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ON PLANE CREMONA MAPS OF SMALL DEGREE AND THEIR QUADRATIC LENGTHS

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Abstract in English and Sintesi in italiano

On plane Cremona maps of small degree and their quadratic lengths

Abstract. The plane Cremona group $\mathrm{Cr}(\mathbb{P}^2)$ is the group of birational transformations of the complex projective plane. By the famous Noether-Castelnuovo theorem, every birational map $\varphi \in \mathrm{Cr}(\mathbb{P}^2)$ is the composition of finitely many (ordinary) quadratic maps. This leads to the notion of (ordinary) quadratic length of a given plane Cremona map. While quadratic maps are classically very well-known, only recently Cerveau and Déserti extensively studied and gave a classification of cubic plane Cremona maps. However, it turns out that their classification is not complete and it contains some inaccuracies.

In this thesis, we first give a fine and complete classification of cubic plane Cremona maps, up to a natural notion of equivalence, by using the so-called enriched weighted proximity graph associated to the base points of the homaloidal net defining the given cubic plane Cremona map. We then classify such enriched weighted proximity graphs also for quartic plane Cremona maps. This allows to compute exactly the ordinary quadratic length and the quadratic length of cubic plane Cremona maps and, in many cases, also of quartic plane Cremona maps.

Sulle trasformazioni piane di Cremona di grado basso e le loro lunghezze quadratiche

Sintesi. Il gruppo di Cremona $\mathrm{Cr}(\mathbb{P}^2)$ è il gruppo di trasformazioni birazionali del piano proiettivo complesso. Per il famoso teorema di Noether-Castelnuovo, ogni trasformazione birazionale $\varphi \in \mathrm{Cr}(\mathbb{P}^2)$ è la composizione di un numero finito di trasformazioni quadratiche (ordinarie). Ciò porta alla nozione di lunghezza quadratica (ordinaria) di una data trasformazione cremoniana. Mentre le trasformazioni quadratiche sono classicamente molto conosciute, solo recentemente Cerveau e Déserti hanno studiato in dettaglio e dato una classificazione delle trasformazioni cremoniane cubiche. Tuttavia, è risultato che la loro classificazione è incompleta e contiene qualche inaccuratezza.

In questa tesi, prima diamo una classificazione fine e completa della trasformazioni cremoniane cubiche, a meno di una nozione naturale di equivalenza, usando il cosiddetto grafo di prossimità pesato e arricchito, associato ai punti base della rete omaloidica che definisce la data trasformazione cremoniana cubica. Poi classifichiamo tali grafi di prossimità pesati e arricchiti anche per le trasformazioni cremoniane quartiche. Ciò ci permette di calcolare esattamente le lunghezze quadratiche (ordinarie) delle trasformazioni cremoniane cubiche e, in molti casi, anche di quelle quartiche.

Keywords and phrases: cubic plane Cremona maps, quartic plane Cremona maps, quadratic length, ordinary quadratic length, de Jonquières maps.

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Introduction

We work over the field $\mathbb{C}$ of complex numbers.

We denote by $\mathbb{P}^2$ the projective plane and by $\mathrm{Cr}(\mathbb{P}^2)$ the *plane Cremona group*, that is the group of birational maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Recall that the celebrated Noether-Castelnuovo Theorem says that $\mathrm{Cr}(\mathbb{P}^2)$ is generated by the automorphisms of $\mathbb{P}^2$ and the elementary quadratic transformation

$$\sigma\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \qquad [x:y:z] \mapsto [yz:xz:xy].$$

Note that a presentation of $\mathrm{Cr}(\mathbb{P}^2)$ involving exactly these generators have been found only very recently by Urech and Zimmermann in [24].

In other words, any plane Cremona map $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ can be written as

$$\varphi = \alpha_n \circ \sigma \circ \alpha_{n-1} \circ \sigma \circ \cdots \circ \alpha_1 \circ \sigma \circ \alpha_0$$

where $\alpha_i \in \mathrm{Aut}(\mathbb{P}^2)$ for any $i = 0, \ldots, n$, for some integer n.

Let us say that a decomposition of φ as above is "minimal" if so is n among all decompositions of φ. Let us call such n the "ordinary quadratic length" of φ and denote it by $\mathrm{oql}(\varphi)$. Recall that a quadratic plane Cremona map is called "ordinary" if it has three proper base points. In other words, $\mathrm{oql}(\varphi)$ is the minimal number of ordinary quadratic maps needed to decompose φ.

Similarly, let us define the "quadratic length" of a plane Cremona map φ as the minimal number of quadratic maps needed to decompose φ and let us denote it by $\mathrm{ql}(\varphi)$.

Let us say that two plane Cremona maps $\varphi, \varphi'\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ are *equivalent* if there exist two automorphisms $\alpha, \alpha' \in \mathrm{Aut}(\mathbb{P}^2)$ such that $\varphi' = \alpha' \circ \varphi \circ \alpha$. The classification of equivalence classes of quadratic plane Cremona maps is very well-known from the beginning of the study of plane Cremona maps more than one hundred years ago.

Nonetheless, a classification of equivalence classes of cubic plane Cremona maps has been described only few years ago by Cerveau and Déserti in [11]: they find 32 types of cubic plane Cremona maps, namely 27 types are a single map whereas 4 types are families of maps depending on one parameter and one type is a family of maps depending on two parameters. Their classification is based on the detailed analysis of those plane curves which are contracted by a cubic plane Cremona map.

However, it turns out that the classification in [11] is not complete and it contains some inaccuracies, see Section 4.2 for a more detailed account:

- we found a map that does not occur in their list;
- we found that their type 17, that is a single map, should be replaced by a one-parameter set of maps;
- we found that their type 19 is equivalent to a particular case of their type 18;
- we found that their type 31 is equivalent to a particular case of their type 30.

One of the main purpose of this thesis is giving a complete classification of equivalence classes of cubic plane Cremona maps. Our classification is based on the study of *enriched weighted proximity graphs* of the base points of the homaloidal net defining a plane Cremona map. Accordingly, we divide cubic plane Cremona maps into 31 types, namely 25 types are single maps, 5 types are families of maps depending on one parameter and 1 type is a family of maps depending on two parameters. Two maps of two different types are not equivalent. Moreover, we find the conditions when two maps of the same type (depending on parameters) are equivalent. Then, using our classification, we compute exactly the quadratic length and ordinary quadratic length of all cubic plane Cremona maps.

Furthermore, we generalize this approach to study quartic plane Cremona maps and we compute their quadratic length and ordinary quadratic length. Concerning quartic plane Cremona maps, recall that they can divided in de Jonquières maps, that have a triple base point and 6 simple base points, and non-de Jonquières maps, that have 3 double base points and 3 simple base points. We give a complete list of all possible enriched weighted proximity graphs of the base points of all quartic plane Cremona maps, namely there are exactly 449 types of enriched weighted proximity graphs of quartic de Jonquières maps and 119 types of enriched weighted proximity graphs of quartic non-de Jonquières maps. Using these classifications, we compute the quadratic lengths and the ordinary quadratic lengths of many quartic Cremona maps.

In details, this thesis is divided into five chapters.

In Chapter 1: a very brief summary of the most relevant results about plane curves, blowing-ups and plane birational maps is provided with little or no proof, simply to fix notation and to set the stage. In particular, we give a way to describe infinitely near points that we call standard coordinates. Some applications to plane conics are presented right after that.

Let us describe the content of Chapter 2: we recall in detail the proximity matrices and the admissible oriented graphs which encode sequences of blowing-ups. It allows us to define the so-called enriched weighted proximity graph for a given plane Cremona map, based on proximity relations among the base points of the map, together with some other properties, for instance collinearity properties of the base points or at least 6 base points are on an irreducible conic and so on.

In Chapter 3: we introduce the notion of quadratic length and ordinary quadratic length. We study their first properties, in particular those related with weighted proximity graphs of de Jonquières maps.

In Chapter 4: we give a complete classification of equivalence classes of cubic plane Cremona maps. This allows us compute the quadratic length and the ordinary quadratic length of all cubic plane Cremona maps.

We finish the thesis with Chapter 5, where we extensively study quartic plane Cremona maps and their quadratic length and ordinary quadratic length.

Contents

Chapter 1

Generalities on plane Cremona maps

A comprehensive understanding of plane Cremona maps requires some background in algebraic geometry. This chapter aims to recall basic concepts, properties and well-known facts of plane Cremona maps, simply to fix notation and to set the stage. Most results in this chapter can be found in almost any introduction to algebraic geometry and, for a more in depth treatment, we suggest some sources on the subject such as [16, 19].

Throughout this thesis, we work over $\mathbb{C}$, the field of complex numbers. To avoid confusion, we adopt the following notational conventions.

Notation 1.1. Any non-zero complex number z can be written uniquely as follows

$$z = re^{i\theta} = r\big(\cos(\theta) + i\sin(\theta)\big), \qquad \text{with } r > 0, \ \text{ and } \theta \in [0, 2\pi).$$

The angle θ is called the *argument* of z and the real number r is the *norm* of z.
Any non-zero complex number $z = r\big(\cos(\theta) + i\sin(\theta)\big)$ has two square roots, namely

$$z_0 = \sqrt{r}\left[\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right], \qquad z_1 = -z_0.$$

From now on, we denote z_0 by $\sqrt{z}$ and z_1 by $-\sqrt{z}$.
For any $t \in \mathbb{C}$ such that $t^2 \neq 4$, set $t^\bullet = \sqrt{t^2 - 4}$, $t_+ = (t + t^\bullet)/2$ and $t_- = (t - t^\bullet)/2$, that is, $t_\pm$ are the roots of the equation $x^2 - tx + 1 = 0$. Note that, if $t^2 \neq 4$ then $t_+ \neq t_-$ and $t_+, t_- \neq 0$.

By a *surface*, we mean a smooth projective irreducible algebraic surface over $\mathbb{C}$.

1.1 Plane curves

The main reference for this section is Chapter 2 in [21].

1.1.1 Affine curves in $\mathbb{C}^2$

Let $f(x, y)$ be a non-constant polynomial in two variables with complex coefficients. One says that $f(x, y)$ has no repeated factors if one cannot write

$$f(x,y) = \big(g(x,y)\big)^2 h(x,y)$$

where $g(x, y)$ and $h(x, y)$ are polynomials and $g(x, y)$ is non-constant.

Definition 1.2. Let $f(x, y)$ be a non-constant polynomial in two variables with complex coefficients and no repeated factors. Then, the *affine curve C in $\mathbb{C}^2$* defined by $f(x, y)$ is

$$C = \big\{(x,y) \in \mathbb{C}^2 \big| f(x,y) = 0\big\}.$$

Remark 1.3. *Two polynomials $f(x, y)$ and $g(x, y)$ define the same affine curves in $\mathbb{C}^2$ if and only if they are scalar multiples of each other, and a polynomial with repeated factors is then thought of as defining a curve with multiplicities attached.*

Definition 1.4. The *degree d* of the curve C defined by $f(x, y) = \sum_{r,s} c_{r,s} x^r y^s$ is the degree of the polynomial f, i.e.

$$d = \max\big\{r + s \big| c_{r,s} \neq 0\big\}.$$

Definition 1.5. An affine curve C defined by a polynomial $f(x, y)$ is called *irreducible* if the polynomial f is irreducible, that is, if $f(x, y)$ has no factors other than constants and scalar multiples of itself.

If the irreducible factors of $f(x, y)$ are

$$f_1(x,y), \ldots, f_k(x,y),$$

then the curves defined by $f_i(x, y)$ are called the *irreducible components* of C for any $i = 1, \ldots, k$.

1.1.2 The projective plane

Definition 1.6. The *complex projective plane* $\mathbb{P}^2$ is the set

$$\mathbb{C}^3 \setminus \big\{(0,0,0)\big\} / \sim$$

where $\sim$ is the equivalence relation

$$(x,y,z) \sim (x',y',z') \Leftrightarrow \exists \lambda \in \mathbb{C}^* : x' = \lambda x, y' = \lambda y, z' = \lambda z.$$

A point of $\mathbb{P}^2$ is denoted by $[x : y : z]$.

Note that $\mathbb{P}^2$ is covered by three affine charts, namely $\mathbb{P}^2 = U_0 \cup U_1 \cup U_2$ where

$$\begin{aligned} U_0 &= \big\{[x:y:z] \in \mathbb{P}^2 \big| x \neq 0\big\}, \\ U_1 &= \big\{[x:y:z] \in \mathbb{P}^2 \big| y \neq 0\big\}, \\ U_2 &= \big\{[x:y:z] \in \mathbb{P}^2 \big| z \neq 0\big\}, \end{aligned}$$

and one can identify U_i with $\mathbb{C}^2$ for each $i = 0, 1, 2$. For instance, one has $U_2 \simeq \mathbb{C}^2$ where an isomorphism $\phi : U_2 \to \mathbb{C}^2_{\overline{x},\overline{y}}$ is defined by

$$\phi([x:y:z]) = \left(\frac{x}{z}, \frac{y}{z}\right) \tag{1.1}$$

with inverse

$$(\overline{x}, \overline{y}) \longmapsto [\overline{x} : \overline{y} : 1].$$

The complement of U_2 in $\mathbb{P}^2$ is the projective line defined by $z = 0$ which we can identify with $\mathbb{P}^1$ via the map

$$[x : y : 0] \longmapsto [x : y].$$

In other words, $\mathbb{P}^2$ is the disjoint union of a copy of $\mathbb{C}^2$ and a copy of $\mathbb{P}^1$ which we think of as "the line at infinity".

1.1.3 Projective curves in $\mathbb{P}^2$

Recall that a polynomial $F(x, y, z)$ is called homogeneous of degree d if

$$F(\lambda x, \lambda y, \lambda z) = \lambda^d F(x, y, z)$$

for all $\lambda \in \mathbb{C}$. Note that the first partial derivatives of F are homogeneous polynomials of degree $d - 1$.

Definition 1.7. Let $F(x, y, z)$ be a non-constant homogeneous polynomial in three variables x, y, z with complex coefficients. Assume that $F(x, y, z)$ has no repeated factors. Then, the *projective curve C in $\mathbb{P}^2$* defined by $F(x, y, z)$ is

$$C = \big\{[x:y:z] \in \mathbb{P}^2 \big| F(x, y, z) = 0\big\}.$$

Note that the condition $F(x, y, z) = 0$ is independent of the choice of homogeneous coordinates $[x : y : z]$ because F is a homogeneous polynomial and hence

$$F(\lambda x, \lambda y, \lambda z) = 0 \Longleftrightarrow F(x, y, z) = 0$$

for any $\lambda \in \mathbb{C}^*$.

Remark 1.8. *Just as for curves in $\mathbb{C}^2$, it is in fact that the case that two homogeneous polynomials $F(x, y, z)$ and $G(x, y, z)$ with no repeated factors define the same projective curves in $\mathbb{P}^2$ if and only if they are scalar multiples of each other, and a homogeneous polynomial with repeated factors can be thought of as defining a curve with multiplicities attached to its components.*

Definition 1.9. The *degree* of a projective curve C in $\mathbb{P}^2$ defined by a homogeneous polynomial $F(x,y,z)$ is the degree of $F(x,y,z)$. The curve C is called *irreducible* if $F(x,y,z)$ is irreducible, i.e. $F(x,y,z)$ has no non-constant polynomial factors other than scalar multiples of itself. An irreducible projective curve D defined by a homogeneous polynomial $G(x,y,z)$ is called a *component* of C if $G(x,y,z)$ divides $F(x,y,z)$.

1.1.4 From affine to projective curves and vice versa

Affine and projective curves are closely related. From an affine curve C one can obtain a projective curve $\tilde{C}$ by adding points at infinity. Vice versa, from a projective curve $\tilde{C}$ one can obtain an affine curve C by discarding points at infinity.

Let $F(x,y,z)$ be a non-constant homogeneous polynomial of degree d. Under the identification (1.1) of U_2 with $\mathbb{C}^2$, the intersection with U_2 of the projective curve $\tilde{C}$ defined by F is the affine curve C in $\mathbb{C}^2$ defined by the (possibly inhomogeneous) polynomial in two variables

$$F(x,y,1).$$

This polynomial has degree d provided that $z=0$ is not a factor of $F(x,y,z)$ (i.e. $\tilde{C}$ does not contain the line $z=0$).

Conversely, if $f(x,y)$ is a polynomial of degree d in two variables x and y, say

$$f(x,y) = \sum_{r+s\leq d} a_{r,s} x^r y^s,$$

then the affine curve C defined by $f(x,y)$ is the intersection of U_2 (identified with $\mathbb{C}^2$) with the projective curve $\tilde{C}$ in $\mathbb{P}^2$ defined by the homogeneous polynomial

$$z^d f\left(\frac{x}{z}, \frac{y}{z}\right) = \sum_{r+s\leq d} a_{r,s} x^r y^s z^{d-r-s}.$$

The intersection of this projective curve with the line at infinity $z=0$ is the set of points

$$\left\{[x:y:0] \in \mathbb{P}^2 \middle| \sum_{0\leq r\leq d} a_{r,d-r} x^r y^{d-r} = 0\right\}.$$

However, the polynomial

$$\sum_{0\leq r\leq d} a_{r,d-r} x^r y^{d-r}$$

can be factorised as a product of linear factors

$$\prod_{1\leq i\leq d} (\alpha_i x + \beta_i y).$$

This factors correspond to points $[-\beta_i : \alpha_i]$ in $\mathbb{P}^1$; when $\mathbb{P}^1$ is identified with the line $z=0$ in $\mathbb{P}^2$, these points are precisely the points of $\tilde{C} \setminus C$.

In this way, we get a bijective correspondence between affine curves C in $\mathbb{C}^2$ and projective curves $\tilde{C}$ in $\mathbb{P}^2$ not containing the line at infinity $z=0$.

1.1.5 Automorphisms of the projective plane

The projective plane is an excellent backdrop for studying the classical algebraic geometry, and so, among other things, it will be useful to understand automorphisms of the projective plane.

Notation 1.10. We denote by $Aut_{\mathbb{C}}(\mathbb{P}^2)$, or simply $Aut(\mathbb{P}^2)$, the *group of automorphisms of* $\mathbb{P}^2$, that it is isomorphic to the quotient PGL_3 of the general linear group GL_3 by the one-dimensional subgroup of scalar matrices $\{\lambda I \mid \lambda \in \mathbb{C}^*\}$, see for instance Proposition 11.46, §11 in [18].
More precisely, an automorphism $\alpha : \mathbb{P}^2 \to \mathbb{P}^2$ is of the following form

$$\alpha([x:y:z]) = \left[a_{11}x + a_{12}y + a_{13}z : a_{21}x + a_{22}y + a_{23}z : a_{31}x + a_{32}y + a_{33}z\right]$$

where $a_{ij} \in \mathbb{C}$ for any $i, j \in \{1,2,3\}$ and the (3×3)-matrix $M = \left(a_{ij}\right)$ satisfies $\det(M) \neq 0$. One says that M is the *associated matrix of the automorphism* α, or simply one says that α is defined by M.

Lemma 1.11. *(The Four Points Lemma) Let* $p_i = [x_i : y_i : z_i]$ $(i = 1,2,3,4)$ *be four points in the projective plane such that no three of them are collinear. Then, there is a unique automorphism of* $\mathbb{P}^2$*, sending* $e_1 = [1:0:0], e_2 = [0:1:0], e_3 = [0:0:1]$ *and* $e_4 = [1:1:1]$*, to* p_1, p_2, p_3 *and* p_4*, respectively.*

Proof. See §11.2 in [17]. □

Definition 1.12. Two projective curves defined respectively by two polynomials F, G in $\mathbb{P}^2$ are called *projectively equivalent* if there exists an automorphism α of $\mathbb{P}^2$ and a scalar $\lambda \in \mathbb{C}^*$, for which $G = \lambda(F \circ \alpha)$.

Note that projective equivalence is an equivalence relation, and that projectively equivalent curves have the same degree. Moreover, F is reduced if and only if so is G.

1.1.6 Plane conics

Any conic C in $\mathbb{P}^2$ is defined by a quadratic polynomial

$$Q(x,y,z) = \begin{pmatrix} x & y & z \end{pmatrix} A \begin{pmatrix} x & y & z \end{pmatrix}^T$$

where A is a (3×3) non-zero symmetric complex matrix.

Note that C is irreducible if and only if $\det(A) \neq 0$.

More precisely, a plane conic C is defined as follows

$$Q(x,y,z) = ax^2 + bxy + cy^2 + dxz + eyz + fz^2,$$

which is associated to the matrix

$$A = \frac{1}{2}\begin{pmatrix} 2a & b & d \\ b & 2c & e \\ d & e & 2f \end{pmatrix}.$$

.

Remark 1.13. *Let C be an irreducible conic and ℓ be a line in $\mathbb{P}^2$. Then, $C \cap \ell$ is nonempty and it consists of at most two points.*

When $C \cap \ell$ is just one point p_0, one says that ℓ is tangent to C at p_0 and we denote ℓ by $T_{p_0}(C)$.

Note that, if $p_0 \in C$, then C has a unique tangent line at p_0, while, if $p_0 \notin C$, then there are exactly two tangent lines to C passing through p_0.

Lemma 1.14 (cf. [23, Lem 1.2.3, Sec 1.2])**.** *Any two irreducible conics can be mapped each other by projective transformations.*

Proof. Let C be an irreducible conic. It suffices to show that there exists a projective transformation that maps C to the conic $C_0 \colon xz - y^2 = 0$. On C, take mutually distinct points p_1, p_2 and p_3. Let p_4 be the intersection point of $T_{p_1}(C)$ and $T_{p_2}(C)$. Clearly, no three among p_1, p_2, p_3, p_4 are collinear. Therefore, by Lemma 1.11, there exists an automorphism α of $\mathbb{P}^2$ that sends p_1, p_2, p_3, p_4 to e_1, e_3, e_4, e_2, respectively. Hence, α sends C to the conic C_0. □

The proof of the previous lemma shows also the following:

Lemma 1.15. *Let $n \in \{1, 2, 3\}$. Let C_1, C_2 be irreducible conics. Let $p_1, \dots, p_n \in C_1$ and let $q_1, \dots, q_n \in C_2$. Then, there exists an auotmorphism α of $\mathbb{P}^2$ such that $\alpha(C_1) = C_2$ and $\alpha(p_i) = q_i$, $i = 1, \dots, n$.*

We recall the following result, taken directly from §5.2 of Chapter 5 in [26]:

Lemma 1.16. *Suppose $p_1, p_2, p_3, p_4, p_5 \in \mathbb{P}^2$ are any five points such that no three of them are collinear. Then, there is a unique irreducible conic passing through $p_1, \dots, p_5$.*

In Section 1.3.1, we will generalize the previous result to infinitely near points, when it is possible.

1.2 Blowing-ups

The notion of blowing-up is the most fundamental one in the subject of birational geometry. In this section, we study the blowing-up map. References for this section are [3] and [19].

1.2.1 Blowing-up of a surface at a point

Firstly, we will construct the blowing-up of $\mathbb{A}^2$ at $\mathbf{0} := (0,0)$.
Consider the product $\mathbb{A}^2 \times \mathbb{P}^1$, suppose that x, y are the affine coordinates of $\mathbb{A}^2$ and u, v are the homogeneous coordinates of $\mathbb{P}^1$. Then,

Definition 1.17. The *blowing-up of* $\mathbb{A}^2$ *at* $\mathbf{0}$ is the closed subset $\mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2)$ of $\mathbb{A}^2 \times \mathbb{P}^1$ defined by

$$\mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2) := \big\{\big((x,y),[u:v]\big) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid xv = uy\big\}.$$

We have a natural morphism $\varphi : \mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2) \to \mathbb{A}^2$ obtained by restricting the projection map pr_1 of $\mathbb{A}^2 \times \mathbb{P}^1$ onto the first factor. In other words, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2) \arrow[r, hook] \arrow[dr, "\varphi"'] & \mathbb{A}^2 \times \mathbb{P}^1 \arrow[d, "\mathrm{pr}_1"] \arrow[dl, phantom, "\circlearrowright"] \\
& \mathbb{A}^2.
\end{tikzcd}$$

Lemma 1.18. (1) *If $p \in \mathbb{A}^2$ and $p \neq \mathbf{0}$, then $\varphi^{-1}(p)$ consists of a single point.*

(2) $\varphi^{-1}(\mathbf{0}) \simeq \mathbb{P}^1$.

(3) *The points of $\varphi^{-1}(\mathbf{0})$ are in one-to-one correspondence with the set of lines through $\mathbf{0}$ in $\mathbb{A}^2$.*

(4) $\mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2) \setminus \varphi^{-1}(\mathbf{0})$ *is isomorphic to* $\mathbb{A}^2 \setminus \{\,\mathbf{0}\,\}$.

(5) $\mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2)$ *is irreducible.*

Proof. (1) Let $p = (x_0, y_0) \in \mathbb{A}^2 \setminus \{\,\mathbf{0}\,\}$, suppose that $x_0 \neq 0$ (resp. $y_0 \neq 0$). Now, if $\big(p,[u:v]\big) \in \varphi^{-1}(p)$ then $v = \dfrac{y_0}{x_0}u$ (resp. $u = \dfrac{x_0}{y_0}v$), so $[u:v]$ is uniquely determined as a point in $\mathbb{P}^1$. By setting $u = x_0$ (resp. $v = y_0$), we have $[u:v] = [x_0 : y_0]$. Thus, $\varphi^{-1}(p)$ consists of a single point.

(2) $\varphi^{-1}(\mathbf{0})$ consists of all points $\big(\mathbf{0},[u:v]\big)$ for any $[u:v] \in \mathbb{P}^1$, subject to no restriction.

(3) A line l through $\mathbf{0}$ in $\mathbb{A}^2$ can be given by parametric equations

$$\big\{x = at, y = bt \mid t \in \mathbb{A}^1\big\}$$

where $a, b \in \mathbb{C}$ are not both zero. Now, consider the line $l' = \varphi^{-1}\big(l \setminus \{\,\mathbf{0}\,\}\big)$ in $\mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2) \setminus \varphi^{-1}(\mathbf{0})$. It is given parametrically by

$$\big\{x = at, y = bt, u = at, v = bt \mid t \in \mathbb{A}^1 \setminus \{\mathbf{0}\}\big\}.$$

Since u, v are homogeneous coordinates in $\mathbb{P}^1$, we can write l' as follows

$$\{x = at, y = bt, u = a, v = b \mid t \in \mathbb{A}^1 \setminus \{\mathbf{0}\}\}.$$

These equations make sense also for $t = 0$, and give the closure $\overline{l'}$ of l in $\mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2)$. Now $\overline{l'}$ meets $\varphi^{-1}(\mathbf{0})$ in the point $q = [u : v] \in \mathbb{P}^1$, so we see that sending l to q gives one-to-one correspondence between lines through $\mathbf{0}$ in $\mathbb{A}^2$ and points of $\varphi^{-1}(\mathbf{0})$.

(4) Let $p = (x_0, y_0) \in \mathbb{A}^2 \setminus \{\mathbf{0}\}$, define $\psi(p) = \big((x_0, y_0), [x_0 : y_0]\big) \in \mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2)$. Then, $\psi : \mathbb{A}^2 \setminus \{0\} \to \mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2) \setminus \varphi^{-1}(\mathbf{0})$ is an isomorphism which is the inverse of the restriction of φ to $\mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2) \setminus \varphi^{-1}(\mathbf{0})$.

(5) $\mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2)$ is the union of $\mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2) \setminus \varphi^{-1}(\mathbf{0})$ and $\varphi^{-1}(\mathbf{0})$. The first piece is isomorphic to $\mathbb{A}^2 \setminus \{\mathbf{0}\}$, hence irreducible. On the other hand, we have just seen that every point of $\varphi^{-1}(\mathbf{0})$ is in the closure of some subset (the line l') of $\mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2) \setminus \varphi^{-1}(\mathbf{0})$. Hence, $\mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2) \setminus \varphi^{-1}(\mathbf{0})$ is dense in $\mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2)$, and $\mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2)$ is irreducible.

□

Definition 1.19. If Y is a closed subvariety of $\mathbb{A}^2$ passing through $\mathbf{0}$, we define the *blowing-up of Y at $\mathbf{0}$* to be $\tilde{Y} = \overline{\varphi^{-1}(Y \setminus \{\mathbf{0}\})}$, where $\varphi : \mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2) \to \mathbb{A}^2$ is the blowing-up of $\mathbb{A}^2$ at the point $\mathbf{0}$ described above. We denote also by $\varphi : \tilde{Y} \to Y$ the morphism obtained by restricting $\varphi : \mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2) \to \mathbb{A}^2$ to $\tilde{Y}$.

Remark 1.20. *Note that φ induces an isomorphism of $\tilde{Y} \setminus \varphi^{-1}(\mathbf{0})$ to $Y \setminus \{\mathbf{0}\}$, so that φ is a birational morphism of $\tilde{Y}$ to Y.*

Remark 1.21. *To blow up any other point p of $\mathbb{A}^2$, make a linear change of coordinates sending p to $\mathbf{0}$.*

Definition 1.22. Let $\varphi : \mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2) \to \mathbb{A}^2$ be the blowing-up of $\mathbb{A}^2$ at $\mathbf{0}$ as in Definition 1.17. Then, we can write $\mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2) = \mathbb{A}^2_{x_1,y_1} \cup \mathbb{A}^2_{x_2,y_2}$ where

$$\begin{aligned}
\mathbb{A}^2_{x_1,y_1} &= \big\{\big((x_1, x_1y_1), [1 : y_1]\big)\big\} \subset \mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2),\\
\mathbb{A}^2_{x_2,y_2} &= \big\{\big((x_2y_2, y_2), [x_2 : 1]\big)\big\} \subset \mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2)
\end{aligned}$$

are called respectively the first and the second chart of the blowing-up. The restriction of φ to the first chart $\mathbb{A}^2_{x_1,y_1}$ is given by

$$\mathbb{A}^2_{x_1,y_1} \longrightarrow \mathbb{A}^2_{x,y}, \qquad \big((x_1, x_1y_1), [1 : y_1]\big) \longmapsto (x_1, x_1y_1),$$

while the restriction of φ to the second chart $\mathbb{A}^2_{x_2,y_2}$ is given by

$$\mathbb{A}^2_{x_2,y_2} \longrightarrow \mathbb{A}^2_{x,y}, \qquad \big((x_2y_2, y_2), [x_2 : 1]\big) \longmapsto (x_2y_2, y_2).$$

Note that $\varphi^{-1}(\mathbf{0}) \simeq \mathbb{P}^1$ is locally defined by $x_1 = 0$ in the first chart $\mathbb{A}^2_{x_1,y_1}$ and by $y_2 = 0$ in the second chart $\mathbb{A}^2_{x_2,y_2}$.

Remark 1.23. *Let Y be an affine curve in $\mathbb{A}^2$ defined by the equation $f(x,y)=0$ and let $m = \operatorname{mult}_{\mathbf{0}}(Y)$ be the multiplicity of the curve Y at $\mathbf{0}$. Then, the strict transform $\tilde{Y}$ of Y is locally defined in the first chart $\mathbb{A}^2_{x_1,y_1}$ by*

$$\frac{f(x_1, x_1y_1)}{x_1^m} = 0$$

and in the second chart $\mathbb{A}^2_{x_2,y_2}$ by

$$\frac{f(x_2y_2, y_2)}{y_2^m} = 0.$$

Definition 1.24. Let S be a surface and $p \in S$. Then, there exist a surface $\tilde{S}$ and a morphism $\pi : \tilde{S} \to S$, which are unique up to isomorphisms, such that

(*i*) the restriction of π to $\pi^{-1}\big(S \setminus \{p\}\big)$ is an isomorphism onto $S \setminus \{p\}$;

(*ii*) $E := \pi^{-1}(p)$, is isomorphic to $\mathbb{P}^1$.

We shall say that π is the *blowing-up of S at p* and E is the *exceptional curve* of π.

Take a neighbourhood U of p on which there exist local coordinates x, y at p (i.e. the curves $x = 0, y = 0$ intersect transversely at p). We can assume that p is the only point of U in the intersection of these two curves. Define the subvariety $\tilde{U}$ of $U \times \mathbb{P}^1$ by

$$\tilde{U} := \big\{\big((x,y),[u:v]\big) \in U \times \mathbb{P}^1 \mid xv = uy\big\}.$$

It is clear that the projection $\pi : \tilde{U} \to U$ is an isomorphism over the points of U where at most one of the coordinates x, y vanishes, while $\pi^{-1}(p) = \{p\} \times \mathbb{P}^1$. We get S by passing $\tilde{U}$ and $S \setminus \{p\}$ along $U \setminus \{p\} \cong \tilde{U} \setminus \pi^{-1}(p)$.

Definition 1.25. Let C be an irreducible curve on S. The closure of $\pi^{-1}\big(C \setminus \{p\}\big)$ in $\tilde{S}$ is an irreducible curve $\tilde{C}$ on $\tilde{S}$, which we call the *strict transform* of C. Let us call $\pi^{-1}(C)$ the *total inverse image* of C and π^*C the *total transform* of C.

Remark 1.26. *Note that $\pi^{-1}(C)$ coincides with $\tilde{C}$ if and only if $p \notin C$, otherwise $\pi^{-1}(C) = \tilde{C} \cup E$.*

Proposition 1.27. *Let S be a surface, $\pi : \tilde{S} \to S$ the blowing-up of a point $p \in S$ and $E \subset \tilde{S}$ the exceptional curve. Then,*

(*i*) *there is an isomorphism $\operatorname{Pic} S \oplus \mathbb{Z} \to \operatorname{Pic} \tilde{S}$ defined by $(C,n) \mapsto \pi^*C + nE$. Hence, $\operatorname{Pic} \tilde{S} = \pi^* \operatorname{Pic} S \oplus \mathbb{Z}E$.*

(*ii*) *for each $C, D \in \operatorname{Pic} S$, one has $\pi^*C.\pi^*D = C.D$. Moreover, $E.\pi^*C = 0$ and $E^2 = -1$.*

(*iii*) *$K_{\tilde{S}} = \pi^*K_S + E$.*

Proof. See Lemma *II*.3 in [3]. □

Lemma 1.28. *Let π be as above and let C be an irreducible curve on S. Setting $m \geqslant 0$ the multiplicity of C at p, one has $\pi^*C = \tilde{C} + mE$, $\tilde{C}.E = m$ and $\tilde{C}^2 = C^2 - m$.*

Proof. See Lemma *II*.2 in [3]. □

1.2.2 A sequence of blowing-ups of points

Definition 1.29. Let $p_1 \in \mathbb{P}^2 = S_0$ be a point. Consider the blowing-up $\pi_1 : S_1 \to \mathbb{P}^2$ at p_1 and denote by $E_1^1 = \pi_1^{-1}(p_1)$ the exceptional curve.

Let $p_2 \in S_1$ and $\pi_2 : S_2 \to S_1$ be the blowing-up of S_1 at p_2. We denote the exceptional curve by E_2^2 and the strict transform of E_1^1 in S_2 by E_1^2. One observes that if $p_2 \notin E_1^1$, then the total transform of E_1^1 in S_2 coincides with the strict transform E_1^2. Otherwise, if $p_2 \in E_1^1$, by Remark 1.26 and Lemma 1.28, it follows:

$$(\pi_1 \circ \pi_2)^{-1}(p_1) = \pi_2^{-1}(E_1^1) = E_1^2 \cup E_2^2 \qquad \text{and} \qquad \pi_2^*(E_1^1) = E_1^2 + E_2^2.$$

Repeating the construction r times, one defines for all $i = 1, \ldots, r$:

- the blowing-up $\pi_i : S_i \to S_{i-1}$ of S_{i-1} at $p_i \in S_{i-1}$;
- the exceptional curve $E_i^i = \pi_i^{-1}(p_i)$ of S_i;
- for any $j > i$, $\pi_{ij} : S_j \to S_{i-1}$ the composition $\pi_i \circ \pi_{i+1} \circ \ldots \circ \pi_j$;
- the total transform $E_i^* = \pi_{i+1,r}^*(E_i^i)$ of E_i^i in $S = S_r$;
- for any $j > i$, the strict transform E_i^j of E_i^i in S_j;
- the strict transform $E_i := E_i^r$ of E_i^i in S;
- $\left(,\right)_i$ and $\left(,\right)$ respectively the intersection number in S_i and in S.

All these data form the sequence of blowing-ups

$$\pi = \pi_{1r} : S = S_r \xrightarrow{\pi_r} S_{r-1} \to \ldots \to S_1 \xrightarrow{\pi_1} S_0 = \mathbb{P}^2$$

at the points $p_1, \ldots, p_r$. From now on, with abuse of notation, we say that E_i and E_i^* are respectively the *strict* and the *total transform* of the point p_i in S.

Remark 1.30. *Note that the strict transform E_i^j for any $j > i$ can be defined inductively:*

$$E_i^j = \begin{cases} \pi_j^*(E_i^{j-1}) & \textit{if } p_j \notin E_i^{j-1}, \\ \pi_j^*(E_i^{j-1}) - E_j^j & \textit{if } p_j \in E_i^{j-1}. \end{cases}$$

Lemma 1.31 (cf. [7, Lem 1.1.8, Chap 1])**.** *Let $\pi : S \to \mathbb{P}^2$ be a sequence of blowing-ups of r points, as above. Then, one has*

$$\operatorname{Pic} S \cong \operatorname{Pic} \mathbb{P}^2 \oplus \mathbb{Z}^r,$$

where $\operatorname{Pic} \mathbb{P}^2 \hookrightarrow \operatorname{Pic} S$ is defined by $C \mapsto \pi^*(C)$ and $\left\{E_i^*\right\}_{1 \le i \le r}$ is a set of generators of $\mathbb{Z}^r$. The intersection numbers of the E_i^* are

$$\left(E_i^*, E_j^*\right) = -\delta_{ij} = \begin{cases} -1 & \textit{if } i = j, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The first part of assertion follows by induction on r and Proposition 1.27.
As for the second part, by definition of E_i^* and by part (ii) of Proposition 1.27, one has

$$\left(E_i^*, E_i^*\right) = \left(\pi_{i+1,r}^*(E_i^i), \pi_{i+1,r}^*(E_i^i)\right) = \left(E_i^i, E_i^i\right)_i = -1.$$

Similarly, if $j > i$, one has

$$\left(E_i^*, E_j^*\right) = \left(\pi_{j+1,r}^*(\pi_{i+1,j}^*(E_i^i)), \pi_{j+1,r}^*(E_j^j)\right) = \left(\pi_{i+1,j}^*(E_i^i), E_j^j\right)_j = 0.$$

$\square$

Remark 1.32 (see e.g. [7, §1.3.7])**.** *One can see that another set of generators of $\mathbb{Z}^r$ in the previous lemma is $\{E_i\}_{1\leq i\leq r}$. Moreover, the basis change matrices $N = (n_{ij})$ and $M = (m_{ij}) = N^{-1}$, such that*

$$E_i = \sum_{j=1}^{r} n_{ij} E_j^*, \qquad E_i^* = \sum_{j=1}^{r} m_{ij} E_j$$

are given by $N = I_r - Q$ where I_r is the $(r \times r)$ identity matrix and $Q = (q_{ij})$ is defined by

$$q_{ij} = \begin{cases} 1 & \textit{if } p_j \in E_i^{j-1}, \\ 0 & \textit{otherwise.} \end{cases}$$

In Chapter 2, Q^T will be called the proximity matrix of π.

Blowing-ups of points are so important because any birational map between surfaces factors through blowing-ups in the following sense:

Theorem 1.33. *Let $\varphi : X \dashrightarrow Y$ be a birational map between surfaces. Then, there is a surface Z and birational morphisms $\pi_X : Z \to X$ and $\pi_Y : Z \to Y$, which are sequences of blowing-ups of points, such that the following diagram commutes:*

$$\begin{tikzcd} & Z \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] \arrow[d, phantom, "\circlearrowleft"] & \\ X \arrow[rr, dashed, "\varphi"] & {} & Y. \end{tikzcd}$$

For the proof see e.g. Theorem 4.9, §3.3, Chapter 4 in [25]. In particular, the theorem is a corollary of the following two results:

- Let X be a surface and $\varphi : X \dashrightarrow \mathbb{P}^n$ a rational map. Then, there exists a sequence of blowing-ups of points of surfaces $X_m \overset{\pi_m}{\to} \ldots \overset{\pi_2}{\to} X_1 \overset{\pi_1}{\to} X$ such that the composite rational map $\psi = \varphi \circ \pi_1 \circ \ldots \circ \pi_m : X_m \to \mathbb{P}^n$ is morphism.

- Let $\varphi : X \to Y$ be a birational morphism between surfaces. Then, there exists a sequence of blowing-ups of points $\pi_i : Y_i \to Y_{i-1}$ for $i = 1, \ldots, r$ where $Y_0 = Y, Y_r = X$ such that $\varphi = \pi_1 \circ \ldots \circ \pi_r$. In other words, any birational morphism between surfaces can be factored in to a sequence of blowing-ups of points.

1.2.3 Bubble space of $\mathbb{P}^2$

Definition 1.34 (cf. [15, §7.3.2])**.** We denote by $\mathcal{B}(\mathbb{P}^2)$ the so-called *bubble space* of $\mathbb{P}^2$, which is defined as follows. Consider all surfaces X *above* $\mathbb{P}^2$, i.e. all surfaces X such that there exists a birational morphism $X \to \mathbb{P}^2$. If X_1, X_2 are two surfaces above $\mathbb{P}^2$, say $\pi_1 \colon X_1 \to \mathbb{P}^2$ and $\pi_2 \colon X_2 \to \mathbb{P}^2$ are birational morphisms, one identifies $p_1 \in X_1$ with $p_2 \in X_2$ if the birational map $(\pi_2)^{-1}\pi_1 \colon X_1 \dashrightarrow X_2$ is a local isomorphism at p_1, that sends p_1 to p_2. The bubble space $\mathcal{B}(\mathbb{P}^2)$ is the union of all points of all surfaces above $\mathbb{P}^2$ modulo the equivalence relation generated by these identifications.

For any birational morphism $X \to \mathbb{P}^2$, there is an injective map $X \to \mathcal{B}(\mathbb{P}^2)$, therefore we will identify points of X with their images in $\mathcal{B}(\mathbb{P}^2)$.

One says that $p_1 \in \mathcal{B}(\mathbb{P}^2)$ is *infinitely near* $p_2 \in \mathcal{B}(\mathbb{P}^2)$, say $p_1 \in X_1$ and $p_2 \in X_2$, with birational morphisms $\pi_1 \colon X_1 \to \mathbb{P}^2$ and $\pi_2 \colon X_2 \to \mathbb{P}^2$, if the birational map $(\pi_2)^{-1}\pi_1 \colon X_1 \dashrightarrow X_2$ is defined at p_1, sends p_1 to p_2, but is not a local isomorphism at p_1. In such a case we write that $p_1 \succ p_2$.

One moreover says that p_1 is *in the first neighbourhood* of p_2, or that p_1 is *infinitely near p_2 of the first order*, if $(\pi_2)^{-1}\pi_1$ corresponds locally to the blow-up of p_2. In such a case we write that $p_1 \succ_1 p_2$.

If $p_1 \succ p_2$ then one can define the *infinitesimal order* of p_1 with respect to p_2 by induction, namely if $p_1 \succ_1 p_3$ and $p_3 \succ_k p_2$ for some k, then p_1 is *infinitely near p_2 of order $k+1$*.

If $p_1 \succ p_2$ and $p_1 \in X_1$, then there is a unique irreducible curve $E_2 \subset X_1$ which corresponds to the exceptional curve of the blowing-up of $p_2 \in X_2$. One says that p_1 is *proximate* to p_2 if $p_1 \in E_2$. In such a case we write that $p_1 \dashrightarrow p_2$. Clearly, if $p_1 \succ_1 p_2$, then $p_1 \dashrightarrow p_2$, but the converse is not always true.

If $p_1 \dashrightarrow p_2$ and $p_1 \succ_k p_2$ with $k > 1$, then we say that p_1 is *satellite* to p_2 and we write $p_1 \odot p_2$. Otherwise, if p_1 is not satellite to p_2, then we denote by $p_1 \not\odot p_2$.

One says that a point $p \in \mathbb{P}^2 \subset \mathcal{B}(\mathbb{P}^2)$ is a *proper point of* $\mathbb{P}^2$.

Remark 1.35. *Each point of $\mathcal{B}(\mathbb{P}^2) \setminus \mathbb{P}^2$ is infinitely near a unique point of $\mathbb{P}^2$.*

Remark 1.36. *If $p_1 \succ_k p_k$, say*

$$p_1 \succ_1 p_2 \succ_1 p_3 \succ_1 \cdots \succ_1 p_{k-1} \succ_1 p_k,$$

and $p_1 \dashrightarrow p_k$, then $p_i \dashrightarrow p_k$ also for each $i = 2, \dots, k-1$.

Notation 1.37. If $p_1 \succ p_2 \in \mathbb{P}^2$ where $p_1 \in X_1$ and $\pi_1 : X_1 \to \mathbb{P}^2$ is a birational morphism, we say that a plane curve C passes through p_1 if C passes through p_2 and the strict transform of C on X_1 via π_1 passes through p_1.

Proposition 1.38 (Proximity inequality)**.** *Let $\varphi : S \to \mathbb{P}^2$ be a birational morphism, that is the composition of the blowing-ups $\pi_1, \dots, \pi_r$ such as in Definition 1.29. Let C be a plane* *curve and let C_i be the strict transform of C in S_i for $i = 1, \ldots, r$. Setting $C_0 = C$ and $m_i = \mathrm{mult}_{p_i}(C_{i-1})$ for $i = 1, \ldots, r$, one has, for each $j = 1, \ldots, r$,*

$$m_j \geqslant \sum_{p_k \dashrightarrow p_j} m_k.$$

Proof. See §2.2 in [1] or Theorem 3.5.3, Corollary 3.5.4 in [9]. □

1.3 Standard coordinates of infinitely near points

In this section, we want to give a way to describe infinitely near points that we call *standard coordinates*.

Let $p_1 = [a : b : c] \in \mathbb{P}^2$. Let us consider three cases:

(*i*) if $c \neq 0$, then $p_1 = \left[\frac{a}{c} : \frac{b}{c} : 1\right] = [\overline{a} : \overline{b} : 1]$;

(*ii*) if $c = 0$ and $b \neq 0$, then $p_1 = \left[\frac{a}{b} : 1 : 0\right] = [\overline{a} : 1 : 0]$;

(*iii*) if $c = b = 0$, then $p_1 = [1 : 0 : 0]$.

In case (*i*), we work on the affine chart $U_2 \simeq \mathbb{C}^2_{\overline{x},\overline{y}}$, so that p_1 corresponds to the point $\overline{p}_1 = (\overline{a}, \overline{b})$, and we define the isomorphism $\alpha_1 \colon \mathbb{C}^2_{\overline{x},\overline{y}} \to \mathbb{C}^2_{x_0,y_0}$ by

$$\alpha_1(\overline{x}, \overline{y}) = (\overline{x} - \overline{a}, \overline{y} - \overline{b}).$$

In case (*ii*), we work on the affine chart $U_1 \simeq \mathbb{C}^2_{\overline{x},\overline{z}}$, so that p_1 corresponds to the point $\overline{p}_1 = (\overline{a}, 0)$, and we define the isomorphism $\alpha_1 \colon \mathbb{C}^2_{\overline{x},\overline{z}} \to \mathbb{C}^2_{x_0,y_0}$ by

$$\alpha_1(\overline{x}, \overline{z}) = (\overline{x} - \overline{a}, \overline{z}).$$

In case (*iii*), we work on the affine chart $U_0 \simeq \mathbb{C}^2_{\overline{y},\overline{z}}$, so that p_1 corresponds to the point $\bar{p}_1 = (0, 0)$, and we define the isomorphism $\alpha_1 \colon \mathbb{C}^2_{\overline{y},\overline{z}} \to \mathbb{C}^2_{x_0,y_0}$ by

$$\alpha_1(\overline{y}, \overline{z}) = (\overline{y}, \overline{z}).$$

In all three cases, we defined α_1 in such a way that $\alpha_1(\overline{p}_1) = (0, 0) \in \mathbb{C}^2_{x_0,y_0}$.

We blow-up $\mathbb{C}^2_{x_0,y_0}$ at $(0, 0)$ and we consider the first chart $\mathbb{C}^2_{x_1,y_1}$ where the blowing-up map is given in coordinates by $x_0 = x_1, y_0 = x_1 y_1$, cf. Definition 1.22.

In this chart, the exceptional curve E_1 has local equation $x_1 = 0$, hence a point $p_2 \succ_1 p_1$ corresponds either to the point $(0, t_2) \in E_1$ with $t_2 \in \mathbb{C}$ or to the point which is the origin of the second chart. In the former case, let us say that p_2 has standard coordinates $p_2 = (p_1, t_2)$, while in the latter case let us say that p_2 has standard coordinates $p_2 = (p_1, \infty)$. Setting $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, let us denote the standard coordinates of p_2 by $p_2 = (p_1, t_2)$ with $t_2 \in \mathbb{P}^1$.

Remark 1.39. *Recall that a point $p_2 \succ_1 p_1$ corresponds to the direction of a line passing through p_1. More precisely, one can see that the point $p_2 = (p_1, t_2)$, with $p_1 = [a : b : c]$, corresponds to the line defined by the following equation*

$$\begin{cases} cy - bz = t_2(cx - az) & \text{when } c \neq 0 \text{ and } t_2 \in \mathbb{C}, \\ cx - az = 0 & \text{when } c \neq 0 \text{ and } t_2 = \infty, \\ bz = t_2(bx - ay) & \text{when } c = 0, b \neq 0 \text{ and } t_2 \in \mathbb{C}, \\ bx = ay & \text{when } c = 0, b \neq 0 \text{ and } t_2 = \infty, \\ z = t_2 y & \text{when } b = c = 0 \text{ and } t_2 \in \mathbb{C}, \\ y = 0 & \text{when } b = c = 0 \text{ and } t_2 = \infty. \end{cases}$$

In other words, the above equations define the unique line passing through p_1 and p_2.

We want to go on by blowing-up at $p_2 = (p_1, t_2)$, with $t_2 \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Either $t_2 \in \mathbb{C}$ or $t_2 = \infty$. In the former case, with notation as above, let $\alpha_2 \colon \mathbb{C}^2_{x_1,y_1} \to \mathbb{C}^2_{\bar{x}_1,\bar{y}_1}$ be the isomorphism defined by

$$\alpha_2(x_1, y_1) = (x_1, y_1 - t_2).$$

In the latter case, p_2 corresponds to the origin of the second chart of the blowing-up of $\mathbb{C}^2_{x_0,y_0}$ at $(0,0)$ that we write $\mathbb{C}^2_{x_1',y_1'}$, where the blowing-up map is given by $x_0 = x_1' y_1', y_0 = y_1'$. Let $\alpha_2 \colon \mathbb{C}^2_{x_1',y_1'} \to \mathbb{C}^2_{\bar{x}_1,\bar{y}_1}$ be the isomorphism

$$\alpha_2(x_1', y_1') = (y_1', x_1').$$

In this way, in both cases, in $\mathbb{C}^2_{\bar{x}_1,\bar{y}_1}$ the exceptional curve E_1 has local equation $\bar{x}_1 = 0$ and the point p_2 corresponds to the origin $(0,0)$.

We blow-up $\mathbb{C}^2_{\bar{x}_1,\bar{y}_1}$ at $(0,0)$ and we consider the first chart $\mathbb{C}^2_{x_2,y_2}$ where the blowing-up map is given in coordinates by $\bar{x}_1 = x_2, \bar{y}_1 = x_2 y_2$. In this chart, the exceptional curve E_2 has local equation $x_2 = 0$, hence a point $p_3 \succ_1 p_2$ corresponds either to the point $(0, t_3) \in E_2$ with $t_3 \in \mathbb{C}$ or to the point which is the origin of the second chart.

Let us say that p_3 has standard coordinates $p_3 = (p_1, t_2, t_3)$, where either $t_3 \in \mathbb{C}$ in the former case or $t_3 = \infty$ in the latter case.

Note that the strict transform of E_1 can be seen only in the second chart and it meets E_2 at the origin of the second chart. In other words, the point with standard coordinates (p_1, t_2, ∞) is satellite to p_1.

More generally, let us proceed by induction of the infinitesimal order. Suppose that we have blown-up the point p_{r-1} with standard coordinates $p_{r-1} = (p_1, t_2, \ldots, t_{r-1})$, with $t_i \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, $i = 2, \ldots, r-1$. Following the procedure described above, we may assume that p_{r-1} is the origin of a chart $\mathbb{C}^2_{\bar{x}_{r-1},\bar{y}_{r-1}}$ in such a way that the exceptional curve E_{r-1} has local equation $\bar{x}_{r-1} = 0$.

In the first chart of the blowing up of $\mathbb{C}^2_{\bar{x}_{r-1},\bar{y}_{r-1}}$ at $(0,0)$, given in coordinates by $\bar{x}_{r-1} = x_r, \bar{y}_{r-1} = x_r y_r$, the exceptional curve E_r has local equation $x_r = 0$, hence a point $p_r \succ_1 p_{r-1}$

corresponds either to the point $(0,t_r) \in E_r$ with $t_r \in \mathbb{C}$ or to the point which is the origin of the second chart, given in coordinates by $\bar{x}_{r-1} = x_r y_r, \bar{y}_{r-1} = y_r$.

Let us say that p_r has standard coordinates $p_r = (p_1, t_2, \dots, t_r)$, where $t_r \in \mathbb{C}$ in the former case and $t_r = \infty$ in the latter case.

The above discussion proves the following:

Lemma 1.40. *Let $p_1 \in \mathbb{P}^2$. Then, there is a one-to-one correspondence between points infinitely near p_1 of order r and $(\mathbb{P}^1)^r = \underbrace{\mathbb{P}^1 \times \ldots \times \mathbb{P}^1}_{r\text{-times}}$.*

Corollary 1.41. *There is a one-to-one correspondence between points infinitely near a proper point of order r and $W = \mathbb{P}^2 \times (\mathbb{P}^1)^r$.*

Definition 1.42. We call *standard coordinates* of an infinitely near point the point of W obtained with the above construction.

Example 1.43. Let C be the conic in $\mathbb{P}^2$ defined by $2xy + 3yz - z^2 = 0$. A point of C is $p_1 = [-1:1:2]$. We claim that C passes through the points with standard coordinates

$$p_2 = \left(p_1, -\frac{1}{2}\right), \quad p_3 = \left(p_1, -\frac{1}{2}, \frac{1}{2}\right), \quad \left(p_1, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right), \quad \left(p_1, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right),$$

and so on.

In the affine chart $U_2 \simeq \mathbb{C}^2_{\bar{x},\bar{y}}$, the point p_1 corresponds to the point $\bar{p}_1 = (-1/2, 1/2)$ and C is locally defined by $2\bar{x}\bar{y} + 3\bar{y} - 1 = 0$. The isomorphism $\alpha_1 \colon \mathbb{C}^2_{\bar{x},\bar{y}} \to \mathbb{C}^2_{x_0,y_0}$ defined by $\alpha_1(\bar{x},\bar{y}) = (\bar{x} + 1/2, \bar{y} - 1/2)$ is such that $\alpha_1(\bar{p}_1) = (0,0)$ and C is locally defined in $\mathbb{C}^2_{x_0,y_0}$ by

$$2x_0y_0 + x_0 + 2y_0 = 0. \tag{1.2}$$

In the first chart of the blow-up of $\mathbb{C}^2_{x_0,y_0}$ at $(0,0)$, given in coordinates by $x_0 = x_1, y_0 = x_1y_1$, the strict transform of C has local equation $2x_1y_1 + 2y_1 + 1 = 0$, so that it passes through the point $(0, -1/2)$ and we say that C passes through the point $p_2 \succ_1 p_1$ with standard coordinates $p_2 = (p_1, -1/2)$.

