A Möbius-type gluing technique for obtaining edge-critical graphs

S. Bonvicini [*], A. Vietri [†]

April 5, 2020

Abstract

Using a technique which is inspired by topology, we construct original examples of 3- and 4-edge critical graphs. The 3-critical graphs cover all even orders starting from 26; the 4-critical graphs cover all even orders starting from 20 and all the odd orders. In particular, the 3-critical graphs are not isomorphic to the graphs provided by Goldberg for disproving the Critical Graph Conjecture. Using the same approach we also revisit the construction of some fundamental critical graphs, such as Goldberg's infinite family of 3-critical graphs, Chetwynd's 4-critical graph of order 16 and Fiol's 4-critical graph of order 18.

Keywords: edge-colouring, critical graph, Möbius strip.
MSC(2010): 05C10, 05C15.

1 Introduction

In the present paper, we deal with graphs that are not necessarily simple, so multiple (or parallel) edges are allowed but loops are excluded. We denote by $\chi'(G)$ the chromatic index of a graph G, namely, the minimum number of colours that are needed for an edge-colouring of G. Vizing, in [12], proved that $\Delta(G) \leq \chi'(G) \leq \Delta(G)+\mu(G)$, where $\Delta(G)$ and $\mu(G)$ are the maximum degree and the maximum multiplicity (the number of parallel edges for two fixed vertices) respectively. A simple graph G is said to be class 1 or 2 according to whether $\chi'(G)$ is $\Delta(G)$ or $\Delta(G) + 1$, respectively. We will restrict our attention to graphs whose chromatic index is at most $\Delta + 1$. *Edge-critical* graphs will be our main object of study:

[*]Dipartimento di Scienze Fisiche, Informatiche e Matematiche, Università di Modena e Reggio Emilia, via Campi 213/b, 41126 Modena (Italy); corresponding author.
[†]Dipartimento di Scienze di Base e Applicate per l'Ingegneria, Sapienza Università di Roma, via Scarpa 16, 00161 Rome (Italy).
E-mail addresses: simona.bonvicini@unimore.it, andrea.vietri@uniroma1.it.

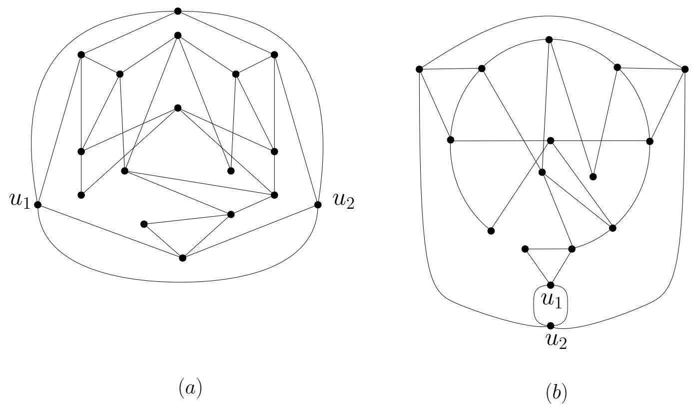

Figure 1: Two remarkable 4-critical graphs.

Definition 1.1. For a given graph G, let $G-e$ denote the graph obtained by removing an edge e; G is Δ-(*edge*)-*critical* if $\chi'(G) = \Delta+1$ and $\chi'(G-e) = \Delta$ for any edge e.

In the literature, three small critical graphs of considerable importance appeared respectively in [9], [7] and [6]. The first graph (see the left side of Figure 10) was constructed by Goldberg as the first counterexample related to the "Critical Graph Conjecture" according to which all critical graphs should have an odd number of vertices (see [6]); such a graph had the smallest number of vertices (22) in an infinite family of graphs of even order constructed by Goldberg. The second graph – see the left side of Figure 1 – was found by Fiol as an example of critical, simple graph of smaller order, namely 18; the last graph – see the right side of the figure – is due to Chetwynd; it has order 16 but it is not simple because of one multiple edge.

It is still unknown whether a simple, critical graph of order 16 exists. As to smaller orders, such a question was settled by a number of contributions over the years. In details, Jacobsen's work (see [10]) ruled out all graphs with 4, 6, 8, and 10 vertices; Fiorini and Wilson (see [8]) added the case 12 to the above list of non-admissible values; Bokal, Brinkmann, and Grünewald (see [2]) proved that also 14 is non-admissible.

In this paper, we push forward the analogy between non-orientable manifolds and class 2 graphs which was introduced in [11] and describe a new method for constructing critical graphs. We show the effectiveness of this method by constructing infinite families of critical simple graphs. The constructions cover all odd and even orders for 4-critical graphs, the odd order starting from 5, the even orders starting from 20, as well as all even orders for 3-critical graphs, including the orders of Goldberg's infinite family starting from 28 (the orders of Goldberg's graphs are all those numbers congruent to 8 (mod 16), and the further value 22). The 3-critical graphs of even order that we construct are not isomorphic to the graphs of Goldberg's infinite family; the graphs are simple, except the 4-critical graph of order 16. According to the literature, our constructions provide in particular the first example of an infinite family of Δ-critical graphs for degree 4. The present approach is expected to yield infinite families also for larger degrees, in the next future, because the key definitions can be easily exported to the general case.

Our method allows to build up critical graphs starting from class 1 graphs with an elementary and "nice" shape (see for instance Figure 2). This is innovative with respect to well-know methods that construct Δ-critical graphs starting from critical graphs with maximum degree not exceeding Δ – see Theorem 4.6 and 4.9 in [14].

Following the mentioned approach in [11], we also show that the infinite family of Goldberg's graphs disproving the "Critical Graph Conjecture" and the other two counterxamples constructed by Fiol and Chetwynd can be obtained by a suitable identification of vertices which is pretty analogous to the topological identification yielding the Möbius strip from a rectangular strip. Details about the change of language – from topology to graph theory – can be found in [11].

Some additional terminology is required; in particular, certain distinguished vertices that play a basic role in the constructions shall be emphasised by suitable adjectives. Leaving details to the next sections, we anticipate that all the constructions will rely on particular pairs of vertices which are analogous to the extremes of a rectangular strip before the identification that leads to a Möbius strip. In our setting, any such pair will undergo a transformation which is similar to the topological identification of the extremes of the rectangular strip. The change from orientability to non-orientability, caused by the identification, is rephrased as the change from class 1 to class 2 as a consequence of the prescribed transformation.

Many standard definitions in this paper are in accordance with the textbook [3] by Bondy and Murty. As a further source, we mention the textbook [5] by Bryant. Edges like $\{u, v\}$ are simply denoted by uv. We use the term *t-colouring* if the colour set has size t. Given a vertex v of a graph G, the *palette of* v, in symbols $P_\gamma(v)$ or simply $P(v)$, is the set of colours that a colouring γ of G assigns to the edges containing v. In some cases, we will need to write γ_G so as to specify the graph we are colouring. The complementary set $\overline{P_\gamma(v)}$ or $\overline{P(v)}$ is the *complementary palette* of v with respect to the colour set of γ. If a colour is missing at a vertex v, we say that v *lacks* that colour. Finally, a vertex of degree h is an *h-vertex*.

For our purposes we also recall Vizing's Adjacency Lemma, concerning the structure of critical (simple) graphs, and the quite elementary, still very useful, Parity Lemma:

Theorem 1.2 (VAL [13])**.** If uv is an edge of a Δ-critical graph, then u is adjacent to at least $\Delta - deg(v) + 1$ Δ-vertices (different from v).

Property 1.3 (PL [1])**.** For any colouring of a graph G, the number of vertices that lack a given colour has the same parity as $|V(G)|$.

Although there exist several generalisations of VAL to multigraphs, for our purposes it suffices to consider the simple graph version (see the lines just above Remark 2.8).

2 Fertile pairs of vertices

As hinted in the Introduction, the constructions of critical graphs that follow can be thought of as identifications of special pairs of vertices which change the colouring class from 1 to 2. Accordingly, the first step in each construction is the choice of a suitable pair of vertices which we are going to define as *fertile pair*. There are three kinds of fertile pairs, but after a little thought all of them can be related to the same kind – as we will soon explain. Conversely, given a critical graph, we will show that it is obtained as a suitable identification of a fertile pair which collapses to a unique vertex. In this reconstruction process, it is important to note that the identification could be arbitrarily performed on every vertex, but the choice of a particular vertex is essential both for proving criticality in a comfortable way, and for generating new critical graphs using a pattern which is readily suggested by the fertile pair.

Definition 2.1. Let u, v be vertices of a graph G. Assume that the following conditions hold:

($*$) u is not adjacent to v, $deg(u) + deg(v) \leq \Delta$ and, for every Δ-colouring, $P(u) \cap P(v) \neq \emptyset$.

($**$) For any edge e, $G - e$ admits a Δ-colouring such that $P(u) \cap P(v) = \emptyset$.

Then, u and v are said to be *conflicting*. Assume, instead, the following:

($*$) $deg(u) = deg(v) = \Delta - 1$ and, for every Δ-colouring, $P(u) = P(v)$.

($**$) For any edge e which does not contain u nor v, $G - e$ admits a Δ-colouring such that $P(u) \neq P(v)$.

In this case, u and v are *same-lacking*. Finally, assume the following:

($*$) $deg(u)$, $deg(v)$ are smaller than Δ and, for every Δ-colouring, $|P(u) \cup P(v)| = \Delta$.

($**$) For any edge e, $G - e$ admits a Δ-colouring such that $|P(u) \cup P(v)| < \Delta$.

In this last case, u and v are said to be *saturating*.

In all of the three cases, we say that (u, v) is a *fertile* pair of vertices.

Remark 2.2. After the removal of e in the same-lacking case, we equivalently require that $|\overline{P(u)} \cup \overline{P(v)}| \geq 2$; this is trivial if e contains one or both vertices u, v. Furthermore, notice that in the saturating case condition $|P(u) \cup P(v)| = \Delta$ is equivalent to $\overline{P(u)} \cap \overline{P(v)} = \emptyset$.

The following lemma is the basic link between topology and graph theory in the present context, and should be considered the starting point for all the next constructions.

Lemma 2.3. Let (u, v) be a fertile pair of a graph G having $\chi'(G) = \Delta \geq 2$. For each of the following cases, the corresponding operation yields a Δ-critical graph.

