetilde{u} =: \partial_t \widetilde{u} + \langle v, \nabla_x \widetilde{u} \rangle - \mathrm{div}_v(\widetilde{A} \nabla_v \widetilde{u}) - \langle \widetilde{b}, \nabla_v \widetilde{u} \rangle = \widetilde{f},$$

where the inverse $(v_0, x_0, t_0)^{-1}$ is defined in (4. 12) and

$$\widetilde{A}(v, x, t) := A \left(d_{1/r}\big((v_0, x_0, t_0)^{-1} \circ (v, x, t)\big) \right), \qquad \widetilde{b}(v, x, t) := b \left(d_{1/r}\big((v_0, x_0, t_0)^{-1} \circ (v, x, t)\big) \right)$$
$$\widetilde{f}(v, x, t) := f \left(d_{1/r}\big((v_0, x_0, t_0)^{-1} \circ (v, x, t)\big) \right).$$

Even though $\widetilde{\mathcal{K}}$ does not agree with $\mathcal{K}$, it satisfies the assumptions of Theorem 4.5 with the same structural constants n, λ and Λ. We then apply Theorem 4.5 to the function $\widetilde{u}$ and we plainly obtain our claim for the function u. $\square$

A useful tool in the proof of our main theorem is the following lemma (Lemma 2.2 in [21]). To give here its statement we introduce a further notation. We choose any $S \in]0, R[$ and we set

$$K^- = [-S, S]^n \times \left[-S^3, S^3\right]^n \times \left\{ -\left(\Delta + \frac{R^2}{2} \right) \right\}.$$

Moreover, for every $(v, x, t) \in \mathbb{R}^{2n+1}$ and $r > 0$ we let

$$K_r^-(v, x, t) = (v, x, t) \circ d_r(K^-).$$

Lemma 4.7 *Let $\gamma : [0,T] \to \mathbb{R}^{2n+1}$ be an $\mathcal{K}-$admissible path and let a, b be two constants s.t. $0 \le a < b \le T$. Then there exists a positive constant h, only depending on $\mathcal{K}$, such that*

$$\int_a^b |\omega(\tau)|^2 \delta\tau \le h \qquad \Longrightarrow \qquad \gamma(b) \in K_r^-(\gamma(a)), \ \text{ with } \ r = \sqrt{\frac{b-a}{(\Delta + 1/2)}}.$$

Remark 4.8 *Note that $K_r^-(v,x,t)$ is a compact subset of $Q_r^-(v,x,t)$ for every $(v,x,t) \in \mathbb{R}^{2n+1}$ and for any $r > 0$. As a consequence of Lemma 4.7, $K_r^-(\gamma(a))$ is an open neighborhood of $\gamma(b)$.*

4.4 Proof of the main results

A useful notion in the proof of our main result is that of Harnack chain.

Definition 4.9 *We say that $\{z_0, \dots, z_k\} \subseteq \Omega$ is a Harnack chain connecting z_0 to z_k if there exist k positive constants $C_1, \dots, C_k$ such that*

$$u(z_j) \le C_j u(z_{j-1})) \qquad j = 1, \dots, k$$

for every non-negative solution u of $\mathcal{K}u = f$ in Ω.

Our first result of this section is a local version of Theorem 4.1.

Proposition 4.10 *For every $(v,x,t) \in \operatorname{int}\big(\mathscr{A}_{(v_0,x_0,t_0)}\big)$, there exist an open neighborhood $U_{(v,x,t)}$ of (v,x,t) and a positive constant $C_{(v,x,t)}$ such that*

$$\sup_{U_{(v,x,t)}} u \le C_{(v,x,t)} \ \Big(u(v_0,x_0,t_0) \ + \ \|f\|_{L^\infty(\Omega)} \Big),$$

for every non-negative solution to $\mathcal{K}u = f$, with $f \in L^\infty(\Omega)$.

Proof. Let (v,x,t) be any point of $\operatorname{int}\big(\mathscr{A}_{(v_0,x_0,t_0)}\big)$. We plan to prove our claim by constructing a finite Harnack chain connecting (v,x,t) to (v_0,x_0,t_0). Because of the very definition of $\mathscr{A}_{(v_0,x_0,t_0)}$, there exists a $\mathcal{K}-$admissible curve $\gamma : [0,T] \to \Omega$ steering (v_0,x_0,t_0) to (v,x,t). Our Harnack chain will be a finite subset of $\gamma([0,T])$.

In order to construct our Harnack chain, we introduce a further notation. Let $\widetilde{Q} :=]-1,1[^{2n+1}$ and note that it is an open neighborhood of the origin of $\mathbb{R}^{2n+1}$. Because of the continuity of the Galilean change of variable "$\circ$" and of the dilation $(d_r)_{r>0}$, for every $(v',x',t') \in \mathbb{R}^{2n+1}$, the family

$$\Big(\widetilde{Q}_r(v',x',t')\Big)_{r>0}, \qquad \widetilde{Q}_r(v',x',t') := (v',x',t') \circ d_r \widetilde{Q},$$

is a neighborhood basis of the point (v',x',t'). Then, again because of the continuity of "$\circ$" and $(d_r)_{r>0}$, for every $s \in [0,T]$ there exists a positive r such that $\widetilde{Q}_r(\gamma(s)) \subseteq \Omega$. Thus we can define

$$r(s) \ := \ \sup\Big\{ r > 0 \ : \ \widetilde{Q}_r(\gamma(s)) \subseteq \Omega \Big\}.$$

Note that the function $r(s)$ is continuous, then it is well defined the positive number

$$r_0 \ := \ \min_{s \in [0,T]} r(s).$$

As $Q_r(\gamma(s)) \subset \widetilde{Q}_r(\gamma(s))$, we conclude that

$$Q_r(\gamma(s)) \subseteq \Omega \quad \text{for every } s \in [0,T] \quad \text{and } r \in]0, r_0]. \tag{4. 16}$$

On the other hand, we notice that the function

$$I(s) \;:=\; \int_0^s |\omega(\tau)|^2 dt,$$

is (uniformly) continuous in $[0,T]$, then there exists a positive δ_0 such that $\delta_0 \le (\Delta + R^2/2) r_0$ and that

$$\int_a^b |\omega(\tau)|^2 dt \le h \qquad \text{for every } a,b \in [0,T], \quad \text{such that } 0 < a-b \le \delta_0, \tag{4. 17}$$

where h is constant appearing in Lemma 4.7.

We are now ready to construct our Harnack chain. Let k be the unique positive integer such that $(k-1)\delta_0 < T$, and $k\delta_0 \ge T$. We define $\{s_j\}_{j\in\{0,1,\dots,k\}} \in [0,T]$ as follows: $s_j = j\delta_0$ for $j = 0,1,\dots,k-1$, and $s_k = T$. As noticed before, the equation (4. 17) allows us to apply Lemma 4.7. We then obtain

$$\gamma(s_{j+1}) \in Q^-_{r_0}(\gamma(s_j)) \quad\; j = 0,\dots,k-2, \qquad \gamma(s_k) \in Q^-_{r_1}(\gamma(s_{k-1})), \tag{4. 18}$$

for some $r_1 \in]0, r_0]$. We next show that $(\gamma(s_j))_{j=0,1,\dots,k}$ is a Harnack chain and we conclude the proof. We proceed by induction. For every $j = 1,\dots,k-2$ we have that $\gamma(s_{j+1}) \in Q^-_{r_0}(\gamma(s_j))$. From (4. 16) we know that $Q_{r_0}(\gamma(s_j)) \subseteq \Omega$, then we apply Theorem 4.5 and we find

$$u(\gamma(s_{j+1})) \le \sup_{Q^-_{r_0}(\gamma(s_j))} u \le \; M\Big(\inf_{Q^+_{r_0}(\gamma(s_j))} u + \; \|f\|_{L^\infty(Q(\gamma(s_j)))}\Big) \;\le M\Big(u(\gamma(s_j)) + \|f\|_{L^\infty(\Omega)}\Big).$$

Here we rely on the fact that u is a continuous function. As a consequence we obtain

$$\begin{aligned} u(\gamma(s_{k-1})) &\le M(u(\gamma(s_{k-2})) + \|f\|_{L^\infty(\Omega)}) \\ &\le M(M(u(\gamma(s_{k-3})) + \|f\|_{L^\infty(\Omega)}) + \|f\|_{L^\infty(\Omega)}) \\ &\;\vdots \\ &\le M^{k-1} u(\gamma(0)) + \|f\|_{L^\infty(\Omega)} \sum_{i=1}^{k-1} M^i. \end{aligned}$$

We eventually apply Theorem 4.5 to the set $Q_{r_1}(\gamma(s_{k-1})) \subseteq \Omega$ and we obtain

$$\sup_{U_{(v,x,t)}} u \le C_{(v,x,t)} \;\Big(u(v_0,x_0,t_0) \;+\; \|f\|_{L^\infty(\Omega)}\Big),$$

where $C_{(v,x,t)} = \sum_{i=1}^k M^i$ and $U_{(v,x,t)} = Q^-_{r_1}(\gamma(s_{k-1}))$. As we noticed in Remark 4.8, $Q^-_{r_1}(\gamma(s_{k-1}))$ is an open neighborhood of $\gamma(T)$. This concludes the Proof of Proposition 4.10. $\square$

Proof of Theorem 4.1. Let K be any compact subset of $\operatorname{int}\big(\mathscr{A}_{(v_0,x_0,t_0)}\big)$. For every $(v,x,t) \in K$ we consider the open set $U_{(v,x,t)}$. Clearly we have

$$K \;\subseteq \bigcup_{(v,x,t)\in K} U_{(v,x,t)}.$$

Because of its compactness, there exists a finite covering of K

$$K \;\subseteq \bigcup_{j=1,\dots,m_K} U_{(v_j,x_j,t_j)},$$

and Proposition 4.10 yields we

$$\sup_{U_{(v_j,x_j,t_j)}} u \le C_{(v_j,x_j,t_j)} \Big(u(v_0,x_0,t_0) \ + \ \|f\|_{L^\infty(\Omega)} \Big) \qquad j = 1, \dots, m_K.$$

This concludes the proof of Theorem 4.1, if we choose

$$C_K \ = \max_{j=1,\dots,m_K} C_{(v_j,x_j,t_j)}.$$

□

Proof of Theorem 4.2. If u is a non-negative solution to $\mathcal{K}u = 0$ in Ω and K is a compact subset of $\mathscr{A}$, then $\sup_K u \le C_K u(v_0,x_0,t_0)$. If moreover $u(v_0,x_0,t_0) = 0$, we have $u(v,x,t) = 0$ for every $(v,x,t) \in K$ and, thus, $u(v,x,t) = 0$ for every $(v,x,t) \in \mathscr{A}_{(v_0,x_0,t_0)}$. The conclusion of the proof then follows from the continuity of u. □

Chapter 5

The Moser's iterative method for weak solutions

This chapter is devoted to the extension of the Moser's iterative scheme to positive weak solutions to the degenerate second order partial differential equation of Kolmogorov type

$$\begin{aligned}\mathcal{K}u(x,t) := \sum_{i,j=1}^{m_0} \partial_{x_i}\big(a_{ij}(x,t)\partial_{x_j}u(x,t)\big) + \sum_{i,j=1}^{N} b_{ij}x_j\partial_{x_i}u(x,t) - \partial_t u(x,t) + \\ + \sum_{i=1}^{m_0} b_i(x,t)\partial_i u(x,t) - \sum_{i=1}^{m_0} \partial_{x_j}\left(a_i(x,t)u(x,t)\right) + c(x,t)u(x,t) = 0\end{aligned} \tag{5. 1}$$

where $(x,t) \in \mathbb{R}^{N+1}$ and $1 \le m_0 \le N$, with measurable coefficients under minimal assumptions on the integrability of the lower order coefficients $a_1, \dots, a_{m_0}, b_1, \dots, b_{m_0}, c$. First of all, we need to introduce some further notation in order to state our main assumptions throughout this chapter. Here and in the sequel we denote by

$$D = (\partial_{x_1}, \dots, \partial_{x_N}), \qquad \langle \cdot, \cdot \rangle, \qquad \text{div} \tag{5. 2}$$

the gradient, the inner product, and and the divergence in $\mathbb{R}^N$, respectively. As the operator $\mathcal{K}$ introduced in (5. 1) is non degenerate with respect to the first m_0 components of x, we also introduce the notation

$$D_{m_0} = (\partial_{x_1}, \dots, \partial_{x_{m_0}}), \tag{5. 3}$$

to denote the gradient in $\mathbb{R}^{m_0}$. In particular, we introduce the $N \times N$ matrix

$$A(x,t) = (a_{ij}(x,t))_{1 \le i,j \le N} = \begin{pmatrix} \overline{A} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}, \tag{5. 4}$$

where a_{ij} is the coefficient appearing in (5. 1) for $i,j = 1, \dots, m_0$, while $a_{ij} \equiv 0$ whenever $i > m_0$, or $j > m_0$. Eventually, we define

$$a(x,t) = (a_1(x,t), \dots, a_{m_0}(x,t), 0, \dots, 0), \qquad b(x,t) = (b_1(x,t), \dots, b_{m_0}(x,t), 0, \dots, 0) \tag{5. 5}$$

$$Y = \sum_{i,j=1}^{N} b_{ij}x_j\partial_{x_i} - \partial_t$$

Then the operator $\mathcal{K}$ of (5. 1) takes the following compact form

$$\mathcal{K}u = \text{div}(ADu) + Yu + \langle b, Du \rangle - \text{div}(au) + cu. \tag{5. 6}$$

We assume the following structural condition on $\mathcal{K}$.

(H1) The matrix $(a_{ij}(x,t))_{i,j=1,\dots,m_0}$ is symmetric with real measurable entries. Moreover, $a_{ij}(x,t) = a_{ji}(x,t)$, $1 \le i,j \le m_0$, and there exists a positive constant λ such that

$$\lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{m_0} a_{ij}(x,t)\xi_i\xi_j \le \lambda|\xi|^2,$$

for every $(x,t) \in \mathbb{R}^{N+1}$ and $\xi \in \mathbb{R}^{m_0}$. The matrix $B = (b_{ij})_{i,j=1,\dots,N}$ is constant.

