UNIVERSITÀ DEGLI STUDI DI MODENA E REGGIO EMILIA
Dipartimento di Scienze Fisiche, Informatiche e Matematiche

Dottorato di Ricerca in Matematica
in convenzione con l'Università degli Studi di Ferrara e l'Università degli Studi di Parma

Ciclo XXXIII

Nowhere-zero Circular Flows and Factors of Graphs
Constructions and Counterexamples

Candidato: Davide Mattiolo
Relatore: Prof. Giuseppe Mazzuoccolo
Coordinatore del Corso di Dottorato: Prof. Cristian Giardinà

Anno Accademico 2019/2020

ACKNOWLEDGMENTS

First of all I would like to thank Giuseppe Mazzuoccolo for his time, patience and all of his good advice. I enjoyed working with him and I have really learned a lot from him.

I am also grateful to Arrigo Bonisoli, for his help and support during these three years of PhD.

A big thanks goes to Jan Goedgebeur, for all his long-distance help with computational problems and also for inviting me to Ghent for a research visit. I really liked working there and enjoyed the stay very much.

During my PhD I spent four months in Paderborn working at PACE, Paderborn Center for Advanced Studies, and I want to thank a lot Eckhard Steffen for giving me this opportunity. It has been a pleasure working with him. Special thanks go also to Chiara, Antje, Astrid, Arnab and Peter, for all their help during my stay and the good time spent together.

Also, I must extend my gratitude to all new friends and colleagues I met in these past three years for making this journey great. In particular I want to thank Jean Paul Zerafa for the nice discussions on common topics of research and study, his company at conferences we attended together and friendship.

Last but not least, my parents, Anna Maria and Renzo, and my brother Riccardo, for your great patience, your encouragement and support.

SUMMARY

This dissertation treats arguments of the theory of nowhere-zero flows in graphs. Most of the presented results are constructions of infinite families of graphs with properties that are interesting with respect to certain open problems. In the first chapter we recall the classical definitions and results that are needed in the next chapters.

The second and third chapters are devoted to the study of the circular flow number of graphs and snarks. In particular we present new methods to construct snarks with circular flow number 5. Indeed, it is known that such graphs are interesting with respect to the famous Tutte's 5-Flow Conjecture, claiming that every bridgeless graphs admits a nowhere-zero 5-flow. Moreover we show that these constructions, together with most of the already known ones, are instances of a more general method. In the third chapter we continue the study of the circular flow number of snarks and present new computational results.

It is well known that nowhere-zero circular flows reflect structural properties of graphs and it is also well known that flow problems are strictly connected with problems on factors, matchings and edge colorings of graphs. A clear example of this connection can be found in the class of cubic graphs: indeed a cubic graph is class 1 if and only if its circular flow number is at most 4. Similar results hold for $(2t+1)$-regular graphs. In the fourth and fifth chapters we study problems of this kind. More precisely, in the fourth chapter, we solve two conjectures regarding the set of circular flow numbers of class 1 and class 2 $(2t+1)$-regular graphs; in the fifth chapter we present infinite families of $4k$-edge-connected $4k$-regular graphs without $4k-2$ pairwise disjoint perfect matchings.

Finally, in the last chapter we consider $\mathbb{Z}$-flow-continuous maps between cubic graphs. These maps induce a quasi-order $\succ_{\mathbb{Z}}$ on the class of finite graphs. The main result of this last chapter is the construction of an infinite antichain in the quasi-order $\succ_{\mathbb{Z}}$.

RIASSUNTO

Questa tesi tratta argomenti della teoria dei nowhere-zero flows su grafi. La maggior parte dei risultati presentati consistono in costruzioni di famiglie infinite di grafi con specifiche proprietà interessanti rispetto a determinati problemi aperti. Nel primo capitolo ricordiamo le definizioni e i risultati principali necessari nei capitoli successivi.

Il secondo e terzo capitolo sono dedicati allo studio del numero di flusso circolare di grafi e snarks. In particolare presentiamo nuovi metodi di costruzione di snarks con numero di flusso circolare 5. Tali grafi, infatti, risultano essere interessanti rispetto alla famosa Congettura dei 5-Flussi di Tutte, secondo cui ogni grafo senza ponti ammette un nowhere-zero 5-flow. Inoltre, mostriamo che questi metodi, e molti tra quelli precedentemente conosciuti, sono particolari istanze di un unico metodo più generale. Nel terzo capitolo, proseguiamo con lo studio del numero di flusso circolare degli snarks e presentiamo nuovi risultati computazionali.

È noto che i nowhere-zero circular flows riflettono proprietà strutturali dei grafi ed è altrettanto ben noto che problemi di flusso sono strettamente collegati a problemi di fattori, matchings e colorazioni sugli spigoli di grafi. Un esempio di questa connessione si può notare nei grafi cubici: un grafo cubico è di classe 1 se e solo se il suo numero di flusso circolare è al più 4. Risultati simili valgono per grafi $(2t+1)$-regolari. Nel quarto e quinto capitolo studiamo problemi di questa natura. Più precisamente, nel quarto capitolo risolviamo due congetture riguardanti l'insieme dei numeri di flusso circolare di grafi $(2t+1)$-regolari di classe 1 e 2; nel quinto capitolo presentiamo famiglie infinite di grafi $4k$-regolari e $4k$-connessi per archi privi di $4k-2$ matchings perfetti a due a due disgiunti.

Infine, nell'ultimo capitolo, studiamo mappe continue per $\mathbb{Z}$-flussi tra grafi cubici. Queste mappe inducono un quasi-ordine $\succ_{\mathbb{Z}}$ sulla famiglia dei grafi finiti. Il risultato principale di quest'ultimo capitolo consiste nella costruzione di un'anticatena infinita nel quasi-ordine $\succ_{\mathbb{Z}}$.

CONTENTS

LIST OF FIGURES

INTRODUCTION

In this dissertation we study nowhere-zero flow problems and related topics such as edge-colorings and matchings. The theory of nowhere-zero flows on finite graphs represents a research area of great interest in graph theory. Certainly one of the main reasons is the fact that nowhere-zero flows generalize face-colorings of planar graphs. A plane graph, that is a drawing of a planar graph into the plane $\mathbb{R}^2$ without edge-crossings, is said to be face-k-colorable if its faces can be colored with k colors in such a way that, for all edges, different colors are assigned to its neighboring faces. Such a coloring is said to be proper. It is easy to see that a plane graph with a bridge does not admit a proper face-k-coloring for any k. The conjecture known as the 4-Color Problem states that every bridgeless plane graph admits a face-4-coloring. It received a great deal of attention since its formulation in the 19^{th} century and remained unsolved for many years. A proof of this conjecture, that now is known as the 4-Color Theorem, was presented in 1976 by Appel and Haken [4]. This result is one of the most outstanding achievements of the 20^{th} century in Graph Theory.

On the other hand, we have nowhere-zero flows, that, at first sight, look like very different objects with respect to face-colorings. A nowhere-zero k-flow on a graph $G = (V, E)$ consists of an orientation together with a function $f\colon E \to \{1, 2, \dots, k-1\}$ such that, at every vertex, the sum of all incoming flow values equals the sum of all outgoing flow values. In analogy with proper face-colorings, it is an easy and known fact that graphs with bridges do not admit any nowhere-zero flow. In 1954 [83], Tutte proved that a plane graph has a face-k-coloring if and only if it admits a nowhere-zero k-flow, showing that, when dealing with planar graphs, face-colorings and nowhere-zero flows are indeed equivalent objects. An immediate consequence is the fact that every bridgeless planar graph admits a nowhere-zero 4-flow. Thus, the problem of finding the minimum integer k such that every bridgeless graph has a nowhere-zero k-flow can be seen as a generalization of the face-coloring problem to non-planar graphs. In 1954 [83], Tutte conjectured that such a minimum integer is 5 (see Conjecture 1.13), i.e. that every bridgeless graph has a nowhere-zero 5-flow. This remarkable conjecture, known as Tutte's 5-Flow Conjecture is still open and is one of the most striking problems in modern graph theory.

The best approximation of this conjecture is Seymour's 6-Flow Theorem [68], see Theorem 1.23. Furthermore some results on the existence of nowhere-zero 5-flows on some classes of graphs are known, see for example [56], [73] and [75]. A classical approach to attack the 5-Flow Conjecture consists in reducing it to a proper subclass of bridgeless graphs. Theorems 1.15 and 1.16 show that Tutte's 5-Flow Conjecture is equivalent to its restriction to cubic graphs that are not 3-edge-colorable. Another approach consists in studying the structure of a minimum counterexample, see Section 1.4.1 for more details. Well known results show that such a minimum counterexample must be a *snark*, that is a non-3-edge-colorable cubic graph with some further properties on the girth and cyclic edge connectivity.

Together with the 5-Flow Conjecture, Tutte proposed two more conjectures regarding nowhere-zero flows, that are still open as well and that are also among the most interesting and remarkable problems in structural graph theory: the 3-Flow Conjecture, that is Conjecture 1.11, and the 4-Flow Conjecture, that is Conjecture 1.12. They are all presented in Chapter 1, where we begin with a brief introduction of the theory of integer flows, recalling the main definitions and all major results. If, for a real number $r \geq 2$, we let f be a function $E \to [1, r-1] \subseteq \mathbb{R}$ in the above definition of a nowhere-zero flow, then we speak of a nowhere-zero circular r-flow. It is easy to see that this definition generalizes the previous one. Circular flows have been introduced in [26] and represent one of the main objects that we study throughout this dissertation. They are introduced in Section 1.2. In the next sections of Chapter 1 we go through the main topics connected with nowhere-zero (circular) flows that will be needed in the consecutive chapters.

In Chapter 2 we present results based on the paper [P.2]. Namely we attack the problem of constructing families of graphs and snarks with circular flow number (at least) 5: we present indeed new construction methods and a description attempting a unified approach involving most of the known methods.

Chapter 3 is dedicated to presenting new results on the circular flow number of snarks. More precisely, there are two main results: first we improve the best known upper bound for the circular flow number of Goldberg snarks and then we present an infinite family of snarks whose circular flow number is the smallest possible with respect to their order. Furthermore we present an implementation of an algorithm that computes the circular flow number of a cubic graph and use it to compute the circular flow number of the most famous snarks. This chapter is mostly based on the manuscript [P.1]. In its final section, that is based on [P.6], we prove a result about the existence of 3-bisections (i.e. vertex-colorings with some additional properties) on sub-cubic graphs.

In Chapter 4 we solve two conjectures from [74]. In that paper the author shows that circular flows reflect structural properties of $(2t+1)$-regular graphs. Theorems 1.43 and 4.1 explain in details the connection between edge-colorings and circular flows in $(2t+1)$-regular graphs. In particular we show that there are $(2t+1)$-regular graphs of class 1 with circular flow number larger than $2 + \frac{2}{t}$ and that, for every $\epsilon > 0$, there is a $(2t + 1)$-regular graph of class 2 with circular flow number inside $(2 + \frac{2}{2t-1}, 2 + \frac{2}{2t-1} + \epsilon)$. The chapter is based on the manuscript [P.3].

In Chapter 5 we construct infinite families of highly edge connected regular graphs that do not contain a certain number of pairwise disjoint perfect matchings. In particular we construct counterexamples to some instances of Problem 5.1, that appears in [80]. In this paper some results regarding the factorization of regular graphs into regular factors are proved. Such results are proved using a weak version of Tutte's 3-Flow Conjecture, proved in [46]. Results of this chapter come from the manuscript [P.4].

Finally in Chapter 6 we consider mappings between edge-sets of two oriented graphs with the property that their inverse mappings preserve flows. These maps have been introduced and studied in [15] as a new approach towards well-known conjectures of structural graph theory such as the Petersen Coloring conjecture [39]. Indeed such conjecture can be equivalently stated in terms of cycle-continuous maps, see Conjecture 6.1. In particular, we are interested in studying $\mathbb{Z}$-flow-continuous maps and the quasi-order that they induce on the class of finite graphs. Indeed, if there is a $\mathbb{Z}$-flow-continuous map $E(G) \to E(H)$ and H has a nowhere-zero (circular) r-flow, it follows that G also has such a flow. Our purpose is to study the existence of such maps between cubic graphs and snarks. The main result of this chapter, Theorem 6.17, presents an infinite antichain of snarks with respect to this quasi-order. All results presented in this chapter are part of the manuscript [P.5].

CONTRIBUTIONS

We give here the list of contributions which this dissertation is based on.

[P.1] J. Goedgebeur, D. Mattiolo, G. Mazzuoccolo, *Computational results and new bounds for the circular flow number of snarks*, Discrete Math., 343 (2020), 112026.

[P.2] J. Goedgebeur, D. Mattiolo, G. Mazzuoccolo, *A unified approach to construct snarks with circular flow number* 5, published as early view in Journal of Graph Theory, `https://doi.org/10.1002/jgt.22641`

[P.3] D. Mattiolo, E. Steffen, *Edge colorings and circular flows on regular graphs*, arXiv:2001.02484 [math.CO] (Submitted).

[P.4] D. Mattiolo, E. Steffen, *Highly edge-connected regular graphs without large factorizable subgraphs*, arXiv:1912.09704 [math.CO] (Submitted).

[P.5] D. Mattiolo, *On $\mathbb{Z}$-flow-continuous maps and oriented colorings of cubic graphs*, accepted for publication in the Australasian Journal of Combinatorics.

[P.6] D. Mattiolo, G. Mazzuoccolo, *On 3-bisections in cubic and subcubic graphs*, Graphs and Combinatorics (2021) `https://doi.org/10.1007/s00373-021-02275-z`

1 BACKGROUND

This chapter is devoted to a brief introduction of the theory of integer and circular flows on graphs. We discuss here some major and classical results on these topics and some of the most important conjectures. We put particular emphasis on those facts and tools that will be needed in the next chapters, where we present new results.

In this dissertation all graphs are finite, i.e. on a finite number of vertices and edges, they may contain multiple edges and loops, unless differently stated. We refer to [17] for basic definitions and notation.

1.1 INTEGER FLOWS

In this section we recall the main results regarding integer flows on graphs. We refer to [90] for a more detailed presentation of the theory of integer flows.

An *orientation* of a graph is an assignment of a direction to each edge. A graph having an orientation is an *oriented* graph.

If D is an orientation of a graph G and $X \subseteq V(G)$, we define $\partial^-(X)$ and $\partial^+(X)$ to be the set of all edges pointing inward and, respectively, outward X in the orientation D. Similarly, we denote by $\partial(X)$ the set of edges of G with exactly one end in X. When $X = \{v\}$ then we omit the set-brackets, i.e. for example we write $\partial(v)$ instead of $\partial(\{v\})$. Moreover, we let the *outdegree* of a vertex $v \in V(D)$ be $d^+(v) = |\partial^+(v)|$ and, similarly, the *indegree* of a vertex $v \in V(D)$ be $d^-(v) = |\partial^-(v)|$.

Definition 1.1. Let A be an additive abelian group with 0 as identity element. An *A-flow* on a graph G is a pair (D, f) where D is an orientation of G and $f\colon E(G) \to A$ such that, for all $v \in V$,

$$\sum_{e \in \partial^-(v)} f(e) = \sum_{e \in \partial^+(v)} f(e). \tag{1}$$

An A-flow on G is a *nowhere-zero* flow, in short an A-NZF, if $f(e) \neq 0$ for all $e \in E$.

If we consider the additive abelian group of integers we get the following definition.

Definition 1.2. Let G be a graph and $k \geq 2$ an integer. A *k-flow* on G is a $\mathbb{Z}$-flow (D, f) such that $f\colon E(G) \to \{0, \pm 1, \dots, \pm(k-1)\}$. We call it k-NZF if it is nowhere-zero.

Sometimes, when the orientation is not relevant and does not play a role in the proofs we just refer to a k-NZF on a graph G by its flow function f. Furthermore, we sometimes consider k-flows on oriented graphs, that are exactly k-flows as defined above but for the fact that the orientation is fixed a priori.

The existence of an A-flow (D, f) on a graph G does not depend on the chosen orientation. Indeed let D' be another orientation of G, and let $f' \colon E(G) \to A$ be such that $f'(e) = f(e)$ if e has the same direction in both orientations, $f'(e) = -f(e)$ otherwise, then (D', f') is again an A-flow. Notice that, if (D, f) is a k-NZF, we can always find an orientation such that all flow values are positive.

This fact tells us that the existence of an A-NZF on a graph G is due to the structure of G itself and not to the chosen orientation. For example, it is not possible to define nowhere-zero flows on graphs containing bridges. This follows from the following proposition.

Proposition 1.3. *Let (D, f) be an A-flow on a graph G. Then, for all $X \subseteq V$,*

$$\sum_{e \in \partial^+(X)} f(e) = \sum_{e \in \partial^-(X)} f(e).$$

It follows from the previous proposition that if (D, f) is a A-flow on a graph G and $e \in E$ is a bridge, then $f(e) = 0$.

It turns out that the existence of a nowhere-zero A-flow on a graph is equivalent to the existence of a $|A|$-NZF.

Lemma 1.4. *Let G be an oriented graph and A, B two abelian groups such that $|A| = |B| = k$. Moreover let $F_{G,A}$ and $F_{G,B}$ be the number of distinct nowhere-zero A-flows and the number of distinct nowhere-zero B-flows on G respectively. Then $F_{G,A} = F_{G,B}$.*

Proof. We use induction on $m = |E|$. If $m = 0$ then the thesis follows. If G contains only loops, then $F_{G,A} = F_{G,B} = (k-1)^m$ and the theorem follows once again. Therefore, we can assume the existence of $e \in E$, such that e is not a loop, and let X be the set of all A-flows on G with $f(e) = 0$ and $f|_{E \setminus \{e\}} \subseteq A \setminus \{0\}$. Moreover, let Y be the set of all nowhere-zero A-flows on G. For each $f \in X \cup Y$, the restriction of f to $G_2 := G - e$ gives rise to a nowhere-zero A-flow in $G_1 := G/e$. Conversely, every nowhere-zero A-flow in G_1 corresponds to a unique A-flow in $X \cup Y$. Therefore, $F_{G_1,A} = |X \cup Y| = |X| + |Y|$ and we have

$$F_{G,A} = |Y| = |Y| + |X| - |X| = F_{G_1,A} - F_{G_2,A}.$$

Using the same argument we conclude that $F_{G,B} = F_{G_1,B} - F_{G_2,B}$. But then the inductive hypothesis applies on G_1 and G_2 and we conclude that

$$F_{G,A} = F_{G_1,A} - F_{G_2,A} = F_{G_1,B} - F_{G_2,B} = F_{G,B}.$$ □

Lemma 1.5. *Let G be a graph and $k \geq 2$ be an integer. Then G has a nowhere-zero $\mathbb{Z}_k$-flow if and only if G has a nowhere-zero k-flow.*

Proof. If f is a k-NZF on G then f is also a $\mathbb{Z}_k$-NZF on G.

Conversely, let f be a $\mathbb{Z}_k$-NZF on G. Define $f(v)^+ = \sum_{e \in \partial^+(v)} f(e)$ and similarly $f(v)^-$, where we compute those summations in $\mathbb{Z}$. Moreover let

$$D(f) := \sum_{v \in V} |f(v)^+ - f(v)^-|$$

and consider a $\mathbb{Z}_k$-flow h such that $D(h)$ is minimal. We claim that h is a nowhere-zero k-flow in G, i.e. $D(h) = 0$.

First notice that we can assume that h is positive on every edge, if not just reverse e in the current orientation of G and set its flow value to be $-h(e)$, for those edges e where it is negative. This operation does not alter the function $D(h)$. Indeed, let $e = uw$, oriented from u to w, be an edge that has been reversed as described above. Let α_u be the sum of the incoming edges minus the outgoing ones different from e at u, computed in $\mathbb{Z}$ and let α_w be defined in the same way. Then

$$|\alpha_u - h(e)| + |\alpha_w + h(e)| = |\alpha_u + (-h(e))| + |\alpha_w - (-h(e))|,$$

hence it follows that the value of D is unchanged.

Now we are ready to prove that $D(h) = 0$. Let $S := \{v \in V(G) \colon h(v)^+ > h(v)^-\}$ and $T := \{v \in V(G) \colon h(v)^+ < h(v)^-\}$.

We claim that there is no directed path from S to T. Otherwise, choose a shortest directed path $P = sx_1 \dots x_p t$ connecting S and T and let h' be the $\mathbb{Z}_k$-flow in G defined as follows: $h'(e) = h(e)$ when $e \notin P$ and $h'(e) = k - h(e)$ for all $e \in P$, where the chosen orientation is obtained by reversing the direction of the edges of P. Since h is a $\mathbb{Z}_k$-flow, $h(v)^+ - h(v)^- \geq k$ for each $v \in S$ and $h(v)^+ - h(v)^- \leq -k$ for each $v \in T$. Thus

$$|h'(s)^+ - h'(s)^-| = |h(s)^+ - h(s)^- - k| < |h(s)^+ - h(s)^-|$$

and

$$|h'(t)^+ - h'(t)^-| = |h(t)^+ - h(t)^- + k| < |h(t)^+ - h(t)^-|.$$

Therefore, noticing that $|h'(v)^+ - h'(v)^-| = |h(v)^+ - h(v)^-|$ for all other vertices different from s and t, we get the following contradiction

$$D(h') = \sum_{v \in V} |h'(v)^+ - h'(v)^-| < \sum_{v \in V} |h(v)^+ - h(v)^-| = D(h).$$

So, since there is no directed path connecting S to T we deduce that there exists a one-way-cut separating S from T, that is a partition of V, say $V = A \cup B$, with $S \subseteq A$ and $T \subseteq B$, such that every A-B edge is oriented from B to A. Then

$$\sum_{v \in A} (h(v)^+ - h(v)^-) = \underbrace{\sum_{e \in \partial^+(A)} h(e)}_{=0} - \sum_{e \in \partial^-(A)} h(e) \leq 0,$$

and so, since $h(v)^+ - h(v)^- \geq 0$ for every $v \in A$, it follows that $S = \emptyset$. A similar argument applies to B and we conclude that $T = \emptyset$ too. Therefore $D(h) = 0$. $\square$

The following result can be deduced from previous proof.

Proposition 1.6 (Tutte [82])**.** *If a graph has a $\mathbb{Z}_k$-NZF (D, f) then it admits a k-NZF (D, f') such that $f'(e) \equiv f(e) \bmod k$, for all $e \in E$.*

We also recall the following theorem claiming that the existence of an A-NZF on a graph does not depend on the group structure of A but on the number of its elements.

Theorem 1.7. *Let A be a finite abelian group and G be a graph. Then G has a nowhere-zero A-flow if and only if G has a nowhere-zero $|A|$-flow.*

Proof. By Lemma 1.4, G has a nowhere-zero A-flow if and only if G has a nowhere-zero $\mathbb{Z}_{|A|}$-flow. By Lemma 1.5, G has a nowhere-zero $\mathbb{Z}_{|A|}$-flow if and only if G has a nowhere-zero $|A|$-flow. □

The problem of finding the minimum k such that a graph G admits a k-NZF is one of the most studied problems in the theory of flows in graphs.

Definition 1.8. Let G be a graph. The *flow number* of G is

$$\phi(G) = \min\{k \in \mathbb{Z} \colon G \text{ has a } k\text{-NZF}\}.$$

It will be clear soon that, if G is bridgeless, $\phi(G)$ is finite, see for example Theorems 1.21 and 1.23. Because of Proposition 1.3 we set $\phi(G) = \infty$, for all graphs G having a bridge.

Now we prove some classical results of the theory of flows in graphs. We recall that a graph is *even* if every vertex has even degree.

Proposition 1.9. *A graph G is even if and only if it has a nowhere-zero 2-flow.*

Proof. If G is even, then the constant function $f(e) = 1$, for all $e \in E$, defines a nowhere-zero $\mathbb{Z}_2$-flow on G. Conversely, having a nowhere-zero 2-flow is equivalent to having a nowhere-zero $\mathbb{Z}_2$-flow. Whence, since every edge has flow value 1, and the sum of all flow values around each vertex is 0 mod 2, it follows that G is even. □

We recall that the *union* of two graphs G and H, is the graph $K = (V(G) \cup V(H), E(G) \cup E(H))$.

Proposition 1.10. *A graph G has a nowhere-zero 4-flow if and only if it is a union of two even graphs.*

Proof. Suppose that $G = H_1 \cup H_2$ is union of even graphs. Proposition 1.9 implies that each H_i admits a nowhere-zero 2-flow f_i. Therefore, the flow $g := (f_1, f_2)$ in G is a nowhere-zero $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-flow, indeed:

- If $e \in H_1 \setminus H_2$ then $g(e) = (1, 0)$;

- If $e \in H_2 \setminus H_1$ then $g(e) = (0,1)$;
- If $e \in H_1 \cap H_2$ then $g(e) = (1,1)$.

So by Theorem 1.7 we have the thesis.

Conversely, let $g\colon E \to \mathbb{Z}_2 \times \mathbb{Z}_2, e \mapsto (f_1(e), f_2(e))$ be a nowhere-zero $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-flow on G. For $i \in \{1,2\}$, let $X_i = \{e \in E\colon f_i(e) \neq 0\}$. Then, $H_i = G[X_i]$ is an even subgraph of G, because f_i is a nowhere-zero $\mathbb{Z}_2$-flow on H_i, and $G = H_1 \cup H_2$ because g is nowhere-zero. □

1.1.1 *Tutte's Conjectures*

In the '60s Tutte proposed some of the most important and well known conjectures in the theory of flows in graphs.

Conjecture 1.11 (3-Flow Conjecture, see unsolved problem 48 in [8])**.** *Every* 4*-edge-connected graph has a nowhere-zero* 3*-flow.*

Conjecture 1.12 (4-Flow Conjecture [85])**.** *Every bridgeless graph containing no subdivision of the Petersen graph has a nowhere-zero* 4*-flow.*

Conjecture 1.13 (5-Flow Conjecture [83])**.** *Every bridgeless graph has a nowhere-zero* 5*-flow.*

In what follows, we recall some of the major results in direction of these outstanding conjectures.

Definition 1.14. Let G be a graph and $v \in V(G)$ a vertex. The *expansion* of v into a new graph H is the operation carried out as follows: delete v from G and replace it by H; moreover, for all edges $vw \in E(G)$, add a new edge connecting w to an arbitrary vertex of H.

We begin with a classical theorem in direction of Tutte's 5-Flow Conjecture.

Theorem 1.15. *Let $k \geq 5$ be an integer. Every bridgeless graph has a nowhere-zero k-flow if and only if every bridgeless cubic graph has a nowhere-zero k-flow.*

Proof. One direction is trivial. So, suppose that every bridgeless cubic graph has a nowhere-zero k-flow and consider a bridgeless graph G. Expand every vertex v of degree higher than 3 into a circuit of length $d_G(v)$ and suppress every vertex of degree 2. The graph H obtained after this procedure is bridgeless and cubic and so has a k-NZF. Therefore, by Proposition 1.3, we can conclude that G has a k-NZF. □

Previous theorem has the following implication: Tutte's 5-Flow Conjecture is equivalent to its restriction to bridgeless cubic graphs. This is the main reason why we will study such class of graphs throughout this dissertation. Tutte proved that edge-colorings are deeply connected with the existence of certain flows in cubic graphs. Recall that, because of Vizing's Theorem [86], the edge-set of a simple k-regular graph can be colored with either k or $k+1$ colors. In the former case, such a graph is said to be *class 1*, otherwise it is said to be *class 2*.

Theorem 1.16 (Tutte [82], [83]). *Let G be a cubic graph.*

- *G is class 1 if and only if G has a 4-NZF;*
- *G is bipartite if and only if G has a 3-NZF.*

Proof. Let G be a cubic graph.

Let G be 3-edge-colorable and $c\colon E \to (\mathbb{Z}_2 \times \mathbb{Z}_2) \setminus (0,0)$ a proper 3-edge-coloring. Then, since elements of the chosen group are self-inverse, we can endow G with any orientation and c turns out to be a nowhere-zero $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-flow on G. So, by Theorem 1.7, G admits a nowhere-zero 4-flow. Conversely the existence of a nowhere-zero 4-flow on G guarantees the existence of a nowhere-zero $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-flow. Therefore, for all $v \in V$, since G is cubic and each element is self inverse, $\partial(v)$ consists of edges with different flow value. Hence G admits a 3-edge-coloring.

Let G be bipartite graph with bipartition $V = S \cup T$. Orient every edge from S to T and let $f(e) = 1 \in \mathbb{Z}_3$ for all $e \in E$. Then G has a 3-NZF. Conversely, fix a nowhere-zero $\mathbb{Z}_3$-flow (D, f) on G, with the property that $f(e) = 1$ for all $e \in \mathbb{Z}_3$. Then D is an orientation having all sources or sinks, i.e. vertices having only outgoing or, respectively, incoming edges. Therefore if S and T are respectively the subsets of sources and sinks, they form a bipartition of G. $\square$

Before stating some further results, we need the following theorem.

Theorem 1.17 (Nash-Williams [60], Tutte [81]). *A graph has k disjoint spanning trees if and only if for every partition $V_1, \dots, V_t$ of V there are at least $k(t-1)$ partition edges.*

Corollary 1.18. *Every $2k$-edge-connected graph has k pairwise disjoint spanning trees.*

Proof. Let $\mathcal{P}$ be a partition of V into t subsets. For all $X \in \mathcal{P}$, $|\partial(X)| \geq 2k$ and so, the number of partition edges is $\frac{1}{2}\sum_{X\in\mathcal{P}} |\partial(X)| \geq kt$. Thus, by previous theorem, the thesis follows. $\square$

Corollary 1.18 plays an important role in the proofs of the following two famous theorems by Jaeger.

Theorem 1.19 (Jaeger [36]). *Every 4-edge-connected graph has a nowhere-zero 4-flow.*

Proof. By Corollary 1.18, G has two edge-disjoint spanning trees T_1 and T_2. Let $\{e^1, \dots, e^t\} = E(G) \setminus E(T_1)$. For each edge $e^j \notin T_1$ there exists a unique circuit C^j inside $T_1 + e^j$ containing e^j. Let $f_1^j(e) = 1$ for all $e \in C^j$ and 0 elsewhere. Moreover, let $f_1 \colon E(G) \to \mathbb{Z}_2, e \mapsto \sum_j f_1^j(e)$ and notice that it defines a $\mathbb{Z}_2$-flow on G. Similarly, we can define a $\mathbb{Z}_2$-flow f_2 following the same procedure with T_2 in place of T_1. Then $f := (f_1, f_2)$ is a nowhere-zero $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-flow in G. □

We define, for a graph G, the *splitting off* operation at a vertex $u \in V$ of degree at least 3 as follows. Let $uv, uw \in \partial(u)$ be two different edges. Delete them and add the new edge vw to G.

We also recall the following result due to Fleischner.

Lemma 1.20 (Fleischner [22]). *Let G be a connected bridgeless graph with a vertex v of degree at least 4 with $u_1, u_2, u_3 \in N(v)$, such that u_1, u_3 belong to different blocks of G if v is a cut-vertex. Moreover, for $j \in \{2, 3\}$, let G_{1j} be the graph obtained by splitting off vu_1 and vu_j at v. Then, at least one between G_{12} and G_{13} is bridgeless and connected. In particular, if v is a cut-vertex, G_{13} has this property.*

Theorem 1.21 (Jaeger [36]). *Every bridgeless graph has a nowhere-zero 8-flow.*

Proof. Let G be a smallest counterexample to the statement. We show that G must be a 3-edge-connected cubic graph. Suppose that G has a 2-edge-cut $\{e_1, e_2\}$ and let H_1, H_2 be the two connected components of $G - e_1 - e_2$. Then the graph $H = G/e_1$ has a nowhere-zero 8-flow (D, f). We can extend this 8-NZF on G by assigning to e_1 the opposite orientation with respect to e_2, i.e. if $e_2 \in \partial^+(V(H_1))$ then $e_1 \in \partial^-(V(H_1))$ and vice versa, and letting $f(e_1) = f(e_2)$. This contradicts the fact that G is a counterexample to the statement and so G must be 3-edge-connected.

On the other hand, since G is bridgeless, it does not have any vertex of degree 1, and, since it is a smallest counterexample, it does not have any vertex of degree 2 as these vertices can be suppressed. Let v be a vertex of degree at least 4 in G. By Lemma 1.20 we can apply the splitting off operation at v in such a way that the resulting graph H' is still bridgeless. Then, since H' has fewer edges than G, it has a 8-NZF. It follows that also G has a 8-NZF and so it is not a smallest counterexample. We conclude that G must be also cubic.

Now, consider the graph $2G$ obtained by substituting every edge of G by a pair of parallel edges. Clearly, $2G$ is 6-regular and 6-edge-connected. Then, by Corollary 1.18, $2G$ has 3 pairwise disjoint spanning trees T_1, T_2 and T_3. They are spanning trees of G as well, but in general they are not disjoint in G. Let us define the $\mathbb{Z}_2$-flows f_i on G, for $i = 1, 2, 3$, exactly in the same way we did in the proof of Theorem 1.19. Namely, let $\{e^1, \dots, e^t\} = E(G) \setminus E(T_1)$ and for each edge $e^j \notin T_1$ let C^j be the unique circuit inside $T_1 + e^j$ containing e^j.

Define $f_1^j(e) = 1$ for all $e \in C^j$ and 0 elsewhere and let $f_1 \colon E(G) \to \mathbb{Z}_2, e \mapsto \sum_j f_1^j(e)$. The flows f_2, f_3 are defined similarly. Then (f_1, f_2, f_3) is a nowhere-zero $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$-flow on G. $\square$

Theorem 1.19 can be seen as an approximation of Tutte's 3-Flow Conjecture. More recently new results towards it have been published, see for example [40], [44], [46], [63] and [78]. Similarly, the best known approximation of Tutte's 5-Flow Conjecture is the famous 6-Flow Theorem by Seymour [68]. We present here a proof of this result that is different from the original one proposed by Seymour and given in [16]. As well as for the 3-Flow Conjecture, there are other recent results towards Tutte's 5-Flow Conjecture that we do not present in this dissertation as we focus on problems of different flavor, see for example [14], [56], [69], [73] and [75].

Let G be a 2-edge-connected graph. For $e_1, e_2 \in E(G)$ we define the following equivalence relation

$$e_1 \sim e_2 \iff e_1 = e_2 \text{ or } \{e_1, e_2\} \text{ is a 2-edge-cut of } G.$$

The equivalence classes with respect to $\sim$ are called *generalized series classes*. Moreover, if f is a function on the edge set of a graph G, we let the *support* of f be $\operatorname{supp}(f) = \{e \in E(G) \colon f(e) \neq 0\}$.

The following lemma plays a central role for the proof of Theorem 1.23.

Lemma 1.22 ([16]). *Let G be an oriented graph with u as a root. Let $S \subseteq \partial(u)$ with $|S| = 2$, $\psi_2 \colon S \to \mathbb{Z}_2$ and $\psi_3 \colon \partial(u) \to \mathbb{Z}_3$ be two functions. Furthermore suppose that*

1. *$d_G(v) = 3$, for every $v \neq u$;*
2. *G is 3-edge-connected;*
3. *$G - u$ is 2-edge-connected;*
4. *$\sum_{e \in \partial^+(u)} \psi_3(e) = \sum_{e \in \partial^-(u)} \psi_3(e)$.*

Then there are two flows $\phi_2 \colon E(G) \to \mathbb{Z}_2$ and $\phi_3 \colon E(G) \to \mathbb{Z}_3$ such that $\phi_2|_S = \psi_2$, $\phi_3|_{\partial(u)} = \psi_3$ and $(\phi_2(e), \phi_3(e)) \neq (0,0)$ for every $e \in E(G) \setminus \partial(u)$.

Proof. We use induction on $|V|$. If $|V| = 2$, then $\partial(u) = E(G)$ and we set $\psi_3 = \phi_3$. Furthermore, choose $\phi_2 \colon E(G) \to \mathbb{Z}_2$ such that $\phi_2|_S = \psi_2$ and $|\operatorname{supp}(\phi_2)|$ is even. Then both ϕ_2 and ϕ_3 are flows in G satisfying the last condition.

Suppose now $|V| > 2$. Choose $uv \in \partial(u) \setminus S$ and define F to be the generalized series class in $G - u$ containing both edges incident to v. Let $G_0, \ldots, G_{k-1}$ be the connected components of $(G - u) - F$, ordered in such a way that G_t is adjacent with G_{t-1} and G_{t+1}, and let $F = \{e_1, e_2, \ldots, e_k\}$ such that e_s connects G_s with G_{s+1}, for each

$0 \le s \le k-1$, where we are doing sums modulo k. In the case where $k = 2$, we let e_0 be different from e_1. Furthermore, assume that G_0 is the component consisting of the only vertex v and that the two edges of S connect u to G_i and G_j in G, with $0 < i \le j$. Denote by G^* the graph obtained from G by shrinking each component G_i. Note that, since $G - u$ is 2-edge-connected, G^* has the structure of a wheel W_k with central vertex u, where multiple edges between u and other vertices are allowed. We can extend the domain of ψ_3 to F in such a way that it becomes a $\mathbb{Z}_3$-flow on G^*. Moreover, adding a suitable constant along F, we can further assume that $\psi_3(e_{i-1}) \neq 0 \neq \psi_3(e_j)$. Then define $\psi_2(e) = 1$ for $e \in F \setminus \{e_{i-1}, e_j\}$.

For every $0 \le s \le k-1$, construct the graph G_s^+ by identifying all vertices of $V(G) \setminus V(G_s)$ to a single node u_s. In each G_s^+, ψ_3 satisfies the flow conservation law at u_s.

If $i = j$, then both edges of S end in G_i. We apply the inductive hypothesis to the graph G_i^+ together with the set S and the functions $\psi_2|_S$ and $\psi_3|_{\partial(u_i)}$ obtaining a pair of flows $\phi_2^i \colon E(G_i^+) \to \mathbb{Z}_2$ and $\phi_3^i \colon E(G_i^+) \to \mathbb{Z}_3$ on G_i^+ (that satisfy the thesis of this Lemma on G_i^+). Set $\psi_2(e_{i-1}) := \phi_2^i(e_{i-1})$ and $\psi_2(e_i) := \phi_2^i(e_i)$. Then we apply again the inductive hypothesis on every G_k^+, $k \neq i$, together with the new set $\{e_{k-1}, e_k\}$ in place of S and the appropriate restrictions of ψ_2 and ψ_3, and we combine the flows obtained as solutions in such a way that two flows on the original graph are constructed. By construction they extend ψ_2 and ψ_3, satisfy the non-zero condition at $E(G) \setminus \partial(u)$ and verify the flow conservation law everywhere.

If $i < j$ we apply the inductive hypothesis to the graph G_i^+ together with the edges e_i and the edge of S incident to G_i, and with the appropriate restrictions to these sets of ψ_2 and ψ_3. Again we are left with two flows $\phi_2^i \colon E(G_i^+) \to \mathbb{Z}_2$ and $\phi_3^i \colon E(G_i^+) \to \mathbb{Z}_3$. Apply induction once again to G_j^+ together with the edges e_{j-1} and the edge of S incident to G_j, and with the appropriate restrictions to these sets of ψ_2 and ψ_3. We have two more solutions $\phi_2^j \colon E(G_j^+) \to \mathbb{Z}_2$ and $\phi_3^j \colon E(G_j^+) \to \mathbb{Z}_3$. Set $\psi_2(e_{i-1}) := \phi_2^i(e_{i-1})$ and $\psi_2(e_j) := \phi_2^j(e_j)$. Then we proceed as above by applying the inductive hypothesis to every other graph G_k^+, $k \notin \{i, j\}$, with $\{e_{k-1}, e_k\}$ in place of S and the right restrictions of ψ_2 and ψ_3. Again we can construct a solution for the original graph by combining the obtained flows. □

Theorem 1.23 (Seymour [68]). *Every bridgeless graph has a nowhere-zero 6-flow.*

Proof. Consider the family of counterexamples to the statement having minimum number of edges contained in a 2-edge-cut. Let G be the graph among all these counterexamples minimizing $\sum_{v \in V_{>3}} (d_G(v) - 3)$, where $V_{>3}$ is the set of vertices of degree higher than 3.

If G has a 2-edge-cut $\{e_1, e_2\}$, then $G_1 = G/e_1$ is still a bridgeless graph and, by minimality of G, has a 6-NZF f. Then G also has a 6-

NZF g by letting $g|_{E(G_1)} = f$ and $g(e_1) = f(e_1)$, where e_1 has opposite orientation with respect to e_2 (in the sense that e_1 points outward a component of $G - \{e_1, e_2\}$ if and only if e_2 points inward that same component). Hence G has no 2-edge-cuts.

If v is a cut-vertex in G, then G is a union of subgraphs H, K such that $E(H) \cap E(K) = \emptyset$ and $V(H) \cap V(K) = \{v\}$. By minimality, both H and K have a nowhere-zero 6-flow, and hence G also has one. Thus, G cannot contain any cut-vertex.

Suppose that G has a vertex v such that $d_G(v) > 3$. Then we can expand v into a cycle of length $d_G(v)$ in such a way that each new vertex is connected to exactly one edge of $\partial_G(v)$. Notice that this operation does not create bridges and, since v is not a cut-vertex, it does not generate new 2-edge-cuts. Therefore, by our second minimality criteria the new expanded graph has a 6-NZF and so G has a 6-NZF as well. We conclude that G must be a 3-edge-connected cubic graph.

Choose an arbitrary vertex u and two functions ψ_2 and ψ_3 such that $\mathrm{supp}(\psi_2) \cup \mathrm{supp}(\psi_3) = \partial(u)$. We are in the hypothesis of Lemma 1.22, thus we can apply it and obtain two flows ϕ_2 and ϕ_3 which combine to a nowhere-zero $(\mathbb{Z}_2 \times \mathbb{Z}_3)$-flow (ϕ_2, ϕ_3) on G. Hence G admits a nowhere-zero 6-flow. A contradiction. □

We conclude here the introduction of integer flows and present circular flows in the next section. Other papers treating integer flows that have not been mentioned yet in this first chapter are for example [18], [45], [54], [84] and [89].

1.2 CIRCULAR FLOWS AND CIRCULAR FLOW NUMBER

Circular flows on graphs have been introduced in [26] as a generalization of integer flows on graphs. This dissertation is mainly focused on the study of such objects.

Circular flows are defined in the same way as integer flows but for the fact that the flow function is a real-valued function.

Definition 1.24. Let $r \geq 2$ be a real number. A *circular nowhere-zero r-flow*, or r-CNZF, on a graph G is a pair (D, f) where D is an orientation of G and $f \colon E \to \mathbb{R}$ such that, for all $e \in E$, $|f(e)| \in [1, r-1]$, and such that for all $v \in V$ equation (1) holds.

Similarly to the case of integer flows, we are interested in studying the following parameter.

Definition 1.25. The *circular flow number* of a graph G is the parameter

$$\phi_c(G) = \inf\{r \in \mathbb{R} \colon \text{ G has an } r\text{-CNZF}\}.$$

Since a k-NZF is also a k-CNZF, we have that $\phi_c(G) \leq \phi(G)$, and so, by Theorem 1.23, $\phi_c(G) \leq 6$, for all bridgeless graphs G. It is also well known that there are graphs such that $\phi_c(G) < \phi(G)$. Similarly to what we previously said, if a graph has a bridge, we set its circular flow number to be ∞.

In what follows we recall some important properties of the circular flow number of a graph. First we need the following result due to Hoffman.

Theorem 1.26 (Hoffman [31]). *Let G be an oriented graph and let $l, u\colon E \to \mathbb{R}^+$ be two functions such that $l(e) \le u(e)$, for all $e \in E$. Then the following statements are equivalent:*

- *there is a flow $f\colon E \to \mathbb{R}$ on G such that $l(e) \le f(e) \le u(e)$;*
- *for all $X \subseteq V$, $\sum_{e\in\partial^+(X)} u(e) \ge \sum_{e\in\partial^-(X)} l(e)$.*

Thank to this well known theorem we can prove the following fact about the circular flow number of bridgeless graphs, that was originally stated in [26] in the more general language of matroids. Recall that the *complement* of a subset of vertices $X \subseteq V(G)$ of a graph G is the set $\bar{X} = V(G) \setminus X$.

Theorem 1.27 (Goddyn et al. [26]). *An oriented graph G has a r-CNZF if and only if*

$$\frac{1}{r-1} \le \frac{|\partial^+(X)|}{|\partial^-(X)|} \le r-1,$$

for all $X \subseteq V$.

Proof. Apply Theorem 1.26 with two functions l, u on E such that, $u(e) = r-1$ and $l(e) = 1$, for all $e \in E$. Then, there is an r-CNZF (D, f) on G having positive flow values if and only if, for all X,

- $(r-1)|\partial^+(X)| = \sum_{e\in\partial^+(X)}(r-1) \ge \sum_{e\in\partial^-(X)} 1 = |\partial^-(X)|$;
- $(r-1)|\partial^-(X)| = \sum_{e\in\partial^+(\bar{X})}(r-1) \ge \sum_{e\in\partial^-(\bar{X})} 1 = |\partial^+(X)|$.

The last two conditions are equivalent to

$$\frac{1}{r-1} \le \frac{|\partial^+(X)|}{|\partial^-(X)|} \le r-1,$$

for all $X \subseteq V$. □

Previous theorem implies that the circular flow number of a bridgeless graph G can be computed as a fraction as the minimum over all orientations of G of the quantities $1 + \max\left\{\frac{|\partial^+(X)|}{|\partial^-(X)|} : X \subseteq V\right\}$. The following corollary holds.