- ⊙ If u and v are non-adjacent and conflicting, *identify* u and v.
- ⊙ If u and v are same-lacking, *add* a new vertex w and edges uw, vw.
- ⊙ If u and v are saturating, *add* the edge uv.

Proof. If we identify a pair of conflicting vertices, we obtain a graph G' having maximum degree Δ and no proper Δ-coloring, since the palettes of two conflicting vertices share at least one color; hence G' is class 2. By definition 2.1, if we remove any edge e from G', we find at lest one Δ-coloring of $G' - e$ such that the two conflicting vertices have disjoint palettes with respect to it; therefore, G' is Δ-critical. The same-lacking and saturating cases can be managed analogously. □

Notice that adding two pendant edges uw, vw' when u and v are same-lacking yields conflicting 1-vertices w, w'. Similarly, adding one pendant edge uw when u and v are saturating yields conflicting vertices w, v. Therefore, the above operations can be regarded as identifications of conflicting vertices in all cases. These procedures could be rephrased in terms of atlases and orientability, as explained in [11]; the prototype of this analogy is given by the odd cycle C_{2n+1} of any fixed length. Such a graph is the result of the identification of two conflicting vertices, namely, the extremes of the path P_{2n+2} having the same number of edges. The path is "orientable" (i.e. 2-colourable) but the identification of conflicting vertices increases the chromatic index and compromises orientability. More precisely, the orientation of P_{2n+2} starts from a "local chart" (a colouring of the 2-star containing a non-extremal vertex v), and the local chart is subsequently extended so as to cover as many edges as possible. In the case of the path, we succeed in covering all the graph (so we have a "global atlas", that is, a global 2-colouring) whereas the cycle does not allow for a global 2-colouring because one edge must be excluded (the atlas cannot be extended to the whole graph). Notice that the hypothesis $(**)$ for conflicting vertices is crucial to prove criticality.

Remark 2.4. The 4-critical graphs in Figure 1 can be obtained in the way described in Lemma 2.3, by considering the graphs G_{17}, G_{19} in Figure

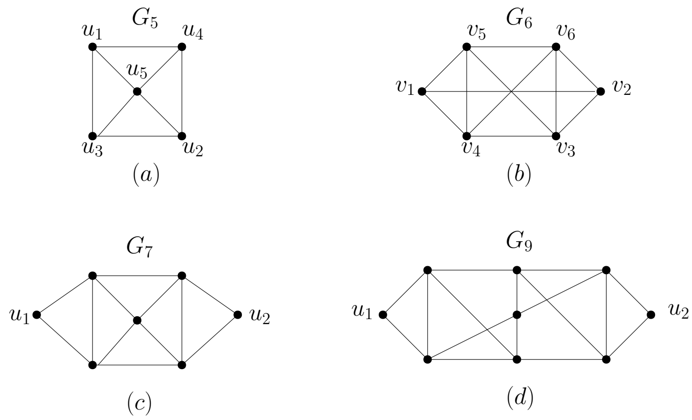

Figure 2: Fertile pairs of vertices: u_1 and u_2 are saturating, v_1 and v_2 are same-lacking.

8(b), 9(a), respectively, and identifying the vertices v, v'. Such vertices are conflicting, as we will show in Section 3.

Here follow some examples as a first step towards the main theorems.

Example 2.5. Let us show that the graph G_5 in Figure 2(a) has saturating vertices u_i, u_j, with $1 \leq i < j \leq 4$. For every 4-colouring the number of vertices that lack a fixed colour is odd, according to PL, whence every 3-vertex lacks a different colour; on the other hand, one can easily verify that the removal of any edge allows for a 4-colouring such that $|P(u_i) \cup P(u_j)| = 3$ for any pair of 3-vertices.

Example 2.6. The graphs G_7 and G_9 in Figure 2(c)-(d) have saturating vertices u_1, u_2, because PL implies that these vertices have disjoint palettes for any 4-colouring, and it remains to make routine checks after the removal of any arbitrary edge.

Example 2.7. The graph G_6 in Figure 2(b) has same-lacking vertices v_1, v_2, because PL forces the palettes to be equal and this is no longer true if we remove any edge not containing one or both vertices v_1, v_2.

Notice that graphs with same-lacking vertices can be replicated so as to form a chain along which a color is “transmitted”. Such a transmission of colour is a fundamental concept in this paper and will be described more thoroughly in the next section.

In the following remark, we consider critical graphs having at least three vertices of maximum degree. VAL implies that this property holds for every simple graph, but in the presence of multiple edges the number of vertices of maximum degree might be smaller than 3. For instance, the complete graph

K_3 with $\Delta - 1$ parallel edges connecting two fixed vertices is Δ-critical and has only two vertices of maximum degree.

Remark 2.8. Let G be a Δ-critical graph having at least three vertices of maximum degree. Let u, v be adjacent vertices that are connected by h parallel edges (possibly $h = 1$). After deleting one of the parallel edges, u and v become saturating and the degree remains equal to Δ.

According to the above remark, Chetwynd's 4-critical graph can also be obtained by inserting an additional edge between the saturating vertices u_1, u_2.

3 Construction of graphs with fertile pairs

Graphs with fertile pairs of vertices can be obtained in several ways from smaller graphs with the same property. The methods we present here will be applied to prove the main theorems.

Lemma 3.1. Let H_1 and H_2 be vertex-disjoint graphs of degree $\Delta \geq 2$ and such that $\chi'(H_1) = \chi'(H_2) = \Delta$. Assume that v_1, v_2 are same-lacking in H_1 and u_1, u_2 are same-lacking (resp. saturating) in H_2. The graph H obtained from H_1 and H_2 by adding the edge u_2v_2 has again maximum degree Δ, chromatic index Δ, and has same-lacking (resp. saturating) vertices u_1, v_1.

Proof. Let us analyse the same-lacking case. A colouring of H can be obtained by assuming that u_2 and v_2 lack the same colour in two given Δ-colourings of H_1 and H_2; by the hypothesis, u_1 and v_1 lack that colour. If we now remove any edge, say in H_1, u_2v_2 can be coloured with a colour which is present at u_1. Such a colour is instead missing at v_1. A similar argument applies to the saturating case.

□

Example 3.2. We consider two copies of G_6 – see Figure 2(b) – as the graphs H_1 and H_2. We can actually iterate the gluing process m times, $m \geq 1$, so as to obtain a graph of order $6m$, of maximum degree 4, whose 3-vertices are still fertile (same-lacking). Let us denote this graph by G_6^m – see Figure 3. This graph will play a basic role in the proofs of Theorem 5.1 and 5.2.

The purpose of the next couple of definitions is twofold. On one hand, they allow to recover Chetwynd and Fiol's counterexamples in the light of our approach via transmission of colours along the edges of a graph. On the other hand, they play an important role in the construction of critical graphs of even order that will follow in the next pages. These definitions involve graphs with maximum degree 4, although they can be extended to graphs with $\Delta > 4$.

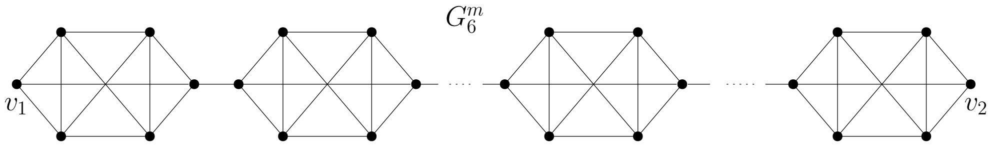

Figure 3: The graph G_6^m in Example 3.2 is a concatenation of graphs with same-lacking pairs.

Before providing the definitions, some further observations are in order. What we refer to as *transmitting* vertices should be regarded as terminal nodes which lend themselves to being connected to other graphs so as to yield a global graph with conflicting vertices and, eventually, a critical graph. The fundamental property of 2- or 3-colour transmitting vertices concerns the complementary palettes, that is, the colours actually missing at each vertex. For, the missing colours can be seen as the admissible colours of any edge which is added to the graph and contains that vertex. In the two definitions, it is the interplay between the colours missing at each distinguished vertex to ensure that the connecting edges, when added, will transmit some prescribed colour across the whole graph, and will eventually increase the chromatic index. Indeed, the vertices we are going to introduce are the first step towards the construction of graphs with conflicting vertices (see Propositions 3.8 and 3.12).

Let $S \ominus T$ denote the symmetric difference between the sets S and T.

Definition 3.3. Let G be a graph having $\chi'(G) = \Delta = 4$, and u, v, u_1, u_2 be distinct vertices of G, where $deg(u) = deg(v) = 2$, $deg(u_1) = deg(u_2) = 3$. We say that G is 3-*colour transmitting with respect to u, v, u_1, u_2* if the following conditions hold:

(1) there exists a 4-colouring such that u_1 and u_2 lack distinct colours A and B, exactly one colour is missing simultaneously in u, v and this colour is either A or B;

(2) for every 4-colouring such that u_1 and u_2 lack distinct colours A and B, $|\{A,B\} \cup (\overline{P(u)} \ominus \overline{P(v)})| \neq 3$ (in particular, in the colouring in (1) the two other colours missing at u and v are different from A and B);

(3) for every edge e there exists a 4-colouring of $G - e$ with colours A, B, C, D satisfying $A \in \overline{P(u_1)}$, $B \in \overline{P(u_2)}$, $C \in \overline{P(u)} \cap \overline{P(v)}$ and the set $\{A, D\}$ or $\{B, D\}$ is contained in $\overline{P(u)} \ominus \overline{P(v)}$.

If we slightly alter the above definition by setting $u_1 = u_2$ and $deg(u_1) = 2$, the resulting graph is said 3-*colour transmitting with respect to u, v, u_1*. In this case, the first requirement in (1) and (2) clearly becomes "u_1 lacks colours A and B", in symbols $A, B \in \overline{P(u_1)}$.

Definition 3.4. Let G be a graph of maximum degree $\Delta = 4$ and $\chi'(G) = 4$. Let w, w_1, w_2 be distinct vertices of G, where $deg(w) = 2$, $deg(w_1) = deg(w_2) = 3$. We say that G is 2*-colour transmitting with respect to* w, w_1, w_2, if the following conditions hold:

(1) for every 4-colouring of G the set $|\overline{P(w_1)} \cup \overline{P(w_2)}|$ contains exactly two colours and coincides with $\overline{P(w)}$;

(2) for every edge e there exists a 4-colouring of $G - e$ with colours A, B, C such that $A \in \overline{P(w_1)}$, $B \in \overline{P(w_2)}$ and $\overline{P(w)}$ contains $\{A, C\}$ or $\{B, C\}$.