Note that the operator $\mathcal{K}$ is uniformly parabolic when $m_0 = N$. Here, we are mainly interested in the case $m_0 < N$, that is the strongly degenerate one. It is known that the first order part of $\mathcal{K}$ may provide it with strong regularity properties. To be more specific, let us consider the operator $\mathcal{K}_0$ defined as follows:

$$\mathcal{K}_0 u(x,t) := \sum_{i,j=1}^{m_0} \partial^2_{x_i} u(x,t) + \sum_{i,j=1}^{N} b_{ij}x_j\partial_{x_i}u(x,t) - \partial_t u(x,t) \tag{5. 7}$$

It is known that if the matrix B satisfies a suitable assumption, then $\mathcal{K}_0$ is hypoelliptic (as it is recalled by Proposition 1.1). This means that if u is a distributional solution to $\mathcal{K}_0 u = f$ in some open set Ω of $\mathbb{R}^{N+1}$ and $f \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$ and it is a classic solution to the equation. The hypoellipticity of $\mathcal{K}_0$ can be tested via the rank condition (9) (recall in the Introduction of this work) introduced by Hörmander in [64]:

$$\operatorname{rank}\operatorname{Lie}(\partial_{x_1},\dots,\partial_{x_{m_0}},Y)(x,t) = N+1, \quad \text{for every } (x,t) \in \mathbb{R}^{N+1},$$

where $\operatorname{Lie}(\partial_{x_1},\dots,\partial_{x_{m_0}},Y)(x,t)$ denotes the Lie algebra generated by the first order differential operators (vector fields) $(\partial_{x_1},\dots,\partial_{x_{m_0}},Y)$, computed at (x,t). We refer to Chapter 1, and the references therein, for a characterization of the hypoellipticity of $\mathcal{K}_0$ in terms of the matrix B.

(H2) The principal part $\mathcal{K}_0$ of $\mathcal{K}$ is hypoelliptic.

We remark that if $\mathcal{K}$ is an uniformly parabolic operator (i.e. $m_0 = N$ and $B \equiv 0$), then **(H2)** is clearly satisfied. Indeed, the principal part of $\mathcal{K}$ simply is the heat operator, which is hypoelliptic and homogeneous with respect to the parabolic dilations $\delta_\lambda(x,t) = (\lambda x, \lambda^2 t)$. In the degenerate setting, $\mathcal{K}_0$ plays the same role that the heat operator plays in the family of parabolic operators. For this reason, $\mathcal{K}_0$ will be referred to as principal part of $\mathcal{K}$.

This chapter is devoted to the proof of L^∞_{loc} estimates for positive weak solutions to the degenerate second order partial differential equation of Kolmogorov type (5. 1) with measurable coefficients under minimal assumptions on the integrability of the lower order coefficients $a_1,\dots,a_{m_0},b_1,\dots,b_{m_0},c$. Our study has been inspired by the article of Nazarov and Uralt'seva [99], who prove L^∞_{loc} estimates and Harnack inequalities for uniformly elliptic and parabolic operators in divergence form that are those with $m_0 = N$ according to our notation. The authors consider uniformly parabolic equations in $\mathbb{R}^{N+1}$

$$\mathscr{L}u = \operatorname{div}(ADu) + \langle b, Du\rangle - \partial_t u = 0,$$

with $b_1,\dots,b_N \in L^q(\mathbb{R}^{N+1})$. They prove that the Moser's iteration can be accomplished provided that $\frac{N+2}{2} < q \le N+2$ relying on the condition $\operatorname{div} b \ge 0$ to relax the integrability assumption on $b_1,\dots,b_{m_0}$. Here and in the sequel, the quantity $\operatorname{div} b$ will be understood in the distributional sense

$$\int_\Omega \varphi(x,t)\operatorname{div} b(x,t)dx\,dt = -\int_\Omega \langle b(x,t), \nabla\varphi(x,t)\rangle dx\,dt,$$

for every $\varphi \in C^\infty_0(\Omega)$. Of course, also the quantity $\operatorname{div} a$ will be understood in the distributional sense. When considering degenerate operators, a suitable dilation group $(\delta_r)_{r>0}$ in $\mathbb{R}^{N+1}$ replaces the usual parabolic dilation $\delta_r(x,t) = (rx, r^2 t)$, and the *parabolic dimension* $N+2$ of $\mathbb{R}^{N+1}$ is replaced by a bigger integer $Q+2$, which is called *homogeneous dimension* of $\mathbb{R}^{N+1}$ with respect to $(\delta_r)_{r>0}$ and it is defined in (5. 19). Our main result will be declared in terms of this quantity, that will be introduced in Section 5.1. As far as it concerns degenerate operators, Wang and Zhang obtain in [119] the local boundedness and the Hölder continuity for weak solutions to $\mathcal{K}u = 0$ by assuming the condition $b_1, \dots, b_{m_0} \in L^q(\mathbb{R}^{N+1})$, with $q = Q+2$. Our assumption on the integrability of the lower order coefficients a_i, b_i, with $i = 1, \dots, m_0$ and c is stated as follows:

(H3) $a_i, b_i, c \in L^q_{\text{loc}}(\Omega)$, with $i = 1, \dots, m_0$, for some $q > \frac{3}{4}(Q+2)$. Moreover,

$$\text{div} a, \text{div} b \geq 0 \qquad \text{in } \Omega.$$

In general, solutions to $\mathcal{K}u = 0$ will be understood in the following weak sense.

Definition 5.1 *Let Ω be an open subset of $\mathbb{R}^{N+1}$. A weak solution to $\mathcal{K}u = 0$ is a function u such that $u, D_{m_0}u, Yu \in L^2_{\text{loc}}(\Omega)$ and*

$$\int\limits_{\Omega} -\langle ADu, D\varphi \rangle + \varphi Y u + \langle b, Du \rangle \varphi + \langle a, D\varphi \rangle u + cu\varphi = 0, \qquad \forall \varphi \in C_0^\infty(\Omega).$$

In the sequel, we will also consider weak sub-solutions to $\mathcal{K}u = 0$, namely functions u such that $u, D_{m_0}u, Yu \in L^2_{\text{loc}}(\Omega)$ and

$$\int_{\Omega} -\langle ADu, D\varphi \rangle + \varphi Y u + \langle b, Du \rangle \varphi + \langle a, D\varphi \rangle u + cu\varphi \geq 0, \qquad \forall \varphi \in C_0^\infty(\Omega), \varphi \geq 0.$$

A function u is a super-solution of $\mathcal{K}u = 0$ if $-u$ is a sub-solution.

We note that if u is both a sub-solution and a super-solution of $\mathcal{K}u = 0$ then it is a solution, i.e. $\mathcal{K}u = 0$ holds. Indeed, for every given $\varphi \in C_0^\infty(\Omega)$, we may consider $\psi \in C_0^\infty(\Omega)$ such that $\psi \geq 0$ and $\psi - \varphi \geq 0$ in Ω. Therefore $\mathcal{K}u = 0$ follows by applying (5.1) to $\pm u$.

A comparison of our result with that of Nazarov and Uralt'seva is in order. It would be natural to expect that the optimal lower bound for the exponent q is $\frac{Q+2}{2}$. Indeed, the difficulty in considering degenerate equations lies in the fact that a Caccioppoli inequality gives an *a priori* L^2 estimate for the derivatives $\partial_{x_1}u, \dots, \partial_{x_{m_0}}u$ of the solution u, that are the derivative with respect to the *non-degeneracy* directions of $\mathcal{K}$. Moreover, the standard Sobolev inequality cannot be used to obtain an improvement of the integrability of the solution as in the non-degenerate case. For this reason we rely on a representation formula for the solution u first used in [104]. Specifically, we represent a solution u to $\mathcal{K}u = 0$ in terms of the fundamental solution of $\mathcal{K}_0$. Indeed, if u is a solution to $\mathcal{K}u = 0$ in Ω, then we have

$$u(x,t) = \int_{\Omega} \Gamma_0(x,t,\xi,\tau) \mathcal{K}_0 u(\xi,\tau) d\xi \, d\tau, \tag{5. 8}$$

where Γ_0 is the fundamental solution to $\mathcal{K}_0$ (see (1. 5) and (1. 6) in Chapter 1), and

$$\mathcal{K}_0 u = (\mathcal{K}_0 - \mathcal{K})\, u = \text{div}\left((A_0 - A)Du\right) - \langle b, Du \rangle + \text{div}(au) - cu, \tag{5. 9}$$

where we denote

$$A_0 = \begin{pmatrix} \mathbb{I}_{m_0} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}, \tag{5. 10}$$

where $\mathbb{I}_{m_0}$ is the identity matrix in $\mathbb{R}^{m_0}$, and $\mathbb{O}$ are zero matrices. This representation formula provides us with a Sobolev type inequality only for weak solutions to the equation $\mathcal{K}u = 0$. Specifically, we find that, for every $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega$, there exist a positive constant $c_1\left(\| b \|_{L^q(\Omega)}, \Omega_1, \Omega_2\right)$ such that

$$\| u \|_{L^{2\alpha}(\Omega_1)} \leq c_1 \left(\| a \|_{L^q(\Omega)}, \| b \|_{L^q(\Omega)}, \| c \|_{L^q(\Omega)}, \Omega_1, \Omega_2\right) \| D_{m_0} u \|_{L^2(\Omega_2)},$$

and, by considering u as a test function, we obtain the following Caccioppoli inequality

$$\| D_{m_0} u \|_{L^2((\Omega_2)} \leq c_2 \left(\| a \|_{L^q(\Omega)}, \| b \|_{L^q(\Omega)}, \| c \|_{L^q(\Omega)}, \Omega_2, \Omega_3 \right) \| u \|_{L^{2\beta}(\Omega_3)},$$

where D_{m_0} is the gradient defined in (5. 3), and

$$\alpha := \frac{q(Q+2)}{q(Q-2)+2(Q+2)}, \qquad \beta := \frac{q}{q-1}. \tag{5. 11}$$

As far as it concerns the Moser's iteration, the above inequalities are applied to a sequence of functions $u_k := u^{p_k}$, with $p_k \to +\infty$, in order to obtain an L^∞_{loc} bound for the solution u. We note that, the Sobolev inequality is useful to the iteration whenever $\alpha > 1$, and this is true if, and only if $q > \frac{Q+2}{2}$. Moreover, the condition $q > \frac{Q+2}{2}$ is required by Nazarov and Uralt'seva in the proof of the Caccioppoli inequality for non-degenerate operators. Since in our work both Sobolev and Caccioppoli inequalities depend on the L^q norm of $a_1, \ldots, a_{m_0}, b_1, \ldots, b_{m_0}, c$, we require a more restrictive condition on q to improve the integrability of u. Specifically, if we combine the Sobolev and the Caccioppoli inequalities, we need to have $\alpha > \beta$, and this is true if, and only if $q > \frac{3}{4}(Q+2)$, as we require in assumption **(H3)**.

We next state our main result. As we have already pointed out in Chapter 1, the natural geometry underlying the operator $\mathcal{K}$ is determined by a suitable homogeneous Lie group structure on $\mathbb{R}^{N+1}$. Our main results reflect this non-Euclidean background. Let "$\circ$" denote the Lie product on $\mathbb{R}^{N+1}$ defined in (1. 12) and $\{\delta_r\}_{r>0}$ the family of dilations defined in (1. 19). Let us consider the cylinder:

$$\mathcal{Q}_1 := \left\{ (x,t) \in \mathbb{R}^N \times \mathbb{R} \ : \ |x| < 1, \ \ |t| < 1 \right\}.$$

For every $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$, we set

$$\mathcal{Q}_r(z_0) := z_0 \circ (\delta_r(\mathcal{Q}_1)) = \left\{ z \in \mathbb{R}^{N+1} \ : \ z = z_0 \circ \delta_r(\zeta), \zeta \in \mathcal{Q}_1 \right\}.$$

Theorem 5.2 *Let u be a non-negative weak solution to $\mathcal{K}u = 0$ in Ω. Let $z_0 \in \Omega$ and $r, \rho, \frac{1}{2} \leq \rho < r \leq 1$, be such that $\overline{\mathcal{Q}_r(z_0)} \subseteq \Omega$. Then there exist positive constants $C = C(p, \lambda)$ and $\gamma = \gamma(p,q)$ such that for every $p \neq 0$, it holds*

$$\sup_{\mathcal{Q}_\rho(z_0)} u^p \leq \frac{C \left(1 + \| a \|^2_{L^q(\mathcal{Q}_r(z_0))} + \| b \|^2_{L^q(\mathcal{Q}_r(z_0))} + \| c \|_{L^q(\mathcal{Q}_r(z_0))} \right)^\gamma}{(r-\rho)^{9(Q+2)}} \int_{\mathcal{Q}_r(z_0)} u^p, \tag{5. 12}$$

where $\gamma = \frac{2\alpha^2\beta}{\alpha - 1}$, with α and β defined as in (5. 11).