Corollary 1.28 (Goddyn et al. [26]). *For all bridgeless graphs G,*

$$\phi_c(G) = \min_{\text{orientations of } G} \max\left\{\frac{|\partial(X)|}{|\partial^-(X)|} : X \subseteq V\right\}.$$

In particular, $\phi_c(G)$ is a minimum and a rational number for all bridgeless graphs G.

Notice that if G has a r-CNZF then G has an r'-CNZF for all $r' > r$. Moreover, if G has a t-CNZF with t irrational number then there is a rational $t' < t$ such that G has a t'-CNZF, since the circular flow number is rational. We end this section with the following results on the structure of circular flows on graphs.

Lemma 1.29. *Let G be a graph and $r \geq 2$ a rational number. The following are equivalent.*

1. *G has an r-CNZF;*
2. *G has an r-CNZF (D, f) such that $f\colon E \to \mathbb{Q}$.*

Proof. Point 2. implies point 1. by definition. Let (D, f) be an r-CNZF on G having positive flow values and let $F = \{e \in E\colon f(e) \in \mathbb{R} \setminus \mathbb{Q}\}$. For $e \in E$, let $m(e) = \min\{f(e) - 1, r - f(e) - 1\}$ and let m be the irrational number defined as $\min\{m(e)\colon e \in F\}$. Moreover let $\tilde{e} \in F$ be an edge such that $m(\tilde{e}) = m$. There is a circuit $\tilde{C} \subseteq F$ containing $\tilde{e}$ because F cannot contain leaves. If $m(\tilde{e}) = f(\tilde{e}) - 1$ (resp. $r - f(\tilde{e}) - 1$) we orient $\tilde{C}$ in such a way that it is a directed circuit and the orientation of $\tilde{e}$ is opposite (resp. the same) with respect to D. Then, adding $m(e)$ along $\tilde{C}$ we get an r-CNZF with one irrational flow value less. We can repeat this procedure and finally construct an r-CNZF having all rational flow values. □

Theorem 1.30 (Steffen [71])**.** *Let G be a graph and $k \geq 2$ a positive integer. The following statements are equivalent.*

1. *there are two integers $1 \leq p < q$ such that $1 < \frac{q}{p} + 1 \leq k$ and G has a $(1 + \frac{q}{p})$-CNZF;*
2. *there two integers $1 \leq p < q$ such that $1 < \frac{q}{p} + 1 \leq k$ and G has a $(1 + \frac{q}{p})$-CNZF (D, f), with the property that, for all $e \in E$, there is a positive integer n such that $f(e) = \frac{n}{p}$;*
3. *G has a k-NZF.*

Proof. Points 2. and 3. imply point 1.

Let (D, f) be a $(1 + \frac{q}{p})$-CNZF on G with $1 < \frac{q}{p} + 1 \leq k$. We show that point 3. follows. By Lemma 1.29 we can choose (D, f) having all rational flow values. We proceed as previous proof by letting $F = \{e \in E\colon f(e) \in \mathbb{Q} \setminus \mathbb{Z}\}$ and, for all $e \in E$, $m(e) = \min\{f(e) - 1, k - f(e) - 1\}$. Moreover, we choose $\tilde{e}$ to be such that $m(\tilde{e})$ is minimum over the edges of F. There is a circuit $\tilde{C} \subseteq F$ containing $\tilde{e}$. If $m(\tilde{e}) = f(\tilde{e}) - 1$ (resp. $k - f(\tilde{e}) - 1$) we add it along the circuit $\tilde{C}$ in the opposite (resp. the same) orientation of $\tilde{e}$. Therefore, we get a new flow with one non-integer flow value less. Repeating this procedure we can construct a k-NZF on G.

Point 2. can be proved in the same way by letting F be the set of edges not being a fraction of the form $\frac{n}{p}$. □

Immediate consequence of previous theorem is the fact that, for all bridgeless graphs,

$$\phi(G) = \lceil \phi_c(G) \rceil, \tag{2}$$

and so circular flows can be interpreted as refinement of integer flows.

1.3 BALANCED VALUATIONS AND BISECTIONS OF CUBIC GRAPHS

Bondy [7] and Jaeger [35] introduced the concept of a balanced valuation. This is a valuable tool that can be used to study circular flows on graphs, and sometimes, when studying flows, it will be more convenient to work with balanced valuations instead. We will mostly make use of them in Chapters 3 and 4. In this section we discuss the relation between such objects and nowhere-zero flows.

We begin with the following result by Hakimi. The proof we propose can be found in [23] and makes use of submodular functions, i.e. set functions g such that $g(X) + g(Y) \geq g(X \cap Y) + g(X \cup Y)$.

Theorem 1.31 (Hakimi [29])**.** *Let G be a graph and $m\colon V \to \mathbb{Z}$ a function such that $\sum_{v \in V} m(v) = |E|$. The following statements are equivalent:*

i) G has an orientation such that, for all $v \in V$, $d^+(v) = m(v)$;

ii) for all $X \subseteq V$, $\sum_{v \in X} m(v) \geq |E(X)|$.

Proof. Suppose that G has such an orientation. Then, for all $X \subseteq V$,

$$\sum_{v \in X} m(v) = \sum_{v \in X} d^+(v) = |E(X)| + |\partial^+(X)| \geq |E(X)|.$$

On the other hand, assume condition *ii)* holds and let g be the set function defined as follows: for all $X \subseteq V$, $g(X) = |E(X)| + |\partial(X)|$. First of all notice that, for every $X \subseteq V(G)$,

$$g(X) = |E(G)| - |E(\bar{X})| \geq |E(G)| - \sum_{v \in \bar{X}} m(v) = \sum_{v \in X} m(v). \tag{3}$$

Claim 1.32. *For all $X, Y \subseteq V$, $g(X) + g(Y) \geq g(X \cap Y) + g(X \cup Y)$.*

Proof of Claim 1.32. The statement follows from the fact that an edge uv such that $u \in X \setminus Y$ and $v \in Y \setminus X$ is counted twice on the left-hand side but only once on the right-hand side of the inequality. □

Call *tight* any set $X \subseteq V$ such that $g(X) = \sum_{v \in X} m(v)$. For example, $V(G)$ and $\emptyset$ are tight.

Claim 1.33. *The intersection and the union of two tight sets is a tight set.*

Proof of Claim 1.33. Let X, Y be two tight sets. We have the following chain of equalities and inequalities

$$\begin{aligned}\sum_{v \in X} m(v) + \sum_{v \in Y} m(v) &= g(X) + g(Y) \\ &\geq g(X \cap Y) + g(X \cup Y) \\ &\geq \sum_{v \in X \cap Y} m(v) + \sum_{v \in X \cup Y} m(v) \\ &= \sum_{v \in X} m(v) + \sum_{v \in Y} m(v).\end{aligned}$$

Therefore, all inequalities must be equalities and we get $g(X \cap Y) + g(X \cup Y) = \sum_{v \in X \cap Y} m(v) + \sum_{v \in X \cup Y} m(v)$. Since $g(Z) \geq \sum_{v \in Z} m(v)$ for all $Z \subseteq V(G)$, we conclude that $X \cup Y$ and $X \cap Y$ must be tight. □

Now we proceed with the proof of the main statement. We argue by induction on $\sum_{v \in V} m(v) = \tilde{m}$. If $\tilde{m} = 0$ the statement is true. So suppose that there is $w \in V$ such that $m(w) > 0$. By Claim 1.33 there is a unique largest tight subset $X \subseteq V$ such that $w \notin X$. There exists $u \notin X$ such that $uw \in E(G)$. Indeed, otherwise we would get the following contradiction with (3): $g(X + w) = g(X) = \sum_{v \in X} m(v) = \left(\sum_{v \in X} m(v) + m(w)\right) - m(w) < \sum_{v \in X} m(v) + m(w)$.

Let $H := G - uw$ and let $m' \colon V \to \mathbb{Z}$ be such that $m'(v) = m(v)$, for all $v \in V \setminus \{w\}$ and $m'(w) = m(w) - 1$.

H meets condition *ii)* together with the new function m'. Otherwise, if there is $Y \subseteq V$ violating condition *ii)* in H, then $w \in Y$ and $u \notin Y$. Moreover $|E(Y)| > \sum_{v \in Y} m'(v)$ implies that $|E(\bar{Y})| + |\partial_H(\bar{Y})| = |E(H)| - |E(Y)| = \sum_{v \in V} m'(v) - |E(Y)| < \sum_{v \in \bar{Y}} m'(v) = \sum_{v \in \bar{Y}} m(v)$. Therefore we conclude that $\bar{Y}$ is a tight set in G with respect to the function m. Since X is the largest tight set not containing w we get $\bar{Y} \subseteq X$, in contradiction with the fact that $u \notin X$.

Hence, by induction, H has an orientation such that $d^+(v) = m'(v)$ for all $v \in V$, and so we can construct the desired orientation on G by orienting the edge uw from w to u. □

Definition 1.34. A *balanced valuation* of a graph G is a function $\omega \colon V \to \mathbb{R}$ such that, for all $X \subseteq V$,

$$\Big|\sum_{v \in X} \omega(v)\Big| \leq |\partial(X)|.$$

Proposition 1.35 (Jaeger [35])**.** *Let G be a graph and $m \colon V \to \mathbb{Z}$ a function with non-negative values. There is an orientation of G such that $d^+(v) = m(v)$ for every $v \in V$ if and only if the function $\omega \colon V \to \mathbb{R}, v \mapsto 2m(v) - d(v)$ is a balanced valuation.*

Proof. By Theorem 1.31, G has and orientation such that $d^+(v) = m(v)$ for every $v \in V$ if and only if $\sum_{v \in V} m(v) = |E|$ and, for all

$X \subseteq V, \sum_{v \in X} m(v) \geq |E(X)|$. These conditions are equivalent to the following inequalities

$$\text{for all } X \subseteq V, |E(X)| \leq \sum_{v \in X} m(v) \leq |E(X)| + |\partial(X)|.$$

Since, for every subset of vertices X, $\sum_{v \in X} d(v) = 2|E(X)| + |\partial(X)|$, previous inequalities hold if and only if,

$$\text{for all } X \subseteq V, -|\partial(X)| \leq 2 \sum_{v \in X} m(v) - \sum_{v \in X} d(v) \leq |\partial(X)|,$$

that is, ω is a balanced valuation. □

The following theorem relates balanced valuations to nowhere-zero flows.

Theorem 1.36 (Jaeger [35]). *Let G be a graph and $r > 2$ a real number. Then G has an r-CNZF if and only if there is a balanced valuation $\omega\colon V \to \mathbb{R}$ of G such that, for every $v \in V$ there is an integer k_v such that $k_v \equiv d(v) \bmod 2$ and $\omega(v) = k_v \frac{r}{r-2}$.*

Proof. By Theorem 1.26, a graph G with orientation D has an r-CNZF if and only if for all X,

$$\begin{cases} (r-1)|\partial^+(X)| \geq |\partial^-(X)|; \\ (r-1)|\partial^-(X)| \geq |\partial^+(X)|. \end{cases} \tag{4}$$

Notice that the first inequality holds if and only if, for all $X \subseteq V$,

$$(r-1)|\partial^+(X)| - |\partial^-(X)| \geq |\partial^-(X)| - (r-1)|\partial^+(X)|$$

if and only if

$$(r-2)\big(|\partial^+(X)| + |\partial^-(X)|\big) \geq r\big(|\partial^-(X)| - |\partial^+(X)|\big).$$

If we apply a similar argument to the second inequality, we get that conditions (4) holds if and only if, for all $X \subseteq V$,

$$\frac{r}{r-2}\big||\partial^+(X)| - |\partial^-(X)|\big| \leq |\partial^+(X)| + |\partial^-(X)| = |\partial(X)|.$$

Now, notice that, for all $X \subseteq V$, $|\partial^+(X)| - |\partial^-(X)| = \sum_{v \in X}(2d_D^+(v) - d_G(v))$, and so the oriented graph D has an r-CNZF if and only if the function $\omega\colon V \to \mathbb{R}$ such that,

$$\omega(v) = \frac{r}{r-2}(2d_D^+(v) - d_G(v)), \text{ for all } v \in V$$

is a balanced valuation.

On the other hand, let ω be a balanced valuation of G with the required properties. Then, there is a non-negative integer function $m\colon V \to \mathbb{Z}$ such that for all $v \in V$, $\omega(v) = \frac{r}{r-2}(2m(v) - d(v))$, where

$m(v) = \frac{k_v + d(v)}{2}$. Then, the function $\tilde{\omega}(v) = 2m(v) - d(v)$, for all $v \in V$, is also a balanced valuation because, for all $X \subseteq V$,

$$\big|\sum_{v\in X} \tilde{\omega}(v)\big| \leq \frac{r}{r-2}\big|\sum_{v\in X} \tilde{\omega}(v)\big| = \big|\sum_{v\in X} \omega(v)\big| \leq |\partial(X)|.$$

Therefore, by Proposition 1.35, G has an orientation D such that $d^+(v) = m(v)$, for all $v \in V$ and so the balanced valuation ω is such that

$$\omega(v) = \frac{r}{r-2}(2d^+_D(v) - d_G(v)) \tag{5}$$

for all vertices. We already proved that this is equivalent to the fact that G has an r-CNZF (D, f). □

1.3.1 *Bisections of cubic graphs*

As we already mentioned, cubic graphs play a special role in the study of flows. In this subsection we discuss balanced valuations of cubic graphs. We remark that all values of a balanced valuation ω are bounded by the degree of a vertex, namely $|\omega(v)| \leq d(v)$, for every vertex v. Thus, balanced valuations of cubic graphs are such that $|\omega(v)| \leq 3$. In particular, by Theorem 1.36, the existence of an r-CNZF on a cubic graph is equivalent to the existence of a balanced valuation with values in $\{\pm\frac{r}{r-2}\}$.

Consider a cubic graph G with such a balanced valuation ω. Its vertex set is naturally partitioned into two subsets $V = \mathcal{B} \cup \mathcal{W}$ defined as follows:

$$\mathcal{W} = \left\{v \in V \colon \omega(v) = \frac{r}{r-2}\right\} \text{ and } \mathcal{B} = \left\{v \in V \colon \omega(v) = -\frac{r}{r-2}\right\}.$$

Vertices of $\mathcal{W}$ are called *white* whereas, vertices of $\mathcal{B}$ are called *black*.

From equation (5) we further get that there is an orientation of G such that

$$d^+(v) = \begin{cases} 1 & \text{if } v \in \mathcal{B}; \\ 2 & \text{if } v \in \mathcal{W}. \end{cases}$$

Moreover note that, all connected components of $G[\mathcal{B}]$ as well as $G[\mathcal{W}]$ are trees. Indeed, if G contains a monochromatic circuit C, then we get the following contradiction $|\sum_{v\in V(C)} \omega(v)| = \frac{r}{r-2}|V(C)| > |\partial(V(C))|$.

These observations motivate the following definition.

Definition 1.37. Let $k \geq 2$ be an integer. A *k-bisection* $(\mathcal{B}, \mathcal{W})$ of a cubic graph G is a bipartition of its vertex set $V = \mathcal{B} \cup \mathcal{W}$ such that $|\mathcal{B}| = |\mathcal{W}|$ and all connected components of both subgraphs $G[\mathcal{B}]$ and $G[\mathcal{W}]$ are trees on at most k vertices. Such components are called *monochromatic components.*

It can be proved that, if G is such that $\phi_c(G) < 5$, then the balanced valuation corresponding to a $\phi_c(G)$-CNZF induces a 2-bisection: consider such a balanced valuation $\omega\colon V \to \{\pm\frac{\phi_c(G)}{\phi_c(G)-2}\}$; as we already proved that G has no monochromatic circuit, we just need to show that every monochromatic path of G is on at most 2 vertices. Let T be a monochromatic path, then $\frac{\phi_c(G)}{\phi_c(G)-2}|V(T)| \leq |\partial(T)| = |V(T)| + 2$, and so $|V(T)| \leq \phi_c(G) - 2 < 3$. Moreover it is also well known that the Petersen graph admits a 3-bisection but no 2-bisections. It is proved in [20] that every bridgeless cubic graph has a 3-bisection and further papers, where the existence of bisections in cubic graphs has been studied, are for instance [2] and [77].

1.4 flows on cubic graphs

By Theorem 1.15, Tutte's 5-Flow Conjecture is equivalent to its restriction to cubic graphs. In particular, Theorem 1.16 shows that a counterexample needs to be a class 2 cubic graph. It is well known that many other conjectures in graph theory, such as, for example, the Berge-Fulkerson Conjecture [24] and the Cycle Double Cover Conjecture [67], [76], are equivalent to their restriction to class 2 cubic graphs (see [21] for a survey on these topics). This motivates the study of the structure of such graphs, that are commonly known as *snarks*. In the literature, see for example [51], [52], [61], [70], [72] one can find different definitions of snarks, depending on the taste and needs of authors: sometimes further restrictions on the girth and cyclic edge-connectivity are required in order to avoid considering trivial cases. There are indeed easy arguments that allow to exclude small induced cycles and edge-cuts from a smallest counterexample to some of the above mentioned conjectures. In this dissertation, a snark will be a graph having the properties presented in the definition below. Recall that a graph G is *cyclically k-edge-connected* if G has no cycle-separating k-edge-cut.

Definition 1.38. A *snark* is a cyclically 4-edge-connected cubic graph of class 2 with girth at least 5.

It is well known, indeed, that a smallest counterexample to the 5-Flow Conjecture must be a snark, and, in the first part of this section we summarize the main results that show this fact. Though, we remark that more restrictive, but also much more technical, properties of a smallest counterexample are known, see Theorem 1.42.

1.4.1 *Smallest counterexamples to the 5-Flow Conjecture*

Here we study the structure of a smallest counterexample H to Conjecture 1.13. As we did in the first part of the proof of Theorem 1.21, we first show that H must be a 3-edge-connected cubic graph.

Indeed, H is bridgeless, thus has not degree 1 vertices and, since it is a smallest counterexample it neither has degree 2 vertices, that can be suppressed. So the minimum degree of H is $\delta(H) \geq 3$. Moreover it must be connected. Suppose that there is $u \in V(H)$ such that $d(u) \geq 4$. By Lemma 1.20 we can apply the splitting off operation at u in such a way that the resulting graph H' is still bridgeless. If H' has a 5-NZF then also H has one, and so it is not a smallest counterexample.

It is not hard to prove that H is 3-edge-connected. Indeed suppose that it has a 2-edge-cut $F = \{e_1, e_2\}$ such that $H - F$ has components H_1 and H_2. Since H is a smallest counterexample, H/e_1 has a 5-NZF (D, f). This 5-flow can be extended to H by orienting e_1 the opposite way with respect to e_2, i.e. if $e_2 \in \partial^+(V(H_1))$ then $e_1 \in \partial^-(V(H_1))$ and vice versa, and setting $f(e_1) = f(e_2)$.

In order to study cyclic edge-cuts of higher order we make use of *networks*, that are graphs G having prescribed vertices $U \subseteq V$ called *terminals* and denoted by the pair (G, U).

Let $\mathcal{A}_n = \{(s_1, \ldots, s_n)\colon$ for all $i = 1, \ldots, n, s_i \neq 0$ and $\sum_{i=1}^n s_i = 0 \in \mathbb{Z}_5\}$. Consider an oriented graph G with terminals $U = \{u_1, \ldots, u_n\}$ and let $s \in \mathcal{A}_n$, we define $F_{(G,U)}(s)$ as the number of $\mathbb{Z}_5$-NZF f on G such that $s = (f(u_1), \ldots, f(u_n))$.

Lemma 1.39. *Let G be an oriented graph with $U = \{u_1, u_2, u_3\} \subseteq V$ as terminal vertices such that $d^+(u_i) = 0$ and $d^-(u_i) = 1$ for all i. There is a non negative integer k such that, for all $s \in \mathcal{A}_3$, $F_{(G,U)}(s) = k$.*

Proof. We use induction over $|E| = m$. If $m = 3$ we have the following two cases:

- G is union of three isolated edges or of one isolated edge and a path on three vertices. In such cases, for all $s \in \mathcal{A}_3$, $F_{(G,U)}(s) = 0$, as we have at least one internal degree 1 vertex;

- G is a claw. In this case, for all $s \in \mathcal{A}_3$, $F_{(G,U)}(s) = 1$.

If $m \geq 4$ there is an edge $xy \in E$ such that $x, y \notin U$. By inductive hypothesis there are non negative integers k_1 and k_2 such that, for all $s \in \mathcal{A}_3$, $F_{(G-xy,U)}(s) = k_1$ and $F_{(G/xy,U)}(s) = k_2$. Therefore, for all $s \in \mathcal{A}_3$, we get that

$$\begin{aligned} F_{(G,U)}(s) &= F_{(G,U)}(s) + F_{(G-xy,U)}(s) - F_{(G-xy,U)}(s) \\ &= F_{(G/xy,U)}(s) - F_{(G-xy,U)}(s) \\ &= k_2 - k_1, \end{aligned}$$

and the thesis follows. □

Theorem 1.40. *Let H be a smallest counterexample to Conjecture 1.13. Then H is cyclically 4-edge-connected.*

Proof. Suppose that H has not a $\mathbb{Z}_5$-NZF and has a cyclic 3-edge-cut C. Let H_1 and H_2 be the two connected components of the graph $H - C$ and let x_1^j, x_2^j, x_3^j be the vertices of degree 2 in H_j, $j \in \{1,2\}$, such that $x_i^1 x_i^2 \in C$, for all $i \in \{1,2,3\}$. Let (G_j, U_j), $j \in \{1,2\}$, be the network constructed as follows: add the set of terminal vertices $U_j = \{u_1^j, u_2^j, u_3^j\}$ and all edges $x_i^j u_i^j$, $i \in \{1,2,3\}$, to H_j. Orient both G_j in such a way that all vertices of U_1 and U_2 have positive indegree and positive outdegree respectively.

By Lemma 1.39, for all $j \in \{1,2\}$, there is k_j such that, for all $s \in \mathcal{A}_3$, $F_{(G_j,U_j)}(s) = k_j$.

If $k_1, k_2 > 0$, fix a $\mathbb{Z}_5$-NZF f_j on G_j such that $(f_j(u_1^j), f_j(u_2^j), f_j(u_3^j)) = s$, for a suitable $s \in \mathcal{A}_3$. Notice that a $\mathbb{Z}_5$-NZF can be constructed in H by identifying the vertex u_i^1 with u_i^2 for all $i \in \{1,2,3\}$ and then suppressing all degree 2 vertices created with this procedure.

Thus, without loss of generality $k_1 = 0$. This means that the graph $H' = H/H_2$ has no $\mathbb{Z}_5$-NZF as well. Therefore, in both cases we conclude that G is not a smallest counterexample to Tutte's 5-Flow Conjecture, that is a contradiction. $\square$

Theorem 1.41. *Let H be a smallest counterexample to Conjecture 1.13. Then H has girth at least* 5.

Proof. We already proved that H is a cyclically 4-edge-connected cubic graph, therefore H has not cycles of length 2 or 3. If, by contradiction, H has a 4-cycle $Q = v_1v_2v_3v_4$, consider the graph H/Q. Since H/Q is smaller than H, it is not a counterexample to Tutte's 5-Flow Conjecture and so we can fix on it a $\mathbb{Z}_5$-NZF (D', f') such that all edges in $\partial_{H/Q}(v_Q)$ point towards v_Q, where v_Q is the unique vertex of degree 4 of H/Q. Set on H a $\mathbb{Z}_5$-flow (D, f) such that $D|_{E(H-Q)} = D'$, $f|_{E(H-Q)} = f'$, $f|_{E(Q)} = 0$ and orient the edge $v_iv_{i+1} \in E(Q)$ from v_i to v_{i+1}, where we sum indices modulo 4. We are going to turn it into a $\mathbb{Z}_5$-NZF on H. Let e_i be the edge of $\partial(Q)$ adjacent to v_i, $i \in \{1,2,3,4\}$. There is $a \in \mathbb{Z}_5 \setminus \{0, -f(e_1), -(f(e_1) + f(e_2)), -(f(e_1) + f(e_2) + f(e_3))\}$. Then we modify f on $E(Q)$ as follows and get a $\mathbb{Z}_5$-NZF on H: $f(v_4v_1) = a, f(v_1v_2) = a + f(e_1), f(v_2v_3) = a + f(e_1) + f(e_2), f(v_3v_4) = a + f(e_1) + f(e_2) + f(e_3)$. $\square$

Further but more technical reductions on the cyclic connectivity and the girth have been done by Kochol, see for instance [41], [42] and [43]. Namely, he proved that a smallest counterexample to the 5-Flow Conjecture is cyclically 6-edge-connected and has girth at least 11.

Theorem 1.42 (Kochol [41],[43]). *A minimal counterexample to Tutte's 5-Flow Conjecture is a cyclically* 6*-edge-connected class* 2 *cubic graph of girth at least* 11.

1.4.2 *Circular flow number of cubic graphs*

Previous observations and results are the main motivations of the study of the existence of nowhere-zero flows on cubic graphs and snarks. In what follows we recall the main results on this topic.

If a graph G has a $(2t+1)$-edge-cut then it follows that $\phi_c(G) \geq 2+\frac{1}{t}$. Therefore, for $t=1$, we get that the circular flow number of a cubic graph is not smaller than 3. Steffen proved the following result for regular graphs of odd degree.

Proposition 1.43 (Steffen [71])**.** *A $(2t+1)$-regular graph G is bipartite if and only if $\phi_c(G) = 2+\frac{1}{t}$. Moreover, if G is not bipartite, $\phi_c(G) \geq 2+\frac{2}{2t-1}$.*

Proof. Let G be a $(2t+1)$-regular graph.

First of all notice that a graph with a $(2t+1)$-edge-cut has circular flow number at least $2+\frac{1}{t}$. Hence $\phi_c(G) \geq 2+\frac{1}{t}$.

If G is bipartite, such that $V = S \cup T$, then G is class 1, meaning that E has a partition into perfect matchings $M_1, M_2, \ldots, M_{2t+1}$. Let $\tilde{M} = \cup_{i=1}^{t} M_i$ and, for all $e \in E$, orient e from S to T if $e \in \tilde{M}$, and from T to S otherwise. The function $f\colon E \to \mathbb{R}$ such that,

$$f(e) = \begin{cases} 1 & \text{if } e \notin \tilde{M}; \\ 1+\frac{1}{t} & \text{if } e \in \tilde{M}, \end{cases}$$

together with the chosen orientation, defines a $(2+\frac{1}{t})$-CNZF on G.

Conversely, if G has a $(2+\frac{1}{t})$-CNZF, by Theorem 1.36, G has a balanced valuation ω such that $\omega(v) \in \{\pm(2t+1)\}$ for all $v \in V$. We show that the subsets S and T consisting of those vertices having positive, and respectively negative, valuation form a bipartition of G. Indeed if $uv \in E$ and $\omega(u) = \omega(v)$, then we get the following contradiction

$$|\omega(u) + \omega(v)| = 4t+2 > 4t \geq |\partial(\{u,v\})|.$$

Using a similar argument we conclude the proof of the theorem by showing that, if $\phi_c(G) = r < 2+\frac{2}{2t-1} = \frac{4t}{2t-1}$ then G is bipartite. Theorem 1.36 implies that G has a balanced valuation ω such that, for all $v \in V$, there is an odd integer k_v such that $\omega(v) = k_v\frac{r}{r-2}$. Since $2+\frac{1}{t} \leq r < 2+\frac{2}{2t-1}$, we get that $2t < \frac{r}{r-2} \leq 2t+1$, and so $k_v \in \{\pm 1\}$. As we previously did, let S be the subset of vertices with positive valuation and T its complement. Then, if $uv \in E$ and $\omega(u) = \omega(v)$ we get the following contradiction:

$$|\omega(u) + \omega(v)| > 4t \geq |\partial(\{u,v\})|. \qquad \square$$

If we focus on cubic graphs, Theorem 1.43 shows that there is no cubic graph with circular flow number inside $(3,4)$, namely, together with Theorem 1.16 we conclude that, for a cubic graph G:

- G is bipartite if and only if $\phi_c(G) = 3$;
- G is class 1 non-bipartite if and only if $\phi_c(G) = 4$;
- G is class 2 if and only if $\phi_c(G) > 4$.

A natural question at this point is the following.

Problem 1.44 (Pan, Zhu [62]). *Is it true that for all rational numbers $q \in (4,5)$ there is a cubic graph G having $\phi_c(G) = q$?*

This question appears in a paper of Pan and Zhu [62] where they proved that for all rational numbers $q \in [2,5]$ there is a graph with q as circular flow number. A positive answer was given by Lukot'ka and Škoviera in [49] where, for all $q \in (4,5)$, they constructed infinitely many snarks with q as circular flow number. Moreover, it is well known that $\phi_c(P_{10}) = 5$, where P_{10} is the Petersen graph.

1.5 FLOWS AND ORIENTATIONS

Let G be a graph and $k \geq 2$ an integer. An orientation of G is called *modulo k-orientation* if, for all $v \in V$, $d^+(v) \equiv d^-(v) \bmod k$.

A graph has a 2-NZF if and only if it has a modulo 2-orientation. Indeed graphs admitting a 2-NZF are exactly even graphs, whose edge-set can be partitioned into circuits and hence admitting a modulo 2-orientation (it suffices to give to all such circuits a clockwise orientation). Moreover a graph G has a 3-NZF if and only if it has a modulo 3-orientation. Indeed, G has a 3-NZF if and only if G has a $\mathbb{Z}_3$-NZF (D, f). We can choose D in such a way that $f(e) = 1$ for all $e \in E$. Then D is a modulo 3-orientation of G.

1.5.1 *The Circular Flow Conjecture*

The connection between flows and modulo orientations was first introduced by Jaeger in [37].

Proposition 1.45 (Jaeger [37]). *Let G be a graph. For all integers $p \geq 1$, G has a modulo $(2p+1)$-orientation if and only if G has a $(2+\frac{1}{p})$-CNZF.*

Proof. Let $\psi \in \mathrm{Aut}(\mathbb{Z}_{2p+1})$ be the group automorphism such that $\psi(1) = p$.

Notice that G has a modulo $(2p+1)$-orientation D if and only if G has a $\mathbb{Z}_{2p+1}$-NZF (D, f) such that $f(e) = 1$ for all $e \in E$. Then $(D, \psi f)$ is a $\mathbb{Z}_{2p+1}$-NZF on G. By Proposition 1.6, G has a $(2p+1)$-NZF (D, g) such that, for all $e \in E$, $g(e) \equiv p \bmod 2p+1$, that is $g(e) \in \{p, -(p+1)\}$. Therefore, $(D, \frac{1}{p}g)$ is a $(2+\frac{1}{p})$-CNZF on G.

The theorem follows by noticing that the same argument can be repeated going backwards by starting from a $(2+\frac{1}{p})$-CNZF on G taking flow values in $\{1, 1+\frac{1}{p}\}$. Such a circular flow exists by Theorem 1.30. □

Another well known result from [44] states that $\phi_c(G) < 2 + \frac{1}{p}$ if and only if G has a strongly connected modulo $(2p+1)$-orientation. Jaeger left the following conjecture known as the Circular Flow Conjecture.

Conjecture 1.46 (Jaeger [37])**.** *For all integers $p \geq 1$, every $4p$-edge-connected graph has a modulo $(2p+1)$-orientation.*

This well known conjecture has been unsolved for many years. However, in 2018 some counterexamples have been constructed for $p \geq 3$, see [30]. We recall the construction here. We are going to use such counterexamples in Section 4.3 in order to disprove a conjecture of Steffen, see Conjecture 4.3.

Construction 1.47. *Let $p \geq 3$ be an integer and let $\{v_1, v_2, \dots, v_{4p}\}$ be the vertex set of the complete graph K_{4p}.*

i. Construct the graph G_1 by adding an additional set of edges T such that $V(T) = \{v_1, v_2, \dots, v_{3(p-1)}\}$ and each component of the edge-induced subgraph $G_1[T]$ is a triangle.

ii. Construct the graph G_2 from G_1 by adding two new vertices z_1 and z_2, adding one edge z_1z_2, adding $p-2$ parallel edges connecting v_{4p} and z_i for both $i \in \{1,2\}$, and adding one edge v_iz_j for each $3p-2 \leq i \leq 4p-1$ and $j \in \{1,2\}$.

iii. Consider $4p+1$ copies $G_2^1, \dots, G_2^{4p+1}$ of G_2. If $v \in V(G_2)$, then we write v^i to refer to the vertex v of the i-th copy of G_2. Construct the graph M_p in the following way. For every $i \in \{1, \dots, 4p+1\}$ identify z_2^i with z_1^{i+1} and call this new vertex c_{i+1}, where we take sums modulo $4p+1$. Finally add a new vertex w and all edges of the form wc_i for all $i \in \{1, \dots, 4p+1\}$.

The following theorem holds.

Theorem 1.48 ([30])**.** *For all $p \geq 3$, M_p does not have a modulo $(2p+1)$-orientation.*

We would like to remark that Conjecture 1.46 is still open for integers $p < 3$. In particular, for $p = 1$ this is Conjecture 1.11 and Jaeger proved that the case $p = 2$ implies Conjecture 1.13. Indeed, suppose Conjecture 1.46 is true and let G be a smallest counterexample to Conjecture 1.13. By Theorem 1.42 G is a cyclically 6-edge-connected cubic graph. Furthermore, let H be the graph obtained by replacing each edge of G by three parallel edges. H is 9-regular and 9-edge-connected and so admits a modulo 5-orientation D_H. Let D be the orientation of G defined as follows: $e = uv \in E(G)$ is oriented from u to v if there are at least two edges in D_H having such an orientation. On the other hand, for all $e = uv \in E(G)$, let $f(e) = 3$ if all parallel edges connecting u to v have the same orientation in D_H, and $f(e) = 1$ otherwise. It follows that (D, f) is a $\mathbb{Z}_5$-NZF on G.

2

CONSTRUCTION OF GRAPHS WITH CIRCULAR FLOW NUMBER 5

This chapter is devoted to the study of the structure of graphs with circular flow number 5. In particular, the main results are construction methods of graphs with circular flow number 5 and most of such results come from a joint work with Jan Goedgebeur and Giuseppe Mazzuoccolo [P.2].

2.1 INTRODUCTION

Flow numbers received much attention in the last decades and some characterizations of graphs having a given circular flow number are known. For example, as we already mentioned in Section 1.4.2, it is proved in [62] that, for any rational value q in the interval $(4,5]$, there exists a graph with circular flow number exactly q, and an analogous result is proven in [49], even if we restrict our attention to the class of snarks. Moreover in [66] and [71] circular flow numbers of regular graphs are studied.

In Section 1.2 we recalled the definition of circular flow number of a graph and remarked that it can be different from its integer flow number. This fact does not hold for the Petersen graph P_{10}, for which $\phi(P_{10}) = \phi_c(P_{10}) = 5$. However, for some time, no other example of snark with such property was known. Mohar asked in 2003 [59] if the Petersen graph is the only possible one. In 2006 Máčajová and Raspaud [53] gave a negative answer to Mohar's question by constructing an infinite family of snarks with circular flow number 5. More recently, Esperet, Mazzuoccolo and Tarsi [19], extending the method proposed in [53], constructed a larger class of snarks with circular flow number 5 and, among other results, showed that deciding whether a given snark has circular flow number less than 5 is an NP-complete problem. Finally, by using methods introduced in [19] another family of snarks having circular flow number 5 was presented in [1].

In this chapter we propose a unified and compact description of all such methods and the new ones introduced here. A summary of these results is Theorem 2.35.

Furthermore, using a computer search we determine all snarks with circular flow number 5 up to order 36. We achieve this making use of an algorithm that computes the circular flow number of a cubic graph. In Chapter 3, we present an implementation of such algorithm and all results of our computations can be found in Tables 1 and 2 of

Section 3.3. We further checked the structure of all snarks on up to 36 vertices: it turns out that all such snarks of order at most 34 fit our description (and hence, more specifically, the description proposed in [19]), and that the same holds for 96 of the 98 snarks of order 36 with circular flow number 5. Even though these computational results seem to suggest that our methods cover a large fraction of snarks with circular flow number 5, we think that this behaviour is due to the fact that we are dealing with relatively small orders.

It is important to stress that our method focuses on the presence of some structures which force the circular flow number of a snark to be large. Indeed, we describe a way to obtain graphs, not necessarily cubic, which are cyclically 4-edge-connected graphs with circular flow number 5. Each such graph can then be transformed into a snark by a suitable expansion of some of its vertices (see Subsection 2.3.1 for a precise description). The construction of such graphs is the main purpose of our method: all snarks obtained starting from them will have circular flow number 5, since the expansion of a vertex does not decrease the circular flow number (cf. Proposition 2.18). A complete description (taken from [19]) of the procedure that allows to construct an example of such snarks is given in the Subsection 2.3.1. To keep things concise, we will not specify every time how we can obtain a snark starting from a given graph. Moreover, along the entire chapter, we only prove that our methods produce graphs with circular flow number *at least* 5 as this is implicitly sufficient to prove that its circular flow number is *exactly* 5 if Tutte's 5-Flow Conjecture is true.

This chapter is organized as follows. In Section 2.2, we present the terminology and notations introduced in [19] that we are going to use in the rest of the chapter. Moreover, we give a complete answer to Problem 7.3 from [19]. In Section 2.3, we summarize known constructions of graphs with circular flow number 5 and propose new ones. Afterwards, we prove that all such methods are nothing but particular instances of a more general construction that we introduce into details here. Section 2.4 and Section 2.5 are devoted to a complete analysis of many possible instances of the introduced method: a summary of all results obtained in these two sections is Theorem 2.35, which is the main result presented in this chapter. Finally, in Section 2.6, we show the results of our computations. On one side, they confirm that our method is a good tool to produce several examples of snarks with circular flow number 5, but, on the other hand, we find two snarks of order 36 which seem to not fit our description. This suggests that the variety of snarks with circular flow number 5 could be very large.

The chapter continues with Section 2.7 where we show that the problem of deciding whether a graph has circular flow number 5 or not can be reduced to cubic graphs. More precisely, we show that, given a non-cubic graph G, $\phi_c(G) \geq 5$ if and only if $\phi_c(H) \geq 5$ for all cubic graphs H constructed applying suitable expansions to G.

Therefore, we get information about $\phi_c(G)$ studying the circular flow numbers of a class of cubic graphs constructed starting from G itself. In doing this we tried to focus on the smallest possible expansions, but we do not pay special attention to the number of constructed graphs. This idea came out when we found out that there were two snarks on 36 vertices not fitting the description given by Theorem 2.35, as explained above. When studying such snarks we tried to describe them as expansions of smaller configurations forcing their circular flow number to be high and, such smaller graphs, necessarily have vertices of higher degree. Results of Section 2.7, together with Algorithm 1, helped us finding out the right configurations and stating theorems of Section 2.8, that give new infinite families of snarks with circular flow number 5.

2.2 GENERALIZED EDGES AND OPEN 5-CAPACITY

This section is mainly devoted to a review of the main results and definitions presented in [19], that will be needed later on. We refer to the same work by Esperet, Mazzuoccolo and Tarsi for a complete proof of the results of this section. Furthermore, we give a complete answer to Problem 7.3 in [19] which was left as an interesting open problem.

First of all we introduce the definition of circular modular flow and we recall that the existence of such a kind of flow is equivalent to the existence of a r-CNZF.

Definition 2.1. A *circular nowhere-zero modular r-flow*, or r-MCNZF, in a graph G, is an assignment $f\colon E \to [1, r-1] \subseteq \mathbb{R}/r\mathbb{Z}$ together with an orientation of G, such that, for every $v \in V$, equation (1) holds modulo r.

Similarly to the case of integer flows the following holds.

Proposition 2.2. *An r-CNZF in a graph G exists if and only if there exists an r-MCNZF.*

The following proposition gives an important tool that will be central in several proofs of the present chapter.

Proposition 2.3. *For a graph G, $\phi_c(G) < r$ if and only if there exists an r-MCNZF f in G such that $f\colon E \to (1, r-1)$.*

Following the notation used in [19], a flow which satisfies the condition in Proposition 2.3 will be called a *sub-r-MCNZF*.

Let $r \in \mathbb{R}$ and consider $\mathbb{R}/r\mathbb{Z}$, the group of real numbers modulo r. This is commonly represented by a circle of length r, where r coincides with 0, and an open interval $(a, b) \subseteq \mathbb{R}/r\mathbb{Z}$ denotes the set of numbers covered when traversing clockwise from a to b, with a, b not included;

closed intervals are denoted in a similar way. In particular (x, x) is defined to be $\mathbb{R}/r\mathbb{Z} - \{x\}$.

We will focus on the case $\mathbb{R}/5\mathbb{Z}$. The set of all *integer open intervals* of $\mathbb{R}/5\mathbb{Z}$, i.e. all intervals (a, b) where $a, b \in \mathbb{Z}$, is denoted by $I_5 := \{(a, b) \subseteq \mathbb{R}/5\mathbb{Z} \colon a, b \in \mathbb{Z}\}$. A subset $X \subseteq \mathbb{R}/5\mathbb{Z}$ is called *symmetric* if and only if $x \in X \iff -x \in X$. We denote by SI_5 the family of all symmetric subsets of $\mathbb{R}/5\mathbb{Z}$ which can be obtained as union of elements of I_5, that is:

$$SI_5 := \{I \subseteq \mathbb{R}/5\mathbb{Z} \colon I \text{ is symmetric and } I = \cup(a, b), (a, b) \in I_5\}.$$

Definition 2.4. The *measure* of $A \in SI_5$, denoted by $\mathrm{Me}(A)$, is the number of unit intervals contained in A.

The description of all constructions in the next sections makes use of the definition of *generalized edge*, G_{xy}, that is a network $(G, \{x, y\})$.

Consider the generalized edge G_{xy} and define a new graph (in general it can have multiple edges) G^+_{xy} by adding a new edge $e^+ = xy$ to G_{xy}.

Definition 2.5. The *open* 5*-capacity* of G_{xy} is

$$CP_5(G_{xy}) := \{f(e^+) \colon f \text{ is a modulo 5-flow in } G^+_{xy} \text{ and } f|_E \subseteq (1, 4)\}.$$

$CP_5(G_{xy})$ is actually the set of values in $\mathbb{R}/5\mathbb{Z}$, that can "pass through" G from the source terminal x to the sink terminal y, under all possible orientations of G, requiring that the flow capacity of every edge is restricted to the open interval $(1, 4)$.

From now on, since we are going to deal only with the case of modular 5-flows, we will for simplicity refer to the open 5-capacity of a generalized edge just as the capacity of that generalized edge.

A strong relation between the concept of capacity of a generalized edge and the set SI_5 is given in the following lemma (see [19]):

Lemma 2.6. *If G_{xy} is a generalized edge, then $CP_5(G_{xy}) \in SI_5$.*

In view of the previous lemma, a generalized edge G_{xy} having 5-capacity $A \in SI_5$ is said to be an A-edge. A standard edge (the graph with two vertices and one edge) is then a $(1, 4)$-edge, but there exist infinitely many $(1, 4)$-edges which are not isomorphic to it.

It is clear that any graph G can be viewed as a union of generalized edges having disjoint vertex-sets except, possibly, for their terminals. Also note that the same graph could admit several different representations with different sets of generalized edges: a trivial representation is obtained by considering every edge of G as a $(1, 4)$-edge; on the opposite side, we can consider the entire graph G and any two of its vertices as a generalized edge itself.

Definition 2.7. Consider a pair (H, σ), where $H = (V(H), E(H))$ is a graph and $\sigma : E(H) \to SI_5$ is a map that associates to each edge

$uv \in E(H)$ a subset $\sigma(uv) \in SI_5$. We denote by H^σ the family of all possible graphs which can be obtained by replacing every edge uv of H with a $\sigma(uv)$-edge with terminals u and v. We will refer to such a σ as the *capacity function* defined on H.

Remark 2.8. *Every graph G belongs to the family G^σ where σ is the constant capacity function: $\sigma(e) = (1,4)$ for each $e \in E(G)$.*

The following proposition will play a crucial role in what follows.

Proposition 2.9. *A graph $G \in H^\sigma$ admits a sub-5-MCNZF if and only if H admits a flow f such that $f(e) \in \sigma(e)$, for all $e \in E(H)$.*

The previous proposition also says that if a graph G in H^σ has circular flow number 5, then all graphs in H^σ have the same property. Hence, in order to find graphs with circular flow number 5, we can work on the pairs (H,σ) instead of working on each specific graph G of the family. Mainly for this reason, we will make use of the following definition in the rest of the chapter.

Definition 2.10. Given the family H^σ, we call a *σ-faithful flow* in H any flow in H which satisfies the condition of Proposition 2.9.

2.2.1 *Every element of SI_5 is graphic*

Lemma 2.6 shows that the open capacity of a generalized edge is an element of SI_5. One of the open problems proposed in [19] (i.e. Problem 7.3) is the determination of all elements of SI_5 which are the open 5-capacity of a generalized edge. In order to study such a problem the following definition naturally arises:

Definition 2.11. Let $A \in SI_5$. If there exists an A-edge, then A is called *graphic*. We denote by $GI_5 \subseteq SI_5$ the set of all graphic elements of $\mathbb{R}/5\mathbb{Z}$.

Two operations to produce new elements of GI_5 starting from the known ones are presented in [19] which are used to prove the following proposition.

Proposition 2.12. *GI_5 is a closed subfamily of SI_5 with respect to sum and intersection.*

Only 5 sets in SI_5 were not proved to be graphic in [19], more specifically those obtained by removing the two elements $\{2,3\}$ from the sets of SI_5 containing them.