Similarly as above, if the vertices w_1, w_2 coincide and $deg(w_1) = 2$, we say that the graph is 2*-colour transmitting with respect to* w, w_1; the requirement in condition (2) becomes "w_1 lacks colours A and B".

Example 3.5. The graph G_{12} in Figure 4(a) is 3-colour transmitting with respect to u, v, u_1, u_2, as we are going to explain by testing the conditions of Definition 3.3. Condition (1) holds as shown in Figure 4(a). Condition (3) can be checked by setting: $P(u) \subseteq \{2, 3\}$, $P(v) \subseteq \{2, 4\}$, and $P(z_1) \subseteq \{1, 4\}$. In the graph $G_{12} - e$, the palettes of the vertices u_1, u_2 take the following values: $P(u_1) \subseteq \{1, 2, 3\}$ and $P(u_2) \subseteq \{1, 3, 4\}$; $P(u_1) \subseteq \{2, 3, 4\}$ and $P(u_2) \subseteq \{1, 2, 3\}$; $P(u_1) \subseteq \{2, 3, 4\}$ and $P(u_2) \subseteq \{1, 2, 4\}$. Notice that $P(u) \subseteq \{2, 3\}$, $P(v) \subseteq \{2, 4\}$ mean that $1 \in \overline{P(u)} \cap \overline{P(v)}$ and $\{3, 4\} \subseteq \overline{P(u)} \ominus \overline{P(v)}$, that is, colour 1 corresponds to colour C in Condition (3) and $\{3, 4\}$ corresponds to one of the sets $\{A, D\}$ or $\{B, D\}$, where $A \in \overline{P(u_1)}$, $B \in \overline{P(u_2)}$. Thus, for instance, if $P(u_1) \subseteq \{1, 2, 3\}$ and $P(u_2) \subseteq \{1, 3, 4\}$, then $A = 4$, $B = 2$ and $D = 3$.

It remains to prove Condition (2). By PL, the number of vertices that lack a given colour is even, and there are 6 vertices of degree smaller than 4. However, a color missing in all these vertices would make the two palettes of degree 3 equal, which is not allowed by assumption. Now let us partition the $2 \cdot 3 + 4 \cdot 2$ colours on the above 6 vertices either as $2 + 2 + 4 + 6$ or as $2 + 4 + 4 + 4$, where each part counts the occurrences of a fixed colour (0 is missing, by the above discussion). Up to permutations of colours there are two colourings of the first type and three of the second type (in the table, palettes of size 4 are not present and we assume that palettes of size 3 are the same in all cases):

$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$
$\{1,2,4\}$	$\{1,2,4\}$	$\{1,2,4\}$	$\{1,2,4\}$	$\{1,2,4\}$
$\{1,2\}$	$\{1,3\}$	$\{1,2\}$	$\{1,4\}$	$\{1,3\}$
$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{1,4\}$	$\{1,4\}$
$\{1,3\}$	$\{1,3\}$	$\{2,4\}$	$\{2,3\}$	$\{3,4\}$
$\{1,4\}$	$\{1,4\}$	$\{3,4\}$	$\{2,4\}$	$\{3,4\}$

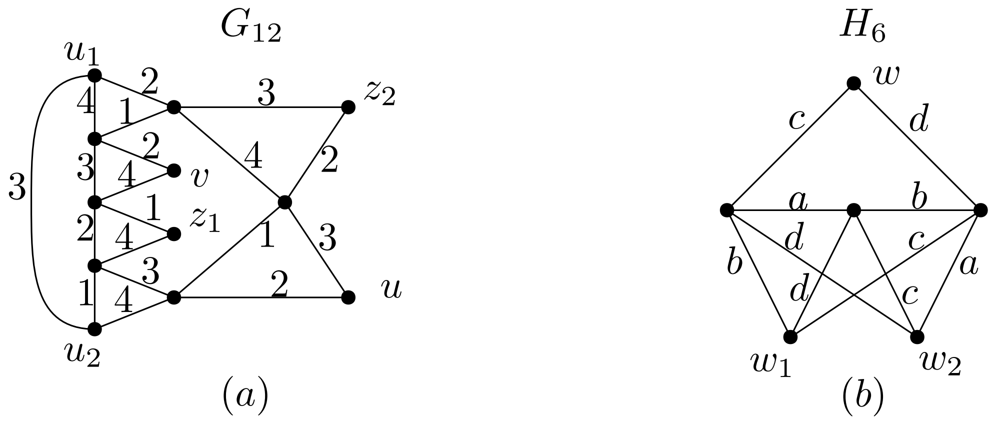

Figure 4: (a) A 4-colouring of the graph G_{12} in Example 3.5 that satisfies conditions (1) and (2) of Definition 3.3. (b) A 4-colouring of the graph H_6 in Example 3.7.

Whatever the assignments of palettes to the 2-vertices, column 2 and column 4 satisfy (2). For the colouring γ_1 in the 1st column, condition $|\overline{P(u_1)} \cup \overline{P(u_2)} \cup (\overline{P(u)} \ominus \overline{P(v)})| \neq 3$ is not satisfied if we choose $\{P(u), P(v)\} = \{\{1,2\},\ \{1,3\}\}$ or $\{P(u), P(v)\} = \{\{1,2\}, \{1,4\}\}$. The permutation of colours 3 and 4 leaves γ_1 invariant and switches the sets $\{\{1,2\},\ \{1,3\}\}$, $\{\{1,2\},\ \{1,4\}\}$. Therefore, in order to show that Condition (2) is satisfied for the colouring γ_1, it suffices to show that the graph G_{12} cannot be coloured according to γ_1 by setting $\{P(u), P(v)\} = \{\{1,2\},\ \{1,3\}\}$.

Suppose, on the contrary, that G_{12} can be coloured according to γ_1 by setting $\{P(u), P(v)\} = \{\{1,2\},\ \{1,3\}\}$. The set of palettes of γ_1 shows that colour 1 induces a perfect matching of the graph G_{12}. As shown in Figure 5, there are exactly four perfect matchings of G_{12}. By the symmetry of the graph and by the fact that the sets $\{\{1,2\},\ \{1,3\}\}$, $\{\{1,2\},\ \{1,4\}\}$ can be obtained one from the other by a permutation of colours 3 and 4, we can consider the first two perfect matchings of Figure 5. The set of palettes of γ_1 also shows that colour 2 induces a matching of cardinality 5, where exactly one of the vertices u, v (respectively, z_1, z_2) is unmatched since we are supposing $\{P(u), P(v)\} = \{\{1,2\},\ \{1,3\}\}$ and $\{P(z_1), P(z_2)\} = \{\{1,2\},\ \{1,4\}\}$. Figure 6 shows how to colour the edges of G_{12} with 1 and 2. In each of the four cases represented in Figure 6, one can see that is not possible to colour to edges of G_{12} according to the colouring γ_1 by setting $\{P(u), P(v)\} = \{\{1,2\},\ \{1,3\}\}$. Therefore, if G_{12} can be coloured by γ_1, then γ_1 satisfies Condition (2). The same can be repeated for the remaining colourings in the 3rd and 5th column. It is thus proved that every 4-colouring of G_{12} with $|\overline{P(u_1)} \ominus \overline{P(u_2)}| = 2$ satisfies Condition (2).

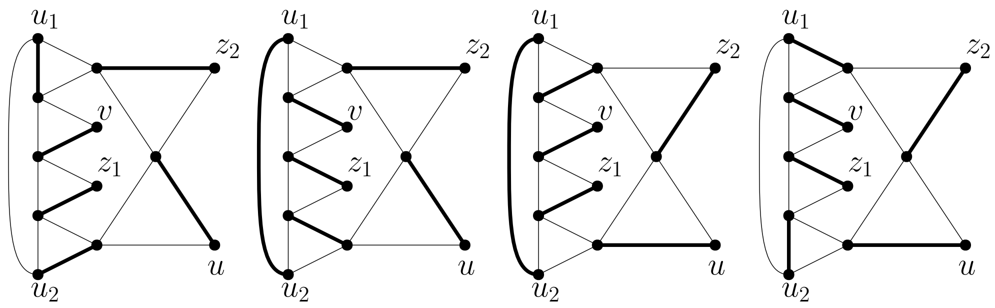

Figure 5: Perfect matchings of the graph G_{12} that are considered in Example 3.5.

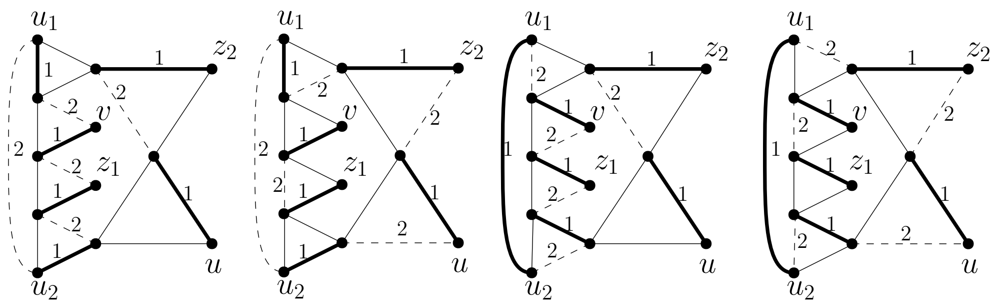

Figure 6: The edges of the graph G_{12} are coloured according to the palettes $\{1, 2, 3, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2\}$, $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$ by setting $\{P(u), P(v)\} = \{\{1, 2\}, \{1, 3\}\}$ and $\{P(z_1), P(z_2)\} = \{\{1, 2\}, \{1, 4\}\}$; colour 1 induces a perfect matching, colour 2 induces a matching of cardinality 5, where exactly one of the vertices u, v (respectively, z_1, z_2) is unmatched (see Example 3.5).

There are several methods for obtaining a 3-colour transmitting graph starting from a smaller one. For instance, in the graph G_{12} of Figure 4(a), we can delete the edge u_1u_2 and connect the remaining graph to the graph G_6^m in Figure 3 by adding the edges u_1v_1, u_2v_2. The resulting graph is 3-colour transmitting with respect to u, v, u_1, u_2. In the next example, we show a more elaborate method for obtaining a 3-colour transmitting graph starting from a smaller one. This method allows to find a graph that will be used to construct Fiol's 4-critical graph of order 18.