Remark 5.3 *Estimate* (5. 12) *is meaningful whenever the integral appearing in its right-hand side is finite. Note that* (5. 12) *is an estimate of the* infimum *of u when $p < 0$. More precisely, we have that*

$$\sup_{\mathcal{Q}_\rho(z_0)} u \leq \frac{C^{\frac{1}{p}} \left(1 + \| a \|^2_{L^q(\mathcal{Q}_r(z_0))} + \| b \|^2_{L^q(\mathcal{Q}_r(z_0))} + \| c \|_{L^q(\mathcal{Q}_r(z_0))} \right)^{\frac{\gamma}{p}}}{(r-\rho)^{\frac{9(Q+2)}{p}}} \left(\int_{\mathcal{Q}_r(z_0)} u^p \right)^{\frac{1}{p}}, \qquad \forall p > 0, \tag{5. 13}$$

$$\inf_{\mathcal{Q}_\rho(z_0)} u \geq \frac{C^{\frac{1}{p}} \left(1 + \| a \|^2_{L^q(\mathcal{Q}_r(z_0))} + \| b \|^2_{L^q(\mathcal{Q}_r(z_0))} + \| c \|_{L^q(\mathcal{Q}_r(z_0))} \right)^{\frac{\gamma}{p}}}{(r-\rho)^{\frac{9(Q+2)}{p}}} \left(\int_{\mathcal{Q}_r(z_0)} \frac{1}{u^{|p|}} \right)^{\frac{1}{p}}, \qquad \forall p < 0, \tag{5. 14}$$

Corollary 5.4 *Let u be a weak solution to $\mathscr{L}u = 0$ in Ω. Then for every $p \geq 1$ we have*

$$\sup_{\mathcal{Q}_\rho(z_0)} |u|^p \leq \frac{C\left(1+ \| a \|^2_{L^q(\mathcal{Q}_r(z_0))} + \| b \|^2_{L^q(\mathcal{Q}_r(z_0))} + \| c \|_{L^q(\mathcal{Q}_r(z_0))}\right)^\gamma}{(r-\rho)^{9(Q+2)}} \int_{\mathcal{Q}_r(z_0)} |u|^p. \tag{5. 15}$$

Proposition 5.5 *Sub and super-solutions also verify estimate* (5. 12) *for suitable values of p. More precisely,* (5. 12) *holds for*

1. *$p > \frac{1}{2}$ or $p < 0$, if u is a non-negative weak sub-solution of* (5. 1)*;*
2. *$p \in]0, \frac{1}{2}[$, if u is a non-negative weak super-solution of* (5. 1).

This chapter is organized as follows. Section 5.1 is devoted to a survey of results on potential estimates for the fundamental solution of the principal part operator $\mathcal{K}_0$. In Section 5.2 we prove Theorem 5.10 and Proposition 5.11 for weak solutions to $\mathcal{K}u = 0$, which is an intermediate result (Caccioppoli type inequality for weak solutions to $\mathcal{K}u = 0$) needed for the bootstrap argument. Finally, in Section 5.3 we deal with the Moser's iterative method. The results we present here appeared for the first time in the paper [7] by the author, Polidoro and Ragusa.

5.1 Potential estimates for the fundamental solution Γ

In this section we briefly recall some notation and preliminary results regarding the non-Euclidean geometry underlying the operators $\mathcal{K}$ and $\mathcal{K}_0$. We refer to Chapter 1 and the references therein for a comprehensive treatment of this subject. By its definition in (5. 7), the operator $\mathcal{K}_0$ is a constant coefficients Kolmogorov operator invariant with respect to the Lie product "$\circ$" defined in (1. 12) on $\mathbb{R}^{N+1}$. Thus, if we denote by $\ell_\zeta, \zeta \in \mathbb{R}^{N+1}$ the left translation $\ell_\zeta(z) = \zeta \circ z$ in the group law we have

$$\mathcal{K}_0 \circ \ell_\zeta = \ell_\zeta \circ \mathcal{K}_0.$$

This means that, if $v(x,t) = u\big((\xi,\tau)\circ(x,t)\big)$ and $g(x,t) = f\big((\xi,\tau)\circ(x,t)\big)$, we have

$$\mathcal{K}_0 u = f \quad \iff \quad \mathcal{K}_0 v = g.$$

Moreover, by Proposition 1.1 assumption **(H2)** is equivalent to assume that for some basis on $\mathbb{R}^N$ the matrix B has the canonical form

$$B = \begin{pmatrix} * & * & \dots & * & * \\ B_1 & * & \dots & * & * \\ \mathbb{O} & B_2 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_\kappa & * \end{pmatrix} \tag{5. 16}$$

where every B_k is a $m_k \times m_{k-1}$ matrix of rank m_j, $j = 1, 2, \dots, \kappa$ with

$$m_0 \geq m_1 \geq \dots \geq m_\kappa \geq 1 \quad \text{and} \quad \sum_{j=0}^{\kappa} m_j = N,$$

and the blocks denoted by "*" are arbitrary. In the sequel, we assume B has the canonical form (5. 16).

As we have already pointed out in Section 1.2, among the operators $\mathcal{K}_0$ where the matrix B is of the form (5. 16) the ones for which the $*-$blocks are equal to zero play a central role (see (1. 27)). Indeed, let us consider the principal part operator $\overline{\mathcal{K}}_0$ associated to $\mathcal{K}_0$

$$\overline{\mathcal{K}}_0 = \Delta_{m_0} + \overline{Y}_0, \tag{5. 17}$$

obtained from(5. 7) by substituting Y with the drift term $\overline{Y}_0 = \langle \overline{B}_0 x, D\rangle - \partial_t$, where the matrix $\overline{B}_0$ is obtained from (5. 16) by choosing the *-blocks equal to zero:

$$\overline{B}_0 = \begin{pmatrix} \mathbb{O} & \mathbb{O} & \dots & \mathbb{O} & \mathbb{O} \\ B_1 & \mathbb{O} & \dots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & B_2 & \dots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_\kappa & \mathbb{O} \end{pmatrix} \tag{5. 18}$$

By Proposition 1.4, the operator $\overline{\mathcal{K}}_0$ is invariant with respect to the family of dilations

$$\delta_r = \mathrm{diag}(r\mathbb{I}_{m_0}, r^3\mathbb{I}_{m_1}, \dots, r^{2\kappa+1}\mathbb{I}_{m_\kappa}, r^2), \qquad r > 0,$$

that we have already introduced in (1. 19). In order to explain the importance of this invariance property we introduce for every positive r the scaled operator

$$\overline{\mathcal{K}}_r = r^2 \left(\delta_r \circ \overline{\mathcal{K}}_0 \circ \delta_{\frac{1}{r}} \right).$$

In order to explicitly write $\overline{\mathcal{K}}_r$ we note that, if

$$B = \begin{pmatrix} B_{0,0} & B_{0,1} & \dots & B_{0,\kappa-1} & B_{0,\kappa} \\ B_1 & B_{1,1} & \dots & B_{\kappa-1,1} & B_{\kappa,1} \\ \mathbb{O} & B_2 & \dots & B_{\kappa-1,2} & B_{\kappa,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_\kappa & B_{\kappa,\kappa} \end{pmatrix},$$

where $B_{i,j}$ are the $m_i \times m_j$ blocks denoted by "$*$" in (5. 16), then we can rewrite $\overline{\mathcal{K}}_r$ as follows

$$\overline{\mathcal{K}}_r = \Delta_{m_0} + \overline{Y}_r,$$

where $\overline{Y}_r := \langle \overline{B}_r\, x, D\rangle - \partial_t$ and $\overline{B}_r := r^2\, D_r\, B\, D_{\frac{1}{r}}$, i.e.

$$\overline{B}_r = \begin{pmatrix} r^2 B_{0,0} & r^4 B_{0,1} & \dots & r^{2\kappa} B_{0,\kappa-1} & r^{2\kappa+2} B_{0,\kappa} \\ B_1 & r^2 B_{1,1} & \dots & r^{2\kappa-2} B_{\kappa-1,1} & r^{2\kappa} B_{\kappa,1} \\ \mathbb{O} & B_2 & \dots & r^{2\kappa-4} B_{\kappa-1,2} & r^{2\kappa-2} B_{\kappa,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_\kappa & r^2 B_{\kappa,\kappa} \end{pmatrix}.$$

Note that

$$\overline{B}_r = B \qquad \text{for every}\, r > 0$$

if, and only if $B_{j,k} = \mathbb{O}$ with $j \le k$. In this case, if $v(x,t) = u\big(\delta_r(x,t)\big)$ and $g(x,t) = f\big(\delta_r(x,t)\big)$, then

$$\mathcal{K}_0 u = f \quad \iff \quad \mathcal{K}_0 v = r^2 g.$$

Since $\overline{K}_0$ is the blow-up limit of $\overline{K}_r$, the dilation group $\{\delta_r\}_{r>0}$ plays a central role also for non-dilation invariant operators, as the case of the operator $\mathcal{K}$ defined in (5. 1). For further discussions on the relationship between dilation invariant and non-dilation invariant Kolmogorov operators we refer to Chapter 1, Section 1.2.

In the following, we consider the homogeneous norm $\|\cdot\|$ of degree 1 with respect to the family of dilations $\{\delta_r\}_{r>0}$ defined in Definition 1.7, and the corresponding invariant quasi-distance $d(z,\zeta)$ introduced in Definition 1.10 for the case of $*-$blocks equal to zero. As it is pointed out in Remark 1.8, every norm is equivalent to any other in $\mathbb{R}^{N+1}$. For this reason, in this chapter we consider the equivalent definition of norm (1. 25), that we report here for the sake of completeness. For every $z = (x_1, \ldots, x_N, t) \in \mathbb{R}^{N+1} \setminus \{0\}$ the norm of z is the unique positive solution r to the following equation

$$\frac{x_1^{q_1}}{r^{2q_1}} + \frac{x_2^{q_2}}{r^{2q_2}} + \ldots + \frac{x_N^{q_N}}{r^{2q_N}} + \frac{t^2}{r^4} = 1.$$

The main advantage of this definition lies in the fact that the set $\{z \in \mathbb{R}^{N+1} : \|z\| = r\}$ is a smooth manifold for every positive r, which is not the case for (1. 24).

Remark 5.6 *The Lebesgue measure is invariant with respect to the translation group associated to $\mathcal{K}$, since* $\det E(t) = e^{t \text{ trace} B} = 1$*, where $E(t)$ is the exponential matrix of equation* (1. 4)*. Moreover, since* $\det \delta_r = r^{Q+2}$*, we also have*

$$\text{meas}\left(\mathcal{Q}_r(z_0)\right) = r^{Q+2} \text{meas}\left(\mathcal{Q}_1(z_0)\right), \qquad \forall\, r > 0, z_0 \in \mathbb{R}^{N+1},$$

where

$$Q = m_0 + 3m_1 + \ldots + (2\kappa + 1)m_\kappa. \tag{5. 19}$$

The natural number $Q+2$ is usually called the homogeneous dimension of $\mathbb{R}^{N+1}$ with respect to $\{\delta_r\}_{r>0}$.

Let us consider again the principal part operator $\mathcal{K}_0$ associated to the operator $\mathcal{K}$. Indeed, it is a constant coefficient operator by definition, and thus it admits a fundamental solution $\Gamma_0(\cdot,\zeta)$ whose explicit expression is given in (1. 6). We remark that

$$\Gamma(z,\zeta) = \Gamma(\zeta^{-1} \circ z, 0), \qquad \text{for every}\, z, \zeta \in \mathbb{R}^{N+1},\, z \neq \zeta.$$

Now, let us consider the principal part operator $\overline{\mathcal{K}}_0$ associated to $\mathcal{K}_0$ and defined in (5. 17). Indeed it is a constant coefficient Kolmogorov operator that, by (5. 18) and Proposition 1.4, is dilation invariant with respect to the family of dilations $\{\delta_r\}_{r>0}$. It is clear that it admits a fundamental solution $\overline{\Gamma}_0$ of the form (1. 6) and also that $\overline{\Gamma}_0$ is a homogeneous function of degree $-Q$ (see Remark 1.5 and Section 1.2), namely

$$\overline{\Gamma}_0\left(D(r)(z), 0\right) = r^{-Q}\, \overline{\Gamma}_0\left(z, 0\right), \text{ for every } z \in \mathbb{R}^{N+1} \setminus \{0\},\, r > 0.$$

This property implies an L^p estimate for Newtonian potential (c. f. for instance [47]).

Proposition 5.7 *Let $\alpha \in]0, Q+2[$ and let $G \in C(\mathbb{R}^{N+1} \setminus \{0\})$ be a $\delta_\lambda-$homogeneous function of degree $\alpha - Q - 2$. If $f \in L^p(\mathbb{R}^{N+1})$ for some $p \in]1, +\infty[$, then the function*

$$G_f(z) := \int_{\mathbb{R}^{N+1}} G(\zeta^{-1} \circ z) f(\zeta) d\zeta,$$

is defined almost everywhere and there exists a constant $c = c(Q,p)$ such that

$$\| G_f \|_{L^q(\mathbb{R}^{N+1}} \leq c \max_{\|z\|=1} |G(z)| \, \| f \|_{L^p(\mathbb{R}^{N+1})},$$

where q is defined by

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q+2}.$$

As we have already pointed out at the beginning of this section, homogeneous operators provide a good approximation of the non-homogeneous ones. Let us consider the homogeneous operator $\overline{\mathcal{K}}_0$ defined in (5. 17) and the non-homogeneous operator $\mathcal{K}_0$ defined in (5. 7), alongside with their fundamental solutions $\overline{\Gamma}_0$ and Γ_0, respectively. Then, for every $M > 0$, there exists a positive constant c such that

$$\frac{1}{c}\,\overline{\Gamma}_0 \,\leq\, \Gamma_0(z) \,\leq\, c\,\overline{\Gamma}_0(z) \tag{5. 20}$$

for every $z \in \mathbb{R}^{N+1}$ such that $\overline{\Gamma}_0(z) \geq M$ (see [84], Theorem 3.1). We define the Γ_0-*potential* of the function $f \in L^1(\mathbb{R}^{N+1})$ as follows

$$\Gamma_0(f)(z) = \int_{\mathbb{R}^{N+1}} \Gamma_0(z,\zeta) f(\zeta) d\zeta, \qquad z \in \mathbb{R}^{N+1}. \tag{5. 21}$$

We also remark that the potential $\Gamma_0(D_{m_0} f) : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^{m_0}$ is well-defined for any $f \in L^p(\mathbb{R}^{N+1})$, at least in the distributional sense, that is

$$\Gamma_0(D_{m_0} f)(z) := - \int_{\mathbb{R}^{N+1}} D^{(\xi)}_{m_0} \Gamma_0(z,\xi)\, f(\xi)\, d\xi, \tag{5. 22}$$

where $D^{(\xi)}_{m_0}\Gamma_0(x,t,\xi,\tau)$ is the gradient with respect to $\xi_1,\dots,\xi_{m_0}$. Based on (5. 20), in [33] are proved potential estimates for non-dilation invariant operators.