Let $\alpha_2 \colon \mathbb{C}^2_{x_1,y_1} \to \mathbb{C}^2_{\bar{x}_1,\bar{y}_1}$ be the isomorphism $\alpha_2(x_1,y_1) = (x_1, y_1 + 1/2)$. In the first chart of the blow-up of $\mathbb{C}^2_{\bar{x}_1,\bar{y}_1}$ at $(0,0)$, given in coordinates by $\bar{x}_1 = x_2, \bar{y}_1 = x_2y_2$, the strict transform of C has local equation $2x_2y_2 - x_2 + 2y_2 = 0$ so that it passes through the point $(0, 1/2)$ and we say that C passes through the point $p_3 \succ_1 p_2$ with standard coordinates $p_3 = (p_1, -1/2, 1/2)$.

Let $\alpha_3 \colon \mathbb{C}^2_{x_2,y_2} \to \mathbb{C}^2_{\bar{x}_2,\bar{y}_2}$ be the isomorphism $\alpha_3(x_2,y_2) = (x_2, y_2 - 1/2)$. In the first chart of the blow-up of $\mathbb{C}^2_{\bar{x}_2,\bar{y}_2}$ at $(0,0)$, given in coordinates by $\bar{x}_2 = x_3, \bar{y}_2 = x_3y_3$, the strict transform of C has local equation

$$2x_3y_3 + x_3 + 2y_3 = 0$$

that is the same equation (1.2), replacing x_3 with x_0 and y_3 with y_0. It follows that the subsequent infinitely near points have standard coordinates $(p_1, -1/2, 1/2, -1/2, 1/2, \ldots)$.

Example 1.44. Let us denote by $F(n)$ the n-th Fibonacci number, starting from $F(0) = F(1) = 1$, then $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$.

For $n \geq 1$, let C_n be the curve in $\mathbb{P}^2$ defined by $x^{F(n)}y^{F(n+1)} - z^{F(n+2)} = 0$. The curve C_n has a singular point of multiplicity $F(n+1)$ at $p_1 = [1:0:0]$. We claim that C_n passes through the points $p_2, p_3, \ldots, p_{n+1}$ with respective standard coordinates

$$p_2 = (p_1, \infty), \quad p_3 = (p_1, \infty, \infty), \quad p_4 = (p_1, \infty, \infty, \infty), \quad \ldots, \quad p_{n+1} = (p_1, \underbrace{\infty, \ldots, \infty}_{n \text{ times}}),$$

with respective multiplicities $F(n)$, $F(n-1)$, ..., $F(0)$. In particular, for $n \geq 3$, one has that $p_n \odot p_{n-2}$.

We prove the claim by induction on n. For $n = 1$, the curve C_1 has equation $xy^2 - z^3 = 0$, so C_1 has a cusp at p_1 with cuspidal tangent the line $y = 0$, so the strict transform of C_1 passes through p_2 with standard coordinates (p_1, ∞) and it passes through $p_3 = (p_1, \infty, \infty)$. Note that $p_3 \odot p_1$.

For $n \geq 2$, in the affine chart $U_0 \simeq \mathbb{C}^2_{\bar{y},\bar{z}}$, the point p_1 corresponds to the origin $\bar{p}_1 = (0,0)$ and C_n is locally defined by $\bar{y}^{F(n+1)} - \bar{z}^{F(n+2)} = 0$. The isomorphism $\alpha_1 \colon \mathbb{C}^2_{\bar{y},\bar{z}} \to \mathbb{C}^2_{x_0,y_0}$ defined by $\alpha_1(\bar{y}, \bar{z}) = (\bar{y}, \bar{z})$ is such that $\alpha_1(\bar{p}_1) = (0,0)$ and C_n is locally defined in $\mathbb{C}^2_{x_0,y_0}$ by

$$x_0^{F(n+1)} - y_0^{F(n+2)} = 0.$$

In the second chart of the blow-up of $\mathbb{C}^2_{x_0,y_0}$ at $(0,0)$, given in coordinates by $x_0 = x_1y_1, y_0 = y_1$, the strict transform of C_n has local equation

$$x_1^{F(n+1)} - y_1^{F(n)} = 0,$$

so that it has multiplicity $F(n)$ at the origin $(0,0)$, that is the point with standard coordinates $p_2 = (p_1, \infty)$.

Let $\alpha_2 \colon \mathbb{C}^2_{x_1,y_1} \to \mathbb{C}^2_{\bar{x}_1,\bar{y}_1}$ be the isomorphism $\alpha_2(x_1, y_1) = (y_1, x_1)$. In $\mathbb{C}^2_{\bar{x}_1,\bar{y}_1}$, the strict transform of C_n has local equation

$$\bar{x}_1^{F(n)} - \bar{y}_1^{F(n+1)} = 0.$$

In the second chart of the blow-up of $\mathbb{C}^2_{\bar{x}_1,\bar{y}_1}$ at $(0,0)$, given in coordinates by $\bar{x}_1 = x_2y_2, \bar{y}_1 = y_2$, the strict transform of C_n has local equation

$$x_2^{F(n)} - y_2^{F(n-1)} = 0,$$

so that it has multiplicity $F(n-1)$ at the origin $(0,0)$, that is the point with standard coordinates $p_3 = (p_1, \infty, \infty)$.

Let $\alpha_3 \colon \mathbb{C}^2_{x_2,y_2} \to \mathbb{C}^2_{\bar{x}_2,\bar{y}_2}$ be the isomorphism $\alpha_3(x_2, y_2) = (y_2, x_2)$. In $\mathbb{C}^2_{\bar{x}_2,\bar{y}_2}$, the strict transform of C_n has local equation

$$x_2^{F(n-1)} - y_2^{F(n)} = 0,$$

and we conclude by the induction hypothesis.

1.3.1 Conics and infinitely near points

Remark 1.45. *If $p_1 \in \mathbb{P}^2, p_3 \succ_1 p_2 \succ_1 p_1$ and $p_3 \odot p_1$, i.e. $p_3 \dashrightarrow p_1$, then there is no smooth curve passing through p_1, p_2, p_3 because of the proximity inequality at p_1.*

Lemma 1.46. *If $p_1 \in \mathbb{P}^2, p_3 \succ_1 p_2 \succ_1 p_1$ and p_1, p_2, p_3 are collinear, namely p_3 lies on the strict transform of the line passing through p_1 and p_2, then there is no irreducible conic passing through p_1, p_2, p_3.*

Proof. Up to automorphisms of $\mathbb{P}^2$, we may assume that $p_1 = [1:0:0]$ and $p_2 = (p_1, 0)$, so p_3 is uniquely determined by p_1, p_2, namely $p_3 = (p_1, 0, 0)$.
Suppose that C is an irreducible conic passing through p_1, p_2. Then, C has equation

$$a_2 y^2 + a_3 xz + a_4 yz + a_5 z^2 = 0$$

where $a_2, a_3, a_4, a_5 \in \mathbb{C}$ and $a_2, a_3 \neq 0$ because C is irreducible.
We work in the affine chart $U_0 \simeq \mathbb{C}^2_{\bar{y},\bar{z}}$ and we consider the isomorphism $\alpha_1 \colon \mathbb{C}^2_{\bar{y},\bar{z}} \to \mathbb{C}^2_{x_0,y_0}$ defined by $\alpha_1(\bar{y}, \bar{z}) = (\bar{y}, \bar{z})$, where the conic C has local equation

$$a_2 x_0^2 + a_3 y_0 + a_4 x_0 y_0 + a_5 y_0^2 = 0.$$

In the first chart of the blowing-up of $\mathbb{C}^2_{x_0,y_0}$ at the origin $(0,0)$, where $x_0 = x_1, y_0 = x_1 y_1$, the strict transform of C has local equation

$$a_2 x_1 + a_3 y_1 + a_4 x_1 y_1 + a_5 x_1 y_1^2 = 0.$$

Note that p_2 is just the origin of $\mathbb{C}^2_{x_1,y_1}$.
Then, the strict transform of C via the blowing-up of $\mathbb{C}^2_{x_1,y_1}$ at the origin $(0,0)$ has local equation in the first chart, where $x_1 = x_2, y_1 = x_2 y_2$,

$$a_2 + a_3 y_2 + a_4 x_2 y_2 + a_5 x_2 y_2^2 = 0.$$

Note that p_3 is just the origin of $\mathbb{C}^2_{y_2,z_2}$ but the strict transform of C does not pass through $(0,0)$ because $a_2 \neq 0$. □

Remark 1.47. *It is easy to check that if $p_1 \in \mathbb{P}^2, p_3 \succ_1 p_2 \succ_1 p_1$, $p_3 \not\odot p_1$ and p_1, p_2, p_3 are not collinear, then there are irreducible conics passing through p_1, p_2, p_3.*

Remark 1.48. *Note that if $p_1 \in \mathbb{P}^2$, $p_2 \succ_1 p_1$, $p_3 \succ_1 p_1$ and $p_2 \neq p_3$, then there is no irreducible conic passing through p_1, p_2, p_3.*

Lemma 1.49. *Let $p_1, p_2, p_3, p_4 \in \mathbb{P}^2$ and $p_5 \succ_1 p_1$ such that no three among $p_1, \ldots, p_5$ are collinear. Then, there exists a unique irreducible conic passing through $p_1, \ldots, p_5$.*

Proof. Up to automorphisms of $\mathbb{P}^2$, we may assume that $p_1 = [1:0:0], p_2 = [0:1:0], p_3 = [0:0:1], p_4 = [1:1:1]$. Then, p_5 has standard coordinates $p_5 = (p_1, t_5)$, namely p_5 is infinitely near p_1 of the first order in the direction of the line $z - t_5 y = 0$, where $t_5 \in \mathbb{C} \backslash \{0, 1\}$:

indeed, if $t_5 = 0$, then p_5, p_2, p_1 would be collinear; if $t_5 = 1$, then p_5, p_4, p_1 would be collinear and finally, if $t_5 = \infty$, then p_5, p_3, p_1 would be collinear. Then, one can check that the conic

$$xz - t_5xy + (t_5 - 1)yz = 0$$

is the unique irreducible conic passing through $p_1, \ldots, p_5$. $\square$

Lemma 1.50. *Let $p_1, p_2, p_3 \in \mathbb{P}^2$ and $p_5 \succ_1 p_4 \succ_1 p_1$ such that $p_5 \not\odot p_1$ and no three among $p_1, \ldots, p_5$ are collinear. Then, there exists a unique irreducible conic passing through $p_1, \ldots, p_5$.*

Proof. Up to automorphisms of $\mathbb{P}^2$, we may assume that $p_1 = [1 : 0 : 0], p_2 = [0 : 1 : 0], p_3 = [0 : 0 : 1]$ and that p_4 has standard coordinates $p_4 = (p_1, 1)$, namely p_4 is infinitely near p_1 of the first order in the direction of the line $y = z$. Then, p_5 has standard coordinates $p_5 = (p_1, 1, t_5)$, where $t_5 \in \mathbb{C}^*$: indeed, if $t_5 = 0$ then p_5, p_4, p_1 would be collinear and if $t_5 = \infty$, then $p_5 \odot p_1$. Then, one can check that the conic

$$xz - xy - t_5yz = 0$$

is the unique irreducible conic passing through $p_1, \ldots, p_5$. $\square$

Lemma 1.51. *Let $p_1, p_2, p_3 \in \mathbb{P}^2$ and $p_4 \succ_1 p_1, p_5 \succ_1 p_2$ such that no three among $p_1, \ldots, p_5$ are collinear. Then, there exists a unique irreducible conic passing through $p_1, \ldots, p_5$.*

Proof. Up to automorphisms of $\mathbb{P}^2$, we may assume that $p_1 = [1 : 0 : 0], p_2 = [0 : 1 : 0], p_3 = [0 : 0 : 1]$ and that the two lines, one through p_1, p_4 and the other one through p_2, p_5, meet at $[1 : 1 : 1]$, namely p_4 is infinitely near p_1 of the first order in the direction of the line $y = z$ and p_5 is infinitely near p_2 of the first order in the direction of the line $x = z$. In other words, p_4 has standard coordinates $p_4 = (p_1, 1)$ and p_5 has standard coordinates $p_5 = (p_2, 1)$. Then, it is clear that the conic

$$xy - yz - xz = 0$$

is the unique irreducible conic passing through $p_1, \ldots, p_5$. $\square$

Lemma 1.52. *Let $p_1, p_2 \in \mathbb{P}^2$ and $p_5 \succ_1 p_3 \succ_1 p_1, p_4 \succ_1 p_2$ such that $p_5 \not\odot p_1$ and no three among $p_1, \ldots, p_5$ are collinear. Then, there exists a unique irreducible conic passing through $p_1, \ldots, p_5$.*

Proof. Up to automorphisms of $\mathbb{P}^2$, we may assume that $p_1 = [1 : 0 : 0], p_2 = [0 : 1 : 0]$, and that the two lines, one through p_1, p_3 and the other one through p_2, p_4, meet at $[0 : 0 : 1]$, namely p_3 is infinitely near p_1 of the first order in the direction of the line $y = 0$ and p_4 is infinitely near p_2 of the first order in the direction of the line $x = 0$. In other words, p_3 has standard coordinates $p_3 = (p_1, \infty)$ and p_4 has standard coordinates $p_4 = (p_2, \infty)$. Then, p_5 has standard coordinates $p_5 = (p_1, \infty, t_5)$ where $t_5 \in \mathbb{C}^*$: indeed, if $t_5 = 0$ then p_5, p_3, p_1 would be collinear and if $t_5 = \infty$, then $p_5 \odot p_1$. One can check that the conic

$$t_5xy - z^2 = 0$$

is the unique irreducible conic passing through $p_1, \ldots, p_5$. $\square$

Remark 1.53. *The previous lemmas are a more precise explanation of Remark* 4.2.1 *in Chapter V in [19].*

Lemma 1.54. *Let $p_1, p_2 \in \mathbb{P}^2$ and $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_1$ such that $p_4 \not\odot p_1, p_5 \not\odot p_3$ and no three among $p_1, \ldots, p_4$ are collinear. Then, there exists a unique irreducible conic passing through $p_1, \ldots, p_5$.*

Proof. Up to automorphisms of $\mathbb{P}^2$, we may assume that $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$ and p_3, p_4 have standard coordinates respectively $p_3 = (p_1, \infty)$ and $p_4 = (p_1, \infty, 1)$, according to the proof of the previous lemma. Then, p_5 has standard coordinates $p_5 = (p_1, \infty, 1, t_5)$ where $t_5 \in \mathbb{C}$: indeed, if $p_5 = \infty$, then we would have $p_5 \odot p_3$, contradicting the hypothesis. One can check that the conic

$$xy + t_5 yz - z^2 = 0$$

is the unique irreducible conic passing through $p_1, \ldots, p_5$. $\square$

Lemma 1.55. *Let $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1 \in \mathbb{P}^2$ such that $p_3 \not\odot p_1, p_4 \not\odot p_2, p_5 \not\odot p_3$ and p_1, p_2, p_3 are not collinear. Then, there exists a unique irreducible conic passing through $p_1, \ldots, p_5$.*

Proof. Up to automorphisms of $\mathbb{P}^2$, we may assume that $p_1 = [1 : 0 : 0]$ and p_2, p_3, p_4 have standard coordinates respectively $p_2 = (p_1, \infty)$, $p_3 = (p_1, \infty, 1)$, $p_4 = (p_1, \infty, 1, 0)$, according to the proof of the previous lemma. Then, p_5 has standard coordinates $p_5 = (p_1, \infty, 1, 0, t_5)$ where $t_5 \in \mathbb{C}$: indeed, if $t_5 = \infty$, then we would have $p_5 \odot p_3$, contradicting the hypothesis. One can check that the conic

$$xy - z^2 + t_5 y^2 = 0$$

is the unique irreducible conic passing through $p_1, \ldots, p_5$. $\square$

1.4 Plane Cremona maps

The *plane Cremona group*, denoted by $\mathrm{Cr}(\mathbb{P}^2)$ or $\mathrm{Bir}(\mathbb{P}^2)$, is the group of birational maps of the projective plane $\mathbb{P}^2$ into itself. Such maps can be written as the following form

$$\varphi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \qquad [x : y : z] \mapsto [\varphi_0(x, y, z) : \varphi_1(x, y, z) : \varphi_2(x, y, z)] \tag{1.3}$$

where $\varphi_i \in \mathbb{C}[x, y, z]_d$ for any $i = 0, 1, 2$ are homogeneous polynomials of the same degree d, that is called the *degree* of φ if $\varphi_0, \varphi_1, \varphi_2$ have no common factor. Usually, abusing of notation, let us write (1.3) as $\varphi = [\varphi_0 : \varphi_1 : \varphi_2]$.

Plane Cremona maps of degree 1 are automorphisms of $\mathbb{P}^2$, i.e. elements of $\mathrm{Aut}(\mathbb{P}^2) \simeq \mathrm{PGL}_3$. Plane Cremona maps of degree 2 (3, 4, resp.) are called *quadratic* (*cubic*, *quartic*, resp.). The *elementary* quadratic transformation is:

$$\sigma \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \qquad [x : y : z] \mapsto [yz : xz : xy]. \tag{1.4}$$

The fundamental result concerning the plane Cremona group is the following theorem:

Theorem 1.56 (Noether-Castelnuovo)**.** *The group* $\mathrm{Cr}(\mathbb{P}^2)$ *is generated by* $\mathrm{Aut}(\mathbb{P}^2)$ *and* σ*.*

Proof. See [10] or [1] for a modern reference. □

Let $\varphi \in \mathrm{Cr}(\mathbb{P}^2)$ be a plane Cremona map of degree d. Then, let $p_1, \dots, p_n \in \mathbb{P}^2$ be the (proper) base points of the net (linear system of dimension 2) Λ defining φ. According to Theorem 1.33, there exist a surface Z and birational morphisms $\pi_1 \colon Z \to \mathbb{P}^2$ and $\pi_2 \colon Z \to \mathbb{P}^2$ such that $\pi_2 = \varphi \circ \pi_1$. The birational morphism $\pi_1 \colon Z \to \mathbb{P}^2$ is the sequence of blowing-up maps at points $p_1, \dots, p_n$ and $p_{n+1}, \dots, p_r \in \mathcal{B}(\mathbb{P}^2)$ as in Section 1.2. Denote by $m_1, \dots, m_r$ the multiplicities of $p_1, \dots, p_r$ of the net Λ, namely the multiplicities at $p_1, \dots, p_r$ (of the strict transform) of a general curve of the net Λ. With a little abuse of notation, let us say that $p_1, \dots, p_r$ are the base points of φ with respective multiplicities $m_1, \dots, m_r$, and let us write $m_i = \mathrm{mult}_{p_i}(\varphi)$ for $i = 1, \dots, r$. Then, it is classically known that (see e.g [1, §2.5]),

$$d^2 - 1 = \sum_{i=1}^{r} m_i^2, \qquad 3(d-1) = \sum_{i=1}^{r} m_i, \tag{1.5}$$

and $(d; m_1, \dots, m_r)$ is called the *characteristic* of φ.

Recall that not all solutions $(d; m_1, \dots, m_r)$ of conditions (1.5) are characteristic of a plane Cremona map (see e.g [1, §5.2]).

Definition 1.57. A plane Cremona map φ is called *de Jonquières* if it has degree d and a base point of multiplicity $d-1$.

Equations (1.5) imply that plane Cremona maps of degree 2 and 3 are de Jonquières.

Definition 1.58. A plane Cremona map φ is called *involutory*, or an *involution*, if $\varphi^{-1} = \varphi$.

Definition 1.59. Let us say that two plane Cremona maps $\varphi, \varphi' \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ are *equivalent* if there exist two automorphisms $\alpha, \alpha' \in \mathrm{Aut}(\mathbb{P}^2)$ such that

$$\varphi' = \alpha' \circ \varphi \circ \alpha.$$

Remark 1.60. *The automorphism* α' *changes the basis of the homaloidal net defining* φ*, while* α *changes the position of the base points of the map. In particular, two plane Cremona maps defined by the same homaloidal net are equivalent.*

1.4.1 Quadratic plane Cremona maps

We have already defined the elementay quadratic transformation σ in (1.4). The map σ is clearly an involution and it has the coordinate points as base points of multiplicity 1.

Definition 1.61. Let us say that a quadratic plane Cremona map φ is *ordinary* if φ has three proper base points.

Remark 1.62. *Let p_1, p_2, p_3 be the proper base points of an ordinary quadratic map φ. Since there exists an automorphism $\alpha\colon \mathbb{P}^2 \to \mathbb{P}^2$ that maps p_1, p_2, p_3 to the coordinate points, it follows that φ is equivalent to σ.*

On the other hand, a plane Cremona map equivalent to σ is clearly ordinary quadratic.

Remark 1.63. *For each $\alpha \in \mathrm{Aut}(\mathbb{P}^2)$, the map $\alpha^{-1} \circ \sigma \circ \alpha$ is involutory ordinary quadratic, but not all involutory ordinary quadratic maps have this form, like e.g. the map $\varphi = [yz : xy : xz]$, cf. [24].*

There are other two fundamental quadratic maps, which are not ordinary.

Example 1.64. The quadratic map

$$\rho\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \qquad [x : y : z] \mapsto [xy : z^2 : yz], \tag{1.6}$$

is an involution which is not ordinary, namely ρ has two proper base points $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$ and the third base point p_3 is the point infinitely near p_1 with standard coordinates $p_3 = (p_1, \infty)$, that is the point in the direction of the line $y = 0$.

Example 1.65. The quadratic map

$$\tau\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \qquad [x : y : z] \mapsto [x^2 : xy : y^2 - xz], \tag{1.7}$$

is an involution which has only one proper base point, that is $p_1 = [0 : 0 : 1]$, while the other two base points p_2 and p_3 are infinitely near p_1 and they have standard coordinates respectively $p_2 = (p_1, \infty)$ and $p_3 = (p_1, \infty, 1)$.

Remark 1.66. *It is classical well-known that any quadratic plane Cremona map is equivalent to one and only one among σ, ρ and τ.*

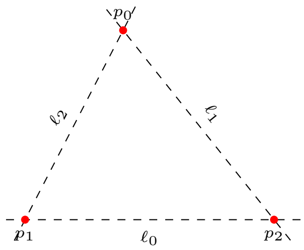
More generally, one can see that the set of quadratic plane Cremona maps has a natural structure of quasi-projective variety of dimension 14 in $\mathbb{P}^{17}$, whose properties have been extensively studied by Cerveau and Déserti in [11].

Definition 1.67. Let us say that a quadratic plane Cremona map φ is

- *of the second type* if φ is equivalent to ρ;
- *of the third type* if φ is equivalent to τ.

In the next sections, we will need to construct examples of quadratic plane Cremona maps with some given property. Let us now see some of these constructions.

Example 1.68. Let p_0, p_1, p_2 be three non-collinear points in $\mathbb{P}^2$. An involutory ordinary quadratic plane Cremona map based at p_0, p_1, p_2 can be easily constructed as follows.

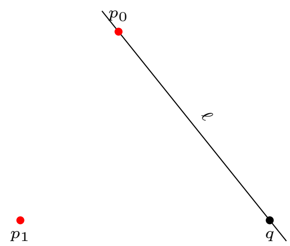


Suppose that the coordinates of p_0, p_1, p_2 are respectively $[a_1 : a_2 : a_3]$, $[b_1 : b_2 : b_3]$ and $[c_1 : c_2 : c_3]$. Let α be the automorphism of $\mathbb{P}^2$ associated to the matrix

$$M = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \in \mathrm{PGL}_3, \tag{1.8}$$

where $\det(M) \neq 0$ because p_0, p_1, p_2 are not aligned. Then, the plane Cremona map φ defined by $\varphi = \alpha \circ \sigma \circ \alpha^{-1}$ is an ordinary, involutory quadratic map based at p_0, p_1, p_2.

Example 1.69. Let p_0, p_1 be two distinct points in $\mathbb{P}^2$ and let p_2 be infinitely near p_0 in the direction of a line ℓ, not passing through p_1. An involutory quadratic plane Cremona map based at p_0, p_1, p_2 can be constructed as follows.



Suppose that the coordinates of p_0, p_1 are respectively $[a_1 : a_2 : a_3]$, $[b_1 : b_2 : b_3]$. Choose a point $q = [c_1 : c_2 : c_3]$ on ℓ different from p_0. Let α be the automorphism of $\mathbb{P}^2$ associated to the matrix M as in (1.8), that has $\det(M) \neq 0$ because p_0, p_1, q are not aligned. Then, the plane Cremona map φ defined by $\varphi = \alpha \circ \rho \circ \alpha^{-1}$ is an involutory quadratic map based at p_0, p_1, p_2.

We need to know the behaviour of plane Cremona maps under the composition with ordinary quadratic maps. A first result is the following classical proposition.

Proposition 1.70. *Let p_1, p_2, p_3 be the base points of an involutory ordinary quadratic plane Cremona map $\varrho \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Let $\varphi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a plane Cremona map of degree $d > 1$ with base points $p_4, \dots, p_r$ and possibly p_1, p_2, p_3. Denote by m_i the multiplicity of φ at p_i, $i = 1, \dots, r$ (that is $m_i = 0$ if p_i is not a base point of φ, $i = 1, 2, 3$). Suppose, moreover, that $p_4, \dots, p_r$ are proper points not lying on the triangle with vertices p_1, p_2, p_3.*

Then, the composite map $\varphi \circ \varrho^{-1} = \varphi \circ \varrho$ has degree $d - \varepsilon$, where

$$\varepsilon = m_1 + m_2 + m_3 - d,$$

and it has $\varrho(p_i)$, $i = 4, \dots, r$, as base points of multiplicity m_i. Furthermore, it has multiplicity $m_i - \varepsilon \geq 0$ at p_i, $i = 1, 2, 3$ (that is, p_i is not a base point of $\varphi \circ \varrho$ when $\varepsilon = m_i$).

Proof. See, e.g., Corollary 4.2.6 in [1]. □

Proposition 1.71. *Let p_1, p_2, p_3 be the base points of a quadratic plane Cremona map $\varrho \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Let $\varphi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a plane Cremona map of degree $d > 1$ with base points*

$p_4, \ldots, p_r$ and possibly p_1, p_2, p_3. Denote by m_i the multiplicity of φ at p_i, $i = 1, \ldots, r$ (that is $m_i = 0$ if p_i is not a base point of φ, $i = 1, 2, 3$).

Then, the composite map $\varphi \circ \varrho^{-1}$ has degree $d - \varepsilon$, where

$$\varepsilon = m_1 + m_2 + m_3 - d.$$

Proof. See, e.g., Proposition 4.2.5 in [1]. □

We will later see what happens when the base points of φ are either infinitely near or belonging to the triangle with vertices p_1, p_2, p_3.

Chapter 2

Weighted proximity graphs of plane Cremona maps

In this chapter we first recall the definition and the main properties of the proximity matrices and the admissible oriented graphs which encode sequences of blowing-ups. We then define the weighted proximity graph of a given plane Cremona map, starting from the proximity properties of the base points of the Cremona map. For small degree maps, we finally introduce the enriched weighted proximity graph that we will use to classify equivalence classes of plane Cremona maps.

2.1 Admissible digraphs

For notation and definitions about directed graphs, see e.g. [2]. For more properties of admissible graphs, we refer to [7, Chap. 1].

Definition 2.1. A *directed graph*, or briefly *digraph*, G is a pair $G = (V, F)$ where V is a finite set of elements, called vertices, and F is a set of ordered pairs of distinct elements of V. An element $(u, v) \in F$ where $u, v \in V$ is denoted by $u \to v$, and it is called an *arc*, or an *arrow*, from u to v.

Remark 2.2. *According to Definition 2.1, a digraph has no loop, i.e. an arrow $u \to u$ where u is a vertex, and it has no multiple arcs between the same vertices.*

Definition 2.3. Let $G = (V, F)$ be a digraph. Then the *external degree* and *internal degree* of a vertex v of G are respectively defined as follows:

$$\mathrm{outdeg}(v) = \sharp\big\{u \in V \big| v \to u\big\}, \qquad \mathrm{indeg}(v) = \sharp\big\{u \in V \big| u \to v\big\}.$$

Definition 2.4. Let $G = (V, F)$ be a digraph. Choose a bijection $\psi : \big\{1, \ldots, n\big\} \to V$, where $n = \sharp V$ is the number of vertices of G. Then the $(n \times n)$-matrix $A_G = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } \psi(i) \to \psi(j), \\ 0 & \text{otherwise} \end{cases}$$

is called the *adjacency matrix of* G with respect to ψ.

Definition 2.5. A digraph $G = (V, F)$ is called *acyclic* if it has no cycle.

Remark 2.6. *Let $G = (V, F)$ be an acyclic digraph. Then, G has at least one vertex of external degree* 0*, see Proposition* 1.4.2 *in [2].*

Remark 2.7. *Let $G = (V, F)$ be an acyclic digraph. Then, there exists an ordering of the vertices of G such that the adjacency matrix A_G is a strictly lower triangular matrix, see Proposition* 1.4.3 *in [2].*

Definition 2.8. Two digraphs $G = (V, F)$ and $G' = (V', F')$ are *isomorphic* if there exists a bijection $\phi : V \to V'$ such that for any $u, v \in V$:

$$(u, v) \in F \iff (\phi(u), \phi(v)) \in F', \quad \text{that is,} \quad u \to v \iff \phi(u) \to \phi(v).$$

Definition 2.9. Let us say that a digraph $G = (V, F)$ is *admissible* if it is acyclic and satisfies the following three properties:

(i) each vertex has the external degree at most two;

(ii) if $\text{outdeg}(u) = 2$, say $u \to v$ and $u \to w$, then either $v \to w$ or $w \to v$;

(iii) fixing two vertices v and w, then there exists at most one vertex u such that $u \to v$ and $u \to w$.

Remark 2.10. *By Property (ii), each vertex u of external degree* 2 *is the vertex of a triangle as in Figure 2.1.(a), up to isomorphisms.*

Figure 2.1: (a) Admissible triangle and (b) non-admissible quadrilateral.

Remark 2.11. *Property (iii) implies that the quadrilateral of Figure 2.1.(b) is not admissible. In fact there are only two types of admissible quadrilaterals, up to isomorphisms, shown in Figure 2.2.*

Figure 2.2: Admissible quadrilaterals.

Lemma 2.12. *An admissible, connected, digraph G has exactly one vertex with external degree* 0.

Proof. By Remark 2.6, there is at least a vertex of external degree 0. Suppose that we have two vertices u and v of external degree 0. Since G is connected, then there exists a (non-directed) path starting at u and ending at v and we can choose one such path of minimum length k. Note that $k \geq 2$ since $u \to v$ and $v \to u$ are not possible, because $\operatorname{outdeg}(u) = \operatorname{outdeg}(v) = 0$. Denote such path from u to v by $\{u = u_0, u_1\}, \{u_1, u_2\}, \ldots, \{u_{k-1}, v = u_k\}$. We claim that there exists a vertex u_j, $1 \leqslant j < k$, of external degree 2 for G in the path, that means that

$$u_j \to u_{j-1}, \qquad u_j \to u_{j+1}.$$

In fact, we know that $u_1 \to u_0$, since $u_0 = u$ has external degree 0. If $u_1 \to u_2$, then u_1 is the vertex we are looking for, otherwise we consider the path starting from u_1 and ending to $u_k = v$. Our claim follows by induction on the length of the path. Then by property (ii) of Definition 2.9, there exists an arrow either $u_{j-1} \to u_{j+1}$ or $u_{j+1} \to u_{j-1}$, so there exists a path that connects u and v with $k-1$ edges, a contradiction with the assumption that k is minimal. □

Corollary 2.13. *The number of connected components of an admissible digraph is equal to the number of vertices with external degree* 0.

2.2 Proximity matrices

The main reference for this section is [7, Chap. 1].

Let $\pi : S \to \mathbb{P}^2$ be a birational morphism. As we saw in Section 1.2.2 of Chapter 1, the morphism π is the composition of finitely many of blowing-ups at single points. Denote by $p_1, \ldots, p_r \in \mathcal{B}(\mathbb{P}^2)$ the blown-up points, so that $\pi = \pi_1 \circ \pi_2 \circ \ldots \circ \pi_r$ where π_i is the blowing-up at the point p_i for each $i = 1, \ldots, r$.

Definition 2.14. Let us associate to a birational morphism $\pi : S \to \mathbb{P}^2$ a digraph G_π with r vertices $p_1, \ldots, p_r$ and there is an arrow $p_i \to p_j$ if and only if p_i is proximate to p_j.

Definition 2.15. With notation as in Section 1.2.2 in Chapter 1, the adjacency matrix

$Q = (q_{ij}) = A_{G_\pi}$ of the digraph G_π is defined by

$$q_{ij} = \begin{cases} 1 & \text{if } p_i \in E_j^{i-1}, \\ 0 & \text{if } p_i \notin E_j^{i-1} \end{cases}$$

and we call Q the *proximity matrix* associated with the birational morphism π, or simply proximity matrix of π.

Remark 2.16. *In [1], the notion of proximity matrix of a cluster is different.*

Remark 2.17. *Note that the order of the blowing-up points is important in the definition of the proximity matrix.*

Example 2.18. Let us blow up $p_1 = [1 : 0 : 0], p_2 = [0 : 1 : 0]$ and $p_3 \succ_1 p_1$ with standard coordinate $p_3 = (p_1, \infty)$, i.e. p_3 corresponds to the line $y = 0$, either in the order p_1, p_2, p_3 or p_1, p_3, p_2. Accordingly, we find isomorphic surfaces S and S' and birational morphisms $\pi : S \to \mathbb{P}^2$ and $\pi' : S' \to \mathbb{P}^2$ with $\pi' = \pi \circ i$ where $i : S' \to S$ is the isomorphism. The digraphs G_π and $G_{\pi'}$ are the same, up to isomorphisms, but the respective proximity matrices $Q = A_{G_\pi}$ and $Q' = A_{G_{\pi'}}$ are different:

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad Q' = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that $Q' = PQP$ where $P = P^{-1}$ is the permutation matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let us recall the properties of a proximity matrix:

Proposition 2.19. *Let Q be the proximity matrix of a birational morphism $\pi : S \to \mathbb{P}^2$. Then*

(1) *Q is a strictly lower triangular matrix;*

(2) *all entries of Q are either 0 or 1;*

(3) *in each row of Q, there are at most two non-zero entries;*

(4) *no row with two non-zero entries is repeated;*

(5) *if $q_{kj} = q_{ki} = 1$ with $i > j$, then $q_{ij} = 1$.*

Proof. Properties (1) and (2) are obvious while Property (3) follows from the fact that a point can belong to at most two strict transforms of distinct exceptional curves. Using notation of Section 1.2.2 in Chapter 1, one observes that

$$p_i \notin E_j^{i-1} \Longrightarrow E_j^i \cap E_i^i = \emptyset \Longrightarrow E_j^k \cap E_i^k = \emptyset \qquad \text{for } k \geqslant i.$$

Therefore, if $p_k \in E_j^{k-1} \cap E_i^{k-1}$ for $k \geqslant i > j$, then $p_i \in E_j^{i-1}$, that is Property (5). Moreover, after blowing-up $p_k = E_i^{k-1} \cap E_j^{k-1}$, it follows that $E_i^k \cap E_j^k = \emptyset$, so that there is no other row with the same two non-zero entries, that is Property (4). □

Lemma 2.20. *In the previous proposition, Properties* (4) *and* (5) *can be replaced by the following formula*

$$q_{ij} \geq \sum_k q_{ki} q_{kj}, \qquad \text{for } \ i > j. \tag{2.1}$$

Proof. Suppose that (4) and (5) hold. Then, the sum in Formula (2.1) is either 0 or 1. If it is 0, then Formula (2.1) is trivially verified, otherwise, if it is 1, Property (5) implies that $q_{ij} = 1$ and Formula (2.1) holds. Vice versa, suppose that Formula (2.1) holds. If $q_{kj} = q_{ki} = 1$ with $i > j$, then Formula (2.1) implies $q_{ij} \geqslant 1$, that is $q_{ij} = 1$ by Property (2). So Property (5) holds. Suppose that Property (4) fails, that means there are two different rows with the same two non-zero entries. Then Formula (2.1) implies $q_{ij} \geqslant 2$, a contradiction with Property (2). □

Remark 2.21. *We now list other properties of the proximity matrix Q associated to a birational morphism π, which is the composition of the blowing-up at points $p_1, \ldots, p_r \in \mathcal{B}(\mathbb{P}^2)$:*

- *if $p_i \succ_1 p_j$, then $q_{ij} = 1$;*
- *the i-th row of Q is zero if and only if $p_i \in \mathbb{P}^2$;*
- *if $E_i \cap E_j \neq \emptyset$ and $i > j$, then $q_{ij} = 1$;*
- *if $q_{ij} = 1$ and $E_i \cap E_j = \emptyset$, then there exists $k > i$ such that $q_{kj} = q_{ki} = 1$;*
- *p_k is satellite if and only if the k-th row of Q has two non-zero entries;*
- *if $q_{ki} = q_{kj} = 1$ with $i > j$, then $p_k \odot p_j$.*

Remark 2.22. *Let $p_k \odot p_j$, namely p_k is satellite to p_j. Then, $p_k \succ_n p_j$ with $n \geqslant 2$, i.e. there exists $p_{j_1}, \ldots, p_{j_{n-1}}$, such that*

$$p_k \succ_1 p_{j_{n-1}} \succ_1 \ldots \succ_1 p_{j_2} \succ_1 p_{j_1} \succ_1 p_j.$$

Note that $p_{j_i} \odot p_j$ for each $i = 2, \ldots, n-1$. Indeed, with notation of Section 1.2.2 in Chapter 1, one has

$$p_k \odot p_j \Longleftrightarrow p_k \dashrightarrow p_j \Longleftrightarrow p_k \in E_j^{k-1}$$

that implies that $p_{j_i} \in E_j^{j_i - 1}$ for each $i = 1, \ldots, n-1$.

Lemma 2.23. *Suppose that Q is a $(r \times r)$-matrix satisfying Properties $(1)-(5)$ of Proposition 2.19. Then, there exists a birational morphism $\pi : S \to \mathbb{P}^2$ such that Q is its proximity matrix.*

Proof. We proceed by induction on r. If $r = 1$, then Q is the zero-matrix and π is just the blowing-up at a point $p_1 \in \mathbb{P}^2$. Let Q' be the $(r-1) \times (r-1)$ submatrix of Q obtained by removing the last row and the last column. Note that Q' also satisfies Properties $(1)-(5)$ of Proposition 2.19 and, by induction hypothesis, there exists a birational morphism $\pi' : S' \to \mathbb{P}^2$ such that Q' is its proximity matrix. Let $p_1, \ldots, p_{r-1} \in \mathcal{B}(\mathbb{P}^2)$ be the blown-up points of π'. Now, there are three cases:

(i) the r-th row has no non-zero entry;

(ii) the r-th row has only one non-zero entry;

(iii) the r-th row has two non-zero entries.

In case (i), choose a general point $p_r \in S'$, that is a general point $p_r \in \mathbb{P}^2$.
In case (ii), one has $q_{rj} = 1$ for some $1 \leqslant j \leqslant r-1$. Choose a general point $p_r \in E_j^{r-1}$.
In case (iii), one has $q_{rj} = q_{ri} = 1$ for some $1 \leqslant j < i \leqslant r-1$. Then Property (5) in Proposition 2.19 implies that $q_{ij} = 1$ so $E_j^i \cap E_i^i \neq \emptyset$. Property (4) implies that the point $E_i^k \cap E_j^k$ has not been blown-up for each $k = i+1, \ldots, r-1$. Choose $p_r = E_i^{r-1} \cap E_j^{r-1}$.
In all three cases, let $\pi_r : S \to S'$ be the blowing-up of S' at p_r and define $\pi = \pi' \circ \pi_r$. □

Corollary 2.24. *A digraph G is admissible if and only if there exists a birational morphism $\pi : S \to \mathbb{P}^2$ such that $G = G_\pi$.*

Proof. If G is admissible, then there exists an ordering of the vertices of G such that the adjacency matrix A_G of G is strictly lower triangular and Properties $(2)-(5)$ in Proposition 2.19 follow from Properties $(i), (ii), (iii)$ of Definition 2.9. Hence, Q is the proximity matrix of a birational morphism $\pi : S \to \mathbb{P}^2$ by the previous lemma. Conversely, if $\pi : S \to \mathbb{P}^2$ is a birational morphism, its proximity matrix Q satisfies Properties $(1)-(5)$ in Proposition 2.19 and hence the corresponding digraph is admissible according to Definition 2.9. □

2.3 Weighted proximity graph of a plane Cremona map

Definition 2.25. Let $G = (V, F)$ be a digraph. Let us say that G is *weighted* if each vertex of G is marked with a positive integer number, namely $G = (V, F, w)$ where $w : V \to \mathbb{N} = \mathbb{Z}_{>0}$ is a map.

Remark 2.26. *In [2], such digraphs are called vertex-weighted. A weighted digraph according to [2] is a digraph where one attaches weights to the arcs. We do not need to do that, so we omit "vertex" in the definition of weighted digraph.*

Definition 2.27. Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a plane Cremona map. Let us associate to φ a *weighted digraph* G_φ, called the *weighted proximity graph of (the base points of)* φ, defined as follows:

- the vertices of G_φ are the base points $p_1, \dots, p_r \in \mathcal{B}(\mathbb{P}^2)$ of φ, cf. Section 1.4;
- there is an arrow $p_i \to p_j$ if and only if p_i is proximate to p_j;
- each vertex p_i is weighted with the multiplicity $\mathrm{mult}_{p_i}(\varphi)$ of φ at p_i.

Example 2.28. A de Jonquières map of degree d has weighted proximity graph with $2d-1$ vertices, one with weight $d-1$ and the other $2d-2$ vertices with weight 1.

Remark 2.29. *Note that the number of connected components of G_φ equals the number of proper base points in $\mathbb{P}^2$ among the base points $p_1, \dots, p_r \in \mathcal{B}(\mathbb{P}^2)$ of φ, by Corollary 2.13.*

Remark 2.30. *Clearly, two equivalent plane Cremona maps have the same weighted proximity graph. The converse is true for quadratic maps but it is not true in general.*

Example 2.31. We will see in Chapter 4 that the two cubic plane Cremona maps

$$\varphi_{10} = [x^3 : y^2 z : xyz], \qquad \text{and} \qquad \varphi_{11} = [x(y^2 + xz) : y(y^2 + xz) : xyz]$$

have the same weighted proximity graph

$$2 \leftarrow 1 \qquad 1 \leftarrow 1 \leftarrow 1$$

but they are not equivalent.

Notation 2.32. When we draw the weighted proximity graph of a plane Cremona map φ, for readers' convenience we write *proper base points* in red and *infinitely near points* in black.

Example 2.33. Let σ, ρ and τ be the quadratic maps defined in Section 1.4 of Chapter 1. Their respective proximity graphs G_σ, G_ρ and G_τ are:

$$G_\sigma = 1 \quad 1 \quad 1 \qquad\qquad G_\rho = 1 \leftarrow 1 \quad 1 \qquad\qquad G_\tau = 1 \leftarrow 1 \leftarrow 1$$

Remark 2.34. *Let*

$$\begin{tikzcd} G = 1 & 1 \arrow[l] & 1 \arrow[ll, bend right] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} G' = 1 & 1 \arrow[l] & 1 \arrow[l] \arrow[ll, bend right] \end{tikzcd},$$

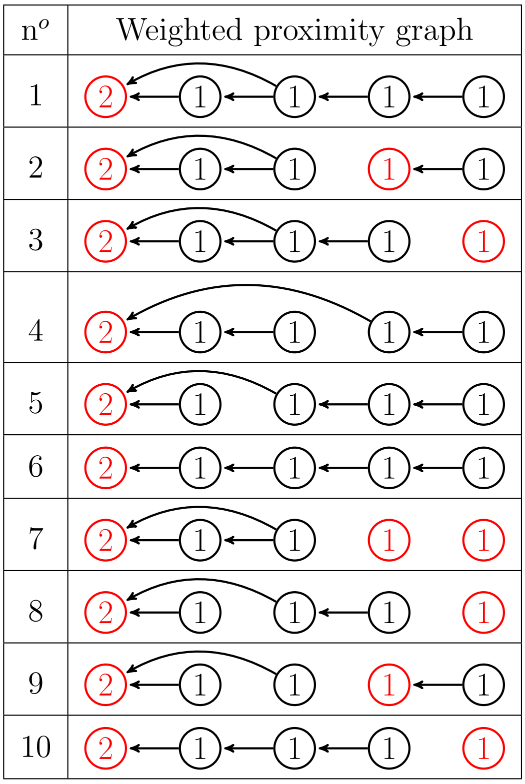
then there is no plane Cremona map with G or G' as weighted proximity graph, cf. the proximity inequality 1.38, Remark 1.45 and Remark 1.48.

In the next chapters we will deal with cubic and quartic plane Cremona maps. Therefore we classify their weighted proximity graphs.

Theorem 2.35. *There are exactly 21 weighted proximity graphs of cubic plane Cremona maps, up to isomorphism, that are listed in Table 2.1.*

There are exactly 143 weighted proximity graphs of quartic plane Cremona maps, up to isomorphism, namely 90 of quartic plane de Jonquières maps, that are listed in Table 2.2 and 53 of quartic plane non-de Jonquières maps, that are listed in Table 2.3.

Table 2.1: Weighted proximity graphs of cubic plane Cremona maps



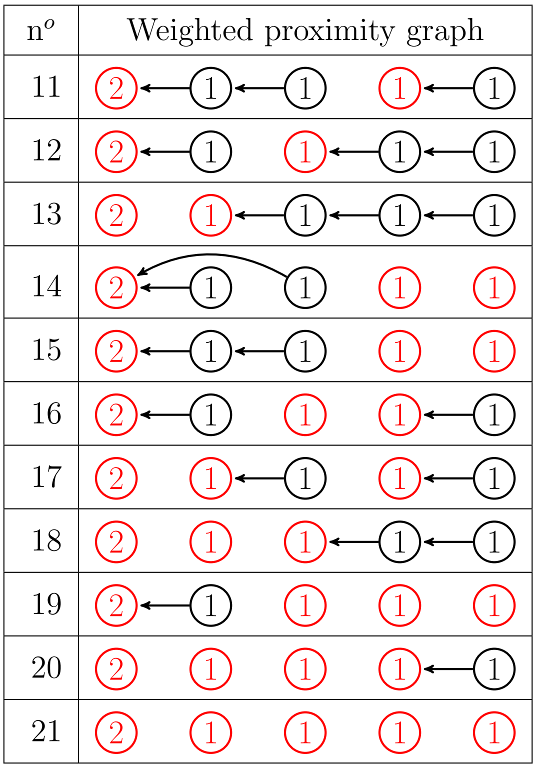
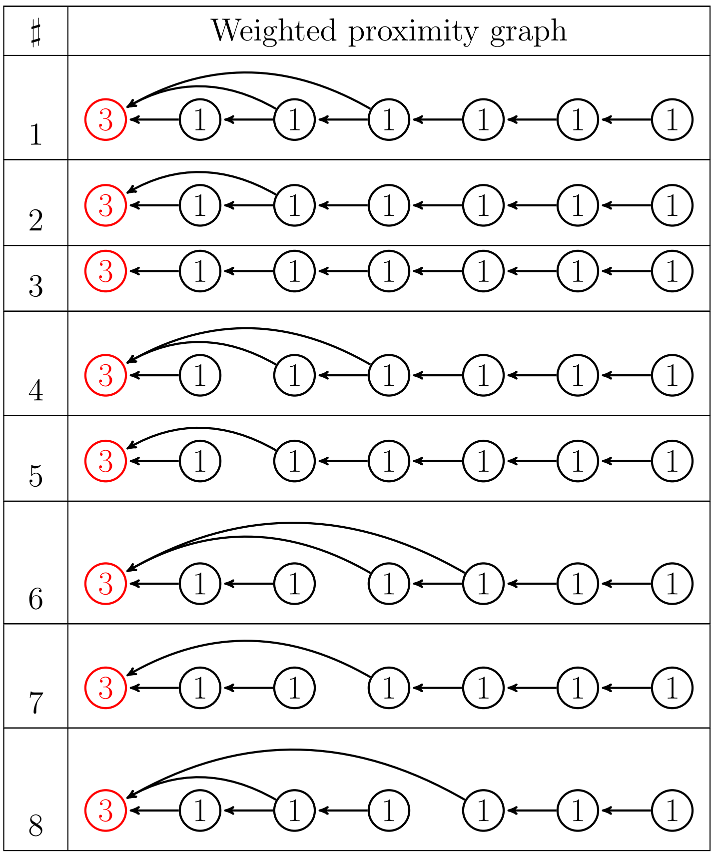
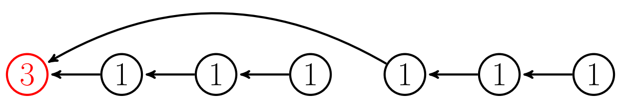
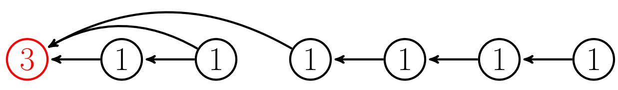
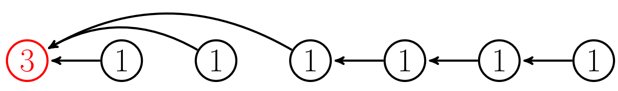
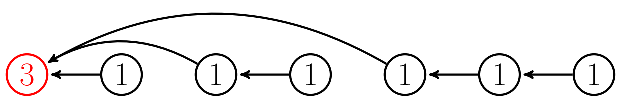
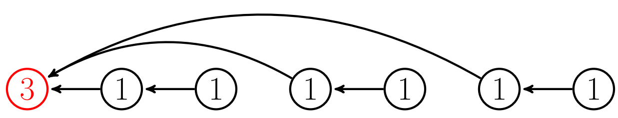
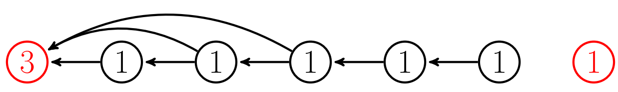
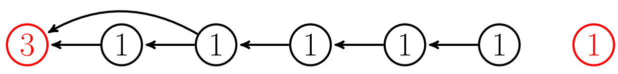
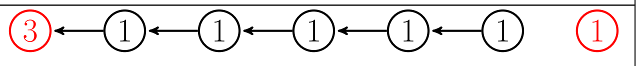
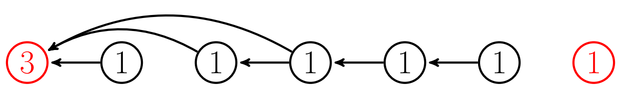
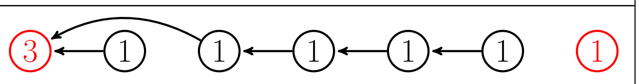
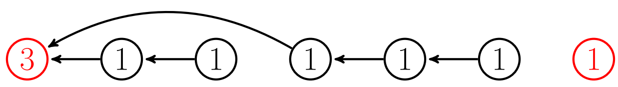
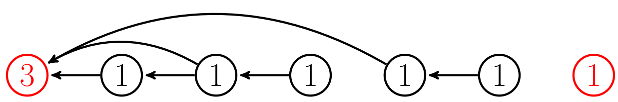
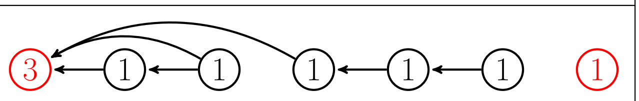
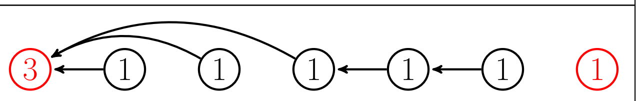
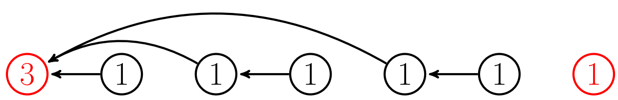
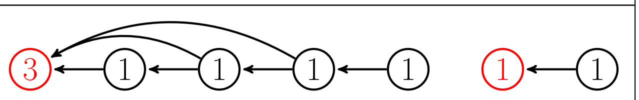
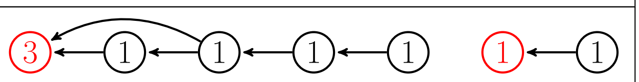
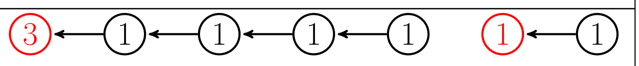
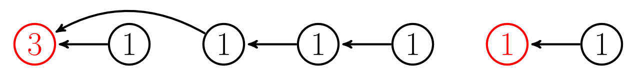
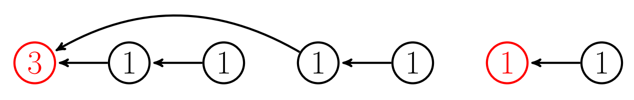
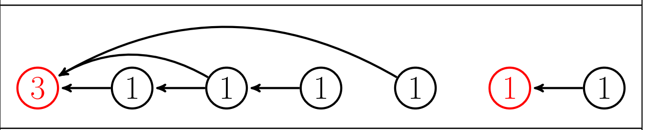
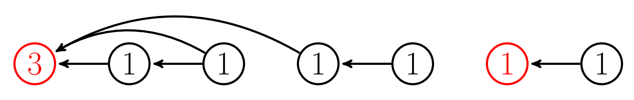
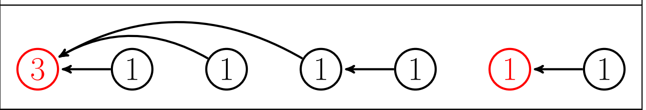
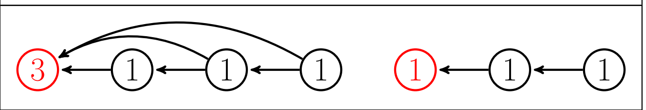
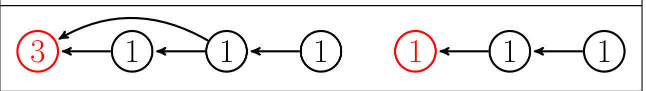
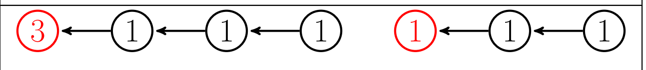
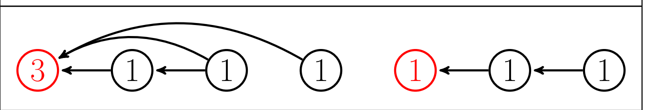
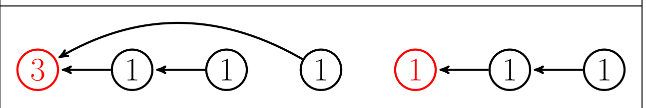
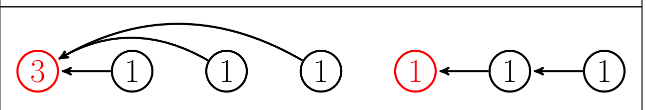
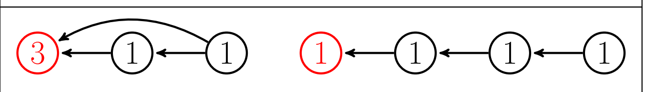
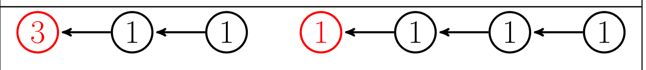
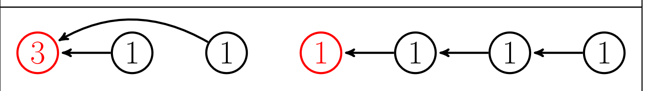
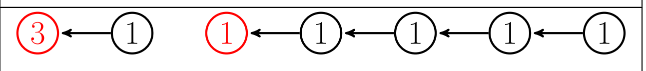
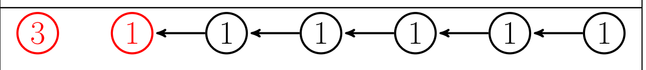
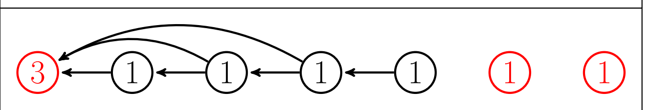
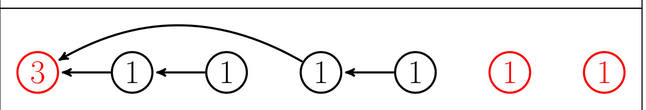
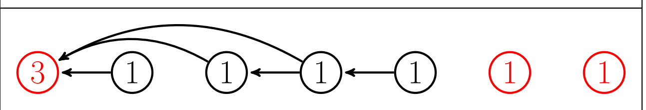
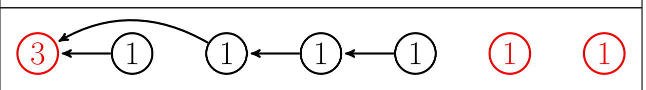
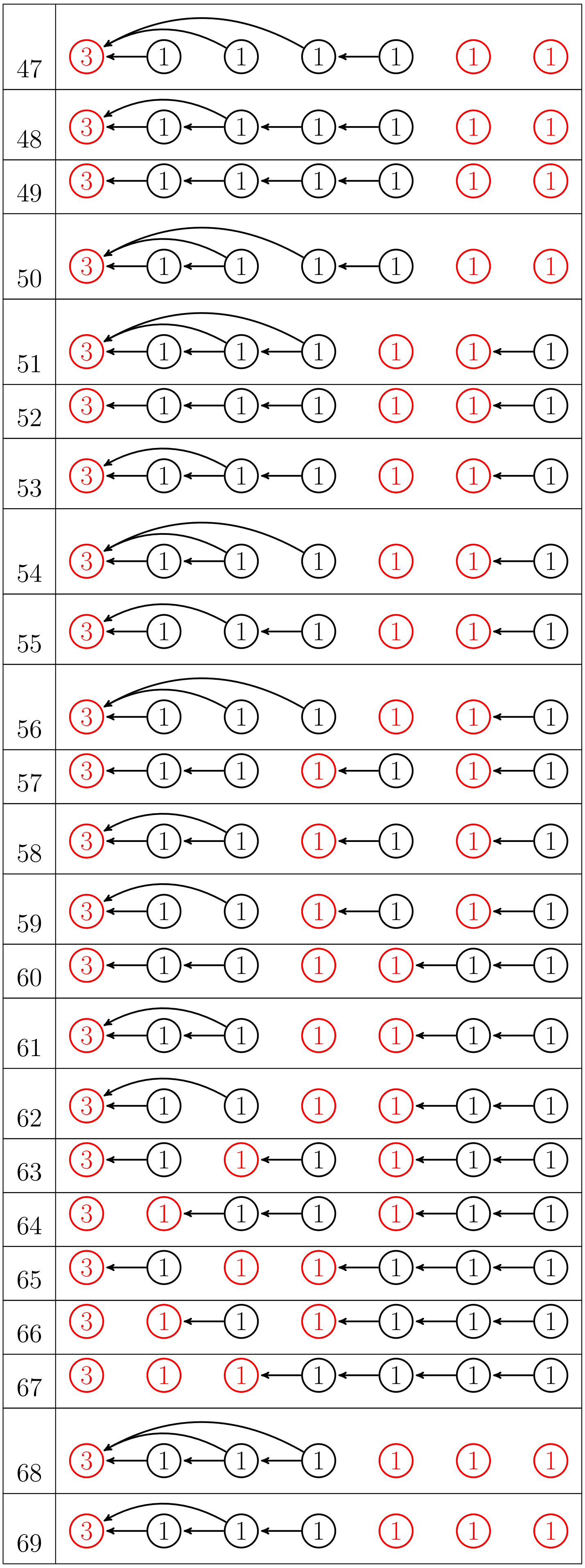


