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Abstract

The aim of this paper is to model trading decisions of financial investors based on a sentiment index. For this purpose, we analyse a dynamical model which includes the sentiment index in the agents' trading behavior. We consider the set up of a Discrete Dynamical System, assuming that in financial markets transactions take place between two groups of fundamentalists that differ in their perception of fundamental value. The proportion of fundamentalists in the two groups is assumed to depend on the sentiment index. The sentiment index used is related to the risk asymmetry index (RAX) enabling us to consider both the variance and the asymmetry of the prediction error between the two groups of fundamentalists. We identify the equilibria of the model and conduct a numerical analysis in order to capture stylized facts documented empirically in the financial literature.

Keywords: Sentiment index; Market risk; Asymmetry; Bifurcation analysis; Numerical simulations

1 Introduction

In the financial literature, it is assumed that investors are loss-adverse in the sense that they are happier with a gain than a loss of the same amount. Furthermore, investors are more sensitive to losses than to gains. Moreover, the distribution of stock returns in financial markets is found to be negatively skewed, in other words, extreme and negative events are more probable than positive ones, in contrast with the assumption of a normal distribution that assigns the same probability to positive and negative returns. If we combine assumptions about investor preferences and empirical evidence on financial returns, we uncover the need for an indicator accounting for these stylized facts. Based on this line of reasoning, some studies have analyzed the role of investor sentiment in stock price formation suggesting that investor sentiment is one of the main determinants of asymmetry in stock returns (Jawadi *et al.* (2018), Verma and Soydemir (2009)).

On the other hand, many theoretical and empirical studies have analysed asset price dynamics in financial markets in order to explain stylized facts of stock returns, such as asymmetry, excess of kurtosis and volatility clustering. An important role in this sense is played by models with heterogeneous agents (see

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Chiarella *et al.* (2002, 2009), Gaunersdorfer (2000) among others). Models with heterogeneous agents exhibit a good performance in capturing market behavior and in the replication of econometric properties and stylized facts of financial time series (see for example He and Li (2007), Franke and Westerhoff (2016)). Moreover, it is well known that models involving heterogeneous agents allow for considerable flexibility in investor behavior because they have the opportunity to switch between different trading rules according to certain fitness measures (Brock and Hommes (1998)). The majority of models with heterogeneous agents consider two types of traders: fundamentalists and chartists. Fundamentalists are aware of market fundamentals and they believe in reversion to the mean, while chartists are considered as a source of instability in the model because with their speculative behavior they destabilize the market leading to intricate scenarios. As a result, these studies consider heterogeneity both in the expectations of agents and their trading decision rules.

The aim of our paper is to highlight the role of a sentiment index in the market with heterogeneous agents. As a proxy for the sentiment index, we adopt the Risk Asymmetry Index (RAX) introduced by Elyasiani *et al.* (2018). The RAX index aims to capture the risk asymmetry in the market, i.e. the higher volatility negative returns, compared to the volatility of positive returns. We examine the joint role of heterogeneity and non-linearity (introduced by the sentiment index) in financial markets. We model an endogenous switching mechanism between traders relying on the sentiment index, that is considered as a benchmark index for all agents. We consider two groups of fundamentalists that adopt the same trading rule but heterogeneous beliefs about fundamental value. In this way, we implicitly introduce a degree of uncertainty about the true fundamental value (which in He and Zheng (2016) is modelled directly in the utility function).

There are a few studies that consider only fundamentalists in asset pricing models. In particular, Naimzada and Ricchiuti (2008) analyze a financial market with two fundamentalists acting as gurus. Traders followed one of the two fundamentalists, by relying on a fitness measure that involves the distance between the fundamental value and the current price. They found that complex dynamics arise when market maker and agents overreact to price misalignment. However, this model does not consider market sentiment: fundamentalists take trading decisions by looking at the distance between the fundamental price and the current price. On the other hand, we assume that all the agents in the market take trading decisions by looking at the same sentiment index. Second, unlike Naimzada and Ricchiuti (2008), we do not consider any kind of imitation, in the sense that in the market there are only two types of agent and no other trader is allowed to enter in the market. The introduction of a sentiment index is the innovative aspect of our model, and it is the main reason that leads us to consider only one group of traders. In this way we can analyse the effect of the sentiment index on the complex scenarios without the destabilization in the market introduced by chartists. Another study considering only fundamentalists is that of Kaltwasser (2010). The author analyzes a heterogeneous agent model of the FOREX market, building on the model of Alfarano *et al.* (2008) considering only fundamentalists. Unlike our study, the author considers the FOREX market and relies on a probabilistic approach, focusing mainly on numerical results. On the other hand, we adopt a qualitative analysis, both analytically and numerically. Moreover, we extend our model by including stochastic shocks to the demand of both fundamentalists. In this way we show that our model is capable to matching the stylized facts observed in financial markets.