Theorem 5.8 *Let $f \in L^p(\mathcal{Q}_r)$. There exists a positive constant $c = c(T,B)$ such that*

$$\| \, \Gamma_0(f) \, \|_{L^{p**}(\mathcal{Q}_r)} \leq c \, \| \, f \, \|_{L^p(\mathcal{Q}_r)}, \tag{5. 23}$$

$$\| \, \Gamma_0(D_{m_0} f) \, \|_{L^{p*}(\mathcal{Q}_r)} \leq c \, \| \, f \, \|_{L^p(\mathcal{Q}_r)}, \tag{5. 24}$$

where $\frac{1}{p*} = \frac{1}{p} - \frac{1}{Q+2}$ and $\frac{1}{p**} = \frac{1}{p} - \frac{2}{Q+2}$.

We can use the fundamental solution Γ_0 as a test function in the definition of sub and super-solution. The following result extends Lemma 2.5 in [104] and Lemma 3 in [33].

Lemma 5.9 *Let v be a non-negative weak sub-solution to $\mathcal{K}u = 0$ in Ω. For every $\varphi \in C^\infty_0(\Omega)$, $\varphi \geq 0$, and for almost every $z \in \mathbb{R}^{N+1}$, we have*

$$\int_\Omega - \langle ADv, D(\Gamma_0(z,\cdot)\varphi)\rangle + \Gamma_0(z,\cdot)\varphi Y v + \\ - \langle a, D(\Gamma_0(z,\cdot)\varphi)\rangle v - \langle b, D(\Gamma_0(z,\cdot)\varphi)\rangle v + cu\Gamma_0(z,\cdot)\varphi \geq 0.$$

An analogous result holds for weak super-solutions to $\mathcal{K}u = 0$.

Proof. We define the cut-off function $\chi_{\rho,r} \in C^\infty(\mathbb{R}^+)$

$$\chi_{\rho,r}(s) = \begin{cases} 0 & \text{if } \ s \geq r, \\ 1 & \text{if } \ 0 \leq s < \rho, \end{cases} \qquad |\chi'_{r,\rho}| \leq \frac{2}{r-\rho} \tag{5. 25}$$

with $\frac{1}{2} \leq \rho < r \leq 1$. Moreover, for every $\varepsilon < 0$ we define

$$\psi_\varepsilon(x,t) = 1 - \chi_{\varepsilon,2\varepsilon}(\| \,(x,t)\, \|).$$

Because v is a weak sub-solution, then by (5.1) for every $\varepsilon > 0$ and $z \in \mathbb{R}^{N+1}$ we have

$$\begin{aligned} 0 \leq & \int_\Omega - [\langle\, ADv, D(\Gamma_0(z,\cdot)\varphi(\zeta)\psi_\varepsilon(z,\cdot))\,\rangle \,+\, \Gamma_0(z,\cdot)\varphi(\zeta)\psi_\varepsilon(z,\cdot)\,Yv]\, d\zeta \\ & + \int_\Omega [\langle b, Dv\rangle\, \Gamma_0(z,\cdot)\varphi(\zeta)\psi_\varepsilon(z,\cdot) \,+ \langle a, D(\Gamma_0(z,\cdot)\varphi(\zeta)\psi_\varepsilon(z,\cdot))\rangle v + cu\,\Gamma_0(z,\cdot)\varphi(\zeta)\psi_\varepsilon(z,\cdot)]\, d\zeta \\ = & - I_{1,\varepsilon}(z) \,+\, I_{2,\varepsilon}(z) \,-\, I_{3,\varepsilon}(z) \,+\, I_{4,\varepsilon}(z) + I_{5,\varepsilon}(z) \end{aligned}$$

where

$$I_{1,\varepsilon}(z) = \int_\Omega \langle\, ADv, D\Gamma_0(z,\cdot)\,\rangle \varphi(\zeta)\psi_\varepsilon(z,\zeta)\; d\zeta$$
$$I_{2,\varepsilon}(z) = \int_\Omega \Gamma_0(z,\cdot)\varphi(\zeta)\psi_\varepsilon(z,\zeta)\;\; (\, -\langle\, ADv, D\varphi(\zeta)\,\rangle + \varphi(\zeta)Yv\,)\;\; d\zeta$$
$$I_{3,\varepsilon}(z) = \int_\Omega \langle\, ADv, D\psi_\varepsilon(z,\cdot)\,\rangle \varphi(\zeta)\Gamma_0(z,\cdot)\; d\zeta$$
$$I_{4,\varepsilon}(z) = \int_\Omega \langle b, Dv\rangle\, \Gamma_0(z,\cdot)\varphi(\zeta)\psi_\varepsilon(z,\cdot)\, d\zeta + \int_\Omega \langle a, D(\Gamma_0(z,\cdot)\varphi)\rangle v\, d\zeta$$
$$I_{5,\varepsilon}(z) = \int_\Omega cu\, \Gamma_0(z,\cdot)\varphi(\zeta)\psi_\varepsilon(z,\cdot)\; d\zeta$$

Keeping in mind Theorem 5.8, it is clear that the integral which defines $I_{i,\varepsilon}(z)$, $i = 1,2,3$ is a potential and it is convergent for almost every $z \in \mathbb{R}^{N+1}$. Thus, by a similar argument to the one used in [104] to prove Lemma 2.5 (pg. 403 – 404), we get that for almost every $z \in \mathbb{R}^{N+1}$

$$\lim_{\varepsilon\to 0^+} I_{1,\varepsilon}(z) = \int_\Omega \langle ADv, D(\Gamma_0(z,\cdot))\rangle\varphi(\zeta)\; d\zeta$$
$$\lim_{\varepsilon\to 0^+} I_{2,\varepsilon}(z) = \int_\Omega \Gamma_0(z,\cdot)\,(-\,\langle ADv, D\varphi(\zeta)\rangle + \varphi(\zeta)Yv)\;\; d\zeta$$
$$\lim_{\varepsilon\to 0^+} I_{3,\varepsilon}(z) = 0.$$

Let us consider the term $I_{4,\varepsilon}$. We integrate by parts and we consider assumption **(H3)**:

$$\begin{aligned} I_{4,\varepsilon} = &- \int_\Omega \mathrm{div} b\, \Gamma_0(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\cdot)v\, d\zeta\; - \int_\Omega \langle\, b, D\,(\Gamma_0(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\cdot))\,\rangle\, v\, d\zeta \\ &- \int_\Omega \mathrm{div} a\, \Gamma_0(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\cdot)v\, d\zeta\; - \int_\Omega \langle\, a, D\,(\Gamma_0(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\cdot))\,\rangle\, v\, d\zeta \\ \le &- \int_\Omega \langle\, b, D\,(\Gamma_0(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\cdot))\,\rangle\, v\, d\zeta\; - \int_\Omega \langle\, a, D\,(\Gamma_0(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\cdot))\,\rangle\, v\, d\zeta \end{aligned}$$

We are left with the estimate of a potential and in order to do so we would like to use Theorem 5.8. Because $a_i, b_i \in L^q_{\mathrm{loc}}(\Omega)$, with $i = 1,\dots,m_0$ and $v \in L^2_{\mathrm{loc}}(\Omega)$, we have that

$$|a|\,|\Gamma_0(z,\cdot)|\,|\varphi|\,|D_{m_0}v|\,,\;\; |b|\,|\Gamma_0(z,\cdot)|\,|\varphi|\,|D_{m_0}v| \in L^{2\alpha}_{\mathrm{loc}}(\Omega)$$

where α is defined as in (5. 11). This yields, for every $\varepsilon > 0$

$$|\langle\, a, D\,(\Gamma_0(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\cdot))\,\rangle\; v| \le |\langle\, a, D\,(\Gamma_0(z,\cdot)\varphi(\zeta))\,\rangle\; v| \in L^1_{\mathrm{loc}}(\Omega),$$
$$|\langle\, b, D\,(\Gamma_0(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\cdot))\,\rangle\; v| \le |\langle\, b, D\,(\Gamma_0(z,\cdot)\varphi(\zeta))\,\rangle\; v| \in L^1_{\mathrm{loc}}(\Omega).$$

Thus, by the Lebesgue convergence theorem, we get for a.e. $z \in \mathbb{R}^{N+1}$

$$\begin{aligned} &\lim_{\varepsilon\to 0^+} \left[\int_\Omega -\langle\, b, D\,(\Gamma_0(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\zeta))\,\rangle\; v\, d\zeta\; - \int_\Omega \langle\, a, D\,(\Gamma_0(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\cdot))\,\rangle\, v\right] d\zeta = \\ &\qquad = - \int_\Omega \langle\, b, D\,(\Gamma_0(z,\cdot)\varphi(\zeta))\,\rangle\; v - \int_\Omega \langle\, a, D\,(\Gamma_0(z,\cdot)\varphi(\zeta))\,\rangle\, v\, d\zeta. \end{aligned}$$

Now, we are left with an estimate of the term $I_{5,\varepsilon}$, which is a Γ_0-potential such that

$$|c|\,|\Gamma_0(z,\cdot)|\,|\varphi|\,|v| \in L^{2\alpha}_{\mathrm{loc}}(\Omega).$$

Thus, we have that

$$|cu\,\Gamma_0(z,\cdot)\varphi(\zeta)\psi_\varepsilon(z,\cdot)| \le |cu\,\Gamma_0(z,\cdot)\varphi(\zeta)| \in L^1_{\text{loc}}(\Omega).$$

Then we can apply the Lebesgue convergence theorem and we get for a. e. $z \in \mathbb{R}^{N+1}$

$$\lim_{\varepsilon\to 0^+} \int_\Omega cv\Gamma_0(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\zeta)\,d\zeta \;= \int_\Omega cv\Gamma_0(z,\cdot)\varphi(\zeta)\,d\zeta.$$

□

5.2 Sobolev and Caccioppoli Inequalities

In this section we give proof of a Sobolev inequality and a Caccioppoli inequality for weak solutions to $\mathcal{K}u = 0$. We start considering the Sobolev inequality and we remark that it holds true for every $q > \frac{Q+2}{2}$.

Theorem 5.10 (Sobolev Type Inequality for sub-solutions) *Let assumptions **(H1)** and **(H2)** hold. Let $a_1, \dots, a_{m_0}, b_1, \dots, b_{m_0}, c \in L^q_{\text{loc}}(\Omega)$, for some $q > (Q+2)/2$, and $\text{div}a, \text{div}b \ge 0$ in Ω. Let v be a non-negative weak sub-solution of $\mathcal{K}u = 0$ in $\mathcal{Q}_1$. Then there exists a constant $C = C(Q,\lambda) > 0$ such that $v \in L^{2\alpha}_{\text{loc}}(\mathcal{Q}_1)$, and the following statement holds*

$$\begin{aligned}\| v \|_{L^{2\alpha}(\mathcal{Q}_\rho(z_0))} \le & C \cdot \left(\| a \|_{L^q(\mathcal{Q}_r(z_0))} + \| b \|_{L^q(\mathcal{Q}_r(z_0))} + 1 + \frac{1}{r-\rho} \right) \| D_{m_0} v \|_{L^2(\mathcal{Q}_r(z_0))} + \\ & + C \cdot \left(\| c \|_{L^q(\mathcal{Q}_r(z_0))} + \frac{\rho+1}{\rho(r-\rho)} \right) \| v \|_{L^2(\mathcal{Q}_r(z_0))}\end{aligned}$$

for every ρ, r with $\frac{1}{2} \le \rho < r \le 1$ and for every $z_0 \in \Omega$, where $\alpha = \alpha(q)$ is defined in (5. 11).

Proof. Let v be a non-negative weak sub-solution to $\mathcal{K}u = 0$. We represent v in terms of the fundamental solution Γ_0. To this end, we consider the cut-off function $\chi_{\rho,r}$ defined in (5. 25) for $\frac{1}{2} \le \rho < r \le 1$. Then we consider the following test function

$$\psi(x,t) = \chi_{\rho,r}(\| (x,t) \|) \tag{5. 26}$$

and the following estimates hold true

$$|Y\psi| \le \frac{c_0}{\rho(r-\rho)}, \qquad |\partial_{x_j}\psi| \le \frac{c_1}{r-\rho} \ \text{ for } j = 1, \dots, m_0 \tag{5. 27}$$

where c_0, c_1 are dimensional constants. For every $z \in \mathcal{Q}_\rho$, we have

$$\begin{aligned} v(z) &= v\psi(z) \qquad (5.\ 28)\\ &= \int_{\mathcal{Q}_r} \left[\langle A_0 D(v\psi), D\Gamma_0(z,\cdot)\rangle - \Gamma_0(z,\cdot) Y(v\psi) \right](\zeta) d(\zeta) \\ &= I_0(z) + I_1(z) + I_2(z) + I_3(z)\end{aligned}$$

where

$$
\begin{aligned}
I_0(z) &= -\int_{\mathcal{Q}_r} [\langle a, D(\psi\Gamma_0(z,\cdot))\rangle v](\zeta)d\zeta \;-\int_{\mathcal{Q}_r} [\langle b, D(\psi\Gamma_0(z,\cdot))\rangle v](\zeta)d\zeta \;+\int_{\mathcal{Q}_r} [cv\Gamma_0(z,\cdot)\psi](\zeta)d\zeta\\
I_1(z) &= \int_{\mathcal{Q}_r} [\langle A_0 D\psi, D\Gamma_0(z,\cdot)\rangle v](\zeta)d\zeta \;-\int_{\mathcal{Q}_r} [\Gamma_0(z,\cdot)vY\psi](\zeta)d\zeta = I_1^{'} + I_1^{''},\\
I_2(z) &= \int_{\mathcal{Q}_r} [\langle (A_0 - A)Dv, D\Gamma_0(z,\cdot)\rangle\psi](\zeta)d\zeta \;-\int_{\mathcal{Q}_r} [\Gamma_0(z,\cdot)\langle ADv, D\psi\rangle](\zeta)d\zeta\\
I_3(z) &= \int_{\mathcal{Q}_r} [\langle ADv, D(\Gamma_0(z,\cdot)\psi)\rangle](\zeta)d\zeta \;-\int_{\mathcal{Q}_r} [(\Gamma_0(z,\cdot)\psi)\,Yv](\zeta)d\zeta +\\
&\quad+\int_{\mathcal{Q}_r} [\langle a, D(\Gamma_0(z,\cdot)\psi)\rangle v](\zeta)d\zeta + \int_{\mathcal{Q}_r} [\langle b, D(\Gamma_0(z,\cdot)\psi)\rangle v](\zeta)d\zeta - \int_{\mathcal{Q}_r} [cv\Gamma_0(z,\cdot)\psi](\zeta)d\zeta
\end{aligned}
$$