Now, we completely answer the question posed in [19] by showing that also the remaining five sets of SI_5 are graphic, that is $GI_5 = SI_5$. We will make use of the following remark:

Remark 2.13. *Let $A \in GI_5$ and H be an A-edge. Then $0 \in CP_5(H)$ if and only if $\phi_c(H) < 5$.*

Theorem 2.14. $GI_5 = SI_5$.

Proof. Consider the generalised edge G_{uv} such that G^+_{uv} is the complete graph with four vertices, and denote by s and t the other two vertices of G. It is sufficient to prove that $CP_5(G_{uv}) = \mathbb{R}/5\mathbb{Z} - \{2,3\}$: indeed this would mean that $\mathbb{R}/5\mathbb{Z} - \{2,3\} \in GI_5$ and, since GI_5 is closed under intersection, all remaining intervals could be conveniently generated. First of all, let us show that 0, 1 and 2.5 are elements of $CP_5(G_{uv})$. Since $\phi_c(G) < 5$, $0 \in CP_5(G_{uv})$ follows from Remark 2.13. For a sufficiently small $\epsilon > 0$, we explicitly construct two flows in Figure 1 such that the flow value of uv is 1 (on the left) or 2.5 (on the right).

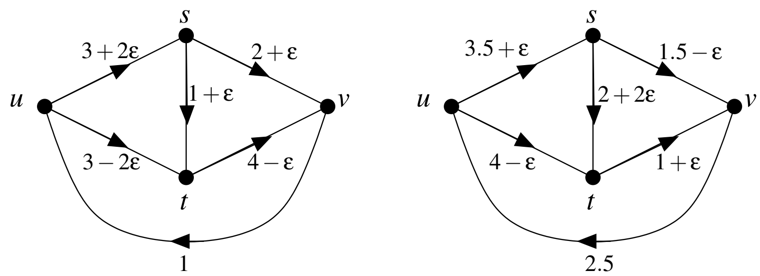

Figure 1: Two flows of K_4 with prescribed flow values on the edge uv.

Hence, thanks to the openness and symmetry of the open capacity, we have proved that $\mathbb{R}/5\mathbb{Z} - \{2,3\} \subseteq CP_5(G_{uv})$. In order to prove our assertion we need to show that $2 \notin CP_5(G_{uv})$ (and by symmetry we also obtain $3 \notin CP_5(G_{uv})$). Take the same orientation of the edges of G^+_{uv} shown in Figure 1 and suppose, by contradiction, that there is a flow f in G^+_{uv} such that $f|_E \subseteq (1,4)$ and $f(uv) = 2$.

From the relation $2 = f(uv) = f(us) + f(ut)$, it follows that both $f(us), f(ut) \in (3,4)$. Similarly from $2 = f(uv) = f(vs) + f(vt)$, we deduce that both $f(vs), f(vt) \in (3,4)$. Then, since $f(us) = f(st) + f(sv)$, it follows that $f(st) \in (4,1)$ a contradiction. $\square$

The main result of this section says that every element of SI_5 is the 5-capacity of a suitable generalized edge. Hence, from now on, every time we will consider a pair (H, σ), we have no general restriction on the values assumed by σ.

For completeness sake, we conclude this section with a proof of the following result, that does not appear in [P.2]. Denote by F_k the k-th number of the well known Fibonacci sequence, defined recursively as follows

$$\begin{cases} \mathrm{F}_1 = \mathrm{F}_2 = 1; \\ \mathrm{F}_k = \mathrm{F}_{k-1} + \mathrm{F}_{k-2}, \text{ for every } k \geq 3. \end{cases}$$

Proposition 2.15. $|SI_k| = \mathrm{F}_{k+3}$, *for every* $k \geq 1$.

Before going to the proof, notice that we can associate to any integer interval of $\mathbb{R}/k\mathbb{Z}$ a binary sequence $a_0 b_0 a_1 b_1 \ldots a_{k-1} b_{k-1}$, where a_j

represents the integer $j \in \mathbb{R}/k\mathbb{Z}$ and b_j represents the integer interval $(j, j+1)$. If A is the associated interval, we set $a_j = 0$ if the node $j \notin A$, $a_j = 1$ otherwise. Similarly $b_j = 0$ if $(j, j+1) \nsubseteq A$, $b_j = 1$ otherwise. For instance, $A = \{0\} \cup [1,2) \cup (3,4]$, in $\mathbb{R}/5\mathbb{Z}$, is associated to 1011000110. Suppose now that k is odd. Since we are interested in symmetric intervals we can stop the sequence at $b_{\frac{k-1}{2}}$, call it a *cut sequence*. If k is even the cut sequence is defined similarly by stopping at $a_{\frac{k}{2}}$. For instance $\{0\} \cup [1,2) \cup (2,3) \cup (3,4]$ in $\mathbb{R}/5\mathbb{Z}$ is associated to 101101. Notice that a symmetric interval corresponding to the cut sequence $a_0 b_0 a_1 b_1 \dots a_{\frac{k-1}{2}} b_{\frac{k-1}{2}}$ (or $a_0 b_0 a_1 b_1 \dots a_{\frac{k}{2}}$) is an element of SI_k if and only if, for all i, the fact that $a_i = 1$ implies that every symbol adjacent to a_i is 1 as well. Call this last property $\mathcal{P}$.

Proof of Proposition 2.15. It is not hard to check that $|SI_1| = 3 = \mathrm{F}_4$ and $|SI_2| = 5 = \mathrm{F}_5$. In order to get the thesis it is enough to show $|SI_k| = |SI_{k-1}| + |SI_{k-2}|$, for $k \geq 3$.

Counting how many elements SI_k contains is equivalent to counting how many cut sequences satisfy property $\mathcal{P}$. So fix the last element of a cut sequence of SI_k to be 0. If k is odd, then necessarily also $a_{\frac{k-1}{2}}$ must be 0, therefore the number of possible sequences satisfying property $\mathcal{P}$ and with last element 0 are $|SI_{k-2}|$. If k is even, the number of possible sequences satisfying property $\mathcal{P}$ and with last element 0 are $|SI_{k-1}|$, since we do not have further restrictions as previous case. On the other hand, the number of sequences satisfying property $\mathcal{P}$ and having last element 1 is $|SI_{k-1}|$, if k is odd and $|SI_{k-2}|$ if k is even, since in this last case we have to fix $b_{\frac{k}{2}-1}$ to be 1. Therefore, for all $k \geq 3$, $|SI_k| = |SI_{k-2}| + |SI_{k-1}|$. □

2.3 UNIFYING KNOWN AND NEW METHODS

We denote by $F_{\geq 5}$ the family of graphs with circular flow number greater than or equal to 5, and by $S_{\geq 5}$ the subfamily of $F_{\geq 5}$ consisting of snarks.

2.3.1 *Known methods*

Some of the constructions of snarks in $S_{\geq 5}$ presented in [19] make use of the following lemma. We report it into detail since it will be used in the next section to describe some new methods.

Lemma 2.16. *Consider a pair (H, σ), where H is a graph and σ a capacity function defined on H. Suppose that P is a path in H with $\sigma(e) = A$ for every $e \in E(P)$ and* $\mathrm{Me}(A) = 2$. *Assume also that each internal vertex v_i of P is adjacent to exactly one vertex v_i' of H not in P. Finally, assume $\sigma(v_i v_i') \subseteq (1,4)$ for every vertex v_i. If P is a directed path in a suitable orientation of H and f is a flow in H such that $f(e) \in \sigma(e)$ for every*

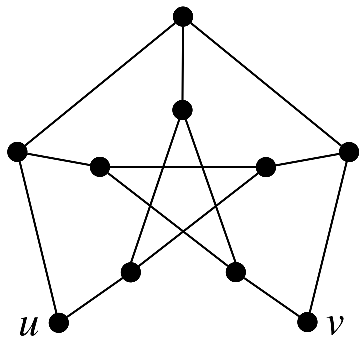

Figure 2: The $(4,1)$-edge $\mathcal{P}_{10}^{*}(u,v)$.

$e \in E(H)$, then f assigns to adjacent edges of P two values which lie in the two different unit intervals of A.

The main method presented in [19] to produce graphs in $F_{\geq 5}$ is a direct application of the following corollary. Also note that the method previously presented in [53] is a particular case of the same corollary.

Corollary 2.17. *Consider a pair (H,σ), where H is a graph and σ a capacity function defined on H. Suppose that C is an odd cycle in H with $\sigma(e) = A$ for every $e \in E(C)$ and $\mathrm{Me}(A) = 2$. Assume also that each vertex v_i of C is adjacent to exactly one vertex v_i' of H not in C. Finally, assume $\sigma(v_iv_i') \subseteq (1,4)$ for every vertex v_i. Then, $\phi_c(G) \geq 5$ for all $G \in H^{\sigma}$.*

In particular, if H has sufficiently large girth and connectivity, we can construct an element of $S_{\geq 5}$ starting from a suitable graph $G \in H^{\sigma}$. As already remarked, the standard trick to obtain a snark starting from an element G of H^{σ} is by applying an expansion operation. We recall that *expanding* a vertex v of a graph G means replacing v with a new graph K such that, for all $y \in N_G(v)$, t new edges connecting y to some vertices of K are added, where t is the number of parallel edges vy in G.

Many different expansions can be performed on the same graph G, but it is well-known that these expansions do not decrease the circular flow number.

Proposition 2.18. *Let G' be a graph obtained with an expansion of a vertex of G. Then, $\phi_c(G') \geq \phi_c(G)$.*

We give here an example which shows how this operation permit to obtain a snark with circular flow number 5.

Denote by C_3 a 3-cycle of K_4. Consider the pair (K_4,σ) where $\sigma(e) = (4,1)$ for all $e \in C_3$ and $\sigma(e) = (1,4)$ otherwise. Thanks to Corollary 2.17 we can say that all graphs in K_4^{σ} have circular flow number at least 5. Now let $\mathcal{P}_{10}^{*}(u,v)$ be the generalized edge obtained removing an edge uv from the Petersen graph P_{10}, with prescribed terminals u and v (see Figure 2). In [19] the authors proved that $\mathcal{P}_{10}^{*}(u,v)$ has capacity $(4,1)$: indeed $CP_5(\mathcal{P}_{10}^{*}(u,v)) \subseteq (4,1)$ since the Petersen graph has circular flow number 5; moreover, notice that $0 \in CP_5(\mathcal{P}^*_{10}(u,v))$ because $P_{10} - uv$ has a 4-NZF and thus, by Lemma 2.6, equality holds. Therefore, if we let G be the graph obtained by replacing every edge of C_3 with a copy of $\mathcal{P}^*_{10}(u,v)$, it follows that $G \in K_4^\sigma$ and so has circular flow number at least 5. The graph G is not a snark as it is not cubic yet. Now, we expand each vertex of degree 5 of G to the graph with two isolated vertices, and we connect these new vertices to the rest of the graph as in Figure 3. This operation produces three vertices of degree 2 that we suppress. None of these operations reduces the circular flow number. Note that the final graph has girth at least 5 and is cyclically 4-edge-connected, so it is a snark. Note also that such a snark is the smallest one larger than the Petersen graph having circular flow number 5 (see [53] and Table 1 of Chapter 3).

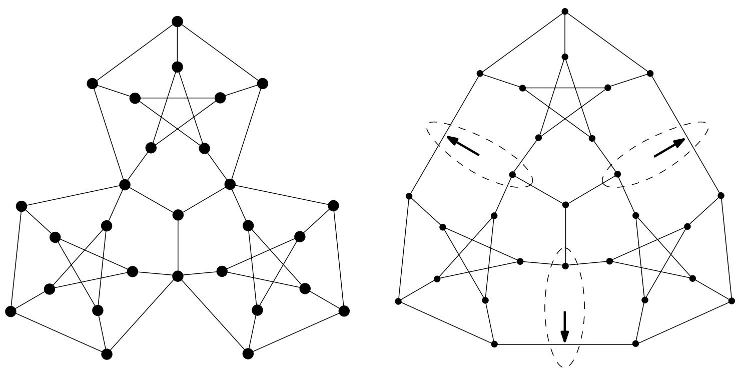

Figure 3: Expansion of the degree 5 vertices of a wheel W_3 having an external cycle of $(4,1)$-edges.

2.3.2 *New methods*

In previous section we have briefly described a method from [19] to generate new graphs in $F_{\geq 5}$. That construction gives ways to generate new members of $F_{\geq 5}$ starting from graphs having particular subgraphs and capacity functions.

Our next goal is to present some new constructions and, after that, to suggest a unified description of all methods described in this section and in the previous one. The final aim is a significant reduction of redundancy and a much better control on the graphs that can be generated. Such a unified description will be studied into detail in the next section and it is the main goal of this chapter.

Let us begin with new corollaries from Lemma 2.16. Each of them produces a new method to generate elements of $F_{\geq 5}$.

Corollary 2.19. *Consider a pair (H,σ), where H is a graph and σ a capacity function defined on H. Suppose that P is a path in H with $\sigma(e) = (4,1)$ for every $e \in E(P)$ and that each internal vertex v_i of P has degree 3 in H.*

Also assume that $\sigma(e) \subseteq (1,4)$ for every edge e not in $E(P)$ incident to an internal vertex of P. If two internal vertices of P at even distance on P are adjacent, i.e. there exists an edge e not in P connecting two internal vertices of P and forming an odd cycle with the edges of P, then $\phi_c(G) \geq 5$ for all $G \in H^\sigma$.

Proof. Give an orientation to P in such a way that it becomes a directed path in H and let f be a flow in H such that $f(e) \in \sigma(e)$ for each $e \in E(H)$. Then, by Lemma 2.16, the edges of P take values alternately from the two unit intervals $(4,0)$ and $(0,1)$: a contradiction arises from the fact that $f(e)$ must stay at the same time in $(4,0) - (0,1) = (3,0)$ and in $(0,1) - (4,0) = (0,2)$. Hence such a flow f cannot exist and so, by Proposition 2.9, every $G \in H^\sigma$ cannot have a sub-5-MCNZF. □

Corollary 2.20. *Consider a pair (H,σ), where H is a graph and σ a capacity function defined on H. Suppose that P_1 and P_2 are distinct paths in H with $\sigma(e) = (4,1)$ for every $e \in E(P_1) \cup E(P_2)$ and that each internal vertex v_i of these paths has degree 3 in H. Also assume that $\sigma(e) \subseteq (1,4)$ for every edge e not in $E(P_i)$ incident to an internal vertex of P_i.*

If two internal vertices of P_1 at even distance (on P_1) are adjacent, respectively, to two internal vertices of P_2 at odd distance (on P_2), i.e. there exist two edges not in $P_1 \cup P_2$ connecting two internal vertices of P_1 to two internal vertices of P_2 and forming an odd cycle with some edges of P_1 and P_2, then $\phi_c(G) \geq 5$ for all $G \in H^\sigma$.

Proof. Give a suitable orientation to H that makes both P_1 and P_2 directed paths and suppose that there exists a flow f in H such that $f(e) \in \sigma(e)$ for each $e \in E(H)$. We can assume without loss of generality that $P_1 = x_0 \ldots x_s$, $P_2 = y_0 \ldots y_t$, with $t > s$, and that the two edges in the hypothesis are x_1y_1 and $x_{s-1}y_{t-1}$. By Lemma 2.16, the edges of each P_j take values alternately from $(4,0)$ and $(0,1)$, but the presence of the edge x_1y_1 between them obliges the paths to start with different intervals, i.e. $x_0x_1 \in (0,1)$ (resp. $(4,0)$) if and only if $y_0y_1 \in (4,0)$ (resp. $(0,1)$). Since $s-1$ and $t-1$ have different parity, the values $x_{s-2}x_{s-1} - x_{s-1}x_s$ and $y_{t-2}y_{t-1} - y_{t-1}y_t$ belong to the same unit interval $(0,1)$ or $(4,0)$, whence there is no orientation of $x_{s-1}y_{t-1}$ such that the flow f does exist. Therefore, by Proposition 2.9, every $G \in H^\sigma$ cannot have a sub-5-MCNZF. □

We conclude this section by proving that all previous results can be slightly generalized when we replace $(4,1)$ with any of its subsets. This fact is an obvious consequence of the following more general proposition.

Proposition 2.21. *Let H be a graph and let σ_1 and σ_2 be two capacity functions defined on H. Assume that $\sigma_2(e) \subseteq \sigma_1(e)$ for all $e \in E(H)$. If $\phi_c(G) \geq 5$ for $G \in H^{\sigma_1}$, then $\phi_c(G') \geq 5$ for $G' \in H^{\sigma_2}$.*

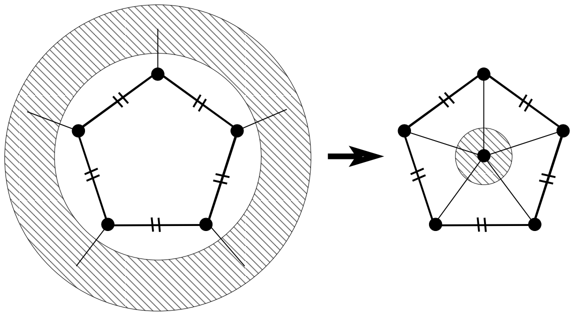

Figure 4: Example of a reduction based on Corollary 2.17.

Proof. It is sufficient to notice that σ_2 is a more restrictive capacity function. Then, starting from a sub-5-MCNZF of G', we can reconstruct a sub-5-MCNZF of G, a contradiction. □

In particular, thanks to Proposition 2.21, both corollaries presented in this section hold as well if some $(4,1)$-edges are replaced with $(4,0) \cup (0,1)$-edges.

2.3.3 *A unified description*

Now we suggest a possible unified description of the three methods arising from Corollary 2.17, Corollary 2.19 and Corollary 2.20.

Our approach is the following: we consider the subgraph of (H,σ) described in the corresponding corollary which forces all graphs in H^σ to have circular flow number at least 5, and we contract the remaining part of H into a unique vertex.

The key point is that the resulting graph is a wheel in all previous reductions (see Figures 4 - 6). We recall that a *wheel* on $n+1$ vertices is the graph W_n consisting of an n-cycle C plus a vertex v_c that is connected to all vertices of $V(C)$.

Moreover, the subgraph induced by the edges with capacity contained in $(4,1)$ (denoted by a double signed edge in the figures) is an even subgraph of the wheel, where an *even subgraph* of a graph H is a subgraph where all vertices have even degree.

More precisely, we can summarize all previous reductions in the following list:

- Corollary 2.17: in case C is a $(2n+1)$-cycle, this generates graphs which belong to W^σ_{2n+1}, where $\sigma(e) \subseteq (4,1)$ for all e of the external cycle of W_{2n+1} and $\sigma(e) \subseteq (1,4)$ for all other edges of the wheel (see Figure 4);

- Corollary 2.19: in case P is a path with $2n+3$ vertices, this generates graphs which belong to W^σ_{2n+1}, where $\sigma(e) \subseteq (4,1)$ for all e of a Hamiltonian cycle of W_{2n+1} and $\sigma(e) \subseteq (1,4)$ for all other edges of the wheel (see Figure 5);

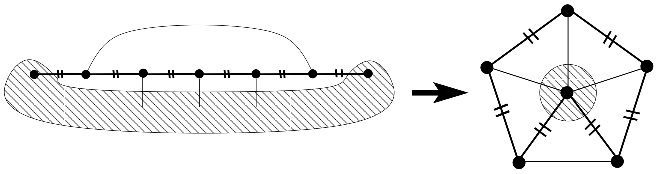

Figure 5: Example of a reduction based on Corollary 2.19.

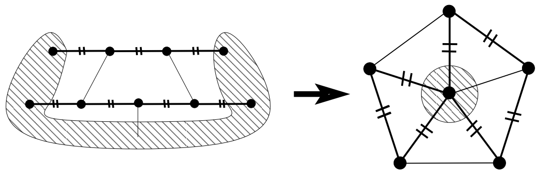

Figure 6: Example of a reduction based on Corollary 2.20.

- Corollary 2.20: in case P_1 and P_2 together have $(2n+5)$ vertices, this generates graphs which belong to W^{σ}_{2n+1}, where $\sigma(e) \subseteq (4,1)$ for all e of a suitable even subgraph of W_{2n+1} and $\sigma(e) \subseteq (1,4)$ for all other edges of the wheel (see Figure 6).

The next section is devoted to an exhaustive analysis of all instances arising from the new approach we have introduced in this section.

2.4 CHARACTERIZING WHEELS

In the previous section, we remark that all construction methods of cubic graphs having circular flow number at least 5 always produce a graph which is a suitable expansion of a graph in W^{σ}_n where σ is a capacity function which assigns a subset of $(4,1)$ to all edges of a given even subgraph of the wheel and $(1,4)$ to all other edges of W_n.

So it is very natural to ask in general whether, given a wheel W_n and a capacity function σ such that $\sigma(e) \subseteq (4,1)$ for all edges of an even subgraph J of W_n and $\sigma(e) = (1,4)$ otherwise, we obtain that a graph G which belongs to W^{σ}_n has circular flow number at least 5.

Remark 2.22. *In this section, we consider only the case in which $\sigma(e) = (4,1)$ for all edges of the even subgraph J and we use the notation $J_{(4,1)}$ to stress the fact that all edges of J have capacity $(4,1)$. Anyway, it follows by Proposition 2.21 that all results we are going to present also hold in the more general case $\sigma(e) \subseteq (4,1)$ for some edges of J.*

More precisely we can consider the following problem:

Problem 2.23. *Given a wheel W_n with $n+1$ vertices and J a non-empty even subgraph of W_n, establish for each integer n and each possible even subgraph J, if a graph $G \in W^{\sigma}_n$ has circular flow number at least 5, where $\sigma(e) = (4,1)$ if $e \in E(J)$ and $\sigma(e) = (1,4)$ otherwise. We will denote such a family of graphs by $(W_n, J_{(4,1)})$.*

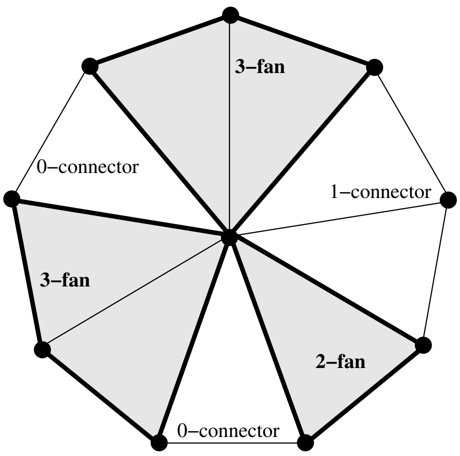

Figure 7: An example of fans and connectors in W_9, where the bold edges represent edges of J and all others are $(1,4)$-edges.

In what follows, we give a complete solution for all possible instances of this problem.

We call $(4,1)$-*edge* an edge of W_n which belongs to J and $(1,4)$-*edge* an edge of W_n which does not belong to J.

In order to describe the structure of J, we introduce the following two useful definitions (see also Figure 7).

Definition 2.24. Given the family $(W_n, J_{(4,1)})$, for an integer $l \geq 2$, an *l-fan F_l* in W_n is a subgraph induced by all vertices of an $(l+1)$-cycle consisting of edges of J and passing through the central vertex v_c of W_n.

Definition 2.25. Given the family $(W_n, J_{(4,1)})$, for an integer $m \geq 0$, an *m-connector C_m* is a subgraph of W_n induced by all the $m+1$ edges of a maximal path P_{m+2} of $(1,4)$-edges of the external cycle of W_n and all $(1,4)$-edges of type uv_c, where u is a degree 2 vertex of P_{m+2} and v_c is the central vertex of W_n.

It is clear that every even subgraph, except if J is the empty graph or the external cycle of W_n, can be described as a sequence of fans and connectors in W_n. From now on, when we refer to the connector (fan) *following* a fan (connector), we are implicitly considering the clockwise order on fans and connectors in W_n.

In what follows, we also use the following terminology:

- We refer to the longest cycle of an m-connector, for $m \geq 2$, as the *external cycle* and, accordingly, we call its edges *external edges*. For all $m \geq 0$, we call *internal edges* all edges of an m-connector incident to v_c which are not external. Finally, we call *lateral edges* of an m-connector the two edges (only one in the case of a 0-connector) that are neither external nor internal.

- We refer to the cycle of edges of J in an l-fan as the *external cycle* of the fan and, accordingly, we call its edges *external edges*.

We will refer to the $(1,4)$-edges of an l-fan as its *internal* edges. Finally, we will call the *first edge of the fan* the unique external edge of J which is incident to v_c and to a lateral edge of the connector preceding the fan. While we call the *last edge of the fan* the unique external edge of J which is incident to v_c and to a lateral edge of the connector following the fan.

2.4.1 *Flows in fans and connectors*

Now we furnish the description of some flows defined in l-fans and m-connectors that will be largely used in the following proofs.

In what follows, take three values $x, y, z \in \mathbb{R}/5\mathbb{Z}$ such that $x \in (1,2)$, $y \in (1,2)$, $z \in (4,0)$, $x+z \in (0,1)$ and $x+2y \in (1,4)$: it is an easy check that such values do exist. Moreover, note also that $2y \in (1,4)$ and $x+y \in (1,4)$, indeed they both lie inside $(2,4)$.

Flow f^+ in an l-fan (l even): consider an l-fan F_l with $l \geq 2$ even, we assign a clockwise orientation to its external cycle and we define f^+ such that it assigns flow value z and $z+x$ alternately to the edges of the external cycle starting from v_c and following the orientation. Moreover, f^+ assigns flow value x to all internal edges of F_l. Now, we consider the unique possible orientation of the internal edges such that f^+ is a zero-sum flow in each vertex of degree 3 in F_l. Indeed, note that if l is even then f^+ is also a zero-sum flow in v_c.

Flow $\tilde{f}^+$ in an l-fan (l odd): consider an l-fan F_l with $l \geq 3$ odd, the flow $\tilde{f}^+$ assigns the orientation and the flow values exactly as f^+, but note that if l is odd then the difference between the inner flow and outer flow of $\tilde{f}^+$ in v_c is exactly $2x$.

Flow g^+ in an m-connector (for $m \neq 1$): consider an m-connector C_m. For $m = 0$ we set the flow value of g^+ equal to x on the unique edge of the connector, which is oriented clockwise in W_n. For $m > 1$ we distinguish two cases according to the parity of m. If $m > 1$ odd, then we define g^+ as in the upper part of Figure 8, while if $m > 0$ even then we define g^+ as in the lower part of Figure 8.

Flow $\tilde{g}^+$ in an m-connector (for $m > 0$): consider an m-connector C_m. For $m > 1$ we distinguish two cases according to the parity of m. If $m > 1$ odd, then we define $\tilde{g}^+$ as on the upper part of Figure 9, while if $m > 0$ even then we define $\tilde{g}^+$ as on the lower part of Figure 9. In particular, for a 1-connector, if we denote by uv_c the unique edge incident to v_c, $\tilde{g}^+$ orients it towards v_c and assigns it flow value $2x$. Moreover $\tilde{g}^+$ assigns to the other two edges flow value x and orients them in such a way that $\tilde{g}^+$ is a zero-sum flow in u.

Flows f^-, g^- and $\tilde{g}^-$: these flows are obtained from f^+, g^+ and $\tilde{g}^+$, respectively, by considering the same flow value on each edge of the corresponding flow and by reversing the orientation of each edge with respect to the orientation in the original one.

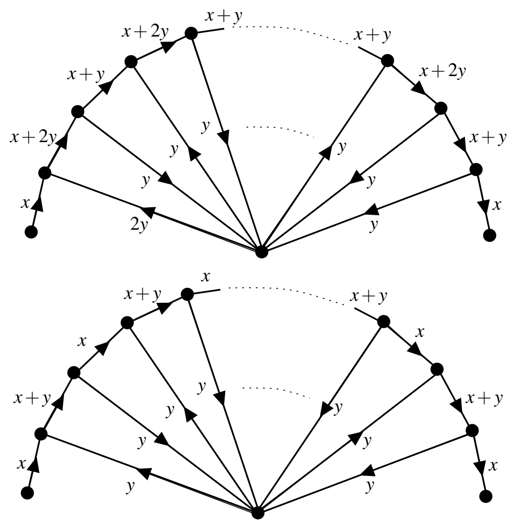

Figure 8: The flow g^+ in an odd connector (up) and in an even connector (down).

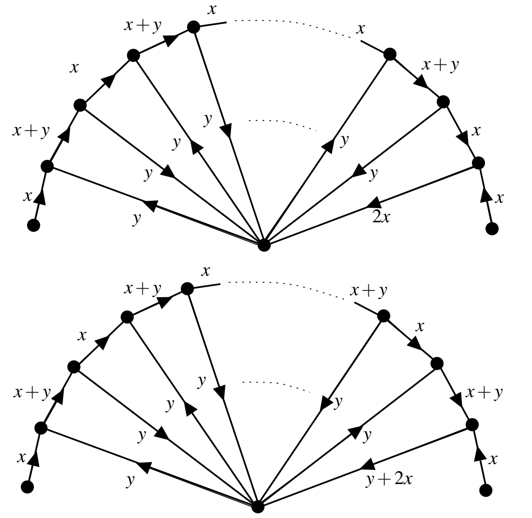

Figure 9: The flow $\tilde{g}^+$ in an odd connector (up) and in an even connector (down).

Consider the family $(W_n, J_{(4,1)})$ and let f be a σ-faithful flow in W_n which coincides with one of the flows f^+,f^-, $\tilde{f}^+$ or $\tilde{f}^+$ when we consider its restriction to an l-fan F_l with $l < n$. Consider the unique two lateral edges in W_n which are incident to a vertex of the l-fan and assume that f has flow value x on both of them. Then, there is a unique way to orient these lateral edges, say e_1 and e_2, in such a way that f is a zero-sum flow in all vertices distinct from v_c of F_l. If we have the flow f^+ (f^-) on the fan, both e_1 and e_2 have a clockwise (anticlockwise) orientation, otherwise in $\tilde{f}^+$ ($\tilde{f}^-$), they both point towards (away from) the fan.

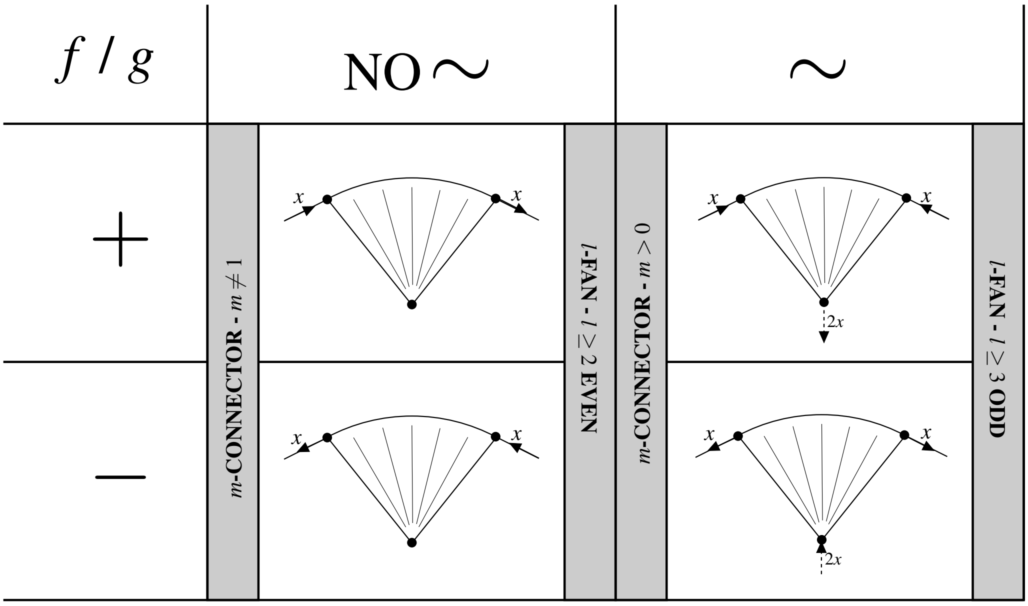

Figure 10: A summary of all flows defined in this section.

More in general, our notation for all previous flows is consistent with the following scheme from Figure 10 which uses the following notation:

- The letters f and g denote flows on fans and connectors, respectively.
- The symbols $+$ and $-$ mean that the edges of the external cycles of W_n are oriented in clockwise and anticlockwise direction in the corresponding flow, respectively.
- The symbol $\sim$ means that, in the vertex v_c, the absolute difference of the inner flow and outer flow in the corresponding subgraph is $2x$. More precisely, it is $2x$ in the case of all flows with symbol $+$, and $-2x$ in the case of all flows with symbol $-$. Whereas, the absence of $\sim$ means that the corresponding flow is a zero-sum flow in v_c.

In other words, in all flows with the symbol $\sim$ the two lateral edges of the connector (or the two lateral edges adjacent to the fan) have the same value x and opposite orientation in the cycle of W_n, while in all flows without the symbol $\sim$ we have the same value x on both edges and the same orientation in the cycle of W_n.

We can briefly say that in the former case the flow *reverses* the orientation, while in the latter case the flow *preserves* the orientation.

2.4.2 *Even subgraphs with edges of capacity* $(4,1)$

In this section, we completely characterize the families $(W_n, J_{(4,1)})$ whose elements are graphs with circular flow number at least 5. When we speak about the sequence of fans and connectors given by the choice of J in W_n in the proofs, we will use the term *component* to speak indifferently about either a fan or a connector of the decomposition.

Proposition 2.26. *If G belongs to $(W_n, J_{(4,1)})$ with n even, then $\phi_c(G) < 5$.*

Proof. Set $n = 2k$. If $J = W_{2k} - v_c$, we construct a σ-faithful flow ψ in W_{2k} and the assertion follows by Proposition 2.3 and Proposition 2.9. Assign alternately the flow values z and $x + z$ to all edges of the external cycle of W_{2k} oriented clockwise and the flow value x to all other edges with the unique orientation that makes ψ a zero-sum flow at each vertex. Hence, we can assume without loss of generality that $v_c \in J$. As already remarked, we can describe J as a sequence of fans and connectors. Now, we recursively assign the flow on the components of such a decomposition. Firstly, select an l-fan and assign it the flow f^+ if l is even and the flow $\tilde{f}^+$ if l is odd. At each further step consider the following component and assign it a flow with these rules:

- if the component is a fan then assign a flow f, if it is a connector then assign a flow g;
- if l (or m) is odd, then assign a flow with $\sim$;
- if the previous component has a flow with $\sim$, then assign a flow with opposite sign with respect to the flow in the previous component, otherwise a flow with the same sign.

Since the wheel is even, the number of odd components is even. Hence, from our construction it follows that there is an even number of components with a flow that reverses the orientation. Hence the flows defined on each subgraph, altogether, induce a zero-sum flow on each vertex of the external cycle, and then also at the vertex v_c. Since we have defined a σ-faithful flow in W_{2k}, the assertion follows. □

Now we complete the characterization of even subgraphs J such that a graph in $(W_n, J_{(4,1)})$, n odd, has circular flow number at least 5.

Theorem 2.27. *Let G be a graph in $(W_n, J_{(4,1)})$, with n odd. Then*

$$\phi_c(G) \geq 5 \text{ if and only if } J \text{ has no } k\text{-connectors, for } k \geq 2.$$

Proof. Assume that J can be described by using only 0-connectors and 1-connectors. If we prove that a σ-faithful flow cannot be defined for a given orientation, then it cannot be defined for an arbitrary orientation. We orient all edges of the external cycle of W_n and all edges of the external cycle of each fan in a clockwise way. Take an arbitrary orientation of all remaining edges and assume, by contradiction, the existence of a σ-faithful flow. First of all, recall that the flow value of the edges in the external cycle of a fan must be in $(4, 1)$. Moreover, by

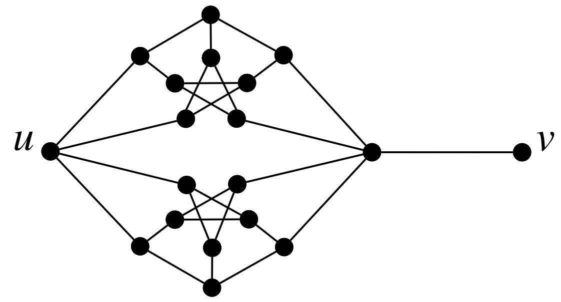

Figure 11: A $(1,2) \cup (3,4)$-edge with terminals u and v.

Lemma 2.16, the flow value on two consecutive external edges of a fan must be in the two disjoint unit intervals $(4,0)$ and $(0,1)$ (except possibly for the two edges sharing the central vertex v_c). It follows that every lateral edge must have a flow value either in the unit interval $(0,1) - (4,0) = (1,2)$ or in the unit interval $(4,0) - (0,1) = (3,4)$, because it is incident to two consecutive external edges of a fan. Note that, due to the chosen orientation, if l is even, lateral edges of an l-fan take values in the same unit interval, whereas, if l is odd, they take values in each one of the two different unit intervals, respectively. Analogously, consider the two lateral edges of a 1-connector. It is easy to verify that these two edges must have flow values in $(1,2)$ and $(3,4)$, respectively. Obviously, the unique lateral edge of a 0-connector takes a flow value in one of those two unit intervals. Now consider an arbitrary lateral edge e, without loss of generality we can assume it has a flow value in $(1,2)$. By previous considerations, starting from e we can establish in which interval between $(1,2)$ and $(3,4)$ each lateral edge lies: if the next component is either an odd fan or a 1-connector, then the flow value of the next lateral edge is in the other unit interval, otherwise it is in the same unit interval. Since in this case the number of odd components is odd, the flow value of e should belong both to $(1,2)$ and $(3,4)$, a contradiction.

For the necessity, assume that there exists an m-connector C_m, with $m \geq 2$. We show that there is a σ-faithful flow in W_n. We assign orientations and flow values to all fans and connectors, except for C_m, following exactly the same rules described in Proposition 2.26. The flow assigned to C_m also follows the same rules described in Proposition 2.26 for the letter and the sign, but we reverse the rule for the presence of $\sim$: more precisely, if m is odd we assign a flow without $\sim$ and if m is even we assign a flow with $\sim$. In this way, we guarantee that the number of components which reverse the orientation is even. Indeed, the number of odd fans plus the number of odd connectors is odd, but we recover the parity by the modification on C_m. Hence a σ-faithful flow is defined in W_n. $\square$

2.4.3 *Even subgraphs with edges of capacity* $(1,2) \cup (3,4)$

Until now we have given a complete characterization of graphs with circular flow number at least 5 arising from wheels having an even subgraph of $(4,1)$-edges (and then also for $(4,0) \cup (0,1)$-edges). Note that several methods in [19] concerned edges with capacity of measure 2. Hence it is natural to ask whether a similar result holds for even subgraphs with the edges of measure 2 we are left with, i.e. $(1,2) \cup (3,4)$-edges. Indeed, we remark that snarks with circular flow number at least 5 can be constructed using $(1,2) \cup (3,4)$-edges. Consider an odd wheel W_{2n+1} and replace every edge of its external cycle with a copy of the $(1,2) \cup (3,4)$-edge depicted in Figure 11 (we refer to [19] for a proof of the fact that its capacity is exactly $(1,2) \cup (3,4)$). By Corollary 2.17 this graph, as well as every snark obtained by expanding its vertices, has circular flow number at least 5. In this section, we prove that the behavior is slightly different in this case.

Select, for the entire section, $x, y \in (1,2) \cup (3,4)$ such that $x = 1 + 2\delta$ and $y = 3.5 + 2\delta$ with $\delta \in (0, 0.25)$. Note that the difference between x and y is exactly 2.5 for every choice of δ.

First of all, we completely solve the case where the even subgraph J is induced by the edges of the external cycle of the wheel W_n.

Theorem 2.28. *Let G be a graph in $(W_n, J_{(1,2)\cup(3,4)})$ where J is the even subgraph induced by the edges of the external cycle of the wheel W_n. Then*

$$\phi_c(G) \geq 5 \text{ if and only if } n \text{ is odd.}$$

Proof. Suppose n is odd. Then, the assertion follows as a direct application of Corollary 2.17. Suppose n is even. Now, take an orientation of W_n such that the external cycle of W_n is clockwise oriented, and assign alternately flow values x and y to the edges of the external cycle. Finally, assign flow value 2.5 to all edges incident to the central vertex (since the flow value is 2.5 modulo 5 the orientation of such edges does not really matter). The defined flow in W_n is a σ-faithful flow, then $\phi_c(G) < 5$ for every $G \in (W_n, J_{(1,2)\cup(3,4)})$ by Proposition 2.9. □

Now, we consider the more general case in which J is an arbitrary even subgraph of W_n. We use the terminology introduced in the previous section to describe J. In order to characterize all even subgraphs J such that $(W_n, J_{(1,2)\cup(3,4)})$ contains graphs with circular flow number at least 5, we need to define some particular flows on l-fans and m-connectors.

Flows f_x and f_y in an l-fan: consider an l-fan F_l, we assign a clockwise orientation to its external cycle of $(1,2) \cup (3,4)$-edges and we assign alternately the values x and y to each $(1,2) \cup (3,4)$-edge. Note that, if l is even, the first edge and the last edge of F_l receive the same flow value, otherwise, if l is odd, they receive distinct values. Finally assign to all further edges, both internal and lateral, the flow value

2.5. Once again, note that the orientation of edges with flow value 2.5 is not relevant, so we can choose lateral edges to be oriented in a clockwise direction. It is important to note that if we add the flow value 2.5 on the external cycle of the l-fan in clockwise direction, we obtain a new flow having edges with flow values x and y exchanged. We denote by f_x and f_y these two flows on an l-fan. More precisely, f_x and f_y are the flow defined as above and having flow value x and y, respectively, on the first edge of F_l.

Flows g_x and g_y in an m-connector, $m \neq 1$: Consider an m-connector, with $m \geq 2$, we proceed exactly as we did above for an l-fan: we assign a clockwise orientation to the external cycle and, since $(1,2) \cup (3,4) \subset (1,4)$, we can assign flow values exactly as for an l-fan. Again, we use the notation g_x and g_y to denote the flows having values x and y, respectively, on the unique edge of the external cycle of the m-connector directed away from v_c. Finally, if $m = 0$, we simply give to the unique edge a clockwise orientation with respect to the external cycle of W_n and we assign flow value 2.5: we will denote such a flow by g_x.

Previous flows are defined in l-fans, for any possible l, and m-connectors, for $m \neq 1$. In order to deal with 1-connectors we are going to define two methods to obtain a flow in them. Both methods partially affect the flow values on some edges of the fans adjacent to the 1-connector, but, with a suitable choice of the parameters, the resulting flows are still σ-faithful flows.

Take x and y as described before, and set $\delta' = 0.5 + \delta$.

Method A: Consider a 1-connector C and let F and F' be the two fans which precede and follow C, respectively. Assume that flows f_x or f_y are assigned on F and F' in such a way that the last edge of F and the first edge of F' have different flow values. That is: one of them has flow value x and the other has flow value y – the two possible cases are presented in Figure 12. Then, according to the flows on F and F', we can modify the flow value of the last edge of F and the first edge of F' as in Figure 12. Moreover, the same figure shows a way to assign a suitable orientation and flow value to all edges of C. The result is a new zero-sum flow for all vertices of the external cycle of the wheel and each flow value belongs to the interval assigned by the capacity function σ.

Method B: Consider a 1-connector C and let F be an l-fan, with $l > 2$, adjacent to C. Assume that a flow f_x or f_y is assigned on F: the two possible cases are presented in Figure 13. Then we can modify the flow value of three edges of F and we can assign an orientation and a flow value to all the edges of C as shown in Figure 13. The result is a zero-sum flow for all vertices of the external cycle of the wheel and each flow value belongs to the interval assigned by the capacity function σ.

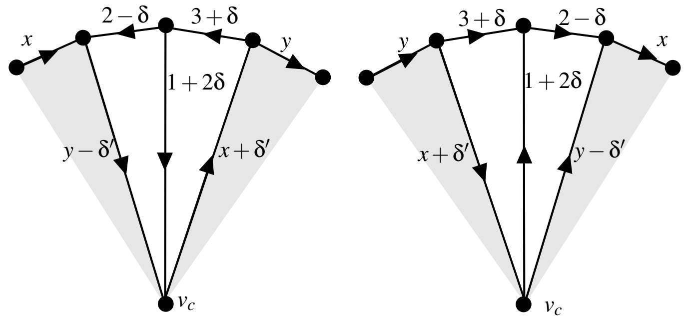

Figure 12: Method A to assign a flow to a 1-connector.

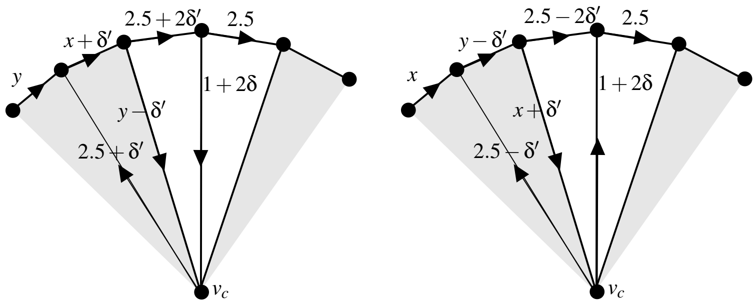

Figure 13: Method B to assign a flow to a 1-connector.

Theorem 2.29. *If $G \in (W_n, J_{(1,2)\cup(3,4)})$, where J is not the external cycle and $n \geq 4$, then $\phi_c(G) < 5$. If $n = 3$, $\phi_c(G) < 5$ if and only if J is not a 3-cycle.*

Proof. If $n = 3$ and J is a 3-cycle of W_3, then the unique vertex of W_3 not in J can always be viewed as the central vertex and J as the external cycle of W_3. In this case, the result follows by Theorem 2.28. Otherwise, if J is a 4-cycle of W_3, we can assign the flow g_x to the unique 0-connector and the flow f_x to the unique 3-fan, thus obtaining the required flow.