Table 2.2: Weighted proximity graphs of quartic plane de Jonquières maps



9	
10	
11	
12	
13	
14	
15	
16	
17	
18	
19	
20	
21	
22	
23	
24	
25	
26	

27	
28	
29	
30	
31	
32	
33	
34	
35	
36	
37	
38	
39	
40	
41	
42	
43	
44	
45	
46	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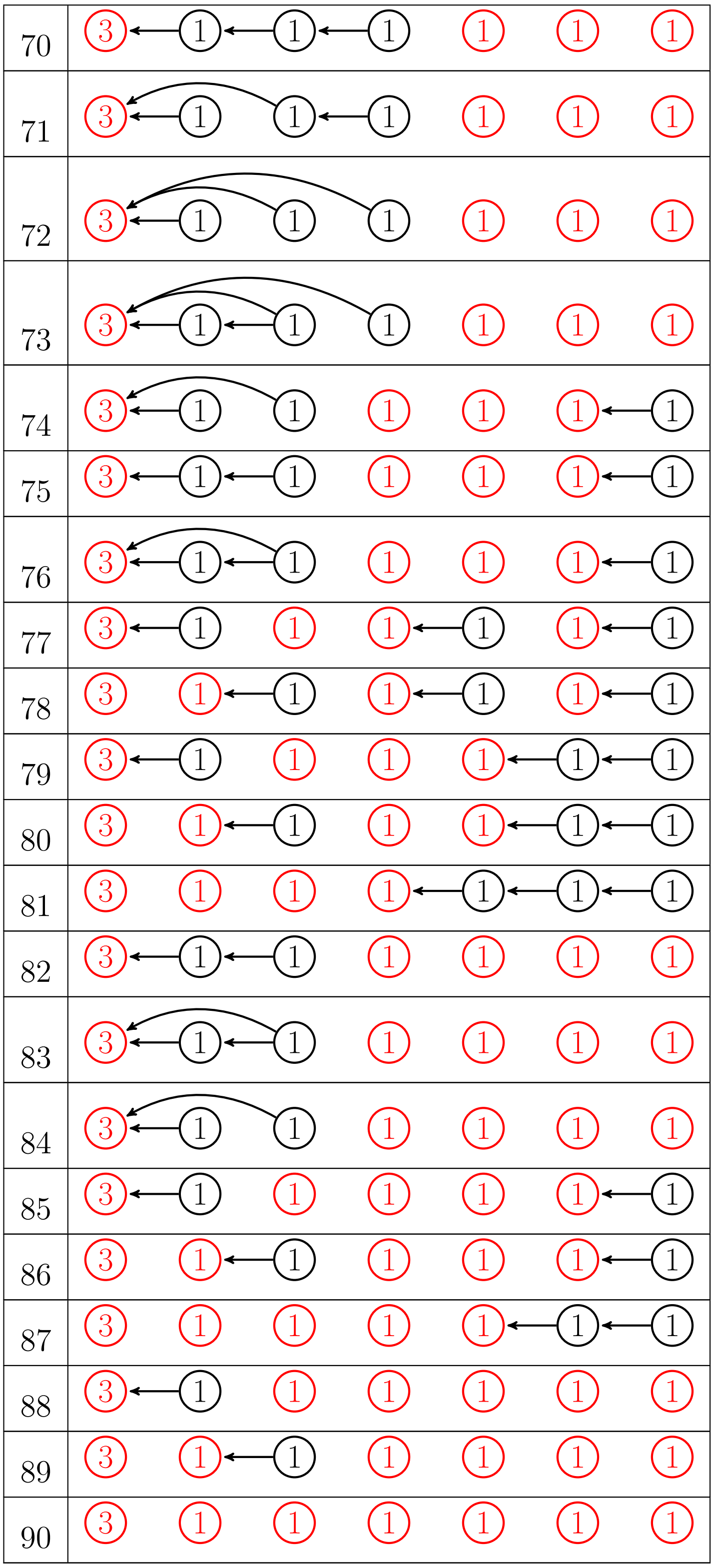


Proof of Theorem 2.35. Weighted proximity graphs of cubic plane Cremona maps can be constructed by hand, recalling the properties of admissible graphs. Indeed, such a graph has 5 vertices, one with weight 2 and the other four with weight 1. Moreover, the proximity inequalities implies that only the double point may have satellite points and there can be at most one of them. For the same reason, a simple base point may have at most one proximate point while the double point may have at most two proximate points. These conditions are

Table 2.3: Weighted proximity graphs of quartic plane non de Jonquières maps

nº	Weighted proximity graph
1	$2 \leftarrow 2 \leftarrow 2 \leftarrow 1 \leftarrow 1 \leftarrow 1$
2	$2 \leftarrow 2 \leftarrow 2 \leftarrow 1 \quad 1 \leftarrow 1$
3	$2 \leftarrow 2 \leftarrow 2 \leftarrow 1 \leftarrow 1 \leftarrow 1$
4	$2 \quad 2 \leftarrow 2 \leftarrow 1 \leftarrow 1 \leftarrow 1$
5	$2 \quad 2 \leftarrow 2 \leftarrow 1 \quad 1 \leftarrow 1$
6	$2 \quad 2 \leftarrow 2 \leftarrow 1 \leftarrow 1 \leftarrow 1$
7	$2 \leftarrow 1 \quad 2 \leftarrow 2 \leftarrow 1 \leftarrow 1$
8	$2 \leftarrow 1 \quad 2 \leftarrow 2 \leftarrow 1 \quad 1$
9	$2 \leftarrow 1 \quad 2 \leftarrow 2 \leftarrow 1 \leftarrow 1$
10	$2 \leftarrow 2 \quad 2 \leftarrow 1 \leftarrow 1 \leftarrow 1$
11	$2 \leftarrow 2 \quad 2 \leftarrow 1 \quad 1 \leftarrow 1$
12	$2 \leftarrow 2 \quad 2 \leftarrow 1 \leftarrow 1 \leftarrow 1$
13	$2 \leftarrow 1 \leftarrow 1 \quad 2 \leftarrow 2 \leftarrow 1$
14	$2 \leftarrow 1 \leftarrow 1 \quad 2 \leftarrow 2 \leftarrow 1$
15	$2 \leftarrow 1 \quad 1 \quad 2 \leftarrow 2 \leftarrow 1$
16	$2 \leftarrow 2 \leftarrow 2 \quad 1 \leftarrow 1 \leftarrow 1$
17	$2 \leftarrow 2 \leftarrow 2 \leftarrow 1 \quad 1 \leftarrow 1$
18	$2 \leftarrow 2 \leftarrow 2 \leftarrow 1 \leftarrow 1 \quad 1$
19	$2 \leftarrow 2 \leftarrow 2 \leftarrow 1 \quad 1 \quad 1$
20	$2 \leftarrow 2 \leftarrow 2 \leftarrow 1 \leftarrow 1 \quad 1$
21	$2 \quad 2 \quad 2 \leftarrow 1 \leftarrow 1 \leftarrow 1$
22	$2 \quad 2 \quad 2 \leftarrow 1 \leftarrow 1 \leftarrow 1$
23	$2 \quad 2 \quad 2 \leftarrow 1 \quad 1 \leftarrow 1$
24	$2 \leftarrow 1 \quad 2 \leftarrow 1 \quad 2 \leftarrow 1$
25	$2 \quad 2 \leftarrow 1 \quad 2 \leftarrow 1 \leftarrow 1$
26	$2 \quad 2 \leftarrow 1 \quad 2 \leftarrow 1 \leftarrow 1$
27	$2 \quad 2 \leftarrow 1 \quad 2 \leftarrow 1 \quad 1$
28	$2 \quad 2 \leftarrow 2 \leftarrow 1 \leftarrow 1 \quad 1$
29	$2 \quad 2 \leftarrow 2 \leftarrow 1 \leftarrow 1 \quad 1$
30	$2 \quad 2 \leftarrow 2 \leftarrow 1 \quad 1 \quad 1$
31	$2 \leftarrow 2 \leftarrow 1 \quad 2 \leftarrow 1 \quad 1$
32	$2 \leftarrow 2 \quad 2 \leftarrow 1 \leftarrow 1 \quad 1$
33	$2 \leftarrow 2 \quad 2 \leftarrow 1 \leftarrow 1 \quad 1$
34	$2 \leftarrow 2 \quad 2 \leftarrow 1 \quad 1 \quad 1$
35	$2 \leftarrow 2 \leftarrow 2 \leftarrow 1 \quad 1 \quad 1$
36	$2 \leftarrow 2 \leftarrow 2 \quad 1 \leftarrow 1 \quad 1$
37	$2 \quad 2 \quad 2 \quad 1 \leftarrow 1 \leftarrow 1$
38	$2 \quad 2 \quad 2 \leftarrow 1 \leftarrow 1 \quad 1$
39	$2 \quad 2 \quad 2 \leftarrow 1 \leftarrow 1 \quad 1$
40	$2 \quad 2 \leftarrow 2 \leftarrow 1 \quad 1 \leftarrow 1$
41	$2 \leftarrow 2 \quad 2 \leftarrow 1 \quad 1 \leftarrow 1$
42	$2 \leftarrow 2 \quad 2 \quad 1 \leftarrow 1 \leftarrow 1$
43	$2 \quad 2 \leftarrow 1 \quad 1 \quad 2 \leftarrow 1$

n^o	Weighted proximity graph
44	$2 \quad 2 \quad 2 \leftarrow 1 \quad 1 \quad 1$
45	$2 \quad 2 \leftarrow 1 \quad 1 \leftarrow 1 \quad 2$
46	$2 \quad 2 \leftarrow 2 \leftarrow 1 \quad 1 \quad 1$
47	$2 \leftarrow 2 \quad 2 \leftarrow 1 \quad 1 \quad 1$
48	$2 \leftarrow 2 \quad 2 \quad 1 \leftarrow 1 \quad 1$

n^o	Weighted proximity graph
49	$2 \leftarrow 2 \leftarrow 2 \quad 1 \quad 1 \quad 1$
50	$2 \quad 2 \quad 2 \quad 1 \quad 1 \leftarrow 1$
51	$2 \quad 2 \quad 2 \leftarrow 1 \quad 1 \quad 1$
52	$2 \quad 2 \leftarrow 2 \quad 1 \quad 1 \quad 1$
53	$2 \quad 2 \quad 2 \quad 1 \quad 1 \quad 1$

enough to find the 21 weighted proximity graphs of cubic plane Cremona maps, that are listed in Table 2.1.

Indeed, we may start from the weighted graph with no arrow, that is number 21 in the list of Table 2.1. We then add one arrow at each time in such a way that the graph is still admissible and the weights satisfy the proximity inequalities for all vertices. For example, if we add one arrow to graph 21, then we find exactly two non-isomorphic weighted proximity graphs, that are numbers 19 and 20. If we add a second arrow, then we find other 5 graphs, that are numbers 14–18. And so on: in the following step we find the graphs with three arrows, that are numbers 7–13. In the next step, we find number 2–6 with four arrows and finally there is only one graph, number 1, with five arrows.

This procedure has also been implemented in Maple, in order to double check the result.

We proceed similarly for weighted proximity graphs of quartic plane Cremona maps.

First, we note that a quartic plane Cremona map either is de Jonquières or it is not de Jonquières. In the former case, the graph has 7 vertices, one with weight 3 and the other six with weight 1. In the latter case, the graph has 6 vertices, three with weight 2 and the other three with weight 1.

Let us consider first the de Jonquières case.

We start from the weighted graph with no arrow (number 90 in the table) and we then add one arrow at each time. We then find:

- 2 graphs with one arrow;
- 5 graphs with two arrows;
- 11 graphs with three arrows;
- 19 graphs with four arrows;
- 24 graphs with five arrows;
- 19 graphs with six arrows;
- 8 graphs with seven arrows;

- finally 1 graph with eight arrows;

that sum up to 90 weighted proximity graphs.

Concerning the non-de Jonquières case, we proceed exactly in the same way.

We start from the weighted graph with no arrow (number 53 in the table) and we then add one arrow at each time. We then find:

- 3 graphs with one arrow;
- 9 graphs with two arrows;
- 16 graphs with three arrows;
- 16 graphs with four arrows;
- 7 graphs with five arrows;
- finally 1 graph with six arrows.

that sum up to 53 weighted proximity graphs. □

Remark 2.36. *By improving the algorithm that computes the number of weighted proximity graphs of de Jonquières maps of degree d, one can check that there are exactly*

- *346 of them in degree 5,*
- *1199 of them in degree 6,*
- *3876 of them in degree 7,*
- *11710 of them in degree 8,*
- *33635 of them in degree 9,*
- *92149 of them in degree 10.*

In degree larger than 10, the computer runs out of memory.

2.4 Enriched weighted proximity graph of a plane Cremona map

We will see that, in order to classify equivalence classes of plane Cremona maps, the position of the base points is also important. Therefore, it is convenient to add to the weighted proximity graph some projective information on the position of the base points of a plane Cremona map. We first consider the case of plane Cremona maps of small degree.

Definition 2.37. Let us add to the weighted proximity graph G_φ of a *cubic* plane Cremona map φ the list of lines passing through *three* base points of φ. Let us call this object the *enriched weighted proximity graph* of φ.

Remark 2.38. *These lines are* unexpected*, in the sense that three points in general position are not aligned.*

A line through three base points of a cubic plane Cremona map φ cannot pass through the (proper) base point of multiplicity 2, otherwise the linear system defining the map would be reducible by Bézout Theorem. For the same reason, a line cannot pass through all four simple base points of φ. Furthermore, there cannot be two different such lines, because they should have two points in common.

Notation 2.39. The line passing through three base points of a cubic plane Cremona map are indicated as dashed green curves in the pictures of weighted proximity graphs.

Theorem 2.40. *There are exactly 31 enriched weighted proximity graphs of cubic plane Cremona maps, up to isomorphism, listed in Table 4.2 at page 57.*

Proof. Recall that a line ℓ passes through an infinitely near point p only if ℓ passes through the proper point q such that $p \succ q$ and the strict transform of ℓ passes through p. Therefore, the enriched weighted proximity graph cannot include a line passing through a base point infinitely near the base point of multiplicity 2, by the previous remark.

Hence, there is no line through three base points in the weighted proximity graphs 1–11, 14 and 15 in Table 2.1.

Let us denote by p_1 the base point of multiplicity 2 and by $p_2, \ldots, p_5$ the other simple base points going from left to right in the pictures of the weighted proximity graphs in Table 2.1.

The weighted proximity graph 12 in Table 2.1 may have a line through the proper simple base point p_3 and both of its infinitely near base points, that are p_4 and p_5. Accordingly, we find the two enriched weighted proximity graphs 10 and 11 in Table 4.2.

Similarly, the weighted proximity graph 13 in Table 2.1 may have a line through p_2, p_3, p_4 and we find the two enriched weighted proximity graphs 8 and 9 in Table 4.2.

Then, the weighted proximity graph 16 in Table 2.1 may have a line through p_3, p_4, p_5 and we find the two enriched weighted proximity graphs 22 and 23 in Table 4.2.

The weighted proximity graph 17 in Table 2.1 may have either a line through p_2, p_4, p_5 or a line through p_2, p_3, p_4, that however give two isomorphic enriched weighted proximity graphs, hence we find the two enriched weighted proximity graphs 20 and 21 in Table 4.2.

The weighted proximity graph 18 in Table 2.1 may have either a line through p_3, p_4, p_5 or a line through p_2, p_3, p_4. Accordingly, we find the three enriched weighted proximity graphs 17, 18 and 19 in Table 4.2.

The weighted proximity graph 19 in Table 2.1 may have a line through p_3, p_4, p_5 and we find the two enriched weighted proximity graphs 28 and 29 in Table 4.2.

The weighted proximity graph 20 in Table 2.1 may have either a line through p_2, p_3, p_4 or a line through p_2, p_4, p_5. (There could be also a line through p_3, p_4, p_5 but the resulting enriched weighted proximity graph would be isomorphic to a previous one.) Accordingly, we find the three enriched weighted proximity graphs 24, 25 and 26 in Table 4.2.

Finally, the weighted proximity graph 21 in Table 2.1 may have four different lines that however give four isomorphic enriched weighted proximity graph. Hence we find the two enriched weighted proximity graphs 30 and 31 in Table 4.2. □

Definition 2.41. Let us add to the weighted proximity graph G_φ of a *quartic de Jonquières* map φ the list of lines passing through *three*, or *four*, base points of φ and the list of (irreducible) conics passing through *six* base points of φ. Let us call this object the *enriched weighted proximity graph* of φ.

Remark 2.42. *These conics are* unexpected*, as well as the lines, in the sense that six points in general position are not contained in any conic.*

Since a quartic plane de Jonquières map has 7 base points, the map φ cannot have two distinct conics by Bézout Theorem.

Similarly, a line cannot pass through five base points of φ; a line and a conic cannot have more than two points in common.

Theorem 2.43. *There are exactly 449 enriched weighted proximity graphs of quartic plane de Jonquières maps, up to isomorphism, listed in Table 5.1 at page 94.*

Proof. The case by case analysis is too long to be presented here.

We first constructed the enriched weighted proximity graphs by adding lines and/or a conic to the 90 weighted proximity graph.

We then checked with the computer that these enriched weighted proximity graphs are pairwise not isomorphic and that they are all. □

Definition 2.44. Let us add to the weighted proximity graph G_φ of a *quartic non-de Jonquières* map φ the list of lines passing through *three* base points of φ. Let us call this object the *enriched weighted proximity graph* of φ.

Remark 2.45. *Recall that φ has three base points of multiplicity 2 and three simple base points. By Bézout Theorem, the lines may pass either through three simple base points or through two simple base points and one base point of multiplicity 2. In other words, a line cannot pass through two base points of multiplicity 2.*

Similarly, there cannot be any conic passing through all six base points.

Theorem 2.46. *There are exactly 119 enriched weighted proximity graphs of quartic non-de Jonquières maps, up to isomorphism, listed in Table 5.3 at page 122.*

Proof. The case by case analysis is again too long to be presented here and the proof is done with the help of the computer. □

More generally, we are interested in de Jonquières of arbitrary degree.

Definition 2.47. Let us add to the weighted proximity graph G_φ of a plane *de Jonquières* map φ of degree $d \geq 3$ the list of *unexpected contractible* rational curves, where "contractible" means that the curve is Cremona equivalent to a line and "unexpected" means that

- the curve is a line passing through three, or more, base points (at most through d base points);
- the curve is an irreducible conic passing through six, or more, base points;
- the curve is an irreducible cubic with a double point passing through at least 7 points...

and so on, until the curve is irreducible of degree at most $d-2$. Let us call this object the *enriched weighted proximity graph* of φ.

Chapter 3

Lengths in the Cremona group

3.1 Decompositions of a plane Cremona map

According to Noether-Castelnuovo Theorem 1.56, any plane Cremona map $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ can be written as

$$\varphi = \alpha_n \circ \sigma \circ \alpha_{n-1} \circ \sigma \circ \cdots \circ \alpha_1 \circ \sigma \circ \alpha_0 \tag{3.1}$$

where $\alpha_i \in \mathrm{Aut}(\mathbb{P}^2)$ for any $i = 0, \ldots, n$, for some integer n.

Definition 3.1. Let us call (3.1) a *decomposition* of φ. Let us say that a decomposition (3.1) is *minimal* if n is minimal among all decompositions of φ. Let us call such n the *ordinary quadratic length* of φ and let us denote it by $\mathrm{oql}(\varphi)$.

Therefore, the ordinary quadratic length of a plane Cremona map φ of degree ≥ 2 is the minimum n such that there exist ordinary quadratic maps $\psi_1, \psi_2, \ldots, \psi_n$ with

$$\varphi = \psi_n \circ \psi_{n-1} \circ \cdots \circ \psi_2 \circ \psi_1. \tag{3.2}$$

Definition 3.2. Let us call the *quadratic length* of plane Cremona map φ the minimum n such that there exists a decomposition (3.2) where ψ_i is a (not necessarily ordinary) quadratic map, for each $i = 1, \ldots, n$, and denote it by $\mathrm{ql}(\varphi)$.

Recall that Blanc and Furter in [6] defined the *length* of a plane Cremona map φ as the minimum n such that there exists a decomposition (3.2) where ψ_i is a *de Jonquières* map, for each $i = 1, \ldots, n$, and denoted it by $\mathrm{lgth}(\varphi)$. Clearly, one has that

$$\mathrm{lgth}(\varphi) \leq \mathrm{ql}(\varphi) \leq \mathrm{oql}(\varphi).$$

Remark 3.3. *In order to compute the ordinary quadratic length of plane Cremona maps, it suffices to work with* involutory *ordinary quadratic maps. Indeed, any decomposition* (3.1) *can be written as the composition of an automorphism and involutory quadratic maps:*

$$\varphi = \alpha'_n \circ \cdots \circ ((\alpha_1 \circ \alpha_0)^{-1} \circ \sigma \circ (\alpha_1 \circ \alpha_0)) \circ (\alpha_0^{-1} \circ \sigma \circ \alpha_0)$$

where $\alpha'_n = \alpha_n \circ \alpha_{n-1} \circ \cdots \circ \alpha_1 \circ \alpha_0$.

Remark 3.4. *Two equivalent plane Cremona maps clearly have the same length, quadratic length and ordinary quadratic length.*

The following lemma is a straightforward application of the definitions.

Lemma 3.5. *Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a plane Cremona map. Then,*

$$\mathrm{oql}(\varphi) = 0 \text{ if and only if } \varphi \in \mathrm{Aut}(\mathbb{P}^2).$$

Moreover, one has

- $\mathrm{oql}(\varphi) = 1$ *if and only if φ is an ordinary quadratic map;*
- $\mathrm{ql}(\varphi) = 1$ *if and only if φ is a quadratic map;*
- $\mathrm{lgth}(\varphi) = 1$ *if and only if φ is a de Jonquières map.*

Corollary 3.6. *Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a plane Cremona map of degree $d \geq 3$. Then,*

$$\mathrm{oql}(\varphi) \geq \mathrm{ql}(\varphi) \geq 2.$$

Example 3.7. Let ρ be the quadratic map defined in (1.6). It is classically well-known that $\mathrm{oql}(\rho) = 2$. A minimal decomposition of ρ is:

$$\rho = [x : z - y : z] \circ \sigma \circ [x : y + z : z] \circ \sigma \circ [x : y - z : z].$$

Example 3.8. Let τ be the quadratic map defined in (1.7). It is classically well-known that τ is the composition of two quadratic maps of the second type and therefore the composition of four ordinary quadratic maps. A decomposition of τ, given in [11], is:

$$\begin{aligned} \tau = [y - x : 2y - x : x - y + z] \circ \sigma \circ [x + z : x : y] \circ \sigma \circ [-y : x - 3y + z : x] \circ \\ \circ \sigma \circ [x + z : x : y] \circ \sigma \circ [y - x : -2x + z : 2x - y]. \end{aligned} \tag{3.3}$$

However, we found no reference with a proof that $\mathrm{oql}(\tau) = 4$, hence that the above decomposition is minimal, even if we believe that it was classically known. On the other hand, we will see in a moment that $\mathrm{oql}(\tau) \geq 3$, because τ has a base point which is infinitely near of order 2. A proof of the fact that $\mathrm{oql}(\tau) = 4$ can be seen as a consequence of the computation of ordinary quadratic lengths of cubic plane Cremona maps in Chapter 4.

Corollary 3.9. *Let $\varphi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a plane Cremona map of degree $d \geq 5$. Then,*

$$\mathrm{oql}(\varphi) \geq \mathrm{ql}(\varphi) \geq 3.$$

Proof. We claim that, if $\mathrm{ql}(\varphi) \leq 2$, then $\deg(\varphi) \leq 4$. This is trivial if $\mathrm{ql}(\varphi) \leq 1$. Suppose that $\mathrm{ql}(\varphi) = 2$, namely $\varphi = \rho_2 \circ \rho_1$, where ρ_1, ρ_2 are quadratic maps. Let p_1, p_2, p_3 be the base points of ρ_2. If m_1, m_2, m_3 are the multiplicities of ρ_1^{-1} at p_1, p_2, p_3, respectively, then

$$\deg(\varphi) = \deg(\rho_2 \circ \rho_1) = 4 - m_1 - m_2 - m_3 \leq 4,$$

that is our claim. □

Lemma 3.10. *Let $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a plane de Jonquières map of degree $d \leq 5$. Then,*

$$\mathrm{oql}(\varphi) \geq \mathrm{ql}(\varphi) \geq d-1.$$

Proof. It is trivial if $d \leq 3$. Let us first consider the case $d = 4$.

By contradiction, suppose that $\mathrm{ql}(\varphi) \leq 2$. Clearly, $\mathrm{ql}(\varphi)$ cannot be less than 2, so we can write $\varphi = \varrho_2 \circ \varrho_1$, where ϱ_1, ϱ_2 are two quadratic plane Cremona maps. In other words, one has that $\varphi \circ \varrho_1^{-1}$ is the quadratic map ϱ_2. We claim that Proposition 1.71 implies that the composition $\varphi \circ \varrho_1^{-1}$ has always degree ≥ 3, that is a contradiction.

We now prove our claim. Suppose that $p_0, p_1, \ldots, p_6$ are the base points of φ, where p_0 is the triple base point and $p_1, \ldots, p_6$ are simple base points.

We distinguish four possibilities:

- if ϱ_1 has base points p_0, p_i, p_j with $0 < i < j \leq 6$, then $\varphi \circ \varrho_1^{-1}$ has degree 3;
- if ϱ_1 has base points p_0, p_i with $0 < i \leq 6$ and p_j is not a base point of ϱ_1 for any j such that $0 \leq j \leq 6$ and $j \neq 0, i$, then $\varphi \circ \varrho_1^{-1}$ has degree 4;
- if ϱ_1 has base point p_0 and $p_1, \ldots, p_6$ are not base points of ϱ_1, then $\varphi \circ \varrho_1^{-1}$ has degree 5;
- if p_0 is not a base point of ϱ_1, then $5 \leq \deg(\varphi \circ \varrho_1^{-1}) \leq 8$.

Our claim is proved.

We are left with the case $d = 5$.

By contradiction, suppose that $\mathrm{ql}(\varphi) \leq 3$, hence, $\mathrm{ql}(\varphi) = 3$ by Corollary 3.9 and we can write $\varphi = \varrho_3 \circ \varrho_2 \circ \varrho_1$, where $\varrho_1, \varrho_2, \varrho_3$ are quadratic plane Cremona maps. In other words, one has that $\varphi \circ \varrho_1^{-1} = \varrho_3 \circ \varrho_2$ has quadratic length 2.

Let p_0 be the base point of multiplicity 4 of φ. There are two cases: either p_0 is a base point of ϱ_1 or p_0 is not a base point of ϱ_1.

In the former case, the map $\varphi \circ \varrho_1^{-1} = \varrho_3 \circ \varrho_2$ is a de Jonquières map of degree d' with $4 \leq d' \leq 6$. If $d' = 5, 6$, then Corollary 3.9 gives a contradiction. Otherwise $d' = 4$, that is another contradiction with the first part of this proof.

In the latter case, the map $\varphi \circ \varrho_1^{-1} = \varrho_3 \circ \varrho_2$ has degree d'' with $7 \leq d'' \leq 10$ and we get again a contradiction with Corollary 3.9. □

Definition 3.11. Let φ be a plane Cremona map. Let us define the *height $\mathrm{h}_\varphi(p)$ of a point $p \in \mathcal{B}(\mathbb{P}^2)$ with respect to φ* as follows:

$$\mathrm{h}_\varphi(p) = \begin{cases} 0 & \text{if } p \text{ is not a base point of } \varphi, \\ 1 & \text{if } p \text{ is a proper base point of } \varphi, \\ n+1 & \text{if } p \text{ is a base point of } \varphi \text{ and } p \succ_n p' \in \mathbb{P}^2. \end{cases}$$

Definition 3.12. Let φ be a plane Cremona map. Let us also define the *load* of a proper base point p with respect to φ as follows:

$$\operatorname{load}_\varphi(p) = \sharp\{q \text{ is a base point of } \varphi \mid q \succ p\} + 1,$$

that is the number of base points of φ which are infinitely near p increased by 1.

Remarks 3.13. (*i*) *If p is a simple proper base point of φ, then the proximity inequality implies that base points that are infinitely near p cannot be satellite; in other words, there is a sequence $p_n \succ_1 p_{n-1} \succ_1 \cdots \succ_1 p_1 \succ_1 p$ where p_i is a base point infinitely near p of order i, $i = 1, \ldots, n$; therefore, $\operatorname{load}_\varphi(p)$ is equal to the maximum height of base points that are infinitely near p.*

(*ii*) *If φ is a de Jonquières map of degree d and it has a unique proper base point p, then $\operatorname{load}_\varphi(p) = 2d - 1$.*

Notation 3.14. Let ϱ be an involutory ordinary quadratic map and let $p_1, p_2, p_3 \in \mathbb{P}^2$ be its base points. Denote by ℓ_1 (ℓ_2, ℓ_3, resp.) the line passing through p_2 and p_3 (p_1 and p_3, p_1 and p_2, resp.) and denote by T the triangle $\ell_1 \cup \ell_2 \cup \ell_3$, as in Figure 3.1 at page 46.

Let us define a bijection $\bar{\varrho}\colon \mathcal{B}(\mathbb{P}^2) \to \mathcal{B}(\mathbb{P}^2)$ induced by ϱ as follows:

- $\bar{\varrho}(p) = p$, if $p = p_i$, $i = 1, 2, 3$;
- $\bar{\varrho}(p) = \varrho(p)$, if $p \in \mathbb{P}^2 \setminus T$;
- $\bar{\varrho}(p)$ is the point infinitely near p_i of order 1 in the direction of the strict transform of the line passing through p_i and p, if $p \in \ell_i \setminus \{p_j, p_k\}$, $\{i, j, k\} = \{1, 2, 3\}$;
- $\bar{\varrho}(p)$ is the point infinitely near p_j of order 1 in the direction of the line ℓ_i, if p is the point infinitely near p_i of order 1 in the direction of the line ℓ_j, where $\{i, j\} \subset \{1, 2, 3\}$;
- $\bar{\varrho}(p)$ is the point $q \in \ell_i$ such that the line passing through p_i and q is the strict transform of the line passing through p_i in the direction of the point p, if p is infinitely near p_i of order 1 (not lying on ℓ_j and ℓ_k, $\{i, j, k\} = \{1, 2, 3\}$);
- $\bar{\varrho}(p)$ is the point infinitely near $\varrho(p')$ of order n in the direction of the strict transform of a plane curve C, if p is infinitely near $p' \in \mathbb{P}^2$ and C is a curve passing through p.

Let us say that $\bar{\varrho}(p) \in \mathcal{B}(\mathbb{P}^2)$ is the point corresponding to $p \in \mathcal{B}(\mathbb{P}^2)$ via ϱ.

The following proposition is a generalization of Proposition 1.70, where now infinitely near base points are allowed.

Proposition 3.15. *Let p_1, p_2, p_3 be the base points of an involutory ordinary quadratic plane Cremona map $\varrho\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Let $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a plane Cremona map of degree $d > 1$ with base points $p_4, \ldots, p_r$ and possibly p_1, p_2, p_3. Denote by m_i the multiplicity of φ at p_i,*

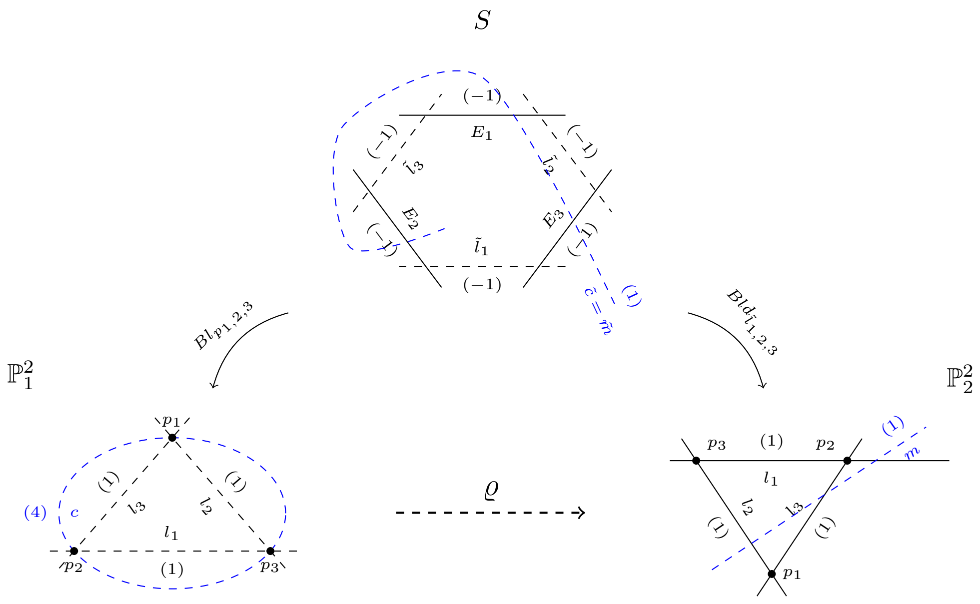


Figure 3.1: The resolution of an involutory ordinary quadratic map ϱ

$i = 1, \ldots, r$ (that is $m_i = 0$ if p_i is not a base point of φ, $i = 1, 2, 3$). Denote by $\bar{\varrho}(p)$ the (possibly infinitely near) point corresponding to p via ϱ as in Notation 3.14.
Then, the composite map $\varphi \circ \varrho^{-1} = \varphi \circ \varrho$ has degree $d - \varepsilon$, where

$$\varepsilon = m_1 + m_2 + m_3 - d,$$

and it has $\bar{\varrho}(p_i)$, $i = 4, \ldots, r$, as base point of multiplicity m_i. Furthermore, it has multiplicity $m_i - \varepsilon \geq 0$ at p_i, $i = 1, 2, 3$ (that is, p_i is not a base point of $\varphi \circ \varrho$ when $\varepsilon = m_i$).

Proof. Cf. Proposition 4.2.5 in [1]. □

Lemma 3.16. *Let φ be a plane Cremona map and ϱ an involutory ordinary quadratic map. If $p \in \mathcal{B}(\mathbb{P}^2)$ and $\bar{p} = \bar{\varrho}(p) \in \mathcal{B}(\mathbb{P}^2)$ as in Notation 3.14, then*

$$-1 \leq \mathrm{h}_\varphi(p) - \mathrm{h}_{\varphi\circ\varrho}(\bar{p}) \leq 1.$$

Proof. Set $\varphi' = \varphi \circ \varrho$. Let us see the possible cases:

- if p is not a base point of φ, that is $\mathrm{h}_\varphi(p) = 0$, then either $\bar{p}$ is not a base point of φ' or $\bar{p}$ is a proper base point of φ' by Proposition 3.15 and Notation 3.14. In the former case, one has $\mathrm{h}_{\varphi'}(\bar{p}) = 0$, whereas in the latter case one has $\mathrm{h}_{\varphi'}(\bar{p}) = 1$, and the assertion follows;

- if p is a proper base point of φ, that is $\mathrm{h}_\varphi(p) = 1$, then Proposition 3.15 and Notation 3.14 imply that three cases may occur:

  (1) $\bar{p}$ is not a base point of φ',

  (2) $\bar{p}$ is still a proper base point of φ',

(3) $\bar{p}$ is a base point of φ' which is infinitely near (of order 1) a proper base point,

accordingly, one has $\mathrm{h}_{\varphi'}(\bar{p}) = 0$, $\mathrm{h}_{\varphi'}(\bar{p}) = 1$, $\mathrm{h}_{\varphi'}(\bar{p}) = 2$, and the assertion follows;

- if p is a base point of φ and p is infinitely near p' of order n, where p' is a proper base point of φ, that is $\mathrm{h}_{\varphi}(p) = n+1$ and $\mathrm{h}_{\varphi}(p') = 1$, then the previous analysis shows that $0 \leq \mathrm{h}_{\varphi'}(\bar{p}') \leq 2$ and accordingly $n \leq \mathrm{h}_{\varphi'}(\bar{p}) \leq n+2$, that is the assertion.

We conclude that the assertion holds in any case. □

Proposition 3.17. *Let φ be a plane Cremona map. Then*

$$\mathrm{oql}(\varphi) \geq \max\{\mathrm{h}_{\varphi}(p) \mid p \in \mathcal{B}(\mathbb{P}^2)\}.$$

Proof. Let us set $n = \mathrm{oql}(\varphi)$ and let

$$\varphi = \alpha \circ \varrho_n \circ \varrho_{n-1} \circ \cdots \circ \varrho_2 \circ \varrho_1$$

be a minimal decomposition of φ, where ϱ_i, $i = 1, \ldots, n$, is an involutory ordinary quadratic map and α is an automorphism of $\mathbb{P}^2$. We proceed by induction on n. Let us set

$$m(\varphi) = \max\{\mathrm{h}_{\varphi}(p) \mid p \in \mathcal{B}(\mathbb{P}^2)\}.$$

The assertion is clearly true for $n = 0, 1$ because an automorphism has no base point and an ordinary quadratic map has exactly three points of height 1.

We then suppose that $n \geq 2$ and we denote $\varphi \circ \varrho_1$ by φ', so that $\mathrm{oql}(\varphi') = n-1$ and by induction hypothesis $n-1 \geq m(\varphi')$. Now Lemma 3.16 implies that

$$\mathrm{h}_{\varphi'}(\bar{\varrho}_1(p)) \geq \mathrm{h}_{\varphi}(p) - 1,$$

for any $p \in \mathcal{B}(\mathbb{P}^2)$, hence $m(\varphi') \geq m(\varphi) - 1$. Therefore, we conclude that

$$n = \mathrm{oql}(\varphi) = (n-1) + 1 \geq m(\varphi') + 1 \geq m(\varphi),$$

that is the assertion. □

3.2 Quadratic length of de Jonquières maps

In this section we give an upper bound for the quadratic length of plane de Jonquières maps of fixed degree d. For this purpose, we will proceed by induction on the degree d.

The following lemma is classically very well-known.

Lemma 3.18. *Let φ be a plane de Jonquières map of degree $d \geq 3$. If some simple base point of φ is proper. Then, there exists an involutory quadratic map ϱ such that $\varphi \circ \varrho^{-1} = \varphi \circ \varrho$ is a plane de Jonquières map of degree $d-1$.*

Proof. Let p_1 be the maximal multiplicity base point of φ and let p_2 be a simple proper base point of φ. Then, there are three possible cases:

- there exists at least another simple proper base point of φ;
- there exists a simple base point of φ infinitely near p_1 of order 1;
- there exists a simple base point of φ infinitely near p_2 of order 1.

In all three cases, let us choose such a point and call it p_3. Then, the point p_3 cannot be aligned with p_1 and p_2, hence there exists an involutory quadratic map ϱ based at p_1, p_2, p_3. We conclude that $\varphi \circ \varrho$ is a plane de Jonquières map of degree $d-1$. □

The following proposition can be found in [1].

Proposition 3.19. *Let φ be a plane de Jonquières map of degree d. Then, there exists a quadratic transformation ϱ such that the composite map $\varrho \circ \varphi^{-1}$ is a plane de Jonquières map of the same degree d and having at least one simple proper base point.*

Proof. See Proposition 8.4.2 in [1]. □

Corollary 3.20. *Let φ be a plane de Jonquières map of degree d. Then, either there exists an involutory quadratic map ϱ_1 such that $\varphi \circ \varrho_1$ has degree $d-1$, or there exist two quadratic maps ϱ_1, ϱ_2 such that $\varrho_2 \circ \varphi \circ \varrho_1$ is a plane de Jonquières map of degree $d-1$.*

Proof. In the former case, we apply Lemma 3.18, while in the latter case we first apply Proposition 3.19 to the de Jonquières map φ^{-1} and we then conclude by Lemma 3.18. □

Lemma 3.18 and Proposition 3.19 have been used in [1] in order to give an easy proof of the classically well-known fact that a plane de Jonquières map can be resolved in (ordinary) quadratic maps. Using the same technique, we prove the following

Theorem 3.21. *Let φ be a plane de Jonquières map of degree d. Then,*

$$\mathrm{ql}(\varphi) \leqslant 2d - 3.$$

Proof. By induction on the degree d. If $d = 2$, then $\mathrm{ql}(\phi) = 1 = 2\cdot 2 - 3$, that is the assertion. If $d > 2$, then Corollary 3.20 implies that there exist two quadratic transformations ϱ_1 and ϱ_2 such that $\varrho_2 \circ \varphi \circ \varrho_1$ is a plane de Jonquières map of degree $d-1$. By induction hypothesis, one has

$$\mathrm{ql}(\varrho_2 \circ \varphi \circ \varrho_1) \leqslant 2(d-1) - 3 = 2d - 5.$$

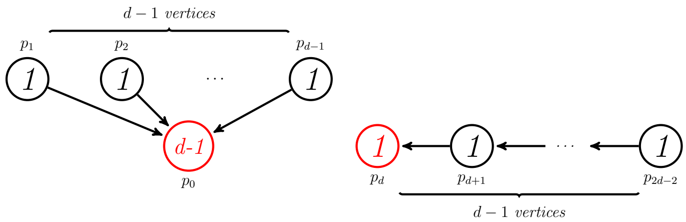
It follows that

$$\mathrm{ql}(\varphi) \leqslant \mathrm{ql}(\varrho_2 \circ \varphi \circ \varrho_1) + 2 \leqslant 2d - 3,$$

that is the assertion. □

For some specific case, we can find a better upper bound.

Lemma 3.22. *Let ϕ be a plane de Jonquières map of degree $d \geqslant 2$. Suppose that the weighted proximity graph of ϕ is*

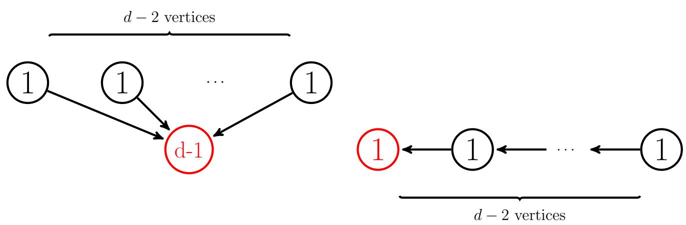


Then, the quadratic length $\mathrm{ql}(\phi)$ *of* ϕ *is at most* $d-1$.

Proof. By induction on the degree d.

If $d = 2$, then ϕ is a quadratic map (of the second type), hence $\mathrm{ql}(\phi) = 1$, that is the assertion.

For $d > 2$, let ϱ be an involutory quadratic map based at p_0, p_d and p_1 (clearly, p_0, p_d, p_1 are not collinear). Then, the composite map $\phi \circ \varrho^{-1} = \phi \circ \varrho$ has degree $d-1$ and its weighted proximity graph is



By induction hypothesis, the quadratic length of $\phi \circ \varrho$ is such that

$$\mathrm{ql}(\phi \circ \varrho) \leqslant d-2.$$

It follows that

$$\mathrm{ql}(\phi) \leqslant \mathrm{ql}(\phi \circ \varrho) + 1 \leqslant d-1,$$

that is the assertion. □

3.3 On ordinary quadratic length of de Jonquières maps

Proposition 3.23. *Let $\varphi \in Cr(\mathbb{P}^2)$ be a de Jonquières map of degree $d \geqslant 3$. Let p_0 be the maximal multiplicity base point and suppose that the simple base points $p_1, \ldots, p_{2d-2}$ are either proper or infinitely near p_0 of order 1. Then, one has*

$$\mathrm{oql}(\varphi) \leqslant d.$$

For convenience, one can reorder the simple base points of φ in such a way $p_1, \ldots, p_i$ are proper and $p_{i+1}, \ldots, p_{2d-2}$ are infinitely near p_0 of order 1, where $d-1 \leqslant i \leqslant 2d-2$.

Proof. By induction on the degree d.

For $d = 3$, the statement holds, as we will see in Table 4.2 at page 57.

Assume as induction hypothesis that within any degree d where $d \geqslant 3$, the statement holds. Now, let consider φ a de Jonquières map of degree $d+1$ under assumptions of the proposition. Since $d \geqslant 3$, then φ has degree $d+1 \geq 4$ and φ has at least three simple proper base points. Let ρ be an ordinary quadratic map centered at p_0, p_1, p_2. Then $\varphi \circ \rho^{-1}$ has degree d and has similar assumption. Therefore, by induction hypothesis, one has

$$\operatorname{oql}(\varphi \circ \rho^{-1}) \leqslant d.$$

Hence,

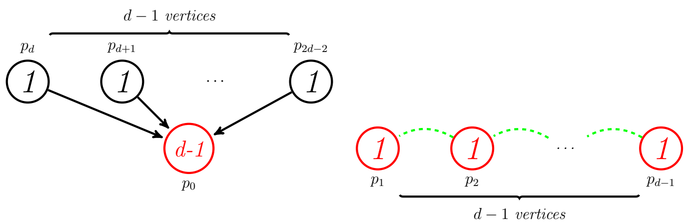
$$\operatorname{oql}(\varphi) \leqslant \operatorname{oql}(\varphi \circ \rho^{-1}) + 1 \leqslant d + 1.$$

$\square$

Remark 3.24. *In the proof of Proposition 3.23, let p'_0 be the maximal multiplicity of $\varphi \circ \rho^{-1}$. If $p_3, \ldots, p_j$ where $3 \leqslant j \leqslant i$ are on the line passing through p_1, p_2 then they correspond to points infinitely near p'_0 of order 1, while $p_{i+1}, \ldots, p_{2d-2}$ correspond to simple proper base points of $\varphi \circ \rho^{-1}$.*

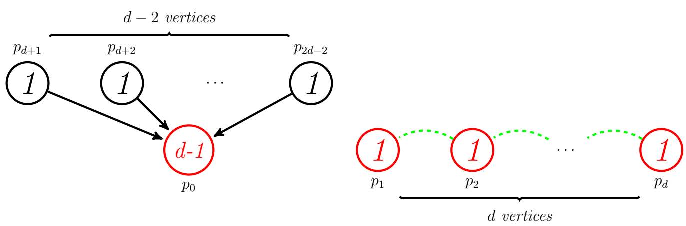
Lemma 3.25. *Let $\varphi \in \mathrm{Cr}(\mathbb{P}^2)$ be a de Jonquières map of degree $d \geq 3$. Suppose that the same assumptions of Proposition 3.23 hold, namely that the simple base points $p_1, \ldots, p_{2d-2}$ are either proper or infinitely near the maximal multiplicity base point p_0 of order 1. Suppose moreover that*

- *if d is **even**, then the enriched weighted proximity graph of φ is **not** isomorphic to the following form*



(3.4)

- *and if d is **odd**, then the enriched weighted proximity graph of φ is **not** isomorphic to the following form*



(3.5)

where the dashed green curves means that $p_1, \ldots, p_{d-1}$ are collinear in the former case and it means $p_1, \ldots, p_d$ are collinear in the latter case, then $\operatorname{oql}(\varphi) \leq d - 1$.

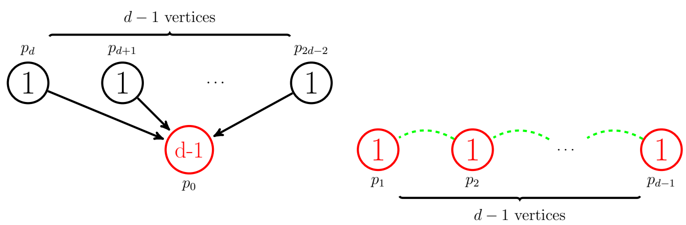
Proof. By induction on the degree d.
In case of degree $d = 3$, the enriched weighted proximity graph of φ is either one of types $27, 29, 30, 31$ in Table 4.2 at page 57, that we will prove in Chapter 4 that $\mathrm{oql}(\varphi) = 2$.
In case of degree $d = 4$, the enriched weighted proximity graph of φ is either one of types $72.1, 84.i$ with $i = 1, \dots, 4$, $88.j$ with $j = 1, \dots, 8$, $90.k$ with $k = 1, \dots, 12$ in Table 5.1 at page 94. Choose two simple proper base points p_1, p_2 and let ρ be an involutory ordinary quadratic map based at p_0, p_1, p_2. Then, $\varphi \circ \rho$ is a de Jonquières map of degree 3 with its enriched weighted proximity graph is one of types $27, 29, 30, 31$ in Table 4.2, and then $\mathrm{oql}(\varphi \circ \rho) = 2$. Note that, type 28 in Table 4.2 can not occur because of the assumption in the statement of the lemma. Therefore, one has

$$\mathrm{oql}(\varphi) \leq \mathrm{oql}(\varphi \circ \rho) + 1 = 3.$$

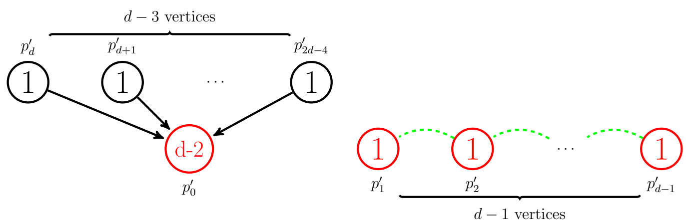
Hence, the assertion holds true for degree $d = 3, 4$.
Suppose by induction that $d \geq 5$ and that the assertion is true for $d - 1$. There are two cases: either d is odd or d is even.

(I) If d is odd, we consider two sub-cases:

(a) if the simple base points are $p_1, \dots, p_{d-1}$ and they are collinear, namely the enriched weighted proximity graph is of the following form



Set ρ_1 an involutory ordinary quadratic map based at p_0, p_1, p_2. Then, $\varphi \circ \rho_1$ has even degree $(d-1)$ and its enriched weighted proximity graph is of the following form



which is not isomorphic to (3.4). By hypothesis induction, one has

$$\mathrm{oql}(\varphi \circ \rho_1) \leq (d-1) - 1 = d - 2.$$

Then,

$$\mathrm{oql}(\varphi) \leq \mathrm{oql}(\varphi \circ \rho_1) + 1 \leq (d-2) + 1 = d - 1.$$

(b) If the simple proper base points of φ are $p_1, \dots, p_i$ with $i \geq d - 1$ and they are not all collinear, namely there exists p_j for some $j = 3, \dots, i$, such that p_j does not lie on the line passing through p_1, p_2. Set ρ_1 an involutory ordinary quadratic map base at p_0, p_1, p_2. Then, $\varphi \circ \rho_1$ has degree $(d-1)$ and either it has at least d simple proper base points if $i = d-1$, or it has at least $(d-1)$ simple proper base points which are not all aligned if $i \geq d$, and if it has infinitely near points then they are infinitely near the maximal multiplicity base point of the first order. That means $\varphi \circ \rho_1$ satisfies the assumption of the lemma, it has even degree and its enriched weighted proximity graph is not isomorphic to (3.4). By hypothesis induction, one has

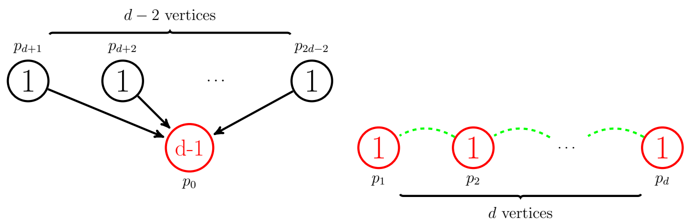
$$\operatorname{oql}(\varphi \circ \rho_1) \leq (d-1) - 1 = d - 2.$$

It follows

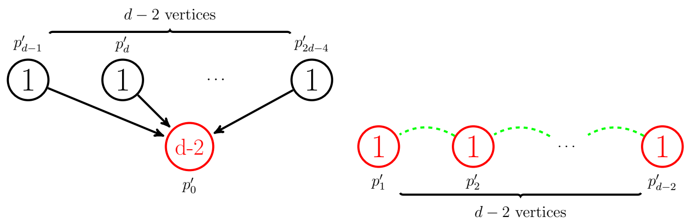
$$\operatorname{oql}(\varphi) \leq \operatorname{oql}(\varphi \circ \rho_1) + 1 \leq (d-2) + 1 = d - 1.$$

(II) If d is even, we consider two sub-cases:

(c) if the simple proper base points of φ are $p_1, \dots, p_d$ and they are collinear, namely the enriched weighted proximity graph of φ is of the following form



Set ρ_1 an involutory ordinary quadratic map based at p_0, p_1, p_2. Then, $\varphi \circ \rho_1$ has odd degree $(d-1)$ and its enriched weighted proximity graph is of the following form



which is not isomorphic to (3.5). By hypothesis induction, one has

$$\operatorname{oql}(\varphi \circ \rho_1) \leq (d-1) - 1 = d - 2.$$

Then,

$$\operatorname{oql}(\varphi) \leq \operatorname{oql}(\varphi \circ \rho_1) + 1 \leq (d-2) + 1 = d - 1.$$

(d) If the simple proper base points of φ are $p_1, \dots, p_i$ with $i \geq d$ and they are not all aligned, namely there exists p_j for some $j = 3, \dots, i$, such that p_j does not lie on the line passing through p_1, p_2. Set ρ_1 an involutory ordinary quadratic map base at p_0, p_1, p_2. Then, $\varphi \circ \rho_1$ has degree $(d-1)$, it has at least $(d-1)$ simple proper base points which are not all aligned and if it has infinitely near points then they are infinitely near the maximal multiplicity base point of the first order.