Example 3.6. Consider the graph N in Figure 7(a). Notice that $\overline{P(w)} = P(w_1) \ominus P(w_2)$ for every 4-colouring of the graph N, as a straightforward consequence of PL. We denote by L the graph obtained from G_{12} in Figure 4 by deleting the edge u_1u_2. Let G_{16} be the graph resulting from the identification of the vertices $w_1 \in V(N)$ with $u_1 \in V(L)$ and of $w_2 \in V(N)$ with $u_2 \in V(L)$. We have that $\chi'(L) = \Delta = 4$ (see the colouring in Figure 7(b)).

Let us show that G_{16} is 3-colour transmitting with respect to u, v, w by testing Definition 3.3 with $u_1 = u_2$. Condition (1) follows from the colouring in Figure 7(b).

Condition (2) is satisfied if every 4-coloring of G_{16} satisfies the relation $|\overline{P(w)} \cup (\overline{P(u)} \ominus \overline{P(v)})| \neq 3$. Suppose that there exists a 4-colouring γ of G_{16} such that $|\overline{P_\gamma(w)} \cup (\overline{P_\gamma(u)} \ominus \overline{P_\gamma(v)})| = 3$, that is, $\overline{P_\gamma(w)} = \{A, B\}$, $\overline{P_\gamma(u)} \ominus \overline{P_\gamma(v)} = \{A, C\}$ or $\{B, C\}$. The colouring γ induces a colouring γ' of G_{12} such that $\overline{P_{\gamma'}(u_1)} \ominus \overline{P_{\gamma'}(u_2)} = \{A, B\}$ and $\overline{P_{\gamma'}(u)} \ominus \overline{P_{\gamma'}(v)} = \{A, C\}$ or $\{B, C\}$, that is, γ' does not satisfies Condition (2) of Definition 3.3. That yields a contradiction, since G_{12} is 3-colour transmitting with respect to u, v, u_1, u_2.

Condition (3) holds if for every edge $e \in E(G_{16})$ there exists a 4-colouring of $G_{16} - e$ such that $\{A, B\} \subseteq \overline{P(w)}$, $C \in \overline{P(u)} \cap \overline{P(v)}$ and $\{A, D\} \subseteq \overline{P(u)} \ominus \overline{P(v)}$ where A, B, D are distinct. Assume $e \in E(G_{12})$. Since G_{12} is 3-colour transmitting with respect to u, v, u_1, u_2, there exists a suitable colouring which can be easily extended to the whole graph G_{16}.

If $e \in E(N)$, we colour the edges of G_{16} belonging to G_{12} by the 4-colouring in Figure 4(a), so that $P(u) = \{2, 3\}$ and $P(v) = \{2, 4\}$. One can verify that the edges of $N - e$ can be coloured in such a way that $P(w) \subseteq \{2, 4\}$. Therefore, $\{1, 3\} \subseteq \overline{P(w)}$, $1 \in \overline{P(u)} \cap \overline{P(v)}$ and $\{3, 4\} \subseteq \overline{P(u)} \ominus \overline{P(v)}$, that is, Condition (3) is satisfied if $e \in E(N)$.

Example 3.7. The graph H_6 in Figure 4(b) is 2-colour transmitting with respect to w, w_1, w_2. The conditions of Definition 3.4 are satisfied: Condition (1) follows from Parity Lemma; Condition (2) can be verified by coluring the edges with A, B, C, D and setting $P(w_1) \subseteq \{B, C, D\}$, $P(w_2) \subseteq \{A, C, D\}$, $P(w) \subseteq \{A, D\}$.

Definitions 3.3 and 3.4 are used to construct graphs having fertile vertices. The next result is a construction of graphs having fertile vertices and

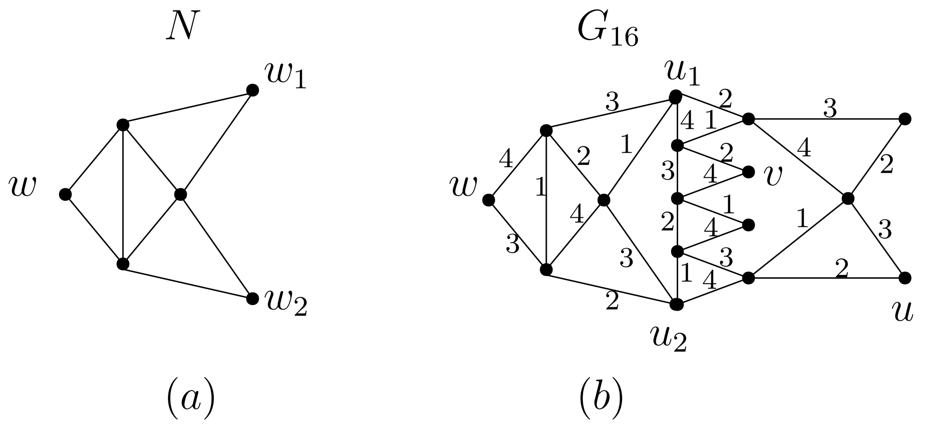

Figure 7: (a) The graph N. (b) A 4-colouring of the graph G_{16} that satisfies conditions (1) and (2) of Definition 3.3; as proved in Example 3.6, the graph G_{16} is 3-colour transmitting with respect to u, v, w.

whose maximum degree Δ is 4. The construction can be extended to graphs whose maximum degree is larger than 4 and having multiple edges. In this context, we limit ourselves to consider $\Delta = 4$.

We recall that a *bowtie* is the graph obtained by identifying two vertices belonging to two distinct 3-cycles, thus obtaining a *centre* of degree 4 and four 2-vertices. If the 3-cycle are (x, y_1, y_2) and (x', y_1', y_2'), then we denote by $B(x, y_1, y_2, y_1', y_2')$ the bowtie resulting from the identification of the vertices x and x'.

Proposition 3.8. Let $\mathbb{B} = B(x, u', v', w, y)$ be a bowtie with centre x and 2-vertices u', v', w, y. Let K and M be graphs of maximum degree 4 and $\chi'(K) = \chi'(M) = 4$, with the following features. The graph K is 3-colour transmitting with respect to u, v, u_1, u_2, where $deg_K(u) = deg_K(v) = 2$, $deg_K(u_1) = deg_K(u_2) = 3$; either M is 2-colour transmitting with respect to w, w_1, w_2, where $deg_M(w) = 2$, $deg_M(w_1) = deg_M(w_2) = 3$, or M is 2-colour transmitting with respect to w, w_1, where $deg_M(w) = deg_M(w_1) = 2$.

Let H be the graph obtained from $\mathbb{B}$, K and M by identifying the vertices u' with u, w' with w and by adding the edges u_1w_1, u_2w_2 or u_1w_1, u_2w_1 according to whether M is 2-colour transmitting with respect to w, w_1, w_2 or with respect to w, w_1, respectively. The graph H has maximum degree 4, $\chi'(H) = 4$ and the vertices v, v' are conflicting.

Proof. We identify the edge u_2w_2 with the edge u_2w_1 if $w_1 = w_2$, that is, if M is 2-colour transmitting with respect to w, w_1. Since the identification of the vertices u, u' and w, w' does not increase the maximum degree of K, M and of the bowtie, the maximum degree of H is still 4. We show that $\chi'(H) = 4$. By Condition (1) of Definition 3.4, there exists a 4-colouring γ_M^*

such that w_1, w_2 lack distinct colours A, B and these colours are missing in w (if $w_1 = w_2$, then w_1 lacks both colours A, B). By Condition (1) and (2) of Definition 3.3, there exists a 4-colouring γ_K^* such that u_1, u_2 lack distinct colours A, B and exactly one of these two colours, say A, is missing simultaneously in u and v; the other two missing colours are different from B, that is, $\overline{P_{\gamma_K^*}(u)} = \{A, C\}$ $\overline{P_{\gamma_K^*}(v)} = \{A, D\}$. We define a 4-colouring γ^* of H such that the restriction of γ^* to the edges of M (respectively, of K) coincides with γ_M^* (respectively, with γ_K^*); the edges of the bowtie and u_1w_1, u_2w_2 are coloured as follows: $\{\gamma^*(u_1w_1), \gamma^*(u_2w_2)\} = \{A, B\}$; $\gamma^*(wx) = A$; $\gamma^*(wy) = B$; $\gamma^*(ux) = C$; $\gamma^*(uv') = A$; $\gamma(v'x) = B$; and $\gamma^*(xy) = D$. In conclusion $\chi'(H) = 4$.

We prove that the vertices v, $v' \in V(H)$ are conflicting. Firstly, we show that for every 4-colouring of H, the palettes of v and v' share at least one colour. Suppose, on the contrary, that there exists a 4-colouring γ_1 of H such that v and v' have disjoint palettes. The restriction of γ_1 to the edges of K (respectively, of M) is a 4-colouring γ_K (respectively, γ_M). The following relations hold: $\overline{P_{\gamma_K}(u_1)} = \overline{P_{\gamma_M}(w_1)} = \gamma_1(u_1w_1) = A$; $\overline{P_{\gamma_K}(u_2)} = \overline{P_{\gamma_M}(w_2)} = \gamma_1(u_2w_2) = B$ (if $w_1 = w_2$ then $A \neq B$ and $\overline{P_{\gamma_M}(w_1)} = \{A, B\}$). Moreover, $\overline{P_{\gamma_K}(v)} = \overline{P_{\gamma_1}(v)} = P_{\gamma_1}(v') = \{\gamma_1(uv'), \gamma_1(v'x)\}$ since we are supposing that v and v' have disjoint palettes with respect to γ_1. Therefore $\overline{P_{\gamma_K}(u)} \ominus \overline{P_{\gamma_K}(v)} = \{\gamma_1(uv'), \gamma_1(ux)\} \ominus \{\gamma_1(uv'), \gamma_1(v'x)\} = \{\gamma_1(ux), \gamma_1(v'x)\}$. By Condition (1) of Definition 3.4, the colours A, B are distinct and $\overline{P_{\gamma_M}(w)} = \{A, B\}$. It follows that $\{\gamma_1(wx), \gamma_1(wy)\} = \{A, B\}$ and $\gamma_1(xy) \neq A, B$, $\gamma_1(ux)$, $\gamma_1(v'x)$. Therefore, exactly one of the colours $\gamma_1(ux)$, $\gamma_1(v'x)$ is in $\{A, B\}$. Consequently, the set $\overline{P_{\gamma_K}(u)} \ominus \overline{P_{\gamma_K}(v)} = \{\gamma_1(ux), \gamma_1(v'x)\}$ contains exactly one of the colours A, B. It follows that $|\overline{P_{\gamma_K}(u_1)} \cup \overline{P_{\gamma_K}(u_2)} \cup (\overline{P_{\gamma_K}(u)} \ominus \overline{P_{\gamma_K}(v)})| = 3$, a contradiction since K is 3-colour transmitting with respect to u, v, u_1, v_1. Hence, for every 4-colouring of H the palettes of the vertices v, v' share at least one colour.