Now, consider $n \geq 4$ and J an even subgraph of W_n distinct from the external cycle of W_n. We will prove that for any possible J there exists a σ-faithful flow in W_n. Then, the assertion will follow by Proposition 2.3 and 2.9. In each step of the proof we construct the σ-faithful flow by first assigning flows in l-fans and m-connectors with $m \neq 1$, and then by applying Method A and Method B to assign a flow in 1-connectors as well.

CLAIM 1: If J has an m-connector C with $m \neq 1$, then $\phi_c(G) < 5$.

Proof of Claim 1: Let F be the fan following C in J. Orient every fan in J in clockwise direction. Starting from the first edge of F, we follow the assigned orientation on the edges of J and we alternately assign flow value x and y to its edges. Assign an arbitrary orientation and flow value 2.5 to all internal edges of every fan of J. In this way, we have defined in each fan a flow which is either f_x or f_y, according to the flow value of the first edge of the fan. Now we define the flow in the connectors. We assign the flow g_x to every m-connector with $m \neq 1$ (connector C included). Finally, we use Method A to assign the flow to every 1-connector. Note that, in this case, we can apply Method A to every 1-connector because C is the unique connector such that flow values of the last edge of the fan preceding it and the flow value of the first edge of the fan following it could be equal. Hence, we have constructed a flow f in W_n with the required properties, then the claim follows by Proposition 2.9. □

Hence, from now on, we can assume that all m-connectors of J are 1-connectors.

CLAIM 2: If J has an l-fan F, with $l > 2$, then $\phi_c(G) < 5$.

Proof of Claim 2: Let C be the 1-connector of J which follows F in J, and F' the fan following C. Orient every fan in J in clockwise direction. Starting from the first edge of F', we follow the assigned orientation on the edges of J and we alternately assign flow value x and y to its edges. Assign an arbitrary orientation and flow value 2.5 to all internal edges of every fan of J. In this way, we have again defined a flow in each fan which is either f_x or f_y, according to the flow value of the first edge of the fan. Now we define the flow on 1-connectors. We can use Method A to assign the flow to every 1-connector except, possibly, to C. Indeed, C is the unique 1-connector for which the flow value of the last edge of the previous fan could be equal to the flow value of the first edge of the next fan. Anyway, since F is an l-fan with $l > 2$, we can apply Method B to obtain a flow in C. Hence, we have constructed a σ-faithful flow in W_n, then the claim follows by Proposition 2.9. □

Hence, from now on, we can assume that all connectors are 1-connectors and all fans are 2-fans. In order to complete the proof, we have to distinguish between two cases according to the parity of the number of 2-fans in J. Assume that J has an even number of 2-fans (and then also an even number of 1-connectors). Orient every fan in J in clockwise direction. Starting from the first edge of an arbitrary 2-fan, we follow the assigned orientation on the edges of J and we alternately assign flow value x and y to its edges. Assign an arbitrary orientation and flow value 2.5 to all internal edges of every fan of J. In this way, we have again defined a flow in each fan which is either f_x or f_y, according to the flow value of the first edge of the fan. Now, we can use Method A to assign the flow to every 1-connector since the flow value of the last edge of a fan is always different from the flow value of the first edge of the next fan in the sequence. Hence, we have constructed a σ-faithful flow in W_n also in this case. Now, assume that J has an odd number of 2-fans (and then also an odd number of 1-connectors). Select a 2-fan F of J. Assign a flow on F and to the two connectors adjacent to F as in Figure 14. Assign alternately flow f_x and f_y to all other 2-fans and use Method A to assign the flow to each 1-connector between them. This defines a σ-faithful flow in W_n. □

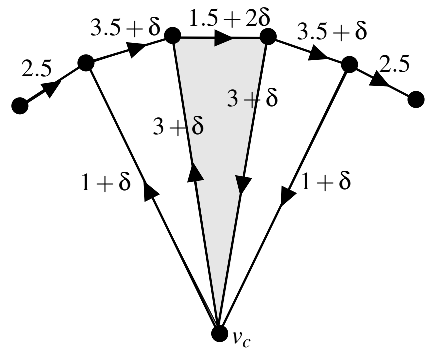

Figure 14: The flow assigned to a 2-fan and its adjacent 1-connectors in the proof of Theorem 2.29.

2.5 even subgraphs with edges having a capacity set of measure different from 2

In Section 2.4, we have completely analyzed some particular instances of the following general problem:

Problem 2.30. *Given a wheel W_n of length $n \geq 3$ with a prescribed even subgraph J and an element $A \in SI_5$, establish if $\phi_c(G) \geq 5$ for $G \in (W_n, J_A)$.*

More precisely, we completely answered to all possible instances with $\mathrm{Me}(A) = 2$. In this section, our goal is to analyze all other cases, that are those with $\mathrm{Me}(A) \neq 2$.

Let us first recall the two methods mentioned in [19] which make use of generalized edges with capacity of measure 0 and 1, respectively.

*M*1. Let G be a graph consisting of simple edges with a degree 3 vertex v, then by replacing two of the edges adjacent to v with $(2,3)$-edges we can generate a graph in $F_{\geq 5}$.

*M*2. Insert an $\emptyset$-edge anywhere in a graph G. Clearly the resulting graph does not admit a sub-5-MCNZF.

We will refer to the first method as *method M*1 and to the second as *method M*2.

2.5.1 *Set A of measure 0*

The unique set in SI_5 of measure 0 is obviously the empty-set. The unique method that involves $\emptyset$-edges is method $M2$. Now we prove that this method indeed does not produce any new examples.

Theorem 2.31. *Let G_{uv} be an $\emptyset$-edge. Denote by G' the (multi-)graph obtained from G_{uv} by identifying u and v, then $\phi_c(G') \geq 5$.*

Proof. Suppose by contradiction that $\phi_c(G') < 5$. Then there is a sub-5-MCNZF ψ in G'. Select an orientation in G' such that all edges which arise from edges incident to v in G_{uv} are oriented towards v and all edges which arise from edges incident to u in G_{uv} are oriented outward from u. Then G_{uv} inherits an orientation and a flow from G', that, with a slight abuse of terminology, we still call ψ, such that $\psi|_{E(G_{uv})} \subseteq (1, 4)$ and

$$\sum_{e \in \partial^+(u)} \psi(e) = \sum_{e \in \partial^-(v)} \psi(e) \mod 5.$$

If $x \in \mathbb{R}/5\mathbb{Z}$ is defined to be the common result of those summations, then $x \in CP_5(G_{uv}) = \emptyset$, a contradiction. □

This last result shows that whenever we generate a graph G in $F_{\geq 5}$ using method $M2$, then G could also be generated by a suitable expansion of a smaller graph $H \in F_{\geq 5}$, where H is obtained by identifying the terminals of the $\emptyset$-edge that has been used to generate G.

2.5.2 *Set A of measure 1*

Similarly as for the case of measure 2, we would like to present this case as an expansion of a suitable wheel. To this purpose let us call a *wheel of length* 2, denoted by W_2, a loopless (multi)graph with exactly 2 vertices of degree 3 and a vertex of degree 2.

Consider a pair (H, σ) that presents the configuration described in method $M1$. Call v the vertex of degree 3 and let $N_H(v) = \{w_1, w_2, w_3\}$ with both w_1v, w_2v $(2, 3)$-edges and w_3v a simple edge. Identify all vertices in $V(H) - \{v\}$ to a unique vertex w, thus obtaining a multigraph with two vertices, v and w, and three parallel edges between them, two of them are $(2, 3)$-edges and one of them, say e, is a simple edge.

Now, if we subdivide the unique simple edge e with a new vertex, then we obtain a wheel of length 2 with the external 2-cycle consisting of $(2, 3)$-edges. The subdivision operation does not alter the circular flow number of the graph because it generates a degree 2 vertex that we can suppress.

Also note that every wheel having an even subgraph of $(2, 3)$-edges presents the configuration described in $M1$, and so it can be reduced by contraction to a wheel of length 2. Moreover, the presence of the configuration described in method $M1$ assures that a graph produced in this way has circular flow number 5 or more.

We would like to stress that this is the unique case in which a wheel of length 2 does produce examples of graphs with circular flow number at least 5 by using methods described in this chapter.

2.5.3 *Set A of measure at least 3*

In this section, we consider sets of measure 3, 4 and 5. Even though, in these cases, no instance of Problem 2.30 produces a graph in $F_{\geq 5}$ we discuss them for completeness sake.

Lemma 2.32. *Consider the families G^σ and G^ρ, such that $\rho(e) \subseteq \sigma(e)$ and $\mathrm{Me}(\rho(e)) = \mathrm{Me}(\sigma(e))$ for every $e \in E$. Let $H_1 \in G^\sigma$ and $H_2 \in G^\rho$. Then a sub-5-MCNZF exists in H_1 if and only if it exists in H_2.*

Proof. The thesis follows from the fact that a sub-5-MCNZF can be taken with no integer values, possibly after adding a small quantity $\epsilon > 0$ to suitable directed cycles. □

It follows by Proposition 2.21 and Lemma 2.32 that we can restrict our analysis of sets of measure 3 to the sets $(1,4)$ and $(4,1) \cup (2,3)$.

Note that an attempt to characterize graphs in $F_{\geq 5}$ with only simple edges and $(1,4)$-edges is equivalent to ask for a direct characterization of $F_{\geq 5}$. Obviously, there is no wheel with all edges of capacity $(1,4)$ having circular flow number at least 5, since it is a planar graph.

Hence, we can focus on $(4,1) \cup (2,3)$-edges. The following theorem holds, of which we give here the complete proof, that does not appear in [P.2].

Theorem 2.33. *For every $n \in \mathbb{N}$, if $G \in (W_n, J_{(4,1)\cup(2,3)})$, then $\phi_c(G) < 5$.*

Proof. Since $(4,1) \subseteq (4,1) \cup (2,3)$, every σ-faithful flow presented in the case of $(4,1)$-edges is a σ-faithful flow also in this case. Hence, by Theorem 2.27, $\phi_c(G) < 5$ when either n is even or n is odd and there is no k-connector with $k > 1$.

So let us assume that n is odd and G only contains 0 and 1-connectors.

We are going to use flow assignments as in Section 2.4.1. Recall that we chose $z \in (4,0)$ and $x \in (1,2)$, such that $z + x$ is in $(0,1)$. Then, there are suitable values of $x \in (1,2)$ and $z \in (4,0)$ such that $z + x + x \in (2,3) \subseteq (4,1) \cup (2,3)$.

Let F be an l-fan. Assign to F the flow f^+ if l is even and the flow $\tilde{f}^+$ if l is odd and apply the procedure described in Proposition 2.26 starting from F. Finally assign to the first edge of F the new flow value $z + x + x$. One can easily check that the defined flow is a zero-sum flow at every vertex and so it is a σ-faithful flow.

The last case to be discussed is the one where the length of the wheel is odd and the chosen even subgraph is the external cycle $x_1 \ldots x_{2t+1}$, for $t \geq 2$.

Here, we define a σ-faithful flow f as follows: we orient the external cycle of the wheel in clockwise direction and we orient every edge $v_c x_i$ with i odd except for $i = 1$ towards the central vertex v_c, and away from v_c otherwise.

We define flow values as follows:

- $f(x_i x_{i+1}) := z$ for i odd $\in \{3, \dots, 2t-1\}$;
- $f(x_i x_{i+1}) := z + x$ for i even $\in \{4, \dots, 2t\}$;
- $f(x_1 x_{2t+1}) := z$;
- $f(x_1 x_2) := z + x$;
- $f(x_2 x_3) := z + x + x$;
- $f(v_c x_3) := x + x$ and $f(v_c x_i) := x$ for $i \neq 3$.

Therefore there is no graph with circular flow number at least 5 belonging to the family $(W_n, J_{(4,1)\cup(2,3)})$. $\square$

If the set A has measure either 4 or 5, a similar result holds, we again include the complete proof.

Theorem 2.34. *For every $n \in \mathbb{N}$ and every $A \in SI_5$ with* $\mathrm{Me}(A) \geq 4$*, if $G \in (W_n, J_A)$, then $\phi_c(G) < 5$.*

Proof. If $\mathrm{Me}(A) = 5$, we can simply observe that each open integer set of measure 5 contains $(4,0) \cup (0,1) \cup (2,3)$, for which Theorem 2.33 holds.

If $\mathrm{Me}(A) = 4$ then Lemma 2.32 says that we can only consider the case $A = (3,2)$. Since $(1,2) \cup (3,4)$ is a subset of $(3,2)$, from Theorem 2.28 and 2.29 we deduce that the only case that could produce a graph in $F_{\geq 5}$ is when the even subgraph J is the external cycle and the wheel has odd length. But now we will show that also in this case we do not obtain any graph in $F_{\geq 5}$.

Consider an odd wheel W_{2t+1} and let J be the even subgraph induced by the edges of the external cycle $x_1 x_2 \dots x_{2t+1}$. Assume $\sigma(e) = (3,2)$ for every edge e of J. Let $x, z \in (1,2)$ such that $y := x + z \in (3,4)$. There exists an $\alpha > 1$, sufficiently close to 1, such that

- $\alpha + z \in (1,4)$.
- $y - \alpha - z = x - \alpha \in (0,1)$.

Hence we can define a σ-faithful flow f in W_{2t+1}. We orient the external cycle of the wheel in a clockwise direction and we orient every edge $v_c x_i$ with i odd except for $i = 1$ towards the central vertex v_c, and away from v_c otherwise. We define flow values as follows:

- $f(x_i x_{i+1}) := x$ for i odd $\in \{1, \dots, 2t-1\}$;
- $f(x_i x_{i+1}) := y$ for i even $\in \{2, \dots, 2t\}$;
- $f(x_{2t+1} x_1) := y - \alpha - z$;
- $f(v_c x_i) := z$ for $i \in \{2, \dots, 2t\}$;
- $f(v_c x_{2t+1}) := \alpha + z$ and $f(v_c x_1) := \alpha$.

Therefore the cases $\mathrm{Me}(A) = 4$ and $\mathrm{Me}(A) = 5$ do not produce any new example of graph with circular flow number 5 or more. $\square$

2.6 SUMMARY AND SOME REMARKS

In this chapter we have considered all possible instances of Problem 2.30 and we have proved that several known constructions of graphs with circular flow number at least 5 can be described as particular instances of this problem.

All our results can be summarized in the following theorem:

Theorem 2.35. *If $G \in (W_n, J_I)$ where J is a (non-empty) even subgraph of W_n and $I \in SI_5$, then $\phi_c(G) \geq 5$ if and only if one the following holds*

- $I = \emptyset$;
- $I = (2, 3)$;
- $I \subseteq (4, 1)$, *n odd and J has no k-connector for $k > 1$;*
- $I \subseteq (1, 2) \cup (3, 4)$ *and either $n > 3$ odd and J is the external cycle of W_n or $n = 3$ and J is a 3-cycle.*

A more general formulation of the previous problem could be considered, where the capacity function σ is not constant in J.

Problem 2.36. *Given a wheel W_n with $n + 1$ vertices and J a non-empty even subgraph of W_n, establish for every integer n, every even subgraph J and every capacity function σ if a graph $G \in W_n^\sigma$ has circular flow number at least 5, where $\sigma(e) = (1, 4)$ for all $e \notin E(J)$.*

In particular, our results say that all methods from [19] can be described as an instance of Problem 2.30, i.e. considering σ constant on all edges of J, except the one arising from Lemma 4.6 in [19] where we need to consider the more general Problem 2.36; indeed, in this last case, we must consider a capacity function σ which could be non-constant on J.

Last observation suggests that new methods can be obtained by looking at Problem 2.36 in its general formulation, and we leave this as a possible research problem.

Máčajová and Raspaud determined all snarks with circular flow number 5 up to 30 vertices in [53]. We designed an algorithm for computing the circular flow number of a cubic graph (the details of this algorithm are described in Chapter 3 and can be found also in [P.1]). By applying this algorithm to the complete list of all snarks up to 36 vertices from [11], we were able to determine all snarks with circular flow number 5 up to that order. The counts of these snarks can be found in Table 1 of Chapter 3, where we also indicate how many snarks with circular flow number 5 fit our description. We can see in this table that nearly all small snarks with circular flow number 5 can be obtained by using methods described in this paper. We do not expect this behavior to maintain for higher orders and it would be interesting to know what fraction of snarks with circular flow number 5 is covered by these methods.

All graphs from Table 1, Chapter 3, can be downloaded from the *House of Graphs* [10] at `http://hog.grinvin.org/Snarks`. The snarks with circular flow number 5 can also be inspected at the database of interesting graphs from the *House of Graphs* by searching for the keywords "snark with circular flow number 5". The two snarks with circular flow number 5 on 36 vertices which cannot be obtained through our unified method, see Table 1 in Chapter 3, can be found by searching for "snark with circular flow number 5 which cannot be obtained" and are depicted on the right-hand side of Figure 18.

All snarks obtained by using our methods as well as the two additional snarks with circular flow number 5 on 36 vertices are cyclically 4-edge-connected but not cyclically 5-edge-connected. Therefore the following open problem remains.

Problem 2.37. *Is the Petersen graph the only cyclically* 5*-edge-connected snark with circular flow number* 5*?*

2.7 A CERTIFICATE FOR NON-CUBIC GRAPHS WITH CIRCULAR FLOW NUMBER AT LEAST 5

In general the problem of establishing the circular flow number of a bridgeless graph is hard to solve. Many flow problems indeed can be reduced to the class of cubic graphs where they can be attacked much more powerfully by making use of known structural properties of cubic graphs. In this section we show that the problem of deciding whether $\phi_c(G) \geq 5$ for a graph G can be reduced to deciding whether $\phi_c(H) \geq 5$ for every $H \in \mathcal{H}$, where $\mathcal{H}$ is a finite class of cubic graphs that can be constructed by applying suitable operations to G.

Definition 2.38. Let G be a graph and $v \in V(G)$ with $d_G(v) \geq 4$. Define $\mathcal{H}^v(G)$ to be the class of all graphs that can be obtained from G by expanding v into a copy of K_2, in such a way that no vertex of degree 1 or 2 is created in the resulting graph.

Note that, if we denote by d the degree of $v \in G$, then

$$|\mathcal{H}^v(G)| \leq 2^{d-1} - d - 1.$$

Indeed a graph in $\mathcal{H}^v(G)$ can be constructed by partitioning the set $\partial(v)$ of all edges incident to v into two disjoint subsets A_1 and A_2 such that both of them consist of at least two elements and letting all edges of A_1 be adjacent to one vertex of K_2 and all edges of A_2 be adjacent to the other one. This way we can notice that the number of graphs in $\mathcal{H}^v(G)$ can be at most half of the number of subsets A_1 of edges incident to v, such that $|A_1| \in \{2, 3, \dots, d-2\}$. Thus

$$|\mathcal{H}^v(G)| \leq \frac{2^d - 2d - 2}{2} = 2^{d-1} - d - 1.$$

It may happen that $\mathcal{H}^v(G)$ contains some graphs with a bridge, even when G is bridgeless. We recall that the circular flow number of a graph with a bridge is set to be ∞.

Now suppose that G has a vertex of degree 4 and let $\mathcal{C}^v(G)$ be the class of all graphs that can be obtained by expanding $v \in G$ in one of the two ways shown in Figure 15 (in such a way that no vertex of degree 2 is created).

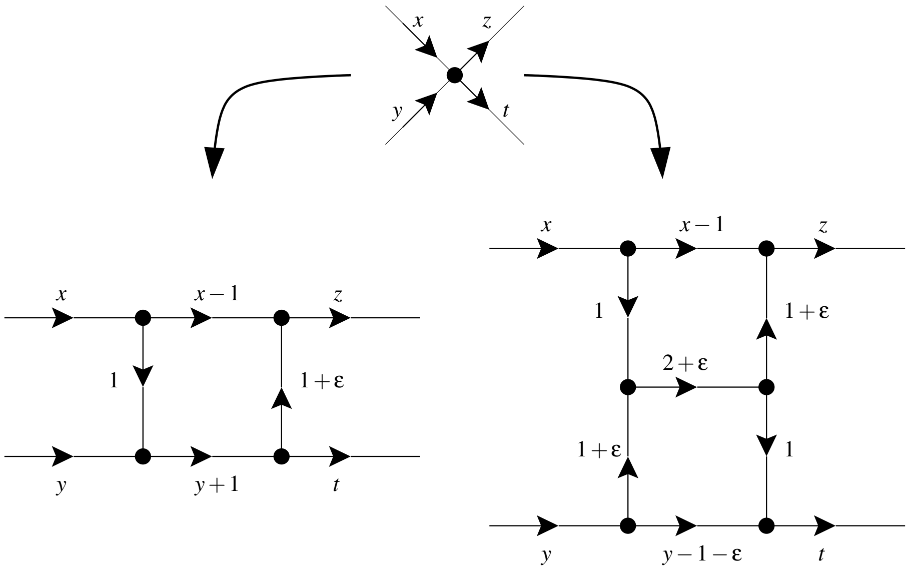

Figure 15: Expansions for a degree 4 vertex.

Definition 2.39. Let G be a graph and $v \in V(G)$ with $d_G(v) = 4$. Define

$$\mathcal{G}^v(G) := \mathcal{C}^v(G) \cup \mathcal{H}^v(G).$$

The following two propositions are the main results of this section.

Proposition 2.40. *Let G be a graph with a vertex v of degree 4. Then*

$$\phi_c(G) \geq 5 \iff \phi_c(H) \geq 5, \forall H \in \mathcal{G}^v(G).$$

Proof. The implication from the left to the right follows by noticing that every graph in $\mathcal{G}^v(G)$ is an expansion of G and by Proposition 2.18.

For the other direction we prove that if G has a circular r-flow f with $r < 5$ then f can be extended to a circular s-flow with $s < 5$ in at least one of the graphs of the family $\mathcal{G}^v(G)$.

Take such an r-flow and orient G in such a way that all flow values are positive. Call x, y, z and t the flow values of the four edges incident to v and let e_x, e_y, e_z and e_t be the corresponding edges.

Suppose first that there are three incoming flow values at v, say x, y and z. Then $x + y + z = t \in [1, 4)$, meaning that $x + y \in [2, 3)$. Then if we expand v into a copy of K_2 with vertices v_1 and v_2, in such a way that e_x, e_y are adjacent to v_1 and e_z, e_t to v_2, then we can orient the new edge from v_1 to v_2 and assign it the flow value $x + y = t - z$. Therefore the flow has been extended properly.

Suppose, on the other hand, that there are two incoming flow values x, y and two outgoing ones z, t (without loss of generality let $x \geq y$). If either $x + y \in [2, 4)$ or $|x - z| \in [1, 4)$ then we can repeat the argument used above, i.e. in the first case let e_x, e_y be adjacent to v_1 and e_z, e_t to v_2, whereas, in the second case, let e_x, e_z be adjacent to v_1 and e_y, e_t to v_2. The r-flow in G naturally extends to an s-flow in this new graph, where $s < 5$.

Hence we can suppose without loss of generality that $x + y \geq 4$ and $z = x + \epsilon$ for a suitable $\epsilon \in [0, 1)$. Indeed, if $z \leq x$ then we can just switch the orientation of all edges of G and keep their flow value unchanged. Note that from $z = x + \epsilon$ we get that $t = y - \epsilon$. Furthermore, since $x + y \geq 4$, at least one of x and y must be at least 2. By the assumption $x \geq y$ it must be $x \geq 2$. Therefore, if $y < 3$ we extend the flow as shown in the left part of Figure 15, while, if $y \geq 3$, we extend the flow as shown in the right part of Figure 15. □

Proposition 2.41. *Let G be a graph with a vertex v of degree at least 5. Then*

$$\phi_c(G) \geq 5 \iff \phi_c(H) \geq 5, \forall H \in \mathcal{H}^v(G).$$

Proof. The implication from the left to the right follows by noticing that every graph in $\mathcal{H}^v(G)$ is an expansion of G and by Proposition 2.18.

For the other direction we prove that if $\phi_c(G) < 5$ then there is at least one graph in $\mathcal{H}^v(G)$ that has circular flow number less than 5.

Take a circular nowhere-zero $\phi_c(G)$-flow f on G with an orientation such that all flow values are positive. Let $n_1 := |\partial^-(v)| > 0$ be the number of incoming edges at v and let $x_1, \ldots, x_{n_1} \in [1, 4)$ be all incoming flow values at v. Let $n_2 := |\partial^+(v)| > 0$ and $y_1, \ldots, y_{n_2} \in [1, 4)$ be all outgoing flow values at v. Without loss of generality we can suppose $n_1 \geq n_2$, because the entire orientation can be reversed. Moreover let us denote by e_{x_i} the incoming arc at v whose flow value is x_i, and, similarly, by e_{y_j} the outgoing arc at v whose flow value is y_j.

Let us first prove the statement in the case of degree exactly 5.

Notice that there cannot be 4 incoming edges at v since their sum should be at the same time greater then or equal to 4 and equal to the outgoing flow value, which is less than 4.

Hence there are three incoming flow values, namely x_1, x_2 and x_3, and two outgoing flow values, namely y_1 and y_2. Let

$$r := x_1 + x_2 + x_3 = y_1 + y_2.$$

Notice that, in order to complete the proof it is enough to exhibit a partition of $E(v) = E_1 \cup E_2$ into two disjoint subsets E_1 and E_2 of edges such that both $|E_i| > 1$ (condition required in order to guarantee that no vertices of degree 1 or 2 appear) and

$$\begin{aligned}&\Big|\sum_{e\in E_1\cap\partial^-(v)} f(e) - \sum_{e\in E_1\cap\partial^+(v)} f(e)\Big| = \\ &\Big|\sum_{e\in E_2\cap\partial^+(v)} f(e) - \sum_{e\in E_2\cap\partial^-(v)} f(e)\Big| \in [1,4).\end{aligned}$$

Indeed, if $H \in \mathcal{H}^v(G)$ is obtained by expanding v into the new edge v_1v_2 and letting all edges of E_i be adjacent only to v_i, we can extend the flow f in G to a suitable s-flow in H ($s < 5$) by letting

$$f(v_1v_2) := \sum_{e\in E_1\cap\partial^-(v)} f(e) - \sum_{e\in E_1\cap\partial^+(v)} f(e),$$

where v_1v_2 is oriented from v_1 to v_2.

If there are x_i and x_j, $i \neq j$, such that $x_i + x_j < 4$ then we are done, by taking $E_1 = \{e_{x_i}, e_{x_j}\}$. Hence, we can suppose that

$$\begin{cases} x_1 + x_2 \geq 4 \\ x_2 + x_3 \geq 4 \\ x_1 + x_3 \geq 4 \end{cases}$$

and by summing the three inequalities together we get

$$r \geq 6.$$

There exists an incoming flow value, say $\hat{x}$, which is at most $\frac{r}{3}$, and there exists an outgoing flow value, say $\hat{y}$, which is at least $1 + \frac{r}{3}$. Otherwise, if both y_1 and y_2 are less than $1 + \frac{r}{3}$, we get the following contradiction:

$$r = y_1 + y_2 < 2\big(1 + \frac{r}{3}\big) = 2 + \frac{2}{3}r \leq \frac{1}{3}r + \frac{2}{3}r = r.$$

Therefore $\hat{y} - \hat{x} \in [1,4)$ and if we take $E_1 = \{\hat{x}, \hat{y}\}$ we obtain a graph in $\mathcal{H}^v(G)$ with circular flow number less than 5.

Let us now suppose $d_G(v) \geq 6$. If

$$\sum_{i\in I} x_i - \sum_{j\in J} y_j \in [1,4),$$

for two suitable subsets $I \subseteq \{1,\dots,n_1\}, J \subseteq \{1,\dots,n_2\}$, such that $1 < |I| + |J| < n_1 + n_2 - 2$, then the thesis follows by taking $E_1 = \{e_{x_i} : i \in I\} \cup \{e_{y_j} : j \in J\}$.

Now let

$$r := \sum_{i=1}^{n_1} x_i = \sum_{j=1}^{n_2} y_j,$$

be the sum of all incoming (or all outgoing) flow values and define

$$\begin{cases} r_1 := \frac{r}{n_1} & \text{mean incoming flow value,} \\ r_2 := \frac{r}{n_2} & \text{mean outgoing flow value.} \end{cases}$$

First of all, notice that $r_1 \leq r_2$. Suppose that $r_2 - r_1 \geq 1$. Then there exists x_α and y_β such that

$$x_\alpha + 1 \leq r_1 + 1 \leq r_2 \leq y_\beta,$$

and we are done again by setting $E_1 = \{e_{x_\alpha}, e_{y_\beta}\}$.

Suppose that $r_2 - r_1 < 1$ and

$$|\sum_{i \in I} x_i - \sum_{j \in J} y_j| \notin [1,4), \tag{6}$$

for every $I \subseteq \{1,\ldots,n_1\}, J \subseteq \{1,\ldots,n_2\}$, with $1 < |I| + |J| < n_1 + n_2 - 2$. We show that this set of hypotheses leads to a contradiction.

There are x_a and x_b such that $x_a + x_b \leq 2r_1 \leq 2r_2$. Hence, if $x_a + x_b \in [5, 2r_2]$ then take $\tilde{y} \in [r_2, 4)$ and notice that

$$x_a + x_b - \tilde{y} \in (1, r_2],$$

which is not possible.

Therefore, for every couple x_a and x_b of different incoming flow values such that $x_a + x_b \leq 2r_1$, the sum $x_a + x_b < 5$, meaning that $x_a + x_b \in [4,5)$, where we deduce that $x_a + x_b \geq 4$ from (6). Then at least one of them, let us say x_a, is such that $x_a \in [1, 2.5)$. As a consequence, since $|x_a - y_j| < 1$, for every j, we get that

$$y_j \in [1, 3.5), \forall j \in \{1,\ldots,n_2\}.$$

If there is a $\hat{y} \in [1,3]$ then $x_a + x_b - \hat{y} \in [1,4)$, a contradiction. If, on the other hand, every $y_j \in (3, 3.5)$ then $y_1 + y_2 \in (6,7)$ and so $y_1 + y_2 - (x_a + x_b) \in (1,3)$, a contradiction again. □

By previous propositions, given a non-cubic graph G, we can construct the class $\mathcal{H}$ of all cubic graphs that can be obtained by repeatedly applying to G the expansions defined above. Then, $\phi_c(G) \geq 5$ is equivalent to $\forall H \in \mathcal{H}, \phi_c(H) \geq 5$.

2.8 FURTHER WORK

We conclude this chapter by presenting new construction methods that generates those two snarks not fitting the description presented in previous sections. All results presented in this final section have been obtained in collaboration with Edita Máčajová. We remark that, results of Section 2.7, have had a crucial role in finding the configurations that are described in the theorems stated below. The following theorem gives the first construction method.

Theorem 2.42. *Let G be a graph with circular flow number 5 and let $C = v_1v_2v_3v_4v_5$ be an induced 5-cycle of G such that $d(v_i) = 3$, for $i \in \{3,4\}$. Subdivide once both edges v_1v_2 and v_1v_5 and call, respectively, w_1 and w_4 the new added vertices. Then, subdivide twice the edge v_3v_4 and*

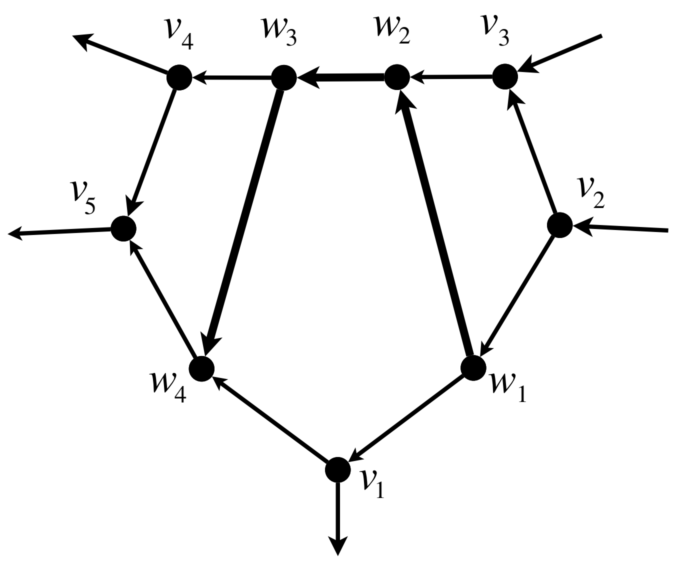

Figure 16: Insertion of the path $w_1w_2w_3w_4$ into an induced 5-cycle of a graph.

call w_2, w_3 the new added vertices, see Figure 16. Finally construct H adding the edges w_1w_2, w_3w_4 to G as depicted in Figure 16.

Let σ be the capacity function defined on H such that $\sigma(w_1w_2)$, $\sigma(w_2w_3)$, $\sigma(w_3w_4) = (4,1)$, and $\sigma(e) = (1,4)$ for all other edges e. Then, for all $K \in H^\sigma$, $\phi_c(K) \geq 5$.

Proof. Fix on H the orientation D depicted in Figure 16, and, by contradiction, let f be a σ-faithful flow in H. By Lemma 2.16, f assigns to adjacent edges of the path $w_1w_2w_3w_4$ values lying in different unit intervals of $(4,1)$. Without loss of generality, let $f(w_1w_2) = \delta \in (0,1)$ and $f(w_3w_4) = \epsilon \in (0,1)$.

Suppose that $\delta \leq \epsilon$. Fix on the cycle $w_3v_4v_5w_4$ a counterclockwise orientation D_ϵ (with respect to Figure 16) and let g_ϵ be the flow that assigns ϵ to its edges. Similarly, fix on the cycle $w_1v_1w_4v_5v_4w_3w_2$ a clockwise orientation D_δ (with respect to Figure 16) and let g_δ be the flow that assigns the value δ to all its edges. Let g be the new flow obtained after adding g_ϵ and g_δ to f. More precisely we keep as final orientation D and $g_\epsilon(e)$ is added to $f(e)$ whenever e has the same orientation in both D_ϵ and D; $g_\epsilon(e)$ is subtracted from $f(e)$ otherwise. The same holds for g_δ.

Notice that $f(w_4v_1) \in (1, 4-\delta)$ for otherwise $f(v_5w_4) \in \epsilon + [4-\delta, 4) = [4-\delta+\epsilon, 4+\epsilon)$, which is not possible. Hence we can add δ to $f(w_4v_1)$. Moreover, $g(w_1v_1) = f(v_2w_1)$ is also fine, as well as $g(v_4w_3) = g(w_3w_2) = f(w_2v_3)$. Then, $g(v_4v_5) = f(v_4v_5) + (\epsilon - \delta) \in (1,3) + [0,1) = (1,4)$, where we use the fact that $f(v_4v_5) \in (1,3)$, coming from the fact that $f(w_3v_4) = f(w_2w_3) - f(w_3w_4) \in ((4,0) - (0,1)) \cap (1,4) = ((4,0) + (4,0)) \cap (1,4) = (3,4)$. Finally $g(v_5w_4) = g(w_4v_1)$ which we already proved to be fine.

Therefore, g is a sub-5-MCNZF on G that is a contradiction.

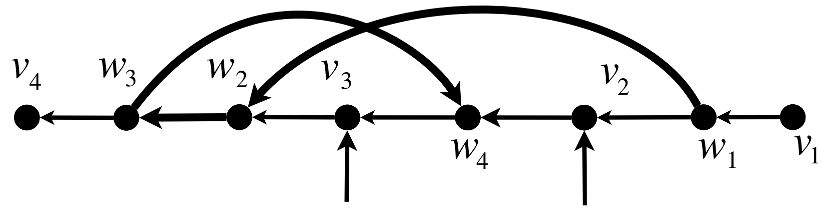

Figure 17: Subdivision of the path $v_1v_2v_3v_4$ and insertion of edges w_1w_2 and w_3w_4.

By the symmetry of the configuration of Figure 16, if $\delta > \epsilon$, we can reorient every edge and use the argument above after choosing the right cycles. □

The second construction method is given by the following theorem.

Theorem 2.43. *Let G be a graph with circular flow number 5 and let $v_1v_2v_3v_4$ be a path of G such that $d(v_3) = 3$. Subdivide once both edges v_1v_2 and v_2v_3 and call, respectively, w_1 and w_4 the new added vertices. Then, subdivide twice the edge v_3v_4 and call w_2, w_3 the new added vertices, see Figure 17. Finally construct H adding the edges w_1w_2, w_3w_4 to G as depicted in Figure 17.*

Let σ be the capacity function defined on H such that $\sigma(w_1w_2)$, $\sigma(w_2w_3)$, $\sigma(w_3w_4) = (4,1)$, and $\sigma(e) = (1,4)$ for all other edges e. Then, for all $K \in H^\sigma$, $\phi_c(K) \geq 5$.

Proof. Suppose by contradiction that f is a σ-faithful flow in H and fix on H the orientation D depicted in the Figure 17. As before we can suppose that $f(w_1w_2) = \epsilon, f(w_3w_4) = \delta$, with $\epsilon, \delta \in (0,1)$ and $f(w_2w_3) \in (4,0)$. This implies that $f(v_3w_2) \in (3,4)$, and so $f(w_4v_3) \in (1,3)$, because $d_H(v_3) = 3$. Fix on the cycle $w_1v_2w_4v_3w_2$ a clockwise orientation D_ϵ (with respect to Figure 17) and let g_ϵ be the flow that assigns ϵ to its edges. Let g be the new flow in the orientation D obtained after adding g_ϵ and to f, that is $g(e)$ is the sum of $g_\epsilon(e)$ and $f(e)$ whenever e has the same orientation in both D_ϵ and D; $g_\epsilon(e)$ is subtracted from $f(e)$ otherwise. First of all notice that $g(w_1v_2) = f(v_1w_1) \in (1,4)$. Then, from the fact that $f(w_4v_3) \in (1,3)$ and $\epsilon \in (0,1)$ we deduce that $f(v_2w_4) \in (1,3)$ and therefore $g(v_2w_4) = f(v_2w_4) + \epsilon \in (1,3) + (0,1) = (1,4)$. Now let H' be the graph obtained after removing w_1w_2 from H and suppressing vertices of degree 2. Notice that $H'/H'[w_4, v_3, w_3]$ is isomorphic to G and has a sub-5-MCNZF, where $H'[w_4, v_3, w_3]$ denotes the subgraph of H' induced by $\{w_4, v_3, w_3\}$ and $H'/H'[w_4, v_3, w_3]$ denotes the graph obtained from H' by contracting $H'[w_4, v_3, w_3]$ to a single vertex. A contradiction. □

The two snarks not fitting the presented description can be constructed as follows.

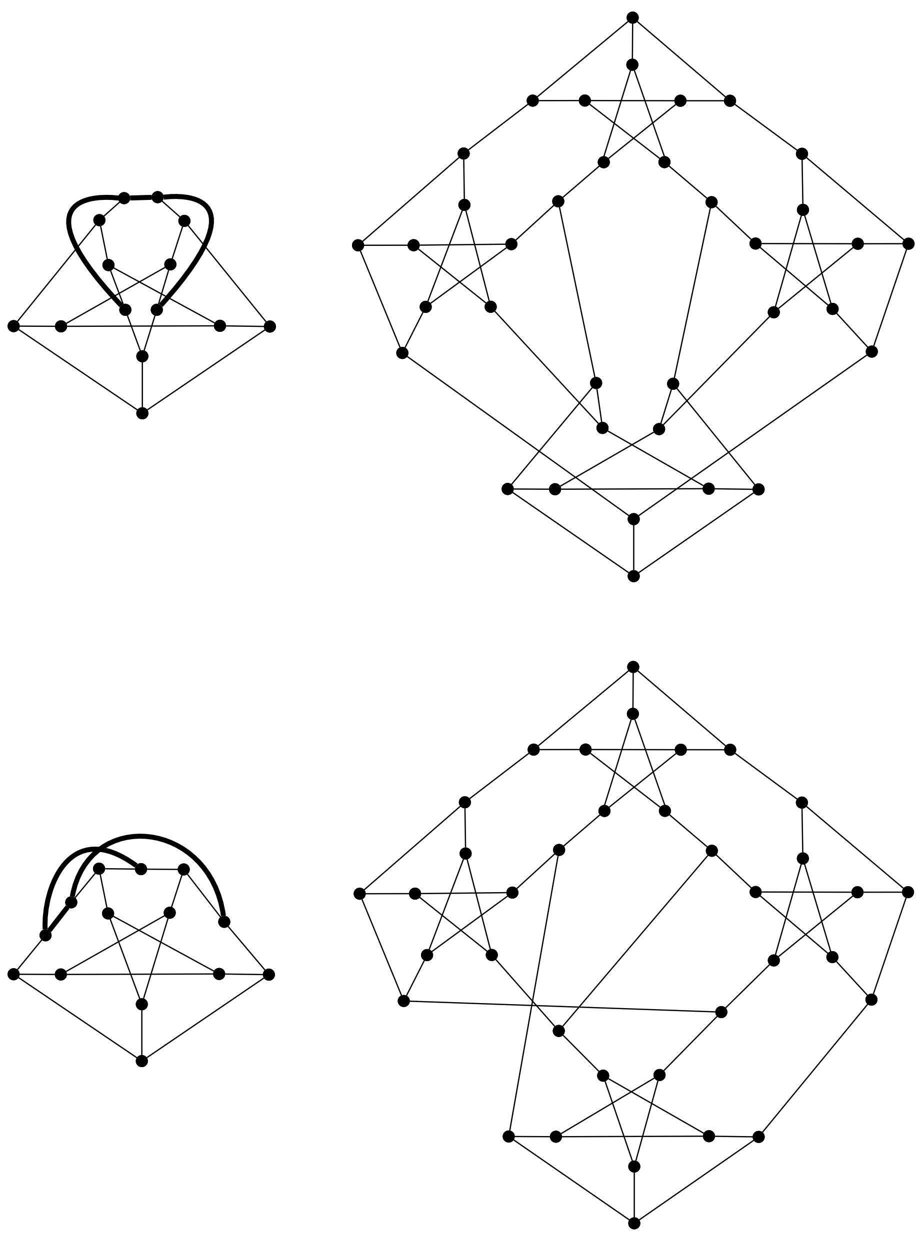

Figure 18: The two snarks on 36 vertices that do not fit the description of Theorem 2.35 are depicted on the right-hand side of the picture. On the left-hand side we show the two configurations that are used to construct them. The bold edges represent edges of capacity $(4, 1)$ and are replaced with a copy of $\mathcal{P}_{10}^*(u, v)$ each during the construction process.

Let C be a 5-cycle of the Petersen graph P_{10}. Apply the construction based on Theorem 2.42 to C using copies of the generalized edge $\mathcal{P}^*_{10}(u,v)$, that is the Petersen graph minus an edge uv, with terminals u and v, as $(4,1)$-edges. This new graph has circular flow number 5 because of Theorem 2.42 and has 38 vertices. Split off all vertices of degree 4 and 5 keeping the graph cyclically 4-edge-connected as shown in the upper part of Figure 18. Finally suppress all vertices of degree 2 that are generated during this process. The obtained cubic graph on 36 vertices is one of the two snarks not fitting the description of Theorem 2.35.

Now we describe how to construct the second one. Let T be a path on 4 vertices on the Petersen graph. Apply the construction based on Theorem 2.43 to T using once again copies of the generalized edge $\mathcal{P}^*_{10}(u,v)$ as $(4,1)$-edges. This non-cubic graph has circular flow number 5 because of Theorem 2.43 and has 38 vertices. As before, split off all vertices of degree 4 and 5 keeping the graph cyclically 4-edge-connected as shown in the lower part of Figure 18. Once again suppress all new vertices of degree 2. The obtained cubic graph on 36 vertices is the second snark not fitting the description of Theorem 2.35.

3 ON THE CIRCULAR FLOW NUMBER OF SNARKS: COMPUTATIONAL RESULTS AND NEW BOUNDS

In this chapter we continue the study of circular flows on snarks. The presented results come from the joint works [P.1] and [P.6], with Jan Goedgebeur and Giuseppe Mazzuoccolo.

3.1 INTRODUCTION

In contrast with Chapter 2, where we focused on constructions of graphs with circular flow number 5, here we attack problems of different flavor. First of all, we implement a practical algorithm that computes the circular flow number of a cubic graph. For reasons explained later, this algorithm works for all bridgeless cubic graphs having circular flow number strictly less than 5; if a bridgeless cubic graph has circular flow number at least 5, the algorithm only says that it has $\phi_c(G) \geq 5$. Clearly, if Tutte's 5-flow Conjecture holds, then it can be applied to all bridgeless cubic graphs. The algorithm works using a well-known relation between nowhere-zero flows and bisections in cubic graphs, discussed in Section 1.3. Recall that, for cubic graphs, a balanced valuation can be viewed as a bisection with, in addition, a weight function which assigns to every vertex a weight in the set $\{-p, p\}$ (where p is a positive real number) such that all vertices in the same bisection class receive the same weight value (see Theorem 1.36). More precisely, a cubic graph has a balanced valuation with weights $\{\pm\frac{r}{r-2}\}$ if and only if it has a r-CNZF, see Theorem 1.36. Determining the maximum possible value of p among all possible balanced valuations of a cubic graph G, denoted by $p_{max}(G)$, is equivalent to determining the circular flow number of the graph. Indeed, the following easy relation holds:

$$\frac{\phi_c(G)}{\phi_c(G) - 2} = p_{max}(G).$$

The chapter begins with Section 3.2 that is devoted to the description of properties of bisections that are useful for the design of the algorithm. We have already seen that such objects are deeply connected with flows and, in the last decades, they have been studied by some authors. For example in [13], the authors extend the main result of [20], stating that all bridgeless cubic graphs admit a 3-bisection, to sub-cubic graphs. In Section 3.6 we give a shorter proof of such extended result. The algorithm is presented afterwards in Section 3.3. Using our implementation of this algorithm, we determine the circular flow number of all snarks on up to 36 vertices as well as the circular flow number of various famous snarks. The results of these computations can also be found in Section 3.3.