That means $\varphi \circ \rho_1$ satisfies the assumption of the lemma, it has odd degree and its enriched weighted proximity graph is not isomorphic to (3.5). By hypothesis induction, one has

$$\operatorname{oql}(\varphi \circ \rho_1) \leq (d-1) - 1 = d - 2.$$

It follows

$$\operatorname{oql}(\varphi) \leq \operatorname{oql}(\varphi \circ \rho_1) + 1 \leq (d-2) + 1 = d - 1.$$

□

Chapter 4

Cubic plane Cremona maps

In this chapter, we classify equivalence classes of cubic plane Cremona maps. Moreover, by using this classification, we compute the quadratic length and the ordinary quadratic length of all cubic plane Cremona maps. The main tool of the classification is the enriched weighted proximity graph of the base points of a plane Cremona map. A previous "classification" was obtained by Cerveau and Déserti, cf. [11] and Section 4.2.

4.1 Classification theorems

Let us set $\mathbb{C}^{**} = \mathbb{C} \setminus \{0, 1\}$ and let us define the following maps:

$$g_1, g_2 \colon \mathbb{C}^{**} \times \mathbb{C}^{**} \to \mathbb{C}^{**} \times \mathbb{C}^{**}, \qquad g_1(a, b) = (b, a), \qquad g_2(a, b) = \left(\frac{1}{a}, \frac{1}{b}\right).$$

Therefore, $g_3 := g_2 \circ g_1 = g_1 \circ g_2$ is the map $(a, b) \mapsto (1/b, 1/a)$. Clearly,

$$G = \{\mathrm{id}, g_1, g_2, g_3\}$$

is a group, under the composition, which is isomorphic to $((\mathbb{Z}/2\mathbb{Z})^2, +)$.

For $a \neq b$ and $a, b \in \mathbb{C}^{**}$, let us denote by S' the following set

$$S' = \left\{ (a, b), \left(\frac{a}{a-1}, \frac{a-b}{a-1}\right), \left(\frac{b}{b-1}, \frac{b-a}{b-1}\right), \right.$$
$$\left. \left(\frac{a-b}{b(a-1)}, \frac{1}{1-a}\right), \left(\frac{b-a}{a(b-1)}, \frac{1}{1-b}\right), \left(\frac{a-1}{b-1}, \frac{b(a-1)}{a(b-1)}\right) \right\}$$

and let us define

$$S = \{g(s) \mid g \in G \text{ and } s \in S'\}. \tag{4.1}$$

Theorem 4.1. *Any cubic plane Cremona map is equivalent to one of the maps in Table 4.1 at page 56, where the first 25 types are single maps, types 26-30 depend on one parameter $\gamma \neq 0, 1$ and type 31 depends on two parameters a, b, where $a, b \neq 0, 1$ and $a \neq b$.*

Two cubic plane Cremona maps of two different types are not equivalent.

Concerning the types depending on parameters:

- $\varphi_{26,\gamma}$, *that is type 26 in Table 4.1 with parameter* $\gamma \neq 0, 1$, *is equivalent to* $\varphi_{26,\gamma'}$ *if and only if either* $\gamma' = \gamma$ *or* $\gamma' = \gamma/(\gamma - 1)$*;*
- $\varphi_{27,\gamma}$, *that is type 27 in Table 4.1 with parameter* $\gamma \neq 0, 1$, *is equivalent to* $\varphi_{27,\gamma'}$ *if and only if either* $\gamma' = \gamma$ *or* $\gamma' = 1/\gamma$*;*
- *for* $n \in \{28, 29, 30\}$, *the map* $\varphi_{n,\gamma}$, *that is type* n *in Table 4.1 with parameter* $\gamma \neq 0, 1$, *is equivalent to* $\varphi_{n,\gamma'}$ *if and only if*
$$\gamma' \in \left\{\gamma, \frac{1}{\gamma}, 1 - \gamma, \frac{1}{1 - \gamma}, \frac{\gamma}{\gamma - 1}, \frac{\gamma - 1}{\gamma}\right\}.$$
- $\varphi_{31,a,b}$, *that is type 31 in Table 4.1 with two parameters* $a, b \neq 0, 1$, $a \neq b$, *is equivalent to* $\varphi_{31,a',b'}$ *if and only if* $(a', b') \in S$, *where* S *is defined in* (4.1)*.*

In Table 4.1 at page 56, the first column lists our type, the second column lists the formula of the maps, the third column lists the corresponding types in [11], cf. Section 4.2, and finally the fourth column lists the types of the inverse maps.

Using the above classification theorem, it is easy to compute the ordinary quadratic length and the quadratic length of all cubic plane Cremona maps:

Theorem 4.2. *Plane Cremona maps equivalent to type 1 in Table 4.1 have quadratic length 3, while all other cubic plane Cremona maps have quadratic length 2.*

A plane Cremona map equivalent to type n, $1 \leq n \leq 31$, *in Table 4.1 has the respective ordinary quadratic length listed in the third column in Table 4.2 at page 57.*

Corollary 4.3. *The ordinary quadratic length of* τ *is* $\mathrm{oql}(\tau) = 4$, *hence the decomposition* (3.3) *of* τ *is minimal.*

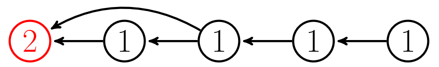
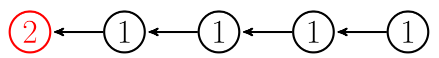
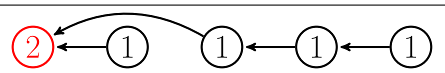
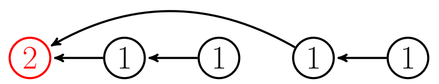
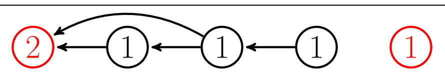
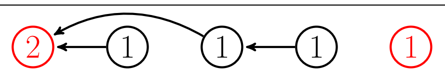
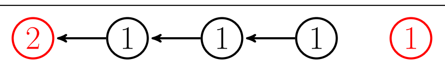
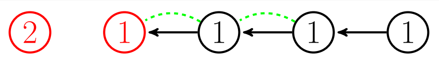
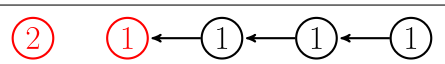
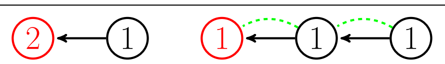
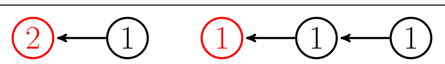
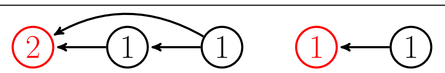
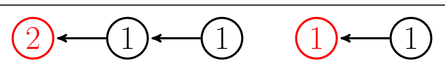
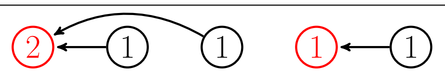
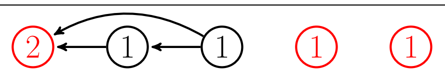
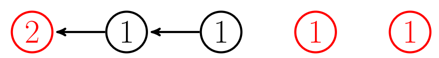
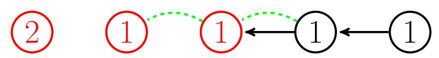
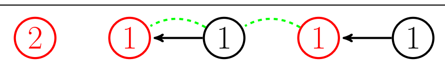
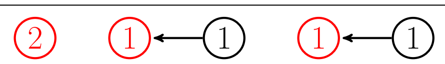
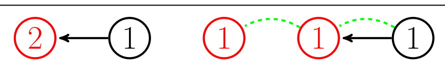
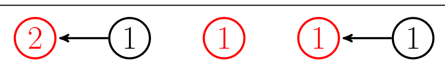
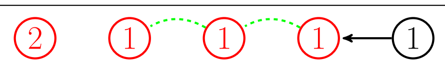
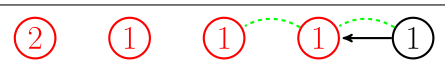
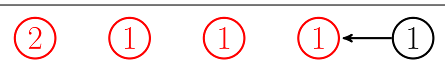
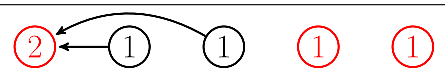
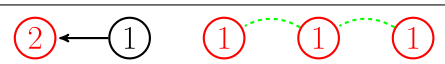
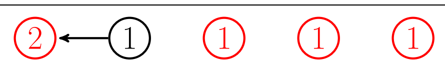
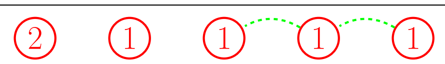
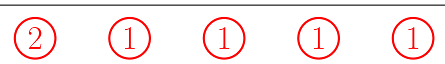
Proof. Let p_1, p_2, p_3 be the base points of τ, where $p_3 \succ p_2 \succ p_1 \in \mathbb{P}^2$ and let ℓ be the line through p_1 and p_2. Proposition 3.17 implies that $\mathrm{oql}(\tau) \geq 3$ and the decomposition (3.3) says that $\mathrm{oql}(\tau) \leq 4$. Suppose by contradiction that $\mathrm{oql}(\tau) = 3$. Then, there exists an involutory ordinary quadratic map ψ such that $\mathrm{oql}(\tau \circ \psi) = 2$. Either p_1 is a base point of ψ or it is not. In the latter case, $\tau \circ \psi$ has a base point of height 3, hence Proposition 3.17 implies $\mathrm{oql}(\tau \circ \psi) \geq 3$, a contradiction. In the former case, if one of the other two base points of ψ lies on the line ℓ, then p_2 corresponds to a base point of the map $\tau \circ \psi$ which is still infinitely near and, therefore, $\tau \circ \psi$ has still a base point of height 3 and we get again the same contradiction. Otherwise, the map $\tau \circ \psi$ has the proximity graph of type 24 in Table 4.2, which has ordinary quadratic length 3, according to Theorem 4.2, a contradiction. $\square$

We are going to prove these theorems in Sections 4.3 and 4.4.

Table 4.1: Types of cubic plane Cremona maps.

♯	Map	[11]	Inv
1	$[xz^2 + y^3 : yz^2 : z^3]$	1	1
2	$[x(x^2 + yz) : y^3 : y(x^2 + yz)]$	20	8
3	$[xz^2 : x^3 + xyz : z^3]$	3	5
4	$[x^2z : x^3 + z^3 + xyz : xz^2]$	4	4
5	$[x^2z : x^2y + z^3 : xz^2]$	5	3
6	$[x^2(x - y) : xy(x - y) : xyz + y^3]$	12	6
7	$[x(x^2 + yz) : y(x^2 + yz) : xy^2]$	24	17
8	$[xyz : yz^2 : z^3 - x^2y]$	6	2
9	$[y^2z : x(xz + y^2) : y(xz + y^2)]$	21	9
10	$[x^3 : y^2z : xyz]$	7	10
11	$[x(y^2 + xz) : y(y^2 + xz) : xyz]$	22	18
12	$[xz^2 : x^2y : z^3]$	2	12
13	$[x(y^2 + xz) : y(y^2 + xz) : xy^2]$	23	20
14	$[x^3 : x^2y : (x - y)yz]$	11	15
15	$[x^2y : xy^2 : (x - y)^2z]$	$(\star)$	14
16	$[x(x^2 + yz) : y(x^2 + yz) : xy(x - y)]$	28	24
17	$[xyz : y^2z : x(y^2 - xz)]$	10	7
18	$[x^2(y - z) : xy(y - z) : y^2z]$	8	11
19	$[x(x^2 + yz + xz) : y(x^2 + yz + xz) : xyz]$	26	19
20	$[x^2z : xyz : y^2(x - z)]$	9	13
21	$[x(xy + xz + yz) : y(xy + xz + yz) : xyz]$	25	21
22	$[xz(x + y) : yz(x + y) : xy^2]$	13	22
23	$[x(x^2 + xy + yz) : y(x^2 + xy + yz) : xyz]$	27	25
24	$[xyz : (y - x)yz : x(x - y)(y - z)]$	15	16
25	$[x(x + y)(y + z) : y(x + y)(y + z) : xyz]$	14	23
26	$[x(\gamma xz - \gamma y^2 - xy + y^2) : \gamma xy(z - y) : \gamma y^2(z - x)]$	29	26
27	$[\gamma x^2y : \gamma xy^2 : (x + y)(x + \gamma y)z]$	16	27
28	$[xy(x - y) : xz(y - \gamma x) : z(y + \gamma x)(y - \gamma x)]$	$17^\dagger$	28
29	$[xy(x - y) : x(xy - \gamma xy + \gamma xz - yz) : x^2y - \gamma^2x^2y + \gamma^2x^2z - y^2z]$	30	30
30	$[x(xy + \gamma xz - xz - \gamma y^2) : \gamma xz(x - y) : \gamma z(x - y)(x + y)]$	18	29
31	$[ax(-abxz + aby^2 - b^2xy + b^2xz + axy - ay^2) : ax(-abxz + abyz + axy$ $-ayz - bxy + bxz) : -a^2bx^2z + a^2by^2z + a^2x^2y - a^2y^2z - b^2x^2y + b^2x^2z]$	32	31

Table 4.2: Enriched weighted proximity graphs and ordinary quadratic lengths of the cubic plane Cremona maps.

♯	Enriched weighted prox. graph	oq
1		6
2		5
3		5
4		4
5		5
6		4
7		4
8		5
9		4
10		3
11		3
12		3
13		3
14		3
15		3
16		3
17		4
18		3
19		3
20		3
21		2
22		3
23		2
24		3
25		2
26		2
27		2
28		3
29		2
30		2
31		2

4.2 Comparison with the classification in [11]

In this section we compare our classification with the one in [11]. The classification in [11] is divided in 32 types, namely 27 types are a single map each, 4 types are families depending on one parameter and 1 type is a family depending on two parameters. Their classification is based on the analysis of plane curves contracted by a cubic plane Cremona map. We will freely use Notation 1.1 at page 1.

Remark 4.4. *The classification in [11] is not complete. Our type* 15 *does not occur in their list, even if it is equivalent to the inverse of their type* 11.

Remarks 4.5.

- ♣ *In [11], their type* 19 *is equivalent to a specific case of their type* 18 *with parameter* $\gamma_0 = -3/\sqrt{2}$. *In particular, let us denote by* ψ_{19} *and* $\psi_{18,\gamma}$ *the two maps of their type* 19 *and their type* 18 *with parameter* γ *respectively, then*

$$\psi_{19} = [2x - z : z : \sqrt{2}y - 2x] \circ \psi_{18,\gamma_0} \circ [x - y : \sqrt{2}x : \sqrt{2}(z - y)].$$

- ♣ *Similarly, in [11], their type* 31 *is equivalent to their type* 30 *with parameter* $\gamma_0 = 3/\sqrt{2}$. *Indeed, let us denote by* ψ_{31} *and* $\psi_{30,\gamma}$ *the two maps of their type* 31 *and their type* 30 *with parameter* γ *respectively, then*

$$\psi_{31} = [y + \sqrt{2}x : -y : 2(z - y)] \circ \psi_{30,\gamma_0} \circ [x + y : -\sqrt{2}y : x + 2z].$$

This explains why the two types 19 *and* 31 *in [11] do not appear in the third column of our Table 4.1.*

Remark 4.6. *Let* ψ_{17} *be type 17 in [11], that is*

$$\psi_{17}([x : y : z]) = [xz(x + y) : yz(x + y) : xy(x - y)].$$

Then, ψ_{17} *is equivalent to our type 28 in Table 4.1 with* $\gamma_0 = -1$, *because*

$$[y : y + z : x] \circ \varphi_{28,\gamma_0} = \psi_{17}.$$

However, it seems that our type 28 with $\gamma \neq -1$ *does not occur in the list in [11]. This explains why we added † at type* 17 *in the third column of Table 4.1.*

Remarks 4.7.

- ♣ *Let* φ_{24} *be the map of type* 24 *in Table 4.1. Then,* φ_{24} *is equivalent to type* 15 *in [11], that is* $\psi_{15}([x : y : z]) = [x(x+y)(z+y+x) : y(x+y)(z+y+x) : xyz]$. *Indeed, one has*

$$\varphi_{24} \circ [x : x + y : x + y + z] = \psi_{15}.$$

- ♣ *Let* $\varphi_{26,\gamma}$ *be the map of type* 26 *with parameter* $\gamma \neq 0, 1$ *in Table 4.1. Let* $\psi_{29,t}$ *be the map of type* 29 *with parameter* $t \neq 0, 1$ *in [11], that is*

$$\psi_{29,t}([x : y : z]) = [x(y^2 + txy + xz + yz) : y(y^2 + txy + xz + yz) : xyz].$$

Then, one has that

$$[ty - x : t(y - tz) : ty] \circ \varphi_{26,\gamma_0} \circ [-tx : y : y + z] = \psi_{29,t},$$

where $\gamma_0 = \dfrac{1}{1-t}$*, that shows that* φ_{26,γ_0} *is equivalent to* $\psi_{29,t}$*.*

♣ *Let* $\varphi_{27,\gamma}$ *be the map of type* 27 *with parameter* $\gamma \neq 0, 1$ *in Table 4.1. Let* $\psi_{16,t}$ *be the map of type* 16 *with parameter* t *such that* $t^2 \neq 4$ *in [11], that is*

$$\psi_{16,t}([x : y : z]) = [x(x^2 + y^2 + txy) : y(x^2 + y^2 + txy) : xyz].$$

Then, one has that $\psi_{16,t}$ *and*

$$\varphi_{27,\gamma_0} \circ [-(t_- x + y) : t_+ x + y : z],$$

where $\gamma_0 = t_-/t_+$*, are defined by the same homaloidal net, therefore* φ_{27,γ_0} *is equivalent to* $\psi_{16,t}$*.*

♣ *Let* $\varphi_{29,\gamma}$ *be the map of type* 29 *with parameter* $\gamma \neq 0, 1$ *in Table 4.1. Let* $\psi_{30,t}$ *be the map of type* 30 *with parameter* t *such that* $t^2 \neq 4$ *in [11], that is*

$$\psi_{30,t}([x : y : z]) = [x(x^2 + y^2 + txy + xz) : y(x^2 + y^2 + txy + xz) : xyz].$$

Then, one has that $\psi_{30,t}$ *and*

$$\varphi_{29,\gamma_0} \circ [t^\bullet y : t_+(y + t_- x) : t_+ y + x + z)],$$

where $\gamma_0 = \dfrac{1}{2} + \dfrac{t}{2t^\bullet}$*, are defined by the same homaloidal net, therefore* φ_{29,γ_0} *is equivalent to* $\psi_{30,t}$*.*

♣ *Let* $\varphi_{30,\gamma}$ *be the map of type* 30 *with parameter* $\gamma \neq 0, 1$ *in Table 4.1. Let* $\psi_{18,t}$ *be the map of type* 18 *with parameter* t *such that* $t^2 \neq 4$ *in [11], that is*

$$\psi_{18,t}([x : y : z]) = [x(x^2 + y^2 + txy + t_+ xz + yz) : y(x^2 + y^2 + txy + t_+ xz + yz) : xyz].$$

Then, one has that φ_{30,γ_0}*, where* $\gamma_0 = tt_+ - 1$*, and*

$$\psi_{18,t} \circ [x : -t_+ y : t_+ y - t_- x - t^\bullet z]$$

are defined by the same homaloidal net, therefore φ_{30,γ_0} *is equivalent to* $\psi_{18,t}$*.*

♣ *Let* $\varphi_{31,a,b}$ *be the map of type* 31 *with two parameters* a, b *such that* $a \neq b$ *and* $a, b \neq 0, 1$ *in Table 4.1. Let* $\psi_{32,t,h}$ *be the map of type* 32 *with two parameters* t, h *such that* $t^2 \neq 4$ *and* $h \neq t_\pm$ *in [11], that is*

$$\psi_{32,t,h}([x : y : z]) = [x(txy + hxz + x^2 + y^2 + yz) : y(txy + hxz + x^2 + y^2 + yz) : xyz].$$

Then, one has that $\psi_{32,t,h}$ *and*

$$\varphi_{31,a_0,b_0} \circ [t^\bullet x : -t_- x - y : -t_- x - y - z], \qquad (a_0, b_0) = \left(\frac{(2 - tt_+)h}{h - t_+}, \frac{h}{h - t_+} \right),$$

are defined by the same homaloidal net, therefore φ_{31,a_0,b_0} *is equivalent to* $\psi_{32,t,h}$*.*

Remarks 4.8. ♣ *Let* ψ_{19} *be type* 19 *in [11], that is*

$$\psi_{19}([x:y:z]) = [y(x-y)(x+z) : x(x-y)(z-y) : yz(x+y)].$$

Then, ψ_{19} *is equivalent to* $\varphi_{30,-1}$*, that is type* 30 *in Table 4.1, with parameter* $\gamma = -1$*, because*

$$[y-x+z : x-y : y-z] \circ \varphi_{30,-1} \circ [-x:y:z] = \psi_{19}.$$

♣ *Let* ψ_{31} *be type* 19 *in [11], that is*

$$\psi_{31}([x:y:z]) = [x(x^2+yz+xz) : y(x^2+yz+xz) : xy(x-y)].$$

Then, ψ_{31} *is equivalent to* $\varphi_{29,-1}$*, that is type* 29 *in Table 4.1, with parameter* $\gamma = -1$*, because*

$$[-y : 2x-y-z : 2x] \circ \varphi_{29,(\gamma=-1)} \circ [x:y:2x+2z] = \psi_{31}.$$

Remark 4.9. *In Section 6.4, Théorème 6.39 in [11], there is a list of decompositions in quadratic maps of their 32 types of cubic plane Cremona maps. Note that the decompositions of types 25 and 26 are exchanged and the decomposition of type 24 is incorrect. A correct decomposition in quadratic maps of their type 24 (our type 7) is:*

$$[x(x^2+yz) : y(x^2+yz) : xy^2] = [x:z:y] \circ \rho \circ [y:x+y:z] \circ \rho \circ [z:y:x].$$

Table 4.3: Decompositions of the 31 cubic plane Cremona maps listed in Table 4.1.

$\sharp$	A decomposition $\alpha_n \circ \sigma \circ \alpha_{n-1} \circ \sigma \circ \cdots \circ \alpha_1 \circ \sigma \circ \alpha_0$
1	$\alpha_6 = [27y+225z : 12y : 8x-8y], \alpha_5 = [2x+5y : 5y-x : 15x+15z],$ $\alpha_4 = [2x+2z : 5x : 3x+10y-2z], \alpha_3 = [x-y : z+2y-x : 2y],$ $\alpha_2 = [z : z-2x : 2x+2y-z], \alpha_1 = [x-y : z-x+y : 2x-y], \alpha_0 = [y : y+z : x]$
2	$\alpha_5 = [8y-8x : x+z : 4x], \alpha_4 = [x+y : y : z-x],$ $\alpha_3 = [2x : -y-2x : y+2x-2z], \alpha_2 = [y-x : x : x+z-y],$ $\alpha_1 = [x : z-x : y], \alpha_0 = [x : z : x+y]$
3	$\alpha_5 = [4y : 4y+3x : 4y+4z], \alpha_4 = [3x-z : z-y : y],$ $\alpha_3 = [9z+3x : y : 3z-y], \alpha_2 = [3y+4z-x : x-z : 3x],$ $\alpha_1 = [y+z : x-y+z : y-z], \alpha_0 = [2y : x+z : x-z],$
4	$\alpha_4 = [y+z : x+2z : z-y], \alpha_3 = [2x : y-z : y+z],$ $\alpha_2 = [y-4x-4z : x : z], \alpha_1 = [y+z : x : y-z], \alpha_0 = [2y : x+z : x-z]$
5	$\alpha_5 = [4y+4z : 12z+x+9y : 6y+8z], \alpha_4 = [2y+z : -2x-z : 2x+2z],$ $\alpha_3 = [2y : 2y-z+x : z-y], \alpha_2 = [2z+2x-y : 2z-y : y],$ $\alpha_1 = [y-x-z : 2z+x : x+z], \alpha_0 = [x-z : y : z]$
6	$\alpha_4 = [-2y-4x : 4x : 2x+y+2z], \alpha_3 = [x-2z : z : y], \alpha_2 = [y : z-2y-x : 2x],$ $\alpha_1 = [y-x : x+y : 2z], \alpha_0 = [x-y : x+y : x+y+2z]$
7	$\alpha_4 = [y-x : y : y-z], \alpha_3 = [x : z : z+y], \alpha_2 = [z : y-x-z : x],$ $\alpha_1 = [x : y : x+z], \alpha_0 = [x : x+z : y-x]$

8	$\alpha_5=[2y:-4x-4y:8x+9y+z], \alpha_4=[2x-2y:2y-x:x+2z]$, $\alpha_3=[x+y:x:z-y-2x], \alpha_2=[x+y:-2x:2x+y+z]$, $\alpha_1=[2x+y:y:2x-y+2z], \alpha_0=[-z:x+z:y+z]$
9	$\alpha_4=[z-y:x:z], \alpha_3=[x:y+z:z], \alpha_2=[x:x-y:z]$, $\alpha_1=[x:y+z:z], \alpha_0=[x:z-y:y]$
10	$\alpha_3=[x+y:-y-z:y], \alpha_2=[z-x:x+y:-y]$, $\alpha_1=[z:x-z:y+z-x], \alpha_0=[x:x+z:x+y]$
11	$\alpha_3=[z:x:y], \alpha_2=[x:x+z-y:z], \alpha_1=[x:y+z:z], \alpha_0=[x:z-y:y]$
12	$\alpha_3=[-x:x-z:x+y], \alpha_2=[y-x:x:y+z]$, $\alpha_1=[y:x+y:z-x-y], \alpha_0=[-z:x+z:y-z]$
13	$\alpha_3=[x:y:z], \alpha_2=[z-y:z:x+z-y], \alpha_1=[x:y+z:z], \alpha_0=[z:x-y:y]$
14	$\alpha_3=[x+z:z:y], \alpha_2=[x:z-y:z-x], \alpha_1=[x:y+z:z], \alpha_0=[y:z-x:x]$
15	$\alpha_3=[z-y:y+z:4y-4x], \alpha_2=[x+y:y:z]$, $\alpha_1=[y-x+2z:x-y:x+y], \alpha_0=[x:y:z]$
16	$\alpha_3=[-x:y:2y-z], \alpha_2=[y:x:x+z]$, $\alpha_1=[x+z:-z:y-2x-2z], \alpha_0=[x:x+z:y-x]$
17	$\alpha_4=[-y:x-y:3y+z], \alpha_3=[x+y:y:z], \alpha_2=[z:x:y-x+z]$, $\alpha_1=[x:y-x:z], \alpha_0=[y:y+z:x-y]$
18	$\alpha_3=[x+z:z:z-y], \alpha_2=[x:y+z:z], \alpha_1=[y-x:z-y-x:x], \alpha_0=[x:y:z]$
19	$\alpha_3=[x:z:-y], \alpha_2=[y:z-y:x], \alpha_1=[x:z:y-x], \alpha_0=[x:x+z:y]$
20	$\alpha_3=[z-y:z:x+z], \alpha_2=[x:y+z:z]$, $\alpha_1=[z-x-y:x-y:y], \alpha_0=[x:y:z]$
21	$\alpha_2=[x:y:z], \alpha_1=[x:y:x+y+z], \alpha_0=[x:y:z]$
22	$\alpha_3=[y-2z:z:x+z], \alpha_2=[x:y+z:z]$, $\alpha_1=[x+y-z:2x+y:-x-y], \alpha_0=[x:y:z]$
23	$\alpha_2=[x:-y:z], \alpha_1=[y+z:z:x+y+z], \alpha_0=[z:x:-x-y]$
24	$\alpha_3=[x:y:z], \alpha_2=[x+z:z-x:6z-4y]$, $\alpha_1=[x:y+z:z], \alpha_0=[y-2x:2z-3y:y]$
25	$\alpha_2=[-x:z:y], \alpha_1=[z:x+y:y+z], \alpha_0=[z:y:-x-y]$
26	$\alpha_2=[\gamma(\gamma x-2x+y)+x+z:\gamma(\gamma x-x+y):\gamma(\gamma x+y)]$, $\alpha_1=[\gamma((\gamma-1)x-\gamma y+z):\gamma(y-x):\gamma x], \alpha_0=[x:y:z]$
27	$\alpha_2=[\gamma(\gamma x+y):-\gamma(x+y):(\gamma-1)^2z], \alpha_1=[\gamma x+y:-x-y:(\gamma-1)z]$, $\alpha_0=[x:y:z]$
28	$\alpha_3=[z:\gamma^2(x+y):\gamma^2(x+\gamma x+\gamma y)], \alpha_2=[x+\gamma y-y:-\gamma y:z]$, $\alpha_1=[x:x-\gamma y:-\gamma z], \alpha_0=[x:y:z]$
29	$\alpha_2=[y+z:y-x:\gamma(y-x-z)-x+y+z]$, $\alpha_1=[x-y:x-\gamma y:(1-\gamma)z-x+\gamma y], \alpha_0=[x:y:z]$
30	$\alpha_2=[\gamma^2x+(1-\gamma)y-z:\gamma(\gamma x-y):\gamma((\gamma+1)x-y)]$, $\alpha_1=[\gamma(y+z)-y:y+z:x+z], \alpha_0=[z-x:y-x:x]$
31	$\alpha_2=[a(a(x+(b-1)^2z)+by):a(ax+y):by-((b-1)z-x)a^2-$ $-(b((1-b)z-x)-y)a], \alpha_1=[ax-by:y-x:(b-1)ax-b(a-1)y+(a-b)z]$, $\alpha_0=[x:y:z]$

Remark 4.10. *For types* $9, 11, 13, 14, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 30$ *and* 31*, the decompositions listed in [11] are already minimal.*

Among 31 cubic plane Cremona maps listed in Table 4.1, for those maps where their ordinary quadratic lengths are at least 3, we give in Table 4.4 other decompositions into (not necessary ordinary) quadratic maps.

Table 4.4: Decomposition into quadratic maps of some types.

$\sharp$	A minimal decomposition into quadratic maps
1	$[x : z : y] \circ \rho \circ [z : y : x] \circ \tau \circ [z : y : -x] \circ \rho \circ [x : z : y]$
2	$[x + z : y : z] \circ \rho \circ [y - x : z : x] \circ \tau \circ [y : x : -z]$
3	$[z : x : y] \circ \rho \circ [z : y : x] \circ \tau \circ [z : x : -y]$
4	$[y : z : x] \circ \tau \circ [y : x : -z] \circ \tau \circ [x : z : -y]$
5	$[x : z : y] \circ \tau \circ [-z : -y : x + z] \circ \rho \circ [y - z : x : z]$
6	$[-y : z : x] \circ \rho \circ [x + y + 2z : y + z : -z] \circ \rho \circ [x + z : x : y - x]$
7	$[-x - z : z : y] \circ \rho \circ [-z - y : x + y + z : z] \circ \rho \circ [z - x : y : x]$
8	$[y : x : -z] \circ \tau \circ [z : x + z : y] \circ \rho \circ [x - z : y : z]$
9	$[y : -x - z : z] \circ \rho \circ [-2z - x : x + y + z : z] \circ \rho \circ [x - y : z : y]$
10	$[y : x + z : z] \circ \rho \circ [z - y : x + z : y] \circ \rho \circ [z - x : y : x]$
11	$[x : z : z - y] \circ \rho \circ [z : x + y + z : y] \circ \rho \circ [z - y : x : y]$
12	$[z : x + z : y] \circ \rho \circ [x + z - y : z : y] \circ \rho \circ [y - z : x : z]$
13	$[z : y : x] \circ \sigma \circ [y + x + z : z : y] \circ \rho \circ [z - y : x : y]$
14	$[y : y + z : -x] \circ \rho \circ [x + z : z - y : y] \circ \rho \circ [z - x : y : x]$
15	$[x : z + x : y] \circ \rho \circ [y : z : x - y] \circ \sigma$
16	$[x : z : y + z] \circ \rho \circ [y : x - z - y : y + z] \circ \sigma \circ [x + z : y - x : x]$
17	$[y : x : -z] \circ \tau \circ \sigma$
18	$[x + z : z : -y] \circ \rho \circ [y - x : y - z : x] \circ \sigma$
19	$[z : x : y + z] \circ \rho \circ [z : x - y - z : y] \circ \sigma \circ [x + z : y : x]$
20	$[y : z : x + z] \circ \rho \circ [z - x - y : x : y] \circ \sigma$
22	$[y - z : z : x + z] \circ \rho \circ [z - x - y : x : x + y] \circ \sigma$
24	$[y + z : -z : x - z] \circ \rho \circ [x - y + z : y - x : x] \circ \sigma$
28	$[x : z - y : 2\gamma z - (1 + \gamma)y] \circ \rho \circ [(1 - \gamma)z : x - y : x - \gamma y] \circ \sigma$

Remark 4.11. *With those maps (including types* $21, 23, 25, 26, 27, 29, 30, 31$ *listed in Table 4.1), whose have the ordinary quadratic length exactly* 2 *(hence their quadratic lengths are also* 2*), then their minimal decompositions into quadratic maps can be found in Table 4.3.*

4.3 Proof of the Classification Theorem 4.1

The results in Chapter 2 imply that any cubic plane Cremona map has an enriched weighted proximity graph of the 31 types in Table 4.2. We will show that a cubic plane Cremona map with enriched weighted proximity graph of type n, $1 \le n \le 31$, in Table 4.2 is equivalent to the map of type n in Table 4.1 at page 56.

Lemma 4.12. *Let φ_1 be the map 1 in Table 4.1 and let ψ_1 be a map with enriched weighted proximity graph 1 in Table 4.2. Then, ψ_1 is equivalent to φ_1.*

Proof. The base points of φ_1 are $p_0 = [1 : 0 : 0]$ of multiplicity 2 and $p_1, \dots, p_4$ with $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1 \succ_1 p_0$ and $p_2 \odot p_0$, whose standard coordinates are $p_1 = (p_0, 0)$, $p_2 = (p_0, 0, \infty)$, $p_3 = (p_0, 0, \infty, -1)$, $p_4 = (p_0, 0, \infty, -1, 0)$.

The base points of ψ_1 are q_0 of multiplicity 2 and $q_1, \dots, q_4$ with $q_4 \succ_1 q_3 \succ_1 q_2 \succ_1 q_1 \succ_1 q_0$ and $q_2 \odot p_0$. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$ such that $\alpha_1(p_0) = q_0$ and $\alpha_1(p_1) = q_1$, so that also $\alpha_1(p_2) = q_2$.

The base points of $\psi_1 \circ \alpha_1$ are then $p_0, p_1, p_2, q'_3, q'_4$ where q'_3 has standard coordinates $q'_3 = (p_0, 0, \infty, u_3)$ for some $u_3 \in \mathbb{C}^*$ because, if u_3 were 0, then q'_3 would be proximate to p_0, a contradiction, and, if u_3 were ∞, then q'_3 would be proximate to p_1, again a contradiction.

An automorphism α_2 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2 and that maps p_3 to q'_3 is

$$\alpha_2([x : y : z]) = [-x : u_3 y : u_3 z].$$

The base points of $\psi_1 \circ \alpha_1 \circ \alpha_2$ are then $p_0, p_1, p_2, p_3, q''_4$ where q''_4 has standard coordinates $q''_4 = (p_0, 0, \infty, -1, u_4)$ for some $u_4 \in \mathbb{C}$ because, if u_4 were ∞, then q''_4 would be proximate to p_2, a contradiction.

An automorphism α_3 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2, p_3 and that maps p_4 to q''_4 is

$$\alpha_3([x : y : z]) = [3x : 3y + u_4 z : 3z].$$

Therefore, the maps φ_1 and $\psi_1 \circ \alpha_1 \circ \alpha_2 \circ \alpha_3$ are defined by the same homaloidal net and, hence, φ_1 and ψ_1 are equivalent. □

Lemma 4.13. *Let φ_2 be the map defined by type 2 in Table 4.1 at page 56. Then, φ_2 has only proper base point $p_0 = [0 : 0 : 1]$ of multiplicity 2 and other base points p_1, p_2, p_3, p_4 satisfy $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1 \succ_1 p_0$ where their standard coordinates respectively are $p_1 = (p_0, 0), p_2 = (p_0, 0, -1), p_3 = (p_0, 0, -1, 0)$ and $p_4 = (p_0, 0, -1, 0, 0)$, that is each p_i in the direction of the conic $c_2 : x^2 + yz = 0$ for any $i = 1, \dots, 4$.*

Proof. Consider $\varphi_2 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by

$$\varphi_2([x : y : z]) = [x(x^2 + yz) : y^3 : y(x^2 + yz)].$$

The map has only proper base point $p_0 = [0 : 0 : 1]$ and its multiplicity 2.
A curve C of the linear system associated to φ_2 is of the following form

$$\lambda_1 x(x^2 + yz) + \lambda_2 y^3 + \lambda_3 y(x^2 + yz) = 0,$$

for some $[\lambda_1 : \lambda_2 : \lambda_3] \in \mathbb{P}^2$.
In the affine chart $U_2 = \{[x : y : z] \in \mathbb{P}^2 | z \neq 0\} \simeq \mathbb{A}_{\overline{x},\overline{y}}$, so that p_0 corresponds to the origin $\mathbf{0} = (0, 0)$, the curve C has local equation

$$C_a : \lambda_1 \overline{x}(\overline{x}^2 + \overline{y}) + \lambda_2 \overline{y}^3 + \lambda_3 \overline{y}(\overline{x}^2 + \overline{y}) = 0$$

and the local equation of the conic c_2 is

$$c_{2a} : \overline{x}^2 + \overline{y} = 0.$$

♦ Blowing-up $\mathbb{A}^2_{\overline{x},\overline{y}}$ at $\mathbf{0}$ and consider the first chart given in coordinates by $\overline{x} = x_1$ and $\overline{y} = x_1 y_1$, one has

- the exception curve E_0 is defined by $x_1 = 0$;
- the strict transform of the curve C_a is given by

$$C_{a1} : \lambda_2 x_1 y_1^3 + \lambda_3 x_1 y_1 + \lambda_3 y_1^2 + \lambda_1 x_1 + \lambda_1 y_1 = 0;$$

- the strict transform of the conic c_{2a} is

$$c_{2a1} : x_1 + y_1 = 0.$$

Then, $p_1 = E_0 \cap C_{a1} \cap c_{2a1}$ is the origin of $\mathbb{A}^2_{x_1,y_1}$. In other words, the standard coordinates of p_1 w.r.t φ_2 is $p_1 = (p_0, 0)$. Moreover, one can easy check that p_1 is the only point infinitely near p_0 of the first order.

♦ Blowing-up $\mathbb{A}^2_{x_1,y_1}$ at $\mathbf{0}$ and consider the first chart given in coordinates by $x_1 = x_2$ and $y_1 = x_2 y_2$, one has

- the exception curve E_1 is defined by $x_2 = 0$;
- the strict transform of the curve C_{a1} is given by

$$C_{a2} : \lambda_2 x_2^3 y_2^3 + \lambda_3 x_2 y_2^2 + \lambda_3 x_2 y_2 + \lambda_1 y_2 + \lambda_1 = 0;$$

- the strict transform of the conic c_{2a1} is

$$c_{2a2} : 1 + y_2 = 0.$$

It follows the local coordinate of $p_2 = E_1 \cap C_{a2} \cap c_{2a2}$ in $\mathbb{A}^2_{x_2,y_2}$ is $p_2 = (0,-1)$. Thus, the standard coordinates of p_2 w.r.t φ_2 is $p_2 = (p_0, 0, -1)$ and clearly $p_2 \not\succ p_0$.

♦ Blowing-up $\mathbb{A}^2_{x_2,y_2}$ at $p_2 = (0,-1)$.
Consider $\alpha : \mathbb{A}^2_{x_2,y_2} \to \mathbb{A}^2_{X,Y}$ a linear change coordinates defined as follows

$$\begin{cases} x_2 & = X, \\ y_2 & = Y - 1. \end{cases}$$

With the new coordinates, p_2 is the origin of $\mathbb{A}^2_{X,Y}$ and the equations of the curve C_{a2} and the conic c_{2a2} respectively are

$$C_{a2} : \lambda_2 X^3 Y^3 - 3\lambda_2 X^3 Y^2 + 3\lambda_2 X^3 Y - \lambda_2 X^3 + \lambda_3 X Y^2 - \lambda_3 XY + \lambda_1 Y = 0,$$

and

$$c_{2a2} : Y = 0.$$

Blowing-up $\mathbb{A}^2_{X,Y}$ at $\mathbf{0}$ and consider the first chart given in coordinates by $X = x_3$ and $Y = x_3 y_3$, one has

- the exception curve E_2 is defined by $x_3 = 0$;
- the strict transform of the curve C_{a2} is given by

$$C_{a3} : \lambda_1 y_3 + (x_3^5 y_3^3 - 3x_3^4 y_3^2 + 3x_3^3 y_3 - x_3^2)\lambda_2 + (x_3^2 y_3^2 - x_3 y_3)\lambda_3 = 0;$$

- the strict transform of the conic c_{2a2} is

$$c_{2a3} : y_3 = 0.$$

Then, $p_3 = E_2 \cap C_{a3} \cap c_{2a3}$ is the origin of $\mathbb{A}^2_{x_3,y_3}$. It follows the standard coordinates of p_3 w.r.t φ_2 is $p_3 = (p_0, 0, -1, 0)$.

♦ Blowing-up $\mathbb{A}^2_{x_3,y_3}$ at $\mathbf{0}$ and consider the first chart given in coordinates by $x_3 = x_4$ and $y_3 = x_4, y_4$, one has

- the exception curve E_3 is defined by $x_4 = 0$;
- the strict transform of the curve C_{a3} is given by

$$C_{a4} : \lambda_1 y_4 + (x_4^7 y_4^3 - 3x_4^5 y_4^2 + 3x_4^3 y_4 - x_4)\lambda_2 + (x_4^3 y_4^2 - x_4 y_4)\lambda_3 = 0.$$

- the strict transform of the conic c_{2a3} is

$$c_{2a4} : y_4 = 0.$$

Then, the local coordinate of $p_4 = E_3 \cap C_{a4} \cap c_{2a4}$ is the origin of $\mathbb{A}^2_{x_4,y_4}$. In other words, the standard coordinates of p_4 w.r.t φ_2 is $p_4 = (p_0, 0, -1, 0, 0)$. $\square$

Lemma 4.14. *Let φ_2 be the map 2 in Table 4.1 and let ψ_2 be a map with enriched weighted proximity graph 2 in Table 4.2. Then, ψ_2 is equivalent to φ_2.*

Proof. The base points of φ_2 are $p_0 = [0 : 0 : 1]$ of multiplicity 2 and $p_1, \ldots, p_4$ with standard coordinates $p_1 = (p_0, 0)$, $p_2 = (p_0, 0, -1)$, $p_3 = (p_0, 0, -1, 0)$, $p_4 = (p_0, 0, -1, 0, 0)$. So there is a unique irreducible conic passing through $p_0, \ldots, p_4$, that is $C_1 \colon x^2 + yz = 0$. Let q_0 be the double base point of ψ_2 and let $q_1, \ldots, q_4$ be the simple base points of ψ_2. According to Lemma 1.55, there is a unique irreducible conic C_2 passing through $q_0, \ldots, q_4$. Moreover, Lemma 1.15 implies that there exists an automorphism α of $\mathbb{P}^2$ such that $\alpha(q_0) = p_0$ and $\alpha(C_2) = C_1$. This forces $\alpha(q_i) = p_i$, $i = 1, 2, 3, 4$. Therefore, ψ_2 is equivalent to φ_2. $\square$

Lemma 4.15. *Let φ_3 be the map 3 in Table 4.1 and let ψ_3 be a map with enriched weighted proximity graph 3 in Table 4.2. Then, ψ_3 is equivalent to φ_3.*

Proof. The base points of φ_3 are $p_0 = [0 : 1 : 0]$ of multiplicity 2 and p_1, p_2, p_3, p_4 where $p_1 \succ_1 p_0$ and $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_0$ with standard coordinates $p_1 = (p_0, \infty), p_2 = (p_0, 0), p_3 = (p_0, 0, -1)$ and $p_4 = (p_0, 0, -1, 0)$.

The base points of ψ_3 are q_0 of multiplicity 2 and $q_1, \ldots, q_4$ where $q_1 \succ_1 q_0$ and $q_4 \succ_1 q_3 \succ_1 q_2 \succ_1 q_0$. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$ such that $\alpha_1(q_i) = p_i$ for $i = 0, 1, 2$.

The base points of $\psi_3 \circ \alpha_1$ are then $p_0, p_1, p_2, q_3', q_4'$ where q_3' has standard coordinates $q_3' = (p_0, 0, u_3)$ for some $u_3 \in \mathbb{C}^*$ because, if u_3 were 0, then q_3' would be aligned with p_0 and p_2, a contradiction, and, if u_3 were ∞, then q_3' would be proximate to p_0, again a contradiction.

An automorphism α_2 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2 and that maps $p_3 = (p_0, 0, -1)$ to $q_3' = (p_0, 0, u_3)$ is

$$\alpha_2([x : y : z]) = [u_3 x : -y : u_3 z].$$

The base points of $\psi_3 \circ \alpha_1 \circ \alpha_2$ are then $p_0, p_1, p_2, p_3, q_4''$ where q_4'' has standard coordinates $q_4'' = (p_0, 0, -1, u_4)$ for some $u_4 \in \mathbb{C}$ because, if u_4 were ∞, then q_4'' would be proximate to p_2, a contradiction.

An automorphism α_3 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2, p_3 and that maps p_4 to q_4'' is

$$\alpha_3([x : y : z]) = [x : y - u_4 x : z].$$

Therefore, the maps φ_3 and $\psi_3 \circ \alpha_1 \circ \alpha_2 \circ \alpha_3$ are defined by the same homaloidal net and, hence, φ_3 and ψ_3 are equivalent. $\square$

Lemma 4.16. *Let φ_4 be the map 4 in Table 4.1 and let ψ_4 be a map with enriched weighted proximity graph 4 in Table 4.2. Then, ψ_4 is equivalent to φ_4.*

Proof. The base points of φ_4 are $p_0 = [0 : 1 : 0]$ of multiplicity 2 and $p_1, \ldots, p_4$ where $p_3 \succ_1 p_1 \succ_1 p_0$ and $p_4 \succ_1 p_2 \succ_1 p_0$, with standard coordinates $p_1 = (p_0, \infty)$, $p_3 = (p_0, \infty, -1)$, $p_2 = (p_0, 0)$ and $p_4 = (p_0, 0, -1)$.

The base points of ψ_4 are q_0 of multiplicity 2 and $q_1, \dots, q_4$ where $q_3 \succ_1 q_1 \succ_1 q_0$ and $q_4 \succ_1 q_2 \succ_1 q_0$. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$ such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2$.

The base points of $\psi_4 \circ \alpha_1$ are then $p_0, p_1, p_2, q_3', q_4'$ where q_3' has standard coordinates $q_3' = (p_0, \infty, u_3)$ for some $u_3 \in \mathbb{C}^*$ because, if u_3 were 0, then q_3' would be aligned with p_0 and p_1, a contradiction, and, if u_3 were ∞, then q_3' would be proximate to p_0, again a contradiction.

An automorphism α_2 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2 and that maps $p_3 = (p_0, \infty, -1)$ to $q_3' = (p_0, \infty, u_3)$ is

$$\alpha_3([x : y : z]) = [-u_3 x : y : z].$$

The base points of $\psi_4 \circ \alpha_1 \circ \alpha_2$ are then $p_0, p_1, p_2, p_3, q_4''$ where q_4'' has standard coordinates $q_4'' = (p_0, 0, u_4)$ for some $u_4 \in \mathbb{C}^*$ because, if u_4 were 0, then q_4' would be aligned with p_0 and p_2, a contradiction, and if u_4 were ∞, then q_4'' would be proximate to p_0, a contradiction.

An automorphism α_3 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2, p_3 and that maps $p_4 = (p_0, 0, -1)$ to $q_4'' = (p_0, 0, u_4)$ is

$$\alpha_3([x : y : z]) = [(-u_4)^{2/3} x : y : (-u_4)^{1/3} z].$$

Therefore, the maps φ_4 and $\psi_4 \circ \alpha_1 \circ \alpha_2 \circ \alpha_3$ are defined by the same homaloidal net and, hence, φ_4 and ψ_4 are equivalent. □

Lemma 4.17. *Let φ_5 be the map 5 in Table 4.1 and let ψ_5 be a map with enriched weighted proximity graph 5 in Table 4.2. Then, ψ_5 is equivalent to φ_5.*

Proof. The base points of φ_5 are $p_0 = [0 : 1 : 0]$ of multiplicity 2, $p_1 = [1 : 0 : 0]$ and p_2, p_3, p_4 where $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_0$ and $p_3 \odot p_0$, with standard coordinates $p_2 = (p_0, \infty)$, $p_3 = (p_0, \infty, \infty)$ and $p_4 = (p_0, \infty, \infty, -1)$.

The base points of ψ_5 are $q_0 \in \mathbb{P}^2$ of multiplicity 2, $q_1 \in \mathbb{P}^2$ and q_2, q_3, q_4 where $q_4 \succ_1 q_3 \succ_1 q_2 \succ_1 q_0$ and $q_3 \odot q_0$. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$ such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2$. It follows that also $\alpha_1(p_3) = q_3$.

The base points of $\psi_5 \circ \alpha_1$ are then p_0, p_1, p_2, p_3, q_4' where q_4' has standard coordinates $q_4' = (p_0, \infty, \infty, u_4)$ for some $u_4 \in \mathbb{C}^*$ because, if u_4 were 0, then q_4' would be proximate to p_0, a contradiction, and, if u_4 were ∞, then q_4' would be proximate to p_2, again a contradiction.

An automorphism α_2 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2, p_3 and that maps $p_4 = (p_0, \infty, \infty, -1)$ to $q_4' = (p_0, \infty, \infty, u_4)$ is

$$\alpha_2([x : y : z]) = [x : -u_4 y : z].$$

Therefore, the maps φ_5 and $\psi_5 \circ \alpha_1 \circ \alpha_2$ are defined by the same homaloidal net and, hence, φ_5 and ψ_5 are equivalent. □

Lemma 4.18. *Let φ_6 be the map 6 in Table 4.1 and let ψ_6 be a map with enriched weighted proximity graph 6 in Table 4.2. Then, ψ_6 is equivalent to φ_6.*

Proof. The base points of φ_6 are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [1 : 1 : -1]$ and p_2, p_3, p_4 where $p_2 \succ_1 p_0$ and $p_4 \succ_1 p_3 \succ_1 p_0$, with standard coordinates $p_2 = (p_0, 0)$, $p_3 = (p_0, \infty)$ and $p_4 = (p_0, \infty, -1)$.

The base points of ψ_6 are $q_0 \in \mathbb{P}^2$ of multiplicity 2, $q_1 \in \mathbb{P}^2$ and q_2, q_3, q_4 where $q_2 \succ_1 q_0$ and $q_4 \succ_1 q_3 \succ_1 q_0$. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$ such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2, 3$.

The base points of $\psi_6 \circ \alpha_1$ are then p_0, p_1, p_2, p_3, q_4' where q_4' has standard coordinates $q_4' = (p_0, \infty, u_4)$ for some $u_4 \in \mathbb{C}^*$ because, if u_4 were 0, then q_4' would be aligned with p_0 and p_3, a contradiction, and, if u_4 were ∞, then q_4' would be proximate to p_0, again a contradiction.

An automorphism α_2 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2, p_3 and that maps $p_4 = (p_0, \infty, -1)$ to $q_4' = (p_0, \infty, u_4)$ is

$$\alpha_2([x : y : z]) = [x : y : (-u_4 - 1)x - u_4 z].$$

Therefore, the maps φ_6 and $\psi_6 \circ \alpha_1 \circ \alpha_2$ are defined by the same homaloidal net and, hence, φ_6 and ψ_6 are equivalent. □

Lemma 4.19. *Let φ_7 be the map 7 in Table 4.1 and let ψ_7 be a map with enriched weighted proximity graph 7 in Table 4.2. Then, ψ_7 is equivalent to φ_7.*

Proof. The base points of φ_7 are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [0 : 1 : 0]$ and p_2, p_3, p_4 where $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_0$ with standard coordinates $p_2 = (p_0, 0)$, $p_3 = (p_0, 0, -1)$, $p_4 = (p_0, 0, -1, \infty)$. So there is a unique irreducible conic passing through $p_0, \ldots, p_4$, that is $C_1\colon x^2 + yz = 0$. The base points of ψ_7 are q_0 of multiplicity 2 and $q_1, \ldots, q_4$ where $q_1 \in \mathbb{P}^2$ and $q_4 \succ_1 q_3 \succ_1 q_2 \succ_1 q_0$. According to Lemma 1.54, there is a unique irreducible conic C_2 passing through $q_0, \ldots, q_4$. Moreover, Lemma 1.15 implies that there exists an automorphism α of $\mathbb{P}^2$ such that $\alpha(C_1) = C_2$ and $\alpha(p_i) = q_i$, $i = 0, 1$. This forces $\alpha(p_i) = q_i$, $i = 2, 3, 4$. Therefore, ψ_7 is equivalent to φ_7. □

Lemma 4.20. *Let φ_8 be the map 8 in Table 4.1 and let ψ_8 be a map with enriched weighted proximity graph 8 in Table 4.2. Then, ψ_8 is equivalent to φ_8.*

Proof. The base points of φ_8 are $p_0 = [0 : 1 : 0]$ of multiplicity 2, $p_1 = [1 : 0 : 0]$ and p_2, p_3, p_4 where $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ with standard coordinates $p_2 = (p_1, \infty), p_3 = (p_1, \infty, 0)$ and $p_4 = (p_1, \infty, 0, 1)$.

The base points of ψ_8 are q_0 of multiplicity 2, $q_1 \in \mathbb{P}^2$ and q_2, q_3, q_4 where $q_4 \succ_1 q_3 \succ_1 q_2 \succ_1 q_1$ and q_3 is aligned with q_1 and q_2. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$ such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2$. It follows that also $\alpha_1(p_3) = q_3$.

The base points of $\psi_8 \circ \alpha_1$ are then p_0, p_1, p_2, p_3, q_4' where q_4' has standard coordinates $q_4' = (p_1, \infty, 0, u_4)$ for some $u_4 \in \mathbb{C}^*$ because, if u_4 were 0, then q_4' would be aligned with p_1, p_2, p_3, a contradiction, and, if u_4 were ∞, then q_4' would be proximate to p_2, again a contradiction.