We show that for every edge $e \in E(H)$ there exists a 4-colouring γ' of $H - e$ such that v and v' have disjoint palettes. We distinguish the cases: $e \in E(K)$; $e \in E(M)$; $e \in E(\mathbb{B})$; and $e \in \{u_1w_1, u_2w_2\}$.

Case $e \in E(K)$.

By Condition (3) of Definition 3.3, there exists a 4-colouring $\tilde{\gamma}$ of $K - e$ such that $A \in \overline{P_{\tilde{\gamma}}(u_1)}$, $B \in \overline{P_{\tilde{\gamma}}(u_2)}$, $C \in \overline{P_{\tilde{\gamma}}(u)} \cap \overline{P_{\tilde{\gamma}}(v)}$, and the set $\{A, D\}$ or $\{B, D\}$ is contained in $\overline{P_{\tilde{\gamma}}(u)} \ominus \overline{P_{\tilde{\gamma}}(v)}$, where A, B, D are distinct. Without loss of generality, we can assume $\{A, D\} \subseteq \overline{P_{\tilde{\gamma}}(u)} \ominus \overline{P_{\tilde{\gamma}}(v)}$. Now $\{A, D\}$ can be contained in exactly one of the complementary palettes $\overline{P_{\tilde{\gamma}}(u)}$, $\overline{P_{\tilde{\gamma}}(v)}$ or in neither of them. The first case occurs only if e contains exactly one of the vertices u, v, and in this case $\{\overline{P_{\tilde{\gamma}}(u)}, \overline{P_{\tilde{\gamma}}(v)}\} = \{\{A, D, C\}, \{B, C\}\}$. If, instead, e does not contain u, v, then $\{\overline{P_{\tilde{\gamma}}(u)}, \overline{P_{\tilde{\gamma}}(v)}\} = \{\{A, C\}, \{D, C\}\}$.

We colour the edges of M according to an arbitrary 4-colouring γ_M of the graph M. By a permutation of the colours and by Condition (1) of Definition

3.4, we can assume that the colours A, B are missing in w and w_1, w_2 lack A, B, respectively (if $w_1 = w_2$, then w_1 lacks both colours A, B). We define a 4-colouring γ' of $H - e$ such that the restriction of γ' to $K - e$ (respectively, to M) corresponds to the colouring $\tilde{\gamma}$ (respectively, γ_M) and $\gamma'(u_1w_1) = A$; $\gamma'(u_2w_2) = B$; $\gamma'(uv') = C$; $\gamma'(xy) = C$. The colouring of the edges ux, $v'x$, wx, wy depends on the set $\{\overline{P_{\gamma'}(u)}, \overline{P_{\gamma'}(v)}\}$. If $\{\overline{P_{\gamma'}(u)}, \overline{P_{\gamma'}(v)}\} = \{\{A, C\}, \{D, C\}\}$, then we set $\gamma'(wx) = B$, $\gamma'(wy) = A$ and the edges ux, $v'x$ are coloured by A, D or D, A, respectively, according to whether $\overline{P_{\gamma'}(u)} = \{A, C\}$ or $\overline{P_{\gamma'}(u)} = \{D, C\}$, respectively. If $\{\overline{P_{\gamma'}(u)}, \overline{P_{\gamma'}(v)}\} = \{\{A, D, C\}, \{B, C\}\}$, then we set $\gamma'(wx) = A$, $\gamma'(wy) = B$ and the edges ux, $v'x$ are coloured by D, B or B, D, respectively, according to whether $\overline{P_{\gamma'}(u)} = \{A, D, C\}$ or $\overline{P_{\gamma'}(u)} = \{B, C\}$, respectively. Notice that $P_{\gamma'}(v') \subseteq \overline{P_{\gamma'}(v)}$, hence v, v' have disjoint palettes with respect to γ'.

Case $e \in E(M)$.

We define a 4-colouring γ' of $H - e$ such that the edges of K are coloured according to the 4-colouring γ_K^* of K defined at the beginning of the proof. We have that $\overline{P_{\gamma'}(u)} = \overline{P_{\gamma_K^*}(u)} = \{A, C\}$, $\overline{P_{\gamma'}(v)} = \overline{P_{\gamma_K^*}(v)} = \{A, D\}$. Since u_1, u_2 lack distinct colours A, B with respect to γ_K^*, we can assume that u_1 lacks A and u_2 lacks B.

By Condition (2) of Definition 3.4, we can colour the edges of $M - e$ according to the 4-colouring γ_M' of M such that the vertices w_1, w_2 lack distinct colours, say A, B, and the colours A, C are missing in w, where A, B, C are distinct (if $w_1 = w_2$, then w_1 lacks both colours A, B). The remaining edges of $H - e$ are coloured as follows: $\gamma'(u_1w_1) = A$; $\gamma'(u_2w_2) = B$; $\gamma'(uv') = A$; $\gamma'(ux) = C$; $\gamma'(v'x) = D$; $\gamma'(wx) = A$; $\gamma'(wy) = C$; and $\gamma'(xy) = B$. The vertices v, v' have disjoint palettes with respect to γ', since $P_{\gamma'}(v') = \overline{P_{\gamma'}(v)} = \{A, D\}$.

Case $e \in E(\mathbb{B})$**.**

We define a 4-colouring γ' of $H-e$ that corresponds to the 4-colouring γ^* of H defined at the beginning of the proof, except on the remaining edges of $\mathbb{B} - e$. The edges of $\mathbb{B} - e$ are coloured in such a way that $P_{\gamma'}(v') \subseteq \{A, D\}$, $P_{\gamma'}(u) \subseteq \{A, C\}$ and $\{\gamma'(wx), \gamma'(wy)\} \subseteq \{A, B\}$. The vertices v, v' have disjoint palettes with respect to γ', since $P_{\gamma'}(v') \subseteq \overline{P_{\gamma'}(v)} = \{A, D\}$.

Case $e \in \{u_1w_1, u_2w_2\}$**.**

We define a 4-colouring γ' of $H - e$ which coincides with γ_K^* on the subgraph K. So we have that $\overline{P_{\gamma'}(u)} = \overline{P_{\gamma_K^*}(u)} = \{A, C\}$, $\overline{P_{\gamma'}(v)} = \overline{P_{\gamma_K^*}(v)} = \{A, D\}$ and $\{\gamma_K^*(u_1w_1), \gamma_K^*(u_2w_2)\} = \{A, B\}$. Without loss of generality, we can assume that the edge e that has been removed is coloured with A. By Condition (1) of Definition 3.4, we can colour the edges of M in such a way that w_1, w_2 lack two distinct colurs, say B, C, and these two colours are missing in w. The edges of $\mathbb{B}$ are coloured as follows: $\gamma'(uv') = A$; $\gamma'(ux) = C$; $\gamma'(v'x) = D$; $\gamma'(wx) = B$; $\gamma'(wy) = C$; and $\gamma'(xy) = A$. The vertices v, v' have disjoint palettes with respect to γ', since $P_{\gamma'}(v') =$

$\overline{P_{\gamma'}(v)} = \{A, D\}$. □

Remark 3.9. The argument of the above proof is still valid if we assume that K is 3-colour transmitting with respect to u, v, u_1, where u, v, u_1 have degree 2 in K.

Example 3.10. We apply Proposition 3.8 to the graphs $K = G_{12}$ and $M = H_6$ in Figure 4. As remarked in Example 3.5, the graph G_{12} is 3-colour transmitting with respect to u, v, u_1, u_2. Similarly, in Example 3.7 we have seen that H_6 is 2-colour transmitting with respect to w, w_1, w_2. By Proposition 3.8, we obtain the graph G_{21} in Figure 8(a). The graph G_{21} has order 21, maximum degree 4, and $\chi'(G_{21}) = 4$. The vertices v, $v' \in V(G_{21})$ are conflicting. Following the proof of Proposition 3.8 we can colour the edges of G_{21} according to the 4-colourings γ_K^* and γ_M^* in Figure 4 by setting $a = 1$, $b = 2$, $c = 3$ and $d = 4$ (or $c = 4$ and $d = 3$). This graph will be used in the proof of Theorem 5.2.

Example 3.11 (Chetwynd's counterexample). We can apply Proposition 3.8 to the graph $K = G_{12}$ in Figure 4(a) and to the dipole $M = D_2$ with two parallel edges even thought the dipole D_2 is not 2-colour transmitting with respect to its vertices. More precisely, as remarked in Example 3.5, the graph G_{12} is 3-colour transmitting with respect to u, v, u_1, u_2. It is easy to see that every 4-colouring of the graph D_2 satisfies conditions (1) and (2) of Definition 3.4 with $w_1 = w_2$. Therefore, we can repeat the proof of Proposition 3.8 and obtain the graph G_{17} in Figure 8(b) having order 17, maximum degree 4 and $\chi'(G_{17}) = 4$. The vertices v, $v' \in V(G_{17})$ are conflicting. By Lemma 2.3, the identification of the vertices v, v' yields a 4-critical graph, namely, Chetwynd's 4-critical graph in Figure 1(b).

Proposition 3.12. Let $\mathbb{B} = B(x, u', v', w, y)$ be a bowtie with centre x and 2-vertices u', v', w, y. Let K and M be graphs of maximum degree 4 and $\chi'(K) = \chi'(M) = 4$ with the following features. The graph K is 3-colour transmitting with respect to u, v, u_1 where $deg_K(u) = deg_K(v) = deg_K(u_1) = 2$. The 2-vertices $w, w_1 \in V(M)$ are saturating and for every $e \in E(M)$ not containing w nor w_1 there exists a 4-colouring of $M - e$ such that w, w_1 lack exactly one colour simultaneously.