Two of the main results are given in Section 3.4. In [48], Lukot'ka and Škoviera prove a general lower bound for the circular flow number of a snark in terms of its order (Theorem 3.8) and, at the end of their paper, they suggest that there might exist an infinite family of snarks of order $8k+2$ with circular flow numbers reaching their lower bound. The first main result we present is Theorem 3.9, where we confirm the existence of such an infinite family of snarks. The second main result is Proposition 3.10 that improves the previous known upper bound from [47] for the circular flow number of Goldberg snarks.

In Section 3.5 we present two new conjectures regarding the circular flow number of snarks. The chapter ends with Section 3.6 where we propose a new proof of the fact that all sub-cubic graphs admit a 3-bisection: this last section is based on [P.6].

3.2 USEFUL PROPERTIES OF GOOD BISECTIONS

Consider a 2-bisection $(\mathcal{B}, \mathcal{W})$ of a cubic graph G. For all $X \subseteq V(G)$, we can define $\Delta(X) = |b_X - w_X|$, where $b_X = |\mathcal{B} \cap X|$ and $w_X = |\mathcal{W} \cap X|$.

Setting a function m to be 1 on black vertices and 2 otherwise, Proposition 1.35 implies that a 2-bisection has an orientation such that black vertices have exactly 1 outgoing edge and white vertices have exactly 2 outgoing edges if and only if

$$\frac{|\partial(X)|}{\Delta(X)} \geq 1$$

for every $X \subseteq V(G)$. A 2-bisection fulfilling such property is said to be *orientable*.

It is easy to check that, for all $r < 5$, every circular nowhere-zero r-flow on a cubic graph G induces a 2-bisection on G. Indeed, it is enough to color a vertex white or black according to the number of incoming edges (1 or 2, respectively) in the orientation of G corresponding to the flow with a positive value for every edge. Moreover, Theorem 1.36 can be reformulated using k-bisections for cubic graphs as follows.

Theorem 3.1 ([35]). *For $r > 2$, a cubic graph G has an r-CNZF if and only if G has a k-bisection $(\mathcal{B}, \mathcal{W})$ such that $\frac{|\partial(X)|}{\Delta(X)} \geq \frac{r}{r-2}$.*

Thus, in order to compute the circular flow number of G, when $\phi_c(G) < 5$, we can compute, for all 2-bisections of G, the minimum ratio

$$\frac{|\partial(X)|}{\Delta(X)} \tag{7}$$

and then search for the maximum among these values. Therefore, the well-known relation between the ratio (7) and the circular flow number of G, if $\phi_c(G) < 5$, is the following.

$$\max_{\text{2-bisection of } G} \left(\min_{X \subset V(G)} \frac{|\partial(X)|}{\Delta(X)} \right) = \frac{\phi_c(G)}{\phi_c(G) - 2} \tag{8}$$

The left term is exactly the parameter $p_{max}(G)$ defined in the introduction. We would like to stress that if G has circular flow number at least 5, it is not true in general that its flow naturally induces a 2-bisection. For instance, the Petersen graph does not admit a 2-bisection at all, and there exist other bridgeless cubic graphs, admitting a 2-bisection, with the property that no 5-flow induces a 2-bisection (see for instance [3, 20, 77] for a more general discussion about bisections in cubic graphs). Theoretically, in order to manage these sporadic cases, we should admit bisections with (at most) three vertices in each connected component induced by a monochromatic class, but this would turn out to be unnecessary if Tutte's 5-flow Conjecture is true. Moreover, for all graphs G with circular flow number at least 5 that we determined (except for the Petersen graph), we could easily establish that $\phi_c(G) = 5$ since they admit a 2-bisection for which the minimum ratio $\frac{|\partial(X)|}{\Delta(X)}$ is equal to $\frac{5}{3}$. Therefore we do not present a more general version of Algorithm 1 considering 3-bisections here.

For any fixed 2-bisection of a graph, if it does exist, we call the subsets that minimize the ratio (7) *good*. Moreover, we call the 2-bisections that maximize the left term in (8) *optimal*.

If $\Delta(X) = 0$, we define its ratio to be ∞, hence we will look for subsets of $V(G)$ such that $\Delta(X) > 0$. In particular, if X is a proper subset of $V(G)$ it follows that $\frac{|\partial(X)|}{\Delta(X)} = \frac{|\partial(\bar{X})|}{\Delta(\bar{X})}$, where $\bar{X}$ denotes $V(G) - X$, and so, for a given 2-bisection, we can always find at least a good subset of order at most $\frac{|V(G)|}{2}$. From now on, we will also assume without loss of generality to have more black vertices than white ones in X, i.e. $b_X > w_X$.

Lemma 3.2. *Consider a graph G having a 2-bisection $V(G) = \mathcal{B} \cup \mathcal{W}$ and a subset $X \subseteq V(G)$. Suppose that there is $X \subseteq V$ such that $G[X]$ is disconnected with components $A_1, \dots, A_n$. Then there is one of those components A such that*

$$\frac{|\partial(A)|}{\Delta(A)} \leq \frac{|\partial(X)|}{\Delta(X)}.$$

Proof. There is $A \in \{A_1, \dots, A_n\}$ such that $\frac{|\partial(A)|}{\Delta(A)} \leq \frac{|\partial(A_i)|}{\Delta(A_i)}$ for each i. Therefore from $|\partial(A)|\Delta(A_i) \leq |\partial(A_i)|\Delta(A)$ and summing up all such inequalities we get

$$\frac{|\partial(A)|}{\Delta(A)} \leq \frac{\sum_{i=1}^{n} |\partial(A_i)|}{\sum_{i=1}^{n} \Delta(A_i)} \leq \frac{|\partial(X)|}{\Delta(X)}. \qquad \square$$

Applying previous lemma we conclude that, if $X \subseteq V$ is a good subset such that $G[X]$ is not connected, then all its connected components are good as well.

Consider a graph G with a 2-bisection. For a subset $X \subseteq V(G)$ let us denote by $\partial_V(X) = \{v \in X \colon \deg_{G[X]}(v) < 3\} = \{v \in X \colon \exists\, w \in V(G) - X$ such that $vw \in \partial(X)\}$.

Lemma 3.3. *Consider a bridgeless cubic graph G having a 2-bisection. Consider a 2-bisection $V(G) = \mathcal{B} \cup \mathcal{W}$ and one of its good subsets $X \subseteq V(G)$, with $b_X > w_X$ and $\frac{|\partial(X)|}{\Delta(X)} > 1$. Then, $\partial_V(X)$ is a subset of black vertices.*

Proof. We want to show that there are no white vertices in $\partial_V(X)$. Suppose by contradiction that there is a white vertex $v \in \partial_V(X)$.

If v is incident to a unique edge of $\partial(X)$, then setting $Y := X - v$,

$$\frac{|\partial(Y)|}{\Delta(Y)} = \frac{|\partial(X)| + 1}{\Delta(X) + 1} < \frac{|\partial(X)|}{\Delta(X)}$$

which is a contradiction since X is good.

If, on the other hand, v is incident to two edges of $\partial(X)$, then setting $Y := X - v$,

$$\frac{|\partial(Y)|}{\Delta(Y)} = \frac{|\partial(X)| - 1}{\Delta(X) + 1} < \frac{|\partial(X)| + 1}{\Delta(X) + 1} < \frac{|\partial(X)|}{\Delta(X)}$$

and again we have a contradiction. □

Remark 3.4. *If $X \subseteq V(G)$ is a good subset of vertices in a 2-bisection, then also $\bar{X}$ is good. In particular, if the 2-bisection is optimal then both $\partial_V(X)$ and $\partial_V(\bar{X})$ are monochromatic (in particular if one is white the other is black).*

Corollary 3.5. *Consider a bridgeless cubic graph G with $\phi_c(G) < 5$. Consider an optimal 2-bisection $V(G) = \mathcal{B} \cup \mathcal{W}$, and let $X \subseteq V(G)$ be a good subset. Then there is no couple of adjacent vertices v, w with the same color such that*

$$v \in X \text{ and } w \in \bar{X}.$$

Remark 3.6. *We have proved that, for a given optimal 2-bisection of a bridgeless cubic graph G with circular flow number less than 5 and among all* ***good*** *subsets of vertices*

- *there is at least one of them, say X, that induces a connected subgraph;*
- *we can search it among all subsets with cardinality at most $\frac{|V(G)|}{2}$, since the ratio of a subset equals the ratio of its complement;*
- *the boundaries $\partial_V(X)$ and $\partial_V(\bar{X})$ are monochromatic of different colors.*

The main idea of the algorithm presented in the following section is to only process sets X which satisfy the three properties in Remark 3.6. In order to assure that this produces consistent results, we need to stress that in every 2-bisection (not necessarily optimal) there exists a set X (not necessarily good) which satisfies all three properties and such that the ratio $\frac{|\partial(X)|}{\Delta(X)}$ is less than or equal to the ratio for a good set in an optimal 2-bisection. The critical property is the one on monochromatic boundaries, since it follows by Lemma 3.3 where we need to assume $\frac{|\partial(X)|}{\Delta(X)} > 1$. Indeed, in principle, it could be that in a non-orientable 2-bisection no good subsets satisfy the third property in Remark 3.6 and, at the same time, all subsets satisfying such properties have a ratio larger than the minimum one in an optimal 2-bisection. The following lemma excludes this possibility.

Lemma 3.7. *Consider a bridgeless cubic graph G having a 2-bisection. Consider a 2-bisection $V(G) = \mathcal{B} \cup \mathcal{W}$ and a good subset $X \subseteq V(G)$, with $b_X > w_X$ and $\frac{|\partial(X)|}{\Delta(X)} \leq 1$. Then, there exists a subset X' of X such that $\partial_V(X')$ is a subset of black vertices and $\frac{|\partial(X')|}{\Delta(X')} \leq 1$.*

Proof. If $\partial_V(X)$ is a subset of black vertices, then trivially we can take $X' = X$. Assume there is a white vertex v in $\partial_V(X)$.

If v is incident to a unique edge of $\partial(X)$, then, since $\frac{|\partial(X)|}{\Delta(X)} \leq 1$:

$$\frac{|\partial(X-v)|}{\Delta(X-v)} = \frac{|\partial(X)|+1}{\Delta(X)+1} \leq 1.$$

If, on the other hand, v is incident to two edges of $\partial(X)$ then,

$$\frac{|\partial(X-v)|}{\Delta(X-v)} = \frac{|\partial(X)|-1}{\Delta(X)+1} < \frac{|\partial(X)|+1}{\Delta(X)+1} \leq 1.$$

By repeatedly removing vertices in this way, we obtain a subset X' of X which satisfies the required properties. □

3.3 ALGORITHM AND COMPUTATIONAL RESULTS

The pseudocode of the algorithm to compute the circular flow number of a bridgeless cubic graph is shown in Algorithm 1. Furthermore, we also use several properties of good subsets from the previous section to speed up the algorithm (cf. Remark 3.6). It is also possible to give an optional input parameter r to the algorithm in case you only want to know if $\phi_c(G) \geq r$ or not. This is usually significantly faster than computing the exact value of $\phi_c(G)$. We implemented Algorithm 1 in the programming language C. The source code of the program can be obtained from [25].

The algorithm is exponential as it takes exponential time to generate all 2-bisections and exponential time to generate all subsets of a given bisection. Though note that the algorithm only generates subsets which satisfy the three properties from Remark 3.6. Our experiments show that for snarks on 32 vertices, on average approximately 180 000 subsets are generated per 2-bisection. This is less than 0.01% of all subsets X for which $|X| \leq \frac{|V(G)|}{2}$. Next to that, the bounding criteria allow to prune several more subset searches. Furthermore, if you only want to know if $\phi_c(G) \geq r$, the algorithm stops as soon as a 2-bisection with a `min_fraction` larger than $\frac{r}{r-2}$ is found.

In [11] Brinkmann et al. determined all snarks on up to 36 vertices. Using our implementation, we determine the circular flow number of all snarks on up to 36 vertices and the results can be found in Table 1. In particular, as we already remarked in Chapter 2, we also determine all snarks of circular flow number 5 on up to 36 vertices [P.2].

Order	Circular flow number					Total
	$4+1/4$	$4+1/3$	$4+1/2$	$4+2/3$	5	
10					1 (1)	1
18			2			2
20			6			6
22			20			20
24			38			38
26		57	223			280
28		1 258	1 641		1 (1)	2 900
30		10 500	17 897		2 (2)	28 399
32		60 008	233 042		9 (9)	293 059
34	3 627	372 708	3 457 227		25 (25)	3 833 587
36	199 338	3 339 506	56 628 773	17	98 (**96**)	60 167 732

Table 1: Counts of all snarks on up to 36 vertices with respect to their circular flow number. We indicate the number of snarks with circular flow number 5 which can be obtained by using methods of Chapter 2 in parentheses.

Moreover, using our implementation of the algorithm, we also determine the circular flow number of various famous named snarks. The results are summarized in Table 2 together with the circular flow number of the Flower snarks, which was already determined by Lukot'ka and Škoviera in [48], of the Generalized Blanuša snarks, which was already determined by Lukot'ka in [47], and of some Goldberg snarks. We remark that the circular flow number of the Goldberg snarks of order $16k+8$ is not completely determined for $k \geq 4$. It is only known to belong to the interval $[4+\frac{1}{2k+1}, 4+\frac{1}{k+1}]$, see Section 3.4.2 for further details.

Algorithm 1 Compute the circular flow number of a (bridgeless) cubic graph G

Optional input: value for r
if r is defined **then**
  `test_lower_bound` := 1 // i.e. only test if $\phi_c(G) \geq r$
else
  `test_lower_bound` := 0 // i.e. compute $\phi_c(G)$
end if
`max_min_fraction` := 0
for every 2-bisection $(\mathcal{B}, \mathcal{W})$ of G **do**
  `min_fraction` := ∞
  for every subset $X \subseteq V(G)$ for which: $2 \leq |X| \leq \frac{|V(G)|}{2}$ **and** $G[X]$ is connected **and** $\partial_V(X)$ and $\partial_V(\bar{X})$ are monochromatic of different colors **do**
    Compute $|\partial(X)|$ and $\Delta(X)$
    if $\frac{|\partial(X)|}{\Delta(X)} <$ `min_fraction` **then**
      `min_fraction` := $\frac{|\partial(X)|}{\Delta(X)}$
      if `min_fraction` $\leq$ `max_min_fraction` **then**
        abort subset search
      end if
      if `test_lower_bound` **and** `min_fraction` $\leq \frac{r}{r-2}$ **then**
        abort subset search // since we are searching for a `min_fraction` $> \frac{r}{r-2}$
      end if
    end if
  end for
  if `test_lower_bound` **and** `min_fraction` $> \frac{r}{r-2}$ **then**
    return $\phi_c(G) < r$
  end if
  if `min_fraction` $>$ `max_min_fraction` **then**
    `max_min_fraction` := `min_fraction`
  end if
end for
if `test_lower_bound` **then**
  return $\phi_c(G) \geq r$ // i.e. `max_min_fraction` $\leq \frac{r}{r-2}$
else
  return $\phi_c(G) = \frac{2 \cdot \texttt{max\_min\_fraction}}{\texttt{max\_min\_fraction} - 1}$
end if

3.4 IMPROVING BOUNDS FOR THE CIRCULAR FLOW NUMBER OF SOME SNARKS

In this section we present the two main results of the chapter, namely, for all positive integers k, we construct a snark of order $8k + 2$ and minimum possible circular flow number and we improve the best

Name	Order	ϕ_c
(Generalized) Blanuša snarks [6], [87]	$8k+2$	$4+1/2$ [47]
Flower snark J_{2k+1} [32]	$8k+4$	$4+1/k$ [48]
Goldberg snark G_3 [27]	24	$4+1/2$
Goldberg snark G_5 [27]	40	$4+1/3$
Goldberg snark G_7 [27]	56	$4+1/4$
Goldberg snark G_{2k+1} [27]	$16k+8$	$[4+\frac{1}{2k+1}, 4+\frac{1}{k+1}]$
Loupekine snark 1 and 2 [33]	22	$4+1/2$
Celmins-Swart snarks 1 and 2 [9]	26	$4+1/2$
Double star snark [32]	30	$4+1/3$
Szekeres snark [76]	50	$4+1/2$
Watkins snark [87]	50	$4+1/3$

Table 2: The values of the circular flow number of various famous snarks.

known upper bound for the circular flow number of the Goldberg snarks.

3.4.1 *Snarks having minimum possible circular flow number*

In [48] a lower bound on the circular flow number that depends only on the order of a graph is given, that is:

Theorem 3.8 (Lukot'ka and Škoviera [48])**.** *Let G be a connected bridgeless cubic graph of order at most $8k+4$ that does not admit any 3-edge-coloring. Then*

$$\phi_c(G) \geq 4 + \frac{1}{k}.$$

In the same paper it is shown that Flower snarks form a family of snarks of order $8k+4$ that attain this bound with equality, more precisely the Flower snark J_{2k+1} has $8k+4$ vertices and circular flow number $4+\frac{1}{k}$, which shows that the upper bound given in [71] was indeed the optimal one.

The paper also reports that Edita Máčajová (using a computer search from [53]) determined that the two Blanuša snarks on $18 = 8 \cdot 2 + 2$ vertices have circular flow number $4+\frac{1}{2}$ and that there are exactly 57 snarks on $26 = 8 \cdot 3 + 2$ vertices with circular flow number $4+\frac{1}{3}$. In [48] Lukot'ka and Škoviera mention that this strongly suggests that there exists an infinite family of snarks of order $8k+2$ with circular flow number $4+\frac{1}{k}$. They also report that they are not aware of any graphs of order $8k$ or $8k-2$ with circular flow number $4+\frac{1}{k}$.

In Table 1 from the previous section, we determined the circular flow number of all snarks on up to 36 vertices. The graphs from Table 1 with the minimum circular flow number for each order can be downloaded from the *House of Graphs* [10] at `http://hog.grinvin.org/Snarks`. As can be seen from that table, none of the snarks on up to 36 vertices of order $8k$ or $8k-2$ has circular flow number $4+\frac{1}{k}$.

We now present a family $\mathcal{S} = \{S_k \colon k \in \mathbb{N}\}$ consisting of snarks of order $8k+2$ and having circular flow number $4+\frac{1}{k}$. Every snark S_k is obtained by performing a dot product of S_{k-1} and a copy of the Petersen graph P_k with two adjacent vertices removed in a suitable way. We recall the definition of dot product of two connected cubic graphs, say G and H, on at least 6 vertices (see also Figure 19). Consider $G' = G - \{ab, cd\}$, where ab and cd are independent edges of G. Let $H' = H - \{x, y\}$, where x and y are adjacent vertices in H, and let u, v and w, z be the other two neighbours of x and y, respectively. Then the dot product $G \cdot H$ is defined as the graph

$$(V(G) \cup V(H'), E(G') \cup E(H') \cup \{au, bv, cw, dz\}).$$

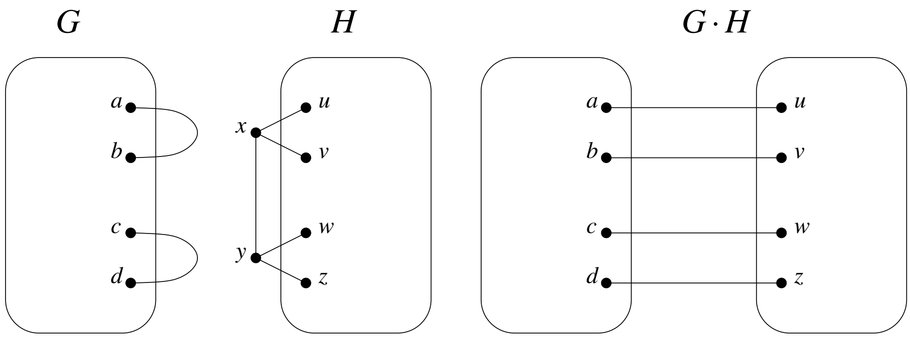

Figure 19: The dot product operation.

Indeed, if the way in which vertices a, b, c, d and u, v, w, z are linked is not specified, there are several ways to form the dot product for selected edges ab, cd and vertices x and y. This order will be relevant in our construction as only one specific way seems to work for our aims.

In what follows, we always consider the edge set of $G \cdot H$ as partitioned in three subsets $E(G')$, $E(H')$ and $\{au, bv, cw, dz\}$, and, with a slight abuse of terminology, we refer to the edges of $G \cdot H$ in $E(G')$, $E(H')$ and $\{au, bv, cw, dz\}$ as edges of G in $G \cdot H$, edges of H in $G \cdot H$, and new edges of $G \cdot H$, respectively.

We inductively define the snark S_k as follows:

- Let S_1 be the Petersen graph.
- Let S_2 be the Blanuša snark obtained by performing the dot product between two copies P_1 and P_2 of the Petersen graph where, we select a pair of edges of P_1 at distance 1 (where by distance we mean the number of edges of the shortest path connecting two ends of those edges) and a pair of adjacent vertices of P_2.

- For $k \geq 3$, S_k is a dot product of S_{k-1} and a copy P_k of the Petersen graph, where we select a pair of adjacent vertices of the Petersen graph (by symmetry every pair) and two independent edges of S_{k-1}: one in the copy of P_{k-1} and the other one in the set of new edges of S_{k-1} as illustrated in Figure 20 (bold edges).

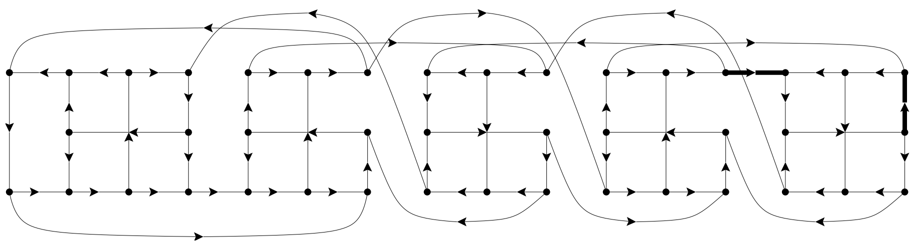

Figure 20: The Snark S_5.

It is easy to check that S_2 has order $18 = 8 \cdot 2 + 2$ and that it has circular flow number $4 + \frac{1}{2}$.

Theorem 3.9. *For any positive integer k, S_k is a snark of order $8k + 2$ with circular flow number $4 + \frac{1}{k}$.*

Proof. It is well known that the dot product of two snarks is a snark (see [32]), and an easy computation shows that the order of S_k is equal to $8k + 2$. Theorem 3.8 gives the lower bound $4 + \frac{1}{k}$ on the circular flow number, for all snarks of order $8k + 2$. Then, we only need to show that a nowhere-zero flow with maximum flow value $3 + \frac{1}{k}$ can be defined on S_k. We construct such a flow in the following way. First, we exhibit a 4-flow (D_k, f_k) on S_k which has flow value zero only for a specific edge e (the dashed edge in Figure 22).

Let (D, f) be the 4-flow on S_1 defined as in Figure 21 (right) and let (D^{-1}, f) be the 4-flow on S_1 obtained from (D, f) by reversing the orientation of every edge. Moreover let $(\mathcal{D}_k, f)$ be the 4-flow in P_k defined as follows:

$$(\mathcal{D}_k, f) = \begin{cases} (D, f) & \text{if } k \text{ is even,} \\ (D^{-1}, f) & \text{otherwise.} \end{cases}$$

We construct such a 4-flow (D_k, f_k) on S_k as follows:

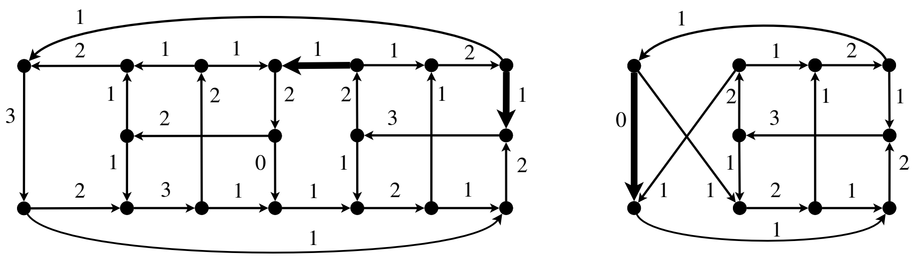

Figure 21: The 4-flow f_2 in S_2 (left) and the 4-flow (D, f) on S_1 (right).

- Fix on S_2 the 4-flow (D_2, f_2) as shown in Figure 21 (left);
- For $k \geq 3$, the dot product $S_{k-1} \cdot P_k$ can be performed in such a way that the vertices x, y such that $xy \in E(P_k)$ and $f(xy) = 0$ are removed. Then, we define (D_k, f_k) to be the unique 4-flow on S_k such that $D_k = D_{k-1}$ and $f_k = f_{k-1}$ when restricted to the edges of S_{k-1} in S_k and such that $D_k = \mathcal{D}_k$ and $f_k = f$ when restricted to the edges of P_k in S_k. The iterative construction works as the edges that will be selected when performing the dot product $S_k \cdot P_{k+1}$ still have flow value 1 and the right orientation.

The 4-flow (D_k, f_k) has the desired properties (Figure 22 shows the flow f_5 in S_5.)

Then, we construct a set of k oriented cycles in the orientation D_k, say $C_1, \ldots, C_k$, in S_k such that:

- the edge e belongs to every C_i;
- every edge of S_k having flow value 3 in f_k belongs to exactly one of the cycles C_i's.

Such properties assure that the flow (D_k, f'_k) obtained by adding $\frac{1}{k}$ along every oriented cycle C_i to (D_k, f_k) is a nowhere-zero $(4 + \frac{1}{k})$-flow on S_k. Indeed, the former property implies that the edge e has flow value $k \cdot \frac{1}{k} = 1$ in f'_k, and the latter one implies that every other edge has flow value in the interval $[1, 3 + \frac{1}{k}]$.

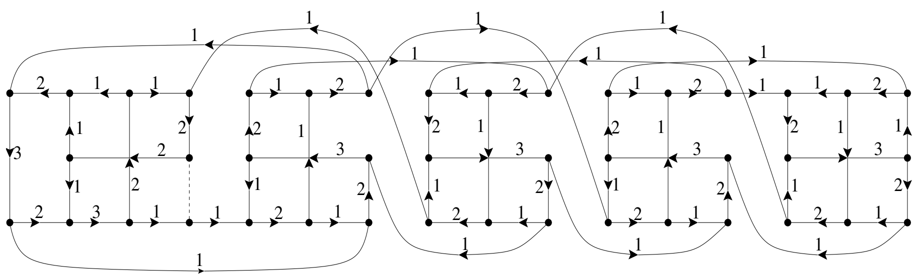

Figure 22: A 4-flow in S_5: the dashed edge is the unique one with flow value zero.

Figure 23 shows the set of five cycles for the case $k = 5$. We refer to this example to briefly explain the general construction of the cycles $C_1, \ldots, C_k$.

For any $k > 1$, the cycle C_1 of S_k is the highlighted cycle in the first graph of Figure 23. Indeed, note that C_1 does not depend on how many times a dot product is performed to obtain S_k. Moreover, it contains the two leftmost edges with flow value 3 in f_k.

Every other C_i, for $1 < i < k$, is constructed analogously as it is shown for the second, third and fourth cycle in Figure 23. In particular, note that also in this case C_i does not depend on the value of k, if $k > i$, and the unique edge of S_k with flow value 3 in the cycle C_i does not belong to any cycle C_j with $i \neq j$.

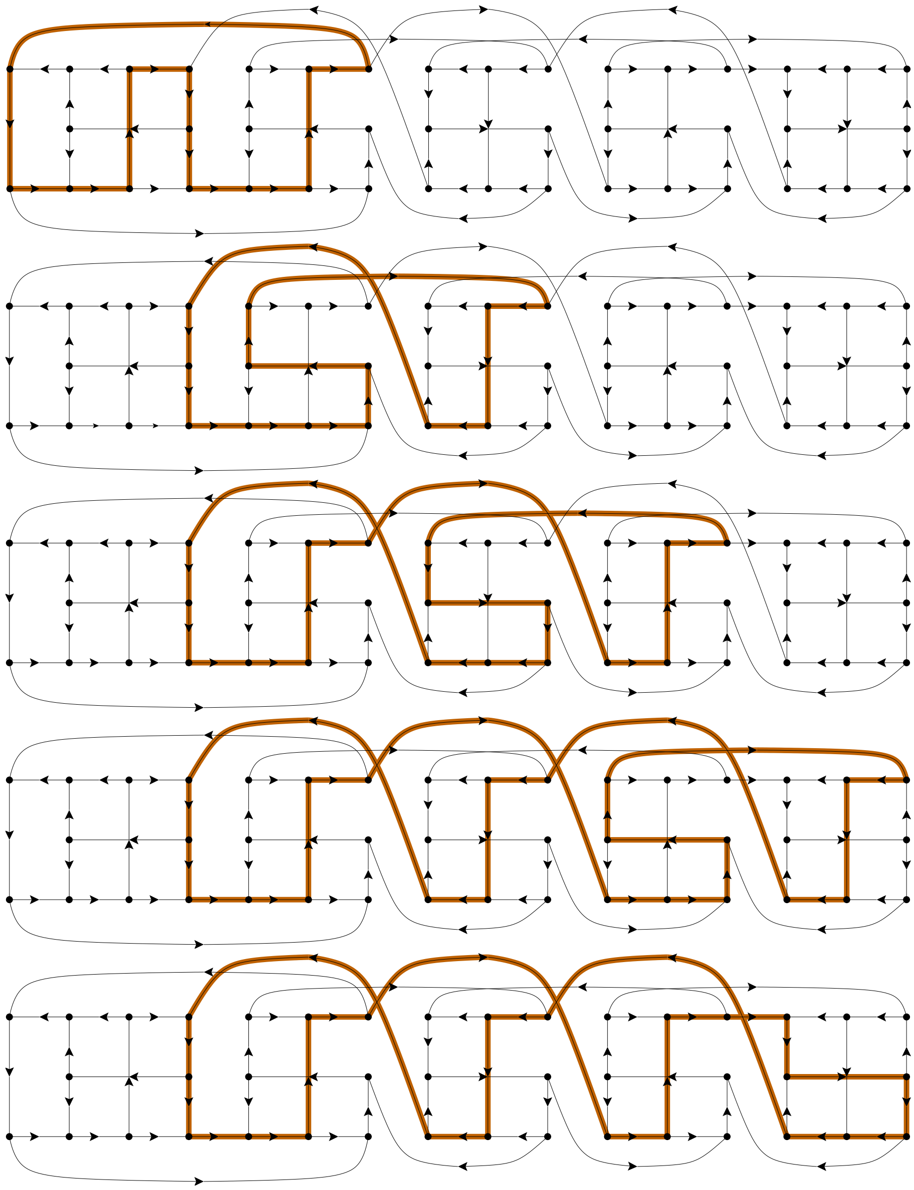

Figure 23: Cycles C_1, C_2, C_3, C_4 and C_5 in S_5.

Finally, we construct the last cycle C_k in analogy to the construction of the fifth cycle in Figure 23. Again the unique edge with flow value 3 in the cycle C_k does not belong to any cycle C_j with $j \neq k$.

All of these k oriented cycles pass through the dashed edge e in Figure 22 and, as remarked, every edge of S_k with flow value 3 belongs to exactly one of them. Then, we can construct a nowhere-zero $(4 + \frac{1}{k})$-flow of S_k and the assertion follows. □

For the sake of completeness, we remark that all snarks S_k constructed here are permutation snarks of order $8k + 2$, i.e. S_k admits a 2-factor consisting of two chordless cycles of length $4k + 1$.

3.4.2 *A new upper bound for the circular flow number of Goldberg snarks*

The Goldberg snarks $\{G_{2k+1}\}_{k\in\mathbb{N}}$ are another classical family of snarks. The snark G_{2k+1} is constructed in the following way. Let P^- be the Petersen graph minus two vertices at distance 2, take $2k+1$ copies $P_1^-, \ldots, P_{2k+1}^-$ of P^- and glue them together as shown in Figure 24.

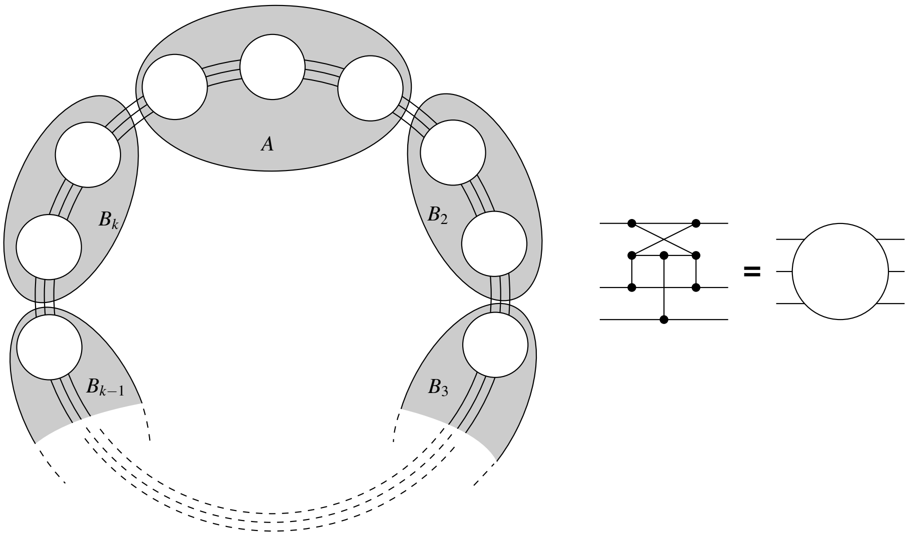

Figure 24: The Goldberg snark G_{2k+1} on $8(2k+1)$ vertices.

In [47] the circular flow number of the Goldberg snark G_{2k+1} is shown to be inside the interval $[4+\frac{1}{2k+1}, 4+\frac{1}{k}]$. By using our implementation of the algorithm described in Section 3.3, we have shown (see Table 2) that $\phi_c(G_{2k+1}) = 4+\frac{1}{k+1}$ for $k = 1, 2, 3$. Here, we show that the value $4+\frac{1}{k+1}$ is an upper bound for the circular flow number of G_{2k+1} for all possible k, thus improving the best known upper bound.

Proposition 3.10. *Let G_{2k+1} be the Goldberg snark of order $8(2k+1)$. Then,*

$$\phi_c(G_{2k+1}) \leq 4 + \frac{1}{k+1}.$$

Proof. The Goldberg snark G_{2k+1} consists of $2k+1$ copies of the Petersen graph minus two vertices at distance 2 glued together as shown in Figure 24. Define the multipole A to be three consecutive blocks of G_{2k+1} and each multipole B_t to be two consecutive blocks of G_{2k+1}, for $t = 2, \ldots, k$. We define on these multipoles the nowhere-zero circular flow represented in Figures 25 and 26. Note that the flow value on each edge of these multipoles is between 1 and $3+\frac{1}{k+1}$.

Moreover, we can glue them together as shown in Figure 24, in such a way that the Goldberg snark G_{2k+1} is constructed. It follows that a nowhere-zero circular $(4+\frac{1}{k+1})$-flow is defined in G_{2k+1}, for every positive integer k. $\square$

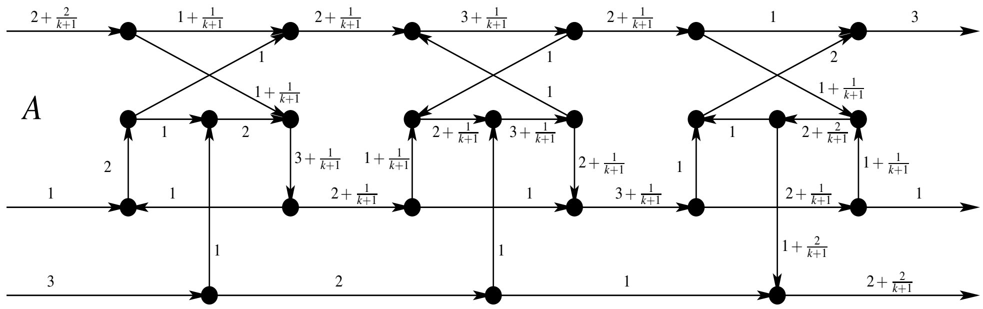

Figure 25: Flow in the multipole A.

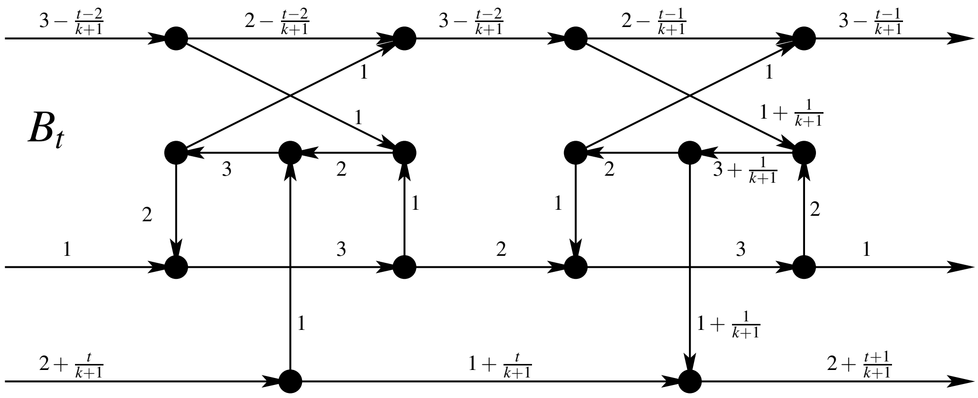

Figure 26: Flow in the multipole B_t.

3.5 OPEN PROBLEMS AND NEW CONJECTURES

We verified by computer that none of the cyclically 4-edge-connected cubic graphs G without a 3-edge-coloring, having girth at least 4 and on at most 32 vertices such that $|V(G)| \equiv 0$ or $6 \mod 8$ has a circular flow number that attains the lower bound of Lukot'ka and Škoviera [48] from Theorem 3.8 (cf. Table 1). This fact implies that none of the non-3-edge-colorable cubic graphs of order at most 32 such that $|V(G)| \equiv 0$ or $6 \mod 8$ has a circular flow number attaining the bound from Theorem 3.8. Indeed, assume that there exists such a graph G having a 3-edge-cut (and the property that $\phi_c(G)$ attains the bound of Theorem 3.8). Then, a non-3-edge-colorable smaller graph can be constructed by contracting one of the two sides of the 3-edge-cut, say H, and $\phi_c(H) \le \phi_c(G)$ holds, because G could be seen as an expansion of H. Hence, either we get a contradiction with Theorem 3.8 if H has much fewer vertices than G, or, iterating this process, we get a cyclically 4-edge-connected cubic graph with no 3-edge-coloring having circular flow number that attains the bound of Theorem 3.8, in contradiction with our computational results. Note that a similar argument applies to 2-edge-cuts.

Computational evidence suggests the following strengthened version of Theorem 3.8:

Conjecture 3.11. *Let G be a connected bridgeless cubic graph of order at most* $8k + 8$ *that does not admit any* 3*-edge-coloring. Then*

$$\phi_c(G) \geq 4 + \frac{1}{k}.$$

By using the algorithm presented here, we verified that the circular flow number of the Goldberg snarks G_3, G_5 and G_7 meet the upper bound given by Proposition 3.10. This seems to suggest that the following conjecture could be true:

Conjecture 3.12. *Let* G_{2k+1} *be the Goldberg snark on* $8(2k+1)$ *vertices. Then*

$$\phi_c(G_{2k+1}) = 4 + \frac{1}{k+1}$$

for every positive integer k.

3.6 3-BISECTIONS IN SUBCUBIC GRAPHS

Esperet, Mazzuoccolo and Tarsi [20] proved in 2017 that every simple cubic graph admits a 3-bisection. Recently, Cui and Liu [13] extended that result to the class of simple subcubic graphs. Their proof is an adaptation of the quite long proof of the cubic case to the subcubic one. In this section, we propose an easier proof of a slightly stronger result. Namely, starting from the result for simple cubic graphs, we prove the existence of a 3-bisection for all cubic graphs (also admitting parallel edges). Then we prove the same result for the larger class of subcubic graphs as an easy corollary.

Along this section all graphs can have parallel edges but not loops. A graph without parallel edges will be called simple. Let G be a graph, a vertex partition of G is a partition $c = (\mathcal{B}, \mathcal{W})$ of its vertex-set $V(G)$ into two disjoint subsets, i.e. $V(G) = \mathcal{B} \cup \mathcal{W}$ and $\mathcal{B} \cap \mathcal{W} = \emptyset$. We often identify a vertex partition with the vertex coloring $c\colon V(G) \to \{black, white\}$ where $c(v) = black$ if and only if $v \in \mathcal{B}$. For that reason, we will refer to each connected component of the two subgraphs induced by B and W as a *monochromatic component* of c.

We define bisections of non-cubic graphs as follows.

Definition 3.13. A vertex partition of G is a *k-bisection* of G if:

i) $|\mathcal{B}|$ and $|\mathcal{W}|$ differ by at most one;

ii) every monochromatic component is a tree on at most k vertices.

Open problems and recent results on the existence of a 2-bisection in cubic graphs can be found in [3], [2], [5] and [12], while the existence of a 3-bisection in a cubic simple graph is proved in [19] (for a stronger result see also [77]):

Theorem 3.14 ([19]). *Every simple cubic graph admits a* 3*-bisection.*

Very recently, Cui and Liu [13] extended this result to the class of subcubic graphs, i.e. graphs having maximum degree at most 3. In this section, we obtain a proof of a slightly stronger version of their result as a corollary of Proposition 3.15 claiming that every cubic graph (not necessarily simple) has a 3-bisection.

Main results

If a graph admits a 3-bisection, each monochromatic component is a path of length at most two. In what follows, we call *very bad* a vertex which is an end of a monochromatic path of length two, and we call *bad* a vertex which is the inner vertex of a monochromatic path of length two. Finally, we call *good* all other vertices, that is all vertices of monochromatic paths of length 0 or 1.

Proposition 3.15. *Every cubic graph admits a* 3*-bisection.*

Proof. By contradiction, let G be a smallest counterexample to the statement. The graph G is connected, otherwise one of its connected components would be a smaller counterexample. If G has three parallel edges, then G is the unique graph with two vertices and three edges which is trivially not a counterexample. By Theorem 3.14, G is not simple, and so it has at least two vertices, say x and y, connected by exactly two parallel edges. Let u and v be the other vertices adjacent to x and y respectively. If u and v are distinct, consider the graph H obtained by removing x and y from G and by adding a new edge with ends u and v. Since H is a connected cubic graph with less vertices than G, it has a 3-bisection c. We extend c to a 3-bisection of G by giving to x and y a color in the following way: set $c(x) = c(v)$ and $c(y) \neq c(x)$. It is clear that property i)) of a bisection is preserved since the additional vertices x and y receive different colors. Moreover, an easy check shows that we are not creating larger monochromatic components. Therefore, x and y must have a common neighbor, say u, and the same holds for every pair of vertices of G joined by two parallel edges. In this case u is a cut vertex and there is an edge uw which is a bridge of G (see left part of Figure 27).

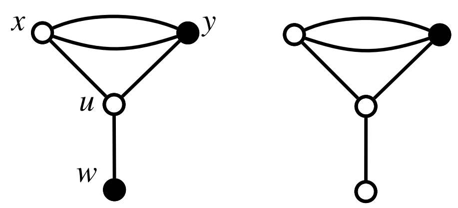

Figure 27: The two colorings c_1 (left) and c_2 (right) of Y.

From now on, we denote by Y the subgraph induced by $\{x, y, u, w\}$. Moreover, we will make use of the two colorings of Y showed in Figure 27 and denoted by c_1 and c_2. Note that c_1 have the same number of black and white vertices, while c_2 has two more white vertices.

Let w_1, w_2 be the other two neighbors of w in G. Note that w_1 and w_2 are distinct, since otherwise w and w_1 are two vertices connected by two parallel edges without a common neighbor. Let H be $G - Y + w_1w_2$. Again, H has a 3-bisection since it is cubic and smaller than G. Among all possible 3-bisections of H, we choose one, say c, with the minimum number of very bad vertices, and from now on we assume w.l.o.g. that w_1 is colored white. If $c(w_1) = c(w_2)$, then we extend c to a 3-bisection of G by assigning to vertices of Y colors as in c_1, that clearly preserves both properties of a bisection. Hence, we can assume $c(w_2)$ is black. If w_2 is good, we can easily extend c to G by giving colors to Y as in c_1 again. Property ii)) of a bisection is preserved since, even if w and w_2 receive the same color, w is in a monochromatic component of c of order at most three. If just w_1 is good, we can switch all colors in H and argue as previous case. Hence, w_1 and w_2 are vertices of a monochromatic path with three vertices. We have the following two cases.

CASE I: one between w_1 and w_2 is a bad vertex, w.l.o.g. we let w_1 be such a vertex (otherwise we switch colors in c and argue considering w_2). Note that in this case $w_1w_2 \notin E(G)$. First we recolor w_1 black and then we assign to the vertices of Y the colors as in c_2. This procedure fixes the gap created when we change the color of w_1 from white to black, and it assures that property i)) is satisfied. Moreover, no monochromatic component of order larger than three is generated. Therefore we have a 3-bisection of G.