An automorphism α_2 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2, p_3 and that maps $p_4 = (p_1, \infty, 0, 1)$ to $q_4' = (p_0, \infty, 0, u_4)$ is

$$\alpha_2([x : y : z]) = [x : u_4 y : z].$$

Therefore, the maps φ_8 and $\psi_8 \circ \alpha_1 \circ \alpha_2$ are defined by the same homaloidal net and, hence, φ_8 and ψ_8 are equivalent. $\square$

Lemma 4.21. *Let φ_9 be the map 9 in Table 4.1 and let ψ_9 be a map with enriched weighted proximity graph 9 in Table 4.2. Then, ψ_9 is equivalent to φ_9.*

Proof. The base points of φ_9 are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [1 : 0 : 0]$ and p_2, p_3, p_4 where $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ with standard coordinates $p_2 = (p_1, 0)$, $p_3 = (p_1, 0, -1)$, $p_4 = (p_1, 0, -1, 0)$. So there is a unique irreducible conic passing through $p_0, \ldots, p_4$, that is $C_1 \colon xz + y^2 = 0$. The base points of ψ_9 are q_0 of multiplicity 2 and $q_1, \ldots, q_4$ where $q_1 \in \mathbb{P}^2$ and $q_4 \succ_1 q_3 \succ_1 q_2 \succ_1 q_1$. According to Lemma 1.54, there is a unique irreducible conic C_2 passing through $q_0, \ldots, q_4$. Moreover, Lemma 1.15 implies that there exists an automorphism α of $\mathbb{P}^2$ such that $\alpha(C_1) = C_2$ and $\alpha(p_i) = q_i$, $i = 0, 1$. This forces $\alpha(p_i) = q_i$, $i = 2, 3, 4$. Therefore, ψ_9 is equivalent to φ_9. $\square$

Lemma 4.22. *Let φ_{10} be the map 10 in Table 4.1 and let ψ_{10} be a map with enriched weighted proximity graph 10 in Table 4.2. Then, ψ_{10} is equivalent to φ_{10}.*

Proof. The base points of φ_{10} are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [0 : 1 : 0]$ and p_2, p_3, p_4 where $p_2 \succ_1 p_0$ and $p_4 \succ_1 p_3 \succ_1 p_1$ with standard coordinates $p_2 = (p_0, 0), p_3 = (p_1, 0)$ and $p_4 = (p_1, 0, 0)$. The base points of ψ_{10} are q_0 of multiplicity 2, $q_1 \in \mathbb{P}^2$ and q_2, q_3, q_4 where $q_2 \succ_1 q_0$ and $q_4 \succ_1 q_3 \succ_1 q_1$ and q_4 is aligned with q_1 and q_3. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$ such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2, 3$. It follows that also $\alpha_1(p_4) = q_4$, so the maps φ_{10} and $\psi_{10} \circ \alpha_1$ are defined by the same homaloidal net, therefore φ_{10} and ψ_{10} are equivalent. $\square$

Lemma 4.23. *Let φ_{11} be the map 11 in Table 4.1 and let ψ_{11} be a map with enriched weighted proximity graph 11 in Table 4.2. Then, ψ_{11} is equivalent to φ_{11}.*

Proof. The base points of φ_{11} are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [1 : 0 : 0]$ and p_2, p_3, p_4 where $p_2 \succ_1 p_0$ and $p_4 \succ_1 p_3 \succ_1 p_1$ with standard coordinates $p_2 = (p_0, \infty)$, $p_3 = (p_1, 0)$, $p_4 = (p_1, 0, -1)$. So there is a unique irreducible conic passing through $p_0, \ldots, p_4$, that is $C_1 \colon xz + y^2 = 0$. The base points of ψ_{11} are q_0 of multiplicity 2 and $q_1, \ldots, q_4$ where $q_1 \in \mathbb{P}^2$ and $q_2 \succ_1 q_0$ and $q_4 \succ_1 q_3 \succ_1 q_1$. According to Lemma 1.52, there is a unique irreducible conic C_2 passing through $q_0, \ldots, q_4$. Moreover, Lemma 1.15 implies that there exists an automorphism α of $\mathbb{P}^2$ such that $\alpha(C_1) = C_2$ and $\alpha(p_i) = q_i$, $i = 0, 1$. This forces $\alpha(p_i) = q_i$, $i = 2, 3, 4$. Therefore, ψ_{11} is equivalent to φ_{11}. $\square$

Lemma 4.24. *Let φ_{12} be the map 12 in Table 4.1 and let ψ_{12} be a map with enriched weighted proximity graph 12 in Table 4.2. Then, ψ_{12} is equivalent to φ_{12}.*

Proof. The base points of φ_{12} are $p_0 = [0:1:0]$ of multiplicity 2, $p_1 = [1:0:0]$ and p_2, p_3, p_4 where $p_3 \succ_1 p_1$, $p_4 \succ_1 p_2 \succ_1 p_0$ and $p_4 \odot p_0$ with standard coordinates $p_2 = (p_0, \infty), p_3 = (p_1, \infty)$ and $p_4 = (p_0, \infty, \infty)$. The base points of ψ_{12} are q_0 of multiplicity 2, $q_1 \in \mathbb{P}^2$ and q_2, q_3, q_4 where $q_3 \succ_1 q_1$, $q_4 \succ_1 q_2 \succ_1 q_0$ and $q_4 \odot q_0$. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$ such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2, 3$. It follows that also $\alpha_1(p_4) = q_4$, so the maps φ_{12} and $\psi_{12} \circ \alpha_1$ are defined by the same homaloidal net, therefore φ_{12} and ψ_{12} are equivalent. □

Lemma 4.25. *Let φ_{13} be the map 13 in Table 4.1 and let ψ_{13} be a map with enriched weighted proximity graph 13 in Table 4.2. Then, ψ_{13} is equivalent to φ_{13}.*

Proof. The base points of φ_{13} are $p_0 = [0:0:1]$ of multiplicity 2, $p_1 = [1:0:0]$ and p_2, p_3, p_4 where $p_2 \succ_1 p_1$ and $p_4 \succ_1 p_3 \succ_1 p_0$ with standard coordinates $p_2 = (p_1, 0)$, $p_3 = (p_0, \infty)$, $p_4 = (p_0, \infty, -1)$. So there is a unique irreducible conic passing through $p_0, \ldots, p_4$, that is $C_1 \colon xz + y^2 = 0$. The base points of ψ_{13} are q_0 of multiplicity 2 and $q_1, \ldots, q_4$ where $q_1 \in \mathbb{P}^2$ and $q_2 \succ_1 q_1$ and $q_4 \succ_1 q_3 \succ_1 q_0$. According to Lemma 1.52, there is a unique irreducible conic C_2 passing through $q_0, \ldots, q_4$. Moreover, Lemma 1.15 implies that there exists an automorphism α of $\mathbb{P}^2$ such that $\alpha(C_1) = C_2$ and $\alpha(p_i) = q_i$, $i = 0, 1$. This forces $\alpha(p_i) = q_i$, $i = 2, 3, 4$. Therefore, ψ_{13} is equivalent to φ_{13}. □

Lemma 4.26. *Let φ_{14} be the map 14 in Table 4.1 and let ψ_{14} be a map with enriched weighted proximity graph 14 in Table 4.2. Then, ψ_{14} is equivalent to φ_{14}.*

Proof. The base points of φ_{14} are $p_0 = [0:0:1]$ of multiplicity 2, $p_1 = [0:1:0]$ and p_2, p_3, p_4 where $p_2 \succ_1 p_0$, $p_3 \succ_1 p_0$ and $p_4 \succ_1 p_1$ with standard coordinates $p_2 = (p_0, 0), p_3 = (p_0, 1)$ and $p_4 = (p_1, 0)$. The base points of ψ_{14} are q_0 of multiplicity 2, $q_1 \in \mathbb{P}^2$ and q_2, q_3, q_4 where $q_2 \succ_1 q_0$, $q_3 \succ_1 q_0$ and $q_4 \succ_1 q_1$. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$ such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2, 4$.

The base points of $\psi_{14} \circ \alpha_1$ are then p_0, p_1, p_2, q_3', p_4 where q_3' has standard coordinates $q_3' = (p_0, u_3)$ for some $u_3 \in \mathbb{C}^*$ because, if u_3 were 0, then q_3' would be equal to p_2, a contradiction, and, if u_3 were ∞, then q_3' would be aligned with p_0 and p_1, again a contradiction.

An automorphism α_2 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2, p_4 and that maps $p_3 = (p_0, 1)$ to $q_3' = (p_0, u_3)$ is

$$\alpha_2([x:y:z]) = [x : u_3 y : z].$$

Therefore, the maps φ_{14} and $\psi_{14} \circ \alpha_1 \circ \alpha_2$ are defined by the same homaloidal net and, hence, φ_{14} and ψ_{14} are equivalent. □

Lemma 4.27. *Let φ_{15} be the map 15 in Table 4.1 and let ψ_{15} be a map with enriched weighted proximity graph 15 in Table 4.2. Then, ψ_{15} is equivalent to φ_{15}.*

Proof. The base points of φ_{15} are $p_0 = [0:0:1]$ of multiplicity 2, $p_1 = [0:1:0]$, $p_2 = [1:0:0]$ and p_3, p_4 where $p_4 \succ_1 p_3 \succ_1 p_0$ and $p_4 \odot p_0$ with standard coordinates

$p_3 = (p_0, 1)$ and $p_4 = (p_0, 1, \infty)$. The base points of ψ_{15} are q_0 of multiplicity 2, $q_1, q_2 \in \mathbb{P}^2$ and q_3, q_4 where $q_4 \succ_1 q_3 \succ_1 q_0$ and $q_4 \odot q_0$. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$ such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2, 3$. It follows that also $\alpha_1(p_4) = q_4$, so the maps φ_{15} and $\psi_{15} \circ \alpha_1$ are defined by the same homaloidal net, therefore φ_{15} and ψ_{15} are equivalent. $\square$

Lemma 4.28. *Let φ_{16} be the map 16 in Table 4.1 and let ψ_{16} be a map with enriched weighted proximity graph 16 in Table 4.2. Then, ψ_{16} is equivalent to φ_{16}.*

Proof. The base points of φ_{16} are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [0 : 1 : 0]$, $p_2 = [1 : 1 : -1]$ and p_3, p_4 where $p_4 \succ_1 p_3 \succ_1 p_0$ with standard coordinates $p_3 = (p_0, 0)$, $p_4 = (p_0, 0, -1)$. So there is a unique irreducible conic passing through $p_0, \dots, p_4$, that is $C_1 \colon x^2 + yz = 0$. The base points of ψ_{16} are q_0 of multiplicity 2 and $q_1, \dots, q_4$ where $q_1, q_2 \in \mathbb{P}^2$ and $q_4 \succ_1 q_3 \succ_1 q_0$. According to Lemma 1.50, there is a unique irreducible conic C_2 passing through $q_0, \dots, q_4$. Moreover, Lemma 1.15 implies that there exists an automorphism α of $\mathbb{P}^2$ such that $\alpha(C_1) = C_2$ and $\alpha(p_i) = q_i$, $i = 0, 1, 2$. This forces $\alpha(p_i) = q_i$, $i = 3, 4$. Therefore, ψ_{16} is equivalent to φ_{16}. $\square$

Lemma 4.29. *Let φ_{17} be the map 17 in Table 4.1 and let ψ_{17} be a map with enriched weighted proximity graph 17 in Table 4.2. Then, ψ_{17} is equivalent to φ_{17}.*

Proof. The base points of φ_{17} are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$ and p_3, p_4 where $p_4 \succ_1 p_3 \succ_1 p_1$ with standard coordinates $p_3 = (p_1, 0)$ and $p_4 = (p_1, 0, 1)$. The base points of ψ_{17} are q_0 of multiplicity 2, $q_1, q_2 \in \mathbb{P}^2$ and q_3, q_4 where $q_4 \succ_1 q_3 \succ_1 q_1$ and q_3 is aligned with q_1 and q_2. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$ such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2$. It follows that also $\alpha_1(p_3) = q_3$.

The base points of $\psi_{17} \circ \alpha_1$ are then p_0, p_1, p_2, p_3, q_4' where q_4' has standard coordinates $q_4' = (p_1, 0, u_4)$ for some $u_4 \in \mathbb{C}^*$ because, if u_4 were 0, then q_4' would be aligned with p_1, p_2 and p_3, a contradiction, and, if u_4 were ∞, then q_4' would be satellite to p_1, again a contradiction.

An automorphism α_2 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2, p_3 and that maps $p_4 = (p_1, 0, 1)$ to $q_4' = (p_1, 0, u_4)$ is

$$\alpha_2([x : y : z]) = [u_4 x : y : z].$$

Therefore, the maps φ_{17} and $\psi_{17} \circ \alpha_1 \circ \alpha_2$ are defined by the same homaloidal net and, hence, φ_{17} and ψ_{17} are equivalent. $\square$

Lemma 4.30. *Let φ_{18} be the map 18 in Table 4.1 and let ψ_{18} be a map with enriched weighted proximity graph 18 in Table 4.2. Then, ψ_{18} is equivalent to φ_{18}.*

Proof. The base points of φ_{18} are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$ and p_3, p_4 where $p_4 \succ_1 p_3 \succ_1 p_1$ with standard coordinates $p_3 = (p_1, 1)$ and $p_4 = (p_1, 1, 0)$. The base points of ψ_{18} are q_0 of multiplicity 2, $q_1, q_2 \in \mathbb{P}^2$ and q_3, q_4 where $q_4 \succ_1 q_3 \succ_1 q_1$ and q_4 is aligned with q_1 and q_3. Clearly, there exists an automorphism α_1 of

$\mathbb{P}^2$ such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2, 3$. It follows that also $\alpha_1(p_4) = q_4$, so the maps φ_{18} and $\psi_{18} \circ \alpha_1$ are defined by the same homaloidal net, therefore φ_{18} and ψ_{18} are equivalent. $\square$

Lemma 4.31. *Let φ_{19} be the map 19 in Table 4.1 and let ψ_{19} be a map with enriched weighted proximity graph 19 in Table 4.2. Then, ψ_{19} is equivalent to φ_{19}.*

Proof. The base points of φ_{19} are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [0 : 1 : 0]$, $p_2 = [1 : 0 : -1]$ and p_3, p_4 where $p_4 \succ_1 p_3 \succ_1 p_1$ with standard coordinates $p_3 = (p_1, 0)$, $p_4 = (p_1, 0, -1)$. So there is a unique irreducible conic passing through $p_0, \ldots, p_4$, that is $C_1 \colon x^2 + xz + yz = 0$. The base points of ψ_{19} are q_0 of multiplicity 2 and $q_1, \ldots, q_4$ where $q_1, q_2 \in \mathbb{P}^2$ and $q_4 \succ_1 q_3 \succ_1 q_1$. According to Lemma 1.50, there is a unique irreducible conic C_2 passing through $q_0, \ldots, q_4$. Moreover, Lemma 1.15 implies that there exists an automorphism α of $\mathbb{P}^2$ such that $\alpha(C_1) = C_2$ and $\alpha(p_i) = q_i$, $i = 0, 1, 2$. This forces $\alpha(p_i) = q_i$, $i = 3, 4$. Therefore, ψ_{19} is equivalent to φ_{19}. $\square$

Lemma 4.32. *Let φ_{20} be the map 20 in Table 4.1 and let ψ_{20} be a map with enriched weighted proximity graph 20 in Table 4.2. Then, ψ_{20} is equivalent to φ_{20}.*

Proof. The base points of φ_{20} are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$ and p_3, p_4 where $p_3 \succ_1 p_1$ and $p_4 \succ_1 p_2$ with standard coordinates $p_3 = (p_1, 0)$ and $p_4 = (p_2, 1)$. The base points of ψ_{20} are q_0 of multiplicity 2, $q_1, q_2 \in \mathbb{P}^2$ and q_3, q_4 where $q_3 \succ_1 q_1$ and $q_4 \succ_1 q_2$ and q_3 is aligned with q_1 and q_2. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$ such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2, 4$. It follows that also $\alpha_1(p_3) = q_3$, so the maps φ_{20} and $\psi_{20} \circ \alpha_1$ are defined by the same homaloidal net, therefore φ_{20} and ψ_{20} are equivalent. $\square$

Lemma 4.33. *Let φ_{21} be the map 21 in Table 4.1 and let ψ_{21} be a map with enriched weighted proximity graph 21 in Table 4.2. Then, ψ_{21} is equivalent to φ_{21}.*

Proof. The base points of φ_{21} are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$ and p_3, p_4 where $p_3 \succ_1 p_1$ and $p_4 \succ_1 p_2$ with standard coordinates $p_3 = (p_1, -1)$, $p_4 = (p_2, -1)$. So there is a unique irreducible conic passing through $p_0, \ldots, p_4$, that is $C_1 \colon xy + xz + yz = 0$. The base points of ψ_{21} are q_0 of multiplicity 2 and $q_1, \ldots, q_4$ where $q_1, q_2 \in \mathbb{P}^2$, $q_3 \succ_1 q_1$ and $q_4 \succ_1 q_2$. According to Lemma 1.51, there is a unique irreducible conic C_2 passing through $q_0, \ldots, q_4$. Moreover, Lemma 1.15 implies that there exists an automorphism α of $\mathbb{P}^2$ such that $\alpha(C_1) = C_2$ and $\alpha(p_i) = q_i$, $i = 0, 1, 2$. This forces $\alpha(p_i) = q_i$, $i = 3, 4$. Therefore, ψ_{21} is equivalent to φ_{21}. $\square$

Lemma 4.34. *Let φ_{22} be the map 22 in Table 4.1 and let ψ_{22} be a map with enriched weighted proximity graph 22 in Table 4.2. Then, ψ_{22} is equivalent to φ_{22}.*

Proof. The base points of φ_{22} are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$ and p_3, p_4 where $p_3 \succ_1 p_0$ and $p_4 \succ_1 p_1$ with standard coordinates $p_3 = (p_0, -1)$ and $p_4 = (p_1, 0)$. The base points of ψ_{22} are q_0 of multiplicity 2, $q_1, q_2 \in \mathbb{P}^2$ and q_3, q_4 where

$q_3 \succ_1 q_0$ and $q_4 \succ_1 q_1$ and q_4 is aligned with q_1 and q_2. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$ such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2, 3$. It follows that also $\alpha_1(p_4) = q_4$, so the maps φ_{22} and $\psi_{22} \circ \alpha_1$ are defined by the same homaloidal net, therefore φ_{22} and ψ_{22} are equivalent. □

Lemma 4.35. *Let φ_{23} be the map 23 in Table 4.1 and let ψ_{23} be a map with enriched weighted proximity graph 23 in Table 4.2. Then, ψ_{23} is equivalent to φ_{23}.*

Proof. The base points of φ_{23} are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [0 : 1 : 0]$, $p_2 = [1 : -1 : 0]$ and p_3, p_4 where $p_3 \succ_1 p_0$ and $p_4 \succ_1 p_1$ with standard coordinates $p_3 = (p_0, 0)$, $p_4 = (p_1, -1)$. So there is a unique irreducible conic passing through $p_0, \ldots, p_4$, that is $C_1 \colon x^2 + xy + yz = 0$. The base points of ψ_{23} are q_0 of multiplicity 2 and $q_1, \ldots, q_4$ where $q_1, q_2 \in \mathbb{P}^2$, $q_3 \succ_1 q_0$ and $q_4 \succ_1 q_1$. According to Lemma 1.51, there is a unique irreducible conic C_2 passing through $q_0, \ldots, q_4$. Moreover, Lemma 1.15 implies that there exists an automorphism α of $\mathbb{P}^2$ such that $\alpha(C_1) = C_2$ and $\alpha(p_i) = q_i$, $i = 0, 1, 2$. This forces $\alpha(p_i) = q_i$, $i = 3, 4$. Therefore, ψ_{23} is equivalent to φ_{23}. □

Lemma 4.36. *Let φ_{24} be the map 24 in Table 4.1 and let ψ_{24} be a map with enriched weighted proximity graph 24 in Table 4.2. Then, ψ_{24} is equivalent to φ_{24}.*

Proof. The base points of φ_{24} are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$, $p_3 = [1 : 1 : 0]$ and p_4 where $p_4 \succ_1 p_1$ with standard coordinates $p_4 = (p_1, 1)$. The base points of ψ_{24} are q_0 of multiplicity 2, $q_1, q_2, q_3 \in \mathbb{P}^2$ and q_4 where $q_4 \succ_1 q_1$ and q_3 is aligned with q_1 and q_2. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$ such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2, 3$.

The base points of $\psi_{24} \circ \alpha_1$ are then p_0, p_1, p_2, p_3, q'_4 where q'_4 has standard coordinates $q'_4 = (p_1, u_4)$ for some $u_4 \in \mathbb{C}^*$ because, if u_4 were 0, then q'_4 would be aligned with p_1, p_2 and p_3, a contradiction, and, if u_4 were ∞, then q'_4 would be aligned with p_0 ad p_1, again a contradiction.

An automorphism α_2 of $\mathbb{P}^2$ that fixes p_0, p_1, p_2, p_3 and that maps $p_4 = (p_1, 1)$ to $q'_4 = (p_1, u_4)$ is

$$\alpha_2([x : y : z]) = [x : y : u_4 z].$$

Therefore, the maps φ_{24} and $\psi_{24} \circ \alpha_1 \circ \alpha_2$ are defined by the same homaloidal net and, hence, φ_{24} and ψ_{24} are equivalent. □

Lemma 4.37. *Let φ_{25} be the map 25 in Table 4.1 and let ψ_{25} be a map with enriched weighted proximity graph 25 in Table 4.2. Then, ψ_{25} is equivalent to φ_{25}.*

Proof. The base points of φ_{25} are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : -1]$, $p_3 = [1 : -1 : 0]$ and p_4 where $p_4 \succ_1 p_1$ with standard coordinates $p_4 = (p_1, 0)$. The base points of ψ_{25} are q_0 of multiplicity 2, $q_1, q_2, q_3 \in \mathbb{P}^2$ and q_4 where $q_4 \succ_1 q_1$ and q_4 is aligned with q_1 and q_2. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$

such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2, 3$. It follows that also $\alpha_1(p_4) = q_4$, so the maps φ_{25} and $\psi_{25} \circ \alpha_1$ are defined by the same homaloidal net, therefore φ_{25} and ψ_{25} are equivalent. $\square$

Lemma 4.38. *Let $\varphi_{26,\gamma}$ be the map 26 in Table 4.1 with parameter γ and let ψ_{26} be a map with enriched weighted proximity graph 26 in Table 4.2. Then, ψ_{26} is equivalent to $\varphi_{26,\gamma}$ for some $\gamma \neq 0, 1$.*

Proof. The base points of $\varphi_{26,\gamma}$ are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$, $p_3 = [1 : 1 : 1]$ and p_4 where $p_4 \succ_1 p_1$ with standard coordinates $p_4 = (p_1, 1/\gamma)$. The base points of ψ_{26} are q_0 of multiplicity 2, $q_1, q_2, q_3 \in \mathbb{P}^2$ and q_4 where $q_4 \succ_1 q_1$. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$ such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2, 3$.

The base points of $\psi_{26} \circ \alpha_1$ are then p_0, p_1, p_2, p_3, q_4' where q_4' has standard coordinates $q_4' = (p_1, u_4)$ for some $u_4 \in \mathbb{C}^{**}$ because, if u_4 were 0, then q_4' would be aligned with p_1 and p_2, a contradiction; if u_4 were ∞, then q_4' would be aligned with p_0 ad p_1, again a contradiction, and, if u_4 were 1, then q_4' would be aligned with p_1 and p_3, still a contradiction. Setting $\gamma = 1/u_4$, the maps $\varphi_{26,\gamma}$ and $\psi_{26} \circ \alpha_1$ are defined by the same homaloidal net, therefore $\varphi_{26,\gamma}$ and ψ_{26} are equivalent. $\square$

Lemma 4.39. *Let $\varphi_{27,\gamma}$ be the map 27 in Table 4.1 with parameter γ and let ψ_{27} be a map with enriched weighted proximity graph 27 in Table 4.2. Then, ψ_{27} is equivalent to $\varphi_{27,\gamma}$ for some $\gamma \neq 0, 1$.*

Proof. The base points of $\varphi_{27,\gamma}$ are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [0 : 1 : 0]$, $p_2 = [1 : 0 : 0]$ and p_3, p_4 where $p_3 \succ_1 p_0$ and $p_4 \succ_1 p_0$ with standard coordinates $p_3 = (p_0, -1)$ and $p_4 = (p_0, -1/\gamma)$. The base points of ψ_{27} are q_0 of multiplicity 2, $q_1, q_2 \in \mathbb{P}^2$ and q_3, q_4 where $q_3 \succ_1 q_0$ and $q_4 \succ_1 q_0$. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$ such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2, 3$.

The base points of $\psi_{27} \circ \alpha_1$ are then p_0, p_1, p_2, p_3, q_4' where q_4' has standard coordinates $q_4' = (p_0, u_4)$ for some $u_4 \in \mathbb{C}^{**}$ because, if u_4 were 0, then q_4' would be aligned with p_0 and p_2, a contradiction; if u_4 were ∞, then q_4' would be aligned with p_0 ad p_1, again a contradiction, and, if u_4 were 1, then q_4' would be equal to p_3, still a contradiction. Setting $\gamma = -1/u_4$, the maps $\varphi_{27,\gamma}$ and $\psi_{27} \circ \alpha_1$ are defined by the same homaloidal net, therefore $\varphi_{27,\gamma}$ and ψ_{27} are equivalent. $\square$

Lemma 4.40. *Let $\varphi_{28,\gamma}$ be the map 28 in Table 4.1 with parameter γ and let ψ_{28} be a map with enriched weighted proximity graph 28 in Table 4.2. Then, ψ_{28} is equivalent to $\varphi_{28,\gamma}$ for some $\gamma \neq 0, 1$.*

Proof. The base points of $\varphi_{28,\gamma}$ are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [0 : 1 : 0]$, $p_2 = [1 : 0 : 0]$, $p_3 = [1 : 1 : 0]$ and p_4 where $p_4 \succ_1 p_0$ with standard coordinates $p_4 = (p_0, \gamma)$. The base points of ψ_{28} are q_0 of multiplicity 2, $q_1, q_2, q_3 \in \mathbb{P}^2$ and q_4 where $q_4 \succ_1 q_0$ and q_1, q_2, q_3 are collinear. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$ such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2, 3$.

The base points of $\psi_{28} \circ \alpha_1$ are then p_0, p_1, p_2, p_3, q'_4 where $q'_4 = (p_0, u_4)$ for some $u_4 \in \mathbb{C}^{**}$ because, if u_4 were 0, then q'_4 would be aligned with p_0 and p_2, a contradiction; if u_4 were ∞, then q'_4 would be aligned with p_0 ad p_1, again a contradiction, and, if u_4 were 1, then q'_4 would be aligned with p_0 and p_3, still a contradiction. Setting $\gamma = u_4$, the maps $\varphi_{28,\gamma}$ and $\psi_{28} \circ \alpha_1$ are defined by the same homaloidal net, therefore $\varphi_{28,\gamma}$ and ψ_{28} are equivalent. □

Lemma 4.41. *Let $\varphi_{29,\gamma}$ be the map 29 in Table 4.1 with parameter γ and let ψ_{29} be a map with enriched weighted proximity graph 29 in Table 4.2. Then, ψ_{29} is equivalent to $\varphi_{29,\gamma}$ for some $\gamma \neq 0, 1$.*

Proof. The base points of $\varphi_{29,\gamma}$ are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [0 : 1 : 0]$, $p_2 = [1 : 0 : 0]$, $p_3 = [1 : 1 : 1]$ and p_4 where $p_4 \succ_1 p_0$ with standard coordinates $p_4 = (p_0, \gamma)$. The base points of ψ_{29} are q_0 of multiplicity 2, $q_1, q_2, q_3 \in \mathbb{P}^2$ and q_4 where $q_4 \succ_1 q_0$. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$ such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2, 3$.

The base points of $\psi_{29} \circ \alpha_1$ are then p_0, p_1, p_2, p_3, q'_4 where $q'_4 = (p_0, u_4)$ for some $u_4 \in \mathbb{C}^{**}$ because, if u_4 were 0, then q'_4 would be aligned with p_0 and p_2, a contradiction; if u_4 were ∞, then q'_4 would be aligned with p_0 ad p_1, again a contradiction, and, if u_4 were 1, then q'_4 would be aligned with p_0 and p_3, still a contradiction. Setting $\gamma = u_4$, the maps $\varphi_{29,\gamma}$ and $\psi_{29} \circ \alpha_1$ are defined by the same homaloidal net, therefore $\varphi_{29,\gamma}$ and ψ_{29} are equivalent. □

Lemma 4.42. *Let $\varphi_{30,\gamma}$ be the map 30 in Table 4.1 with parameter γ and let ψ_{30} be a map with enriched weighted proximity graph 30 in Table 4.2. Then, ψ_{30} is equivalent to $\varphi_{30,\gamma}$ for some $\gamma \neq 0, 1$.*

Proof. The base points of $\varphi_{30,\gamma}$ are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [0 : 1 : 0]$, $p_2 = [1 : 0 : 0]$, $p_3 = [\gamma : 1 : 0]$ and $p_4 = [1 : 1 : 1]$. The base points of ψ_{30} are q_0 of multiplicity 2, $q_1, q_2, q_3, q_4 \in \mathbb{P}^2$ where q_1, q_2, q_3 are collinear. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$ such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2, 4$.

The base points of $\psi_{30} \circ \alpha_1$ are then p_0, p_1, p_2, q'_3, p_4 where $q'_3 = [u_3 : 1 : 0]$ for some $u_3 \in \mathbb{C}^{**}$ because, if u_3 were 0, then q'_3 would be equal to p_1, a contradiction, and, if u_3 were 1, then q'_3 would be aligned with p_0 and p_4, again a contradiction. Setting $\gamma = u_3$, the maps $\varphi_{30,\gamma}$ and $\psi_{30} \circ \alpha_1$ are defined by the same homaloidal net, therefore $\varphi_{30,\gamma}$ and ψ_{30} are equivalent. □

Lemma 4.43. *Let $\varphi_{31,a,b}$ be the map 31 in Table 4.1 with parameters a, b and let ψ_{31} be a map with enriched weighted proximity graph 31 in Table 4.2. Then, ψ_{31} is equivalent to $\varphi_{31,\gamma}$ for some $a, b \neq 0, 1$, $a \neq b$.*

Proof. The base points of $\varphi_{31,\gamma}$ are $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [0 : 1 : 0]$, $p_2 = [1 : 0 : 0]$, $p_3 = [1 : 1 : 1]$ and $p_4 = [a : b : 1]$. The base points of ψ_{31} are q_0 of multiplicity 2 and $q_1, q_2, q_3, q_4 \in \mathbb{P}^2$. Clearly, there exists an automorphism α_1 of $\mathbb{P}^2$ such that $\alpha_1(p_i) = q_i$ for $i = 0, 1, 2, 3$.

The base points of $\psi_{31} \circ \alpha_1$ are then p_0, p_1, p_2, p_3, q'_4 where $q'_4 = [t_4 : u_4 : v_4]$ with $t_4, u_4, v_4 \in \mathbb{C}^*$: indeed,

- $v_4 \neq 0$ because otherwise q_4' would be aligned with p_1 and p_2;
- $u_4 \neq 0$ because otherwise q_4' would be aligned with p_0 and p_1;
- $t_4 \neq 0$ because otherwise q_4' would be aligned with p_0 and p_2.

Moreover, t_4/v_4 and u_4/v_4 satisfy the following conditions:

- $t_4/v_4 \neq 1$ because otherwise q_4' would be aligned with p_1 and p_3;
- $u_4/v_4 \neq 1$ because otherwise q_4' would be aligned with p_2 and p_3;
- $t_4/v_4 \neq u_4/v_4$ because otherwise q_4' would be aligned with p_0 and p_3.

Setting $a = t_4/v_4$ and $b = u_4/v_4$, it follows that $a, b \in \mathbb{C}^{**}$ and $a \neq b$, the maps $\varphi_{31,a,b}$ and $\psi_{31} \circ \alpha_1$ are defined by the same homaloidal net, therefore $\varphi_{31,a,b}$ and ψ_{31} are equivalent. $\square$

Lemma 4.44. *Set $\varphi_{26,\gamma}$ the map of type 26 in Table 4.1 with parameter γ where $\gamma \neq 0, 1$. Then, $\varphi_{26,\gamma}$ is equivalent to $\varphi_{26,\gamma'}$ if and only if either $\gamma' = \gamma$ or $\gamma' = \gamma/(\gamma - 1)$.*

Proof. Let $p_0, p_1, \ldots, p_4$ be the base points of $\varphi_{26,\gamma}$ as in the proof of Lemma 4.38.
An automorphism α of $\mathbb{P}^2$ that fixes the homaloidal net defining $\varphi_{26,\gamma}$, and that is different from the identity, is such that $\alpha(p_i) = p_i$, $i = 0, 1$, $\alpha(p_2) = p_3$ and $\alpha(p_3) = p_2$. Therefore, α is unique and it is defined by

$$\alpha([x : y : z]) = [y - x : y : y - z].$$

so $\alpha(p_4)$ has standard coordinates $(p_1, (\gamma - 1)/\gamma)$, hence $\varphi_{26,\gamma/(\gamma-1)}$ is equivalent to $\varphi_{26,\gamma}$. $\square$

Lemma 4.45. *Set $\varphi_{27,\gamma}$ the map of type 27 in Table 4.1 with parameter γ where $\gamma \neq 0, 1$. Then, $\varphi_{27,\gamma}$ is equivalent to $\varphi_{27,\gamma'}$ if and only if either $\gamma' = \gamma$ or $\gamma' = 1/\gamma$.*

Proof. Let p_0, p_1, p_2, p_3, p_4 be the base points of $\varphi_{27,\gamma}$ as in the proof of Lemma 4.39.
The base points of $\varphi_{27,\gamma'}$ are $q_i = p_i$, $i = 0, 1, 2, 3$, and $q_4 = (q_0, -1/\gamma')$.
Suppose that $\varphi_{27,\gamma'}$ is equivalent to $\varphi_{27,\gamma}$. This implies that there exist automorphisms $\alpha_1, \ldots, \alpha_4$ of $\mathbb{P}^2$ with the following properties:

(1) α_1 is such that $\alpha_1(p_i) = q_i$, $i = 0, 1, 2, 3, 4$;

(2) α_2 is such that $\alpha_2(p_i) = q_i$, $i = 0, 1, 2$, $\alpha_2(p_3) = q_4$ and $\alpha_2(p_4) = q_3$;

(3) α_3 is such that $\alpha_3(p_i) = q_i$, $i = 0, 3, 4$, $\alpha_3(p_1) = q_2$ and $\alpha_3(p_2) = q_1$;

(4) α_4 is such that $\alpha_4(p_0) = q_0$, $\alpha_4(p_1) = q_2$, $\alpha_4(p_2) = q_1$, $\alpha_4(p_3) = q_4$ and $\alpha_4(p_4) = q_3$.

Then, Case (1) occurs only if $\gamma' = \gamma$ and α_1 is the identity. Case (2) occurs only if $\gamma' = 1/\gamma$ and $\alpha_2([x : y : z]) = [x : \gamma y : -\gamma z]$. Case (3) occurs only if $\gamma' = 1/\gamma$ and $\alpha_3([x : y : z]) = [y : x : -z]$. Case (4) occurs only if $\gamma' = \gamma$ and $\alpha_4([x : y : z]) = [\gamma y : x : z]$. $\square$

Before moving on other types, let us first recall several definitions from permutations with cycle notation.

$\mathfrak{S}_n$ denotes the set of permutations of $\{1, 2, \ldots, n\}$. $\mathfrak{s} \in \mathfrak{S}_n$ is a one-to-one and onto mapping from $\{1, 2, \ldots, n\}$ to itself. An explicit representation of $\mathfrak{s}$ can be given by the $2 \times n$ matrix:

$$\begin{bmatrix} 1 & 2 & 3 & \ldots & n \\ \mathfrak{s}(1) & \mathfrak{s}(2) & \mathfrak{s}(3) & \ldots & \mathfrak{s}(n) \end{bmatrix}$$

or simply by $\{\mathfrak{s}(1), \mathfrak{s}(2), \mathfrak{s}(3), \ldots, \mathfrak{s}(n)\}$. Every permutation of a finite set can be written as a cycle or a product of disjoint cycles. More precisely, the elements in each cycle are put inside parentheses, ordered so that $\mathfrak{s}(i)$ immediately follows i. Without any confusion, one can consider a cycle as fixing any element not appearing in it and particularly, the permutation which fixes all elements is denoted by (1). We list in Tables 4.5 and 4.6 all permutations and their cycle notations of $\mathfrak{S}_3$ and $\mathfrak{S}_4$ respectively.

Table 4.5: 6 permutations of $\mathfrak{S}_3$.

i	permutation $\mathfrak{s}_i$ of $\mathfrak{S}_3$	cycle notation
1	$\{1, 2, 3\}$	(1)
2	$\{1, 3, 2\}$	(23)
3	$\{2, 1, 3\}$	(12)
4	$\{2, 3, 1\}$	(123)
5	$\{3, 2, 1\}$	(13)
6	$\{3, 1, 2\}$	(132)

Table 4.6: 24 permutations of $\mathfrak{S}_4$.

i	permutation $\mathfrak{s}_i$ of $\mathfrak{S}_4$	cycle notation	i	permutation $\mathfrak{s}_i$ of $\mathfrak{S}_4$	cycle notation	i	permutation $\mathfrak{s}_i$ of $\mathfrak{S}_4$	cycle notation
1	$\{1, 2, 3, 4\}$	(1)	9	$\{3, 1, 2, 4\}$	(132)	17	$\{3, 2, 4, 1\}$	(134)
2	$\{2, 1, 3, 4\}$	(12)	10	$\{3, 2, 1, 4\}$	(13)	18	$\{3, 1, 4, 2\}$	(1342)
3	$\{1, 2, 4, 3\}$	(34)	11	$\{4, 1, 2, 3\}$	(1432)	19	$\{4, 2, 3, 1\}$	(14)
4	$\{2, 1, 4, 3\}$	(12)(34)	12	$\{4, 2, 1, 3\}$	(143)	20	$\{4, 1, 3, 2\}$	(142)
5	$\{1, 3, 2, 4\}$	(23)	13	$\{2, 3, 4, 1\}$	(1234)	21	$\{3, 4, 1, 2\}$	(13)(24)
6	$\{2, 3, 1, 4\}$	(123)	14	$\{1, 3, 4, 2\}$	(234)	22	$\{3, 4, 2, 1\}$	(1324)
7	$\{1, 4, 2, 3\}$	(243)	15	$\{2, 4, 3, 1\}$	(124)	23	$\{4, 3, 1, 2\}$	(1423)
8	$\{2, 4, 1, 3\}$	(1243)	16	$\{1, 4, 3, 2\}$	(24)	24	$\{4, 3, 2, 1\}$	(14)(23)

Lemma 4.46. *For $n \in \{28, 29, 30\}$, set $\varphi_{n,\gamma}$ the map of type n in Table 4.1 with parameter γ where $\gamma \neq 0, 1$. Then, $\varphi_{n,\gamma'}$ is equivalent to $\varphi_{n,\gamma}$ if and only if*

$$\gamma' \in \left\{\gamma, \frac{1}{\gamma}, 1-\gamma, \frac{1}{1-\gamma}, \frac{\gamma}{\gamma-1}, \frac{\gamma-1}{\gamma}\right\}.$$

Proof. We first consider the case $n = 28$.

The map $\varphi_{28,\gamma}$ has base points $p_0 = [0 : 0 : 1]$ of multiplicity 2, $p_1 = [0 : 1 : 0]$, $p_2 = [1 : 0 : 0]$, $p_3 = [1 : 1 : 0]$ and p_4 where $p_4 \succ_1 p_0$ with standard coordinates $p_4 = (p_0, \gamma)$.

The base points of $\varphi_{28,\gamma'}$ are $q_0, \dots, q_4$ where $q_i = p_i$, $i = 0, 1, 2, 3$ and $q_4 = (p_0, \gamma')$.

Suppose that $\varphi_{28,\gamma'}$ is equivalent to $\varphi_{28,\gamma}$. This implies that there exist automorphisms $\alpha_1, \dots, \alpha_6$ of $\mathbb{P}^2$ such that, for $i = 1, \dots, 6$, one has $\alpha_i(p_j) = q_j$, $j = 0, 4$, and

$$\alpha_i(p_j) = q_{\mathfrak{s}_i(j)} \quad \text{for } j = 1, 2, 3,$$

where $\mathfrak{s}_1, \dots, \mathfrak{s}_6$ are the six elements of $\mathfrak{S}_3$ given in Table 4.5.

- Case $i = 1$ occurs only if $\gamma' = \gamma$ and α_1 is the identity.
- Case $i = 2$ occurs only if $\gamma' = 1 - \gamma$ and $\alpha_2 = [x : x - y : z]$.
- Case $i = 3$ occurs only if $\gamma' = 1/\gamma$ and $\alpha_3 = [y : x : z]$.
- Case $i = 4$ occurs only if $\gamma' = 1/(1 - \gamma)$ and $\alpha_4 = [x - y : x : z]$.
- Case $i = 5$ occurs only if $\gamma' = \gamma/(\gamma - 1)$ and $\alpha_5 = [x - y : -y : z]$.
- Case $i = 6$ occurs only if $\gamma' = \gamma/(\gamma - 1)$ and $\alpha_6 = [y : y - x : z]$.

We proceed similarly for $n = 29$. The map $\varphi_{29,\gamma}$ has the same base points p_i, $i = 0, 1, 2, 4$, of $\varphi_{28,\gamma}$ but $p_3 = [1 : 1 : 1]$. The base points of $\varphi_{29,\gamma'}$ are $q_0, \dots, q_4$ where $q_i = p_i$, $i = 0, 1, 2, 3$ and $q_4 = (q_0, \gamma')$.

If $\varphi_{28,\gamma'}$ is equivalent to $\varphi_{28,\gamma}$, then there exist automorphisms $\alpha_1, \dots, \alpha_6$ of $\mathbb{P}^2$ with the same above properties that occur exactly when γ' is as above and α_1 is the identity,

$$\alpha_2 = [x : x - y : x - z], \qquad \alpha_3 = [y : x : z], \qquad \alpha_4 = [x - y : x : x - z],$$
$$\alpha_5 = [y - x : y : y - z], \qquad \alpha_6 = [y : y - x : y - z].$$

Finally, for $n = 30$, the map $\varphi_{30,\gamma}$ has the same base points p_i, $i = 0, 1, 2$, of $\varphi_{28,\gamma}$ but $p_3 = [\gamma : 1 : 0]$ and $p_4 = [1 : 1 : 1]$. The base points of $\varphi_{30,\gamma'}$ are $q_0, \dots, q_4$ where $q_i = p_i$, $i = 0, 1, 2, 4$ and $q_3 = [\gamma' : 1 : 0]$.

If $\varphi_{30,\gamma'}$ is equivalent to $\varphi_{30,\gamma}$, then there exist automorphisms $\alpha_1, \dots, \alpha_6$ of $\mathbb{P}^2$ with the same above properties that occur exactly when γ' is as above and α_1 is the identity,

$$\alpha_2 = [(\gamma - 1)x : \gamma y - x : (\gamma - 1)z], \qquad \alpha_3 = [y : x : z],$$
$$\alpha_4 = [\gamma y - x : (\gamma - 1)x : (\gamma - 1)z], \qquad \alpha_5 = [\gamma y - x : (\gamma - 1)y : (\gamma - 1)z],$$
$$\alpha_6 = [(\gamma - 1)y : \gamma y - x : (\gamma - 1)z].$$

□

Remark 4.47. *One may check that the numbers in the set of Lemma 4.46 are all different if and only if*

$$\gamma \notin \left\{-1, 2, \frac{1}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}, \frac{1}{2} + i\frac{\sqrt{3}}{2}\right\}.$$

Lemma 4.48. *Set $\varphi_{31,a,b}$ the map of type 31 in Table 4.1 with two parameters a, b where $a \neq b$ and $a, b \neq 0, 1$. Then, $\varphi_{31,a',b'}$ is equivalent to $\varphi_{31,a,b}$ if and only if $(a', b') \in S$, where S is defined in* (4.1).

Proof. The base points of $\varphi_{31,a,b}$ are $p_0 = [0 : 0 : 1]$ of multiplicity 2 and four simple base points $p_1 = [0 : 1 : 0], p_2 = [1 : 0 : 0], p_3 = [1 : 1 : 1], p_4 = [a : b : 1]$. Similarly, the base points of $\varphi_{31,a',b'}$ are $q_0, \ldots, q_4$ where $q_i = p_i$, $i = 0, 1, 2, 3$ and $q_4 = [a' : b' : 1]$.

Suppose that $\varphi_{31,a',b'}$ is equivalent to $\varphi_{31,a,b}$. Then, there exists an automorphism, says γ, of $\mathbb{P}^2$ such that $\gamma(p_0) = q_0$ and γ maps $p_1, \ldots, p_4$ to a permutation of q_1, q_2, q_3, q_4. Therefore, for each element $\mathfrak{s}_i$, $i = 1, \ldots, 24$, of $\mathfrak{S}_4$ there is an automorphism γ_i, $i = 1, \ldots, 24$, of $\mathbb{P}^2$ such that

$$\gamma_i(p_j) = q_{\mathfrak{s}_i(j)} \quad \text{for } j = 1, \ldots, 4,$$

and, accordingly, we find the values of (a', b') for each one of the 24 cases. In Table 4.7, we list the automorphisms γ_i, $i = 1, \ldots, 24$ and their corresponding values of (a', b').

Table 4.7: Automorphisms $\gamma_1, \ldots, \gamma_{24}$ of $\mathbb{P}^2$ and their corresponding values of (a', b')

i	$\gamma_i([x : y : z])$	(a', b')
1	$[x : y : z]$	(a, b)
2	$[y : x : z]$	(b, a)
3	$[bx : ay : abz]$	$\left(\frac{1}{a}, \frac{1}{b}\right)$
4	$[ay : bx : abz]$	$\left(\frac{1}{b}, \frac{1}{a}\right)$
5	$[x : x - y : x - z]$	$\left(\frac{a}{a-1}, \frac{a-b}{a-1}\right)$
6	$[x - y : x : x - z]$	$\left(\frac{a-b}{a-1}, \frac{a}{a-1}\right)$
7	$\left[\frac{x}{a} : \frac{x-y}{a-b} : \frac{x-z}{a-1}\right]$	$\left(\frac{a-1}{a}, \frac{a-1}{a-b}\right)$
8	$\left[\frac{x-y}{a-b} : \frac{x}{a} : \frac{x-z}{a-1}\right]$	$\left(\frac{a-1}{a-b}, \frac{a-1}{a}\right)$
9	$[y : y - x : y - z]$	$\left(\frac{b}{b-1}, \frac{b-a}{b-1}\right)$
10	$[y - x : y : y - z]$	$\left(\frac{b-a}{b-1}, \frac{b}{b-1}\right)$

11	$\left[\dfrac{y}{b}:\dfrac{x-y}{a-b}:\dfrac{y-z}{b-1}\right]$	$\left(\dfrac{b-1}{b},\dfrac{b-1}{b-a}\right)$
12	$\left[\dfrac{x-y}{a-b}:\dfrac{y}{b}:\dfrac{y-z}{b-1}\right]$	$\left(\dfrac{b-1}{b-a},\dfrac{b-1}{b}\right)$
13	$[bx-ay:bx:b(x-az)]$	$\left(\dfrac{b-a}{b(1-a)},\dfrac{1}{1-a}\right)$
14	$[bx:bx-ay:b(x-az)]$	$\left(\dfrac{1}{1-a},\dfrac{b-a}{b(1-a)}\right)$
15	$\left[\dfrac{ay-bx}{a-b}:x:\dfrac{az-x}{a-1}\right]$	$\left(\dfrac{b(a-1)}{a-b},1-a\right)$
16	$\left[x:\dfrac{ay-bx}{a-b}:\dfrac{az-x}{a-1}\right]$	$\left(1-a,\dfrac{b(a-1)}{a-b}\right)$
17	$[ay-bx:ay:a(y-bz)]$	$\left(\dfrac{a-b}{a(1-b)},\dfrac{1}{1-b}\right)$
18	$[ay:ay-bx:a(y-bz)]$	$\left(\dfrac{1}{1-b},\dfrac{a-b}{a(1-b)}\right)$
19	$\left[\dfrac{ay-bx}{a-b}:y:\dfrac{bz-y}{b-1}\right]$	$\left(\dfrac{a(1-b)}{a-b},1-b\right)$
20	$\left[y:\dfrac{ay-bx}{a-b}:\dfrac{bz-y}{b-1}\right]$	$\left(1-b,\dfrac{a(1-b)}{a-b}\right)$
21	$\left[y-x:\dfrac{ay-bx}{a}:\dfrac{(1-b)x}{a-1}+\dfrac{(b-a)z}{a-1}+y\right]$	$\left(\dfrac{a-1}{b-1},\dfrac{b(a-1)}{a(b-1)}\right)$
22	$\left[\dfrac{ay-bx}{a}:y-x:\dfrac{(1-b)x}{a-1}+\dfrac{(b-a)z}{a-1}+y\right]$	$\left(\dfrac{b(a-1)}{a(b-1)},\dfrac{a-1}{b-1}\right)$
23	$\left[y-x:\dfrac{ay-bx}{b}:\dfrac{(a-1)y}{b-1}+\dfrac{(b-a)z}{b-1}-x\right]$	$\left(\dfrac{b-1}{a-1},\dfrac{a(b-1)}{b(a-1)}\right)$
24	$\left[\dfrac{ay-bx}{b}:y-x:\dfrac{(a-1)y}{b-1}+\dfrac{(b-a)z}{b-1}-x\right]$	$\left(\dfrac{a(b-1)}{b(a-1)},\dfrac{b-1}{a-1}\right)$

$\square$

Remark 4.49. *One may check that the pairs in S are all different if and only if (a,b) does not belong to the following set:*

$$\left\{\left(a,\frac{1}{a}\right)\middle|a\neq -1\right\}\cup\left\{\left(a,\frac{2a-1}{a}\right)\middle|a\neq\frac{1}{2}\right\}\cup\left\{\left(\frac{2b-1}{b},b\right)\middle|b\neq\frac{1}{2}\right\}$$
$$\cup\left\{(a,b)\middle|a=\frac{3}{2}\pm\frac{i\sqrt{3}}{6},b=-\frac{1}{2}\pm\frac{i\sqrt{3}}{6}\right\}\cup\left\{(a,\overline{a})\middle|a\in\left\{\frac{1}{2}\pm\frac{i\sqrt{3}}{6},\frac{3}{2}\pm\frac{i\sqrt{3}}{2}\right\}\right\}$$
$$\cup\left\{(a,-\overline{a}),\left(a,\overline{-\overline{a}}\right)\middle|a\in\left\{-\frac{1}{2}\pm\frac{i\sqrt{3}}{2},\frac{1}{2}\pm\frac{i\sqrt{3}}{2},-\frac{1}{2}\pm\frac{i\sqrt{3}}{6}\right\}\right\}.$$

4.4 Ordinary quadratic length of cubic plane Cremona maps

In this section we prove Theorem 4.2. Theorem 4.1 implies that it suffices to compute the lengths of the cubic plane Cremona maps listed in Table 4.1 at page 56.

Recall that the quadratic length, and hence the ordinary quadratic length, of cubic plane Cremona maps is at least 2 (Corollary 3.6). On the other hand, in Table 4.3 at page 60 and Table 4.4 at page 62 there are decompositions of all types of plane cubic maps, but type 1, in exactly two quadratic maps. So, in order to complete the proof of the first assertion of Theorem 4.2, it remains to prove the following lemma.

Lemma 4.50. *Let $\varphi_1 \in \mathrm{Cr}(\mathbb{P}^2)$ be the map 1 in Table 4.1. Then, φ_1 has quadratic length* 3.

Proof. Let p_1 be the double base point of φ_1 and let $p_2, \ldots, p_5$ be its simple base points, that are all infinitely near p_1, namely $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ where $p_3 \odot p_1$. Hence, a quadratic map can be based at p_1 and at p_2, but not at p_3, cf. Remark 2.34.

The decomposition in Table 4.4 at page 62 implies that $q(\varphi_1) \leq 3$. By contradiction, suppose that $\mathrm{ql}(\varphi_1) = 2$. Then, there should exist a quadratic map ρ such that $\mathrm{ql}(\varphi_1 \circ \rho^{-1}) = 1$, so $\varphi_1 \circ \rho^{-1}$ should be a quadratic map by Lemma 3.5. However,

- if ρ is not based at p_1, then $\varphi_1 \circ \rho^{-1}$ has degree 6, a contradiction;
- if ρ is based at p_1, but not at p_2, then $\varphi_1 \circ \rho^{-1}$ has degree 4, again a contradiction;
- finally, if ρ is based at p_1 and p_2, then $\varphi_1 \circ \rho^{-1}$ has degree 3, a contradiction.

Hence, we conclude that $\mathrm{ql}(\varphi_1) = 3$. □

We now prove the second assertion of Theorem 4.2, that is that the cubic plane Cremona map of type n, $1 \leq n \leq 31$, in Table 4.1 at page 56 has the respective ordinary quadratic length listed in the third column in Table 4.2 at page 57.

The decompositions in Table 4.3 at page 60 show that the maps of types 21, 23, 25, 26, 27, 29, 30, 31 have the ordinary quadratic length exactly 2.

Recall that Proposition 3.17 says the ordinary quadratic length of a plane Cremona map is at least the maximum height of its base points. In particular, the maps φ_n, $n = 10, 11, 12, 13, 15, 16, 18, 19$, have $\mathrm{oql}(\varphi_n) \geq 3$ and the decompositions in Table 4.3 at page 60 show that indeed $\mathrm{oql}(\varphi_n) = 3$. Similarly, the maps φ_n, $n = 2, 7, 9$, have $\mathrm{oql}(\varphi_n) \geq 4$ and the decompositions in Table 4.3 show that $\mathrm{oql}(\varphi_n) = 4$.

We now consider the maps of the remaining types, going backwards from the last types to the first ones.

Lemma 4.51. *Let φ_{28} be the map 28 in Table 4.1. Then,* $\mathrm{oql}(\varphi_{28}) = 3$.

Proof. Let p_1 be the double base point of φ_{28} and p_3, p_4, p_5 the proper simple base points of φ_{28}, which are collinear. The decomposition of φ_{28} in Table 4.3 shows that $\mathrm{oql}(\varphi_{28}) \leq 3$. Suppose by contradiction that $\mathrm{oql}(\varphi_{28}) = 2$. Therefore, there should exist an ordinary quadratic map ρ such that $\mathrm{oql}(\varphi_{28} \circ \rho^{-1}) = 1$, i.e. the map $\varphi_{28} \circ \rho^{-1}$ should be an ordinary quadratic map. Since $\varphi_{28} \circ \rho^{-1}$ should have degree 2, the map ρ must be based at p_1 and two proper simple base points of φ_{28}, say p_3, p_4. However, in that case, the quadratic map $\varphi_{28} \circ \rho^{-1}$ is not ordinary, because p_5 would correspond to an infinitely near base point of $\varphi_{28} \circ \rho^{-1}$, a contradiction. □

Remark 4.52. *The same argument used in the proof of Lemma 4.51 shows that the maps 20, 22, 24 in Table 4.1 have ordinary quadratic length exactly* 3.

Lemma 4.53. *Let φ_{17} be the map 17 in Table 4.1. Then,* $\mathrm{oql}(\varphi_{17}) = 4$.