Let H be the graph obtained from $\mathbb{B}$, K and M by identifying the vertices u' with u; w' with w; and u_1 with w_1. The graph H has maximum degree 4, $\chi'(H) = 4$ and the vertices v, v' are conflicting.

Proof. The argument is the same as in the proof of Proposition 3.8. It is different in the case $e \in E(M)$. We show that if we remove an edge $e \in E(M)$, then there exists a 4-colouring γ' of $H - e$ such that v, v' have disjoint palettes with respect to it. As in the proof of Proposition 3.8, the restriction of γ' to the edges of K corresponds to a 4-colouring γ_K^* of K

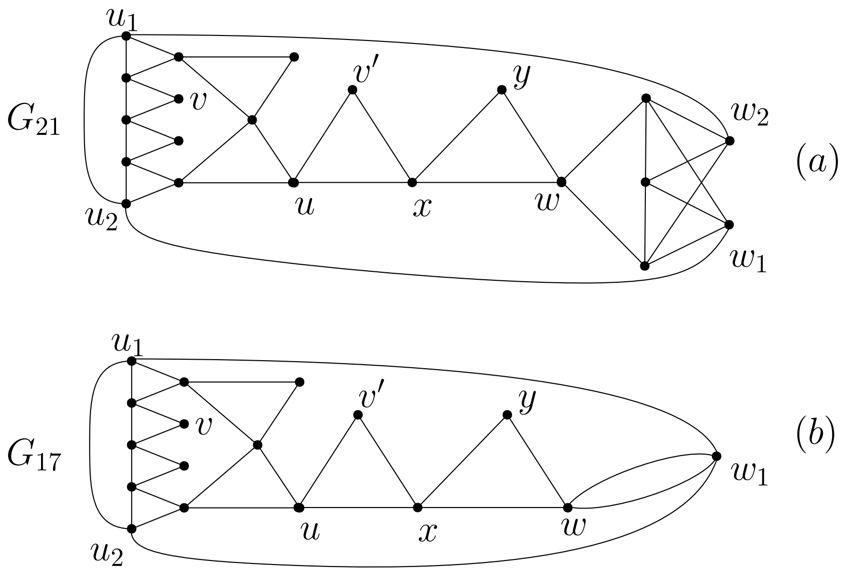

Figure 8: (a) The graph G_{21} constructed in Example 3.10. (b) The graph G_{17} constructed in Example 3.11.

such that $\overline{P_{\gamma_K^*}(u_1)} = \{A, B\}$, $\overline{P_{\gamma_K^*}(u)} = \{A, C\}$, $\overline{P_{\gamma_K^*}(v)} = \{A, D\}$. We set $\gamma'(uv') = A$, $\gamma'(ux) = C$, $\gamma'(v'x) = D$. The restriction of γ' to the edges of $M - e$ corresponds to a 4-colouring γ'_M of $M - e$. Since u_1 and w_1 are identified, the palette of w_1 with respect to γ'_M is contained in $\{A, B\}$. We define γ'_M on the other edges of $M - e$ as follows.

If $e \in E(M)$ does not contain w nor w_1, then $P_{\gamma'_M}(w_1) = \{A, B\}$. By the assumptions, there exists a 4-colouring of $M - e$ such that w, w_1 lack exactly one colour simultaneously. By a permutation of the colours, we can set $P_{\gamma'_M}(w) = \{A, C\}$. We can colour the remaining edges of $H - e$ as follow: $\gamma'(wx) = B$, $\gamma'(wy) = D$, $\gamma'(xy) = A$. The colouring γ' of $H - e$ is thus defined and v, v' have disjoint palettes with respect to it, since $P_{\gamma'}(v') = \overline{P_{\gamma'}(v)} = \{A, D\}$. We can repeat similar arguments if the edge $e \in E(M)$ contains w but not w_1.

If $e \in E(M)$ contains w_1, then we can assume that $P_{\gamma'_M}(w_1) = \{A\}$. We can permute the colours in $M - e$ so that $P_{\gamma'_M}(w) \subseteq \{B, C\}$ or $P_{\gamma'_M}(w) \subseteq \{B, D\}$. The remaining edges of $H - e$ are coloured as follows: $\gamma'(wx) = A$, $\gamma'(xy) = B$ and $\gamma'(wy) = D$ or C according to whether $P_{\gamma'_M}(w) \subseteq \{B, C\}$ or $P_{\gamma'_M}(w) \subseteq \{B, D\}$, respectively. The colouring γ' of $H - e$ is thus defined and v, v' have disjoint palettes with respect to it, since $P_{\gamma'}(v') = \overline{P_{\gamma'}(v)} = \{A, D\}$. □

Example 3.13. The graph G_{25} in Figure 9(b) has order 25, maximum degree 4 and $\chi'(G_{25}) = 4$. The vertices v, v' are conflicting. It is obtained

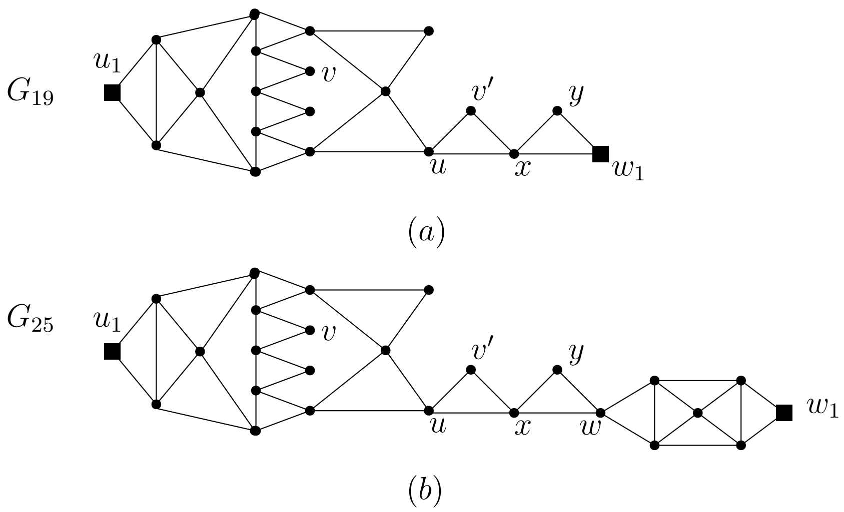

Figure 9: u_1 and w_1 should be identified in both graphs. (a) The graph G_{19} has order 19, maximum degree 4 and $\chi'(G_{19}) = 4$. (b) The graph G_{25} has order 25, maximum degree 4 and $\chi'(G_{25}) = 4$. As shown in Example 3.14, the vertices v, v' are conflicting.

by applying Proposition 3.12 to the graphs $K = G_{16}$ in Figure 7(b) and $M = G_7$ in Figure 2(c). The vertices u_1, w_1 are identified. As remarked in Example 3.6, the graph G_{16} is 3-colour transmitting with respect to u, v, u_1. As remarked in Example 2.6, the 2-vertices w, $w_1 \in V(G_{16})$ are saturating. Moreover, for every $e \in G_7$ not containing w nor w_1 there exists a colouring of $G_7 - e$ such that $P(w_1) \subseteq \{A, B\}$ and $P(w) \subseteq \{A, C\}$, that is, the assumption in Proposition 3.12 is satisfied. By Lemma 2.3, the identification of the conflicting vertices v, v' yields a 4-critical graph of order 24.

Example 3.14 (**Fiol's counterexample**). Proposition 3.12 is still true if we assume that M consists of exactly one vertex. For instance, consider the graph G_{19} in Figure 9(a) obtained from the graph G_{16} in Figure 7(b) and M consisting of exactly one vertex. The vertices u_1 and w_1 are identified. The vertices v, $v' \in V(G_{19})$ are conflicting (we can repeat the proof of Proposition 3.8 without considering the case $e \in E(M)$). By Lemma 2.3, the identification of the vertices v, v' yields a 4-critical graph, namely, Fiol's 4-critical graph in Figure 1(a).

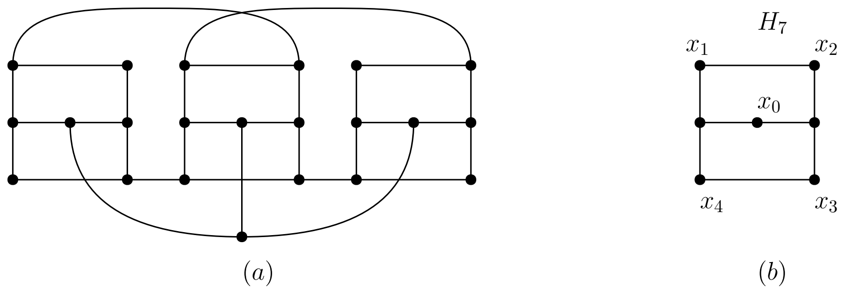

Figure 10: (a) The 3-critical graph of order 22 constructed by Goldberg. (b) The graph H_7 which is used to construct 3-critical graphs of order $n \equiv 8 \pmod{16}$, $n \geq 24$.

4 Counterexamples to the Critical Graph Conjecture

In 1971, Jacobsen showed that there are no 3-critical graphs of order ≤ 10 and no 3-critical multigraphs of order ≤ 8. This led him to formulate the Critical Graph Conjecture. As we already mentioned, the first counterexamples to the conjecture were constructed by Goldberg [9], and afterwards by Chetwynd [6] and Fiol [7]. In this section we show that also Goldberg's counterexample can be obtained by a Möbius-type technique. Furthermore, combining our technique with Goldberg's construction we show that for every even value value of n, $n \geq 22$, there exists a 3-critical graph of order n.

Goldberg was the first to disprove the Critical Graph Conjecture by constructing an infinite family of 3-critical graphs of even order, the smallest of which has order 22 [9]. The graph of order 22 is represented in Figure 10(a). A 3-critical graph of the infinite family can be obtained from the 3-critical graph of order 22 in Figure 10(a) by adding in pairs the graph H_7 of order 7 in Figure 10(b). The result is the graph in Figure 11(a). A 3-critical graph of the infinite family has order $n \equiv 8 \pmod{16}$, $n \geq 24$.