Our contribution to the literature is twofold. First, for the first time we introduce a new sentiment index relying on the work of Elyasiani *et al.* (2018), that takes into consideration the difference between the market price and the two fundamental prices, representing market sentiment. Thanks to the sentiment index, traders modify their expectations about fundamental value and consequently they switch to the strategy that they believe to be performing better. Second, by using the sentiment index, we model theoretically the empirical evidence on the fact that periods of declining prices (fear scenario) have a greater impact on volatility than periods rising prices (greed scenario).

Our analysis confirms the findings of Elyasiani *et al.* (2018), that is, the RAX index has the advantage of signaling prevailing market sentiment. In particular it is a sentiment index of fear in the sense that it is able to signal downturn moves to investors that can modify their strategies in order to avoid huge losses. Moreover, we find that a period of stability in the market is possible only when the fraction of one type of trader is very low.

The paper proceeds as follows. In Section (2) we describe the model setup. Section (3) focuses on the study of fixed points and their stability analysis. Section (4) analyses the economic implications of our model with a bifurcation analysis and a global analysis. In Section (5) we extend the deterministic model introducing stochastic shock to the fundamental demand of both types of traders and we carry out an in-depth statistical analysis comparing our results with those of the S&P500 index. Section (6) concludes our work.

2 The model

We outline a financial market model which describes price dynamics in the presence of traders with different beliefs about fundamental value. The model includes a *market maker* who adjusts the price based on *order imbalances*, two fundamentalists who believe in reversion to the mean (i.e. they expect the price to return to the fundamental value F). As a result, fundamentalists place orders to buy when the price is below F because in this case they believe that the market is undervalued; they place orders to sell when the price is above F because they believe that the market is overvalued.

Our model incorporates two types of fundamentalists: in particular, we assume that type-2 fundamentalists (f_2) underestimate the fundamental value with respect to type-1 fundamentalists (f_1), i.e. $F_2 < F_1$. This implies that when the price (P_t) is lower than F_2, type-2 fundamentalists overestimate the price less than type-1 fundamentalists. On the other hand, when price P_t is higher than F_1, type-2 fundamentalists underestimate the price P_t more than type-1 fundamentalists. We assume that both types of fundamentalist have the same excess demand function:

$$D_t^{f_1} = \lambda(F_1 - P_t) \tag{1}$$

$$D_t^{f_2} = \lambda(F_2 - P_t) \tag{2}$$

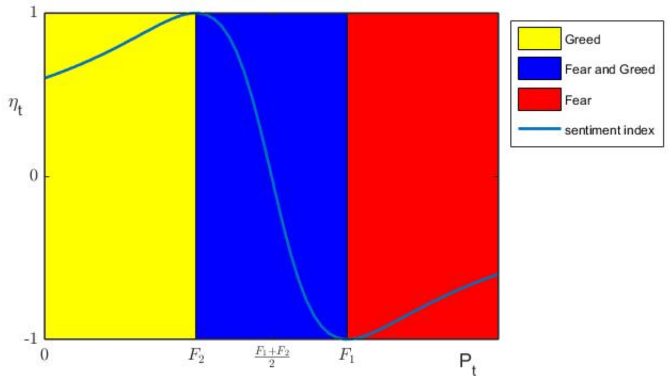

Figure 1: Fear and Greed predominance regions

where $D_t^{f_i}$ is the excess demand function for type-i fundamentalists; $i = 1, 2$, λ is a positive parameter and indicates how aggressively the fundamentalist reacts to the distance of the price to the corresponding fundamental value (F_1, F_2).

Within this framework, when the price is below F_2 type-2 fundamentalists buy less than type-1 fundamentalists. When price is above F_1, type-2 fundamentalists sell more than type-1 fundamentalists.

Depending on the price, we can distinguish the following fear or greed predominance regions (see Fig. 1):

a) when $P_t > F_1 > F_2$, in this case both fundamentalists sell, type-2 fundamentalists sell more than type-1 (fear predominance region).

b) $F_2 < P_t < F_1$, type-2 fundamentalists sell, whereas type-1 fundamentalists buy. This is similar to the bull and bear regime described in Day and Huang (1990) or Tramontana *et al.* (2009) (fear and greed mixed predominance region).

c) $P_t < F_2 < F_1$, both types of fundamentalists buy, but type-2 fundamentalists buy less than type-1 fundamentalists (greed predominance region).