Since v is a non-negative weak sub-solution to $\mathcal{K}u = 0$, it follows from Lemma 5.9 that $I_3 \leq 0$, then

$$0 \leq v(z) \leq I_0(z) + I_1(z) + I_2(z) \quad \text{for a.e. } z \in \mathcal{Q}_\rho.$$

To prove our claim is sufficient to estimate v by a sum of Γ_0-potentials. We start by estimating I_0. In order to do so, we recall that

$$\langle a, Dv\rangle,\, \langle b, Dv\rangle,\, cv \,\in L^{2\frac{q}{q+2}} \quad \text{for } b \in L^q,\;\; q > \frac{Q+2}{2} \;\text{ and } D_{m_0}v \in L^2.$$

Thus by Theorem 5.8 we get

$$\Gamma_0 * \langle a, Dv\rangle, \Gamma_0 * \langle b, Dv\rangle, \Gamma_0 * (cv) \in L^{2\alpha},$$

where $\alpha = \alpha(q)$ is defined in (5. 11). When $q \leq (Q+2)$ we have that $\alpha \leq 2^{**}$. Moreover, thanks to estimate (5. 23), we have

$$
\begin{aligned}
\| I_0(\zeta) \|_{L^{2\alpha}(\mathcal{Q}_\rho)} &\leq \text{meas}(\mathcal{Q}_\rho)^{2/Q} \| I_0(\zeta) \|_{L^{2^{**}}(\mathcal{Q}_\rho)}\\
&= \text{meas}(\mathcal{Q}_\rho)^{2/Q} \| \Gamma_0 * (\langle a, Dv\rangle\psi) + \Gamma_0 * (\langle b, Dv\rangle\psi) + \Gamma_0 * (cv\psi) \|_{L^{2^{**}}(\mathcal{Q}_\rho)}\\
&\leq C\cdot \big(\| a \|_{L^q(\mathcal{Q}_\rho)} + \| b \|_{L^q(\mathcal{Q}_\rho)}\big) \| D_{m_0}v \|_{L^2(\mathcal{Q}_\rho)} + C\cdot \| c \|_{L^q(\mathcal{Q}_\rho)} \| v \|_{L^2(\mathcal{Q}_\rho)}.
\end{aligned}
$$

We prove an estimate for the term I_1. I_1' can be estimated by (5. 24) of Theorem 5.8 as follows

$$\| I_1' \|_{L^{2\alpha}(\mathcal{Q}_\rho)} \leq C \| I_1' \|_{L^{2^*}(\mathcal{Q}_\rho)} \leq C \| vD_{m_0}\psi \|_{L^2(\mathbb{R}^{N+1})} \leq \frac{C}{r-\rho} \| v \|_{L^2(\mathcal{Q}_\rho)},$$

where the last inequality follows from (5. 27). To estimate I_1'' we use (5. 23)

$$
\begin{aligned}
\| I_1'' \|_{L^{2\alpha}(\mathcal{Q}_\rho)} &\leq C \| I_1'' \|_{L^{2^*}(\mathcal{Q}_\rho)} \leq \text{meas}(\mathcal{Q}_\rho)^{2/Q} \| I_1'' \|_{L^{2^{**}}(\mathcal{Q}_\rho)}\\
&\leq C \| vY\psi \|_{L^2(\mathbb{R}^{N+1})} \leq \frac{C}{\rho(r-\rho)} \| v \|_{L^2(\mathcal{Q}_\rho)}.
\end{aligned}
$$

We can use the same technique to prove that

$$\| I_2 \|_{L^{2\alpha}(\mathcal{Q}_\rho)} \leq C\left(1 + \frac{1}{r-\rho}\right) \| D_{m_0}v \|_{L^2(\mathcal{Q}_\rho)},$$

for some constant $C = C(Q, \lambda)$.

A similar argument proves the thesis when v is a super-solution to $\mathcal{K}u = 0$. In this case we introduce the following auxiliary operator

$$\widetilde{K} = \operatorname{div}(A_0\, D) + \widetilde{Y}, \qquad\qquad \widetilde{Y} \equiv -\langle x, BD\rangle - \partial_t\,. \tag{5. 29}$$

Then we proceed analogously as in [104], Section 3, proof of Theorem 3.3. □

Finally, we give proof of a Caccioppoli inequality for weak solutions to $\mathcal{K}u = 0$.

Proposition 5.11 *Let* ***(H1)-(H3)*** *hold. Let u be a non-negative weak solution of $\mathcal{K}u = 0$ in $\mathcal{Q}_1$. Let $p \in \mathbb{R}$, $p \neq 0$, $p \neq 1/2$ and let r, ρ be such that $\frac{1}{2} \le \rho < r \le 1$. Then there exists a constant C such that*

$$\begin{aligned}
&\frac{1}{\lambda} \parallel D_{m_0} v \parallel^2_{L^2(\mathcal{Q}_\rho)} \le \\
&\le \left[\frac{C\,p}{2\lambda} \frac{1}{(r-\rho)^2} + \frac{C}{r-\rho} \big(1 + \parallel a \parallel_{L^q(\mathcal{Q}_r)} + \parallel b \parallel_{L^q(\mathcal{Q}_r)}\big) + \frac{p}{2} \parallel c \parallel_{L^q(\mathcal{Q}_r)} \right] \parallel v \parallel^2_{L^{2\beta}(\mathcal{Q}_r)},
\end{aligned}$$

where $\beta = \beta(q)$ is defined in (5. 11).

Proof. We consider the case $p < 1$, $p \neq 0$, $p \neq 1/2$. First of all, we consider an uniformly positive weak solution u to $\mathcal{K}u = 0$, that is $u \ge u_0$ for some constant $u_0 > 0$. For every $\psi \in C_0^\infty(\mathcal{Q}_r)$ we consider the function $\varphi = u^{2p-1}\psi^2$. Note that $\varphi, D_{m_0}\varphi \in L^2(\mathcal{Q}_r)$, then we can use φ as a test function in (5.1):

$$0 = \int_{\mathcal{Q}_r} \big(-\langle ADu, D(u^{2p-1}\psi^2)\rangle + u^{2p-1}\psi^2 Y u + \langle a, D(u^{2p-1}\psi^2)\rangle u + \langle b, Du\rangle u^{2p-1}\psi^2 + c u^{2p}\psi^2 \big)$$

Let $v = u^p$. Since u is a weak solution to $\mathcal{K}u = 0$ and $u \ge u_0$, then $v, D_{m_0}v, Yv \in L^2(\mathcal{Q}_r)$:

$$\begin{aligned}
0 = &- \int_{\mathcal{Q}_r} \left(1 - \frac{1}{2p}\right) \langle ADv, Dv\rangle \psi^2 - \int_{\mathcal{Q}_r} \langle ADv, D\psi\rangle v\psi + \frac{1}{4}\int_{\mathcal{Q}_r} Y(v^2)\psi^2 \\
&- \int_{\mathcal{Q}_r} \operatorname{div} a\, v^2\psi^2 - \frac{1}{4}\int_{\mathcal{Q}_r} \langle a, D(v^2)\rangle\psi^2 + \frac{1}{4}\int_{\mathcal{Q}_r} \langle b, D(v^2)\rangle\psi^2 + \frac{p}{2}\int_{\mathcal{Q}_r} c v^2\psi^2.
\end{aligned}$$

Because of assumption **(H1)** and by definition (5. 26) of the cut-off function ψ, we get the following inequality

$$\begin{aligned}
&\frac{1}{\lambda}\left(\frac{2p-1}{2p} + \varepsilon\right)\int_{\mathcal{Q}_\rho} |D_{m_0} v|^2 \le \\
&\le \frac{1}{4\varepsilon\lambda}\frac{C}{(r-\rho)^2}\int_{\mathcal{Q}_r} |v|^2 \underbrace{\boxed{-\int_{\mathcal{Q}_r} \operatorname{div} a\, v^2\psi^2 - \frac{1}{4}\int_{\mathcal{Q}_r} \langle a, D(v^2)\rangle \psi^2}}_{A} + \\
&+ \underbrace{\boxed{\frac{1}{4}\int_{\mathcal{Q}_r} \langle b, D(v^2)\rangle\psi^2}}_{B} + \underbrace{\boxed{\frac{p}{2}\int_{\mathcal{Q}_r} c v^2\psi^2}}_{C} + \underbrace{\boxed{\frac{1}{4}\int_{\mathcal{Q}_r} Y(v^2)\psi^2}}_{D}
\end{aligned} \tag{5. 30}$$

where ε is a positive constant coming from the application of the Young's inequality. In the following we are going to consider exponents $\alpha = \alpha(q)$ and $\beta = \beta(q)$ defined in (5. 11). Now we need to estimate the boxed terms. Let us consider the term A. By Assumption **(H3)** and a classic Hölder estimate we have that

$$\begin{aligned}
\underbrace{\boxed{-\int_{\mathcal{Q}_r} \operatorname{div} a\, v^2\psi^2 - \frac{1}{4}\int_{\mathcal{Q}_r} \langle a, D(v^2)\rangle\psi^2}}_{A} &\le -\frac{3}{4}\int_{\mathcal{Q}_r} \operatorname{div} a\, v^2\psi^2 + \frac{1}{2}\int_{\mathcal{Q}_r} |\langle a, D\psi\rangle|\, |\psi|\, v^2 \\
&\le \frac{C}{r-\rho} \parallel a \parallel_{L^q(\mathcal{Q}_r)} \parallel v \parallel^2_{L^{2\beta}(\mathcal{Q}_r)}\,.
\end{aligned}$$

Let us consider the term B. Thus, by Assumption **(H3)** and a classic Hölder estimate we have that

$$\boxed{\frac{1}{4}\int_{\mathcal{Q}_r}\psi^2\langle b, D(v^2)\rangle}_B \le -\frac{1}{4}\int_{\mathcal{Q}_r} v^2\psi^2 \mathrm{div} b + \frac{1}{2}\int_{\mathcal{Q}_r}|\langle b, D\psi\rangle||\psi|v^2$$
$$\le \frac{C}{r-\rho}\parallel b\parallel_{L^q(\mathcal{Q}_r)}\parallel v\parallel^2_{L^{2\beta}(\mathcal{Q}_r)}.$$

Let us consider the linear term C. We estimate it via a classical Hölder estimate:

$$\boxed{\frac{p}{2}\int_{\mathcal{Q}_r} cv^2\psi^2}_C \le \frac{p}{2}\parallel c\parallel_{L^q(\mathcal{Q}_r)}\parallel v\parallel^2_{L^{2\beta}(\mathcal{Q}_r)}.$$

As far as it concerns the term D, we begin considering the following equality:

$$\psi^2 Y(v^2) = Y(\psi^2 v^2) - 2v^2\psi Y\psi.$$

Since by the divergence theorem $D_1 = 0$ ($v^2\psi^2$ is null on the boundary of $\mathcal{Q}_r$), we get

$$\boxed{\frac{1}{4}\int_{\mathcal{Q}_r} Y(v^2)\psi^2}_D = D_1 + D_2 = \int_{\mathcal{Q}_r}\frac{1}{4}Y(v^2\psi^2) + \int_{\mathcal{Q}_r}\frac{v^2\psi}{2}Y\psi \le \frac{C}{\rho(r-\rho)}\parallel v\parallel^2_{L^2(\mathcal{Q}_r)}.$$

Thus we have

$$\frac{1}{\lambda}\left(\frac{2p-1}{2p}+\varepsilon\right)\parallel D_{m_0}v\parallel^2_{L^2(\mathcal{Q}_\rho)} \le \left(\frac{c}{4\varepsilon\lambda}\frac{1}{(r-\rho)^2}+\frac{C}{\rho(r-\rho)}\right)\parallel v\parallel^2_{L^2(\mathcal{Q}_r)} +$$
$$+\frac{C}{r-\rho}\left(\parallel a\parallel_{L^q(\mathcal{Q}_r)}+\parallel b\parallel_{L^q(\mathcal{Q}_r)}\right)\parallel v\parallel^2_{L^{2\beta}(\mathcal{Q}_r)} + \frac{p}{2}\parallel c\parallel_{L^q(\mathcal{Q}_r)}\parallel v\parallel^2_{L^{2\beta}(\mathcal{Q}_r)}.$$

By choosing $\varepsilon = \frac{1}{2p}$ and considering that $\beta > 2$ we have that

$$\frac{1}{\lambda}\parallel D_{m_0}v\parallel^2_{L^2(\mathcal{Q}_\rho)} \le \tag{5. 31}$$
$$\le \left[\frac{C\,p}{2\lambda}\frac{1}{(r-\rho)^2}+\frac{C}{r-\rho}\left(1+\parallel a\parallel_{L^q(\mathcal{Q}_r)}+\parallel b\parallel_{L^q(\mathcal{Q}_r)}\right)+\frac{p}{2}\parallel c\parallel_{L^q(\mathcal{Q}_r)}\right]\parallel v\parallel^2_{L^{2\beta}(\mathcal{Q}_r)}.$$

The previous argument can be adapted to the case of a non-negative weak solution to $\mathcal{K}u = 0$. Indeed, we may consider the estimate (5. 31) for the solution $u + \frac{1}{n}$, $n \in \mathbb{N}$,

$$\frac{1}{\lambda}\int_{\mathcal{Q}_\rho}\left|D_{m_0}\left(u+\frac{1}{n}\right)^p\right|^2 \le$$
$$\le \left[\frac{C\,p}{2\lambda}\frac{1}{(r-\rho)^2}+\frac{C}{r-\rho}\left(1+\parallel a\parallel_{L^q(\mathcal{Q}_r)}+\parallel b\parallel_{L^q(\mathcal{Q}_r)}\right)+\frac{p}{2}\parallel c\parallel_{L^q(\mathcal{Q}_r)}\right]\left(\int_{\mathcal{Q}_r}\left(u+\frac{1}{n}\right)^{2\beta}\right)^{\frac{1}{\beta}}.$$

We let n go to infinity. The passage to the limit in the first integral is allowed because

$$\left|D_{m_0}\left(u+\frac{1}{n}\right)^p\right| = p\left(u+\frac{1}{n}\right)^{p-1}|D_{m_0}u| \nearrow |D_{m_0}u^p|, \quad \forall p<1,\ n\to\infty.$$

For the second integral we rely on the assumptions $u^p \in L^2(\mathcal{Q}_r)$ and $u^p \in L^{2\frac{q}{q-1}}(\mathcal{Q}_r)$.