In what follows, we show that the 3-critical graphs constructed by Goldberg can be obtained by a Möbius type technique, namely, by identifying a pair of conflicting vertices in the case of the graph in Figure 10(a), or by connecting a pair of saturating vertices in the case of the graph in Figure 11(a). In Lemma 4.2, we will show that the vertices u, v of the graph H_{23} in Figure 11(b) are conflicting. We give a proof of the fact that u, v are conflicting showing that the structure of the graph H_7 forces to colour the edges of the graph in Figure 10(a) in a prescribed way, thus determining which vertex has to be split into two conflicting vertices. Analogously, for the proof of Lemma 4.3. The proofs of Lemmas 4.2 and 4.3 are based on

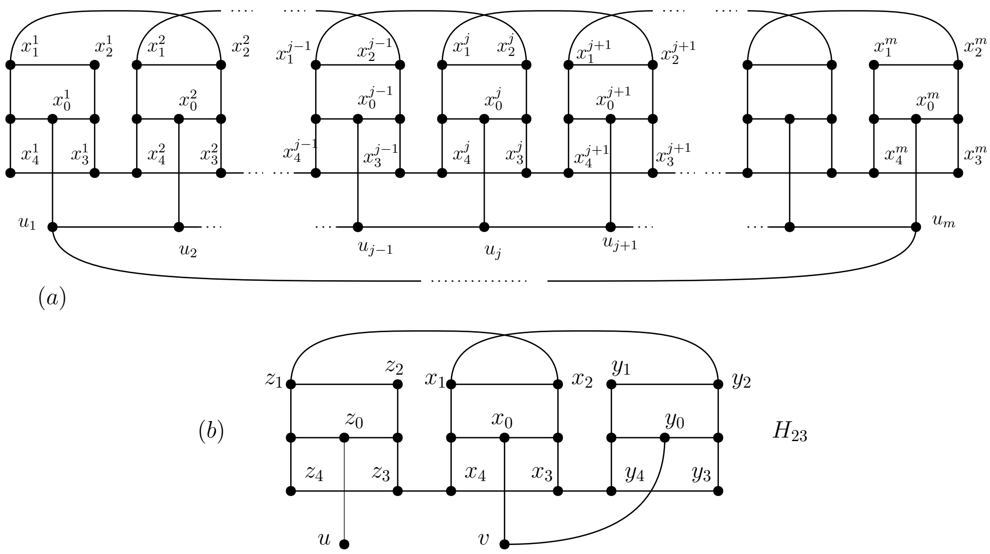

Figure 11: (a) The infinite family of 3-critical graphs of order $8m$, $m \geq 3$, m odd, constructed by Goldberg. (b) The graph H_{23} that yields the 3-critical graph of order 22 constructed by Goldberg by identifying the conflicting vertices u, v.

the following result.

Lemma 4.1. Every 3-colouring of the graph H_7 in Figure 10(b) satisfies the following condition:

$$|\overline{P(x_0)} \cup \overline{P(x_i)} \cup \overline{P(x_{i+2})}| = 3 \text{ and } P(x_{i+1}) = P(x_{i+3}) = P(x_r)$$

where $i = 1$ or $i = 2$, $r \in \{0, i, i+2\}$ and the subscripts are (mod 4).

Proof. Since the colour set has cardinality 3 and PL holds, exactly three vertices of H_7 lack the same colour A and the remaining 2-vertices of H_7 lack distinct colours B, C, both different from A. A direct inspection on the graph shows that the vertices lacking the same colours are x_{i+1}, x_{i+3} and x_r, where $i = 1$ or $i = 2$ and $r \in \{0, i, i+2\}$. □

Lemma 4.2. The graph H_{23} in Figure 11 (b) is class 1 and the vertices $u, v \in V(H_{23})$ are conflicting.

The 3-critical graph of order 22 in Figure 10 (a) constructed by Goldberg can be obtained from the graph H_{23} by identifying the conflicting vertices u, $v \in V(H_{23})$.

Proof. It is easy to see that H_{23} is class 1. We show that the vertices $u, v \in V(H_{23})$ are conflicting. Firstly, we prove that $P(u) \cap P(v) \neq \emptyset$ for every 3-colouring of the the graph H_{23}.

Let γ be a 3-colouring of H_{23}. Since γ induces a 3-colouring of the subgraphs of H_{23} that are isomorphic to H_7 and Lemma 4.1 holds, it is either $|\{\gamma(x_1y_2), \gamma(x_3y_4), \gamma(x_0v)\}| = 3$ or $|\{\gamma(x_2z_1), \gamma(x_4z_3), \gamma(x_0v)\}| = 3$. If $|\{\gamma(x_1y_2), \gamma(x_3y_4), \gamma(x_0v)\}| = 3$, then $\gamma(x_0v) = \gamma(y_0v)$, since Lemma 4.1 on the subgraph of H_{23} which is isomorphic to H_7 and contains the vertices y_i, $0 \leq i \leq 4$. That yields a contradiction, hence $|\{\gamma(x_2z_1), \gamma(x_4z_3), \gamma(x_0v)\}| = 3$. Since Lemma 4.1 holds on the subgraph of H_{23} which is isomorphic to H_7 and contains the vertices z_i, $0 \leq i \leq 4$, we have $\gamma(x_0v) = \gamma(z_0u)$. It is thus proved that $P(u) \cap P(v) \neq \emptyset$ for every 3-colouring of H_{23}.

It remains to prove that for every edge $e \in E(H_{23})$ there exists a 3-colouring γ' of $H_{23} - e$ such that the vertices u, v have disjoint palettes with respect to it. The existence is straightforward if e is incident to u, since u has degree 1. Let $\{1, 2, 3\}$ be the colour set of γ'. To define γ', it suffices to define γ' on the edges in $\{x_0v, y_0v,\ z_0u, x_iy_{i+1},\ x_{i+1}z_i : i = 1, 3\}$ and colour the remaining edges according to Lemma 4.1. For instance, if e is incident to the vertices in $\{x_i, y_i : 0 \leq i \leq 4\}$, $e \notin \{x_0v, y_0v,\ z_0u, x_iy_{i+1},\ x_{i+1}z_i : i = 1, 3\}$, then we set $\gamma'(x_1y_2) = \gamma'(z_0u) = 1$; $\gamma'(x_3y_4) = \gamma'(x_0v) = 2$; $\gamma'(y_0v) = 3$; $\gamma'(x_2z_1) = \gamma'(x_4z_3) = a \in \{1, 2\}$. The remaining cases can be managed in a similar way. It is thus proved that u, v are conflicting. Now the assertion follows from Lemma 2.3 by identifying the vertices u, v. $\square$

Lemma 4.3. Let H_{8m}, $m \geq 3$, m odd, be the graph obtained from the graph in Figure 11(a) by deleting the edge u_1u_m. The graph is class 1 and the vertices u_1, u_m are saturating. The 3-critical graphs of the infinite family constructed by Goldberg can be obtained by connecting a pair of saturating vertices.

Proof. One can easily verify that the graph H_{8m} is class 1. We prove that u_1, u_m are saturating. Firstly, we show that $|P(u_1) \cup P(u_m)| = 3$ for every 3-colouring of the graph H_{8m}. For $1 \leq j \leq m$, let H_j be the subgraph of H_{8m} which is isomorphic to the graph H_7 in Figure 10 (b) and contains the vertices x_i^j, $0 \leq i \leq 4$. Every 3-colouring γ of H_{8m} induces a 3-colouring γ' of the graph H_7, that is, Lemma 4.1 holds. By the symmetry of the graph, we can assume that $|\overline{P_{\gamma'}(x_0^1)} \cup \overline{P_{\gamma'}(x_2^1)} \cup \overline{P_{\gamma'}(x_4^1)}| = 3$ and $P_{\gamma'}(x_1^1) = P_{\gamma'}(x_3^1)$. Consequently, $P_{\gamma'}(x_2^2) = P_{\gamma'}(x_4^2)$ and $|\overline{P_{\gamma'}(x_0^2)} \cup \overline{P_{\gamma'}(x_1^2)} \cup \overline{P_{\gamma'}(x_3^2)}| = 3$. From this we deduce that $|\overline{P_{\gamma'}(x_0^j)} \cup \overline{P_{\gamma'}(x_2^j)} \cup \overline{P_{\gamma'}(x_4^j)}| = 3$ and $P_{\gamma'}(x_1^j) = P_{\gamma'}(x_3^j)$ if j is odd, $1 \leq j \leq m$; $|\overline{P_{\gamma'}(x_0^j)} \cup \overline{P_{\gamma'}(x_1^j)} \cup \overline{P_{\gamma'}(x_3^j)}| = 3$ and $P_{\gamma'}(x_2^j) = P_{\gamma'}(x_4^j)$ if j is even, $1 \leq j \leq m$. It follows that $\gamma(x_0^ju_j) = \gamma(x_0^{j+1}u_{j+1})$ for every $2 \leq j \leq m-1$, j even. We colour the edges of H_{8m} by $\{1, 2, 3\}$ and set $\gamma(x_0^2u_2) = \gamma(x_0^3u_3) = 3$. Without loss of generality we can set $\gamma(u_2u_3) = 1$, whence $\gamma(u_1u_2) = 2$. One can see that $\{\gamma(x_0^ju_j), \gamma(u_ju_{j+1})\} = \{\gamma(x_0^ju_{j+1}), \gamma(u_ju_{j+1})\} = \{1, 3\}$ for every $2 \leq j \leq m-1$, j even. As a consequence, $P(u_m) = \{1, 3\}$. It is thus proved that $|P(u_1) \cup P(u_m)| = 3$ for every 3-colouring of H_{8m}, since $2 \in P(u_1)$.

We omit the routine proof that for every $e \in E(H_{8m})$ there exists a colouring of H_{8m} such that $|P(u_1) \cup P(u_m)| < 3$. It is thus proved that u_1, u_m are saturating and the assertion follows from Lemma 2.3. □

It is known that the 3-critical graph of order 22 constructed by Goldberg is the smallest 3-critical graph [4]. Combining our construction with that one of Goldberg, we can prove the following result.

Theorem 4.4. For every even value of n, $n \geq 22$, there exists a 3-critical graph of order n.