Proof. The enriched weighted proximity graph of φ_{17} is listed in Table 4.2 at page 57. Let p_1 be the double base point, p_2, p_3 the two proper simple base points and p_4, p_5 such that $p_5 \succ_1 p_4 \succ_1 p_3$ where p_2, p_3, p_4 are aligned. Then,

$$3 \leqslant \mathrm{oql}(\varphi_{17}) \leqslant 4$$

because of the decomposition of φ_{17} in Table 4.3 and the fact that the height of p_5 with respect to φ_{17} is 3, cf. Proposition 3.17.

Suppose by contradiction that $\mathrm{oql}(\varphi_{17}) = 3$. Then, there should exist an ordinary quadratic map ρ such that $\mathrm{oql}(\varphi_{17} \circ \rho^{-1}) = 2$. In particular, ρ must be based at p_3, otherwise, the maximum height of the base points of the map $\varphi_{17} \circ \rho^{-1}$ would be still 3 and Proposition 3.17 would give a contradiction.

If ρ is based also at p_2 (or at another point on the line passing through p_3 and p_2), then p_4 would correspond to an infinitely near base point of $\varphi_{17} \circ \rho^{-1}$ and the maximum height of the base points of $\varphi_{17} \circ \rho^{-1}$ would be again 3, a contradiction.

There are now two cases: either p_1 is a base point of ρ or p_1 is not a base point of ρ.

In the former case, the map $\varphi_{17} \circ \rho^{-1}$ would have the enriched weighted proximity graph 24 in Table 4.2, and therefore would have ordinary quadratic length 3, as we noted in Remark 4.52, a contradiction.

In the latter case, the map $\varphi_{17} \circ \rho^{-1}$ would have degree 5, and therefore its ordinary quadratic length cannot be 2 by Corollary 3.9, a contradiction.

Hence, we conclude that $\mathrm{oql}(\varphi_{17}) = 4$. □

Lemma 4.54. *Let φ_{14} be the map 14 in Table 4.1. Then,* $\mathrm{oql}(\varphi_{14}) = 3$.

Proof. The decomposition of φ_{14} in Table 4.3 shows that $\mathrm{oql}(\varphi_{14}) \leq 3$. Suppose by contradiction that $\mathrm{oql}(\varphi_{14}) = 2$. Therefore, there should exist an ordinary quadratic map ρ such that $\mathrm{oql}(\varphi_{14} \circ \rho^{-1}) = 1$, i.e. the map $\varphi_{14} \circ \rho^{-1}$ should be an ordinary quadratic map. In other words, ρ should be based at the double base point of φ_{14} and other two proper simple base points of φ_{14}, theat however do not exist. $\square$

Lemma 4.55. *Let φ_8 be the map 8 in Table 4.1. Then,* $\mathrm{oql}(\varphi_8) = 5$.

Proof. The enriched weighted proximity graph of φ_8 is listed in Table 4.2. Let p_1 be the double base point, p_2 the proper simple base point and p_3, p_4, p_5 the other infinitely near base points such that $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2$ where p_2, p_3, p_4 are aligned. Then,

$$4 \leqslant \mathrm{oql}(\varphi_8) \leqslant 5$$

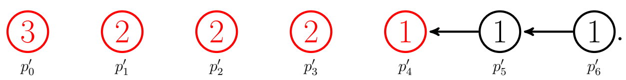
because of the decomposition of φ_8 in Table 4.3 and the fact that the height of p_5 with respect to φ_8 is 4, cf. Proposition 3.17.

Suppose by contradiction that $\mathrm{oql}(\varphi_8) = 4$. Then, there should exist an ordinary quadratic map ρ_1 such that $\mathrm{oql}(\varphi_8 \circ \rho_1^{-1}) = 3$. In particular, ρ_1 must be based at p_2, otherwise, the maximum height of the base points of the map $\varphi_8 \circ \rho_1^{-1}$ would be still 4 and Proposition 3.17 would give a contradiction. For the same reason, ρ_1 cannot be based at p_2 and also at a point on the line passing through p_2 and p_3.

There are now two cases: either p_1 is a base point of ρ_1 or p_1 is not a base point of ρ_1.

In the former case, the map $\varphi_8 \circ \rho^{-1}$ would have the enriched weighted proximity graph 17 in Table 4.2, and therefore it would have ordinary quadratic length 4, as we proved in Lemma 4.53, a contradiction.

In the latter case, the map $\varphi_8 \circ \rho^{-1}$ would have degree 5 and the following weighted proximity graph:



where p'_0, p'_4, p'_5 are aligned. Furthermore, there should exist an ordinary quadratic map ρ_2 such that $\mathrm{oql}(\varphi_8 \circ \rho_1^{-1} \circ \rho_2^{-1}) = 2$. In particular, ρ_2 must be based at p'_4, otherwise the maximum height of the base points of the map $\varphi_8 \circ \rho_1^{-1} \circ \rho_2^{-1}$ would be still 3 and Proposition 3.17 would give a contradiction. For the same reason, ρ_2 cannot be based at p'_4 and also at p'_0 or at another point on the line passing through p'_4 and p'_5. Therefore, ρ_2 is based at p'_4 and other two points where $\varphi_8 \circ \rho^{-1}$ has multiplicity ≤ 2, hence the map $\varphi_8 \circ \rho_1^{-1} \circ \rho_2^{-1}$ would have degree ≥ 5 and we get a contraction with Corollary 3.9.

We conclude that $\mathrm{oql}(\varphi_8) = 5$. $\square$

Lemma 4.56. *Let φ_6 be the map 6 in Table 4.1. Then,* $\mathrm{oql}(\varphi_6) = 4$.

Proof. The enriched weighted proximity graph of φ_6 is listed in Table 4.2. Let p_1 be the double base point, p_5 the proper simple base point and p_2, p_3, p_4 the other infinitely near base points such that $p_2 \succ_1 p_1$ and $p_4 \succ_1 p_3 \succ_1 p_1$. Then,

$$3 \leqslant \mathrm{oql}(\varphi_6) \leqslant 4$$

because of the decomposition of φ_6 in Table 4.3 and the fact that the height of p_4 with respect to φ_6 is 3, cf. Proposition 3.17.

Suppose by contradiction that $\operatorname{oql}(\varphi_6) = 3$. Then, there should exist an ordinary quadratic map ρ such that $\operatorname{oql}(\varphi_6 \circ \rho^{-1}) = 2$. In particular, ρ must be based at p_1, otherwise the maximum height of the base points of the map $\varphi_6 \circ \rho^{-1}$ would be still 3 and Proposition 3.17 would give a contradiction. For the same reason, ρ_1 cannot be based at p_1 and also at a point on the line passing through p_1 and p_3.

There are now two cases: either ρ is based at p_5 or ρ is not based at p_5.

In the former case, the map $\varphi_6 \circ \rho^{-1}$ would have the enriched weighted proximity graph 24 in Table 4.2, and therefore it would have ordinary quadratic length 3 (cf. Remark 4.52), a contradiction.

In the latter case, the map $\varphi_6 \circ \rho^{-1}$ would be a de Jonquières map of degree 4, a contradiction with Lemma 3.10.

Therefore, we conclude that $\operatorname{oql}(\varphi_6) = 4$. □

Lemma 4.57. *Let φ_5 be the map 5 in Table 4.1. Then,* $\operatorname{oql}(\varphi_5) = 5$.

Proof. The enriched weighted proximity graph of φ_5 is listed in Table 4.2. Let p_1 be the double base point, p_5 the proper simple base point and p_2, p_3, p_4 the other infinitely near base points such that $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ with $p_3 \odot p_1$. Then,

$$4 \leqslant \operatorname{oql}(\varphi_5) \leqslant 5$$

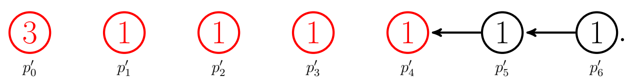
because of the decomposition of φ_5 in Table 4.3 and the fact that the height of p_4 with respect to φ_5 is 4, cf. Proposition 3.17.

Suppose by contradiction that $\operatorname{oql}(\varphi_5) = 4$. Then, there should exist an ordinary quadratic map ρ_1 such that $\operatorname{oql}(\varphi \circ \rho_1^{-1}) = 3$. This implies that ρ_1 must be based at p_1, otherwise the maximum height of the base points of the map $\varphi_5 \circ \rho_1^{-1}$ would be still 4 and Proposition 3.17 would give a contradiction. For the same reason, ρ_1 cannot be based at p_1 and at a point on the line passing through p_1 and p_2.

There are now two cases: either p_5 is a base point of ρ_1 or p_5 is not a base point of ρ_1.

In the former case, the map $\varphi_5 \circ \rho_1^{-1}$ would have enriched weighted proximity graph of type 17 in Table 4.2 and, therefore, it would have ordinary quadratic length 4 by Lemma 4.53, a contradiction.

In the latter case, the map $\varphi_5 \circ \rho_1^{-1}$ would be a de Jonquières map of degree 4 and its weighted proximity graph would be



where p'_2, p'_3, p'_4, p'_5 are aligned.

Then, there should exist an ordinary quadratic map ρ_2 such that $\mathrm{oql}(\varphi_5 \circ \rho_1^{-1} \circ \rho_2^{-1}) = 2$. The map ρ_2 must be based at p'_4, and not at p'_2, p'_3, otherwise the maximum height of the base points of the map $\varphi_5 \circ \rho_1^{-1} \circ \rho_2^{-1}$ would be at least 3, a contradiction with Proposition 3.17. If ρ_2 is not based at p'_0, then $\deg(\varphi_5 \circ \rho_1^{-1} \circ \rho_2^{-1}) \geq 6$ and we get a contradiction with Corollary 3.9. Otherwise ρ_2 is based at p'_0 and, furthermore, either p'_1 is a base point of ρ_2 or p'_1 is not a base point of ρ_2.

In the latter case, the map $\varphi_5 \circ \rho_1^{-1} \circ \rho_2^{-1}$ would be a de Jonquières map of degree 4 and we get a contradiction with Lemma 3.10.

In the former case, the map $\varphi_5 \circ \rho_1^{-1} \circ \rho_2^{-1}$ would have the enriched weighted proximity graph of type 24 in Table 4.2 and its ordinary quadratic length would be 3, a contradiction.

Hence, we conclude that $\mathrm{oql}(\varphi_5) = 5$. □

Lemma 4.58. *Let φ_4 be the map 4 in Table 4.1. Then,* $\mathrm{oql}(\varphi_4) = 4$.

Proof. The enriched weighted proximity graph of φ_4 is listed in Table 4.2. Let p_1 be the double base point, p_2, p_3, p_4, p_5 the infinitely near simple base points such that $p_3 \succ_1 p_2 \succ_1 p_1$ and $p_5 \succ_1 p_4 \succ_1 p_1$. Then,

$$3 \leqslant \mathrm{oql}(\varphi_4) \leqslant 4$$

because of the decomposition of φ_4 in Table 4.3 and the fact that the heights of p_3 and of p_5 with respect to φ_4 are 3, cf. Proposition 3.17.

Suppose by contradiction that $\mathrm{oql}(\varphi_4) = 3$. Then, there should exist an ordinary quadratic map ρ such that $\mathrm{oql}(\varphi \circ \rho^{-1}) = 2$. In particular, ρ must be based at p_1. Then, the map $\varphi \circ \rho^{-1}$ is a de Jonquières map of degree 4 and we get a contradiction with Lemma 3.10. □

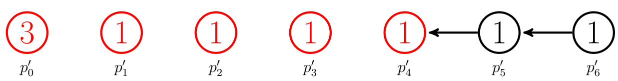
Lemma 4.59. *Let φ_3 be the map 3 in Table 4.1. Then,* $\mathrm{oql}(\varphi_3) = 5$.

Proof. The enriched weighted proximity graph of φ_3 is listed in Table 4.2. Let p_1 be the double base point, p_2, p_3, p_4, p_5 the infinitely near simple base points such that $p_2 \succ_1 p_1$ and $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_1$. Then,

$$4 \leqslant \mathrm{oql}(\varphi_3) \leqslant 5$$

because of the decomposition of φ_3 in Table 4.3 and the fact that the height of p_4 with respect to φ_3 is 4, cf. Proposition 3.17.

Suppose by contradiction that $\mathrm{oql}(\varphi_3) = 4$. Then, there should exist an ordinary quadratic map ρ_1 such that $\mathrm{oql}(\varphi_3 \circ \rho_1^{-1}) = 3$. In particular, ρ_1 must be based at p_1 and not at a point lying on the line passing through p_1 and p_3, otherwise the maximum height of the base points with respect to $\varphi_3 \circ \rho_1^{-1}$ would be still 4. Then, $\varphi_3 \circ \rho_1^{-1}$ is a de Jonquières map of degree 4 and its weighted proximity graph is:



(4.2)

where p'_1, p'_2, p'_3, p'_4 are aligned.

Then, there should exist an ordinary quadratic map ρ_2 such that $\mathrm{oql}(\varphi_3 \circ \rho_1^{-1} \circ \rho_2^{-1}) = 2$. The map ρ_2 must be based at p_4', otherwise the maximum height of the base points of the map $\varphi_3 \circ \rho_1^{-1} \circ \rho_2^{-1}$ would be at least 3, a contradiction with Proposition 3.17. Furthermore, the map ρ_2 must be based also at p_0', otherwise the degree of $\varphi_3 \circ \rho_1^{-1} \circ \rho_2^{-1}$ would be larger than 4, a contradiction with Corollary 3.9.

There are now two cases: either ρ_2 is based at p_i', for some $i \in \{1, 2, 3\}$, or ρ_2 is not based at $p_1'.p_2', p_3'$.

In the former case, the map $\varphi_3 \circ \rho_1^{-1} \circ \rho_2^{-1}$ would have the enriched weighted proximity graph of type 14 in Table 4.2, a contradiction with Lemma 4.54.

In the latter case, the map $\varphi_3 \circ \rho_1^{-1} \circ \rho_2^{-1}$ is a de Jonquières map of degree 4, a contradiction with Lemma 3.10.

Hence, we conclude that $\mathrm{oql}(\varphi_3) = 5$. $\square$

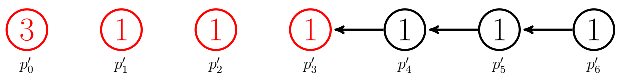
Lemma 4.60. *Let φ_1 be the map 1 in Table 4.1. Then,* $\mathrm{oql}(\varphi_1) = 6$.

Proof. The enriched weighted proximity graph of φ_1 is listed in Table 4.2. Let p_1 be the double base point, p_2, p_3, p_4, p_5 the infinitely near simple base points such that $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ with $p_3 \odot p_1$. Then,

$$5 \leqslant \mathrm{oql}(\varphi_1) \leqslant 6$$

because of the decomposition of φ_1 in Table 4.3 and the fact that the height of p_5 with respect to φ_1 is 5, cf. Proposition 3.17.

Suppose by contradiction that $\mathrm{oql}(\varphi_1) = 5$. Then, there should exist an ordinary quadratic map ρ_1 such that $\mathrm{oql}(\varphi_1 \circ \rho_1^{-1}) = 4$. In particular, ρ_1 must be based at p_1 and not at a point lying on the line passing through p_1 and p_2, otherwise the maximum height of the base points with respect to $\varphi_1 \circ \rho_1^{-1}$ would be still 5. So the map $\varphi_1 \circ \rho_1^{-1}$ is a de Jonquières map of degree 4 and its weighted proximity graph is:



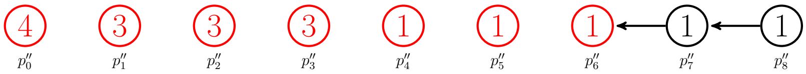
where p_1', p_2', p_3', p_4' are aligned.

Then, there should exist an ordinary quadratic map ρ_2 such that $\mathrm{oql}(\varphi_1 \circ \rho_1^{-1} \circ \rho_2^{-1}) = 3$. In particular, the map ρ_2 must be based at p_3' and not at p_1', p_2' (or at another point lying on the line passing through p_3' and p_4'), otherwise the maximum height of the base points of the map $\varphi_1 \circ \rho_1^{-1} \circ \rho_2^{-1}$ is 4, a contradiction with Proposition 3.17.

There are now two cases: either ρ_2 is based at p_0' or ρ_2 is not based at p_0'.

In the former case, the map $\varphi_1 \circ \rho_1^{-1} \circ \rho_2^{-1}$ is a de Jonquières map of degree 4 and its enriched weighted proximity graph is (4.2) and we reach a contradiction as in the proof of Lemma 4.59.

In the latter case, the map $\varphi_1 \circ \rho_1^{-1} \circ \rho_2^{-1}$ has degree 7 and its weighted proximity graph is:



where p''_2, p''_3, p''_6 are aligned and also $p''_0, p''_4, p''_5, p''_6$ are collinear.

Then, there should exist an ordinary quadratic map ρ_3 such that $\text{oql}(\varphi_1 \circ \rho_1^{-1} \circ \rho_2^{-1} \circ \rho_3^{-1}) = 2$. Thus, ρ_3 must be based at p''_6, otherwise the maximum height of the base points of $\varphi_1 \circ \rho_1^{-1} \circ \rho_2^{-1} \circ \rho_3^{-1}$ is 3, a contradiction with Proposition 3.17. This implies that $\varphi_1 \circ \rho_1^{-1} \circ \rho_2^{-1} \circ \rho_3^{-1}$ would have degree ≥ 6, a contradiction with Corollary 3.9.

Hence, we conclude that $\text{oql}(\varphi) = 6$. $\square$

Chapter 5

Quartic plane Cremona maps

In this chapter, we deal with quartic plane Cremona maps. In Chapter 2 we classified enriched weighted proximity graphs of quartic plane Cremona maps. In principle, one could get a finer classification of equivalence classes of quartic plane Cremona maps by applying the techniques already used in Chapter 4 for cubic plane Cremona maps.

However, this would require a lot of time and patience. Furthermore, for the purpose to compute the quadratic length and the ordinary quadratic length of a map, it is sufficient to know its enriched weighted proximity graph. Recall that a quartic plane Cremona map may or may not be a de Jonquières, thus it is natural to give two separate classifications, one for quartic plane de Jonquières maps and the other for quartic plane non-de Jonquières maps.

5.1 Quartic plane de Jonquières maps

In this section we describe the results contained in Table 5.1. The ordinary quadratic lengths and quadratic lengths of quartic plane de Jonquières maps associated to the graphs are given in the third and the fourth columns, respectively. In many cases, the computation of the exact ordinary quadratic length requires a case by case analysis that we have not yet carried out. In that case, we put a lower bound and an upper bound for the ordinary quadratic length: the number written in **bold** means that we found a decomposition with that number of ordinary quadratic maps and we believe that it is the correct number. Finally, the types of the inverse maps are listed in the fifth column of Table 5.1.

On the other hand, a list of examples of quartic plane de Jonquières maps with enriched weighted proximity graphs in Table 5.1 is also given in Table 5.2.

Theorem 5.1. *Let $\varphi_n \in \mathrm{Cr}(\mathbb{P}^2)$ be a quartic plane de Jonquières map with enriched weighted proximity graph of type n in Table 5.1. Then, the ordinary quadratic length of φ_n (or a lower bound and an upper bound of it) is listed in the third column of Table 5.1 and the quadratic length of φ_n is listed in the fourth column of Table 5.1.*

Proof. The upper bound for the ordinary quadratic length has been obtained by constructing a decomposition with that number of ordinary quadratic maps. The lower bound follows from the height of proper base points either of the map or of its inverse. $\square$

Next, let us consider some particular quartic plane de Jonquières maps.

Lemma 5.2. *Let φ_1 be quartic plane Cremona map defined by $\sharp 1.1$ in Table 5.2 at page 108. Then, φ has only a proper base point $p_0 = [1:0:0]$ of multiplicity 3 and other base points $p_1, p_2, p_3, p_4, p_5, p_6$ satisfy $p_6 \succ_1 p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1 \succ_1 p_0$ where there standard coordinates respectively are $p_1 = (p_0, 0), p_2 = (p_0, 0, \infty), p_3 = (p_0, 0, \infty, \infty), p_4 = (p_0, 0, \infty, \infty, -1), p_5 = (p_0, 0, \infty, \infty, -1, 0)$ and $p_6 = (p_0, 0, \infty, \infty, -1, 0, 0)$.*

Proof. Let φ_1 be a quartic plane Cremona map defined by

$$\varphi_1([x:y:z]) = [xz^3 + y^4 : yz^3 : z^4],$$

that is given by $\sharp 1.1$ in Table 5.2. The map has only proper base point $p_0 = [1:0:0]$ with multiplicity 3.
A curve Q of the linear system associated to φ_1 is of the following form:

$$\lambda_1(xz^3 + y^4) + \lambda_2 yz^3 + \lambda_3 z^4 = 0,$$

for some $[\lambda_1 : \lambda_2 : \lambda_3] \in \mathbb{P}^2$.
In the affine chart $U_1 = \{[x:y:z] \in \mathbb{P}^2 | x \neq 0\} \simeq \mathbb{A}^2_{\overline{y},\overline{z}}$, so that p_0 corresponds to the origin $\mathbf{0} = (0,0)$, the curve Q has local equation

$$Q_a : \lambda_1(\overline{z}^3 + \overline{y}^4) + \lambda_2 \overline{y}\overline{z}^3 + \lambda_3 \overline{z}^4 = 0.$$

- Blowing-up $\mathbb{A}^2_{\overline{y},\overline{z}}$ at $\mathbf{0}$ and consider the first chart given in coordinates by $\overline{y} = y_1, \overline{z} = y_1 z_1$, one has
  - the exception curve E_0 is defined by $y_1 = 0$;
  - the strict transform of the curve Q_a is given by

  $$Q_{a1} : \lambda_1(z_1^3 + y_1) + \lambda_2 y_1 z_1^3 + \lambda_3 y_1 z_1^4 = 0.$$

  Then, $p_1 = E_0 \cap Q_{a1} = \mathbf{0}$ the origin of $\mathbb{A}^2_{y_1,z_1}$. In other words, the standard coordinates of p_1 w.r.t φ_1 is $p_1 = (p_0, 0)$. Moreover, one can check that p_1 is the only point infinitely near p_0 of the first order.

- Blowing-up $\mathbb{A}^2_{y_1,z_1}$ at $\mathbf{0}$ and consider the second chart given in coordinates by $y_1 = y_2 z_2, z_1 = z_2$, one has
  - the exception curve E_1 is defined by $z_2 = 0$;
  - the strict transform of Q_{a1} is given by

  $$Q_{a2} : \lambda_1(z_2^2 + y_2) + \lambda_2 y_2 z_2^3 + \lambda_3 y_2 z_2^4 = 0.$$

It follows the local coordinates of $p_2 = E_1 \cap Q_{a2} = \mathbf{0}$ is the origin of $\mathbb{A}^2_{y_2,z_2}$. In other words, the standard coordinates of p_2 w.r.t φ_1 is $p_2 = (p_0, 0, \infty)$ and one can check $p_2 \not\succ p_0$.

♦ Blowing-up $\mathbb{A}^2_{y_2,z_2}$ at $\mathbf{0}$ and consider the second chart given in coordinates by $y_2 = y_3z_3, z_2 = z_3$, one has

- the exceptional curve E_2 is defined by $z_3 = 0$;
- the strict transform of Q_{a2} is given by

$$Q_{a3} : \lambda_1(z_3 + y_3) + \lambda_2 y_3 z_3^3 + \lambda_3 y_3 z_3^4 = 0.$$

Then, $p_3 = E_2 \cap Q_{a3} = (0, 0)$ is the origin of $\mathbb{A}^2_{y_3,z_3}$. It follows the standard coordinates of p_3 w.r.t φ_1 is $p_3 = (p_0, 0, \infty, \infty)$.

♦ Blowing-up $\mathbb{A}^2_{y_3,z_3}$ at $\mathbf{0}$ and consider the first chart given in coordinates $y_3 = y_4, z_3 = y_4z_4$, one has

- the exceptional curve E_3 is defined by $y_4 = 0$;
- the strict transform of Q_{a3} is given by

$$Q_{a4} : \lambda_1(z_4 + 1) + \lambda_2 y_4^3 z_4^3 + \lambda_3 y_4^4 z_4^4 = 0.$$

Then, the local coordinates of $p_4 = E_3 \cap Q_{a4}$ in $\mathbb{A}^2_{y_4,z_4}$ is $p_4 = (0, -1)$. Therefore, the standard coordinates of p_4 w.r.t φ_1 is $p_4 = (p_0, 0, \infty, \infty, -1)$.

♦ Blowing-up $\mathbb{A}^2_{y_4,z_4}$ at $p_4 = (0, -1)$.
Consider $\alpha : \mathbb{A}^2_{y_4,z_4} \to \mathbb{A}^2_{Y,Z}$ a linear change coordinates defined as follows

$$\begin{cases} y_4 &= Y, \\ z_4 &= Z - 1. \end{cases}$$

With the new coordinates, p_4 is the origin of $\mathbb{A}^2_{Y,Z}$ and the curve Q_{a4} becomes

$$Q_{a4} : \lambda_1 Z + \lambda_2 Y^3(Z-1)^3 + \lambda_3 Y^4(Z-1)^4 = 0.$$

Blowing-up $\mathbb{A}^2_{Y,Z}$ at $\mathbf{0}$ and consider the first chart given in coordinates by $Y = y_5, Z = y_5z_5$, one has

- the exceptional curve E_4 is defined by $y_5 = 0$;
- the strict transform of Q_{a4} is given by

$$Q_{a5} : \lambda_1 z_5 + \lambda_2 y_5^2(y_5z_5 - 1)^3 + \lambda_3 y_5^3(y_5z_5 - 1)^4 = 0.$$

Then, the point $p_5 = E_4 \cap Q_{a5} = (0, 0)$ is the origin of $\mathbb{A}^2_{y_5,z_5}$. It follows the standard coordinates of p_5 w.r.t φ_1 is $p_5 = (p_0, 0, \infty, \infty, -1, 0)$.

♦ Blowing-up $\mathbb{A}^2_{y_5,z_5}$ at $\mathbf{0}$ and consider the first given in coordinates by $y_5 = y_6, z_5 = y_6 z_6$, one has

- the exceptional curve E_5 is defined by $y_6 = 0$;
- the strict transform of Q_{a5} is given by

$$Q_{a6} : \lambda_1 z_6 + \lambda_2 y_6 (y_6^2 z_6 - 1)^3 + \lambda_3 y_6^2 (y_6^2 z_6 - 1)^4 = 0.$$

Therefore $p_6 = E_5 \cap Q_{a6} = (0,0)$ the origin of $\mathbb{A}^2_{y_6,z_6}$ and then its standard coordinates w.r.t φ_1 is $p_6 = (p_0, 0, \infty, \infty, -1, 0, 0)$.

♦ Blowing-up $\mathbb{A}^2_{y_6,z_6}$ at $\mathbf{0}$ and consider the first chart given in coordinates by $y_6 = y_7, z_6 = y_7 z_7$, one has

- the exceptional curve E_6 is defined by $y_7 = 0$;
- the strict transform of Q_{a6} is given by

$$Q_{a7} : \lambda_1 z_7 + \lambda_2 (y_7^3 z_7 - 1)^3 + \lambda_3 y_7 (y_7^3 z_7 - 1)^4 = 0,$$

which is a smooth curve.

□

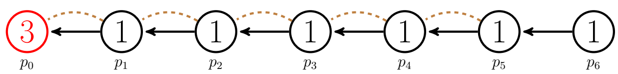
Example 5.3. A decomposition of φ_1 in 8 ordinary quadratic maps is

$$\begin{aligned}\varphi_1([x:y:z]) = \quad & [12x + 56y - 81z : -12x - 8y : 12x + 4y]\sigma[-y + 6z : y - 2z : 8x + 2y]\sigma \\ & [-9x - 13y - 12z : y + 3z : 3y]\sigma[3x + z - y : y - z : 3z]\sigma \\ & [4x + y + 2z : 2y : -y - 2z]\sigma[x + z : 3y - z : -2y]\sigma \\ & [2x + 2z + y : y + 2z : -y]\sigma[x - z + y : 2y - z : z - y]\sigma[x : y : y + z].\end{aligned}$$

Example 5.4. Let φ_4 be the quartic plane de Jonquières map listed in Table 5.2 at page 108 with number 3.2:

$$\varphi_4([x:y:z]) = [y^2(xz - y^2) - z^4 : yz(xz - y^2) : z^2(xz - y^2)].$$

One can show that its enriched weighted proximity graph is of the following form



where the brown dashed curve is a conic. Moreover, a decomposition of φ_4 in 8 ordinary quadratic maps is

$$\begin{aligned}\varphi_4([x:y:z]) = \quad & [2x + 19y + 2z : 8y : 4x + 4y]\sigma[y - x : x : z]\sigma[-y : x : y - 8x + z]\sigma \\ & [-z : z - 2x : 4x + 2y]\sigma[2z : x + y - z : x]\sigma[-2y : 2x + 2y - z : z - 2y]\sigma \\ & [y + x + z : z + y : y]\sigma[x : y : z + y]\sigma[x : y : z - y].\end{aligned}$$

The following figure simulates the process of resolution of φ_4:

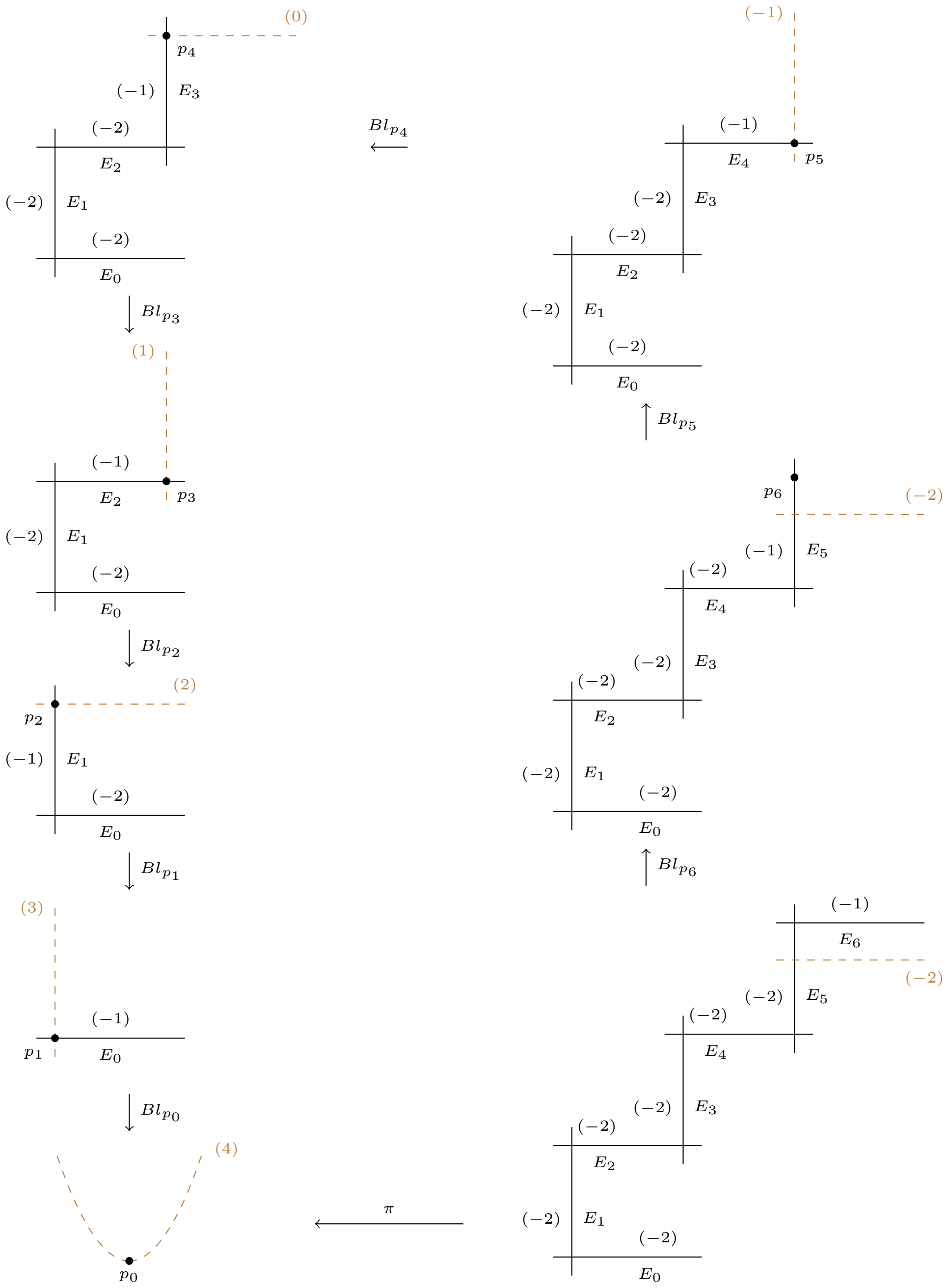
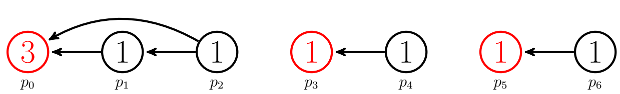


Figure 5.1: The resolution of φ_4

We now give an example of how to give a finer classification of quartic plane de Jonquières maps of a given type.

Example 5.5. ♣ Let consider quartic plane de Jonquières maps with enriched weighted proximity graph of Type 58.1 in Table 5.1, that is



.

Up to automorphisms, one may suppose that $p_0 = [1:0:0], p_3 = [0:1:0], p_5 = [0:0:1] \in \mathbb{P}^2$, $p_2 \succ_1 p_1 \succ_1 p_0$, $p_2 \odot p_0$ where their standard coordinates are $p_1 = (p_0, 1), p_2 = (p_0, 1, \infty)$, in other words $p_1 \succ_1 p_0$ in the direction of the line $\{y - z = 0\}$, and $p_4 \succ_1 p_3$ where the standard coordinates of p_4 is $p_4 = (p_3, -1)$, that is $p_4 \succ_1 p_3$ in the direction of the line $\{x + z = 0\}$. The point $p_6 \succ_1 p_5$ is the only base point which is not fixed, the standard coordinates of p_6 is $p_6 = (p_5, t)$ for some $t \in \mathbb{C}^*$ (note that, $t \neq 0, \infty$, otherwise either p_0, p_5, p_6 or p_3, p_5, p_6 are collinear, contradiction), that means $p_6 \succ_1 p_5$ in the direction of the line $\{y - tx = 0\}$. These maps depend on 1 parameter $t \in \mathbb{C}^*$ and have the following form:

$$\varphi_{58.1,t}([x:y:z]) = [y(xy^2 - 2xyz + xz^2 + y^2z) : y^2z^2 : -z(txy^2 - 2txyz + txz^2 - yz^2)].$$

In particular, one has $\varphi_{58.1,t}$ is equivalent to $\varphi_{58.1,t'}$ if and only if either $t' = t$ or $t' = 1/t$. More precisely, one has

$$\varphi_{58.1,t} = [z:y:x] \circ \varphi_{58.1,1/t} \circ [-tx:z:y].$$

Moreover, when $t = -1$ then $\varphi_{58.1,-1}$ is given by Type 58.1 in Table 5.2.

♣ Quartic plane de Jonquières maps with enriched weighted proximity graph of Type 58.2 in Table 5.1, that is the graph as Type 58.1 such that p_3, p_4, p_5 are collinear. Up to automorphisms, one may suppose that $p_0 = [1:0:0], p_3 = [0:1:0], p_5 = [0:0:1] \in \mathbb{P}^2$, $p_2 \succ_1 p_1 \succ_1 p_0$, $p_2 \odot p_0$ where their standard coordinates are $p_1 = (p_0, 1), p_2 = (p_0, 1, \infty)$, $p_4 \succ_1 p_3$ where the standard coordinates of p_4 is $p_4 = (p_3, \infty)$, namely $p_4 \succ_1 p_3$ in the direction of the line passing through p_3, p_5 that is $\{x = 0\}$, and $p_6 \succ_1 p_5$ where the standard coordinates of p_6 is $p_6 = (p_5, -1)$, in other words $p_6 \succ_1 p_5$ in the direction of the line $\{x + y = 0\}$. The map is given by Type 58.2 in Table 5.1, that is

$$\varphi_{58.2}([x:y:z]) = [xy(y-z)^2 : y^2z^2 : 2xy^3 - 3xy^2z + xz^3 + yz^3].$$

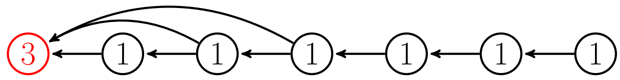
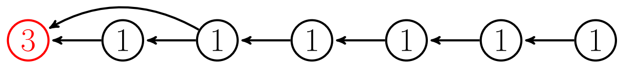
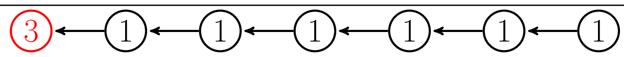
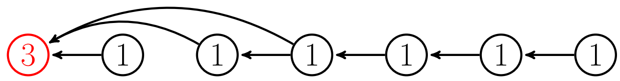
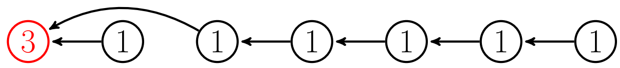
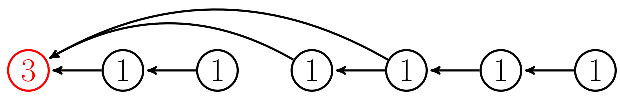
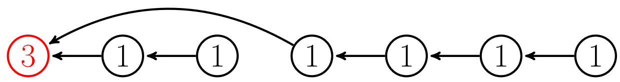
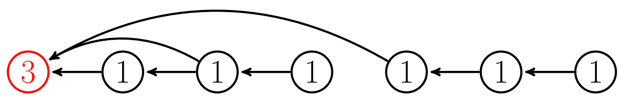
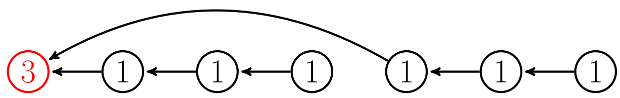
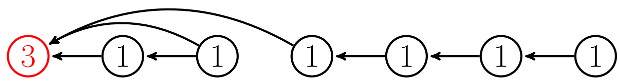
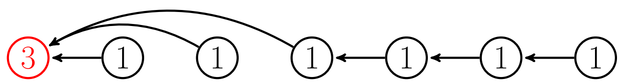
♣ Quartic plane de Jonquières maps with enriched weighted proximity graph of Type 58.3 in Table 5.1, that is the graph as Type 58.1 such that p_3, p_4, p_5, p_6 are collinear. Up to automorphisms, one may suppose that $p_0 = [1:0:0], p_3 = [0:1:0], p_5 = [0:0:1] \in \mathbb{P}^2$, $p_2 \succ_1 p_1 \succ_1 p_0$, $p_2 \odot p_0$ where their standard coordinates are $p_1 = (p_0, 1), p_2 = (p_0, 1, \infty)$, $p_4 \succ_1 p_3$ where the standard coordinates of p_4 is $p_4 = (p_3, \infty)$, and $p_6 \succ_1 p_5$ where the standard coordinates of p_6 is $p_6 = (p_5, \infty)$, in other words, $p_4 \succ_1 p_3$ and $p_6 \succ_1 p_5$ in the direction of the line passing through p_3, p_5 that is $\{x = 0\}$. The map is given by Type 58.3 in Table 5.1, that is

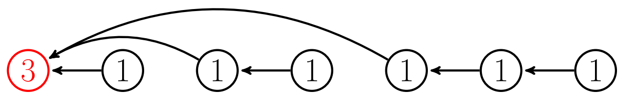
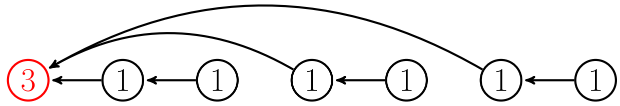
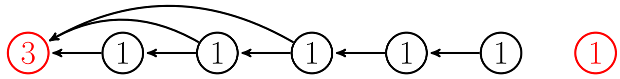
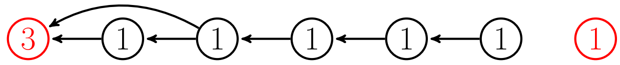
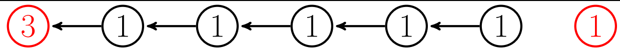
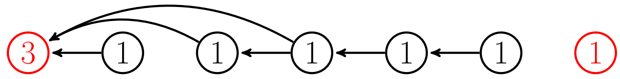
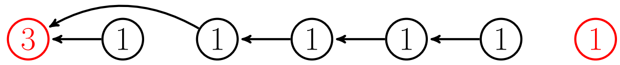
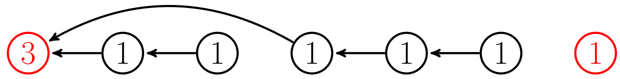
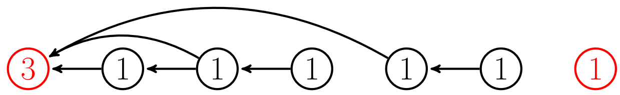
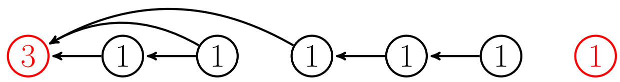
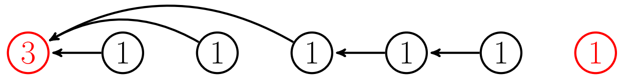
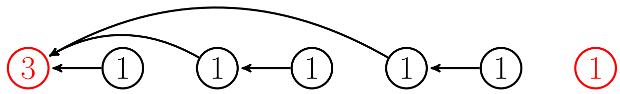
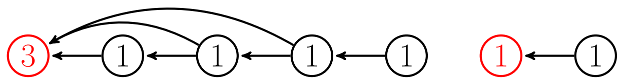
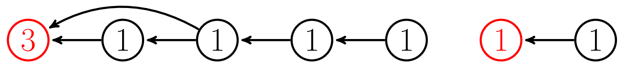
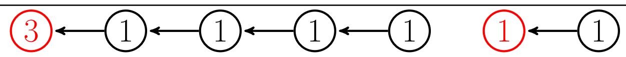
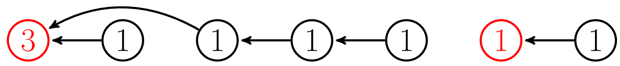
$$\varphi_{58.3}([x:y:z]) = [xy(y-z)^2 : x(2y+z)(y-z)^2 : y^2z^2].$$

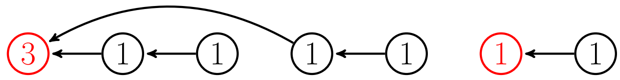
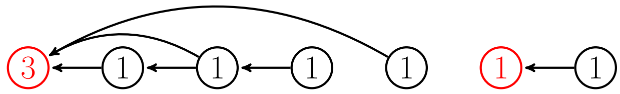
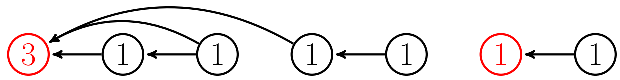
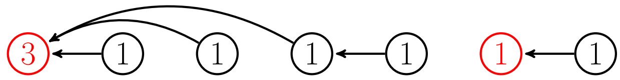
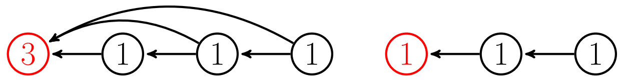
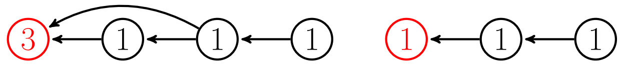
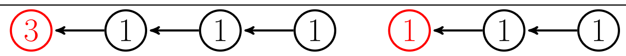
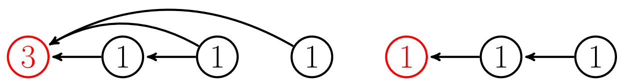
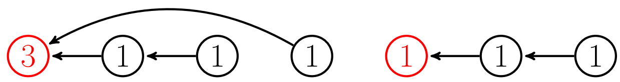
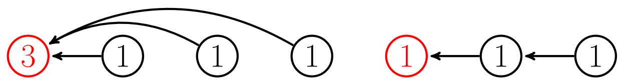
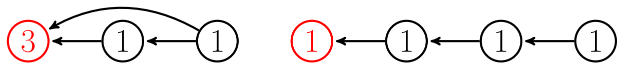
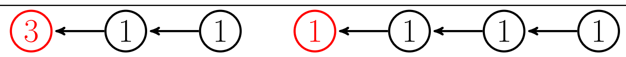
♣ Quartic plane de Jonquières maps with enriched weighted proximity graph of Type 58.4 in Table 5.1, that is the graph as Type 58.1 such that there exists a conic passing through $p_0, p_1, p_3, p_4, p_5, p_6$. Up to automorphisms, one may suppose that $p_0 = [1 : 0 : 0], p_3 = [0 : 1 : 0], p_5 = [0 : 0 : 1] \in \mathbb{P}^2$, $p_2 \succ_1 p_1 \succ_1 p_0$, $p_2 \odot p_0$ where their standard coordinates are $p_1 = (p_0, -1), p_2 = (p_0, -1, \infty)$, that means $p_2 \succ_1 p_1$ in the direction of the line $y + z = 0$, $p_4 \succ_1 p_3$ where the standard coordinates of p_4 is $p_4 = (p_3, -1)$, namely $p_4 \succ_1 p_3$ in the direction of the line $\{x + z = 0\}$, and $p_6 \succ_1 p_5$. Since there exists a unique conic passing through $p_0, p_1, p_2, p_3, p_4, p_5$, that is $\{xy + xz + yz = 0\}$, then p_6 is uniquely determined and its standard coordinates is $p_6 = (p_5, -1)$, that means $p_6 \succ_1 p_5$ in the direction of the line $\{x + y = 0\}$. The map is given by Type 58.3 in Table 5.1

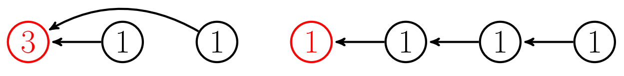
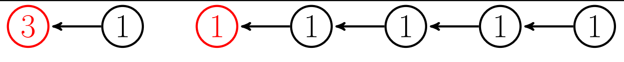
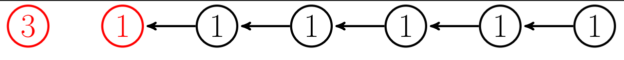
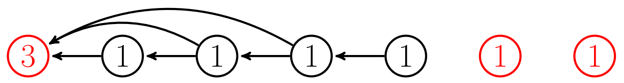
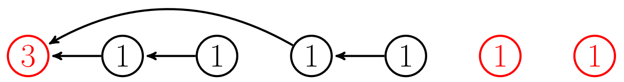
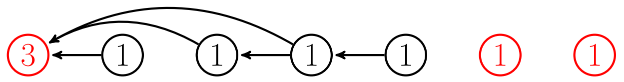
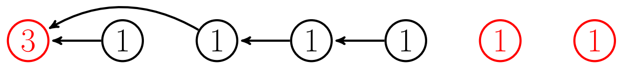
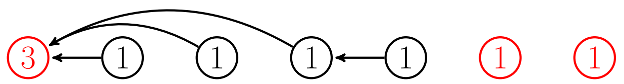
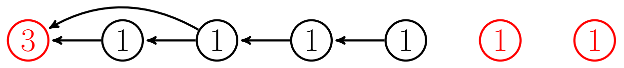
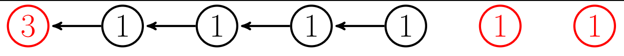
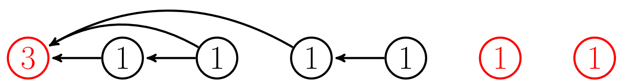
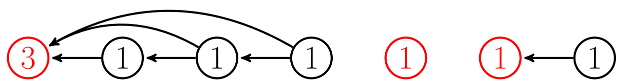
$$\varphi_{58.4}([x : y : z]) = [y(xy^2 + 2xyz + xz^2 + y^2z) : y^2z^2 : z(xy^2 + 2xyz + xz^2 + yz^2)].$$

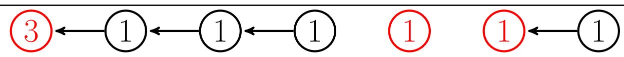
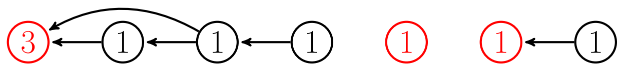
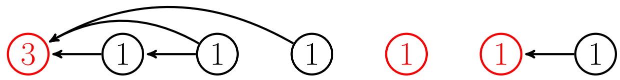
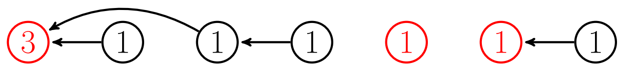
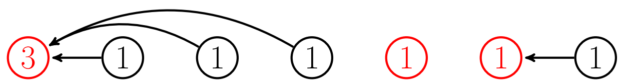
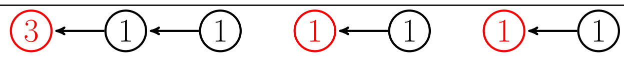
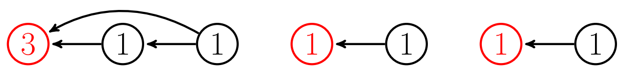
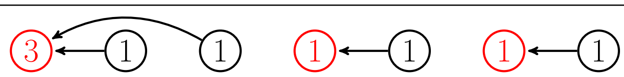
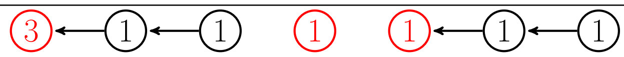
Table 5.1: Enriched weighted proximity graphs and ordinary quadratic lengths of quartic plane de Jonquières maps

♯	Enriched weighted proximity graph	oql	ql	Inv
1.1		7-**8**	5	1.1
2.1		7	4	42.3
3.1		7	3	42.5
3.2	(2;1,2,3,4,5,6)	7-**8**	3	3.2
4.1		6-**7**	4	14.1
5.1		6	3	67.5
5.2	(2;1,3,4,5,6,7)	6	3	41.3
6.1		6	4	6.1
7.1		5	3	66.9
8.1		4-**6**	3	8.1
9.1		4-**5**	3	64.7
10.1		5-**6**	3	24.1
11.1		5-**6**	3	43.1

12.1		4-**5**	3	20.1
13.1		3-**6**	3	13.1
14.1		6-**7**	4	4.1
15.1		6	3	67.4
16.1		6	3	67.7
16.2	(2;1,2,3,4,5,6)	6	3	41.5
16.3	(2;1,2,3,4,5,7)	6-**7**	3	16.3
17.1		5-**6**	3	17.1
18.1		5	3	81.9
18.2	(2;1,3,4,5,6,7)	5	3	65.5
19.1		4	3	80.17
20.1		4-**5**	3	12.1
21.1		4-**5**	3	29.1
22.1		4-**5**	3	45.1
23.1		3-**4**	3	23.1
24.1		5-**6**	3	10.1
25.1		5	3	66.8
26.1		5	3	66.12
26.2	(2;1,2,3,4,5,6)	5	3	39.5
26.3	(2;1,2,3,4,6,7)	5-**6**	3	26.3
27.1		4	3	80.18
27.2	(2;1,3,4,5,6,7)	4	3	63.6

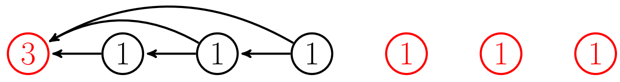
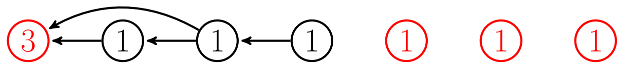
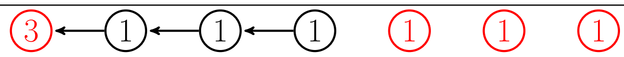
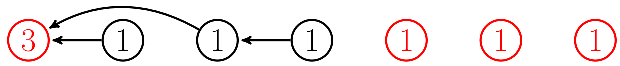
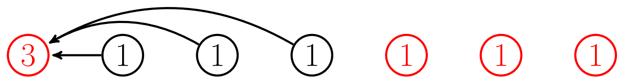
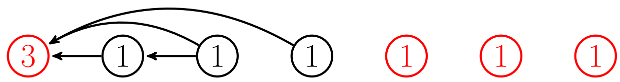
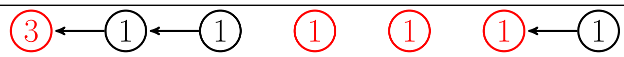
28.1		3-**4**	3	78.8
29.1		4-**5**	3	21.1
30.1		3-**4**	3	30.1
31.1		3-**4**	3	50.1
32.1		4	3	32.1
32.2	(1;5,6,7)	4	3	32.2
33.1		4	3	64.6
33.2	(1;5,6,7)	4	3	64.8
34.1		4	3	64.10
34.2	(1;5,6,7)	4	3	64.11
34.3	(2;1,2,3,4,5,6)	4	3	34.3
34.4	(2;1,2,3,5,6,7)	4-**5**	3	34.4
34.5	(1;5,6,7), (2;1,2,3,4,5,6)	4	3	34.5
35.1		4	3	51.1
35.2	(1;5,6,7)	4	3	51.2
36.1		3-**4**	3	80.19
36.2	(1;5,6,7)	3-**4**	3	80.22
36.3	(2;1,2,3,5,6,7)	3-**4**	3	63.5
37.1		4	3	68.1
37.2	(1;5,6,7)	4	3	68.2
38.1		4	3	66.5
38.2	(1;4,5,6)	4	3	66.7
38.3	(1;4,5,6,7)	4	3	38.3
38.4	(2;1,2,4,5,6,7)	4	3	39.3
39.1		4	3	66.11
39.2	(1;4,5,6)	4	3	66.13
39.3	(1;4,5,6,7)	4	3	38.4

39.4	(2;1,2,4,5,6,7)	4	3	39.4
39.5	(2;1,2,3,4,5,6)	5	3	26.2
40.1		4	3	81.5
40.2	(1;4,5,6)	4	3	81.7
40.3	(1;4,5,6,7)	4	3	40.3
40.4	(2;1,2,4,5,6,7)	4	3	65.4
41.1		5	3	67.9
41.2	(1;3,4,5)	5	3	67.8
41.3	(1;3,4,5,6)	6	3	5.2
41.4	(2;1,3,4,5,6,7)	5	3	41.4
41.5	(2;1,2,3,4,5,6)	6	3	16.2
42.1		6	3	42.1
42.2	(1;2,3,4)	6	3	42.2
42.3	(1;2,3,4,5)	7	4	2.1
42.4	(2;2,3,4,5,6,7)	6	3	42.4
42.5	(2;1,2,3,4,5,6)	7	3	3.1
43.1		5-**6**	3	11.1
44.1		3	3	86.6
45.1		4-**5**	3	22.1
46.1		4	3	87.13
46.2	(2;1,3,4,5,6,7)	4	3	79.7
47.1		3-**4**	3	47.1
48.1		5	3	81.8
49.1		5	3	81.12
49.2	(2;1,2,3,4,5,6)	5	3	65.8
49.3	(2;1,2,3,4,6,7)	5-**6**	3	49.3
50.1		3-**4**	3	31.1
51.1		4	3	35.1