Next, we consider the case $p \geq 1$. For any $n \in \mathbb{N}$, we define the function $g_{n,p}$ on $]0,+\infty[$ as follows

$$g_{n,p}(s) = \begin{cases} s^p, & \text{if } 0 < s \leq n, \\ n^p + pn^{p-1}(s-n), & \text{if } s > n, \end{cases}$$

then we let

$$v_{n,p} = g_{n,p}(u).$$

Note that

$$g_{n,p} \in C^1(\mathbb{R}^+), \quad g^{'}_{n,p} \in L^\infty(\mathbb{R}^+).$$

Thus since u is a weak solution to $\mathcal{K}u = 0$, we have

$$v_{n,p} \in L^2_{\text{loc}}, \;\; Dv_{n,p} \in L^2_{\text{loc}}, \;\; Yv_{n,p} \in L^2_{\text{loc}}.$$

We also note that the function

$$g^{''}_{n,p}(s) = \begin{cases} p(p-1)s^{p-2}, & \text{if } 0 < s < n \\ 0, & \text{if } s \geq n, \end{cases}$$

is the weak derivative of $g^{'}_{n,p}$, then $Dg^{'}_{n,p}(u) = g^{''}_{n,p}(u)D(u)$ (for the detailed proof of this assertion, we refer to [57], Theorem 7.8). Hence, by considering

$$\varphi = g_{n,p}(u)\; g^{'}_{n,p}(u)\; \psi^2, \quad \psi \in C^\infty_0(\mathcal{Q}_r)$$

as a test function in Definition 5.1, we find

$$\begin{aligned} 0 &= \int_{\mathcal{Q}_1} -\langle ADu, D\varphi\rangle + \varphi Yu - \text{div}a\, u\varphi - \langle a, Du\rangle\varphi + \langle b, Du\rangle\varphi + cu\varphi \\ &= \int_{\mathcal{Q}_1} -\left(g^{'}_{n,p}(u)\right)^2 \psi^2\langle ADu, Du\rangle - g^{''}_{n,p}(u)\, g_{n,p}(u)\psi^2\langle ADu, Du\rangle - 2\psi\langle ADu, D\psi\rangle g_{n,p}(u)\, g^{'}_{n,p}(u) + \\ &+ \int_{\mathcal{Q}_1} g_{n,p}(u)\, g^{'}_{n,p}(u)\psi^2\, Yu - \text{div}a\, u\, g_{n,p}(u)\, g^{'}_{n,p}(u)\psi^2 - \langle a, Du\rangle\psi^2 g_{n,p}(u)\, g^{'}_{n,p}(u) + \\ &+ \int_{\mathcal{Q}_1} \langle b, Du\rangle\psi^2 g_{n,p}(u)\, g^{'}_{n,p}(u) + cug_{n,p}(u)\, g^{'}_{n,p}(u)\psi^2. \end{aligned}$$

Since $v = g_{n,p}(u)$ we have that the following equality holds:

$$\begin{aligned} 0 = &\int_{\mathcal{Q}_r} -\psi^2\langle ADv_{n,p}, Dv_{n,p}\rangle - \boxed{g^{''}_{n,p}(u)\, g_{n,p}(u)\psi^2\langle ADu, Du\rangle}_A - 2\psi\langle ADv_{n,p}, D\psi\rangle v_{n,p} + \\ + &\int_{\mathcal{Q}_r} \frac{1}{2}\psi^2\, Y(v^2_{n,p}) + \boxed{\text{div}a\left(\frac{1}{2}v^2_{n,p}\psi^2 - u\, g_{n,p}(u)\, g^{'}_{n,p}(u)\psi^2\right) - \text{div}b\, v^2_{n,p}\psi^2}_B \\ + &\int_{\mathcal{Q}_r} \frac{1}{2}\langle a, D(\psi^2)\rangle v^2_{n,p} - \langle b, D(\psi^2)\rangle v^2_{n,p} + cug_{n,p}(u)\, g^{'}_{n,p}(u)\psi^2. \end{aligned}$$

Since $g_{n,p}^{''}(u) \geq 0$ we have that the boxed term A is non-negative. Moreover, by Assumption **(H3)** the boxed term B is non-positive. Thus, by considering Assumption **(H1)** and by choosing $\varepsilon = \frac{1}{2p}$ we have that

$$\frac{1}{\lambda}\int\limits_{\mathcal{Q}_r} |D_{m_0} v_{n,p}|^2 \leq \frac{C\,p}{2\lambda}\frac{1}{(r-\rho)^2}\int\limits_{\mathcal{Q}_r} |v_{n,p}|^2 + \frac{1}{2}\langle a, D(\psi^2)\rangle v_{n,p}^2 - \langle b, D(\psi^2)\rangle v_{n,p}^2 + cug_{n,p}(u)\, g_{n,p}^{'}(u)\psi^2$$

Since $0 < v_{n,p} \leq u^p$ and

$$|D_{m_0} v_{n,p}| \uparrow |D_{m_0} u^p|, \qquad \text{as}\, n \to \infty,$$

we get from the above inequality

$$\frac{1}{\lambda}\int\limits_{\mathcal{Q}_r} |D_{m_0} u^p|^2 \leq \frac{C\,p}{2\lambda}\frac{1}{(r-\rho)^2}\int\limits_{\mathcal{Q}_r} |u^p|^2 + \frac{1}{2}\langle a, D(\psi^2)\rangle u^{2p} - \langle b, D(\psi^2)\rangle u^{2p} + cu^{2p}\psi^2$$

and we conclude the proof as in the previous case. □

5.3 The Moser's Iteration

In this section we use the classical Moser's iteration scheme to prove Theorem 5.2. We begin with some preliminary remarks. First of all, we recall the following lemma, whose proof can be found in [33], Lemma 6.

Lemma 5.12 *There exists a positive constant* $\overline{c} \in]0,1[$ *such that*

$$z \circ \mathcal{Q}_{\overline{c}r(r-\rho)} \subseteq \mathcal{Q}_r, \tag{5. 32}$$

for every $0 < \rho < r \leq 1$ *and* $z \in \mathcal{Q}_\rho$.

We are now in position to prove Theorem 5.2.

Proof of Theorem 5.2. It suffices to give proof in the case $z_0 = 0$, $r \in]0,1]$ and $0 < \rho < r$. Combining Theorems 5.10 and Proposition 5.11, we obtain the following estimate: if $s, \delta > 0$ verify the condition

$$|s - 1/2| \geq \delta,$$

then, for every ρ, r such that $\frac{1}{2} \leq \rho < r \leq 1$, there exists a positive constant $\widetilde{C}$ such that

$$\| u^s \|_{L^{2\alpha}(\mathcal{Q}_\rho)} \leq \widetilde{C}\left(s, \lambda, \| a \|_{L^q(\mathcal{Q}_r)}, \| b \|_{L^q(\mathcal{Q}_r)}, \| c \|_{L^q(\mathcal{Q}_r)}\right) \| u^s \|_{L^{2\beta}(\mathcal{Q}_r)} \tag{5. 33}$$

where

$$\begin{aligned} \widetilde{C}\left(s, \lambda, \| a \|_{L^q(\mathcal{Q}_r)}, \| b \|_{L^q(\mathcal{Q}_r)}, \| c \|_{L^q(\mathcal{Q}_r)}\right) \;&=\; C(s,\lambda)\left(1+ \| a \|_{L^q(\mathcal{Q}_r)} + \| b \|_{L^q(\mathcal{Q}_r)}\right) \| c \|_{L^q(\mathcal{Q}_r)}^{\frac{1}{2}} \;+ \\ &+\; \frac{C(\lambda)\left(1+ \| a \|_{L^q(\mathcal{Q}_r)} + \| b \|_{L^q(\mathcal{Q}_r)}\right)^{\frac{3}{2}}}{(r-\rho)^{\frac{1}{2}}} \;+\; \frac{C}{(r-\rho)^{\frac{3}{2}}}\left(1+ \| a \|_{L^q(\mathcal{Q}_r)} \;+\; \| b \|_{L^q(\mathcal{Q}_r)}\right)^{\frac{1}{2}} + \\ &+\; \frac{C(s)}{r-\rho}\left(1+ \| a \|_{L^q(\mathcal{Q}_r)} + \| b \|_{L^q(\mathcal{Q}_r)} + \lambda^{\frac{1}{2}} \| c \|_{L^q(\mathcal{Q}_r)}^{\frac{1}{2}}\right) \;+\; \frac{C(s)}{(r-\rho)^2}. \end{aligned}$$

We remark that the previous constant $\widetilde{C}$ can be estimated as follows

$$\begin{aligned} &\widetilde{C}(s, \lambda, \| a \|_{L^q(\mathcal{Q}_r)}, \| b \|_{L^q(\mathcal{Q}_r)}, \| c \|_{L^q(\mathcal{Q}_r)}) \leq \\ &\qquad\qquad \leq \frac{K(\lambda,s)\left(1+ \| a \|_{L^q(\mathcal{Q}_r)}^2 + \| b \|_{L^q(\mathcal{Q}_r)}^2 + \| c \|_{L^q(\mathcal{Q}_r)}\right)}{\left(\rho_n - \rho_{n+1}\right)^2}. \end{aligned} \tag{5. 34}$$

Fixed a suitable $\delta > 0$, we shall specify later on, and $p > 0$ we iterate inequality (5. 33) by choosing

$$\rho_n = \rho + \frac{1}{2^n}(r-\rho), \qquad p_n = \alpha^n \frac{p}{2\beta}, \qquad n \in \mathbb{N} \cup \{0\}.$$

Then we set $v = u^{\frac{p}{2\beta}}$. If $p > 0$ is such that

$$|p\,\alpha^n - \beta| \geq 2\beta\delta, \qquad \forall n \in \mathbb{N} \cup \{0\}, \tag{5. 35}$$

by (5. 33) and estimate (5. 34) we obtain the following inequality for every $n \in \mathbb{N} \cup \{0\}$

$$\| v^{\alpha^n} \|_{L^{2\alpha}(\mathcal{Q}_{\rho_{n+1}})} \leq \frac{K(\lambda,p)\left(1 + \| a \|^2_{L^q(\mathcal{Q}_r)} + \| b \|^2_{L^q(\mathcal{Q}_r)} + \| c \|_{L^q(\mathcal{Q}_r)}\right)}{(\rho_n - \rho_{n+1})^2} \| v^{\alpha^n} \|_{L^{2\beta}(\mathcal{Q}_{\rho_n})}. \tag{5. 36}$$

Since

$$\| v^{\alpha^n} \|_{L^{2\alpha}} = \left(\| v \|_{L^{2\alpha^{n+1}}}\right)^{\alpha^n} \qquad \text{and} \qquad \| v^{\alpha^n} \|_{L^{2\beta}} = \left(\| v \|_{L^{2\alpha^n}}\right)^{\alpha^n}$$

we can rewrite equation (5. 36) in the following form for every $n \in \mathbb{N} \cup \{0\}$

$$\| v \|_{L^{2\alpha^{n+1}}(\mathcal{Q}_{\rho_{n+1}})} \leq \left(\frac{K(\lambda,p)\left(1 + \| a \|^2_{L^q(\mathcal{Q}_r)} + \| b \|^2_{L^q(\mathcal{Q}_r)} + \| c \|_{L^q(\mathcal{Q}_r)}\right)}{(\rho_n - \rho_{n+1})^2}\right)^{\frac{1}{\alpha^n}} \| v \|_{L^{2\beta\,\alpha^n}(\mathcal{Q}_{\rho_n})}.$$

Iterating this inequality, we obtain

$$\| v \|_{L^{2\alpha^{n+1}}(\mathcal{Q}_{\rho_{n+1}})} \leq \prod_{j=0}^{n} \left(\frac{2^{2(j+1)}}{(r-\rho)^2}\right)^{\frac{1}{\alpha^j}} \cdot$$

$$\cdot \left(K(\lambda,p)\left(1 + \| a \|^2_{L^q(\mathcal{Q}_r)} + \| b \|^2_{L^q(\mathcal{Q}_r)} + \| c \|_{L^q(\mathcal{Q}_r)}\right)\right)^{\frac{1}{\alpha^j}} \| v \|_{L^{2\beta}(\mathcal{Q}_r)},$$

and letting n go to infinity, we get

$$\sup_{\mathcal{Q}_\rho} v \leq \frac{\widetilde{K}}{(r-\rho)^\mu} \| v \|_{L^{2\beta}(\mathcal{Q}_r)},$$

where $\mu = \frac{2\alpha}{\alpha - 1}$ and

$$\widetilde{K} = \prod_{j=0}^{n} \left(K(\lambda,p)\left(1 + \| a \|^2_{L^q(\mathcal{Q}_r)} + \| b \|^2_{L^q(\mathcal{Q}_r)} + \| c \|_{L^q(\mathcal{Q}_r)}\right)\right)^{\frac{1}{\alpha^j}}$$

is a finite constant dependent on δ. Thus, we have proved that

$$\sup_{\mathcal{Q}_\rho} u^p \leq \left(\frac{\widetilde{K}}{(r-\rho)^\mu}\right)^{2\beta} \int_{\mathcal{Q}_r} u^p, \tag{5. 37}$$

for every p which verifies condition (5. 35). Because

$$(Q+2) \leq 2\beta\mu < 9(Q+2)$$

we get estimate (5. 12). We now make a suitable choice of $\delta > 0$, only dependent on the homogeneous dimension Q, in order to show that (5. 35) holds for every positive p. We remark that, if p is a number of the form

$$p_m = \frac{\alpha^m(\alpha+1)}{2\beta}, \qquad m \in \mathbb{Z},$$

then (5. 35) is satisfied with

$$\delta = \frac{|q - \frac{(Q+2)}{2}|}{(Q+2)^2}, \qquad \forall\, m \in \mathbb{Z}.$$