CASE II: w_1, w_2 are very bad. Since c is a 3-bisection with minimum number of very bad vertices, it follows that every connected component of the subgraph induced by all very bad vertices in H is a path. Indeed, the maximum degree of such a subgraph is at most two, and, if it contains a circuit, we reduce the number of very bad vertices by switching colors along it. Consider the path P containing w_1 and w_2 and denote by z the end of P closer, in the path length, to w_1 than w_2. Note that z could coincide with w_1. Call $Q \subseteq P$ the subpath connecting w_1 to z. If z is adjacent to a good vertex, we switch colors to all vertices of Q. Otherwise, z is adjacent to a bad vertex b contained in a monochromatic path of length two: let v_1 and v_2 be the two very bad vertices in such a path. If $\{v_1, v_2\} \cap Q \neq \emptyset$ we switch colors to all vertices of Q as before, otherwise we switch colors to all vertices of $Q + b$. Now all monochromatic components are paths on at most three vertices, but for the component containing w_1 and w_2 that could be a path of length (at most) five. We extend such a coloring to a 3-bisection of G as follows: if the number of black and white vertices in H is the same, we color vertices in Y the opposite way as they are colored in c_1. Otherwise, H has two more black vertices and we color vertices in Y as in the coloring c_2. In all cases, both property i)) and ii)) are satisfied and so we have a 3-bisection of G. □

Now, we use previous proposition to prove the following corollary. As already remarked, the same result limited to the class of simple subcubic graphs was already proved in [13].

Corollary 3.16. *Let G be a subcubic graph. Then, G has a 3-bisection.*

Proof. It is straightforward to check that if we prove the assertion for connected subcubic graphs then it holds in general. Hence, let us assume G connected from now on. We argue by contradiction: let G be a counterexample having the minimum possible number of vertices of degree 1, and among them let G be one with the minimum number of vertices of degree two. Denote by $\{u_1, \dots, u_s\}$ and $\{v_1, \dots, v_t\}$ the set of vertices of G having degree 1 and 2, respectively. If $s = t = 0$, then G is cubic and it admits a 3-bisection by Proposition 3.15. If $s > 0$, we construct the graph G' by adding to G two new vertices x and y, two parallel edges xy among them and the new edges xu_1 and yu_1. The graph G' is subcubic and it has less than s vertices of degree one, then it admits a 3-bisection, say c. Since x and y have different colors in every 3-bisection of G', then c naturally induces a 3-bisection of G, that is a contradiction. Hence, we can assume $s = 0$. If $t \geq 2$, then we can add an edge v_1v_2 to G thus obtaing a graph with $s = 0$ and less than t vertices of degree two, and so admitting a 3-bisection. The very same coloring induces a 3-bisection also for G, that is a contradiction again. Hence it remains to consider the case $s = 0$, $t = 1$: construct the cubic graph G' starting from G by adding three new vertices u, v, w and by adding two parallel edges uv and the three edges v_1w, uw, vw. The graph G' is cubic and then it admits a 3-bisection by Proposition 3.15. Moreover, $V(G') = V(G) \cup \{u, v, w\}$. The numbers of black and white vertices in $\{u, v, w\}$ differ exactly by one in every 3-bisection of G', and so the numbers of black and white vertices in $V(G)$ differ by one too. We conclude that G admits a 3-bisection in this case as well. □

4

EDGE COLORINGS AND CIRCULAR FLOWS ON REGULAR GRAPHS

In this chapter we attack problems of nowhere-zero flows on $(2t+1)$-regular graphs. All results presented here are from a joint work with Eckhard Steffen [P.3].

4.1 INTRODUCTION

In Section 1.1 we recall a theorem of Tutte claiming that a cubic graph G has a nowhere-zero 3-flow if and only if G is bipartite and that G has a nowhere-zero 4-flow if and only if G is a class 1 graph. Theorem 1.43 shows that there is no cubic graph H with $3 < \phi_c(H) < 4$ and that a $(2t+1)$-regular graph G is bipartite if and only if $\phi_c(G) = 2 + \frac{1}{t}$ and has $\phi_c(G) \geq 2 + \frac{2}{2t-1}$ otherwise. Moreover, a characterization of $(2t+1)$-regular graphs with circular flow number equal to $2 + \frac{2}{2t-1}$ is given in [74].

Theorem 4.1 ([74]). *A $(2t+1)$-regular graph G has a 1-factor M such that $G - M$ is bipartite if and only if $\phi_c(G) \leq 2 + \frac{2}{2t-1}$.*

In [74] it is further shown that, in contrast with the cubic case, for every $t \geq 2$, there is no flow number that separates $(2t+1)$-regular class 1 graphs from class 2 ones. In particular Theorem 4.1 implies that a $(2t+1)$-regular graph G having $\phi_c(G) \leq 2 + \frac{2}{2t-1}$ is class 1 and it was conjectured that this is the biggest flow number r such that every $(2t+1)$-regular graph H with $\phi_c(H) \leq r$ is class 1. If we define the parameter $\Phi^{(2)}(2t+1) := \inf\{\phi_c(G) \colon G \text{ is a } (2t+1)\text{-regular class 2 graph}\}$, such a conjecture can be stated as follows.

Conjecture 4.2 ([74]). *For every integer $t \geq 1$*

$$\Phi^{(2)}(2t+1) = 2 + \frac{2}{2t-1}.$$

In Section 4.2 we prove Conjecture 4.2. Moreover, let us define $\mathcal{G}_{2t+1} := \{G \colon G \text{ is a } (2t+1)\text{-regular class 1 graph such that there is no perfect matching } M \text{ of } G \text{ such that } G-M \text{ is bipartite}\}$ and consider the following parameter:

$$\Phi^{(1)}(2t+1) := \inf\{\phi_c(G) \colon G \in \mathcal{G}_{2t+1}\}.$$

In Section 4.2 we further prove that $\Phi^{(1)}(2t+1) = 2 + \frac{2}{2t-1}$, for all positive integers t.

If a graph G has a small odd edge cut, say of cardinality $2k+1$, then $\phi_c(G) \geq 2 + \frac{1}{k}$. Recall that an *r-graph* is an r-regular graph G such that $|\partial_G(X)| \geq r$, for every $X \subseteq V(G)$ with $|X|$ odd. The circular flow number of the complete graph K_{2t+2} on $2t+2$ vertices is $2+\frac{2}{t}$ [71] and K_{2t+2} is a class 1 graph. In [74] the following two conjectures are proposed.

Conjecture 4.3 ([74])**.** *Let G be a $(2t+1)$-regular class 1 graph. Then $\phi_c(G) \leq 2+\frac{2}{t}$.*

Conjecture 4.4 ([74])**.** *Let G be a $(2t+1)$-graph. Then $\phi_c(G) \leq 2+\frac{2}{t}$.*

Since $(2t+1)$-regular class 1 graphs are $(2t+1)$-graphs Conjecture 4.4 implies Conjecture 4.3. We show that both these conjectures are false by constructing $(2t+1)$-regular class 1 graphs with circular flow number greater than $2+\frac{2}{t}$. The construction of the counterexamples relies on a family of counterexamples to Jaeger's Circular Flow Conjecture (Conjecture 1.46) which was given by Han, Li, Wu, and Zhang in [30], see Construction 1.47.

4.2 CIRCULAR FLOW NUMBER OF $(2t+1)$-REGULAR GRAPHS

Let G be a graph and let $M \subseteq E(G)$. We denote by $G+M$ the graph obtained by adding a copy of M to G. Such a graph has vertex set $V(G+M) = V(G)$ and edge set $E(G+M) = E(G) \cup M'$, where M' is a copy of M. Let G_1 and G_2 be cubic graphs having perfect matching M_1 and M_2 respectively. Let G be a dot product $G_1 \cdot G_2$ where we remove from G_1 two non-adjacent edges $e_1, e_2 \in E(G_1 - M_1)$ and from G_2 two adjacent vertices x, y such that $xy \in M_2$. Then we say that G is an (M_1, M_2)-dot-product of G_1 and G_2. We recall that Figure 19 represents the dot product operation.

Moreover, let H be a cubic graph with a perfect matching M_3 such that, for all positive integers t, $H+(2t-2)M_3$ is a $(2t+1)$-regular class 1 (resp. class 2) graph. Then we say that H has the M_3-class-1 (resp. M_3-class-2) property.

We recall now the following well known result (see Izbicki [34]).

Lemma 4.5 (Parity Lemma)**.** *Let G be a $(2t+1)$-regular graph of class 1 and $c\colon E(G) \to \{1,2,\dots,2t+1\}$ a proper edge-coloring of G. Then, for every edge-cut $C \subseteq E(G)$ and color i, the following relation holds*

$$|C \cap c^{-1}(i)| \equiv |C| \mod 2.$$

We give now two lemmas that will be used in the next subsection in order to prove Conjecture 4.2.

Lemma 4.6. *For $i = 1,2$, let G_i be a cubic graph having the M_i-class-2 property, where M_i is a perfect matching of G_i. Moreover let G be an (M_1, M_2)-dot-product of G_1 and G_2 and $x, y \in V(G_2)$ the two adjacent vertices that have been removed from G_2 when constructing G. Then $M = M_1 \cup M_2 \setminus \{xy\}$ is a perfect matching of G and G has the M-class-2 property.*

Proof. Let $e_1 = v_1v_2$, $e_2 = v_3v_4$ be the edges that have been removed from G_1 in order to obtain G. Define $H = G + (2t-2)M$ and $H_i = G_i + (2t-2)M_i$, $i \in \{1,2\}$, and let a_1, a_2, a_3, a_4 be the added edges incident to v_1, v_2, v_3 and v_4 respectively. Then $C = \{a_1, a_2, a_3, a_4\}$ is a 4-edge-cut in H which separates $H[V(G_2) - \{x,y\}]$ and $H[V(G_1)]$.

Suppose to the contrary that H is a class 1 graph. By the Parity Lemma, either C intersects only one color class, or it intersects two color classes in exactly two edges each. Moreover, if $c(a_1) = c(a_2)$, then $c(a_3) = c(a_4)$ and a $(2t+1)$-edge-coloring is defined naturally on H_1 by the coloring of H in contradiction to the fact that H_1 is a class 2 graph. Therefore, $c(a_1) \neq c(a_2)$, and so $\{c(a_3), c(a_4)\} = \{c(a_1), c(a_2)\}$. In this case a $(2t+1)$-edge-coloring is naturally defined on H_2 by the coloring of H leading to a contradiction again. □

As we mentioned in Section 1.3, if G has a nowhere-zero r-flow, then G has always an orientation D such that all flow values are positive. Thus, if G is cubic, $V(G) = (\mathcal{B}, \mathcal{W})$ is naturally partitioned into two subsets of equal cardinality according to the number of their incoming edges in D. We say that v is black (resp. white) if $v \in \mathcal{B}$ (resp. $v \in \mathcal{W}$). The balanced valuation ω of G corresponding to the all-positive nowhere-zero r-flow (D, f) is defined as follows: $\omega(v) = -\frac{r}{r-2}$ if v is black and $\omega(v) = \frac{r}{r-2}$ if v is white. Like previous chapter, for $X \subseteq V(G)$ we define $b_X = |X \cap \mathcal{B}|$ and $w_X = |X \cap \mathcal{W}|$. We call the partition $(\mathcal{B}, \mathcal{W})$ of $V(G)$ an r-bipartition of G.

Lemma 4.7. *Let $i \in \{1,2\}$, and $\{G_n : n \in \mathbb{N}\}$ be a family of cubic class 2 graphs such that for each $n \geq 1$:*

- *G_n has a r_n-bipartition $(\mathcal{B}_n, \mathcal{W}_n)$ with $r_n \in (4,5)$;*
- *G_n has a perfect matching M_n with the following properties:*
  - *G_n has the M_n-class-i-property;*
  - *if $ab \in M_n$, then $a \in \mathcal{B}_n$ if and only if $b \in \mathcal{W}_n$.*

If $\lim_{n\to\infty} r_n = 4$, then $\Phi^{(i)}(2t+1) = 2 + \frac{2}{2t-1}$, for every integer $t \geq 1$.

Proof. Fix an integer $t \geq 1$ and let $H_n := G_n + (2t-2)M_n$. Since G_n has the M_n-class-i property $\{H_n : n \in \mathbb{N}\}$ is an infinite family of $(2t+1)$-regular class i graphs.

By Theorem 1.36, $|\partial_{G_n}(X)| \geq \frac{r_n}{r_n-2}|b_X - w_X|$, for every $X \subseteq V(G_n)$. Let $Y \subseteq V(H_n)$. Since M_n pairs black and white vertices of H_n we have that $d = |M_n \cap \partial_{H_n}(Y)| \geq |b_Y - w_Y|$. Therefore, for every $Y \subseteq V(H_n)$, we get the following inequalities:

$$|\partial_{H_n}(Y)| \geq \frac{r_n}{r_n-2}|b_Y - w_Y| + (2t-2)d \geq (\frac{r_n}{r_n-2} + 2t-2)|b_Y - w_Y|.$$

Hence, H_n has a nowhere-zero $(2 + \frac{2(r_n-2)}{r_n+(2t-3)(r_n-2)})$-flow. Notice that, if $i = 1$, then $H_n \in \mathcal{G}_{2t+1}$, for every n, because G_n is a class 2 cubic graph, and so it cannot have a 1-factor whose removal gives rise to a bipartite graph. On the other hand if $i = 2$, then H_n is class 2. Therefore, since the sequence $\{r_n\}_{n\in\mathbb{N}}$ tends to 4 from above, we have that

$$\Phi^{(i)}(2t+1) \leq \lim_{n\to\infty}\Big(2 + \frac{2(r_n-2)}{r_n+(2t-3)(r_n-2)}\Big) = 2 + \frac{2}{2t-1},$$

and thus, equality holds from Theorem 4.1. $\square$

4.2.1 *A family of snarks fulfilling the hypothesis of Lemma 4.7*

In this subsection we present two infinite families of snarks that fulfill all requirements of Lemma 4.7. This proves Conjecture 4.2.

Class 1 *regular graphs*

Consider the family of Flower snarks $\{J_{2n+1}\}_{n\in\mathbb{N}}$, introduced in [32]. The Flower snark J_{2n+1} is the non-3-edge-colorable cubic graph having:

- vertex set $V(J_{2n+1}) = \{a_i, b_i, c_i, d_i \colon i \in \mathbb{Z}_{2n+1}\}$
- edge set $E(J_{2n+1}) = \{b_ia_i, b_ic_i, b_id_i, a_ia_{i+1}, c_id_{i+1}, c_{i+1}d_i \colon i \in \mathbb{Z}_{2n+1}\}$

The following lemma holds true. Since its proof requires some case analysis and lies outside the intent of this section we omit it here and add it in Section 4.4. We remark that the very same result has been obtained independently by Máčajová et al. in [50].

Lemma 4.8. *Let M be a 1-factor of J_{2n+1}, $n \geq 2$. Then $J_{2n+1} + M$ is a class 1 4-regular graph.*

Theorem 4.9. *The graph J_{2n+1} has a $(4+\frac{1}{n})$-bipartition $(\mathcal{B}_n, \mathcal{W}_n)$ and a perfect matching M_n such that:*

- *J_{2n+1} has the M_n-class-1-property;*
- *for all $xy \in M_n$, $x \in B_n$ if and only if $y \in W_n$.*

Proof. We construct explicitly a nowhere-zero $(4+\frac{1}{n})$-flow (D_n, f_n) in J_{2n+1} as sum of an integer 4-flow (D, f) with exactly one edge having flow value 0 and n flows $(D'_1, f'_1), \ldots, (D'_n, f'_n)$ having value $\frac{1}{n}$ each on a different circuit.

Define (D, f) on the directed edges of J_{2n+1} as follows, when we write an edge connecting two vertices u, v in the form uv we assume it to be oriented from u to v in the orientation D:

- $f(a_0b_0) = 0$ and $f(b_0c_0) = f(d_0b_0) = 2$;
- $f(b_ia_i) = f(a_{i+1}b_{i+1}) = 1$, for all $i \in \{1, 3, \ldots, 2n-1\}$;
- $f(b_ic_i) = 2$ and $f(b_{i+1}c_{i+1}) = 3$, for all $i \in \{1, 3, \ldots, 2n-1\}$;

- $f(d_ib_i) = 3$ and $f(d_{i+1}b_{i+1}) = 2$, for all $i \in \{1, 3, \ldots, 2n-1\}$;
- $f(a_ia_{i+1}) = f(c_{i+1}d_i) = 1$, for all $i \in \{0, 2, \ldots, 2n\}$;
- $f(a_ia_{i+1}) = f(c_{i+1}d_i) = 2$, for all $i \in \{1, 3, \ldots, 2n-1\}$;
- $f(c_id_{i+1}) = 1$, for all $i \in \mathbb{Z}_{2n+1}$;

For $j \in \{1, \ldots, n\}$ let (D'_j, f'_j) be the flow on the directed circuit $C_j = a_0b_0c_0d_1 \ldots d_lc_{l+1}d_{l+2} \ldots d_{2j-1}b_{2j-1}a_{2j-1}a_{2j}a_{2j+1} \ldots a_0$ (where $l < j$ and l odd), with $f'_j(e) = \frac{1}{n}$ if $e \in C_j$ and $f'_j(e) = 0$ otherwise.

The sum $(D, f) + \sum_{i=1}^{n}(D'_i, f'_i)$ gives a nowhere-zero $(4 + \frac{1}{n})$-flow (D_n, f_n) in J_{2n+1}. Let $(\mathcal{B}_n, \mathcal{W}_n)$ be the bipartition induced by such a flow and consider the 1-factor $M_n = \{a_ib_i, c_{i+1}d_i : i \in \mathbb{Z}_{2n+1}\}$. Notice that $D_n = D$, and for all $xy \in M_n$: $x \in \mathcal{B}_n$ if and only if $y \in \mathcal{W}_n$. By Lemma 4.8, for all $n \geq 2$, J_{2n+1} has the M_n-class-1 property and it is an easy check that also J_3 has the M_1-class-1 property. □

Class 2 regular graphs

Let P_{10} denote the Petersen graph. We recall now the following result, which follows from Theorem 3.1 of [28] and that will be used in the next subsection in order to prove one of the main theorems.

Lemma 4.10. *Let $M_1, \ldots, M_k$ be perfect matchings of P_{10}. Then $P_{10} + \sum_{i=1}^{k} M_i$ is a $(k+3)$-regular class 2 graph.*

We give now the construction of a family of snarks $\mathcal{G}$ fulfilling the hypothesis of Lemma 4.7.

Construction of $\mathcal{G} = \{G_n : n \in \mathbb{N}\}$

G_1: The graph G_1 is the Blanuša snark, see Figure 28.

$G_{n+1} = G_n \cdot G_1$: The dot product of these two graphs will be carried out as follows. If $v \in V(G_1)$, then the vertex $v^i \in V(G_n)$ corresponds to the vertex v of the i-th copy of G_1 that has been added in order to construct G_n. Consider the bold circuit $C = x_0x_1 \ldots x_8 \subseteq G_1$ as depicted in Figure 29. Delete the vertices x_0, x_1 of G_1 and edges $x_4^nx_5^n, x_7^nx_8^n$ from G_n. Perform the dot product $G_n \cdot G_1$ by adding the edges $x_4^nx_8, x_5^ny_0, x_7^ny_1$ and $x_8^nx_2$, where y_0, y_1 are vertices of G_1 which are not in C and adjacent to x_0, x_1 respectively, see Figure 29. The snark G_2 is depicted in Figure 30.

The following theorem shows that the family $\mathcal{G}$ has the properties that we want.

Theorem 4.11. *Let $n \in \mathbb{N}$ and $G_n \in \mathcal{G}$. The graph G_n has a $(4 + \frac{1}{n+1})$-bipartition $(\mathcal{B}_n, \mathcal{W}_n)$ and a perfect matching M_n such that:*

- *G_n has the M_n-class-2-property;*
- *for all $xy \in M_n$, $x \in B_n$ if and only if $y \in W_n$.*

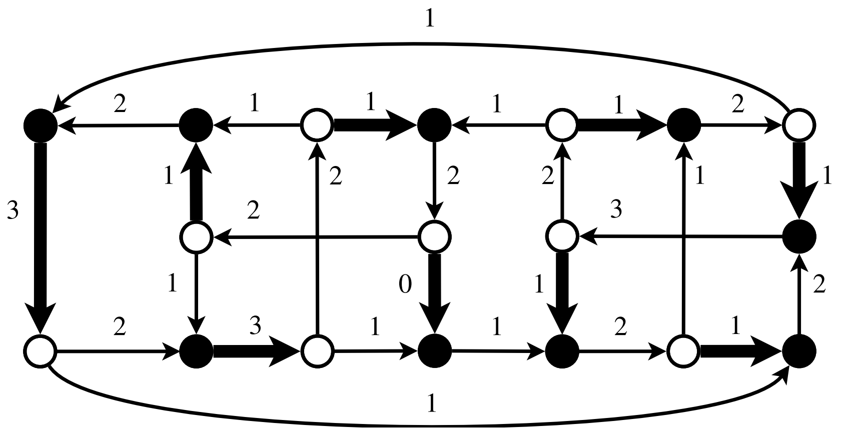

Figure 28: A 4-flow in the Blanuša snark G_1 having just one edge with flow value 0. The perfect matching consisting of all bold edges pairs black vertices with white vertices.

Proof. First we show that for every n there is a nowhere-zero $(4+\frac{1}{n+1})$-flow in G_n. We argue by induction over $n \in \mathbb{N}$. Fix on G_1 the 4-flow (D_1, f_1) as depicted in Figure 28. When we write D_1^{-1} we will refer to the orientation constructed by reversing each edge in D_1, similarly D_1^1 will be the orientation D_1. A nowhere-zero $(4+\frac{1}{2})$-flow in G_1 can be constructed by adding $\frac{1}{2}$ along the two directed circuits C_1, C_2 depicted in Figure 29. Indeed they have the following two properties:

(C.1) the unique edge having flow value 0 belongs to all circuits;

(C.2) every edge with flow value 3 belongs to at most one of the circuits.

Notice also that $f_1(x_7x_8) = 1$ and $f_1(x_4x_5) = 2$. Moreover there is a unique circuit in $\{C_1, C_2\}$ containing the path $x_4 \dots x_8$ and the other one does not intersect it.

Now we proceed with the inductive step. By the inductive hypothesis there is a 4-flow (D_n, f_n) in G_n having a unique edge with flow value 0 and $n+1$ directed circuits $\{C_1, \dots, C_{n+1}\}$ in D_n satisfying properties (C.1) and (C.2). It holds $f_n(x_7^n x_8^n) = 1$, $f_n(x_4^n x_5^n) = 2$. Furthermore, there is a unique circuit $C \in \{C_1, \dots, C_{n+1}\}$ containing the path $\tilde{P} = x_4^n \dots x_8^n$ and such that no other circuit intersects $\tilde{P}$. If n is odd, then $\tilde{P}$ is a directed path in D_n, if n is even, then $x_8^n x_7^n \dots x_4^n$ is a directed path in D_n.

Let $H_n = G_n - \{x_4^n x_5^n, x_7^n x_8^n\}$ and $H' = G_1 - \{x_0, x_1\}$. Then G_{n+1} is constructed by adding edges $x_4^n x_8$, $x_5^n y_0$, $x_7^n y_1$ and $x_8^n x_2$. Let (D_{n+1}, f_{n+1}) be the unique 4-flow in G_{n+1} such that

- $D_{n+1}|_{H_n} = D_n|_{H_n}$ and $D_{n+1}|_{H'} = D_1^{(-1)^n}|_{H'}$;
- $f_{n+1}|_{H_n} = f_n|_{H_n}$ and $f_{n+1}|_{H'} = f_1|_{H'}$.

We show that there exists a set of $\tilde{\mathcal{C}}$ of $n+2$ circuits satisfying properties (C.1) and (C.2). In particular, we are going to construct two

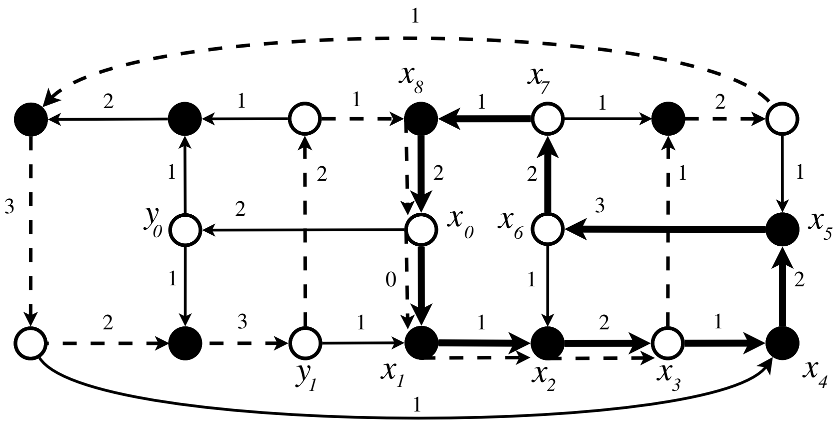

Figure 29: A nowhere-zero $(4+\frac{1}{2})$-flow in the Blanuša snark G_1 can be constructed by adding $\frac{1}{2}$ along the bold and dotted circuits.

circuits out of C. First notice that there are exactly two paths $\tilde{P}_1 = x_8^{n+1}w_1 \dots w_t x_2^{n+1}$ and $\tilde{P}_2 = x_8^{n+1}x_7^{n+1}x_6^{n+1}x_5^{n+1}x_4^{n+1}x_3^{n+1}x_2^{n+1}$ in $H' \subseteq G_{n+1}$, which are directed in $D_{n+1}|_{H'}$ and such that $\tilde{P}_1 \cap \tilde{P}_2 = x_2^{n+1}x_3^{n+1}$, for some vertices $w_1, \dots, w_t \in V(G_{n+1})$ (see Figure 30 for an example in the case of $n = 1$). In particular, if n is odd, then $\tilde{P}_1$ and $\tilde{P}_2$ are both directed from x_8^{n+1} to x_2^{n+1} and vice versa if n is even. We can suppose without loss of generality that $C = v_0v_1 \dots v_k\tilde{P} = v_0 \dots v_k x_4^n \dots x_8^n$ in G_n. Define $\tilde{C}_i$ to be the circuit $v_0 \dots v_k x_4^n \tilde{P}_i x_8^n$, $i \in \{1,2\}$. It follows by construction that $\tilde{C}_1$ and $\tilde{C}_2$ are both directed circuits in D_{n+1}. Therefore, the family $\tilde{\mathcal{C}} = (\mathcal{C} \setminus \{C\}) \cup \{\tilde{C}_1, \tilde{C}_2\}$ consists of $n+2$ circuits satisfying properties (C.1) and (C.2) and so a nowhere-zero $(4+\frac{1}{n+2})$-flow can be constructed in G_{n+1}.

Now we show that for every n there is a perfect matching M_n of G_n satisfying the statement. We argue again by induction. Choose as a perfect matching M_1 of G_1, which is indicated by bold edges in Figure 28. Consider two copies of the Petersen graph P_{10}^1, P_{10}^2 together with a perfect matching N_i of P_{10}^i, $i \in \{1,2\}$. Recall that G_1 is constructed by performing an (N_1, N_2)-dot-product $P_{10}^1 \cdot P_{10}^2$. In particular we can choose N_1, N_2 and perform the dot product in such a way that $M_1 = N_1 \cup N_2 \setminus \{x'y'\}$, where x', y' are the vertices we removed from P_{10}^2 in order to perform the dot product itself. Therefore, by Lemmas 4.6 and 4.10, it follows that G_1 has the M_1-class-2-property. Figure 28 shows that the chosen $(4+\frac{1}{2})$-flow in G_1 and the perfect matching M_1 are related in the following way: let $(\mathcal{B}_1, \mathcal{W}_1)$ be the $(4+\frac{1}{2})$-bipartition of $V(G_1)$ induced by D_1, then for all $xy \in M_1$, $x \in \mathcal{B}_1$ if and only if $y \in \mathcal{W}_1$. Therefore, the statement is true for $n = 1$. Notice that $x_0x_1 \in M_1$ and that $x_4x_5, x_7x_8 \notin M_1$.

For the inductive step, we assume that G_n has a perfect matching M_n fulfilling the inductive hypothesis and $x_4^nx_5^n, x_7^nx_8^n \notin M_n$. There is a unique perfect matching M_{n+1} of G_{n+1} such that $M_{n+1} \cap E(H_n) = M_n$

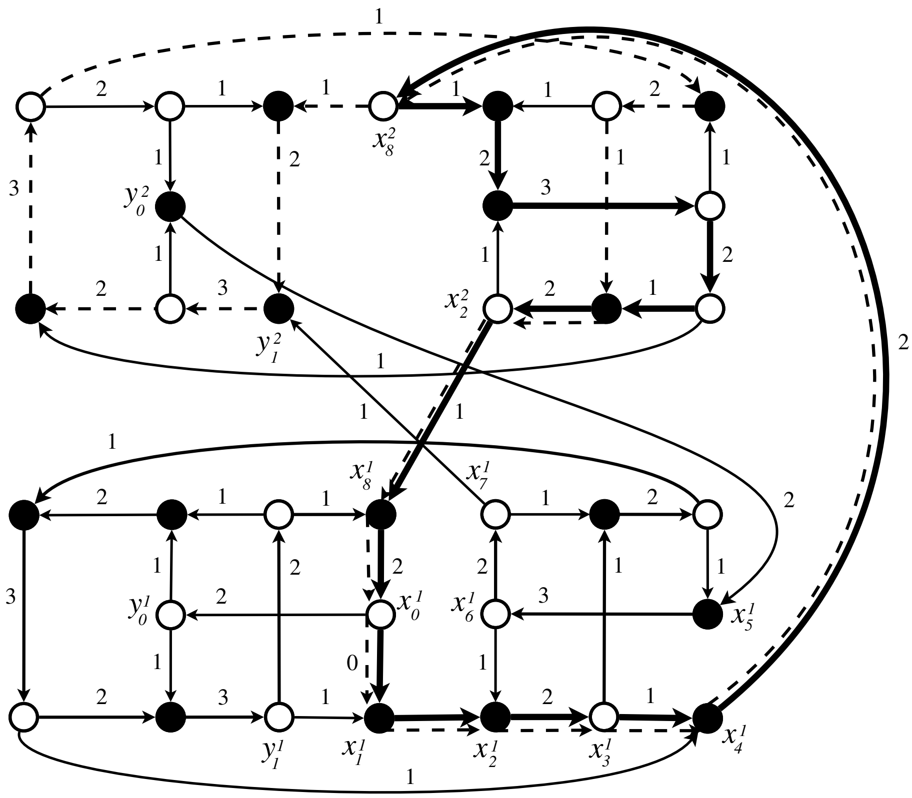

Figure 30: Construction of two more circuits (dotted and bold ones) in G_2.

and $M_{n+1} \cap E(H') = M_1 \setminus \{x_0x_1\}$. Thus, by Lemma 4.6, we get that G_{n+1} has the M_{n+1}-class-2-property.

Define $\mathcal{B}_{n+1} = \mathcal{B}_n \cup \mathcal{B}$ and $\mathcal{W}_{n+1} = \mathcal{W}_n \cup \mathcal{W}$, where $(\mathcal{B}, \mathcal{W})$ is the bipartition induced by $(D_1^{(-1)^n}, f_1)$ in G_1. The bipartition $(\mathcal{B}_{n+1}, \mathcal{W}_{n+1})$ of $V(G_{n+1})$ is a $(4 + \frac{1}{n+1})$-bipartition. Since both $M_n = M_{n+1} \cap E(H_n)$ and $M_1 \setminus \{x_0x_1\} = M_{n+1} \cap E(H')$ pair black and white vertices, it follows that M_{n+1} pairs black and white vertices too. Notice that $x_4^{n+1}x_5^{n+1}, x_7^{n+1}x_8^{n+1} \notin M_{n+1}$, this concludes the inductive step. □

From Lemma 4.7 and Theorems 4.9 and 4.11 we deduce the following corollary.

Corollary 4.12. *For every $t \geq 2$ and $i \in \{1, 2\}$: $\Phi^{(i)}(2t+1) = 2 + \frac{2}{2t-1}$.*

4.3 REGULAR CLASS 1 GRAPHS WITH HIGH FLOW NUMBER

Jaeger's Circular Flow Conjecture is disproved in [30]: indeed, for all integers $p \geq 3$, a counterexample M_p can be constructed as presented in Construction 1.47. Namely, M_p is a $4p$-edge-connected graph which does not admit a nowhere-zero $(2 + \frac{1}{p})$-CNZF.

Notice that, for every $i \in \{1, \dots, 4p+1\}$ we have $d_{M_p}(v^i_{4p}) = 4p - 1 + 2(p-2) = 6p - 5$, $d_{M_p}(c_i) = 2(p - 2 + 4p - 1 - (3p-2) + 1) + 3 = 4p + 3$, and all other vertices have degree $4p + 1$.

Let $k = 2p$. The graph M_p does not have a nowhere-zero $(2 + \frac{2}{k})$-flow. We will use M_p in order to construct a $(2k+1)$-regular graph of class 1 which does not admit a nowhere-zero $(2 + \frac{2}{k})$-flow, thus disproving Conjecture 4.3.

We consider odd integers $p \geq 3$, say $p = 2t + 1$.

Construction of M'_p

The copy of K_{4p} which is used when constructing G_2^i in Construction 1.47 is denoted by K_{4p}^i. Construct the graph M'_p by expanding each vertex v_{4p}^i of K_{4p}^i in M_p to a vertex x^i of degree $4p+1$ and $p-3$ divalent vertices, where x^i is adjacent to every vertex of $V(K_{4p}^i) \setminus \{v_{4p}^i\}$ and to c_i and c_{i+1} and each divalent vertex is adjacent to both c_i and c_{i+1}. After that, suppress the divalent vertices. Note that the construction can also be seen as a edge splitting at v_{4p}^i. We have $d_{M'_p}(c_i) = 4p+3$ for all $i \in \{1, \ldots, 4p+1\}$ and all other vertices of M'_p have degree $4p+1$. Notice that M'_p remains a bridgless graph.

Lemma 4.13. *The graph M'_p admits a $(4p+1)$-edge-coloring c such that for all $i \in \{0, \ldots, 4p\}$ and all $v \in V(M'_p) : |c^{-1}(i) \cap \partial(v)|$ is odd. Furthermore, $\phi_c(M'_p) > 2 + \frac{1}{p}$.*

Proof. All operations performed in order to construct M'_p do not decrease the circular flow number of graphs. Thus $\phi_c(M'_p) \geq \phi_c(M_p) > 2 + \frac{1}{p}$.

Now we show that M'_p can be colored using $8t+5 = 4p+1$ colors in such a way that every vertex sees each color an odd number of times. We say that a vertex v sees a color i, if there is an edge e which is incident to v and $c(e) = i$.

Each copy G_1^i can be constructed by considering the complete graph K_{4p} with vertex set $\mathbb{Z}_{8t+3} \cup \{\infty\}$ and adding the edges of all following triangles:

- $(t+2+j), (t+3+j), (t+4+j)$ for every $j \in \{0, 3, 6, 9, \ldots, 3(t-1)\}$;
- $-(t+2+j), -(t+3+j), -(t+4+j)$ for every $j \in \{0, 3, 6, 9, \ldots, 3(t-1)\}$.

Since p is an odd number, the number of such triangles is even. More precisely there are $p - 1 = 2t$ added triangles. Consider the following 1-factorization of K_{4p}. Let the edges of color 0 be all edges of the set $M_0 = \{0\infty\} \cup \{-ii \colon i \in \mathbb{Z}_{8t+3}\}$ and the edges of color $j \in \{0, 1, \ldots, 8t-2\}$ be all edges of the set $M_j = M_0 + j = \{j\infty\} \cup \{(-i+j)(i+j) \colon i \in \mathbb{Z}_{8t+3}\}$. We can color this way all copies K_{4p}^i of K_{4p} inside G_1^i. Notice that we have used $8t+3 = 4p-1$ colors so far.

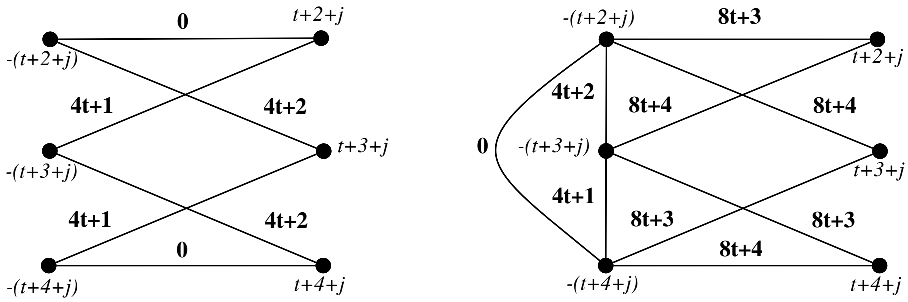

Figure 31: Color the added triangles using colors of the selected circuit. Color the selected circuit with two new colors. Colors are depicted in bold.

Consider the even circuits $t+2+j, -(t+2+j), t+3+j, -(t+4+j), t+4+j, -(t+3+j), t+2+j$ for every $j \in \{0, 3, 6, 9, \dots, 3(t-1)\}$ inside G_1^i. We perform the operation in Figure 31 in order to color all triangles $(t+2+j), (t+3+j), (t+4+j)$ and $-(t+2+j), -(t+3+j), -(t+4+j)$ using two more colors $8t+3$ and $8t+4$.

Consider the even circuit $C = 0, 1, 2, \dots, t+1, \infty, -(t+1), -t, \dots, -2, -1$ inside G_1^i. Notice that these are all vertices of K_{4p}^i in G_1^i that are connected with both c_i and c_{i+1}. Moreover, there are no two edges of C belonging to the same color class. First assign colors $8t+3$ and $8t+4$ to the edges of C alternately. This way we can assign the previous colors of the edges of C to edges of the type $c_i v$ and $c_{i+1} v$, with $v \in V(C)$, in such a way that both $c_i v$ and $c_{i+1} v$ see different colors (notice that c_i and c_{i+1} see the very same set of colors), see Figure 32. Since the length of C is $2t+4$, up to a permutation of colors, we can suppose that c_i receives colors $i + \{1, 3, 5, 7, \dots, 4t+7\} = i + \{2j+1 \colon j = 0, 1, 2, \dots, 2t+3\}$ from the copy G_1^i, where now we are doing sums modulo $8t+5 = 4p+1$.

At this point, every vertex not in $\{w\} \cup \{c_i : i \in \{1, \dots, 4p+1\}\}$ sees every color exactly once.

For every $i \in \{1, \dots, 8t+5\}$, color all $p-2$ parallel edges connecting c_i with c_{i+1} with colors $i + \{4t+8, 4t+10, \dots, 8t+2, 8t+4\}$, where sums are taken modulo $8t+5$ (notice that these are exactly $2t-1 (= p-2)$ colors). Finally, color $c_i w$ with color $i+4t+7$ modulo $8t+5$. The central vertex w sees each color exactly once, whereas the vertex c_i sees all colors once but for $i+4t+7$ which is seen three times. □

At this point we are going to further modify M'_p in order to obtain a $(4p+1)$-regular graph of class 1.

Construction of $\tilde{M}_p$

Consider the graph M'_p. By Lemma 4.13, there is a $(4p+1)$-edge-coloring c such that $|c^{-1}(i) \cap \partial(v)|$ is odd for every color $i \in \{0, \dots, 4p\}$ and vertex v. Construct $\tilde{M}_p$ by expanding each c_i into a vertex of de-

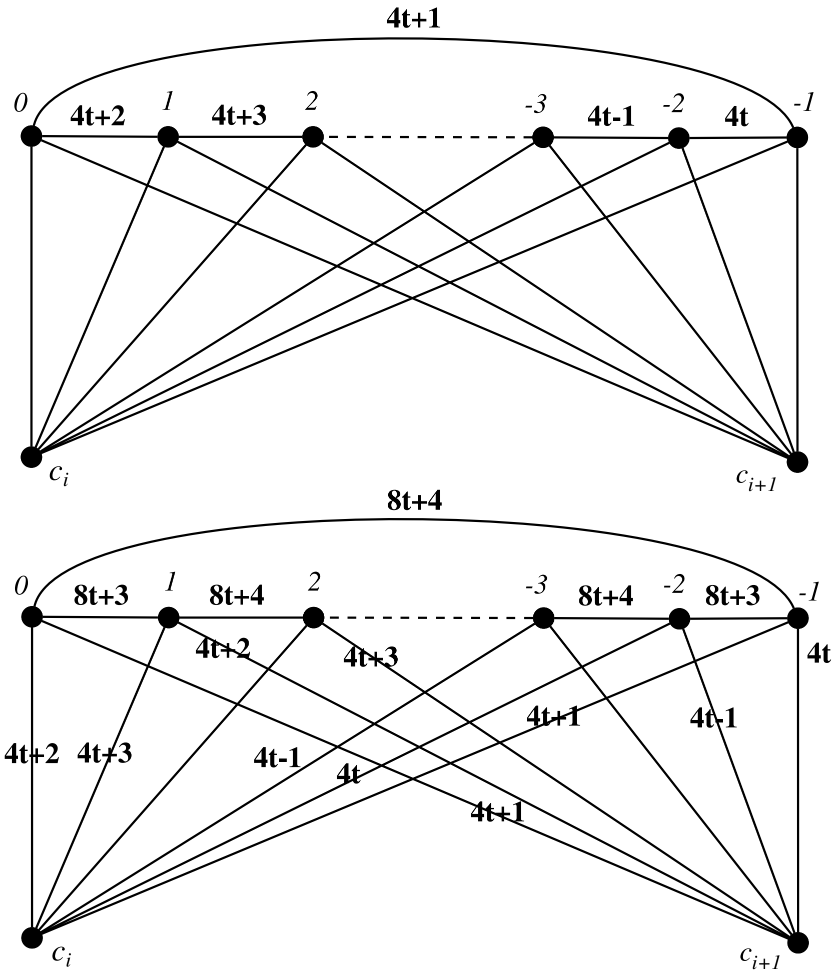

Figure 32: Assign to uncolored edges colors of the selected even circuit and assign to the edges of the selected circuit two new colors. Colors are depicted in bold.

gree $4p+1$ that receives all colors and into a vertex of degree 2 that receives the same color from both its adjacent edges. Finally suppress all such vertices of degree 2.

Theorem 4.14. *$\tilde{M}_p$ is a $(4p+1)$-regular class 1 graph such that $\phi_c(\tilde{M}_p) > 2 + \frac{1}{p}$.*

Proof. Since expanding and suppressing vertices does not decrease the circular flow number of graphs $\phi_c(\tilde{M}_p) \geq \phi_c(M'_p) > 2 + \frac{1}{p}$. Furthermore a natural $(4p+1)$-edge-coloring is defined on $\tilde{M}_p$ by the edge-coloring of M'_p. $\square$

The following corollary holds.

Corollary 4.15. *Let $p \geq 3$ be any odd integer and let $k = 2p$. There is a $(2k+1)$-regular class 1 graph G such that $\phi_c(G) > 2 + \frac{2}{k}$.*

4.4 ADDING PERFECT MATCHINGS TO FLOWER SNARKS

Here we prove that adding any perfect matching to a Flower snark J_{2n+1} with $n \geq 2$, results in a class 1 regular graph.

Proof of Lemma 4.8. We use induction on n. We checked by computer that the statement holds true for $n = 2$. Let $n \geq 3$. For all $i \in \mathbb{Z}_{2n+1}$, let J^i_{2n+1} be $G[\{a_i, b_i, c_i, d_i\}]$ and let $E_{i,i+1} = E(J^i_{2n+1}, J^{i+1}_{2n+1}) = \{a_i a_{i+1}, c_i d_{i+1}, c_{i+1} d_i\}$. Suppose that there is $i \in \mathbb{Z}_{2n+1}$ such that $E_{i,i+1} \cap M = \emptyset$. Then it follows that $E_{i+2,i+3} \cap M = \emptyset$ as well. Remove J^{i+1}_{2n+1} and J^{i+2}_{2n+1} from J_{2n+1} and add the edges $\{a_i a_{i+3}, c_i d_{i+3}, c_{i+3} d_i\}$. This new graph H is isomorphic to the Flower snark J_{2n-1} and so $H + M'$ is class 1, where $M' = M \setminus (E(J^{i+1}_{2n+1}) \cup E(J^{i+2}_{2n+1}) \cup E_{i+1,i+2})$. Therefore, a proper 4-edge-coloring c is naturally defined on $E(J_{2n+1} + M) \setminus (E(J^{i+1}_{2n+1}) \cup E(J^{i+2}_{2n+1}) \cup E_{i+1,i+2})$, such that $c(a_i a_{i+1}) = c(a_{i+2} a_{i+3})$, $c(d_i c_{i+1}) = c(d_{i+2} c_{i+3})$ and $c(c_i d_{i+1}) = c(c_{i+2} d_{i+3})$. Let

- $h_1 = c(c_i d_{i+1}) = c(c_{i+2} d_{i+3})$;
- $h_2 = c(d_i c_{i+1}) = c(d_{i+2} c_{i+3})$;
- $h_3 = c(a_i a_{i+1}) = c(a_{i+2} a_{i+3})$.

Since $M \cap E_{i,i+1}$ is empty, $|E_{i+1,i+2} \cap M| = 2$ and we can assume without loss of generality that $E_{i+1,i+2} \cap M = \{c_{i+1} d_{i+2}, c_{i+2} d_{i+1}\}$ (and so $(E(J^{i+1}_{2n+1}) \cup E(J^{i+2}_{2n+1})) \cap M = \{a_{i+1} b_{i+1}, a_{i+2} b_{i+2}\}$). Furthermore it cannot happen that $h_1 = h_2 = h_3$. Indeed, we can assume w.l.o.g. that $M \cap E(J^i_{2n+1}) = \{b_i c_i\}$. Let $h_4 = c(b_i d_i)$ and $h_5 = c(b_i a_i)$. And so either $h_1 = h_4 \neq h_2$ or $h_1 = h_5 \neq h_3$. Now we extend the coloring on $J_{2n+1} + M$ to a proper 4-edge-coloring.