51.2	(1;5,6,7)	4	3	35.2
52.1		4	3	80.26
52.2	(1;5,6,7)	4	3	80.32
52.3	(2;1,2,3,4,5,6)	4	3	60.6
52.4	(2;1,2,3,4,6,7)	4	3	63.10
52.5	(2;1,2,3,5,6,7)	4-**5**	3	52.5
52.6	(1;5,6,7),(2;1,2,3,4,5,6)	4	3	60.8
52.7	(1;5,6,7),(2;1,2,3,4,6,7)	4	3	63.11
53.1		4	3	80.16
53.2	(1;5,6,7)	4	3	80.20
54.1		3	3	54.1
54.2	(1;5,6,7)	3-**4**	3	54.2
55.1		3	3	86.7
55.2	(1;5,6,7)	3	3	86.7
55.3	(2;1,3,4,5,6,7)	3	3	77.5
56.1		3	3	73.1
56.2	(1;5,6,7)	3-**4**	3	73.2
57.1		3	3	78.10
57.2	(1;4,5,6)	3	3	78.12
57.3	(1;4,5,6,7)	3-**4**	3	58.4
57.4	(2;1,2,3,4,5,6)	3	3	57.4
57.5	(2;1,2,4,5,6,7)	3-**4**	3	57.5
57.6	(1;4,5,6),(2;1,2,3,4,6,7)	3-**4**	3	57.6
58.1		3	3	78.3
58.2	(1;4,5,6)	3	3	78.6
58.3	(1;4,5,6,7)	3-**4**	3	58.3
58.4	(2;1,2,4,5,6,7)	3-**4**	3	57.3
59.1		3	3	86.9
59.2	(1;4,5,6)	3	3	86.20
59.3	(1;4,5,6,7)	3	3	59.3
59.4	(2;1,2,4,5,6,7)	3	3	77.6
60.1		3	3	80.23

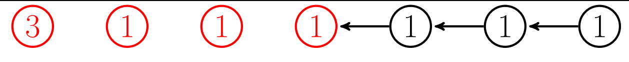
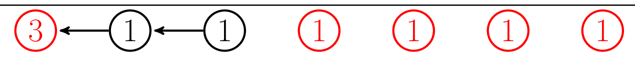
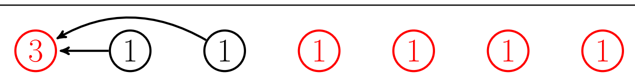
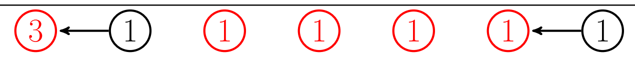
60.2	(1;5,6,7)	3	3	80.31
60.3	(1;4,5,6)	3-**4**	3	80.33
60.4	(1;4,5,6,7)	3-**4**	3	61.5
60.5	(2;1,2,3,5,6,7)	3	3	63.9
60.6	(2;1,2,3,4,5,6)	4	3	52.3
60.7	(2;1,2,4,5,6,7)	3-**4**	3	60.7
60.8	(1;5,6,7),(2;1,2,3,4,5,6)	4	3	52.6
60.9	(1;4,5,6),(2;1,2,3,5,6,7)	3-**4**	3	63.12
61.1	$3 \leftarrow 1 \leftarrow 1 \quad 1 \quad 1 \leftarrow 1 \leftarrow 1$	3	3	80.8
61.2	(1;5,6,7)	3	3	80.15
61.3	(1;4,5,6)	3-**4**	3	80.21
61.4	(1;4,5,6,7)	3-**4**	3	61.4
61.5	(2;1,2,4,5,6,7)	3-**4**	3	60.4
62.1	$3 \leftarrow 1 \quad 1 \quad 1 \quad 1 \leftarrow 1 \leftarrow 1$	3	3	87.6
62.2	(1;5,6,7)	3	3	87.11
62.3	(1;4,5,6)	3-**4**	3	87.15
62.4	(1;4,5,6,7)	3-**4**	3	62.4
62.5	(2;1,2,4,5,6,7)	3	3	79.5
63.1	$3 \leftarrow 1 \quad 1 \leftarrow 1 \quad 1 \leftarrow 1 \leftarrow 1$	3	3	80.24
63.2	(1;5,6,7)	3	3	80.28
63.3	(1;3,5,6)	3-**4**	3	80.27
63.4	(1;3,4,5)	3	3	80.29
63.5	(1;3,5,6,7)	3-**4**	3	36.3
63.6	(1;3,4,5,6)	4	3	27.2
63.7	(1;3,4,5),(1;5,6,7)	3	3	80.30
63.8	(2;1,3,4,5,6,7)	3	3	63.8
63.9	(2;1,2,3,5,6,7)	3	3	60.5
63.10	(2;1,2,3,4,5,6)	4	3	52.4
63.11	(1;5,6,7),(2;1,2,3,4,5,6)	4	3	52.7
63.12	(1;3,4,5),(2;1,2,3,5,6,7)	3-**4**	3	60.9
64.1	$3 \quad 1 \leftarrow 1 \leftarrow 1 \quad 1 \leftarrow 1 \leftarrow 1$	3	3	64.1
64.2	(1;2,5,6)	3-**4**	3	64.2
64.3	(1;5,6,7)	3	3	64.3
64.4	(1;2,3,4),(1;2,5,6)	3-**4**	3	64.4
64.5	(1;2,3,4),(1;5,6,7)	3	3	64.5
64.6	(1;2,5,6,7)	4	3	33.1

64.7	(1;2,3,5,6)	4-**5**	3	9.1
64.8	(1;2,3,4),(1;2,5,6,7)	4	3	33.2
64.9	(2;2,3,4,5,6,7)	3	3	64.9
64.10	(2;1,2,3,4,5,6)	4	3	34.1
64.11	(1;5,6,7),(2;1,2,3,4,5,6)	4	3	34.2
65.1	$3 \leftarrow 1 \quad 1 \quad 1 \leftarrow 1 \leftarrow 1 \leftarrow 1$	4	3	81.11
65.2	(1;3,4,5)	4	3	81.14
65.3	(1;4,5,6)	4	3	81.13
65.4	(1;4,5,6,7)	4	3	40.4
65.5	(1;3,4,5,6)	5	3	18.2
65.6	(2;1,3,4,5,6,7)	4	3	65.6
65.7	(2;1,2,4,5,6,7)	4	3	65.7
65.8	(2;1,2,3,4,5,6)	5	3	49.2
65.9	(1;3,4,5),(2;1,2,4,5,6,7)	4-**5**	3	65.9
66.1	$3 \quad 1 \leftarrow 1 \quad 1 \leftarrow 1 \leftarrow 1 \leftarrow 1$	4	3	66.1
66.2	(1;2,3,4)	4	3	66.4
66.3	(1;2,4,5)	4	3	66.3
66.4	(1;4,5,6)	4	3	66.2
66.5	(1;4,5,6,7)	4	3	38.1
66.6	(1;2,3,4),(1;4,5,6)	4	3	66.6
66.7	(1;2,3,4),(1;4,5,6,7)	4	3	38.2
66.8	(1;2,3,4,5)	5	3	25.1
66.9	(1;2,4,5,6)	5	3	7.1
66.10	(2;1,2,4,5,6,7)	4	3	39.1
66.11	(2;2,3,4,5,6,7)	4	3	173
66.12	(2;1,2,3,4,5,6)	5	3	26.1
66.13	(1;2,3,4),(2;1,2,4,5,6,7)	4	3	39.2
67.1	$3 \quad 1 \quad 1 \leftarrow 1 \leftarrow 1 \leftarrow 1 \leftarrow 1$	5	3	67.1
67.2	(1;2,3,4)	5	3	67.3
67.3	(1;3,4,5)	5	3	67.2
67.4	(1;2,3,4,5)	6	3	15.1
67.5	(1;3,4,5,6)	6	3	5.1
67.6	(2;2,3,4,5,6,7)	5	3	67.6
67.7	(2;1,2,3,4,5,6)	6	3	16.1
67.8	(1;2,3,4),(2;1,3,4,5,6,7)	5	3	41.2
67.9	(2;1,3,4,5,6,7)	5	3	41.1

68.1		4	3	37.1
68.2	(1;5,6,7)	4	3	37.2
69.1		4	3	87.12
69.2	(1;5,6,7)	4	3	87.14
70.1		4	3	87.18
70.2	(1;5,6,7)	4	3	87.22
70.3	(2;1,2,3,4,5,6)	4	3	79.10
70.4	(2;1,2,3,5,6,7)	4-**5**	3	70.4
70.5	(1;5,6,7),(2;1,2,3,4,5,6)	4	3	79.13
71.1		3	3	89.5
71.2	(1;5,6,7)	3	3	89.13
71.3	(2;1,3,4,5,6,7)	3	3	85.5
72.1		3	3	72.1
72.2	(1;5,6,7)	3-**4**	3	72.2
73.1		3	3	56.1
73.2	(1;5,6,7)	3-**4**	3	56.2
74.1		3	3	89.7
74.2	(1;4,5,6)	3	3	89.14
74.3	(1;5,6,7)	3	3	89.15
74.4	(1;4,5,6,7)	3	3	74.4
74.5	(2;1,2,4,5,6,7)	3	3	85.6
75.1		3	3	86.21
75.2	(1;4,5,6)	3	3	86.29
75.3	(1;5,6,7)	3	3	86.31
75.4	(1;4,5,6,7)	3-**4**	3	76.5
75.5	(2;1,2,3,4,5,6)	3	3	75.5
75.6	(2;1,2,3,4,6,7)	3	3	77.11
75.7	(2;1,2,4,5,6,7)	3-**4**	3	75.7
75.8	(1;4,5,6),(2;1,2,3,5,6,7)	3-**4**	3	77.16
75.9	(1;5,6,7),(2;1,2,3,4,6,7)	3	3	77.15
75.10	(1;5,6,7),(2;1,2,3,4,5,6)	3	3	75.10

76.1	$3 \leftarrow 1 \leftarrow 1 \quad 1 \quad 1 \quad 1 \leftarrow 1$	3	3	86.8
76.2	(1;4,5,6)	3	3	86.19
76.3	(1;5,6,7)	3	3	86.18
76.4	(1;4,5,6,7)	3-**4**	3	76.4
76.5	(2;1,2,4,5,6,7)	3-**4**	3	75.4
77.1	$3 \leftarrow 1 \quad 1 \quad 1 \leftarrow 1 \quad 1 \leftarrow 1$	3	3	86.22
77.2	(1;3,4,6)	3	3	86.27
77.3	(1;3,4,5)	3	3	86.28
77.4	(1;4,5,6)	3	3	86.26
77.5	(1;3,4,5,6)	3	3	55.3
77.6	(1;4,5,6,7)	3	3	59.4
77.7	(1;3,4,5),(1;3,6,7)	3	3	86.32
77.8	(1;3,4,5),(1;4,6,7)	3	3	86.30
77.9	(2;1,3,4,5,6,7)	3	3	77.9
77.10	(2;1,2,4,5,6,7)	3	3	77.10
77.11	(2;1,2,3,4,5,6)	3	3	75.6
77.12	(1;3,4,6),(2;1,2,4,5,6,7)	3-**4**	3	77.12
77.13	(1;3,4,5),(2;1,2,4,5,6,7)	3	3	77.13
77.14	(1;3,4,5),(1;3,6,7),(2;1,2,4,5,6,7)	3	3	77.14
77.15	(1;3,4,5),(2;1,2,3,4,6,7)	3	3	75.9
77.16	(1;4,5,6),(2;1,2,3,4,6,7)	3-**4**	3	75.8
78.1	$3 \quad 1 \leftarrow 1 \quad 1 \leftarrow 1 \quad 1 \leftarrow 1$	3	3	78.1
78.2	(1;4,5,6)	3	3	78.2
78.3	(1;4,5,6,7)	3	3	58.1
78.4	(1;2,3,4),(1;4,5,6)	3	3	78.4
78.5	(1;2,3,6),(1;4,5,6)	3	3	78.5
78.6	(1;2,3,4),(1;4,5,6,7)	3	3	58.2
78.7	(1;2,4,6)	3-**4**	3	78.7
78.8	(1;2,4,6,7)	4	3	28.1
78.9	(2;2,3,4,5,6,7)	3	3	78.9
78.10	(2;1,2,3,4,5,6)	3	3	57.1
78.11	(1;2,3,4),(1;2,6,7),(1;4,5,6)	3	3	78.11
78.12	(1;4,5,6),(2;1,2,3,4,6,7)	3	3	57.2
79.1	$3 \leftarrow 1 \quad 1 \quad 1 \quad 1 \leftarrow 1 \leftarrow 1$	3	3	87.17
79.2	(1;3,4,5)	3	3	87.21
79.3	(1;4,5,6)	3	3	87.20
79.4	(1;5,6,7)	3	3	87.19

79.5	(1;4,5,6,7)	3	3	62.5
79.6	(1;3,4,5),(1;5,6,7)	3	3	87.23
79.7	(1;3,4,5,6)	4	3	46.2
79.8	(2;1,3,4,5,6,7)	3	3	79.8
79.9	(2;1,2,3,5,6,7)	3	3	79.9
79.10	(2;1,2,3,4,5,6)	4	3	70.3
79.11	(1;3,4,5),(2;1,2,4,5,6,7)	3	3	79.12
79.12	(1;4,5,6),(2;1,2,3,5,6,7)	3	3	79.11
79.13	(1;5,6,7),(2;1,2,3,4,5,6)	4	3	70.5
80.1	3 1 ← 1 1 1 ← 1 ← 1	3	3	80.1
80.2	(1;2,4,5)	3	3	80.6
80.3	(1;2,3,4)	3	3	80.7
80.4	(1;2,3,5)	3	3	80.5
80.5	(1;4,5,6)	3	3	80.4
80.6	(1;2,5,6)	3	3	80.2
80.7	(1;5,6,7)	3	3	80.3
80.8	(1;4,5,6,7)	3	3	61.1
80.9	(1;2,4,5),(1;5,6,7)	3	3	80.14
80.10	(1;2,3,4),(1;5,6,7)	3	3	80.10
80.11	(1;2,3,4),(1;4,5,6)	3	3	80.12
80.12	(1;2,3,5),(1;5,6,7)	3	3	80.11
80.13	(1;2,3,5),(1;4,5,6)	3	3	80.13
80.14	(1;2,3,4),(1;2,5,6)	3	3	80.9
80.15	(1;2,3,4),(1;4,5,6,7)	3	3	61.2
80.16	(1;2,3,4,5)	4	3	53.1
80.17	(1;2,4,5,6)	4	3	19.1
80.18	(1;2,3,5,6)	4	3	27.1
80.19	(1;2,5,6,7)	4	3	36.1
80.20	(1;2,3,4,5),(1;5,6,7)	4	3	53.2
80.21	(1;2,3,5),(1;4,5,6,7)	3-**4**	3	61.3
80.22	(1;2,3,4),(1;2,5,6,7)	4	3	36.2
80.23	(2;1,2,4,5,6,7)	3	3	60.1
80.24	(2;1,2,3,5,6,7)	3	3	63.1
80.25	(2;2,3,4,5,6,7)	3	3	80.25
80.26	(2;1,2,3,4,5,6)	4	3	52.1
80.27	(1;2,4,5),(2;1,2,3,5,6,7)	3-**4**	3	63.3
80.28	(1;2,3,4),(2;1,2,3,5,6,7)	3	3	63.2
80.29	(1;4,5,6),(2;1,2,3,5,6,7)	3	3	63.4

80.30	(1;2,3,4),(1;4,5,6),(2;1,2,3,5,6,7)	3	3	63.7
80.31	(1;2,3,4),(2;1,2,4,5,6,7)	3	3	60.2
80.32	(1;5,6,7),(2;1,2,3,4,5,6)	4	3	52.2
80.33	(1;2,3,5),(2;1,2,4,5,6,7)	3-**4**	3	60.3
81.1		4	3	81.1
81.2	(1;2,3,4)	4	3	81.4
81.3	(1;3,4,5)	4	3	81.3
81.4	(1;4,5,6)	4	3	81.2
81.5	(1;4,5,6,7)	4	3	40.1
81.6	(1;2,3,4),(1;4,5,6)	4	3	81.6
81.7	(1;2,3,4),(1;4,5,6,7)	4	3	40.2
81.8	(1;2,3,4,5)	5	3	48.1
81.9	(1;3,4,5,6)	5	3	18.1
81.10	(2;2,3,4,5,6,7)	4	3	81.10
81.11	(2;1,2,4,5,6,7)	4	3	65.1
81.12	(2;1,2,3,4,5,6)	5	3	49.1
81.13	(1;2,3,4),(2;1,3,4,5,6,7)	4	3	65.3
81.14	(1;3,4,5),(2;1,2,4,5,6,7)	4	3	65.2
82.1		3	3	89.17
82.2	(1;5,6,7)	3	3	89.22
82.3	(1;4,5,6,7)	3-**4**	3	83.4
82.4	(2;1,2,3,4,5,6)	3	3	85.10
82.5	(2;1,2,4,5,6,7)	3-**4**	3	82.5
82.6	(1;5,6,7),(2;1,2,3,4,5,6)	3	3	85.16
83.1		3	3	89.6
83.2	(1;5,6,7)	3	3	89.16
83.3	(1;4,5,6,7)	3-**4**	3	83.3
83.4	(2;1,2,4,5,6,7)	3-**4**	3	82.3
84.1		3	3	90.3
84.2	(1;5,6,7)	3	3	90.6
84.3	(1;4,5,6,7)	3	3	84.3
84.4	(2;1,2,4,5,6,7)	3	3	88.3
85.1		3	3	89.18
85.2	(1;3,4,5)	3	3	89.23
85.3	(1;4,5,6)	3	3	89.20
85.4	(1;5,6,7)	3	3	89.21

85.5	(1;3,4,5,6)	3	3	71.3
85.6	(1;4,5,6,7)	3	3	74.5
85.7	(1;3,4,5),(1;5,6,7)	3	3	89.25
85.8	(1;3,6,7),(1;4,5,6)	3	3	89.24
85.9	(2;1,3,4,5,6,7)	3	3	85.9
85.10	(2;1,2,3,4,5,6)	3	3	82.4
85.11	(2;1,2,3,4,6,7)	3	3	85.11
85.12	(1;3,4,5),(2;1,2,3,4,6,7)	3	3	85.14
85.13	(1;4,5,6),(2;1,2,3,5,6,7)	3	3	85.13
85.14	(1;5,6,7),(2;1,2,3,4,6,7)	3	3	85.12
85.15	(1;3,4,5),(1;5,6,7),(2;1,2,3,4,6,7)	3	3	85.15
85.16	(1;5,6,7),(2;1,2,3,4,5,6)	3	3	82.6
86.1	3 1 ← 1 1 1 1 ← 1	3	3	86.1
86.2	(1;4,5,6)	3	3	86.5
86.3	(1;2,5,6)	3	3	86.3
86.4	(1;5,6,7)	3	3	86.4
86.5	(1;2,6,7)	3	3	86.2
86.6	(1;2,4,5,6)	3	3	44.1
86.7	(1;2,5,6,7)	3	3	55.1
86.8	(1;4,5,6,7)	3	3	76.1
86.9	(1;2,3,6,7)	3	3	59.1
86.10	(1;2,3,4),(1;4,5,6)	3	3	86.16
86.11	(1;2,3,6),(1;4,5,6)	3	3	86.11
86.12	(1;2,6,7),(1;4,5,6)	3	3	86.12
86.13	(1;2,3,4),(1;2,5,6)	3	3	86.13
86.14	(1;2,3,4),(1;5,6,7)	3	3	86.14
86.15	(1;2,3,5),(1;5,6,7)	3	3	86.15
86.16	(1;2,3,6),(1;5,6,7)	3	3	86.10
86.17	(1;2,3,4),(1;2,5,6,7)	3	3	55.2
86.18	(1;2,3,4),(1;4,5,6,7)	3	3	76.3
86.19	(1;2,3,6),(1;4,5,6,7)	3	3	76.2
86.20	(1;2,3,6,7),(1;4,5,6)	3	3	59.2
86.21	(2;1,2,3,4,5,6)	3	3	75.1
86.22	(2;1,2,3,4,6,7)	3	3	77.1
86.23	(2;2,3,4,5,6,7)	3	3	86.23
86.24	(1;2,3,4),(1;2,6,7),(1;4,5,6)	3	3	86.24
86.25	(1;2,3,5),(1;2,4,6),(1;5,6,7)	3	3	86.25
86.26	(1;4,5,6),(2;1,2,3,5,6,7)	3	3	77.4

86.27	(1;2,5,6),(2;1,2,3,4,6,7)	3	3	77.2
86.28	(1;5,6,7),(2;1,2,3,4,6,7)	3	3	77.3
86.29	(1;2,6,7),(2;1,2,3,4,5,6)	3	3	75.2
86.30	(1;2,3,4),(1;4,5,6),(2;1,2,3,5,6,7)	3	3	77.8
86.31	(1;5,6,7),(2;1,2,3,4,5,6)	3	3	75.3
86.32	(1;2,3,4),(1;4,6,7),(2;1,2,3,5,6,7)	3	3	77.7
87.1	$3 \quad 1 \quad 1 \quad 1 \quad 1 \leftarrow 1 \leftarrow 1$	3	3	87.1
87.2	(1;2,3,4)	3	3	87.5
87.3	(1;3,4,5)	3	3	87.4
87.4	(1;4,5,6)	3	3	87.3
87.5	(1;5,6,7)	3	3	87.2
87.6	(1;4,5,6,7)	3	3	62.1
87.7	(1;2,3,4),(1;5,6,7)	3	3	87.7
87.8	(1;2,3,4),(1;4,5,6)	3	3	87.10
87.9	(1;2,3,5),(1;4,5,6)	3	3	87.9
87.10	(1;3,4,5),(1;5,6,7)	3	3	87.8
87.11	(1;2,3,4),(1;4,5,6,7)	3	3	62.2
87.12	(1;2,3,4,5)	4	3	69.1
87.13	(1;3,4,5,6)	4	3	46.1
87.14	(1;2,3,4,5),(1;5,6,7)	4	3	69.2
87.15	(1;2,3,5),(1;4,5,6,7)	3-**4**	3	62.3
87.16	(2;2,3,4,5,6,7)	3	3	87.16
87.17	(2;1,2,3,5,6,7)	3	3	79.1
87.18	(2;1,2,3,4,5,6)	4	3	70.1
87.19	(1;2,3,4),(2;1,2,3,5,6,7)	3	3	79.4
87.20	(1;3,4,5),(2;1,2,4,5,6,7)	3	3	79.3
87.21	(1;4,5,6),(2;1,2,3,5,6,7)	3	3	79.2
87.22	(1;5,6,7),(2;1,2,3,4,5,6)	4	3	70.2
87.23	(1;2,3,4),(1;4,5,6),(2;1,2,3,5,6,7)	3	3	79.6
88.1	$3 \leftarrow 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1$	3	3	90.8
88.2	(1;5,6,7)	3	3	90.12
88.3	(1;4,5,6,7)	3	3	84.4
88.4	(1;3,4,5),(1;5,6,7)	3	3	90.11
88.5	(2;1,3,4,5,6,7)	3	3	88.5
88.6	(2;1,2,3,4,5,6)	3	3	88.6
88.7	(1;5,6,7),(2;1,2,3,4,5,6)	3	3	88.7
88.8	(1;3,4,5),(1;5,6,7),(2;1,2,3,4,6,7)	3	3	88.8

89.1	3 1 ← 1 1 1 1 1	3	3	89.1
89.2	(1;2,4,5)	3	3	89.2
89.3	(1;5,6,7)	3	3	89.4
89.4	(1;2,3,4)	3	3	89.3
89.5	(1;2,4,5,6)	3	3	71.1
89.6	(1;4,5,6,7)	3	3	83.1
89.7	(1;2,3,4,5)	3	3	74.1
89.8	(1;2,4,5),(1;5,6,7)	3	3	89.12
89.9	(1;2,4,5),(1;2,6,7)	3	3	89.9
89.10	(1;2,3,4),(1;5,6,7)	3	3	89.10
89.11	(1;2,3,4),(1;4,5,6)	3	3	89.11
89.12	(1;2,3,4),(1;2,5,6)	3	3	89.8
89.13	(1;2,3,4),(1;2,5,6,7)	3	3	71.2
89.14	(1;2,3,4,5),(1;2,6,7)	3	3	74.2
89.15	(1;2,3,4,5),(1;5,6,7)	3	3	74.3
89.16	(1;2,3,4),(1;4,5,6,7)	3	3	83.2
89.17	(2;1,2,4,5,6,7)	3	3	82.1
89.18	(2;1,2,3,4,5,6)	3	3	85.1
89.19	(2;2,3,4,5,6,7)	3	3	89.19
89.20	(1;2,4,5),(2;1,2,3,5,6,7)	3	3	85.3
89.21	(1;5,6,7),(2;1,2,3,4,5,6)	3	3	85.4
89.22	(1;2,3,4),(2;1,2,4,5,6,7)	3	3	82.2
89.23	(1;2,3,4),(2;1,2,3,5,6,7)	3	3	85.2
89.24	(1;2,4,5),(1;5,6,7),(2;1,2,3,4,6,7)	3	3	85.8
89.25	(1;2,3,4),(1;4,5,6),(2;1,2,3,5,6,7)	3	3	85.7
89.26	(1;2,3,4),(1;2,5,6),(1;4,6,7)	3	3	89.26
90.1	3 1 1 1 1 1 1	3	3	90.1
90.2	(1;5,6,7)	3	3	90.2
90.3	(1;4,5,6,7)	3	3	84.1
90.4	(1;2,3,4),(1;5,6,7)	3	3	90.4
90.5	(1;3,4,5),(1;5,6,7)	3	3	90.5
90.6	(1;2,3,4),(1;4,5,6,7)	3	3	84.2
90.7	(2;2,3,4,5,6,7)	3	3	90.7
90.8	(2;1,2,3,4,5,6)	3	3	88.1
90.9	(1;2,3,4),(1;3,5,7),(1;4,5,6)	3	3	90.9
90.10	(1;2,3,4),(1;2,6,7),(1;3,5,7),(1;4,5,6)	3	3	90.10
90.11	(1;2,3,4),(1;4,5,6),(2;1,2,3,5,6,7)	3	3	88.4
90.12	(1;5,6,7),(2;1,2,3,4,5,6)	3	3	88.2

Table 5.2: List of 449 quartic plane de Jonquières maps with their enriched weighted proximity graphs listed in Table 5.1 respectively

♯	Map
1.1	$[xz^3+y^4 : yz^3 : z^4]$
2.1	$[-xyz^2-2xz^3+y^4+y^3z : -xyz^2-xz^3+y^4+y^2z^2-2yz^3 : z^4]$
3.1	$[-z^2(xz-y^2) : -y(xyz-xz^2-y^3+y^2z-z^3) : -z(xyz-y^3-z^3)]$
3.2	$[y^2(xz-y^2)-z^4 : yz(xz-y^2) : z^2(xz-y^2)]$
4.1	$[-xyz^2-xz^3+y^4+y^2z^2 : yz^3 : z^4]$
5.1	$[z(y+z)(xz+y^2) : y(xyz+xz^2+y^3+y^2z+z^3) : z^4]$
5.2	$[y(y+z)(xz+y^2) : -(y-z)(y+z)(xz+y^2) : z^4]$
6.1	$[-xyz^2-xz^3+y^4-2y^2z^2 : yz^2(z+y) : -z^2(y-z)(z+y)]$
7.1	$[13xyz^2-13xy^3-3xy^2z+3xz^3-y^4-6y^3z-25y^2z^2 : y(3z^3-4xy^2+$ $+4xz^2-y^3-3y^2z-7yz^2) : 8xy^3-8xyz^2-y^4+14y^2z^2+3z^4]$
8.1	$[-xyz^2-xz^3+y^4-2y^2z^2 : yz(z+y)^2 : -z(2y-z)(z+y)^2]$
9.1	$[y(2xy^2-2xz^2-y^3+5yz^2) : z(2xy^2-2xz^2-y^3+5yz^2) : -(y-z)^2(z+y)^2]$
10.1	$[3xyz^2+3xz^3+y^4-6y^2z^2-8yz^3 : z(3yz^2-xyz-xz^2+y^3+3y^2z) : z^2(xy+xz+z^2)]$
11.1	$[xz^3-xy^2z+4y^4+11y^3z+9y^2z^2 : y(z+y)^3 : (z-3y)(z+y)^3]$
12.1	$[xy^2z-xz^3+y^4+y^2z^2 : (z+y)(xyz-xz^2+y^3-y^2z+yz^2) : xy^2z-xz^3+y^4+z^4]$
13.1	$[xz^3-xy^2z-y^4-2y^3z+y^2z^2 : yz(z-y)(z+y) : xz^3-xy^2z-y^4-2y^3z+z^4]$
14.1	$[xz^3-y^4+y^3z : y^2z^2 : yz^3]$
15.1	$[y^3z-xyz^2-2xz^3+y^4 : y^4-xyz^2-xz^3+y^2z^2 : yz^3]$
16.1	$[xyz^2-xy^2z+xz^3-y^4+y^3z : z^2(xz+y^2) : (z+y)(xyz-xz^2+y^3-y^2z+yz^2)]$
16.2	$[xy^2z-xz^3+y^4 : z^2(xz+y^2) : yz(xz+y^2+z^2)]$
16.3	$[yz(xz+y^2) : z^2(xz+y^2) : y(xyz+y^3+z^3)]$
17.1	$[y^4-xyz^2-xz^3 : y^2z^2 : yz^3]$
18.1	$[xy^2z-xz^3+y^4 : z(z+y)(xz+y^2) : yz^3]$
18.2	$[y(z+y)(xz+y^2) : (z-y)(z+y)(xz+y^2) : yz^3]$
19.1	$[y(2xy^2-2xz^2-y^3+5yz^2) : 2xy^2z-2xz^3+5y^4+4y^3z-5y^2z^2 : y(y-z)(z+y)^2]$
20.1	$[y^4-xyz^2-xz^3+2y^3z : y^2z(z+y) : yz(z-y)(z+y)]$
21.1	$[4yz^3-3xyz^2-3xz^3+y^4 : z(2xyz+2xz^2+y^3-3yz^2) : z^2(-xy-xz+y^2+2yz)]$
22.1	$[xz^3-xy^2z-5y^4-7y^3z : y^2(z+y)^2 : y(z-2y)(z+y)^2]$
23.1	$[z(xy^2-xz^2+2y^3) : y^2(y-z)(z+y) : yz(y-z)(z+y)]$
24.1	$[y^4-xz^3 : y^3z : y^2z^2]$
25.1	$[y^4-xyz^2-xz^3 : z(y^3-xz^2) : y^2z^2]$
26.1	$[yz(xz+y^2) : y^2(xz+y^2+z^2) : (z-y)(xyz+xz^2+y^3+y^2z+yz^2)]$
26.2	$[y^2(xz+y^2) : yz(xz+y^2) : z^2(xz+y^2+yz)]$
26.3	$[xy^2z+xz^3+y^4 : yz(xz+y^2) : z^2(xz+y^2)]$
27.1	$[y(z+y)(xz+y^2) : y^2z^2 : (z-y)(xyz+xz^2+y^3+y^2z+yz^2)]$

27.2	$[xy^2z - xz^3 + y^4 : z(xyz + xz^2 + y^3) : y^2z^2]$
28.1	$[y(xy^2 - xz^2 + 2y^3) : z(xy^2 - xz^2 + 2y^3) : y^2(y - z)(z + y)]$
29.1	$[y^4 - xyz^2 - xz^3 : y^3z : y^2z^2]$
30.1	$[xyz^2 + xz^3 + y^4 : z(y^3 - xyz - xz^2) : z^2(xy + xz + y^2)]$
31.1	$[xy^2z - xz^3 + y^4 : y^3z : y^2z^2]$
32.1	$[y^4 : y^3z : z^2(xz + y^2)]$
32.2	$[xz^3 : y^4 : y^3z]$
33.1	$[y(y^3 - xz^2) : y^3z : z^2(xz + y^2)]$
33.2	$[xz^3 : y(y^3 - xz^2) : y^3z]$
34.1	$[y^2(xz + y^2) : yz(xz + y^2 + yz) : -z(2xyz - xz^2 + 2y^3 - yz^2)]$
34.2	$[y^2(xz + y^2) : yz(xz + y^2 + yz) : z^3(y + x)]$
34.3	$[y^2(xz + y^2) : yz(xz + y^2) : z^2(y^2 - xz)]$
34.4	$[y(xyz - xz^2 + y^3) : yz(xz + y^2) : z^2(xz + y^2)]$
34.5	$[x^3z : xy(y^2 - xz) : y^2(y^2 - xz)]$
35.1	$[y^4 : y^3z : z^2(xy + xz + 3y^2 + yz)]$
35.2	$[y^4 : y^3z : z^2(y + z)(x + y)]$
36.1	$[y^3z : y(xyz + xz^2 + y^3 + yz^2) : yz^3 - 3xy^2z - 2xyz^2 + xz^3 - 3y^4]$
36.2	$[xz^2(z + y) : y(xyz + xz^2 + y^3) : y^3z]$
36.3	$[y(xyz + xz^2 + y^3) : y^3z : z^2(xy + xz + y^2)]$
37.1	$[y^4 : y^3z : -z(xy^2 - xz^2 - y^2z)]$
37.2	$[xz(z - y)(y + z) : y^4 : y^3z]$
38.1	$[y^4 : y(xy^2 - 2xyz + xz^2 + yz^2) : z(xy^2 - 2xyz + xz^2 + y^3 + yz^2)]$
38.2	$[y^4 : y(xy^2 - 2xyz + xz^2 - 2y^2z + yz^2) : (2y + z)(xy^2 - 2xyz + xz^2 - 2y^2z + yz^2)]$
38.3	$[y^4 : y(xy^2 - 2xyz + xz^2 - 2y^2z + yz^2) : 2xy^3 - 3xy^2z + xz^3 - 3y^3z + yz^3]$
38.4	$[y^4 : y(y - z)(xy - xz - yz) : (y - z)(y + z)(xy - xz - yz)]$
39.1	$[y^2(-xy + xz - 3y^2 + yz) : y(y - z)(xy - xz - y^2 - yz) : xy^3 - 2xyz^2 + xz^3 + y^4 + yz^3]$
39.2	$[y^2(xy - xz + 3y^2 - yz) : y(xy^2 - xz^2 + 5y^3 - yz^2) : 2xy^3 - xy^2z - xz^3 + 9y^4 - yz^3]$
39.3	$[y^2(xy - xz + 3y^2 - yz) : y(xy^2 - xz^2 + 5y^3 - yz^2) : xy^3 - xz^3 + 7y^4 - yz^3]$
39.4	$[y^2(xy - xz + 3y^2 - yz) : y(xyz - xz^2 + 3y^3 - yz^2) : xyz^2 - xz^3 + 3y^4 - yz^3]$
39.5	$[y^2(xy - xz - yz) : yz(xy - xz - yz) : 2xy^3 - 2xy^2z + xyz^2 - xz^3 - 2y^4 - yz^3]$
40.1	$[y^4 : -y(xy^2 - xz^2 - 2y^2z - yz^2) : (y - z)(y + z)(2xy - xz - yz)]$
40.2	$[y^4 : -y(xy^2 - xz^2 - yz^2) : -z(xy^2 - xz^2 + 2y^3 - yz^2)]$
40.3	$[y^4 : -y(xy^2 - xz^2 - yz^2) : -z(y - z)(y + z)(x + y)]$
40.4	$[y^4 : -y(y - z)(xy + xz + yz) : -(y - z)(y + z)(xy + xz + yz)]$
41.1	$[x(2x^2z - 2xy^2 + xyz - 3y^3 - y^2z) : (y + x)(x^2z - xy^2 - y^3) :$ $-(y + x)(x^2z - xy^2 - xyz + y^2z)]$
41.2	$[xy^2z : -y(xz^2 - y^3) : z(xyz - xz^2 + y^3)]$
41.3	$[xy^2z : xyz^2 : -xz^3 + y^4]$
41.4	$[y^2(xz + y^2) : yz(xy + xz + y^2) : z^2(xy + xz + y^2)]$

41.5	$[y^2(xz+y^2) : yz(xz+y^2) : z(xy^2+xz^2+y^2z)]$
42.1	$[y^2(xz+y^2) : -y(xy^2-xz^2-y^2z) : xy^3-xy^2z+xz^3+y^2z^2]$
42.2	$[y^3x : y(xyz-xz^2+y^3) : -z(2xy^2-xyz+xz^2-y^3)]$
42.3	$[y^3x : xy^2z : xyz^2-xz^3+y^4]$
42.4	$[y^2(xz+y^2) : -y(xy^2-xz^2-y^2z) : -z(xy^2-xz^2-y^2z)]$
42.5	$[y^2(xz+y^2) : yz(xz+y^2) : xy^3+xz^3+y^2z^2)]$
43.1	$[xy^3-3xy^2z+3xyz^2-xz^3+y^3z : xy^3-3xy^2z+3xyz^2-xz^3+y^2z^2 :$ $xy^3-3xy^2z+3xyz^2-xz^3+yz^3]$
44.1	$[y(xy^2-xz^2+2y^2z) : z(xy^2-xz^2+2y^2z) : yz(y-z)(z+y)]$
45.1	$[xy^3-xy^2z-xyz^2+xz^3-2y^3z : -y^2z(y-z) : -yz(y-z)(z+y)]$
46.1	$[-xy(x-y)^2 : x(3x^2y+x^2z-xy^2-y^2z) : y(x+y)(x^2+xz-yz)]$
46.2	$[-y(xy^2-xz^2+y^3+3yz^2) : yz(y-z)^2 : z(xy^2-xz^2+3y^2z+z^3)]$
47.1	$[y^2z(z+y) : -yz(y-z)(z+y) : -2xy^3+xy^2z+2xyz^2-xz^3-5y^3z+z^4]$
48.1	$[-xy^3+4xy^2z-5xyz^2+2xz^3+y^3z : -2xy^3+7xy^2z-8xyz^2+3xz^3+y^2z^2 :$ $-3xy^3+10xy^2z-11xyz^2+4xz^3+yz^3]$
49.1	$[y(z+y)(xy-xz+yz) : -y(5xy^2-8xyz+3xz^2+y^3+2y^2z-z^3) :$ $-14xy^3+23xy^2z-10xyz^2+xz^3-4y^4-3y^3z+z^4]$
49.2	$[-xy^3-2xy^2z+4xyz^2-xz^3+y^4-4y^3z : yz(xy-xz+yz) : z^2(xy-xz+yz)]$
49.3	$[yz(xy-xz+yz) : -y(xy^2-4xyz+3xz^2+y^3-2y^2z-z^3) :$ $-2xy^3+5xy^2z-2xyz^2-xz^3-2y^4+3y^3z+z^4]$
50.1	$[xy^3-xy^2z-xyz^2+xz^3-8y^3z : y^2z(z+y) : -yz(y-z)(z+y)]$
51.1	$[y^3z : y^2z^2 : -xy^3+3xy^2z-3xyz^2+xz^3+yz^3]$
51.2	$[y^3z : y^2z^2 : -(y-z)(xy^2-2xyz+xz^2+y^3+y^2z+yz^2)]$
52.1	$[x(34x^2y+34x^2z-32xy^2+35xyz-33y^3+y^2z) :$ $36x^3y+36x^3z-35x^2y^2+36x^2yz-34xy^3+y^4 : (x+y)(y-2x)(xy+xz+yz)]$
52.2	$[-y(y-z)(xy-xz+y^2) : yz(xy-xz+yz) : 2xy^3-3xyz^2+xz^3+2y^4+yz^3]$
52.3	$[y^2(xy-xz+yz) : yz(xy-xz+yz) : -2xy^3+6xy^2z-5xyz^2+xz^3+yz^3]$
52.4	$[-y(xy^2-xz^2-y^3+3y^2z) : yz(xy-xz+yz) : z^2(xy-xz+yz)]$
52.5	$[-y(xy^2-3xyz+2xz^2-y^2z) : -y(2xy^2-5xyz+3xz^2-yz^2) :$ $-2xy^3+4xy^2z-xyz^2-xz^3+yz^3]$
52.6	$[x^3(z-y) : xy(y^2-xz) : y^2(y^2-xz)]$
52.7	$[x^3z : xy(y^2-xy-xz) : y^2(y^2-x^2-xz)]$
53.1	$[y(xy^2-2xyz+xz^2+y^2z) : y(xy^2-2xyz+xz^2+yz^2) : z(xy^2-2xyz+xz^2+yz^2)]$
53.2	$[y(xy^2-2xyz+xz^2+y^3) : -y^2z(y-z) : z(xy^2-2xyz+xz^2+yz^2)]$
54.1	$[xy^3-xy^2z-xyz^2+xz^3+yz^3 : y^3z : y^2z^2]$
54.2	$[(z+y)(xy^2-2xyz+xz^2+y^3-y^2z+yz^2) : y^3z : y^2z^2]$
55.1	$[xy^3+3xy^2z-4xz^3+22y^3z-4yz^3 : z(xy^2+xyz-2xz^2+8y^3-2yz^2) : -y^2z(y-z)]$
55.2	$[xy^3+3xy^2z-4xz^3+18y^3z : z(xy^2+xyz-2xz^2+6y^3) : -y^2z(y-z)]$
55.3	$[y(xy^2-xz^2+2y^2z) : -y^2z(z+y) : z(xy^2-xz^2-y^3-yz^2)]$
56.1	$[-16xy^2z+16xz^3+3y^4 : -z(8xy^2-8xz^2-3y^3) : -z(4xy^2-4xz^2-3y^2z)]$

56.2	$[xz(y-z)(z+y) : y^2(y-2z)(y+2z) : y^2z(y-2z)]$
57.1	$[z(3xy^2-5xyz+2xz^2+2yz^2) : y(2xy^2-3xyz+xz^2+2y^2z) : yz(xy-xz+2yz)]$
57.2	$[xz(y-z)^2 : y(2xy^2-3xyz+xz^2+2y^2z) : yz(xy-xz+2yz)]$
57.3	$[x(2y+z)(y-z)^2 : xy(y-z)^2 : y^2(xy-xz+2z^2)]$
57.4	$[-y(xy^2-xz^2+2y^2z) : xy^3+xz^3+2y^3z+yz^3 : y^2(xy+xz+2yz+z^2)]$
57.5	$[yz(xy+xz-yz) : -z(xy^2-xz^2-3y^2z-yz^2) : y^2(xy+xz+yz+2z^2)]$
57.6	$[x^2(y-z)(x-y) : x(x^2y-x^2z-y^2x+y^3) : y^2(y^2-xz)]$
58.1	$[y(xy^2-2xyz+xz^2+y^2z) : y^2z^2 : -z(-xy^2+2xyz-xz^2-yz^2)]$
58.2	$[xy(y-z)^2 : y^2z^2 : 2xy^3-3xy^2z+xz^3+yz^3]$
58.3	$[xy(y-z)^2 : x(2y+z)(y-z)^2 : y^2z^2]$
58.4	$[y(xy^2+2xyz+xz^2+y^2z) : y^2z^2 : z(xy^2+2xyz+xz^2+yz^2)]$
59.1	$[y(xy^2-xz^2+2y^2z) : z(xy^2-xz^2-yz^2) : y^2z^2]$
59.2	$[y(xy^2-xz^2+2y^2z) : xz(y-z)(z+y) : y^2z^2]$
59.3	$[xy(y-z)(z+y) : xz(y-z)(z+y) : y^2z^2]$
59.4	$[y(xy^2-xz^2+y^2z) : z(xy^2-xz^2-yz^2) : y^2z^2]$
60.1	$[y(3xy^2-5xyz+2xz^2+2yz^2) : 7xy^3-9xy^2z+2xz^3+4y^2z^2+2yz^3 :$ $y^2(xy-xz+2yz)]$
60.2	$[y(3xy^2-5xyz+2xz^2+2yz^2) : 5xy^3-7xy^2z+2xz^3+2yz^3 : y^2(xy-xz+2yz)]$
60.3	$[xy(y-z)^2 : 5xy^3-7xy^2z+2xz^3+2y^2z^2 : y^2(xy-xz+2yz)]$
60.4	$[xy(y-z)^2 : x(2y+z)(y-z)^2 : y^2(xy-xz+2yz)]$
60.5	$[y(xy^2-3xyz+2xz^2-2yz^2) : xy^3-3xy^2z+2xz^3-4y^2z^2-2yz^3 : y^2(xy-xz+2yz)]$
60.6	$[-yz(xy-xz+yz) : -xy^3+xz^3-2y^2z^2-yz^3 : y^2(xy-xz+yz)]$
60.7	$[y(xy^2-3xyz+2xz^2-2yz^2) : xy^3-2xy^2z+xz^3-y^2z^2-yz^3 : y^2(xy-xz+2yz)]$
60.8	$[x^2z(x-2y) : xy(xz+y^2) : y^2(xz+y^2)]$
60.9	$[(x-2y)(x+2y)(xz+y^2) : x^2yz : y^2(xy-2xz-2y^2)]$
61.1	$[(y+2z)(xy^2-2xyz+xz^2+yz^2) : z(xy^2-2xyz+xz^2+yz^2) : y^3z]$
61.2	$[xy^3-3xyz^2+2xz^3-3y^2z^2+2yz^3 : z(xy^2-2xyz+xz^2-2y^2z+yz^2) : y^3z]$
61.3	$[xy^3-3xyz^2+2xz^3+2y^2z^2 : z(xy^2-2xyz+xz^2+y^2z) : y^3z]$
61.4	$[x(y+2z)(y-z)^2 : xz(y-z)^2 : y^3z]$
61.5	$[xy^3-3xyz^2+2xz^3+y^2z^2-2yz^3 : z(y-z)(xy-xz+yz) : y^3z]$
62.1	$[x(8x^2z-8xy^2+8xyz-5y^3) : x(18x^2z-18xy^2+13xyz-5y^2z) :$ $16x^3z-16x^2y^2+16x^2yz-5y^4]$
62.2	$[y(xy^2-xz^2-yz^2) : z(xy^2-xz^2-yz^2) : y^3z]$
62.3	$[yx(y-z)(z+y) : z(xy^2-xz^2-y^2z) : y^3z]$
62.4	$[yx(y-z)(z+y) : xz(y-z)(z+y) : y^3z]$
62.5	$[y(xy^2-xz^2-yz^2) : z(y-z)(xy+xz+yz) : y^3z]$
63.1	$[xy^3-xz^3+y^3z-2y^2z^2-yz^3 : z(xy^2-xz^2-2y^2z-yz^2) : z^2(xy-xz-y^2-yz)]$
63.2	$[(y-z)(xy^2+xyz+xz^2+y^2z+yz^2) : z(xy^2-xz^2-yz^2) : z^2(y-z)(y+x)]$
63.3	$[(z+y)(xy^2-xyz+xz^2+y^2z) : z(xy^2-xz^2-y^2z) : z^2(xy+xz+y^2)]$

63.4	$[xy^3 - xz^3 - 2y^2z^2 - yz^3 : z(xy^2 - xz^2 - 2y^2z - yz^2) : z^2(xy - xz - y^2 - yz)]$
63.5	$[xy^3 + xz^3 + y^3z : xz(y - z)(z + y) : xz^2(z + y)]$
63.6	$[xy^3 + xz^3 + y^2z^2 : z(xy^2 - xz^2 - y^2z) : z^2(xy + xz + y^2)]$
63.7	$[xy^3 + xz^3 + yz^3 : z(xy^2 - xz^2 - yz^2) : z^2(z + y)(y + x)]$
63.8	$[xy^3 + xz^3 + y^3z - y^2z^2 - yz^3 : z(z + y)(xy - xz + yz) : z^2(xy + xz - 2y^2 - yz)]$
63.9	$[xy^3 + xz^3 - y^3z - y^2z^2 + yz^3 : z(y - z)(xy + xz + yz) : z^2(xy + xz + yz)]$
63.10	$[xy^3 + xz^3 + y^3z + y^2z^2 + yz^3 : z(z + y)(xy - xz - yz) : z^2(xy + xz + 2y^2 + yz)]$
63.11	$[x^2(xy + xz + yz) : -x(x - y)(xy + xz + yz) : x^3y + x^3z + xy^3 + y^3z]$
63.12	$[x(x - y)(xz - y^2) : x(x^2z - xy^2 - xyz + y^2z) : 2x^3z - 2x^2y^2 - x^2yz + y^4]$;
64.1	$[(z + y)(xy^2 - xyz + xz^2 + y^2z + yz^2) : z(2xy^2 + xz^2 + 2y^2z + yz^2) : z^2(2xy - xz - yz)]$
64.2	$[-y^2(xy - 2xz + yz) : xyz^2 : xy^3 + 2xz^3 + y^3z + 2y^2z^2]$
64.3	$[y^3(z + x) : yz(xy + xz + yz) : z(2xy^2 + xz^2 + 2y^2z + yz^2)]$
64.4	$[y^3(z + x) : z(xy^2 + xz^2 + y^2z) : xyz^2]$
64.5	$[y^3(z + x) : yz(xy + xz + yz) : z^3(y + x)]$
64.6	$[y^2(xy - 2xz + yz) : xyz^2 : xz^3]$
64.7	$[xy^3 + xz^3 + y^2z^2 : xy^2z : xyz^2]$
64.8	$[y^3(z + x) : xyz^2 : xz^3]$
64.9	$[xy^3 + xz^3 - y^3z + 2y^2z^2 - yz^3 : z(2xy^2 - xz^2 - 2y^2z + yz^2) : z^2(2xy + xz - yz)]$
64.10	$[xy^3 - xz^3 - y^3z - y^2z^2 + yz^3 : z(y - z)(xy + xz - yz) : z^2(xy + xz - yz)]$
64.11	$[x(x - y)(xz - y^2) : (x - y)(x + y)(xz - y^2) : x^3z - x^2y^2 + x^2yz - 2xy^2z + y^3z]$
65.1	$[yz(2xy - xz - y^2) : z(3xy^2 - xz^2 - y^3 - y^2z) : y^2(xz - y^2)]$
65.2	$[-y^2xz : z(2xyz - xz^2 + y^3 - y^2z) : y(xz^2 - y^3 + y^2z)]$
65.3	$[xyz(y - z) : -z(xz^2 - y^3) : y^2(xz - y^2)]$
65.4	$[xyz(y - z) : xz(y - z)(z + y) : y^2(xz - y^2)]$
65.5	$[y^2xz : xyz^2 : xz^3 + y^4 - y^3z]$
65.6	$[yz(xy - xz - y^2) : z(z + y)(xy - xz - y^2) : -y^2(2xz + y^2)]$
65.7	$[-y(3xy^2 - xyz - 2xz^2 - 2y^2z) : 3xy^3 - 5xyz^2 + 2xz^3 - 3y^3z + 2y^2z^2 :$ $y(8xy^2 - 8xz^2 + y^3 - 8y^2z)]$
65.8	$[yz(xz + y^2) : -z(z + y)(xy - xz - y^2) : y^2(xz + y^2)]$
65.9	$[xy^2z : y(x^2z - y^2x + y^3) : x(x - y)(xz - y^2)]$
66.1	$[y^2(xy + xz + yz) : y(2xy^2 + xz^2 + 2y^2z + yz^2) : 2xy^3 + xz^3 + 2y^3z + 2y^2z^2 + yz^3]$
66.2	$[y^3x : -yz(2xy - xz - yz) : -z(2xy^2 - xz^2 - 2y^2z - yz^2)]$
66.3	$[(y - z)(xy^2 + xyz + xz^2 + y^2z) : xy^2z : z^2(xy + xz + y^2)]$
66.4	$[(-z + y)(xy^2 + xyz + xz^2 + y^2z + yz^2) : z(xy^2 + xz^2 + yz^2) : yz^2(y + x)]$
66.5	$[y^2(xy + xz + yz) : yz^2(y + x) : z^3(y + x)]$
66.6	$[y^3x : z(xy^2 + xz^2 + yz^2) : yz^2(y + x)]$
66.7	$[y^3x : yz^2(y + x) : z^3(y + x)]$
66.8	$[y^3x : xy^2z : z^2(xy + xz + y^2)]$
66.9	$[xy^3 + xz^3 + y^3z : xy^2z : xyz^2]$

66.10	$[y^2(xy+xz+yz) : y(xy^2+xz^2+y^2z+yz^2) : xy^3+xz^3+y^3z+y^2z^2+yz^3]$
66.11	$[y^2(xy+xz+yz) : -y(2xy^2-xz^2+2y^2z-yz^2) : 6xy^3+xz^3+6y^3z-2y^2z^2+yz^3]$
66.12	$[xy^3-xz^3+y^3z+y^2z^2-yz^3 : z(xy^2+xz^2-y^2z+yz^2) : z^2(xy-xz+2y^2-yz)]$
66.13	$[x(x-y)(xz-y^2) : y(x-y)(xz-y^2) : y(x^2z-y^2x-xyz+y^2z)]$
67.1	$[-y^2(2xy-xz-y^2) : -y(y-z)(xy+xz+y^2) : -4xy^3+xz^3+3y^4-y^3z+y^2z^2]$
67.2	$[y^3x : y(xz^2-y^3+y^2z) : -z(3xy^2-xz^2+y^3-y^2z)]$
67.3	$[-y^2x(y-z) : -xy^3+xyz^2-xz^3+y^3z : -(z+y)(xy^2-xyz+xz^2-y^3)]$
67.4	$[y^3x : xy^2z : -2xyz^2+xz^3+y^4-y^3z]$
67.5	$[-y^2x(y-z) : -yx(y-z)(z+y) : xz^3-y^4]$
67.6	$[y^2(3xy+xz+y^2) : -y(11xy^2-xz^2+y^3-y^2z) : 41xy^3+xz^3+3y^4-y^3z+y^2z^2]$
67.7	$[y^2(xy+xz+y^2) : -y(y-z)(xy+xz+y^2) : -xy^3+xz^3+3y^4-y^3z+y^2z^2]$
67.8	$[x(x-y)(xz-y^2) : y(x-y)(xz-y^2) : y^3z]$
67.9	$[7x(2x-y)(xz+y^2) : 7(2x-y)(2x+y)(xz+y^2) : 24x^3z+24x^2y^2-7y^3z]$
68.1	$[3xz^3+8y^4-24y^3z+16y^2z^2 : y(z-y)(2y-z)(y-2z) :$ $(z-y)(2y-z)(y-2z)(2z+7y)]$
68.2	$[xz^3 : y^2(2y-z)(y-z) : y(y-z)(2y-z)(z+3y)]$
69.1	$[-8xyz^2+7xz^3+8y^4-24y^3z+16y^2z^2 : y(6xz^2-6y^3+25y^2z-33yz^2+14z^3) :$ $6xyz^2-6y^4+21y^3z-19y^2z^2+4z^4]$
69.2	$[xz^3 : -y(2xz^2-2y^3+3y^2z-yz^2) : -y(6xz^2-6y^3+7y^2z-z^3)]$
70.1	$[x(3x^2y+4x^2z-6xy^2+4xyz-5y^3) : -4x(x-y)(2xy+xz+y^2+yz) :$ $7x^3y+4x^3z-10x^2y^2-5xy^3+4y^3z]$
70.2	$[xz^3 : y(4xyz-8xz^2+4y^3-4y^2z-yz^2+z^3) :$ $(y-z)(4xyz+4y^3+4y^2z-yz^2-z^3)]$
70.3	$[y(y+2z)(xz+y^2+yz-2z^2) : yz(3xz+3y^2+2yz-5z^2) :$ $z(y-z)(xz+y^2+yz-z^2)]$
70.4	$[2xy^2z-7xz^3+2y^4+2y^3z+7yz^3+2z^4 : yz(xz+y^2) : z^2(xz+y^2)]$
70.5	$[x(x+y)(xz-y^2) : y(x+y)(y^2-xz) : xy(x^2-xy+xz-y^2)]$
71.1	$[-y(26xyz-38xz^2+26y^3+11y^2z-37yz^2) : -y(18xyz-69xz^2+18y^3+$ $19y^2z-37z^3) : -64xy^2z+307xyz^2-64y^4-10y^3z+74z^4]$
71.2	$[xyz^2 : -y(2xyz+2y^3-7y^2z+7yz^2-2z^3) : -14xy^2z-14y^4+45y^3z-35y^2z^2+4z^4]$
71.3	$[y^2(xz+y^2+z^2) : yz(xz+y^2+z^2) : xy^2z+4xyz^2+y^4+5y^3z+4z^4]$
72.1	$[-y(2xyz+2xz^2+10y^3-29y^2z+15yz^2) : y(y-z)(2y-z)(y-2z) :$ $(y-z)(2y-z)(y-2z)(2z+7y)]$
72.2	$[y(2xyz+2xz^2-6y^3-15y^2z-7yz^2) : y(y+2z)(3y+z)(2y+z) :$ $(2z-11y)(y+2z)(3y+z)(2y+z)]$
73.1	$[xyz^2+xz^3-12y^4+12y^3z : y^2(2y-z)(y-z) : y(y-z)(2y-z)(z+3y)]$
73.2	$[xz^2(z+y) : y^2(2y-z)(y-z) : y(y-z)(2y-z)(z+3y)]$
74.1	$[-31xy^2z+3xyz^2+34xz^3-6y^4 : -(y-z)(8xyz+8xz^2+y^3) :$ $-(y-z)(z+y)(4xz+y^2)]$