Therefore (5. 37) holds for such a choice of p, with $\widetilde{K}$ only dependent on Q, λ and $\| a \|_{L^q(\mathcal{Q}_r)}$, $\| b \|_{L^q(\mathcal{Q}_r)}$, $\| c \|_{L^q(\mathcal{Q}_r)}$. On the other hand, if p is an arbitrary positive number, we consider $m \in \mathbb{Z}$ such that

$$p_m \le p < p_{m+1}. \tag{5. 38}$$

Hence, by (5. 37) we have

$$\sup_{\mathcal{Q}_\rho} u \le \left(\frac{\widetilde{K}}{(r-\rho)^\mu} \right)^{\frac{2\beta}{p_m}} \left(\int_{\mathcal{Q}_r} u^{p_m} \right)^{\frac{1}{p_m}} \le \left(\frac{\widetilde{K}}{(r-\rho)^\mu} \right)^{\frac{2\beta}{p_m}} \left(\int_{\mathcal{Q}_r} u^{p} \right)^{\frac{1}{p}}$$

so that, by (5. 38), we obtain

$$\sup_{\mathcal{Q}_\rho} u^p \le \left(\frac{\widetilde{K}}{(r-\rho)^\mu} \right)^{2\alpha\beta} \int_{\mathcal{Q}_r} u^p$$

This concludes the proof of (5. 12) for $p > 0$. We next consider $p < 0$. In this case, assuming that $u \ge u_0$ for some positive constant u_0, estimate (5. 12) can be proved as in the case $p > 0$ or even more easealy since condition (5. 35) is satisfied for every $p < 0$. On the other hand, if u is a non-negative solution, it suffices to apply (5. 12) to $u + \frac{1}{n}$, $n \in \mathbb{N}$, and let n go to infinity, by the monotone convergence theorem. □

As far as we are concerned with the proof of Corollary 5.4, it can be straightforwardly accomplished proceeding as in [104, Corollary 1.4]. Moreover, Proposition 5.5 can be obtained by the same argument used in the proof of Theorem 5.2. For this reason, we do not give here the proof of these two results.

We close this Section recalling that Theorem 5.2 also holds true in the sets

$$\mathcal{Q}_r^-((x_0,t_0)) := \mathcal{Q}_r((x_0,t_0)) \cap \{t < t_0\}, \tag{5. 39}$$

in the case of non-negative exponents p. This result is analogous to [97], Theorem 3 (see also inequality (6^-) of Lemma 1 in [98]) and states that, in some sense, every point of $\overline{\mathcal{Q}_\rho^-(z_0)}$ can be considered as an interior point of $\overline{\mathcal{Q}_r^-(z_0)}$, when $\rho < r$, even though it belongs to its topological boundary.

Proposition 5.13 *Let u be a non-negative weak sub-solution to $\mathcal{K}u = 0$ in Ω. Let $z_0 \in \Omega$ and $r, \rho, \frac{1}{2} \le \rho < r \le 1$, such that $\overline{\mathcal{Q}_\rho^-(z_0)} \subseteq \Omega$ and $p < 0$. Then there exist positive constants $C = C(p,\lambda)$ and $\gamma = \gamma(p,q)$ such that*

$$\sup_{\mathcal{Q}_\rho^-(z_0)} u^p \le \frac{C \left(1 + \| a \|^2_{L^q(\mathcal{Q}_r(z_0))} + \| b \|^2_{L^q(\mathcal{Q}_r(z_0))} + \| c \|_{L^q(\mathcal{Q}_r(z_0))} \right)^\gamma}{(r-\rho)^{9(Q+2)}} \int_{\mathcal{Q}_r^-(z_0)} u^p, \tag{5. 40}$$

where $\gamma = \frac{2\alpha^2\beta}{\alpha-1}$, with α and β defined in (5. 11), provided that the integral is convergent.

The proof of the above Proposition can be straightforwardly accomplished proceeding as in Proposition 5.1 in [104], and therefore is omitted.

Bibliography

[1] F. Abedin and G. Tralli, *Harnack inequality for a class of Kolmogorov–Fokker–Planck equations in non-divergence form*, Arch. Ration. Mech. Anal., 233 (2019), pp. 867–900.

[2] A. Aimi, L. Diazzi and C. Guardasoni, *Numerical pricing of geometric Asian options with barriers*, Math. Methods Appl. Sci., 41 (2018), pp. 7510–7529.

[3] K. Amano, *Maximum principles for degenerate elliptic-parabolic operators*, Indiana Univ. Math. J., 28 (1979), pp. 545–557.

[4] F. Anceschi, M. Eleuteri and S. Polidoro, *A geometric statement of the Harnack inequality for a degenerate Kolmogorov equation with rough coefficients*, Communications in Contemporary Mathematics, 21 (2018), pp. 1–17.

[5] F. Anceschi, S. Muzzioli and S. Polidoro, *Existence of a fundamental solution of partial differential equations associated to asian options*, arXiv:2007.09037, (2020).

[6] F. Anceschi and S. Polidoro, *A survey on the classical theory for kolmogorov equation*, Le Matematiche, LXXV (2020), p. 221–258.

[7] F. Anceschi, S. Polidoro and M. A. Ragusa, *Moser's estimates for degenerate Kolmogorov equations with non-negative divergence lower order coefficients*, Nonlinear Analysis, 189 (2019), pp. 1–19.

[8] S. Armstrong and J. Mourrat, *Variational methods for the kinetic Fokker-Planck equation*, Pre-print, (2019).

[9] A. Arnold, P. Markowich, G. Toscani and A. Unterreiter, *On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations*, Comm. Partial Differential Equations, 26 (2001), pp. 43–100.

[10] D. G. Aronson and P. Besala, *Uniqueness of positive solutions of parabolic equations with unbounded coefficients*, Colloq. Math., 18 (1967), pp. 125–135.

[11] D. G. Aronson and J. Serrin, *Local behavior of solutions of quasilinear parabolic equations*, Arch. Rational Mech. Anal., 25 (1967), pp. 81–122.

[12] L. V. Ballestra, G. Pacelli and F. Zirilli, *A numerical method to price exotic path-dependent options on an underlying described by the heston stochastic volatility model*, Journal of Banking & Finance, 31 (2007), pp. 3420–3437.

[13] V. Bally and A. Kohatsu-Higa, *Lower bounds for densities of Asian type stochastic differential equations*, J. Funct. Anal., 258 (2010), pp. 3134–3164.

[14] C. Bardos, F. Golse and C. D. Levermore, *Fluid dynamic limits of kinetic equations. II. Convergence proofs for the Boltzmann equation*, Comm. Pure Appl. Math., 46 (1993), pp. 667–753.

[15] E. Barucci, S. Polidoro and V. Vespri, *Some results on partial differential equations and Asian options*, Math. Models Methods Appl. Sci., 11 (2001), pp. 475–497.

[16] T. Björk, *Arbitrage Theory in Continuous Time*, Oxford University Press, 2005.

[17] F. Black and M. Scholes, *The pricing of options and corporate liabilities [reprint of J. Polit. Econ.* **81** *(1973), no. 3, 637–654]*, in Financial risk measurement and management, vol. 267 of Internat. Lib. Crit. Writ. Econ., Edward Elgar, Cheltenham, 2012, pp. 100–117.

[18] F. Bolley and I. Gentil, *Phi-entropy inequalities and Fokker-Planck equations*, in Progress in analysis and its applications, World Sci. Publ., Hackensack, NJ, 2010, pp. 463–469.

[19] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni, *Stratified Lie groups and potential theory for their sub-Laplacians*, Springer Monographs in Mathematics, Springer, Berlin, 2007.

[20] J.-M. Bony, *Principe du maximum, inégalite de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés*, Ann. Inst. Fourier (Grenoble), 19 (1969), pp. 277–304 xii.

[21] U. Boscain and S. Polidoro, *Gaussian estimates for hypoelliptic operators via optimal control*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 18 (2007), pp. 333–342.

[22] M. Bramanti, M. Cerutti and M. M., *L^p estimates for some ultraparabolic operators with discontinuous coefficients*, J. Math. Anal. Appl., 200 (1996), pp. 332–354.

[23] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer Universitext Series, Springer Science & Business Media, 2010, 2010.

[24] L. A. Caffarelli and X. Cabré, *Fully nonlinear elliptic equations*, vol. 43 of American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, RI, 1995.

[25] S. Cameron, L. Silvestre and S. Snelson, *Global a priori estimates for the inhomogeneous Landau equation with moderately soft potentials*, Annales de l'Institut Henri Poincaré, Analyse non linéaire, 35 (2018), pp. 625–642.

[26] P. Carr and M. Schröder, *Bessel processes, the integral of geometric Brownian motion, and Asian options*, Teor. Veroyatnost. i Primenen., 48 (2003), pp. 503–533.

[27] S. Chandresekhar, *Stochastic problems in physics and astronomy*, Rev. Modern Phys., 15 (1943), pp. 1–89.

[28] G. Cibelli and S. Polidoro, *Harnack inequalities and bounds for densities of stochastic processes*, Modern problems of stochastic analysis and statistics, Springer Proc. Math. Stat., 208 (2017), pp. 67–90.

[29] G. Cibelli, S. Polidoro and F. Rossi, *Sharp estimates for Geman-Yor processes and applications to arithmetic average Asian options*, J. Math. Pures Appl. (9), 129 (2019), pp. 87–130.

[30] C. Cinti, K. Nyström and S. Polidoro, *A note on Harnack inequalities and propagation sets for a class of hypoelliptic operators*, Potential Anal., 33 (2010), pp. 341–354.

[31] C. Cinti, K. Nyström and S. Polidoro, *A boundary estimate for non-negative solutions to Kolmogorov operators in non-divergence form*, Ann. Mat. Pura Appl., 4 (2012), pp. 1–23.

[32] C. Cinti, K. Nyström and S. Polidoro, *A Carleson-type estimate in Lipschitz type domains for non-negative solutions to Kolmogorov operators*, Ann. Sc. Norm. Super. Pisa Cl. Sci., 5 (2013), pp. 439–465.

[33] C. Cinti, A. Pascucci and S. Polidoro, *Pointwise estimates for a class of non-homogeneous Kolmogorov equations*, Math. Ann., 340 (2008), pp. 237–264.

[34] C. Cinti and S. Polidoro, *Bounds on short cylinders and uniqueness in Cauchy problem for degenerate Kolmogorov equations*, J. Math. Anal. Appl., 359 (2009), pp. 135–145.

[35] G. Csató, B. Dacorogna and O. Kneuss, *The Pullback Equation for Differential Forms*, Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser Basel, 2012.

[36] G. Cupini and E. Lanconelli, *On Mean Value formulas for solutions to linear second order PDEs*, submitted pre-print, http://cvgmt.sns.it/paper/4323/ (2019).

[37] M. Curran, *Valuing asian and portfolio options by conditioning on the geometric mean price*, Management science, 40 (1994), pp. 1705–1711.

[38] F. Delarue and S. Menozzi, *Density estimates for a random noise propagating through a chain of differential equations*, J. Funct. Anal., 259 (2010), pp. 1577–1630.

[39] J. Dewynne and W. Shaw, *Differential equations and asymptotic solutions for arithmetic asian options: "Black–Scholes formulae" for asian rate calls*, European Journal of Applied Mathematics, 19 (2008), pp. 353–391.

[40] M. Di Francesco and A. Pascucci, *On a class of degenerate parabolic equations of Kolmogorov type*, AMRX Appl. Math. Res. Express, (2005), pp. 77–116.

[41] M. Di Francesco and S. Polidoro, *Schauder estimates, Harnack inequality and Gaussian lower bound for Kolmogorov-type operators in non-divergence form*, Adv. Differential Equations, 11 (2006), pp. 1261–1320.

[42] J. Dolbeault and X. Li, *φ-entropies: convexity, coercivity and hypocoercivity for Fokker-Planck and kinetic Fokker-Planck equations*, Math. Models Methods Appl. Sci., 28 (2018), pp. 2637–2666.

[43] J. Dolbeault, C. Mouhot and C. Schmeiser, *Hypocoercivity for linear kinetic equations conserving mass*, Trans. Amer. Math. Soc., 367 (2015), pp. 3807–3828.

[44] D. Dufresne, *Asian and basket asymptotics*, university of Montreal, Research Paper, (2002).

[45] N. El Ghani and N. Masmoudi, *Diffusion limit of the Vlasov-Poisson-Fokker-Planck system*, Commun. Math. Sci., 8 (2010), pp. 463–479.

[46] R. Esposito, Y. Guo, C. Kim and R. Marra, *Non-isothermal boundary in the Boltzmann theory and Fourier law*, Comm. Math. Phys., 323 (2013), pp. 177–239.

[47] G. B. Folland, *Sub-elliptic estimates and function spaces on nilpotent lie groups*, Ark. Mat., 13 (1975), pp. 161–207.

[48] G. B. Folland and E. M. Stein, *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math., 27 (1974), pp. 429–522.