Proof. A critical graph of the infinite family constructed by Goldberg has order $n \equiv 8 \pmod{16}$, $n \geq 24$. We construct a 3-critical graph of order $n \equiv 2 \pmod 4$, $n \geq 26$; and $n \equiv 0 \pmod 4$, $n \geq 28$. We define the auxiliary graphs H', K' and H'' that will be used in the construction. The graph H' is defined as follows. Consider $m \geq 1$ copies of the complete graph $K_4 - e$; the 2-vertices of $K_4 - e$ are same-lacking. For $1 \leq i \leq m-1$, connect the ith copy of $K_4 - e$ to the $(i+1)$th by adding exactly one edge joining a 2-vertex in the ith copy to a 2-vertex in the $(i+1)$th copy. The resulting graph H' has exactly two 2-vertices, say v_1, v_2. By Lemma 3.1, the graph H' has maximum degree 3, $\chi'(H') = 3$ and the vertices v_1, v_2 are same-lacking. Let K' be the graph of order 6 that can be obtained from the graph G_6 in Figure 2(b) by deleting the edges v_1v_2, v_3v_5, v_4v_6. The graph K' has maximum degree 3, $\chi'(K') = 3$ and the vertices v_1, v_2 are same-lacking. The graph H'' is obtained from the graphs H' and K' by connecting the vertex $v_2 \in V(K')$ to the vertex $v_1 \in V(H')$. By Lemma 3.1, the graph H'' has maximum degree 3, $\chi'(H'') = 3$ and the vertices v_1, v_2 are same-lacking. Let H be the graph obtained from the graph H_{23} in Figure 11(b) and the graph Γ, where $\Gamma \in \{H', K', H''\}$, by deleting the edge $z_0u \in E(H_{23})$ and adding the edges z_0v_1, uv_2. As remarked in Example 2.7, a graph with same-lacking vertices is able to transmit a color, therefore the graph H has maximum degree 3, $\chi'(H) = 3$ and the vertices $u, v \in V(H)$ are conflicting. Notice the following: $|V(H)| = 23 + 4m \geq 27$ if $\Gamma = H'$; $|V(H)| = 29$ if $\Gamma = K'$; $|V(H)| = 29 + 4m \geq 33$ if $\Gamma = H''$. By Lemma 2.3, the identification of the conflicting vertices $u, v \in V(H)$ yields a 3-critical graph of order $|V(H)| - 1$. Hence, the assertion follows. □

The 3-critical graphs of order $n \equiv 0 \pmod 4$, $n \geq 28$, that are constructed in the proof of Theorem 4.4, include the orders of Goldberg's infinte family but are not isomorphic to them. In fact, Goldberg's graphs have girth larger than 3; the 3-critical graphs in the proof of Theorem 4.4 have girth 3 as K' contains a 3-cycle.

5 From graphs with fertile vertices to 4-critical graphs.

We show that it is possible to obtain 4-critical graphs of order n, for every $n \geq 5$, starting from the four graphs in Figure 2, the two graphs in Figure 1 and the graph G_{21} in Figure 8(a); these graphs have a pair of fertile vertices.

Theorem 5.1. *For every odd integer $n \geq 5$ there exists a 4-critical simple graph of order n.*

Proof. For every odd integer $n \geq 5$, we exhibit a graph H of maximum degree 4, $\chi'(H) = 4$ and order n having a pair of saturating vertices u_1, v_1. The assertion follows from Lemma 2.3 by adding the edge u_1v_1.

The graph H is obtained from Lemma 3.1 as follows. We take the graph G_6^m in Figure 3 as the graph H_1 in Lemma 3.1, where $m \geq 1$. As remarked in Example 3.2, it has order $6m \geq 6$, maximum degree 4 and the vertices $v_1, v_2 \in V(G_6^m)$ are same-lacking. We define the graph H_2 in Lemma 3.1 as follows: if $n \equiv 1 \pmod 6$, then H_2 is the graph G_7 in Figure 2(c); if $n \equiv 3 \pmod 6$, then H_2 is the graph G_9 in Figure 2(d); if $n \equiv 5 \pmod 6$, then H_2 is the graph G_5 in Figure 2(a). By the remarks in Examples 2.5 and 2.6, the vertices $u_1, u_2 \in V(H_2)$ are saturating. By Lemma 3.1, the graph H obtained from $H_1 = G_6^m$ and H_2 by adding the edge u_2v_2 has maximum degree 4, $\chi'(H) = 4$ and the vertices $u_1, v_1 \in V(H)$ are saturating. Notice that $|V(H)| = 6m + |V(H_2)| \geq 11$, where $m \geq 1$ and $|V(H_2)| \in \{5, 7, 9\}$. The graph G obtained from H by adding the edge u_1v_1 is 4-critical, since Lemma 2.3 holds. By construction, the graph G is simple. Since $|V(G)| = |V(H)|$, for every odd integer $n \geq 11$ there exists a 4-critical simple graph of order n. For $n = 5, 7, 9$, the assertion follows from Lemma 2.3 by setting $H = G_5, G_7, G_9$, respectively, and by adding the edge u_1u_2. □

Theorem 5.2. *For every even integer $n \geq 16$ there exists a 4-critical graph of order n. The graph is simple unless n is equal to 16.*

Proof. For $n = 16, 18$, we resort to the well known graphs in Figure 1. For $n = 20$ we consider the graph G_{21} in Figure 8(a). As remarked in Example 3.10, the vertices $v, v' \in G_{21}$ are conflicting. The existence of a 4-critical graph of order 20 follows from Lemma 2.3 by identifying the vertices v and v'. Notice that the graph is simple.

For every even integer $n \geq 22$, we exhibit a graph H of maximum degree 4, $\chi'(H) = 4$ and order n having a pair of saturating vertices u_1, v_1. The assertion follows from Lemma 2.3 by adding the edge u_1v_1. The graph H is obtained from Lemma 3.1 as follows. We take G_6^m in Figure 3 as the graph H_1 in Lemma 3.1, where $m \geq 1$. The graph H_2 in Lemma 3.1 has even order and its definition depends on the congruence class of n modulo 6.

Case $n \equiv 0 \pmod 6$**,** $n > 18$:

the graph H_2 is obtained from the 4-critical graph of order 18 in Figure 1(a) by the deletion of the edge u_1u_2. Alternatively, we can consider the 4-critical graph arising from the graph G_{25} in Figure 9(b) by identifying the vertices v, v' (see Example 3.13); H_2 can be obtained by deleting one of the two edges containing u_1.

Case $n \equiv 2 \pmod 6$**,** $n > 20$:

consider the 4-critical graph G_{20} of order 20 obtained from the graph G_{21} in Figure 8(a) by identifying the vertices v, v'. Let H_2 be the graph obtained from G_{20} by deleting the edge u_1u_2.

Case $n \equiv 4 \pmod 6$**,** $n > 16$:

the graph H_2 is obtained from the 4-critical graph of order 16 in Figure 1(b) by the deletion of one parallel edge connecting the vertices u_1, u_2. For each congruence class of n, the vertices $u_1, u_2 \in V(H_2)$ are saturating, since Remark 2.8 holds. Moreover, H_2 is a simple graph of maximum degree 4, $\chi'(H_2) = 4$ and $|V(H_2)| = 18$, 20, 16 according to whether $n \equiv 0, 2, 4 \pmod 6$, respectively. By Lemma 3.1, the graph H obtained from $H_1 = G_6^m$ and H_2 by adding the edge u_2v_2 has maximum degree 4, $\chi'(H) = 4$ and the vertices $u_1, v_1 \in V(H)$ are saturating. Notice that $|V(H)| = 6m + |V(H_2)| \geq 22$, where $m \geq 1$ and $|V(H_2)| \in \{16, 18, 20\}$. By Lemma 2.3, the graph G obtained from H by adding the edge u_1v_1 is 4-critical. Since $|V(G)| = |V(H)|$, for every even integer $n \geq 22$ there exists a 4-critical graph of order n. Notice that these graphs are simple. Combining this result with the remarks on the existence of 4-critical graphs of order 16, 18 and 20, the assertion follows. □

There are alternative methods for constructing 4-critical graphs. For instance, consider the 4-critical graph G of order 20 obtained from the graph G_{21} in Figure 8(a) by identifying the vertices v, v'. Delete the edge $u_1u_2 \in E(G)$ and connect the remaining graph to the graph G_6^m in Figure 3. For every $m \geq 1$ we obtain a 4-critical graph of order $6m + 20$.

A concluding remark.

We are confident that the present work will provide suggestions and tools for constructing infinite families of critical graphs even beyond degree 4. The next step should be inevitably the degree 5. The key definitions are compatible with the general case, and we believe that the method is versatile enough. With some effort and further investigation, new infinite families are expected to be found in the near future.

Acknowledgements.

This manuscript was prepared with the funding support of *Progetti di Ateneo, Sapienza Università di Roma.*

References

[1] D. D. Blanuša., *Problem ceteriju boja (The problem of four colors)*, Hrvatsko Prirodoslovno Društvo Glasnik Mat.-Fiz. Astr. Ser. II, **1** (1946), pp. 31-42.

[2] D. Bokal, G. Brinkmann, and S. Grünewald *Chromatic-index-critical graphs of orders* 13 *and* 14, Discrete Math. **300** (2005), pp. 16-29.

[3] J. A. Bondy and U. S. R. Murty, *Graph theory*, Springer, 2008.

[4] G. Brinkmann, E. Steffen, 3- *and* 4-*critical graphs of small order*, Discrete Math. **169** (1997), pp. 193-197.

[5] V. Bryant, *Aspects of Combinatorics, A Wide-ranging Introduction*, Cambridge University Press, 1993.

[6] A. G. Chetwynd, R. J. Wilson, *The rise and fall of the critical graph conjecture*, J. Graph Theory **7** (1983), pp. 153-157.

[7] A. Fiol, 3-*grafos criticos*, Doctoral dissertation, Barcelona University, Spain (1980).

[8] S. Fiorini and R. J. Wilson, *Edge-colourings of graphs*, Res. Notes Math., vol.16, Pitman, 1977.

[9] M. K. Goldberg, *Construction of class* 2 *graphs with maximum vertex degree* 3, J. Combin. Theory **B 31** (1981), pp. 282-291.

[10] I. T. Jacobsen, *On critical graphs with chromatic index* 4, Discr. Math. **9** (1974), pp. 265-276.

[11] A. Vietri, *An analogy between edge colourings and differentiable manifolds, with a new perspective on* 3-*critical graphs*, Graphs Comb. **31** (2015), pp. 2425-2435.

[12] V. G. Vizing, *On an estimate of the chromatic class of a p-graph*, Diskret. Analiz **3** (1964), pp. 25-30.

[13] V. G. Vizing, *Critical graphs with a given chromatic class*, Diskret. Analiz **5** (1965), pp. 9-17.

[14] H. P. Yap, *Some topics in graph theory*, London Math. Soc. LNS 108, University Press, Cambridge, England 1986.