74.2	$[(y-z)(xyz+xz^2-6y^3) : xyz^2+xz^3+8y^4-10y^3z : y^2(2y-z)(y-z)]$
74.3	$[xz(y-z)(z+y) : xz^2(z+y) : -y^2(y-z)(z+y)]$
74.4	$[xz(y-z)(z+y) : xz^2(z+y) : y^2(2y-z)(y-z)]$
74.5	$[xyz^2+xz^3+2y^4 : (y-z)(xyz+xz^2-y^3) : -(y-z)(z+y)(xz+y^2)]$
75.1	$[y(y+2z)(xz+y^2-2yz) : -z^2(5xy-2xz-3y^2) : -yz(6xz-y^2-5yz)]$
75.2	$[y(2xyz+xz^2+2y^3-5y^2z) : -z(3xyz-xz^2-2y^3) : -yz^2(x-y)]$
75.3	$[-4x^2z(3x-2y) : 2x(9x^2z-8xy^2+4y^3) : 81x^3z-32x^2y^2-8xy^2z+8y^4]$
75.4	$[xyz^2 : xz^3 : -y^2(xz+y^2-z^2)]$
75.5	$[-2xy^2z+xyz^2-6xz^3-2y^4 : y^2(xz+y^2+yz) : xy^2z+6xz^3+y^4+y^2z^2]$
75.6	$[y(xyz+10xz^2+y^3-10yz^2) : z^2(5xy+2xz-3y^2) : yz(6xz+y^2-5yz)]$
75.7	$[xy^2z-3xyz^2+xz^3+y^4 : -yz(xz-y^2) : y(xyz-3xz^2+y^3+yz^2)]$
75.8	$[x(x^2z-y^3) : -yx^2(y-z) : -y^2(x^2-xy+xz-y^2)]$
75.9	$[x^2z(x-y) : xy(y^2+xy-2xz) : -y(x-y)(xy-xz+y^2)]$
75.10	$[x^2(y-z)(-y+2x) : -x(2x^2y-2x^2z-xy^2-y^3) : y^2(xz+y^2)]$
76.1	$[-z^2(5xy-2xz-3y^2) : y(6xz^2+y^3-7yz^2) : -yz(6xz-y^2-5yz)]$
76.2	$[-z(3xyz-xz^2-2y^3) : y(xz^2+2y^3-3y^2z) : -yz^2(x-y)]$
76.3	$[y(xz^2+4y^3-4y^2z) : xz^3+8y^4-8y^3z : y^2(2y-z)(y-z)]$
76.4	$[xyz^2 : xz^3 : y^2(2y-z)(y-z)]$
76.5	$[3xyz^2+xz^3-2y^4 : yz(xz+y^2); -y(3xz^2-2y^3-yz^2)]$
77.1	$[-9x^3y-9x^3z+85x^2y^2-18xy^3+9y^3z : y(22x^2y-9x^2z-18xy^2+9y^2z) :$ $y^2(10x^2-18xy-9xz+9yz)]$
77.2	$[-x^2(2xy+6xz-y^2+6yz) : xyz(x+y) : 8x^3y+24x^3z+23x^2yz-2xy^3+y^3z]$
77.3	$[6x(x^2y+x^2z+2xyz-3y^2z) : 6xy(xy+3xz-3yz) : y(10x^2z+6xy^2-7xyz-3y^2z)]$
77.4	$[xy^3+xz^3+y^3z-3y^2z^2 : xz(y-z)(z+y) : z^2(xy+xz-2y^2)]$
77.5	$[xz(y-z)(z+y) : xz^2(z+y) : -xy^3-xz^3-y^3z+y^2z^2]$
77.6	$[xy^3+xz^3-2y^2z^2 : xz(y-z)(z+y) : z^2(xy+xz-2y^2)]$
77.7	$[2yz(xy+2xz+3yz) : 2y(xy^2-4xz^2+y^2z-4yz^2) : z^2(y+2z)(y+x)]$
77.8	$[-3xy^3+xy^2z-4xz^3-3y^3z : xy^3+xyz^2+2xz^3+y^3z : 4xy^3+4xz^3+4y^3z+y^2z^2]$
77.9	$[(2z-y)(xy^2-xyz-2xz^2-y^2z-2yz^2) : z(2xy^2-xyz-3xz^2-3yz^2) :$ $z(z+y)(xy-xz-yz)]$
77.10	$[4xy^3+4xz^3+4y^3z+17y^2z^2+4yz^3 : z(2xy^2-2xz^2-7y^2z-2yz^2) :$ $z^2(xy+xz+3y^2+yz)]$
77.11	$[xy^3+5xy^2z-4xz^3-y^3z-4yz^3 : z^2(xy+xz+yz) : z(y-z)(xy+xz+yz)]$
77.12	$[-yzx(y+z) : y^2(2xy+2xz+2yz-z^2) : 4xy^3+5xy^2z-xz^3+4y^3z-yz^3]$
77.13	$[z(xy^2-xyz-2xz^2-2yz^2) : xy^3-3xyz^2-2xz^3+y^3z-2yz^3 :$ $z^2(4xy+4xz+y^2+4yz)]$
77.14	$[xy^3+xz^3+y^3z+yz^3 : -z(xy^2-xyz-2xz^2-2yz^2) : z(y-z)(xy+xz+yz)]$
77.15	$[x^2(x+y)(z-y) : xy(xy+yz+xz) : y(y-x)(xy+yz+xz)]$
77.16	$[x^2(2xz+y^2+2yz) : -xz(x+y)(2x-y) : 2x^3z+x^2yz+xy^3+y^3z]$

78.1	$[(x-y)(x^2y+2x^2z+3xy^2-13xyz+4xz^2+3y^2z):2(y-z)(x-y)(xy-2xz+yz):$ $2(y-z)(3x^2y-5x^2z-xy^2+2xyz+3xz^2-y^2z-yz^2)]$
78.2	$[xy^3+3xz^3-7y^2z^2+3yz^3:z(xy^2+2xz^2-5y^2z+2yz^2):z^2(xy+xz-3y^2+yz)]$
78.3	$[y(xy^2-3xz^2+2yz^2):yz(xy-2xz+yz):z^2(xz-y^2)]$
78.4	$[y(xy^2+xz^2-2yz^2):y^2z(x-z):-z^2(3xy-xz-y^2-yz)]$
78.5	$[y^2(xy-2xz+z^2):yz^2(x-y):z(2xy^2+xz^2-4y^2z+yz^2)]$
78.6	$[y(xy^2+xz^2-2yz^2):y^2z(x-z):-z^2(2xy-xz-y^2)]$
78.7	$[-xy^2z+xyz^2:-xy^3-y^3z+y^2z^2:-xy^3-2xy^2z+xz^3-y^3z+yz^3]$
78.8	$[-xy^2z+xyz^2:-xy^2z+xz^3:-xy^3-y^3z+y^2z^2]$
78.9	$[y(xy^2+5xz^2+y^2z-7yz^2):yz(xy+xz-2yz):-z^2(5xy-xz-3y^2-yz)]$
78.10	$[y(y-z)(xy+xz-2yz):yz(xy-2xz+yz):-z^2(2xy-xz-3y^2+2yz)]$
78.11	$[x(y-z)^2(x-y):(x-y)(x^2z+2xy^2-5xyz+xz^2+yz^2):$ $x^3z+5x^2y^2-11x^2yz+3x^2z^2-4xy^3+7xy^2z-yz^3]$
78.12	$[(x+2z)^2(xy+xz+yz):(y+2z)(x+2z)(xy+xz+yz):$ $-(y+2z)(x^2z-xy^2-5xyz-2xz^2-2yz^2)]$
79.1	$[y^2(6xz-5y^2-7yz):y(12xz^2-7y^3-17y^2z):24xz^3-29y^4-43y^3z+24y^2z^2]$
79.2	$[z(xy^2-4xyz+2xz^2+2y^2z):3xz^3-5xyz^2+y^4+3y^2z^2:z(xz^2-2xyz+y^3+y^2z)]$
79.3	$[xyz(y+2z):11xyz^2-2xz^3+2y^4-2y^2z^2:-z(13xyz+2xz^2-2y^3+2y^2z)]$
79.4	$[y^2(6xz-5y^2-7yz):y(12xz^2-7y^3-17y^2z):24xz^3-17y^4-31y^3z]$
79.5	$[xz(y-2z)(y+2z):xz^2(y-2z):8xz^3-y^4+y^3z]$
79.6	$[-y(3xyz-xz^2-2y^3):-7xy^2z+xz^3+6y^4:-y^2z(x-y)]$
79.7	$[xy^2z:xyz^2:2xz^3+y^4-3y^3z+2y^2z^2]$
79.8	$[-y(xyz-xz^2+6y^3):xy^2z+xz^3+6y^4+y^2z^2:y^2(2xz+6y^2+yz)]$
79.9	$[yz(5xy+2xz-3y^2):z(5xy^2+4xz^2-5y^3+4y^2z):y^2(6xz+y^2-5yz)]$
79.10	$[z(4xy^2-6xyz+xz^2+y^2z):-y^2(xz-y^2):-yz(xz-y^2)]$
79.11	$[x^3z+x^2y^2-18xy^3-9y^4:y(x^2z-5xy^2-3y^3):-xy^2(2y-z)]$
79.12	$[x(x-y)(xz+y^2):-xyz(2x+y):x^3z+x^2y^2+2x^2yz-y^4]$
79.13	$[xy(xz+y^2):-xz(2x+y)(3x-y):6x^3z+x^2yz+y^4]$
80.1	$[y(xy^2+8xz^2+y^2z+8yz^2):yz(xy-2xz-2yz):-z^2(7xy-2xz+3y^2-2yz)]$
80.2	$[y(xy^2+2xz^2+y^2z+2yz^2):xy^2z:-z^2(3xy-2xz-y^2-2yz)]$
80.3	$[y^3(z+x):z^2(xz+2y^2+yz):yz(2xy+xz+yz)]$
80.4	$[y(xy^2+4xz^2+4yz^2):yz(xy-2xz-2yz):-z^2(7xy-2xz+3y^2-2yz)]$
80.5	$[y^2(xy+2xz+yz):yz^2(y+x):-z(2xy^2-xz^2-2y^2z-yz^2)]$
80.6	$[y(xy^2-8xz^2+y^2z):xyz(y+2z):z^2(5xy+2xz+2y^2)]$
80.7	$[xy^3-16xz^3+y^3z-16yz^3:z(xy^2+4xz^2+4yz^2):z^2(x+y)(y+2z)]$
80.8	$[y^2(2xy+3xz+2yz):yz^2(y+x):z^3(y+x)]$
80.9	$[xy^2z:-y(y-z)(xy+xz+yz):-(y-z)(xy^2+xyz+xz^2+y^2z+yz^2)]$
80.10	$[y^3(z+x):z(3xy^2-4xz^2-4yz^2):z^2(y+x)(y+2z)]$

80.11	$[y^3(z+x) : yz^2(y+x) : z(2xy^2+xz^2+2y^2z+yz^2)]$
80.12	$[xy^3-8xz^3-8yz^3 : z(xy^2+4xz^2+4yz^2) : z^2(x+y)(y+2z)]$
80.13	$[y^2x(z+y) : 2xy^3+xz^3+2y^2z^2+yz^3 : yz^2(y+x)]$
80.14	$[y^3(z+x) : z^2(xz+y^2) : xyz(z+y)]$
80.15	$[y^3(z+x) : yz^2(y+x) : z^3(y+x)]$
80.16	$[y^3x : xy^2z : -z^2(3xy-xz+y^2-yz)]$
80.17	$[xy^2z : xyz^2 : -(y-z)(xy^2+xyz+xz^2+y^2z)]$
80.18	$[yx(y-2z)(y+2z) : xyz(y+2z) : z^2(5xy+2xz+2y^2)]$
80.19	$[xy^3+16xz^3+y^3z : xz(y-2z)(y+2z) : xz^2(y+2z)]$
80.20	$[y^3x : xy^2z : z^2(y+x)(2y+z)]$
80.21	$[y^2x(2y+z) : yz^2(y+x) : z^3(y+x)]$
80.22	$[y^3(z+x) : xz(y-2z)(y+2z) : xz^2(y+2z)]$
80.23	$[-y^2(xy-2xz+yz) : yz(2xy-xz-yz) : z(2y+z)(2xy-xz-yz)]$
80.24	$[y^3(x-z) : yz(xy-2xz-2yz) : -z^2(3xy-2xz+y^2-2yz)]$
80.25	$[y^2(xz-2xy-2yz) : y(8xy^2+xz^2+8y^2z+yz^2) :$ $(z-2y)(4xy^2+2xyz+xz^2+4y^2z+yz^2)]$
80.26	$[xy^3-2xz^3+y^3z-4y^2z^2-2yz^3 : z(xy^2+2xz^2+4y^2z+2yz^2) :$ $z^2(xy-2xz-3y^2-2yz)]$
80.27	$[-x^2yz : x(x-y)(xy+xz+yz) : y^2(x^2-xy-yz)]$
80.28	$[x^3(y+z) : y^2(x^2+2xy+3xz+2yz) : y(x^2z-4xy^2-4xyz-4y^2z)]$
80.29	$[x^2(2xy+2xz+yz) : xy^2(x+z) : 4x^3y+4x^3z-x^2y^2+xy^3+y^3z]$
80.30	$[x^3(y+z) : xy^2(x+z) : y(y-x)(xy+xz+yz)]$
80.31	$[x^3(y+3z) : xy(xy+xz+yz) : y(y-x)(xy+xz+yz)]$
80.32	$[-x^2(xy+xz+yz) : x(x-y)(xy+xz+yz) : 2x^3y+2x^3z-xy^3-y^3z]$
80.33	$[x^2z(2x-y) : -x(2x^2z+xy^2+y^2z) : 2x^3z+x^2y^2-xy^3-y^3z]$
81.1	$[y(9xy^2-13xyz+2xz^2+2y^2z) : (y-z)(27xy^2-4xyz-4xz^2-4y^2z) :$ $y^2(xy-2xz+y^2)]$
81.2	$[y(3xy^2-5xyz+xz^2+y^2z) : (y-z)(15xy^2-xyz-xz^2-y^2z) : -y^3(x-y)]$
81.3	$[-y^2x(3y-z) : -21xy^3+xz^3-y^3z+y^2z^2 : y(7xy^2-xz^2+y^3-y^2z)]$
81.4	$[4y^3z-3xy^3+3xy^2z-4xz^3 : 3xyz^2-3xy^3-10xz^3+10y^3z :$ $8y^3z-3xy^3-8xz^3+3y^4]$
81.5	$[yx(3y-2z)(y-z) : x(2z+5y)(y-z)(3y-2z) : y^2(xy-2xz+y^2)]$
81.6	$[yx(2y-z)(y-z) : 6xy^3-6xy^2z+xz^3-y^3z : -y^3(x-y)]$
81.7	$[yx(2y-z)(y-z) : x(y-z)(2y-z)(z+3y) : -y^3(x-y)]$
81.8	$[y^3x : y^2xz : -(2y-z)(xz^2-y^3+y^2z)]$
81.9	$[-y^2x(2y-z) : -yx(2y-z)(2y+z) : -7xy^3+xz^3+y^4-y^3z]$
81.10	$[y^2(3xy+2xz+3y^2) : -y(15xy^2-4xz^2+9y^3-4y^2z) :$ $75xy^3+8xz^3+45y^4-8y^3z+8y^2z^2]$
	$[y(9xy^2+xyz-5xz^2-5y^2z) : 9xy^3-5xyz^2+xz^3-6y^3z+y^2z^2 :$

81.11	$y(y-z)(xy+xz+y^2)]$
81.12	$[yz(xz+y^2) : 2xy^3+3xy^2z+xz^3-y^3z+y^2z^2 : y^2(xz+y^2)]$
81.13	$[x(3x-y)(xz+y^2) : (9x^2-y^2)(xz+y^2) : 18x^3z+18x^2y^2-2xy^2z-y^3z]$
81.14	$[x(x-y)(xz+y^2) : (x-y)(x+y)(xz+y^2) : -y^2z(2x+y)]$
82.1	$[-xy^2z+23xyz^2+26xz^3-y^4+y^3z : -(y+z)(xyz-10xz^2+y^3-y^2z) :$ $-xy^2z+17xyz^2+20xz^3-y^4+yz^3]$
82.2	$[xz^2(y+2z) : -xy^2z-8xz^3-y^4+y^2z^2 : z(6xz^2-y^3+yz^2)]$
82.3	$[xyz^2 : xz^3 : -y(xyz+y^3+2y^2z-yz^2-2z^3)]$
82.4	$[yz(2xz+3y^2+yz) : y(y-z)(xz+y^2-yz-z^2) : -(y-z)(z+y)(xz+y^2+z^2)]$
82.5	$[-y(y-z)(4xz+4y^2+3yz) : y(2y-z)(2xz+2y^2+yz-z^2) : z^2(2xz+y^2+z^2)]$
82.6	$[x^2(2x^2-3xy+2xz+y^2+yz) : -x(2x-y)(2x^2-3xy+2xz+y^2+yz) :$ $8x^4-20x^3y+8x^3z+16x^2y^2-4xy^3+y^3z]$
83.1	$[26xyz^2+28xz^3-y^4+y^3z : (y+z)(12xz^2-y^3+y^2z) : 20xyz^2+22xz^3-y^4+yz^3]$
83.2	$[xz^2(y+2z) : -12xz^3-y^4+y^2z^2 : z(6xz^2-y^3+yz^2)]$
83.3	$[xyz^2 : xz^3 : -y(y-z)(y+2z)(z+y)]$
83.4	$[-y(2xz^2+y^3+4y^2z-yz^2) : yz(xz+y^2+z^2) : 2xyz^2+xz^3+y^4+4y^3z+z^4]$
84.1	$[y(594xyz-312xz^2+466y^3-1233y^2z+485yz^2) : y(314xyz+178xz^2+$ $+586y^3-1563y^2z+485z^3) : 1558xy^2z-544xyz^2+1106y^4-2411y^3z+291z^4]$
84.2	$[(z-y)(y+2z)(y-2z)(z+y) : yz(y+2z)(x-y) : z(xy^2-4xyz+2y^2z-yz^2+2z^3)]$
84.3	$[-5xyz^2+y^4+4z^4 : -y^2z(x-y) : -yz^2(x-y)]$
84.4	$[y^2(xz+y^2+z^2) : yz(xz+y^2+z^2) : -28xy^2z+13xyz^2-22y^4-10y^3z+2z^4]$
85.1	$[y(93xyz-150xz^2-94y^3+208yz^2) : -645xyz^2+558xz^3-212y^4+386y^2z^2 :$ $y(42xz^2+66y^3+31y^2z-181yz^2)]$
85.2	$[x(6x^2y+6x^2z-22xy^2+11xyz-y^3) : x(12x^2y+12x^2z-45xy^2+22xyz-y^2z) :$ $-y(5x^2y-6x^2z+y^3)]$
85.3	$[y(5xyz+17xz^2-6y^3-18yz^2) : -7xyz^2+5xz^3+36y^4-12y^2z^2 :$ $y(12xz^2-6y^3+5y^2z-13yz^2)]$
85.4	$[y^2(21xz+26y^2+19yz-87z^2) : y(42xz^2+58y^3+23y^2z-165yz^2) :$ $84xz^3+110y^4+61y^3z-339y^2z^2]$
85.5	$[7xy^2z-9xyz^2+2xz^3+6y^4 : -y^2z(x-y) : -yz^2(x-y)]$
85.6	$[xyz(y-z) : xz(y-z)(z+y) : -y^2(3xz+2y^2+yz-6z^2)]$
85.7	$[y^2z(x+2y-3z) : -z(4xyz-3xz^2+2y^3-3y^2z) : -y(3xz^2-2y^3+11y^2z-12yz^2)]$
85.8	$[y^2(xz-6y^2+17yz-12z^2) : y(xz^2-4y^3+12y^2z-9yz^2) :$ $2xz^3-6y^4+19y^3z-15y^2z^2]$
85.9	$[5xy^2z+5xyz^2+2xz^3+6y^4 : yz(xy+xz+y^2) : -y(5xyz+3xz^2+6y^3-2yz^2)]$
85.10	$[7xy^2z+xyz^2-2xz^3+3y^4 : -z(4xy^2+xyz-2xz^2-3y^3) :$ $z(4xy^2+4xyz-2xz^2+3y^2z)]$
85.11	$[z(5xy^2+15xyz-2xz^2-12y^2z) : 5y^4-15xyz^2-18xz^3-8y^2z^2 : z(y-2z)(xz+y^2)]$
85.12	$[x(4x+y)(xz+y^2) : -x(4x^2z-2xy^2+xyz-y^2z) : -12x^3z-18x^2y^2+x^2yz+y^4]$
85.13	$[2x(x+2y)(xz+y^2) : 2x(y^2z-6x^2z-10xyz-9y^3) : 12x^3z+21x^2yz+19xy^3+2y^4]$

85.14	$[x(x-y)(xz+y^2) : xz(2x+y)(3x-y) : x^2y^2-5x^3z-x^2yz-y^4]$
85.15	$[3x^2y^2+4x^2yz-x^3z-3y^4 : y(4x^2y+3x^2z-xy^2-3y^3) : y(4x^2y+4x^2z-xyz-4y^3)]$
85.16	$[3xy(xz+y^2) : -x(x-y)(4xy+4xz+3yz) : 4x^3y+4x^3z-4x^2y^2-x^2yz+3y^4]$
86.1	$[y^2(xy-7xz+yz+11z^2) : -yz(5xy-2xz-6yz) : -z(13xy^2-4xz^2-14y^2z-4yz^2)]$
86.2	$[x^3y-2x^3z-62x^2y^2-2x^2yz+6x^2z^2-52xy^3+62xy^2z+xyz^2-6xz^3-156y^3z+2z^4 :$ $-y^2(x-z)(x+z) : y(6x^2y+x^2z+4xy^2-6xyz-2xz^2+12y^2z+z^3)]$
86.3	$[-zxy(z+y) : (z+y)(xy^2+xyz-xz^2+y^2z-yz^2) : y^2(xy+yz-z^2)]$
86.4	$[-y^2(xy-5xz+yz+7z^2) : yz(3xy-xz-4yz) : z(3xy^2-xz^2-3y^2z-yz^2)]$
86.5	$[-y^2(xy-7xz+yz+11z^2) : yz(5xy-2xz-6yz) : z(7xy^2-4xz^2-6y^2z)]$
86.6	$[y^2xz : xyz^2 : -(y-z)(xy^2+xyz+xz^2+y^2z+yz^2)]$
86.7	$[-y^2(2xy-5xz+2yz+4z^2) : zyx(y-z) : xz(y-z)(z+y)]$
86.8	$[y(xy^2+4xyz+2xz^2+y^2z) : z^3(y+x) : z^2y(y+x)]$
86.9	$[x(y-z)(y+2z)^2 : xz(y+2z)(y-z) : -z(7xy^2-4xz^2-6y^2z)]$
86.10	$[xy^3+2xy^2z-8xz^3+y^3z-8yz^3 : z(xy^2+xyz+2xz^2+2yz^2) : -y^2z(x-z)]$
86.11	$[4x^2(3xz+2y^2-5yz) : 4xz(3x-2y)(x-y) : -15x^3z+19x^2yz-12xy^3+8y^3z]$
86.12	$[xy^3-40xyz^2+64xz^3+y^3z : xz(y-2z)(y-4z) : -yz^2(2x-y)]$
86.13	$[-zxy(z+y) : (z+y)(xy^2+xyz-xz^2+y^2z-yz^2) : y^2(xy+3xz+yz-z^2)]$
86.14	$[y(xy^2-2xyz-4xz^2+y^2z) : -z(3xy^2+3xyz-2xz^2-2yz^2) : yz(3xy+4xz+yz)]$
86.15	$[x^2(3y-2z)(y+4x) : 2x(192x^2y-128x^2z+3xy^2-y^2z) :$ $-1536x^3y+1024x^3z+6xy^3-2y^3z]$
86.16	$[81xy^2z-10xy^3-324xz^3-10y^3z : xz(y+2z)(y-3z) : z(17xy^2-108xz^2-10y^2z)]$
86.17	$[xy^3+18xz^3+y^3z+2y^2z^2 : xz(y-3z)(y+3z) : xz^2(y-3z)]$
86.18	$[-y(xy^2+xyz-xz^2+y^2z) : z^3(y+x) : y^2(z+y)(z+x)]$
86.19	$[y^3(z+x) : -xz(y+2z)(y-3z) : y^2z(z+x)]$
86.20	$[x(-6z+y)(y+2z)(y+4z) : xz(y+4z)(y+2z) : z^2y(2x+y)]$
86.21	$[-y(xy^2-3xz^2+y^2z) : z^2(xy-xz-yz) : yz(xy-xz-yz)]$
86.22	$[y(4xy^2+3xyz+xz^2+4y^2z) : yz^3-5xy^3-4xy^2z+xz^3-5y^3z :$ $y^2(z^2-3xy-xz-3yz)]$
86.23	$[9xy^3-40xz^3+9y^3z+26y^2z^2-40yz^3 : z(3xy^2-8xz^2+7y^2z-8yz^2) :$ $z^2(xy-2xz+2y^2-2yz)]$
86.24	$[x^2(y+z)(y-2x) : xy^2(x+z) : 4x^3y+4x^3z+x^2y^2+y^3z]$
86.25	$[x^2(y+z)(2x-y) : x(2x^2y+2x^2z-xy^2-y^2z) : 4x^3y+4x^3z-xy^3-y^3z]$
86.26	$[x^3y+x^3z+5x^2y^2-2xy^3-2y^3z : y(3x^2y+x^2z-2xy^2-2y^2z) : xy^2(2x+z)]$
86.27	$[x^2(xy+xz-y^2+2yz) : -xyz(2x+y) : x^3y+x^3z+3x^2yz-xy^3-y^3z]$
86.28	$[x^2(y+z)(2x+y) : -x(3x^2y+3x^2z+xyz-y^2z) : 2x^3y+2x^3z+xy^3+y^3z]$
86.29	$[x^2(xy+xz+yz) : x(y-x)(xy+xz+yz) : y^3z-x^3y-x^3z-5x^2y^2]$
86.30	$[x^2(y+z)(2x+y) : xy^2(2x+z) : 4x^3y+4x^3z+2x^2y^2+xy^3+y^3z]$
86.31	$[-3x^2(xy+xz+yz) : 2x^3y+2x^3z-x^2y^2-3xy^3-6y^3z : 3x(x-y)(xy+xz+yz)]$
86.32	$[x^2(y+z)(-y+2x) : -x(x^2y+x^2z-xyz+y^2z) : 2x^3y+2x^3z-xy^3-y^3z]$

87.1	$[-yz(5xy-2xz-3y^2) : z(5xy^2+4xz^2-5y^3-4y^2z) : y^2(xy-10xz+y^2+8yz)]$
87.2	$[-5x^3z+5x^2y^2+9x^2yz-27y^4+18y^3z : 5y^3(x-3y+2z) :$ $5y(5x^2z+xyz-18y^3+12y^2z)]$
87.3	$[-yz(5xy+2xz-3y^2) : z(19xy^2-4xz^2-11y^3+4y^2z) : y^2(y-2z)(x-y)]$
87.4	$[yx(z+y)(2y+z) : -z(2xy^2-xz^2+y^2z) : y^2(4xy+3xz+yz)]$
87.5	$[xz(y+2z)(y-z) : 3xy^3+26xy^2z-32xz^3+3y^4 : -z(7xy^2-4xz^2-3y^3)]$
87.6	$[xz(y-2z)(y+2z) : xz^2(y+2z) : -xy^3-32xz^3-y^4+y^3z]$
87.7	$[xz(y-z)(3y-4z) : y(xy^2-10xyz+8xz^2+y^3) : -yz(5xy-4xz-y^2])$
87.8	$[yx(y-z)(z+y) : z(2xy^2-xz^2+y^2z) : -y^3(z+x)]$
87.9	$[-yx(y-z)(z+y) : xy^3-xy^2z+xz^3-y^2z^2 : -y^2z(x-y)]$
87.10	$[y(4xy^2-3xyz+2xz^2-3y^2z) : 24xy^3-7xy^2z+4xz^3-21y^3z : -y^2(y-2z)(x-y)]$
87.11	$[xzy(3y-2z) : y^3(x+y-z) : xz(3y-2z)(3y+2z)]$
87.12	$[z(3xy^2+xyz-xz^2+y^2z) : y^3(x-y) : y^2z(x-y)]$
87.13	$[y^2x(2y+z) : -yx(2y-z)(2y+z) : -10xy^3-xz^3+y^3z+y^2z^2]$
87.14	$[-x(y-z)(2y-z)(y-2z) : -y^3(x-y) : -y^2z(x-y)]$
87.15	$[xz(y-2z)(y+2z) : xz^2(y-2z) : xy^3-4xz^3-y^4+y^3z]$
87.16	$[y(8xy^2-3xyz-2xz^2+6y^3) : 13xy^2z-4xz^3-10y^4+4y^2z^2 :$ $y^2(2xy-2xz+4y^2-yz)]$
87.17	$[y(2xy^2-xyz+xz^2-4y^3) : 3xy^3-2xy^2z+xz^3-3y^4-y^2z^2 : y^2(xy+xz-5y^2+yz)]$
87.18	$[18xy^3+27xy^2z+12xyz^2+xz^3-y^2z^2 : y^2(xy+xz+y^2) : yz(xy+xz+y^2)]$
87.19	$[x(y-3x)(xz+y^2) : (y^2-9x^2)(3x+y)(xz+y^2) : 12x^3z+12x^2y^2-4x^2yz+y^3z]$
87.20	$[3x(2x+3y)(xz+y^2) : (9y^2-4x^2)(xz+y^2) : 9y^2z(x+y)-2x^2(4xz+4y^2+9yz)]$
87.21	$[3x(x-3y)(xz+y^2) : y(3y-x)(xz+y^2) : 3yz(2x+y)(y-x)]$
87.22	$[3xy(xz+y^2) : 3y^2(xz+y^2) : 3z(x-y)(3x-y)(y+2x)]$
87.23	$[x(6x+y)(xz+y^2) : (y-6x)(6x+y)(xz+y^2) : yz(2x+y)(y-3x)]$
88.1	$[-7866xy^2z+1155xyz^2+3699xz^3-2144y^4+5156y^2z^2 : 1434xy^2z-936xz^3+106y^4+$ $1155y^3z-1759y^2z^2 : 8256xy^2z-4374xz^3+2524y^4-7561y^2z^2+1155yz^3]$
88.2	$[-y(51xyz-18xz^2-20y^3-51y^2z+38yz^2) : -33xy^2z+12xz^3+4y^4+21y^3z-4y^2z^2 :$ $y(15xyz-8y^3-21y^2z+8yz^2+6z^3)]$
88.3	$[24xy^2z+10xyz^2+9xz^3-4y^4+41y^2z^2 : y^2z(2y-4x-3z) :$ $15yz^3-48xy^2z-18xz^3+8y^4-92y^2z^2]$
88.4	$[x(6x^3-6xy^2+5xyz-7y^2z) : 6xy(x^2+7xz-y^2-5yz) : 6yz(y-3x)(2x-y)]$
88.5	$[-97xy^2z+41xyz^2-6xz^3+6y^4-41y^3z+97y^2z^2 : -yz(xy-z^2) : -z^2(xy-z^2)]$
88.6	$[-y^2(xz-y^2) : -yz(xz-y^2) : z(6xy^2-23xyz-2xz^2+30y^2z-13yz^2+2z^3)]$
88.7	$[4x^4-6x^3y+4x^3z+2x^2y^2-35xy^2z-25y^3z : yz(2x+5y)(x+y) :$ $12x^4-14x^3y+12x^3z+2xy^3-109xy^2z-85y^3z]$
88.8	$[x(6x^3-x^2y-6xy^2-6xyz+y^3) : x(x+y)(2xy+yz-2x^2) :$ $10x^2y^2-10x^4+11x^2yz+y^3z]$
89.1	$[y(970xy^2+53xyz-699xz^2+1072y^3-1396y^2z) : 11156xy^3-7847xyz^2+159xz^3+$ $12024y^4-15492y^3z : -y(1188xy^2-822xz^2+1262y^3-1575y^2z-53yz^2)]$

89.2	$[-y(172xy^2-439xyz-105xz^2-172y^3+544y^2z):572xy^3-1919xy^2z+315xz^3-572y^4+1604y^3z:-y^2(34xy-88xz-34y^2+123yz-35z^2)]$
89.3	$[-y(128xy^2-20xyz-45xz^2-34y^3-29y^2z):-1028xy^3+395xyz^2+45xz^3+264y^4+324y^3z:y(36xy^2-15xz^2-8y^3-18y^2z+5yz^2)]$
89.4	$[y(286xy^2+35xyz-159xz^2+280y^3-280y^2z):3572xy^3-1943xyz^2+105xz^3+3360y^4-3360y^3z:-y(384xy^2-201xz^2+350y^3-315y^2z-35yz^2)]$
89.5	$[z(6xy^2-13xyz+3xz^2+4y^2z):-y(y-3z)(y+3z)(x-y):-yz(y-3z)(x-y)]$
89.6	$[-24xy^3-92xy^2z+30xyz^2+27xz^3+64y^3z:-y^3(4x-2y+3z):y^2z(4x-2y+3z)]$
89.7	$[-yx(2y+z)(y-3z):x(-3z+5y)(y-3z)(2y+z):y^2(4xy+23xz+14y^2-21yz+7z^2)]$
89.8	$[z(10xy^2+5xyz-3xz^2-12y^2z):y(4xy^2-4xyz-9xz^2-4y^3+13yz^2):yz(5yz-3xz-2y^2)]$
89.9	$[z(xy^2+3xyz+2xz^2-6y^3):-y(2xy^2-4xyz-xz^2-2y^3+5y^2z):yz(y-z)(x-2y)]$
89.10	$[-y(116xy^2-8xyz-9xz^2-64y^3+64y^2z):-1060xy^3+85xyz^2+18xz^3+576y^4-576y^3z:y(36xy^2-3xz^2-20y^3+18y^2z+2yz^2))$
89.11	$[y(8xy^2+9xyz-9xz^2+10y^3-10y^2z):14xy^3+81xy^2z-27xz^3+40y^4-40y^3z:y^2(y-3z)(x-y+z)]$
89.12	$[y(58xy^2-55xyz+3xz^2+40y^3-40y^2z):910xy^3-811xy^2z+9xz^3+640y^4-640y^3z:-y^2(19xy-17xz+13y^2-12yz-z^2)]$
89.13	$[x^2y(z-y):2x^2y^2-x^3z+xy^3-2y^3z:2x^2y^2-x^3z+xy^2z-2y^3z]$
89.14	$[x^2(x^2+xz-y^2-2yz):xyz(2x-y):yz(2x-y)(2x+y)]$
89.15	$[-yx(2y+z)(2y-3z):x(-3z+4y)(2y-3z)(2y+z):-y^2(y-z)(2x-2y+z)]$
89.16	$[x(2y-z)(y-3z)(z+y):y^3(x-y+z):y^2(z+y)(x-y+z)]$
89.17	$[-12xy^3-10xy^2z+5xyz^2+2xz^3+6y^4:y^2(xz+y^2+yz):-y(y-z)(xz+y^2+yz)]$
89.18	$[2xy^3-11xy^2z-53xyz^2-6xz^3+70y^3z:y(2y+z)(xz-y^2):y(2xy^2-11xyz-41xz^2+58y^2z-6yz^2)]$
89.19	$[xy^2z-4xz^3-30y^4:-y(10xy^2-4xz^2-3y^2z):-12xz^3-100y^4+y^2z^2]$
89.20	$[4x^2(5x^2+17xz-5y^2+yz):4x(5x^3+11x^2z-5xy^2+y^2z):50x^4+5x^3y+152x^3z-50x^2y^2-5xy^3+4y^3z]$
89.21	$[x(2x-y)(xz+y^2):(4x^2-y^2)(xz+y^2):z(x^2y-2x^3-y^3)-2xy^2(2x+z)]$
89.22	$[x^2yz+xy^3:xy^2z+y^4:y^3z-6x^3y-6x^3z-7x^2y^2-7x^2yz-xy^2z]$
89.23	$[x^3z+x^2y^2:x^2yz+xy^3:6x^3z-5x^2yz-2xy^2z+y^3z]$
89.24	$[x^3z+x^2y^2:x^2yz+xy^3:6x^3z-5x^2yz-xy^2z+y^4+y^3z]$
89.25	$[x^3z+x^2y^2:x^2yz+xy^3:12x^3z-4x^2yz-3xy^2z+y^3z]$
89.26	$[x(5xyz+2y^3-6x^2z-3xy^2):x(6x^2z-xy^2-9xyz+2y^2z):24x^3z-6x^2y^2-32x^2yz+2y^3z]$
90.1	$[-y(586xy^2-315xyz-619xz^2+884y^2z-536z^3):-548xy^3+767xyz^2+105xz^3+668yz^3-992y^3z:y(416xy^2-734xz^2+634y^2z+315yz^2-631z^3)]$
90.2	$[y(198xy^2-885xyz+103xz^2+972y^2z-388z^3):30xy^3+203xy^2z+103xz^3+322yz^3-658y^3z:-y(66xy^2-398xyz+633y^2z-103yz^2-198z^3)]$
	$[-2xy^3+8xy^2z-11xyz^2+4xz^3+y^3z:-6xy^3+25xy^2z-33xyz^2+12xz^3+2y^2z^2:$

90.3	$-18xy^3 + 75xy^2z - 97xyz^2 + 36xz^3 + 4yz^3]$
90.4	$[y(20xy^2 + 36xyz + 49xz^2 - 48y^2z + 48z^3) : 28xy^3 + 47xyz^2 + 9xz^3 - 48y^3z + 48yz^3 :$ $-y(4xy^2 + 17xz^2 - 6y^2z - 9yz^2 + 15z^3)]$
90.5	$[44xy^3 + 7xyz^2 + 159xz^3 - 68y^3z - 49yz^3 + 117z^4 : z(44xy^2 - 96xyz + 241xz^2 - 4y^3 +$ $201z^3 - 197yz^2) : -z(4xyz + 17xz^2 + 2y^3 - 11y^2z - 6yz^2 + 15z^3)]$
90.6	$[3xy^3 - 8xy^2z + 11xyz^2 - 4xz^3 - 3y^4 + y^3z : 9xy^3 - 23xy^2z + 33xyz^2 - 12xz^3 +$ $2y^2z^2 - 9y^4 : 27xy^3 - 69xy^2z + 101xyz^2 - 36xz^3 - 27y^4 + 4yz^3]$
90.7	$[y(6xy^2 + 2xyz + 2xz^2 + 13y^2z + 7yz^2) : 100xy^2z - 318xy^3 - 773y^3z - 347y^2z^2 +$ $+8xz^3 : y(42xy^2 - 12xyz + 103y^2z + 53yz^2 + 4z^3)]$
90.8	$[yz(y^2 - xz) : z^2(y^2 - xz) : 6xy^3 - 35xy^2z + 62xyz^2 - 35xz^3 + 6yz^3]$
90.9	$[10x^4 - 15x^3y + 10x^3z + 5x^2y^2 - 32x^2yz + 12y^3z : (x - y)(30x^3 - 5x^2y +$ $+30x^2z - 5xy^2 - 14xyz - 14y^2z) : 5yz(x + y)(3x - 2y)]$
90.10	$[2x^4 - 3x^3y + 2x^3z + x^2y^2 - 4y^3z : (x - y)(6x^3 - x^2y + 6x^2z - xy^2 + 6xyz + 6y^2z) :$ $yz(x + y)(2y - x)]$
90.11	$[2x^2(2x^2 - 3xy + 2xz + y^2 - 4yz) : 2(x - y)(6x^3 - x^2y + 6x^2z - xy^2 - 6xyz - 6y^2z) :$ $yz(x + y)(3y - x)]$
90.12	$[x^2(2x^2 - 3xy + 2xz + y^2 + 5yz) : 6x^4 - 7x^3y + 6x^3z + 22x^2yz + xy^3 - 10y^3z :$ $yz(x + y)(2y - x)]$

5.2 Quartic plane non-de Jonquières maps

In this section we compute the ordinary quadratic length of quartic plane non-de Jonquières maps, starting from the classification of enriched weighted proximity graph of such maps that we found in Chapter 2. Totally, we found 119 different enriched weighted proximity graphs, listed in Table 5.3.

Theorem 5.6. *Let $\varphi_n \in \mathrm{Cr}(\mathbb{P}^2)$ be a quartic plane non-de Jonquières map with enriched weighted proximity graph of type n in Table 5.3. Then, the ordinary quadratic length of φ_n is listed in the third column of Table 5.3.*

Concerning the quadratic length, it is straightforward to show that quartic plane non-de Jonquières maps have quadratic length 2.

We give two examples of quartic plane non-de Jonquières maps. The first one, that has enriched weighted proximity graph of type 1.1 in Table 5.3, is given by

$$\begin{aligned}\varphi_{1,t}([x:y:z]) \;&:=\; [2tx^2yz - tx^4 - 2tx^3y - tx^2y^2 + xy^3 + 2txy^2z - ty^2z^2 : y^4 : \\ &\quad\; 2tx^2yz - tx^4 - 2tx^3y - (t+1)x^2y^2 + 2txy^2z + y^3z - ty^2z^2]\end{aligned}$$

where $t \in \mathbb{C}$, while the second one, that has enriched weighted proximity graph of type 3.1 in Table 5.3, is given by

$$\begin{aligned}\varphi_{3,t,h}([x:y:z]) \;&:=\; [y(x^3 + x^2y + ((t-1)y^2 - yz)x + (2 - 3t + h)y^3) : \\ &\quad\; y^2(x^2 + xy + (t-1)y^2 - yz) : (y^2 + 2yz)x^2 - x^4 + \\ &\quad\; (2t-2)y^3x + (t^2 - 8t + 2h + 5)y^4 - y^2z^2]\end{aligned}$$

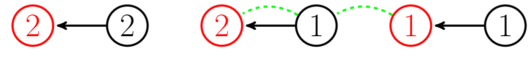
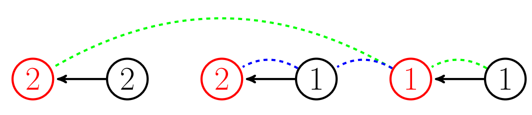
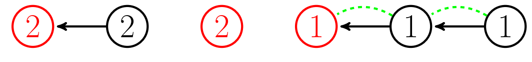
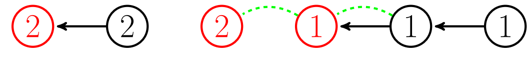
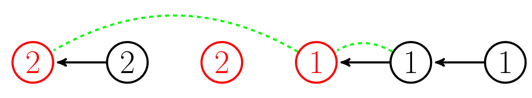
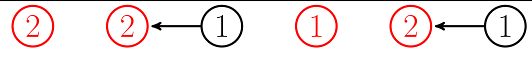
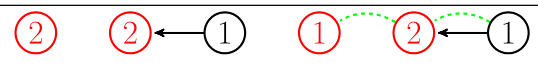
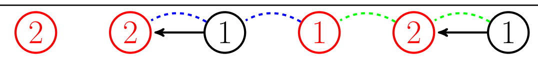
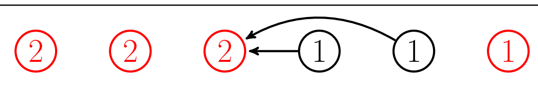
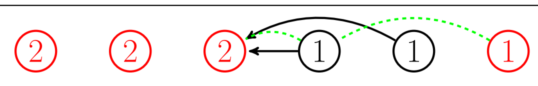
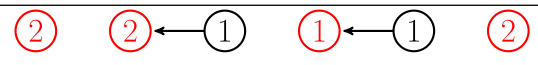
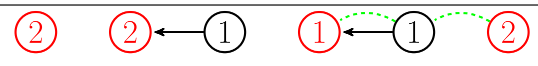
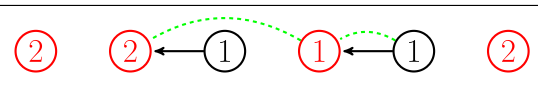
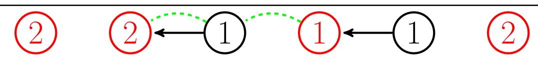
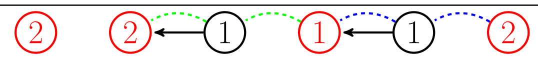
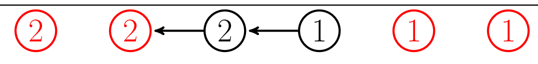
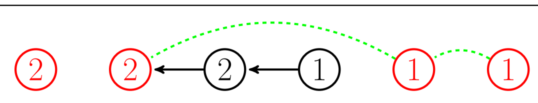
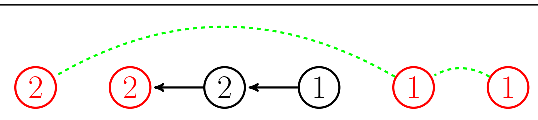
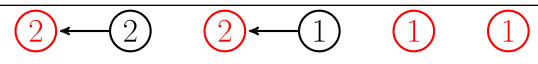
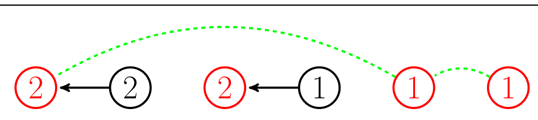
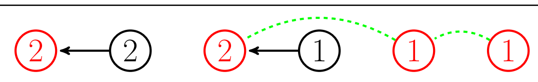
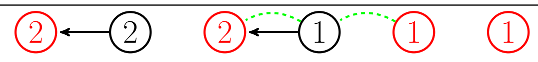
where $t, h \in \mathbb{C}$.

Table 5.3: Enriched weighted proximity graphs and ordinary quadratic lengths of quartic plane non-de Jonquières maps

$\sharp$	Enriched weighted proximity graph	oql
1.1	2 ← 2 ← 2 ← 1 ← 1 ← 1	7
2.1	2 ← 2 ← 2 ← 1 1 ← 1	6
3.1	2 ← 2 ← 2 ← 1 ← 1 ← 1	7
4.1	2 2 ← 2 ← 1 ← 1 ← 1	6
5.1	2 2 ← 2 ← 1 1 ← 1	4
6.1	2 2 ← 2 ← 1 ← 1 ← 1	6
7.1	2 ← 1 2 ← 2 ← 1 ← 1	4
8.1	2 ← 1 2 ← 2 ← 1 1	3
9.1	2 ← 1 2 ← 2 ← 1 ← 1	4
10.1	2 ← 2 2 ← 1 ← 1 ← 1	5
11.1	2 ← 2 2 ← 1 1 ← 1	4
11.2	2 ← 2 2 ← 1 1 ← 1	4
12.1	2 ← 2 2 ← 1 ← 1 ← 1	5
12.2	2 ← 2 2 ← 1 ← 1 ← 1	5
13.1	2 ← 1 ← 1 2 ← 2 ← 1	4
13.2	2 ← 1 ← 1 2 ← 2 ← 1	4
14.1	2 ← 1 ← 1 2 ← 2 ← 1	4
15.1	2 ← 1 1 2 ← 2 ← 1	3
16.1	2 ← 2 ← 2 1 ← 1 ← 1	4
16.2	2 ← 2 ← 2 1 ← 1 ← 1	4
17.1	2 ← 2 ← 2 ← 1 1 ← 1	4

17.2	$2 \leftarrow 2 \leftarrow 2 \leftarrow 1 \quad 1 \leftarrow 1$	5
18.1	$2 \leftarrow 2 \leftarrow 2 \leftarrow 1 \leftarrow 1 \quad 1$	5
19.1	$2 \leftarrow 2 \leftarrow 2 \leftarrow 1 \quad 1 \quad 1$	5
20.1	$2 \leftarrow 2 \leftarrow 2 \leftarrow 1 \leftarrow 1 \quad 1$	5
21.1	$2 \quad 2 \quad 2 \leftarrow 1 \leftarrow 1 \leftarrow 1$	5
21.2	$2 \quad 2 \quad 2 \leftarrow 1 \leftarrow 1 \leftarrow 1$	5
22.1	$2 \quad 2 \quad 2 \leftarrow 1 \leftarrow 1 \leftarrow 1$	5
23.1	$2 \quad 2 \quad 2 \leftarrow 1 \quad 1 \leftarrow 1$	3
23.2	$2 \quad 2 \quad 2 \leftarrow 1 \quad 1 \leftarrow 1$	3
24.1	$2 \leftarrow 1 \quad 2 \leftarrow 1 \quad 2 \leftarrow 1$	2
25.1	$2 \quad 2 \leftarrow 1 \quad 2 \leftarrow 1 \leftarrow 1$	3
25.2	$2 \quad 2 \leftarrow 1 \quad 2 \leftarrow 1 \leftarrow 1$	3
26.1	$2 \quad 2 \leftarrow 1 \quad 2 \leftarrow 1 \leftarrow 1$	3
27.1	$2 \quad 2 \leftarrow 1 \quad 2 \leftarrow 1 \quad 1$	2
28.1	$2 \quad 2 \leftarrow 2 \leftarrow 1 \leftarrow 1 \quad 1$	4
29.1	$2 \quad 2 \leftarrow 2 \leftarrow 1 \leftarrow 1 \quad 1$	4
30.1	$2 \quad 2 \leftarrow 2 \leftarrow 1 \quad 1 \quad 1$	3
31.1	$2 \leftarrow 2 \leftarrow 1 \quad 2 \leftarrow 1 \quad 1$	3
31.2	$2 \leftarrow 2 \leftarrow 1 \quad 2 \leftarrow 1 \quad 1$	3
32.1	$2 \leftarrow 2 \quad 2 \leftarrow 1 \leftarrow 1 \quad 1$	3
32.2	$2 \leftarrow 2 \quad 2 \leftarrow 1 \leftarrow 1 \quad 1$	3
32.3	$2 \leftarrow 2 \quad 2 \leftarrow 1 \leftarrow 1 \quad 1$	4
33.1	$2 \leftarrow 2 \quad 2 \leftarrow 1 \leftarrow 1 \quad 1$	3
33.2	$2 \leftarrow 2 \quad 2 \leftarrow 1 \leftarrow 1 \quad 1$	4

34.1	2 ← 2 2 ← 1 1 1	3
34.2	2 ← 2 2 ← 1 1 1	3
35.1	2 ← 2 ← 2 ← 1 1 1	4
35.2	2 ← 2 ← 2 ← 1 1 1	5
36.1	2 ← 2 ← 2 1 ← 1 1	4
36.2	2 ← 2 ← 2 1 ← 1 1	5
36.3	2 ← 2 ← 2 1 ← 1 1	5
36.4	2 ← 2 ← 2 1 ← 1 1	4
37.1	2 2 2 1 ← 1 ← 1	4
37.2	2 2 2 1 ← 1 ← 1	4
37.3	2 2 2 1 ← 1 ← 1	4
38.1	2 2 2 ← 1 ← 1 1	3
38.2	2 2 2 ← 1 ← 1 1	3
38.3	2 2 2 ← 1 ← 1 1	3
39.1	2 2 2 ← 1 ← 1 1	3
39.2	2 2 2 ← 1 ← 1 1	3
40.1	2 2 ← 2 ← 1 1 ← 1	4
40.2	2 2 ← 2 ← 1 1 ← 1	3
40.3	2 2 ← 2 ← 1 1 ← 1	3
41.1	2 ← 2 2 ← 1 1 ← 1	3
41.2	2 ← 2 2 ← 1 1 ← 1	3
41.3	2 ← 2 2 ← 1 1 ← 1	3

41.4		4
41.5		3
42.1		4
42.2		4
42.3		4
42.4		4
43.1		2
43.2		2
43.3		2
44.1		2
44.2		2
45.1		3
45.2		3
45.3		3
45.4		3
45.5		3
46.1		3
46.2		3
46.3		3
47.1		3
47.2		3
47.3		3
47.4		3

48.1	2 ← 2 2 1 ← 1 1	3
48.2	2 ← 2 2 1 ← 1 1	4
48.3	2 ← 2 2 1 ← 1 1	4
48.4	2 ← 2 2 1 ← 1 1	3
48.5	2 ← 2 2 1 ← 1 1	4
48.6	2 ← 2 2 1 ← 1 1	4
48.7	2 ← 2 2 1 ← 1 1	3
49.1	2 ← 2 ← 2 1 1 1	4
49.2	2 ← 2 ← 2 1 1 1	5
49.3	2 ← 2 ← 2 1 1 1	4
50.1	2 2 2 1 1 ← 1	3
50.2	2 2 2 1 1 ← 1	3
50.3	2 2 2 1 1 ← 1	3
50.4	2 2 2 1 1 ← 1	3
50.5	2 2 2 1 1 ← 1	3
51.1	2 2 2 ← 1 1 1	2
51.2	2 2 2 ← 1 1 1	2
51.3	2 2 2 ← 1 1 1	2
51.4	2 2 2 ← 1 1 1	2
51.5	2 2 2 ← 1 1 1	2
52.1	2 2 ← 2 1 1 1	3
52.2	2 2 ← 2 1 1 1	3
52.3	2 2 ← 2 1 1 1	3

52.4	2 2 ← 2 1 1 1	3
52.5	2 2 ← 2 1 1 1	3
53.1	2 2 2 1 1 1	2
53.2	2 2 2 1 1 1	2
53.3	2 2 2 1 1 1	2
53.4	2 2 2 1 1 1	2

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Glossary of Notations

$Aut(\mathbb{P}^2)$	group of automorphisms of $\mathbb{P}^2$
$\mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2)$	blowing-up of $\mathbb{A}^2$ at $\mathbf{0}$
$\mathcal{B}(\mathbb{P}^2)$	bubble space of $\mathbb{P}^2$
$p \succ q$	p is infinitely near q
$p \succ_k q$	p is infinitely near q of order k
$p \dashrightarrow q$	p is proximate to q
$p \odot q$	p is satellite to q
$p \not\odot q$	p is not satellite to q
$\mathrm{Cr}(\mathbb{P}^2)$	plane Cremona group
$\mathrm{oql}(\varphi)$	ordinary quadratic length of plane Cremona map φ
$\mathrm{ql}(\varphi)$	quadratic length of plane Cremona map φ
$\mathrm{lgth}(\varphi)$	length of plane Cremona map φ
$\mathrm{h}_\varphi(p)$	height of a point $p \in \mathcal{B}(\mathbb{P}^2)$ w.r.t plane Cremona map φ
$\mathrm{load}_\varphi(p)$	load of a proper base point p w.r.t plane Cremona map φ
$G = (V, E)$	a graph where V-set of vertices and E-set of edges
$\langle u, v\rangle$ or $u \to v$	arc (oriented edge) from u to v
$\deg(u)$	degree of vertex u
$\mathrm{outdeg}(u)$	external degree of vertex u
$\mathrm{indeg}(u)$	internal degree of vertex u
A_G	adjacency matrix of oriented graph G
G_φ	weighted directed graph associated to plane Cremona map φ
$\mathrm{mult}_\varphi(p)$	multiplicity of a plane Cremona map φ at the base point p