[49] P. Foschi, S. Pagliarani and A. Pascucci, *Approximations for Asian options in local volatility models*, J. Comput. Appl. Math., 237 (2013), pp. 442–459.

[50] M. Fu, D. Madan and T. Wang, *Pricing continuous time Asian options: a comparison of monte carlo and laplace transform inversion methods*, J. Comput. Finance, (1998), pp. 49–74.

[51] N. Garofalo and E. Lanconelli, *Level sets of the fundamental solution and Harnack inequality for degenerate equations of Kolmogorov type*, Trans. Amer. Math. Soc., 321 (1990), pp. 775–792.

[52] N. Garofalo and G. Tralli, *A class of nonlocal hypoelliptic operators and their extensions*, arXiv:1811.02968, (2018).

[53] N. Garofalo and G. Tralli, *Functional inequalities for a class of nonlocal hypoelliptic equations of hörmander type*, arXiv:1905.08887, (2019).

[54] N. Garofalo and G. Tralli, *Hardy-littlewood-sobolev inequalities for a class of non-symmetric and non-doubling hypoelliptic semigroups*, arXiv:1904.12982, (2019).

[55] N. Garofalo and G. Tralli, *Nonlocal isoperimetric inequalities for kolmogorov-fokker-planck operators*, arXiv:1907.02281, (2019).

[56] H. Geman and M. Yor, *Quelques relations entre processus de Bessel, options asiatiques et fonctions confluentes hypergéométriques*, C. R. Acad. Sci. Paris Sér. I Math., 314 (1992), pp. 471–474.

[57] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin-New York, 1977. Grundlehren der Mathematischen Wissenschaften, Vol. 224.

[58] F. Golse, C. Imbert, C. Mouhot and A. F. Vasseur, *Harnack inequality for kinetic Fokker-Planck equations with rough coefficients and application to the Landau equation*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 19 (2019), pp. 253–295.

[59] F. Golse, C. D. Levermore and L. Saint-Raymond, *La méthode de l'entropie relative pour les limites hydrodynamiques de modèles cinétiques*, in Séminaire: Équations aux Dérivées Partielles, 1999–2000, Sémin. Équ. Dériv. Partielles, École Polytech., Palaiseau, 2000, pp. Exp. No. XIX, 23.

[60] Y. Guo, *The Landau equation in a periodic box*, Comm. Math. Phys., 231 (2002), pp. 391–434.

[61] J. Hadamard, *Extension à l'équation de la chaleur d'un théorème de A. Harnack*, Rend. Circ. Mat. Palermo (2), 3 (1954), pp. 337–346 (1955).

[62] C. Henderson, S. Snelson and A. Tarfulea, *Local existence, lower mass bounds, and a new continuation criterion for the landau equation*, Journal of Differential Equations, 266 (2019), pp. 1536–1577.

[63] C. Henderson, S. Snelson and A. Tarfulea, *Self-generating lower bounds and continuation for the boltzmann equation*, arXiv preprint arXiv:2005.13668, (2020).

[64] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math., 119 (1967), pp. 147–171.

[65] J. C. Hull, *Options, Futures, and Other Derivatives*, Prentice Hall, 1997.

[66] F. Hérau, *Introduction to hypocoercive methods and applications for simple linear inhomogeneous kinetic models*, Lectures on the Analysis of Nonlinear Partial Differential Equations Vol. 5, MLM5, 119-147 (2017).

[67] A. M. Il′ in, *On a class of ultraparabolic equations*, Dokl. Akad. Nauk SSSR, 159 (1964), pp. 1214–1217.

[68] C. Imbert and C. Mouhot, *The schauder estimate in kinetic theory with application to a toy nonlinear model*, Pre-print, arXiv:1801.07891v2 (2019).

[69] C. Imbert and L. Silvestre, *Global regularity estimates for the boltzmann equation without cut-off*, Pre-print, arXiv:1909.12729v1 (2019).

[70] C. Imbert and L. Silvestre, *The schauder estimate for kinetic integral equations*, Pre-print, arXiv:1812.11870v2 (2019).

[71] J. Kim, Y. Guo, and H. J. Hwang, *A L^2 to L^∞ approach for the landau equation*, arXiv preprint arXiv:1610.05346, (2016).

[72] A. E. Kogoj, *On the Dirichlet problem for hypoelliptic evolution equations: Perron-Wiener solution and a cone-type criterion*, J. Differential Equations, 262 (2017), pp. 1524–1539.

[73] A. E. Kogoj and E. Lanconelli, *One-side Liouville theorems for a class of hypoelliptic ultraparabolic equations*, Contemp. Math., Amer. Math. Soc., Providence, RI, 369 (2005), pp. 305–312.

[74] A. E. Kogoj and E. Lanconelli, *Liouville theorems for a class of linear second-order operators with nonnegative characteristic form*, Bound. Value Probl., Art. ID 48232 (2007), p. 16.

[75] A. E. Kogoj, E. Lanconelli and G. Tralli, *Wiener-landis criterion for kolmogorov-type operators*, Discrete & Continuous Dynamical Systems, 38 (2018), pp. 24–67.

[76] A. E. Kogoj, Y. Pinchover and S. Polidoro, *On liouville-type theorems and the uniqueness of the positive cauchy problem for a class of hypoelliptic operators*, J. Evol. Equ., 16, no. 4 (2016), pp. 905–943.

[77] A. E. Kogoj and S. Polidoro, *Harnack Inequality for Hypoelliptic Second Order Partial Differential Operators*, Potential Analysis, 45,3 (2016), pp. 545–555.

[78] A. Kolmogorov, *Zufllige bewegungen. (zur theorie der brownschen bewegung.).*, Ann. of Math., II. Ser., 35 (1934), pp. 116–117.

[79] V. Konakov, S. Menozzi and S. Molchanov, *Explicit parametrix and local limit theorems for some degenerate diffusion processes*, Ann. Inst. H. Poincaré Probab. Statist., 46 (2010), pp. 908–923.

[80] L. P. Kuptsov, *The mean value property and the maximum principle for second order parabolic equations*, Dokl. Akad. Nauk SSSR, 242 (1978), pp. 529–532.

[81] L. P. Kuptsov, *On parabolic means*, Dokl. Akad. Nauk SSSR, 252 (1980), pp. 296–301.

[82] L. P. Kuptsov, *Fundamental solutions of some second-order degenerate parabolic equations*, Mat. Zametki, 31 (1982), pp. 559–570, 654.

[83] A. Lanconelli, A. Pascucci and S. Polidoro, *Gaussian lower bounds for non-homogeneous kolmogorov equations with measurable coefficients*, arXiv:1704.07307, (2018).

[84] E. Lanconelli and S. Polidoro, *On a class of hypoelliptic evolution operators*, Rend. Sem. Mat. Univ. Politec. Torino, 52 (1994), pp. 29–63.

[85] F. Lascialfari and D. Morbidelli, *A boundary value problem for a class of quasilinear ultraparabolic equations*, Commun. Partial Differ. Equations, 23 N. 5-6 (1998), pp. 847–868.

[86] E. B. Lee and L. Markus, *Foundations of optimal control theory*, John Wiley & Sons, Inc., New York-London-Sydney, 1967.

[87] J. Liao, Q. Wang and X. Yang, *Global existence and decay rates of the solutions near Maxwellian for non-linear Fokker-Planck equations*, J. Stat. Phys., 173 (2018), pp. 222–241.

[88] L. Lorenzi, *Schauder estimates for degenerate elliptic and parabolic problems with unbounded coefficients in* $\mathbb{R}^N$, Differential Integral Equations, 18 (2005), pp. 531–566.

[89] A. Lunardi, *Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in* $\mathbf{R}^n$, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 24 (1997), pp. 133–164.

[90] M. Manfredini, *The Dirichlet problem for a class of ultraparabolic equations*, Adv. Differential Equations, 2 (1997), pp. 831–866.

[91] M. Manfredini and S. Polidoro, *Interior regularity for weak solutions of ultraparabolic equations in divergence form with discontinuous coefficients*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat., 8 (1998), pp. 651–675.

[92] I. Markou, *Hydrodynamic limit for a Fokker-Planck equation with coefficients in Sobolev spaces*, Netw. Heterog. Media, 12 (2017), pp. 683–705.

[93] H. Matsumoto and M. Yor, *Exponential functionals of Brownian motion. I. Probability laws at fixed time*, Probab. Surv., 2 (2005), pp. 312–347.

[94] H. Matsumoto and M. Yor, *Exponential functionals of Brownian motion. II. Some related diffusion processes*, Probab. Surv., 2 (2005), pp. 348–384.

[95] R. C. Merton, *Theory of rational option pricing*, Bell J. Econom. and Management Sci., 4 (1973), pp. 141–183.

[96] L. Monti and A. Pascucci, *Obstacle problem for arithmetic Asian options*, C. R. Math. Acad. Sci. Paris, 347 (2009), pp. 1443–1446.

[97] J. Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math., 17 (1964), pp. 101–134.

[98] J. Moser, *Correction to "a Harnack inequality for parabolic differential equations"*, Comm. Pure Appl. Math., 20 (1967), pp. 231–236.

[99] A. I. Nazarov and N. N. Ural'tseva, *The Harnack inequality and related properties of solutions of elliptic and parabolic equations with divergence-free lower-order coefficients*, Algebra i Analiz, 23 (2011), pp. 136–168.

[100] K. Nyström, A. Pascucci and S. Polidoro, *Regularity near the initial state in the obstacle problem for a class of hypoelliptic ultraparabolic operators*, J. Differential Equations, 249 (2010), pp. 2044–2060.

[101] S. Pagliarani, A. Pascucci, and M. Pignotti, *Intrinsic Taylor formula for Kolmogorov-type homogeneous groups*, J. Math. Anal. Appl., 435 (2016), pp. 1054–1087.

[102] A. Pascucci, *PDE and martingale methods in option pricing*, vol. 2 of Bocconi & Springer Series, Springer, Milan; Bocconi University Press, Milan, 2011.

[103] A. Pascucci and A. Pesce, *On stochastic Langevin and Fokker-Planck equations: the two-dimensional case*, arXiv:1910.05301, (2019).

[104] A. Pascucci and S. Polidoro, *The Moser's iterative method for a class of ultraparabolic equations*, Commun. Contemp. Math., 6 (2004), pp. 395–417.

[105] R. Peszek, *Oscillations of solutions to the two-dimensional Broadwell model, an H-measure approach*, SIAM J. Math. Anal., 26 (1995), pp. 750–760.

[106] B. Pini, *Sulla soluzione generalizzata di Wiener per il primo problema di valori al contorno nel caso parabolico*, Rend. Sem. Mat. Univ. Padova, 23 (1954), pp. 422–434.

[107] S. Polidoro, *On a class of ultraparabolic operators of Kolmogorov-Fokker-Planck type*, Matematiche (Catania), 49 (1994), pp. 53–105 (1995).

[108] S. Polidoro, *Uniqueness and representation theorems for solutions of Kolmogorov-Fokker-Planck equations*, Rend. Mat. Appl. (7), 15 (1995), pp. 535–560 (1996).

[109] S. Polidoro and M. A. Ragusa, *Sobolev-Morrey spaces related to an ultraparabolic equation*, Manuscripta Math., 96 (1998), pp. 371–392.

[110] S. Polidoro and M. A. Ragusa, *Hölder regularity for solutions of ultraparabolic equations in divergence form*, Potential Anal., 14 (2001), pp. 341–350.

[111] E. Priola, *Formulae for the derivatives of degenerate diffusion semigroups*, J. Evol. Equ., 6 (2006), pp. 577–600.

[112] L. P. Rothschild and E. M. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math., 137 (1976), pp. 247–320.

[113] W. T. Shaw, *Modelling financial derivatives with Mathematica*, Cambridge University Press, Cambridge, 1998. Mathematical models and benchmark algorithms, With 1 CD-ROM (Windows, Macintosh and UNIX).

[114] I. M. Sonin, *A class of degenerate diffusion processes*, Teor. Verojatnost. i Primenen, 12 (1967), pp. 540–547.

[115] J. Sun and X. Chen, *Asian option pricing formula for uncertain financial market*, Journal of Uncertainty Analysis and Applications, 3 (2015), p. 11.

[116] J. Vecer, *A new pde approach for pricing arithmetic average asian options*, Journal of computational finance, 4 (2001), pp. 105–113.

[117] C. Villani, *A review of mathematical topics in collisional kinetic theory*, in Handbook of mathematical fluid dynamics, Vol. I, North-Holland, Amsterdam, 2002, pp. 71–305.

[118] C. Villani, *Hypocoercivity*, Mem. Amer. Math. Soc., 202 (2009), pp. iv+141.

[119] W. Wang and L. Zhang, *The C^α regularity of a class of non-homogeneous ultraparabolic equations*, Sci. China Ser. A, 52 (2009), pp. 1589–1606.

[120] W. Wang and L. Zhang, *The C^α regularity of weak solutions of ultraparabolic equations*, Discrete Contin. Dyn. Syst., 29 (2011), pp. 1261–1275.

[121] M. Weber, *The fundamental solution of a degenerate partial differential equation of parabolic type*, Trans. Amer. Math. Soc., 71 (1951), pp. 24–37.

[122] H.-T. Yau, *Relative entropy and hydrodynamics of Ginzburg-Landau models*, Lett. Math. Phys., 22 (1991), pp. 63–80.

[123] M. Yor, *On some exponential functionals of Brownian motion*, Adv. in Appl. Probab., 24 (1992), pp. 509–531.

[124] M. Yor, *Exponential functionals of Brownian motion and related processes*, Springer Finance, Springer-Verlag, Berlin, 2001. With an introductory chapter by Hélyette Geman, Chapters 1, 3, 4, 8 translated from the French by Stephen S. Wilson.

[125] J. Zabczyk, *Mathematical control theory: an introduction*, Systems & Control: Foundations & Applications, Birkhäuser Boston, Inc., Boston, MA, 1992.

[126] Y. Zhu, *Velocity averaging and Hölder regularity for kinetic Fokker-Planck equations with general transport operators and rough coefficients*, preprint arXiv:2010.03867, (2020).