Consider the following auxiliary graph $G' = G[V(J^{i+1}_{2n+1}) \cup V(J^{i+2}_{2n+1})] + \tilde{M}$, where $\tilde{M}$ is the set of edges $((E(G[V(J^{i+1}_{2n+1}) \cup V(J^{i+2}_{2n+1})]) \cap M) \cup \{a_{i+1} a_{i+2},\ c_{i+1} d_{i+2},\ c_{i+2} d_{i+1}\}$. Figure 33 represents G'. Extending c to a proper 4-edge-coloring of J_{2n+1} is equivalent to finding a proper edge-coloring of G' for all possible $h_1 = c(c_{i+1} d_{i+2}), h_2 = c(c_{i+2} d_{i+1}), h_3 = c(a_{i+1} a_{i+2})$ non pairwise equal. Such a coloring is depicted in Figure 33.

Now we can assume that, for all $i \in \mathbb{Z}_{2n+1}$, $E_{i,i+1} \cap M \neq \emptyset$. In particular $|E_{i,i+1} \cap M| = 1$. Indeed $|E_{i,i+1} \cap M| \neq |E_{i,i+1}|$ for otherwise the vertices b_i and b_{i+1} would not be matched. On the other hand, if $|E_{i,i+1} \cap M| = 2$, then $E_{i-1,i} \cap M$ and $E_{i+1,i+2} \cap M$ would both be empty, a case that we already discussed.

Define the following function

$$t(E_{i,i+1}) = \begin{cases} x_1 = (1,0,0) & \text{if } E_{i,i+1} \cap M = \{c_i d_{i+1}\} \\ x_2 = (0,1,0) & \text{if } E_{i,i+1} \cap M = \{c_{i+1} d_i\} \\ x_3 = (0,0,1) & \text{if } E_{i,i+1} \cap M = \{a_i a_{i+1}\} \end{cases}$$

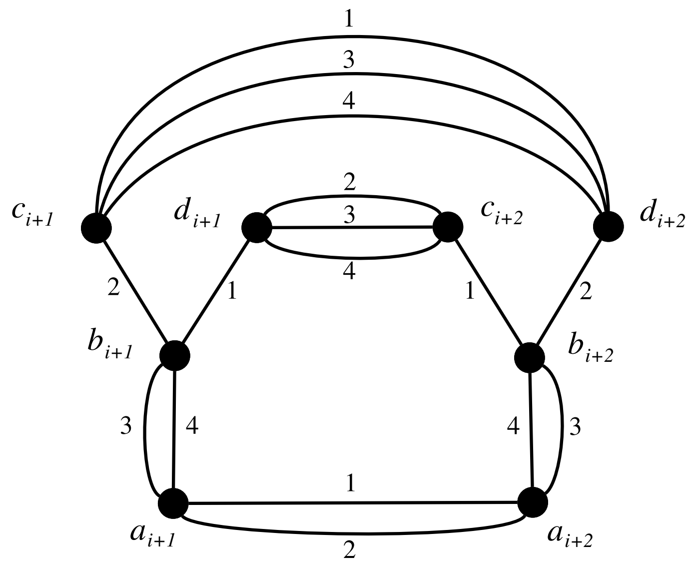

Figure 33: A coloring of the graph G'.

Claim 4.16. *There is $j \in \mathbb{Z}_{2n+1}$ such that $t(E_{j,j+1}) = t(E_{j+2,j+3})$.*

Proof of the Claim. Notice that, for all $i \in \mathbb{Z}_{2n+1}$,

- if $t(E_{i,i+1}) = x_1$, then $t(E_{i+1,i+2}), t(E_{i-1,i}) \in \{x_1, x_3\}$;
- if $t(E_{i,i+1}) = x_2$, then $t(E_{i+1,i+2}), t(E_{i-1,i}) \in \{x_2, x_3\}$;
- if $t(E_{i,i+1}) = x_3$, then $t(E_{i+1,i+2}), t(E_{i-1,i}) \in \{x_1, x_2\}$.

Consider the circuit C_{2n+1} on $2n+1$ vertices, let $E(C_{2n+1}) = \{e_1, e_2, \dots, e_{2n+1}\}$ such that for every i, e_i is adjacent to e_{i+1}. Let $\tilde{c}\colon E(C_{2n+1}) \to \{1,2,3\}$ be a coloring such that there are no adjacent edges e_i, e_{i+1} with either $\tilde{c}(e_i) = \tilde{c}(e_{i+1}) = 3$ or $\{\tilde{c}(e_i), \tilde{c}(e_{i+1})\} = \{1,2\}$. Proving the statement of the claim is equivalent to proving that there are two edges $e_j, e_{j+2} \in E(C_{2n+1})$ such that $\tilde{c}(e_j) = \tilde{c}(e_{j+2})$.

Suppose by contradiction that there are not such edges. Then edges of color 3 must be adjacent to exactly one edge of color 2 and one of color 1. On the other hand, edges of color 2 and, respectively 1, must be adjacent to exactly one edge of color 3 and one of color 2, respectively 1. Let

$$m_s = \begin{cases} \text{number of pairs of adjacent edges of color } s & \text{if } s \in \{1,2\} \\ \text{number of edges of color } s & \text{if } s = 3, \end{cases}$$

then we can count the length of C_{2n+1} as follows

$$2n+1 = 2m_1 + 2m_2 + m_3.$$

Thus, m_3 is an odd number and we conclude that there is a $j \in \mathbb{Z}_{2n+1}$ such that $\tilde{c}(e_{j+1}) = 3$ and $\tilde{c}(e_j) = \tilde{c}(e_{j+2}) \in \{1,2\}$, a contradiction. $\square$

By Claim 4.16, there is j such that $t(E_{j,j+1}) = t(E_{j+2,j+3})$. The graph K obtained from J_{2n+1} by removing J^{j+1}_{2n+1} and J^{j+2}_{2n+1} and adding the edges $\{a_j a_{j+3},\ c_j d_{j+3},\ c_{j+3} d_j\}$ is isomorphic to J_{2n-1}. Thus, $K + M''$ has a proper 4-edge-coloring, where $M'' = M \setminus (E(J^{j+1}_{2n+1}) \cup E(J^{j+2}_{2n+1}) \cup E_{j+1,j+2})$. Therefore, a natural proper 4-edge-coloring is defined on $E(J_{2n+1} + M) \setminus (E(J^{j+1}_{2n+1}) \cup E(J^{j+2}_{2n+1}) \cup E_{j+1,j+2})$. Without loss of generality we assume $t(E_{j,j+1}) = t(E_{j+2,j+3}) = x_1$. Then the edge-coloring can be extended to a proper 4-edge-coloring of J_{2n+1} just as in the previous case. $\square$

5 HIGHLY EDGE CONNECTED REGULAR GRAPHS WITHOUT LARGE FACTORIZABLE SUBGRAPHS

In this chapter we construct infinite families of highly edge connected r-regular graphs without $r-2$ pairwise disjoint perfect matchings. This chapter is based on a joint work with Eckhard Steffen [P.4].

5.1 INTRODUCTION

A well known theorem of Tutte states that the 4-Color Theorem is equivalent to the fact that every planar graph has a 4-NZF. This is also well known to be equivalent to the statement that every bridgeless planar cubic graph is class 1. Thomassen proved in [80] that this last statement holds if and only if every bridgeless 9-regular planar graph (note that it can have multiple edges) can be decomposed into three 3-factors.

Moreover, a weaker version of the 3-Flow Conjecture was proved in [46], that is: let G be a $(2k^2+k)$-edge-connected graph such that $V(G) = \{v_1, \dots, v_n\}$ and let $d_1, \dots, d_n$ integers such that $\sum_{i=1}^{n} d_i \equiv |E(G)| \mod k$, then G has an orientation such that, for all $i \in \{1, \dots, n\}$, $d^+(v_i) \equiv d_i \mod k$. This result was improved in [78] for the case of odd integers k by lowering the edge-connectivity bound to $3k-3$. In [79] such result is reformulated in the language of factors.

The main result proved in [80] is the following: let q, r be natural numbers such that $q \geq 3$ is odd and $r = kq \geq 4$, and let G be an r-regular graph. If k is odd and G has no odd edge-cuts of cardinality less than $3k-2$ then G has a decomposition into q-factors. If k is even, G has an even number of vertices and G has no edge-cut of cardinality less than k^2+2k, then G has a decomposition into q-factors. The proof is carried out using the above weaker version of the 3-Flow Conjecture and furthermore, for $r=9$ and $k=q=3$, it is shown that, if Tutte's 3-Flow Conjecture is true, the bound $3k-2=7$ can be dropped to 5.

The following problem was left for further research.

Problem 5.1 (Thomassen [80])**.** *Is every r-regular r-edge-connected graph of even order the union of $r-2$ disjoint 1-factors and a 2-factor?*

The statement is true for $r=3$. Recall that an r-regular graph G is an r-graph, if $|\partial_G(S)| \geq r$ for every $S \subseteq V(G)$ with $|S|$ odd.

An r-graph is *poorly matchable* if it does not contain two disjoint 1-factors. Clearly, every bridgeless cubic graph with edge chromatic number 4 is poorly matchable. Rizzi [64] constructed poorly matchable r-graphs for each $r \geq 4$. All of them contain an edge of multiplicity

$r - 2$ and therefore, they have a 4-edge cut. However, the poorly matchable 4-graphs are 4-edge-connected and therefore, they provide a negative answer to Problem 5.1. The 4-graphs constructed in [55] are also poorly matchable. The following theorem is the main result of this chapter and it provides a negative answer to the question of Problem 5.1 for every positive integer which is a multiple of 4.

Theorem 5.2. *Let t, r be positive integers and $r \geq 4$. There are infinitely many t-edge-connected r-graphs which do not contain $r - 2$ pairwise disjoint 1-factors, where*

- $t = r$, *if* $r \equiv 0 \mod 4$;
- $t = r - 1$, *if* $r \equiv 1 \mod 2$;
- $t = r - 2$, *if* $r \equiv 2 \mod 4$.

Indeed, we prove that for any $r \geq 4$, there are r-graphs of order 60 and simple r-graphs of order $66r - 8$ with these properties.

5.2 CONSTRUCTIONS

In this section we construct counterexamples to Problem 5.1 and prove Theorem 5.2.

As defined in Chapter 4, adding a subset of edges $F \subseteq E(G)$ to a graph G means adding a copy of each edge of F to G, resulting in a graph having multiple edges. Moreover, when we write $G + kF$, for a positive integer k, we mean the graph obtained from G by adding k copies of each edge of F.

5.2.1 *Perfect matchings of the Petersen graph*

The edge set of a 1-factor of a graph G is a perfect matching of G. A collection $\mathcal{C}$ is a set of objects where repetitions are allowed. Namely we can formally define it as a set $C = \{C_1, \dots, C_n\}$ together with a function $m \colon C \to \mathbb{N}$ which gives the multiplicity of each object C_j in $\mathcal{C}$, that is the number of occurrences of C_j in $\mathcal{C}$. A subcollection $\mathcal{C}'$ of $\mathcal{C}$ is a subset $C' \subseteq C$ with a function $m' \colon C' \to \mathbb{N}$ such that $m'(C_j) \leq m(C_j)$ for all $j \in \{1, \dots, n\}$. In this case we will write $\mathcal{C}' \subseteq \mathcal{C}$.

Our construction heavily relies on the properties of the Petersen graph P_{10}, that, for simplicity, will be denoted by P for the entire chapter. Let $v_1 \dots v_5$ and $u_1u_3u_5u_2u_4$ be the two disjoint 5-cycles of P such that u_1v_1, u_2v_2, u_3v_3, $u_4v_4, u_5v_5 \in E(P)$. Let $M_0 = \{u_iv_i \colon i \in \{1, \dots, 5\}\}$ and, for $i \in \{1, \dots, 5\}$, let M_i be the only other perfect matching containing u_iv_i, see Figure 34. Let $\mathcal{M} = \{N_1, \dots, N_k\}$ be a collection of perfect matchings of P and let $P^{\mathcal{M}} = P + \sum_{j=1}^{k} N_j$. We say that a perfect matching N of $P^{\mathcal{M}}$ is of type j, if N is a copy of M_j. In this case we write $t(N) = j$. We will use the following results of Rizzi [64].

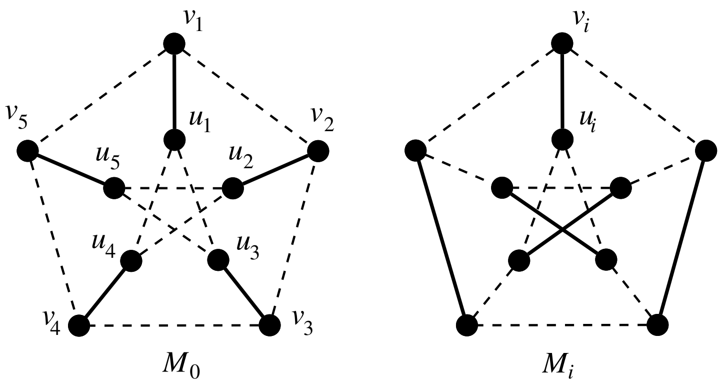

Figure 34: The perfect matchings $M_0, \ldots, M_5$ of P.

Proposition 5.3 (Rizzi [64]). *The function associating to every pair of perfect matchings M_i, M_j of P the unique edge $e \in M_i \cap M_j$ is a bijection.*

Lemma 5.4 (Rizzi [64]). *Consider a perfect matching M_j of P and let $P' = P + M_j$. Furthermore let N_1 and N_2 be two disjoint perfect matchings of P'. Then $j \in \{t(N_1), t(N_2)\}$.*

Previous lemma has the following generalization that will be central in this section.

Lemma 5.5. *Let $\mathcal{M}$ be a collection of k perfect matchings of P. If $\mathcal{M}' = \{M'_1, \ldots, M'_k, M'_{k+1}\}$ is a collection of $k+1$ pairwise disjoint perfect matchings of $P^{\mathcal{M}}$, then $\mathcal{M} \subseteq \mathcal{M}'$.*

Proof. We argue by induction over $k \in \mathbb{N}$. If $k = 1$, then the statement holds by Lemma 5.4. So let $k \geq 2$ and $\mathcal{M}' = \{M'_1, \ldots, M'_k, M'_{k+1}\}$ be pairwise disjoint perfect matchings of $P^{\mathcal{M}}$.

If $M'_i = M'_j$ for all $i, j \in \{1, \ldots, k+1\}$, then $\mathcal{M}$ must contain a unique perfect matching repeated k times, and so $\mathcal{M} \subseteq \mathcal{M}'$.

Otherwise there are $i, j \in \{1, \ldots, k+1\}$ such that $M'_i \neq M'_j$. There is a unique edge $e \in P$ such that $\{e\} = M'_i \cap M'_j$. Such an edge must be a multiedge in $P^{\mathcal{M}}$. Then either M'_i or M'_j has been added to P in order to obtain $P^{\mathcal{M}}$. This means that both $\mathcal{M}$ and $\mathcal{M}'$ contain a copy M of the same perfect matching. Therefore, by the inductive hypothesis, $\mathcal{M} \setminus \{M\} \subseteq \mathcal{M}' \setminus \{M\}$, and so $\mathcal{M} \subseteq \mathcal{M}'$. □

5.2.2 $4k$-edge-connected $4k$-graphs without $4k-2$ pairwise disjoint perfect matchings

For $k \geq 1$, let $P_k = P + kM_0 + (k-1)(M_1 + M_3 + M_4)$.

Lemma 5.6. *For all $k \geq 1$: P_k is $4k$-edge-connected and $4k$-regular.*

Proof. By definition, P_k is $4k$-regular. Let $X \subseteq V(P)$. If $|X|$ is odd, then every perfect matching intersects $\partial(X)$. Hence, $|\partial(X)| \geq 3 + k + 3(k-$

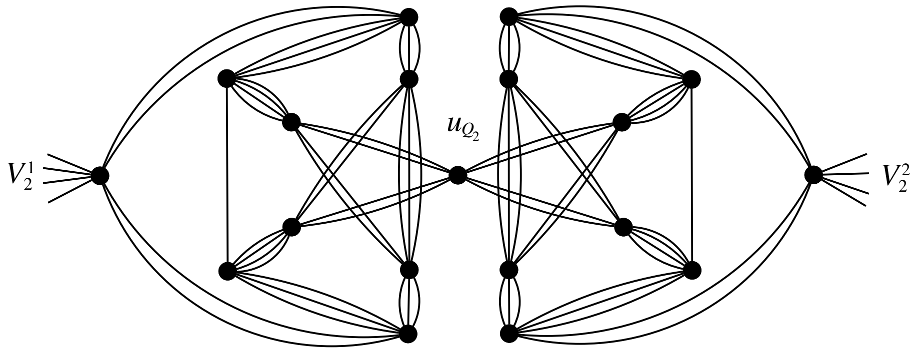

Figure 35: The subgraph Q_2.

$1) = 4k$. If $|X|$ is even, then it suffices to consider the cases $|X| \in \{2, 4\}$. Since P has girth 5, the subgraph induced by X is a path P_X on either 2 or 4 vertices, having some multiple edge. In both cases, since the maximum multiplicity of an edge is $2k$, we have that $\partial(X)$ contains $2k$ edges per both end vertices of P_X, namely $|\partial(X)| \geq 4k$. □

Consider two copies P_k^1 and P_k^2 of P_k. If u is a vertex (or edge) of P_k, then we denote u^i the corresponding vertex (or edge) inside P_k^i. For $i \in \{1, 2\}$ remove the multiedge $u_1^i v_1^i$ from P_k^i and let Q_k be the graph obtained by identifying u_1^1 and u_1^2 to the (new) vertex u_{Q_k}. It holds $d_{Q_k}(u_{Q_k}) = 4k$, and the set of $2k$ edges of $\partial(u_{Q_k})$ which are incident to vertices of $V(P_k^i)$ is denoted by U_k^i. If Q_k is a subgraph of a graph G, then let $V_k^i = \{xv_1^i \in E(G) \colon x \notin V(Q_k)\}$.

Let $\{N_1, \ldots, N_n\}$ be a collection of perfect matchings of a graph G. Define the function $\psi \colon E(G) \to \mathbb{Z}_2^n, e \mapsto (\psi_1(e), \ldots, \psi_n(e))$ such that

$$\psi_j(e) = \begin{cases} 1 & \text{if } e \in N_j; \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, if $W \subseteq E(G)$, then let $\psi(W) = \sum_{e \in W} \psi(e)$. For a vector $x = (x_1, \ldots, x_n) \in \mathbb{Z}_2^n$ the number of its non-zero entries is denoted by $\omega(x)$.

Lemma 5.7. *Let Q_k be a subgraph of a graph G with $\partial(V(Q_k)) = V_k^1 \cup V_k^2$. If, for all $i \in \{1, 2\}, d_G(v_1^i) = 4k$ and $\mathcal{N} = \{N_1, \ldots, N_{4k-2}\}$ is a family of pairwise disjoint perfect matchings of G, then*

$$\omega(\psi(V_k^1)) = \omega(\psi(V_k^2)) = 2k - 1.$$

Proof. Every perfect matching of G intersects $\partial(V(Q_k))$ precisely once since $|V(Q_k)|$ is odd.

It remains to show that V_k^i intersects precisely $2k - 1$ elements of $\mathcal{N}$. Recall that Q_k is constructed by using two copies of $P + \sum_{M \in \mathcal{M}} M$, where

$$\mathcal{M} = \{M_0, \underbrace{M_0, M_1, M_3, M_4, \ldots, M_0, M_1, M_3, M_4}_{(k-1)\text{-times}}\}.$$

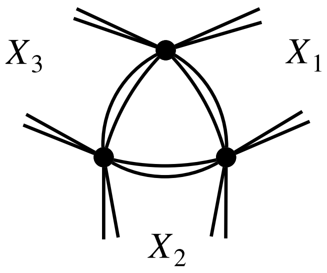

Figure 36: The subgraph T_2.

Since $|V_k^i| = 2k$ and $\mathcal{N}$ contains $4k-2$ perfect matchings, it follows that $\omega(\psi(V_k^i)) \in \{2k-2, 2k-1, 2k\}$. Suppose to the contrary that $\omega(\psi(V_k^1)) = 2k-2$, which is equivalent to $\omega(\psi(V_k^2)) = 2k$. Furthermore, U_k^1 (U_k^2) intersects the same matchings of $\mathcal{N}$ as V_k^2 (V_k^1). Therefore, there is a family $\mathcal{N}_P$ of $4k-2$ pairwise disjoint perfect matching in P_k^1 such that $\mathcal{M} \nsubseteq \mathcal{N}_P$, a contradiction with Lemma 5.5. Hence, $\omega(\psi(V_k^1)) = \omega(\psi(V_k^2)) = 2k-1$. $\square$

Let T_k be the graph on three vertices x_1, x_2, x_3 such that, for all $i \neq j$, there are k parallel edges connecting x_i to x_j.

Let G be a cubic graph. Construct the graph $S_k(G)$ as follows: replace every node $v \in V(G)$ by a copy T_k^v of the graph T_k and every edge $e \in E(G)$ by a copy Q_k^e of the graph Q_k. If the vertex v' is adjacent with the edge e', then the graphs $T_k^{v'}$ and $Q_k^{e'}$ are connected by $2k$ edges. More precisely, add k edges connecting v_1^1 (or v_1^2) together with x_i and k edges connecting v_1^1 (or v_1^2) together with x_{i+1}, for suitable $i \in \mathbb{Z}_3$. Connect those graphs in such a way that the resulting graph $S_k(G)$ is $4k$-regular.

Let $p = w_1 e_1 \dots w_n e_n$ be a path in G, for $w_j \in V(G)$ and $e_j \in E(G)$, then the chain $C = T_k^{w_1} Q_k^{e_1} \dots T_k^{w_n} Q_k^{e_n}$ consists of graphs which are connected to the previous and the next one, with respect to the chain order, in $S_k(G)$. In this case, we will say that the chain of a graph C forms a path in $S_k(G)$.

Lemma 5.8. *Let G be a bridgeless cubic graph. For all $k \geq 1$: $S_k(G)$ is a $4k$-edge-connected $4k$-regular graph.*

Proof. $S_k(G)$ is $4k$-regular by construction. We show that there are $4k$ pairwise disjoint paths between any two vertices of $S_k(G)$. Consider the graph $R_k = P_k - u_1 v_1$, where we remove from P_k all ($2k$) edges connecting u_1 to v_1.

Claim 5.9. *The following statements hold:*

i. there are $2k$ edge-disjoint $u_1 v_1$-paths in R_k;

ii. for all $w \in V(P_k) \setminus \{u_1, v_1\}$ there are $2k$ wu_1-paths and $2k$ wv_1-paths which are pairwise edge-disjoint in R_k;

iii. for all $w_1 \neq w_2 \in V(P_k) \setminus \{u_1, v_1\}$, there exists $t \in \{0, 1, \dots, 2k\}$ such that

- *there are t edge-disjoint w_1w_2-paths containing u_1v_1 in P_k;*
- *there are $4k - t$ edge-disjoint w_1w_2-paths in R_k, which are moreover edge-disjoint from the previous ones.*

iv. for all $x_i \neq x_j \in V(T_k)$, there are $2k$ edge-disjoint x_ix_j-paths in T_k.

Proof. By Lemma 5.6 there are $4k$ edge-disjoint u_1v_1-paths in P_k. Since $\mu(u_1v_1) = 2k$ there are $2k$ edge-disjoint u_1v_1-paths in R_k and *i.* is proved.

Let $w \in V(P_k) \setminus \{u_1, v_1\}$. By Lemma 5.6, there are $4k$ edge-disjoint wv_1-paths in P_k. Then, since P_k is $4k$-regular and u_1v_1 is an edge of multiplicity $\mu(u_1v_1) = 2k$, there are $2k$ of those paths ending with the edge u_1v_1. Thus, there are $2k$ wu_1-paths and $2k$ wv_1-paths which are pairwise edge-disjoint in R_k and so *ii.* is proved.

In order to prove statement *iii.*, pick two different vertices w_1 and w_2 in $V(P_k) \setminus \{u_1, v_1\}$. Since P_k is $4k$-edge-connected there are $4k$-edge-disjoint w_1w_2-paths in P_k. Let t be the number of such paths containing the edge u_1v_1. Then $t \leq \mu(u_1v_1) = 2k$.

The last statement holds since there are k pairwise edge-disjoint paths x_ix_j and there are k pairwise edge-disjoint paths $x_ix_tx_j$, for $t \neq i, j$. Thus, the claim is proved. □

Let $y_1 \neq y_2 \in V(S_k(G))$. There are copies of T_k or Q_k, say Y_1, Y_2, such that $y_i \in Y_i$.

Case 1: Y_1 and Y_2 correspond to two vertices w_1 and w_2 of G, that is, they both are copies of T_k. If $w_1 = w_2$, then $Y_1 = Y_2$ and the statement is trivial. If $w_1 \neq w_2$, since G is bridgeless, there are two edge-disjoint w_1w_2-paths in G. These paths correspond to two chains of (internally) different subgraphs $C = Y_1N_1 \dots N_pY_2$ and $C' = Y_1N'_1 \dots N'_qY_2$ that both form a path in $S_k(G)$. Let s_j, t_j the nodes of N_j which are adjacent to N_{j-1} and N_{j+1} respectively. Let s_1 be adjacent to Y_1 and t_p be adjacent to Y_2. Define in the very same way the vertices s'_j, t'_j in C'. By Claim 5.9, there are $2k$ pairwise edge-disjoint s_1t_p-paths in C and $2k$ pairwise edge-disjoint $s'_1t'_q$-paths in C'. Notice that $s_1 \neq s'_1$ and $t_p \neq t'_q$. By Claim 5.9, there are $2k$ pairwise edge-disjoint $s_1s'_1$-paths passing through Y_1 and $2k$ edge-disjoint $t_pt'_q$-paths passing through Y_2. Therefore, all these paths combine to $4k$ edge-disjoint y_1y_2-paths in $S_k(G)$.

Case 2: If Y_1 and Y_2 correspond to a vertex and an edge, or to two edges of G, say a_1 and a_2, then there is a circuit in G which contains a_1 and a_2. By a similar argumentation as above we deduce that there are $4k$-edge-disjoint y_1y_2-paths in $S_k(G)$. □

Theorem 5.10. *Let G be a bridgeless cubic graph with an even number of edges. For all $k \geq 1 : S_k(G)$ is a $4k$-edge-connected $4k$-graph without $4k-2$ pairwise disjoint perfect matchings.*

Proof. By Lemma 5.8, $S_k(G)$ is $4k$-edge-connected, $4k$-regular and it holds $|V(S_k(G))| = 19|E(G)| + 3|V(G)| \equiv 0 \mod 2$. Thus, $S_k(G)$ is a $4k$-graph.

Suppose to the contrary that $S_k(G)$ has $4k-2$ pairwise disjoint perfect matchings. Consider a vertex $v \in V(G)$ and the corresponding subgraph T_k^v. Since T_k^v has three vertices it follows that no component of $\psi(\partial_{S_k(G)}(V(T_k^v)))$ is 0. Let $\partial_G(v) = \{e_1, e_2, e_3\}$ and X_i be the set of $2k$ edges connecting T_k^v to $Q_k^{e_i}$, see Figure 36. By Lemma 5.7, for all $i \in \{1,2,3\}$, we have that $\omega(\psi(X_i)) = 2k-1$. Since the cardinality of the symmetric difference of three odd sets is odd it follows that there is a $j \in \{1,2,\dots,4k-2\}$ such that $0 = \sum_{i=1}^{3} \psi_j(X_i) = \psi_j(\partial_{S_k(G)}(V(T_k^v)))$, a contradiction. □

5.2.3 *Highly connected r-graphs on 60 vertices*

To prove the other cases of Theorem 5.2, we continue with the construction of regular graphs on 60 vertices.

For $k \geq 1$: Let H_k be the graph which is obtained from three copies Q_k^1, Q_k^2, Q_k^3 of Q_k. In order to simplify the description let z_j be the vertex v_1^j of Q_k, for $j \in \{1,2\}$. For $i \in \{1,2,3\}$, if u is a vertex of Q_k we denote by u^i the corresponding vertex of the copy Q_k^i. Glue them together with the graph T_k as follows: for all $i \in \{1,2,3\}$,

- add k edges connecting x_{i+1} of T_k to z_1^i of Q_k^i;
- add k edges connecting x_{i+2} of T_k to z_1^i of Q_k^i;
- add k edges connecting z_2^i of Q_k^i to z_2^{i+1} of Q_k^{i+1};

where the indices are added modulo 3. The graph H_2 is depicted in Figure 37.

Lemma 5.11. *For all $k \geq 1$: H_k is a $4k$-edge-connected $4k$-graph of order 60 without $4k-2$ pairwise disjoint perfect matchings.*

Proof. Let K_2^3 be the unique (loopless) cubic graph on two vertices. H_k is obtained from $S_k(K_2^3)$ by removing one T_k and then connecting the vertices of degree $2k$ pairwise by k (parallel) edges. Clearly, H_k is $4k$-regular. Note that $\{z_2^1, z_2^2, z_2^3\}$ induce a triangle T in H_k where any two vertices are connected by k edges. Furthermore, for any $2k$ pairwise edge-disjoint paths which connect two vertices of $\{z_2^1, z_2^2, z_2^3\}$ in $S_k(K_2^3)$ and do not contain any edge of $S_k(K_2^3) - T_k$ there are $2k$ corresponding paths in H_k. Hence, H_k is $4k$-edge-connected.

Suppose to the contrary that H_k has $4k-2$ pairwise disjoint perfect matchings $\mathcal{N} = \{N_1, \dots, N_{4k-2}\}$. Let $Z_{i,i+1}$ be the collection of parallel edges of type $z_2^i z_2^{i+1}$. By Lemma 5.7, for each $i \in \{1,2,3\}$, $\partial(z_2^i) \cap E(T)$ intersects $2k-1$ elements of $\mathcal{N}$, implying that $\omega(\psi(Z_{i,i+1}))$ and $\omega(\psi(Z_{i-1,i}))$ have different parity. This is not possible since there are three such vertices. □

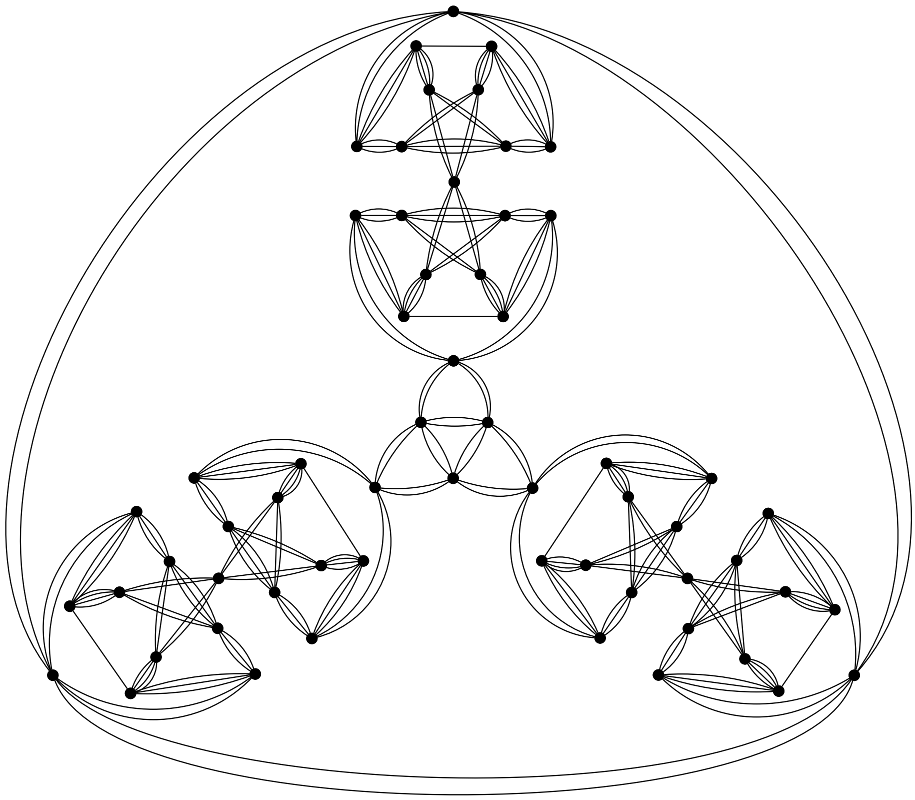

Figure 37: H_2 is an 8-edge-connected 8-graph on 60 vertices without 6 pairwise disjoint perfect matchings.

Next we will identify 4 pairwise disjoint matchings in H_2. These matchings will be used to complete the proof of Theorem 5.2.

Consider a copy of Q_k inside a graph G, such that both V_k^1 and V_k^2 are non-empty. Let M be a perfect matching of G. Then w.l.o.g. $|V_k^1 \cap M| = 1$ and $|V_k^2 \cap M| = 0$. The unique perfect matching in $P_k = P_k^1 + 2ku_1^1v_1^1$ containing the edges of $M \cap E(P_k^1)$ is of type 0 or 1, suppose of type 0. In the same way, the unique perfect matching in $P_k = P_k^2 + 2ku_1^2v_1^2$ containing the edges of $M \cap E(P_k^2)$ is of type 3 or 4, suppose of type 3. In this case we say that Q_k is of type $(0,3)$. For example, the bold perfect matching depicted in Figure 38 is such that all Q_k^is are of type $(0,4)$. We call N_0 such a perfect matching in H_k. Moreover, for $i \in \{1,2,3\}$, let N_i be the perfect matching of H_k such that:

- Q_k^i is of type $(1,3)$;
- Q_k^{i+1} is of type $(3,0)$;
- Q_k^{i+2} is of type $(4,1)$;

where sums of indices are taken modulo 3. In Figure 38 N_1 is depicted using normal lines, N_2 is depicted using dotted lines and N_3 is depicted using dashed lines.

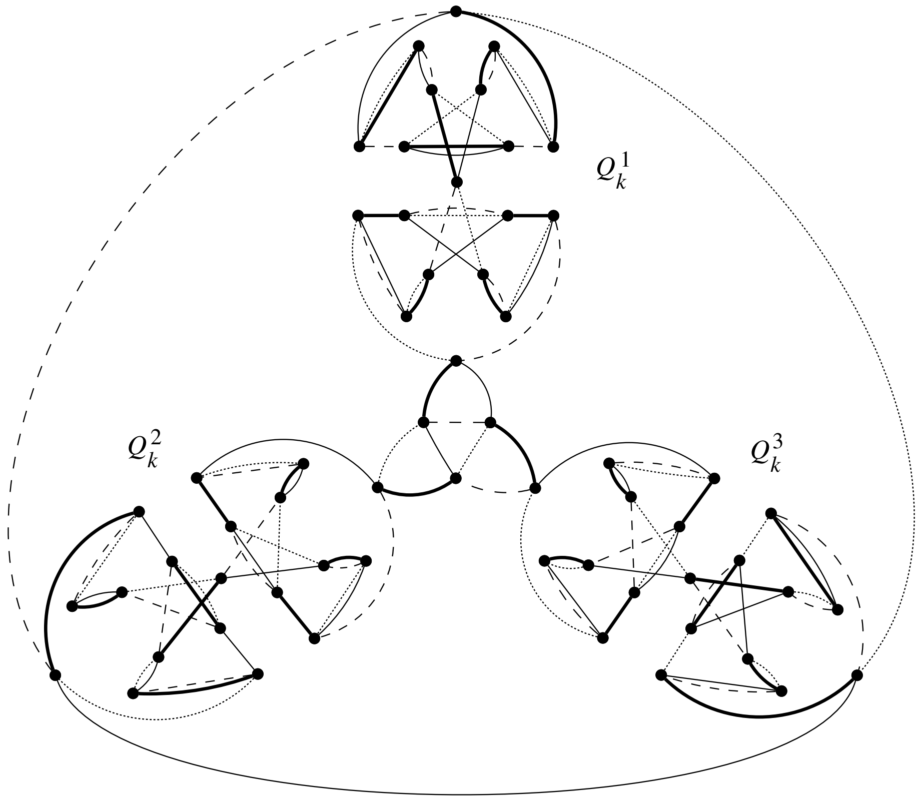

Figure 38: Four pairwise disjoint perfect matchings in H_k.

By construction of the perfect matchings N_0, N_1, N_2, N_3, the following lemma, which will be needed for the proof of Theorem 5.2, follows.

Lemma 5.12. *For all $k \geq 1$: $H_{k+1} = H_k + (N_0 + N_1 + N_2 + N_3)$.*

Lemma 5.13. *For all $t \geq 1$, there is a $2t$-edge-connected $(2t+1)$-graph on 60 vertices without $2t-1$ pairwise disjoint perfect matchings.*

Proof. Case 1: $t = 2k+1$ for a $k \geq 1$. Let $H_k' = H_k + (N_0 + N_1 + N_2)$. Since the graph $\tilde{H} = H_k[N_0 + N_1 + N_2]$ is a 3-edge-colorable connected cubic graph, we have that for all $X \subseteq V(\tilde{H})$, $|\partial_{\tilde{H}}(X)| \geq 3$, if X is odd and $|\partial_{\tilde{H}}(X)| \geq 2$, if X is even. Then H_k' is $(4k+2)$-edge-connected $(4k+3)$-graph. From the equality $H_k' = H_{k+1} - N_3$ we deduce that it has no $4k+1$ pairwise disjoint perfect matchings.

Case 2: $t = 2k$ for a $k \geq 1$. The graph $H_k'' = H_k + N_0$ is a $4k$-edge-connected $(4k+1)$-graph. Since $H_k'' = H_{k+1} - (N_1 + N_2 + N_3)$, it follows that H_k'' has no $4k-1$ pairwise disjoint perfect matchings. □

Lemma 5.14. *For all $k \geq 1$, there is a $4k$-edge-connected $(4k+2)$-graph on 60 vertices without $4k$ pairwise disjoint perfect matchings.*

Proof. The graph $H_k''' = H_k + N_0 + N_1$ is a $4k$-edge-connected $(4k+2)$-graph. It has no $4k$ pairwise disjoint perfect matchings because $H_k''' = H_{k+1} - (N_2 + N_3)$. □

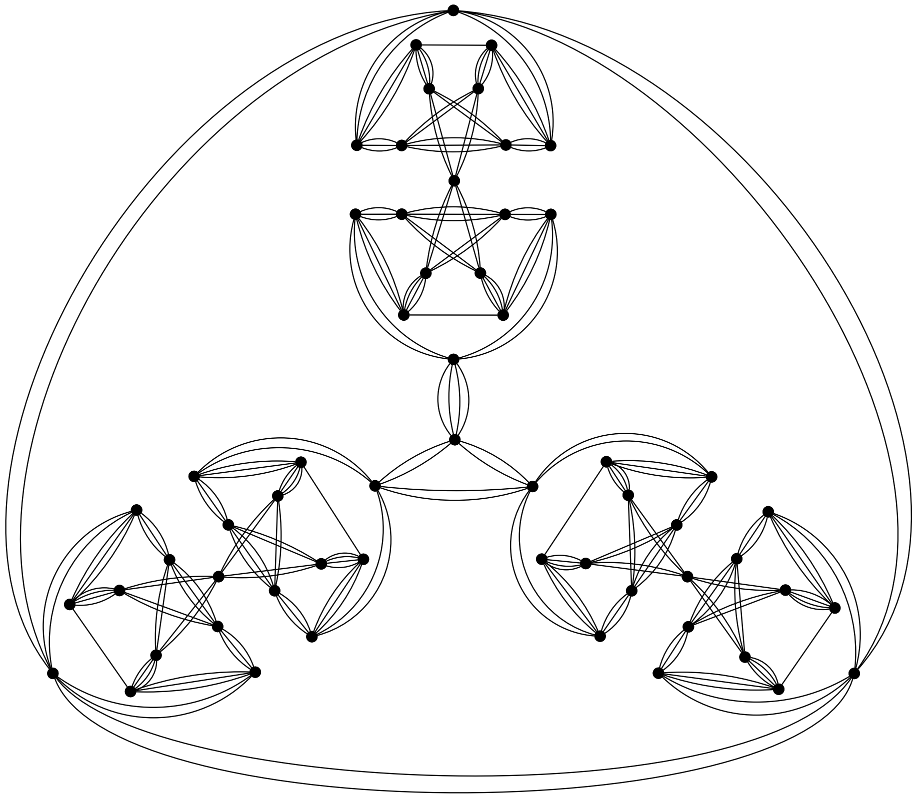

Figure 39: The graph $\tilde{H}_2$.

Theorem 5.2 now follows from Lemmas 5.11, 5.13, and 5.14. It remains to construct simple graphs with the desired property and to show how to expand vertices.

Finally we remark that the smallest example of r-edge-connected r-graph, satisfying the hypothesis of Theorem 5.2, that we can produce is on 58 vertices. For $r = 4k$, we call it $\tilde{H}_k$. It can be constructed as follows: remove the copy of T_k from H_k and add a new vertex x; moreover add $2k$ edges connecting x to z_1^1 and, for $i \in \{2, 3\}$, k edges connecting x to z_1^i. Finally add k parallel edges from z_1^2 to z_1^3, see Figure 39. For $r \equiv 1, 2, 3 \mod 4$ we add or remove perfect matchings from $\tilde{H}_k$ in the same fashion as we did for H_k in this section. We call such graphs $\tilde{H}'_k$, $\tilde{H}''_k$ and $\tilde{H}'''_k$ when $r = 4k + 1, 4k + 2, 4k + 3$.

5.2.4 *Simple graphs*

Let v be a vertex of a graph G such that $d_G(v) = t$. Moreover let $v_1, \ldots, v_t$ be the not necessarily distinct neighbors of v and $u_1, \ldots, u_t$ be the vertices of degree $t - 1$ of $K_{t,t-1}$. The Meredith extension [57] applied to G at v produces the graph G_v obtained from $G - v$ and a copy of the complete graph $K_{t,t-1}$ by adding all edges $v_i u_i$, for $i \in \{1, \ldots, t\}$. Notice that G is t-edge-connected if and only if G_v is t-edge-connected. Furthermore, it is easy to see that for $t \geq 2$, G does not have t pairwise disjoint perfect matchings if and only if G_v does not have t pairwise disjoint perfect matchings.

Let $\mathcal{V} \subset V(\tilde{H}_k)$ be a vertex cover of $\tilde{H}_k$. If Meredith extension is applied on every vertex of $\mathcal{V}$, then we obtain simple r-edge-connected r-graphs without $r-2$ pairwise disjoint perfect matchings. In particular, there is a vertex cover $\mathcal{V}$ of $\tilde{H}_k$ such that $|\mathcal{V}| = 33$. Thus, expanding the vertices of $\mathcal{V}$ at the graphs $\tilde{H}_k, \tilde{H}'_k, \tilde{H}''_k$ and $\tilde{H}'''_k$ yields simple t-edge-connected r-graphs of order $58 + 33(2r-2) = 66r - 8$ with the desired properties. Repeated application of Meredith extension yields infinite families of such graphs.

6 FLOW-CONTINUOUS MAPS AND ORIENTED COLORINGS OF CUBIC GRAPHS

In this chapter we study maps $E(\vec{G}) \to E(\vec{H})$ on the edge-sets of two oriented cubic graphs with the property that every flow on H with orientation $\vec{H}$ is *lifted* to a flow on G with orientation $\vec{G}$. In this setting, it is convenient to work with graphs where an orientation has been fixed a priori.

This chapter is based on [P.5].

6.1 INTRODUCTION

Let M be an abelian group, a map $f\colon E(\vec{G}) \to E(\vec{H})$ between the edge sets of two oriented graphs G and H is called *M-flow-continuous* if every M-flow of H can be lifted to an M-flow of G in the given orientations, i.e. for every M-flow $\psi\colon E(\vec{H}) \to M$ the composition $\psi \circ f$ is still an M-flow.

$$\begin{tikzcd} E(\vec{G}) \arrow[r, "f"] \arrow[dr, "\psi\circ f"'] & E(\vec{H}) \arrow[d, "\psi"] \\ & M \end{tikzcd}$$

It is well known that $\mathbb{Z}_2$-flow-continuous maps are exactly cycle-continuous maps, i.e. maps having the property that the pre-image of every cycle is a cycle, where, in this chapter, by *cycle* we mean an even graph. The interest for these maps comes from an outstanding conjecture by Jaeger claiming that every bridgeless graph has a cycle-continuous map to the Petersen graph P_{10} [39]. Indeed a positive answer to this conjecture would imply many other very important ones like the Cycle Double Cover [67], [76] and Berge-Fulkerson Conjectures [24].

M-flow-continuous maps are introduced in [15] and naturally define quasi-orders on the class of finite graphs. We say that $G \succ_M H$ if there is an M-flow-continuous map between an orientation of G and an orientation of H (we remark that our notation is slightly different from [15] since we only need to specify the group on which the flow function takes values). Using this notation Jaeger's Conjecture can be stated as follows

Conjecture 6.1 (Jaeger [39]). *Every bridgeless graph G satisfies $G \succ_{\mathbb{Z}_2} P_{10}$.*

Jaeger's Conjecture can be reduced to cubic graphs. In this context it is also known as the Petersen Coloring Conjecture since it can be naturally stated in terms of graph colorings. A map $f\colon E(G) \to E(H)$ between two cubic graphs G and H is called an *H-coloring* of G if for every $v \in V(G)$ there is $v_h \in V(H)$ such that $f(\partial(v)) = \partial(v_h)$. If G is cubic then $G \succ_{\mathbb{Z}_2} P_{10}$ if and only if G has a P_{10}-coloring. Hence Jaeger's Conjecture can be equivalently stated as follows

Conjecture 6.2 (Jaeger [39]). *Every bridgeless cubic graph has a P_{10}-coloring.*

In [65] an infinite antichain of cubic graphs in the $\mathbb{Z}_2$-flow-continuous order was presented, and the problem of finding an infinite antichain (in the same quasi-order) of cyclically 4-edge-connected cubic graph was left for further research. Since $\mathbb{Z}$-flow-continuous maps are also cycle-continuous, the problem of finding such an infinite antichain in the $\mathbb{Z}$-flow-continuous order would be a weaker version of the previous one. In this chapter we study the quasi-order $\succ_{\mathbb{Z}}$. In particular we first give an operative description of $\mathbb{Z}$-flow-continuous maps $f\colon E(\vec{G}) \to E(\vec{H})$, when H is a cyclically 4-edge-connected cubic graph, in terms of graph colorings plus an additional requirement on the orientation around each vertex. Finally we show that such quasi-order contains an infinite antichain of snarks containing P_{10}, where we recall that a *snark* is a cyclically 4-edge-connected cubic graph with girth at least 5 and not admitting a 3-edge-coloring.

6.2 ORIENTED COLORINGS

From now on, all graphs considered in this chapter are cubic. Given a positive integer k, a *multipole* consists of a set of vertices V and a set of edges E, which may contain also dangling edges, i.e. edges adjacent just to one vertex and having a dangling side. We call *k-pole* a multipole containing k dangling edges. A graph is a multipole having no dangling edge.

Let G be a multipole and $\vec{G}$ an orientation of G. Moreover let $x \in V(G)$ be a vertex incident to the edges $e_1, e_2 \in \partial(x)$. We say that x *reverses* the orientation of the path $e_1 x e_2$ in $\vec{G}$ if e_1 and e_2 are both incoming or outgoing at x in $\vec{G}$. Otherwise we say that x *preserves* the orientation of the path $e_1 x e_2$ in $\vec{G}$.

Definition 6.3. Let G and H be two multipoles on which we have fixed the orientations $\vec{G}$ and $\vec{H}$ respectively. A map $f\colon E(\vec{G}) \to E(\vec{H})$ is an *H-oriented-coloring* of G if

- for every vertex $v \in V(G)$ there is a vertex $v_h \in V(H)$ such that $f(\partial(v)) = \partial(v_h)$;
- for every $v \in V(G)$ the mutual orientation of pairs of edges $e_1, e_2 \in \partial(v)$ is the same with respect to $f(e_1), f(e_2) \in \partial(v_h)$; in other words if v preserves (resp. reverses) the orientation of the

path e_1ve_2 in $\vec{G}$ then v_h preserves (resp. reverses) the orientation of $f(e_1)v_hf(e_2)$ in $\vec{H}$.

An H-oriented-coloring is first of all an H-coloring. Furthermore if, for an orientation $\vec{H}$ of H, there is an orientation $\vec{G}$ of G and a map $f\colon E(\vec{G}) \to E(\vec{H})$ that is an H-oriented-coloring then, for every orientation of H, there is an orientation of G and a map that is H-oriented-coloring of G. Indeed, given such a map f, just notice that if we reverse the orientation of $e \in E(\vec{H})$ then it suffices to reverse the orientation of the set of edges $f^{-1}(e)$ and f remains an H-oriented-coloring of G.

Previous property holds also for $\mathbb{Z}$-flow-continuous maps $f\colon E(\vec{G}) \to E(\vec{H})$ from an orientation of G and an orientation of H. Indeed, if we reverse the orientation of an edge $e \in E(\vec{H})$, then f is still a $\mathbb{Z}$-flow-continuous map provided that we reverse the orientation of every edge of $f^{-1}(e)$.

Notice also that an oriented coloring map is a $\mathbb{Z}$-flow-continuous map and therefore, for such a map $f\colon E(\vec{G}) \to E(\vec{H})$ between two graphs G and H, the following necessary condition holds $\phi_c(G) \leq \phi_c(H)$.

In [15] the authors prove that a map $f\colon E(G) \to E(P_{10})$, where G is cubic, is a P_{10}-coloring if and only if it is cycle-continuous. A central role is played by the fact that P_{10} has only trivial 3-edge-cuts. Indeed this property still holds for cycle-continuous maps $G \to H$ of cubic graphs, whenever H has only trivial 3-edge-cuts. It is well known that an H-coloring of G is a cycle-continuous map. On the other hand, if $f\colon E(G) \to E(H)$ is a cycle-continuous map and H is cyclically 4-edge-connected, consider three different edges e_1, e_2, e_3 incident with $v \in V(G)$. If the set of edges $F = f(\{e_1, e_2, e_3\})$ does not form a (trivial) edge-cut, there is a cycle C containing an edge $e \in F$ and avoiding all edges of $F - e$. Then $f^{-1}(C)$ is not a cycle because contains a vertex, that is v, with odd degree.

Our interest to oriented colorings of cubic graphs is motivated by the following proposition.

Proposition 6.4. *Let G and H be two bridgeless cubic graphs and let H be cyclically 4-edge-connected. Suppose that they are endowed with the orientations $\vec{G}$ and $\vec{H}$ respectively. Then $f\colon E(\vec{G}) \to E(\vec{H})$ is an H-oriented-coloring of G if and only if f is a $\mathbb{Z}$-flow-continuous map.*

Proof. The necessity holds from the definition of oriented coloring. Let us prove the sufficiency. By previous observations we can choose on H an orientation $\vec{H}$ that admits a positive nowhere-zero $\mathbb{Z}$-flow. Since f is cycle-continuous and H is cyclically 4-edge-connected we get that f is an H-coloring of G. We have to show that for every $v \in G$, the orientation of $\partial(v)$ is the same or opposite to $\partial(v_h)$. Suppose by contradiction that this is not the case meaning that there is a vertex v that does not satisfy the required property. Without loss of generality we can suppose that v_h has one incoming edge a and two outgoing edges b, c. The set $\partial(v)$ is mapped onto the set $\{a, b, c\}$ by f, let us call e_i the edge of $\partial(v)$ such that $f(e_i) = i \in \partial(v_h)$. By our contradictory hypothesis v reverses the orientation of at least one couple between e_a, e_b and e_a, e_c, assume that this holds for e_a and e_b. Let $\psi\colon \vec{H} \to \mathbb{Z}$ be a $\mathbb{Z}$-flow such that $\psi(e) = 1$ for every $e \in C$, where C is a directed cycle of H containing the edges a and b, and $\psi = 0$ everywhere else. Notice that such a directed cycle exists thank to the chosen orientation on H. Then $\psi \circ f$ is not a $\mathbb{Z}$-flow in G as the flow-conservation law does not hold at v, a contradiction. □

In [15] the authors prove that a graph G has flow number at most 4 if and only if $G \succ_{\mathbb{Z}} K_4$. We will show in Section 6.4 an alternative proof of the same result for the case of cubic graphs, that makes use of oriented colorings.

Theorem 6.5 (DeVos et al. [15]). *Let G be a bridgeless cubic graph. Then $\phi_c(G) \leq 4$ if and only if there is a K_4-oriented-coloring of G.*

For the case of bipartite cubic graphs another characterization is proved in [15]: a cubic graph G is bipartite if and only if $G \succ_{\mathbb{Z}} K_2^3$, where K_2^3 is the cubic loopless multigraph on 2 vertices and 3 edges. The following generalization, stated using oriented colorings, holds.

Theorem 6.6 (DeVos et al. [15]). *Let G be a bridgeless cubic multipole. Then G is bipartite if and only if there is a K_2^3-oriented-coloring of G.*

Let G be a multipole. The multipole *induced* by $X \subseteq V(G)$ in G is the multipole whose vertex set is X and edge set consists of all edges adjacent to at least one vertex of X. In the following part, if $f\colon E(\vec{G}) \to E(\vec{H})$ is an oriented coloring of a cubic multipole G, consider the subgraph K of $\vec{H}$ induced by $f(E(\vec{G}))$. With a slight abuse of terminology, we will denote by $f(G)$ the undirected multipole induced by the vertices of degree 3 of K.

Corollary 6.7. *Let G be a cubic multipole containing an odd cycle, H a cubic graph and $f\colon E(\vec{G}) \to E(\vec{H})$ an oriented coloring. Then $f(G)$ contains an odd cycle.*

Remark 6.8. *In the hypothesis of previous corollary, if G is the cubic multipole consisting of a k-cycle and k dangling edges, for an odd number k, then the girth of H is at most k. In particular, if it is exactly k, then $f(G)$ is isomorphic to G and its dangling edges are mapped to dangling edges of its image.*

We are interested in studying the cubic 4-pole N obtained from the Petersen graph P_{10} by removing two adjacent vertices and generating this way 4 dangling edges and, in particular, we are interested in understanding how its image under an oriented coloring map looks like.

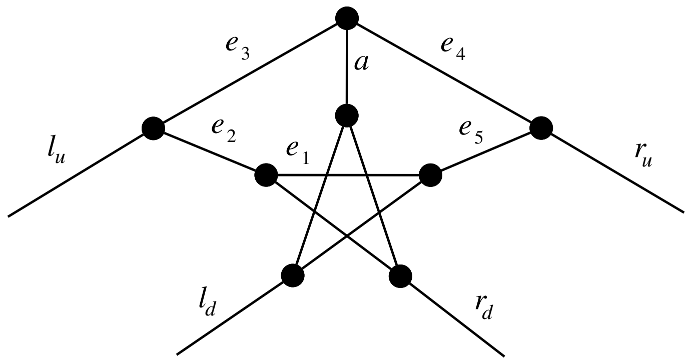

Figure 40: The multipole N.

We say that two (dangling) edges of a multipole are at *distance* k if the shortest path connecting two of their endvertices has length k, where the length of a path is the number of its edges.

Lemma 6.9. *Suppose that f is an H-oriented-coloring of N where H is a cyclically 4-edge-connected cubic graph of girth at least 5. Then $f(N)$ is isomorphic to a copy of N. Moreover dangling edges are mapped to dangling edges of the multipole $f(N)$ in such a way that both pairs $f(l_u), f(l_d)$ and $f(r_u), f(r_d)$ are at distance 3 in $f(N)$ (see Figure 40 as reference for the considered edges).*

Proof. Consider two 5-cycles C_1, C_2 in N such that $C_2 = e_1e_2e_3e_4e_5$, see Figure 40, and C_1 intersects C_2 just in e_1. Let M_i be the multipole induced by $V(C_i)$. By Corollary 6.7, $f(M_1)$ and $f(M_2)$ are both isomorphic to M_1 (and also to M_2), and dangling edges of M_1 are mapped to dangling edges of $f(M_1)$. Notice that $C_2 \cap M_1 = \{e_1, e_2, e_5\}$, and so $f(e_2)$ and $f(e_5)$ are dangling edges of $f(M_1)$. Therefore e_3 and e_4 are mapped to a couple of adjacent edges which are both adjacent to $f(a)$ and such that they are also adjacent to $f(e_2)$ and $f(e_5)$ respectively. Finally, the unique possibility is that the remaining two dangling edges l_u and r_u are mapped to dangling edges adjacent respectively to $f(e_2), f(e_3)$ and $f(e_4), f(e_5)$. □

The following corollary follows immediately from the main result of the note [58] by Mkrtchyan, claiming that if P_{10} has a G-coloring, for a connected bridgeless cubic graph G, then $P_{10} = G$.

Corollary 6.10 (Mkrtchyan [58])**.** *Suppose that f is an H-oriented-coloring of P_{10}, where H is a bridgeless cubic graph. Then $f(P_{10})$ is isomorphic to P_{10}.*

Proof. Follows from the fact that f is an H-coloring of P_{10}. □

Other than N, we want to focus on the 5-pole N' shown in Figure 41.

Lemma 6.11. *There is no oriented coloring $f \colon E(\vec{N'}) \to E(\vec{P_{10}})$.*

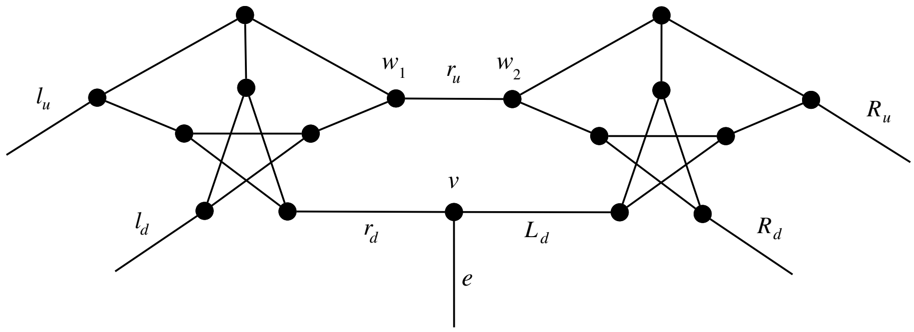

Figure 41: The multipole N'.

Proof. Suppose by contradiction that there is a P_{10}-oriented-coloring f of N'. There are two distinct copies N_1 and N_2 of N inside N', which have a common dangling edge r_u and an other one adjacent to a new vertex v, see Figure 41 as reference for the considered edges. Without loss of generality we say that N_1 is the left copy of N and N_2 is the right one with respect to Figure 41. By previous lemma N_1 is sent to a copy isomorphic to N where l_u, l_d and r_u, r_d are mapped to pairwise adjacent edges. Let $z \in E(P_{10}) \setminus f(N_1)$, i.e. z is adjacent to $f(l_u), f(l_d), f(r_u)$ and $f(r_d)$. In an analogous way N_2 is mapped to a copy isomorphic to N. Hence $f(L_d)$ must be adjacent to $f(r_u)$ and to $f(r_d)$ and therefore $f(L_d) = z$ and $f(e) = f(r_u)$. Thus N_2 is sent to $P_{10} - f(r_d)$.

By definition of oriented coloring we have that the mutual orientation of every possible couple of edges of $\vec{C} = \{r_u, r_d, l_u, l_d\} \subseteq E(\vec{N'})$ must be the same with respect to its image $f(\vec{C})$, and the same property holds for the set of edges $\{r_u, L_d, R_u, R_d\}$, in particular for r_u and L_d. The contradiction arises from the fact that, due to the presence of the vertex v, the mutual orientation of r_u and L_d is different from the mutual orientation of $f(r_u)$ and $f(L_d)$. $\square$

All previous results lead us to the following theorem.

Theorem 6.12. *Let G be a bridgeless cubic graph obtained by joining dangling edges of a 5-pole C with dangling edges of N'. Then $G \not\prec_{\mathbb{Z}} P_{10}$.*

Construction methods described in [19], [P.2] and [53] show that there are many snarks with circular flow number 5 that have the structure described by previous theorem. Those snarks are examples of cubic graphs that are incomparable with P_{10} in the $\mathbb{Z}$-flow-continuous order by Corollary 6.10 and Theorem 6.12. Every such snark S is also incomparable with K_4, indeed $S \not\prec_{\mathbb{Z}} K_4$ since $\phi_c(S) > \phi_c(K_4)$ and $K_4 \not\prec_{\mathbb{Z}} S$ since the girth of S is greater that the girth of K_4.

In the following part we will show that some of the snarks with circular flow number 5 constructed in Chapter 2, together with the Petersen graph, form an infinite antichain in the $\mathbb{Z}$-flow-continuous order.

Definition 6.13. Consider $n \geq 3$ copies $N_1, N_2, \dots, N_n$ of the multipole N. For $i = 1 \dots, n$, let us denote by $l_{u,i}, l_{d,i}, r_{u,i}$ and $r_{d,i}$ the dangling edges of N_i with reference to Figure 40. Consider an n-cycle $c_1c_2 \dots c_n$. Call $\tilde{W}_n$ the graph obtained by identifying $r_{u,i}$ with $l_{u,i+1}$ and by making the vertex c_i and both dangling edges $r_{d,i}, l_{d,i+1}$ be adjacent to a new vertex v_i, where we compute the sum of indices modulo n. We will refer to the copy N_i inside $\tilde{W}_n$ as N_i^n.

Notice that $\tilde{W}_n$ can be constructed as follows: replace each edge of the external cycle of a wheel W_n with a copy of the generalized edge $\mathcal{P}_{10}^*(u, v)$. We recall that this is a $(4, 1)$-edge, with terminals u and v, constructed by removing from the Petersen graph the edge $uv \in E(P_{10})$. Then split off properly each vertex of degree 5 different form the central vertex of the wheel, and expand properly the central vertex into a n-cycle. By Theorem 2.35, $\tilde{W}_n$ has circular flow number 5 whenever n is odd.

In order to make use of Lemma 6.9 and the equivalence given by Proposition 6.4, we will focus on graphs $\tilde{W}_n$ with $n \geq 5$ and, in particular, we will be interested in graphs $\tilde{W}_n, \tilde{W}_m$ such that n and m are coprime.

Proposition 6.14. *Consider two positive integers $n, m \geq 5$. There is a $\tilde{W}_m$-oriented-coloring of $\tilde{W}_n$ if and only if m divides n.*

Proof. Let f be the $\tilde{W}_m$-oriented-coloring of $\tilde{W}_n$.

Claim 6.15. *Let R_i be the multipole isomorphic to N' induced by $V(N_i^n \cup N_{i+1}^n) \cup \{v_i\}$ in $\tilde{W}_n$. Then $f(R_i)$ is isomorphic to R_i.*

Proof of Claim 6.15. We take Figure 41 as a reference when considering edges and vertices, in particular we consider N_i^n to be the left copy of N and N_{i+1}^n the other one.

Since $\tilde{W}_m$ is cyclically 4-edge-connected with girth at least 5, by Lemma 6.9 we deduce that $f(N_i^n)$ is isomorphic to N. Suppose that N_{i+1}^n is sent to the same copy of N. Then $f(L_d) = f(r_d)$ and we get a contradiction because adjacent edges cannot have the same image. Notice that, if $f(\partial(w_2)) = f(\partial(w_1))$ then $f(N_i^n) = f(N_{i+1}^n)$, and we get the same contradiction. Hence we conclude that $f(\partial(w_2))$ is different from $f(\partial(w_1))$. Notice that, because of the structure of $\tilde{W}_m$, $f(\partial(w_2))$ does not contain $f(r_d)$. Therefore N_i^n and N_{i+1}^n are sent to different copies of N having just $f(r_u)$ in common, and the unique possibility for the oriented coloring to be defined is that $f(L_d)$ and $f(r_d)$ are adjacent to the unique vertex v_j in $\tilde{W}_m$ which is adjacent to a dangling edge of both $f(N_i^n)$ and $f(N_{i+1}^n)$. □

Claim 6.16. *Let Q_i be the multipole induced by $V(N_i^n \cup N_{i+1}^n \cup N_{i+2}^n) \cup \{v_i, v_{i+1}\}$ in $\tilde{W}_n$. Then $f(Q_i)$ is isomorphic to Q_i.*

Proof of Claim 6.16. Consider the multipole Q' isomorphic to N' induced by $V(N_i^n \cup N_{i+1}^n) \cup \{v_i\}$ inside $\tilde{W}_n$. Then, by Claim 6.15, $f(Q')$

is isomorphic to N' and so N_i^n and N_{i+1}^n must be sent to two adjacent copies N_j^m, N_{j+1}^m. Without loss of generality we can suppose that they are sent to N_i^m and N_{i+1}^m, and in particular that $f(N_k^n) = N_k^m$, for $k = i, i+1$. Using the same argument we notice that N_{i+1}^n and N_{i+2}^n must be sent to adjacent copies of N. In particular $f(N_{i+2}^n) = N_{i+2}^m$ for otherwise, if $f(N_{i+2}^n) = N_i^m$ we would get that dangling edges l_d and r_d (as well as l_u and r_u) of N_{i+1}^n would be mapped to the same edge a contradiction with Lemma 6.9. □

By previous claims we notice that $f(N_1^n), \ldots, f(N_n^n)$ must be pairwise consecutive copies of N in $\tilde{W}_m$ such that $f(N_i^n)$ is different from both $f(N_{i+1}^n)$ and $f(N_{i+2}^n)$, for every i. Therefore a necessary condition for f to be defined is that n is a multiple of m.

On the other hand a $\tilde{W}_m$-oriented-coloring of $\tilde{W}_{km}$ can be constructed in the natural way identifying via identity map the multipoles M_{h+i}^{km} induced by $V(N_{h+i}^{km} \cup N_{h+i+1}^{km}) + c_{h+i} + c_{h+i+1} + v_{h+i} + v_{h+i+1}$ in $\tilde{W}_{km}$ with the multipoles M_i^m induced by $V(N_i^m \cup N_{i+1}^m) + c_i + c_{i+1} + v_i + v_{i+1}$ in $\tilde{W}_m$, where $h \in \{0, m, 2m, 3m, \ldots, (k-1)m\}$ and $i \in \{1, \ldots, m\}$. The orientation is naturally defined on every M_{hi}^{km} (just set the very same orientation of M_i^m) by the chosen orientation on $\tilde{W}_m$. □

Theorem 6.17. *Let $\{p_j\}_{j\in\mathbb{N}}$ be the sequence of prime numbers greater than 3. The family $\mathcal{F} = \{P_{10}, \tilde{W}_{p_1}, \tilde{W}_{p_2}, \ldots\}$ is an antichain in the $\mathbb{Z}$-flow-continuous order $\succ_{\mathbb{Z}}$.*

Proof. By Proposition 6.14 for every couple of different prime numbers p_s and p_t we have that $\tilde{W}_{p_s}$ and $\tilde{W}_{p_t}$ are incomparable. Moreover P_{10} is incomparable with every other graph of $\mathcal{F}$ by Theorem 6.12. □

6.3 FURTHER EXAMPLES OF ORIENTED COLORINGS

As an example we show here that Goldberg snarks, introduced in [27], form an increasing chain in the $\mathbb{Z}$-flow-continuous order. We have already studied such a family of snarks in Chapter 3, where we improve the upper bound on their circular flow number and propose a new conjecture about it, see Sections 3.4.2 and 3.5. We recall here the construction of the Goldberg snark G_{2k+1} as we need it to define the oriented coloring map. Consider a cycle $v_1 v_2 \ldots v_{2k+1}$ of length $2k+1$. Remove each vertex v_i and substitute it with a copy P_i^- of the 6-pole obtained from the Petersen graph after the removal of two vertices at distance 2. Then, for each couple of adjacent vertices $v_i v_j$ of the initial cycle glue together P_i^- and P_j^- as shown in Figure 24.

We show that $G_{2k+3} \succ_{\mathbb{Z}} G_{2k+1}$, for every positive integer k. Call $Z_1, \ldots, Z_{2k+3}$ and $Q_1, \ldots, Q_{2k+1}$ the consecutive 6-poles of G_{2k+3} and G_{2k+1} respectively. First we can map the subgraph induced by Z_1, Z_2 and Z_3 to Q_1 as shown in Figure 42, as well as fix on them the shown orientation. Then fix on the isomorphic subgraphs induced by

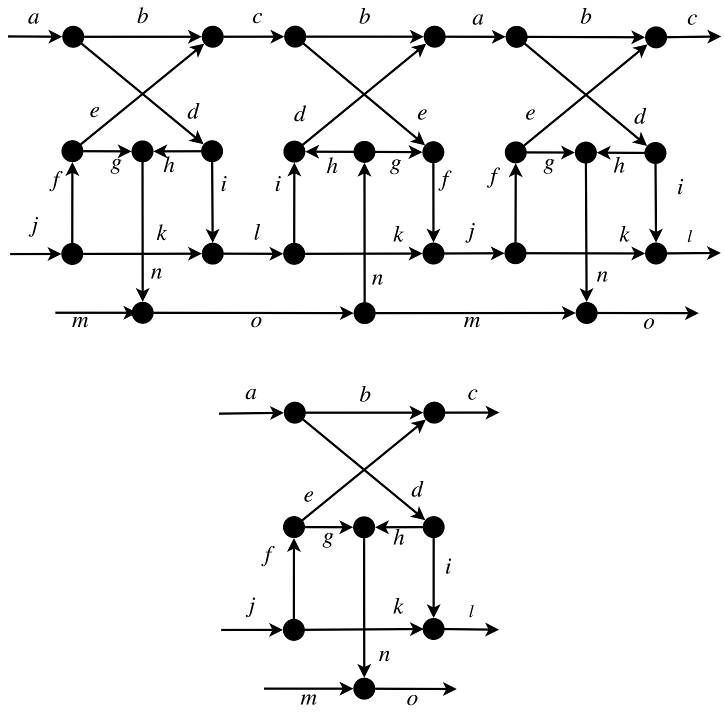

Figure 42: Oriented coloring of 6-poles of Goldberg snarks.

$Z_4, \ldots, Z_{2k+3}$ and $Q_2, \ldots, Q_{2k+1}$ respectively the same orientation and map edges of the multipole Z_{i+2} identically on the edges of Q_i, in the natural way. The map defined is a G_{2k+1}-oriented-coloring of G_{2k+3} and so a $\mathbb{Z}$-flow-continuous map as well. Hence we conclude that the family of Goldberg's snarks $\{G_{2k+1}\}_{k\in\mathbb{N}}$ forms an increasing chain in $\succ_{\mathbb{Z}}$.

By following the very same method one can show that the family of Flower's snarks $\{J_{2k+1}\}_{k\in\mathbb{N}}$, introduced in [32], forms an increasing chain $J_3 \prec_{\mathbb{Z}} J_5 \prec_{\mathbb{Z}} J_7 \prec_{\mathbb{Z}} \ldots$ as well.

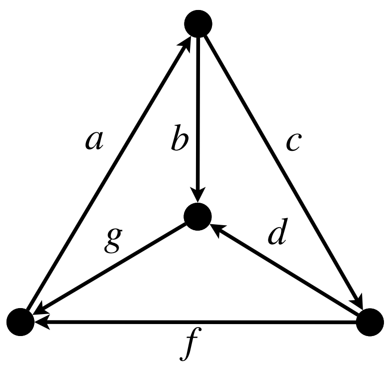

Figure 43: Orientation of K_4.

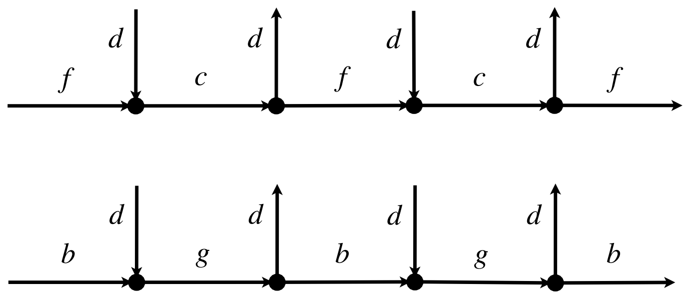

Figure 44: Possible assignments for sequences of d-edges.

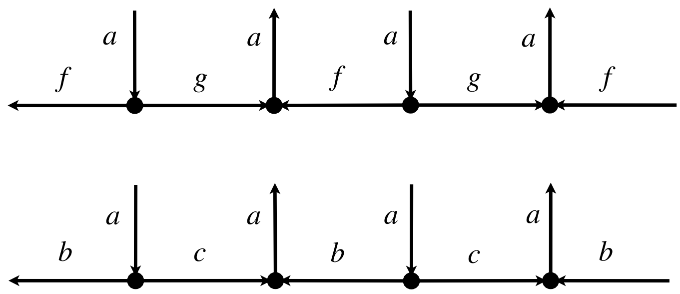

Figure 45: Possible assignments for sequences of a-edges.

6.4 CUBIC GRAPHS ADMITTING A K_4-ORIENTED-COLORING

In this final section we show an alternative proof of Theorem 6.5 that makes use of oriented colorings.

Let C be a connected 2-regular graph and let $\vec{C}$ be an orientation of C. The set of oriented edges $E(\vec{C})$ can be partitioned into two disjoint subsets A and B of edges oriented respectively clockwise and counterclockwise. We say that two edges of A (or B) have the same direction, but an edge of A and an edge of B have opposite direction. If one between A and B is empty we say that $\vec{C}$ is a *directed* cycle. Suppose that $\vec{C} = (A, B)$ is an oriented cycle and let $x, y, z \in V(C)$ be different vertices. Similarly to previous definitions, if $xy, yz \in A$ (or $xy, yz \in B$), we say that y *preserves* the orientation, vice versa if $xy \in A$ and $yz \in B$ (or $xy \in B$ and $yz \in A$) then we say that y *reverses* the orientation.

Proof of Theorem 6.5. If there is an oriented coloring $G \to K_4$ we get $\phi_c(G) \leq \phi_c(K_4) = 4$.

On the other hand if G is 3-edge-colorable there are three disjoint perfect matchings M_1, M_2 and M_3 that partition its edge set. We are going to define an orientation $\vec{G}$ on G and a map $\eta \colon E(\vec{G}) \to E(\vec{K_4})$ with the required properties. Fix on K_4 the orientation shown in Figure 43.

Let $M_{ij} = M_i \cup M_j$ for every $i, j \in \{1, 2, 3\}$. Consider the two 2-factors M_{12} and M_{13}. Orient the edges of M_{12} in such a way that it becomes union of disjoint directed cycles. Then orient the remaining

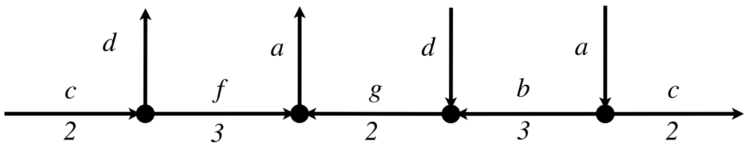

Figure 46: Extention of the map η to the 2-factor M_{23}.

edges in such a way that the edges with color 3 of each component of M_{13} have the same direction. Thus, for every connected component $\vec{C}$ of M_{13}, its edge set $E(\vec{C})$ is partitioned into two subsets of edges A and B with respect to their orientation such that all edges of $\vec{C}$ colored by 3 are contained A. For every connected component of M_{13} set

$$\begin{cases} \eta(e) = a, & \text{for every } e \in A \cap M_1, \\ \eta(e) = d, & \text{for every } e \in B \cap M_1. \end{cases}$$

Every connected component C of the 2-factor M_{23} is an oriented even cycle, hence C must contain an even number of vertices that reverse the orientation. Therefore it contains also an even number of vertices that preserve the orientation. Furthermore notice that vertices that reverse the orientation are incident to edges that are assigned a, call them a-edges. So the number of a-edges pointing towards C equals the number of a-edges pointing outwards C. The same property holds for d-edges incident to C since they are incident to vertices that preserve the orientation and since d-edges have the same orientation of 2-colored edges in M_{12}.

Now we prescribe an assignment also for edges of color 2 and 3. Notice that we can suppose without loss of generality that there are no even sequences of d-edges or a-edges since they can be labeled as in Figures 44 and 45, respectively. Hence the problem translates to the task of finding a proper assignment for the edges of an even cycle C' where there are no adjacent a-edges nor adjacent d-edges. By construction these edges have pairwise the same orientation, just notice that this holds for every edge e of color 3 of C', since they are adjacent to an a-edge (oriented coherently with respect to e) and a d-edge (having reversed orientation with respect to e). Then define the assignment as in Figure 46 and the statement follows. $\square$

BIBLIOGRAPHY

[1] M. Abreu, T. Kaiser, D. Labbate, G. Mazzuoccolo, *Tree–like snarks*, Electron. J. Combin., 23 (2016), Paper 3.54.

[2] M. Abreu, J. Goedgebeur, D. Labbate, G. Mazzuoccolo, *A note on* 2*-bisections of claw-free cubic graphs*, Discrete Appl. Math., 244 (2018), 214-217.

[3] M. Abreu, J. Goedgebeur, D.Labbate, G. Mazzuoccolo, *Colourings of cubic graphs inducing isomorphic monochromatic subgraphs*, J. Graph Theory, 92 (2019), 415-444.

[4] K. Appel, W. Haken, *Every Planar Map is Four Colorable*, Bulletin of the American Mathematical Society, 82 (1976), 711-712.

[5] A. Ban, N. Linial, *Internal Partitions of Regular Graphs*, J. Graph Theory, 83 (2016), 5-18.

[6] D. Blanuša, *Problem cetiriju boja*, Glasnik Mat. Fiz. Astr. Ser. II. 1 (1946), 31-42.

[7] J. A. Bondy, *Balanced Colourings and Graph Orientation*, Congressus Numerantium, 14 (1975), 109-114.

[8] J. A. Bondy, U. S. R. Murty, *Graph theory with applications*, Macmillan, London (1976).

[9] U. A. Celmins, E. R. Swart, *The Constructions of Snarks*, Research Report CORR 79-18, Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada, 1979.

[10] G. Brinkmann, K. Coolsaet, J. Goedgebeur, H. Mélot, House of Graphs: a database of interesting graphs, *Disc. Appl. Math.*, 161 (2013), 311–314. Available at `http://hog.grinvin.org/`.

[11] G. Brinkmann, J. Goedgebeur, J. Hägglund, K. Markström, *Generation and properties of snarks*, J. Combin. Theory Ser. B., 103 (2013), 468-488.

[12] Q. Cui, Q. Liu, 2*-bisections in claw-free cubic multigraphs*, Discrete Appl. Math., 257 (2019), 325-330.

[13] Q. Cui, W. Liu, *A note on 3-bisections in subcubic graphs*, Discrete Appl. Math., 285 (2020), 147-152.

[14] B. L. de Freitas, C. N. da Silva, C. L. Lucchesi, *Hypohamiltonian snarks have a 5-flow*, Electron. Notes Discret. Math., 50 (2015), 199-204.

[15] M. DeVos, J. Nešetřil, A. Raspaud, *On edge-maps whose inverse preserves flows and tensions*, Graph Theory in Paris: Proceedings of a Conference in Memory of Claude Berge (J. A. Bondy, J. Fonlupt, J.-L. Fouquet, J.-C. Fournier, and J. L. Ramirez Alfonsin, eds.), Trends in Mathematics, Birkhäuser, 2006.

[16] M. DeVos, E. Rollová, R. Šámal, *A new proof of Seymour's 6-flow theorem*, J. Combin. Theory Ser. B, 122 (2017), 187-195.

[17] R. Diestel, *Graph Theory*, Springer, New York, 1997.

[18] Z. Dvořák, B. Mohar, R. Šámal, *Exponentially many nowhere-zero* $\mathbb{Z}_3$*-,* $\mathbb{Z}_4$*- and* $\mathbb{Z}_6$*-flows*, Combinatorica, 39 (2019), 1237-1253.

[19] L. Esperet, G. Mazzuoccolo, M. Tarsi, *The structure of graphs with circular flow number 5 or more, and the complexity of their recognition problem*, J. Comb., 7 (2016), 453-479.

[20] L. Esperet, G. Mazzuoccolo, M. Tarsi, *Flows and bisections in cubic graphs*, J. Graph Theory, 86 (2017), 149-158.

[21] M. Fiol, G. Mazzuoccolo, E. Steffen, *Measures of Edge-Uncolorability of Cubic Graphs*, Electron. J. Comb., 25 (2018), P454.

[22] H. Fleischner, *Eulerian Graphs and Related Topics*, Part 1, Vol. 1, Ann. Discrete Math. 45 (North Holland, Amsterdam, 1990).

[23] A. Frank, *Connections in Combinatorial Optimization*, Oxford Lecture Series in Mathematics and Its Applications, Vol. 38. Oxford University Press, 2011.

[24] D. R. Fulkerson, *Blocking and anti-blocking pairs of polyhedra*, Math. Programming, 1 (1971), 168-194.

[25] J. Goedgebeur, D. Mattiolo, G. Mazzuoccolo, Program to compute the circular flow number of a cubic graph, `https://caagt.ugent.be/cfn/`, 2019.

[26] L. A. Goddyn, M. Tarsi, C. Q. Zhang, *On (k,d)-colourings and fractional nowhere-zero flows*, J. of Graph Theory, 28 (1998), 155-161.

[27] M. K. Goldberg, *Construction of class 2 graphs with maximum vertex degree 3*, J. Combin. Theory Ser. B, 31 (1981), 282-291.

[28] S. Grünewald, E. Steffen, *Chromatic-index-critical graphs of even order*, J. Graph Theory, 30 (1999), 27-36.

[29] S. L. Hakimi, *On the degrees of the vertices of a directed graph*, J. Franklin Inst. 279 (1965), 290-308.

[30] M. Han, J. Li, Y. Wu, C.-Q. Zhang, *Counterexamples to Jaeger's Circular Flow Conjecture*, J. Comb. Theory Ser. B, 131 (2018), 1-11.

[31] A.J. Hoffman, *Some recent applications of the theory of linear inequalities to extremal combinatorial analysis*, Combinatorial Analysis: Proceedings of the Tenth Symposium in Applied Mathematics of the American Mathematical Society, R. Bellman and M. Hall Jr., Eds., American Math. Soc. (1960), 113-128.

[32] R. Isaacs, *Infinite families of non-trivial trivalent graphs which are not Tait colorable*, Am. Math. Monthly 82 (1975), 221–239.

[33] R. Isaacs, *Loupekhine's snarks: a bifamily of non-Tait-colorable graphs*, Technical Report No. 263, Department of Mathematical Sciences, Johns Hopkins University, 1976.

[34] H. Izbicki, *Zulässige Kantenfärbungen von pseudo-regulären Graphen 3. Grades mit der Kantenfarbenzahl 3*, Monatsh. Math. 66 (1962), 424-430.

[35] F. Jaeger, *Balanced valuations and flows in multigraphs*, Proc. Amer. Math. Soc. 55 (1976), 237-242.

[36] F. Jaeger, *Flows and generalized coloring theorems in graphs*, J. Comb. Theory Ser. B, 26 (1979), 205-216.

[37] F. Jaeger, *On circular flows in graphs*, Finite and Infinite Sets, Colloquia Mathematica Societatis János Bolyai, vol. 37, Eger, 1981, North-Holland (1984), 391-402.

[38] F. Jaeger, *On five-edge-colorings of cubic graphs and nowhere-zero flow problems*, Ars Combin., 20, 1985, 229-244.

[39] F. Jaeger, *Nowhere-zero flow problems*, Selected topics in graph theory 3, Academic Press, San Diego, CA, 1988, 71-95.

[40] M. Kochol, *An equivalent version of the* 3*-flow Conjecture*, J. Comb. Theory Ser. B, 83 (2001), 258-261.

[41] M. Kochol, *Reduction of the 5-flow conjecture to cyclically 6-edge-connected snarks*, J. Comb. Theory Ser. B, 90 (2004), 139-145.

[42] M. Kochol, *About counterexamples to the 5-flow Conjecture*, Electron. Notes Discret. Math., 22 (2005), 21-24.

[43] M. Kochol, *Smallest counterexample to the 5-flow conjecture has girth at least eleven*, J. Comb. Theory Ser. B, 100 (2010), 381-389.

[44] J. Li, C. Thomassen, Y. Wu, C.-Q. Zhang, *The flow index and strongly connected orientations*, European J. Comb., 70 (2018), 164-177.

[45] H. C. Little, W. T. Tutte, D. H. Younger, *A theorem on integer flows*, Ars Combin., 26A, 1988, 109-112.

[46] L. M. Lovász, C. Thomassen, Y. Wu, C.-Q. Zhang, *Nowhere-zero 3-flows and modulo k-orientations*, J. Comb. Theory Ser. B, 103 (2013), 587-598.

[47] R. Lukot'ka, *Circular flow number of generalized Blanuša snarks*, Discrete Math., 313 (2013), 975-981.

[48] R. Lukot'ka, M. Škoviera, *Real flow number and the cycle rank of a graph*, J. Graph Theory, 59 (2008), 11-16.

[49] R. Lukot'ka, M. Škoviera, *Snarks with given real flow numbers*, J. Graph Theory, 68 (2011), 189-201.

[50] E. Máčajová, G. Mazzuoccolo, V. V. Mkrtchyan, J. P. Zerafa, *Some snarks are worse than others*, on arXiv:2004.14049 [math.CO].

[51] E. Máčajová, M. Škoviera, *Critical and flow-critical snarks coincide*, Discuss. Math. Graph Theory, article in press.

[52] E. Máčajová, M. Škoviera, *Perfect matching index vs. circular flow number of a cubic graph*, on arXiv:2008.04775 [math.CO].

[53] E. Máčajová, A. Raspaud, *On the Strong Circular* 5*-Flow Conjecture*, J. Graph Theory, 52 (2006), 307-316.

[54] E. Máčajová, A. Raspaud, M. Tarsi, X. Zhu, *Short cycle covers of graphs and nowhere-zero flows*, J. Graph Theory, 68 (2011), 340-348.

[55] G. Mazzuoccolo, *An upper bound for the excessive index of an r-graph*, J. Graph Theory, 73 (2013), 377-385.

[56] G. Mazzuoccolo, E. Steffen, *Nowhere-zero 5-flows on cubic graphs with oddness 4*, J. Graph Theory, 85 (2017), 363-371.

[57] G. H. J. Meredith, *Regular n-valent n-connected non-hamiltonian non-n-edge colorable graphs*, J. Comb. Theory Ser. B, 14 (1973), 55-60.

[58] V. Mkrtchyan, *A remark on the Petersen coloring conjecture of Jaeger*, Australas. J. Combin., 56 (2013), 145-151.

[59] B. Mohar, http://www.fmf.uni-lj.si/ mohar/, Problem of the Month, March and April 2003.

[60] C. St. J. A. Nash-Williams, *Edge-disjoint spanning trees of finite graphs*, J. London Math. Soc. 36 (1961), 445-450.

[61] R. Nedela, M. Škoviera, *Decompositions and reductions of snarks*, J. Graph Theory, 22 (1996), 253-279.

[62] Z. Pan, X. Zhu, *Construction of graphs with given circular flow numbers*, J. Graph Theory, 43 (2003), 304-318.

[63] P. Prałat, N. Wormald, *Almost all* 5*-regular graphs have a* 3*-flow*, J. Graph Theory, 93 (2020), 147-156.

[64] R. Rizzi, *Indecomposable r-graphs and some other counterexamples*, J. Graph Theory, 32 (1999), 1-15.

[65] R. Šàmal, *Cycle-Continuous Mappings—Order Structure*, J. Graph Theory, 85 (2016), 56-73.

[66] M. Schubert, E. Steffen, *The set of circular flow numbers of regular graphs*, J. Graph Theory, 76 (2014), 297-308.

[67] P.D. Seymour, Sums of circuits in Graph Theory and Related Topics edited by J.A. Bondy and U.S.R. Murty, Academic Press, New York/Berlin (1979), 341-355.

[68] P. D. Seymour, *Nowhere-zero* 6*-flows*, J. Combin. Theory Ser. B, 30 (1981), 130-135.

[69] E. Steffen, *Tutte's 5-flow conjecture for graphs of nonorientable genus 5*, J. Graph Theory, 22 (1996), 309-319.

[70] E. Steffen, *Classifications and characterizations of snarks*, Discrete Math., 188 (1998), 183-203.

[71] E. Steffen, *Circular flow numbers of regular multigraphs*, J. Graph Theory, 36 (2001), 24-34.

[72] E. Steffen, *Measurements of edge-uncolorability*, Discrete Math., 280 (2004), 191-214.

[73] E. Steffen, *Tutte's 5-flow conjecture for highly cyclically connected cubic graphs*, Discrete Math., 310 (2010), 385-389.

[74] E. Steffen, *Edge-colorings and circular flow numbers of regular graphs*, J. Graph Theory, 79 (2015), 1-7.

[75] E. Steffen, *Intersecting 1-factors and nowhere-zero 5-flows*, Combinatorica, 35 (2015), 633-640.

[76] Szekeres, G. *Polyhedral Decompositions of Cubic Graphs*, Bull. Austral. Math. Soc., 8 (1973), 367-387.

[77] M. Tarsi, *Bounded-excess flows in cubic graphs*, J. Graph Theory, 95 (2020), 138-159.

[78] C. Thomassen, *The weak 3-flow conjecture and the weak circular flow conjecture*, J. Comb. Theory Ser. B, 102 (2012), 521-529.

[79] C. Thomassen, *Graph factors modulo k*, J. Comb. Theory Ser. B, 106 (2014), 174-177.

[80] C. Thomassen, *Factorizing regular graphs*, J. Comb. Theory Ser. B, 141 (2020), 343-351.

[81] W. T. Tutte, *On the problem of decomposing a graph into n connected factors*, J. London Math. Soc., 36 (1961), 221-230.

[82] W. T. Tutte, *On the imbedding of linear graphs in surfaces*, Proc. London Math. Soc., Ser. 2, 51, 1949, 474-483.

[83] W. T. Tutte, *A contribution on the theory of chromatic polynomial*, Canad. J. Math., 6, 1954, 89-91.

[84] W. T. Tutte, *A class of abelian groups*, Canad. J. Math., 8 (1956), 13-28.

[85] W. T. Tutte, *On the algebraic theory of graph colorings*, J. Combinatorial Theory 1 (1966), 15-50.

[86] V. G. Vizing, *On an estimate of the chromatic class of a p-graph*, Diskret. Analiz. 3 (1964), 25-30 (in Russian).

[87] J. J. Watkins, *Snarks*, Ann. New York Acad. Sci. 576 (1989), 606-622.

[88] H. Wielandt, *Finite permutation groups*, Academic Press, New York/London, 1964.

[89] D.H. Younger, *Integer flows*, J. Graph Theory, 7 (1983), 349-357.

[90] C.-Q. Zhang, *Integer flows and cycle covers of graphs*, Marcel Dekker, Inc. New York, Basel, Hong Kong, 1997.