The stock market is characterized by the presence of a market maker that sets the stock price P_{t+1} according to total excess demand:

$$P_{t+1} = P_t + \left(w_1 D_t^{f_1} + w_2 D_t^{f_2} \right) \tag{3}$$

where w_i is the proportion of fundamentalists of type i, $i = 1, 2$ and $w_1 + w_2 = 1$.

We consider a fixed number of traders equal to $2N$, n_i is the number of fundamentalists of type $i = 1, 2$:

$$n_1 + n_2 = 2N \tag{4}$$

Therefore

$$w_1 = \frac{n_1}{2N} \qquad \text{and} \qquad w_2 = \frac{n_2}{2N} \tag{5}$$

We assume that the number of type-1 and type-2 fundamentalists varies according to the following market sentiment index:

$$\eta_t = \frac{(F_1 - P_t)^2 - (F_2 - P_t)^2}{(F_1 - P_t)^2 + (F_2 - P_t)^2} \tag{6}$$

The sentiment index measures in relative terms the distance between the price P_t and the two fundamental prices, F_1 and F_2. In particular the numerator measures how close the price is to the fundamental value F_1 and subtracts how close the price is to the fundamental value F_2. This difference is normalized by the sum of the two distances (the denominator). In this way the sentiment index is bounded between $[-1, 1]$ (see Fig.(1)). In particular, if the price is close to the fundamental value F_1 then the index is negative since for prices higher than F_1 both types of investors sell and for prices $P_t > \frac{F_1+F_2}{2}$ and lower than F_1 type-1 investors buy with a lower intensity than that which characterizes the selling behaviour of type-2 investors, yielding an overall sell signal. In fact, when $P_t > \frac{F_1+F_2}{2}$ then the sentiment index is negative. When $P_t = F_1$ then the sentiment index reaches its minimum value equal to -1. The sentiment index is such that if the price is higher than F_1 it gradually returns to zero, since the selling behavior of both types of fundamentalists determines an oversold market. On the other hand, if the price is close to the fundamental value F_2 then the index is positive since for prices lower than F_2 both types of investors buy and for prices $P_t < \frac{F_1+F_2}{2}$ and higher than F_2 type-1 investors buy with a higher intensity than that which characterizes the selling behaviour of type-2 investors, yielding an overall buy signal. In fact, when $P_t < \frac{F_1+F_2}{2}$ then the sentiment index is positive. When $P_t = F_2$ then the sentiment index reaches its maximum value equal to 1. The sentiment index is such that if the price is lower than F_2 it gradually returns to zero, since the buying behavior of both types of fundamentalists determines an overbought market. Last, if $P_t = \frac{F_1+F_2}{2}$ the sentiment index is equal to zero, indicating no buy or sell signal.

The number of type-1 and type-2 fundamentalists varies according to the sentiment index in the following way[1].

$$n_1 = n_2 - 2N\eta \tag{7}$$

As a result, the number of type-1 fundamentalists is equal to the number of type-2 fundamentalists when $\eta = 0$. The number of type-1 fundamentalists is greater than the number of type-2 fundamentalists when $\eta < 0$. The number of type-1 fundamentalists is smaller than the number of type-2 fundamentalists when $\eta > 0$. The closer the price to the fundamental value F_1 the more the proportion of type-1 fundamentalists

[1] see Lux (1995)

increases since they performed better than the other group in forecasting the equilibrium price (and in the market we have an overall fear predominance since investors expect the price to decrease). On the other hand, tthe closer the price to the fundamental value F_2 the more the proportion of type-2 fundamentalists increases since they performed better than the other group in forecasting the equilibrium price (and in the market we have an overall greed predominance since investors expect prices to increase).

The final map: Based on the above considerations, we obtain the first order nonlinear discrete dynamical equation which describes the price evolution over time. It takes the following form:

$$P_{t+1} = P_t + \Big\{ \Big[\frac{n_2}{2N} - \frac{(F_1 - P_t)^2 - (F_2 - P_t)^2}{(F_1 - P_t)^2 + (F_2 - P_t)^2} \Big] \lambda(F_1 - P_t) + \Big[\frac{n_1}{2N} + \frac{(F_1 - P_t)^2 - (F_2 - P_t)^2}{(F_1 - P_t)^2 + (F_2 - P_t)^2} \Big] \lambda(F_2 - P_t) \Big\} \tag{8}$$

3 Stability analysis of equilibrium points

In this section we explore the qualitative properties of Map (8). We first consider, for the sake of completeness, the case of homogeneous fundamentalists, then the more interesting case of heterogeneous fundamentalists.

Homogeneous fundamentalists. If both fundamentalists are of the same type, they have equal beliefs in the fundamental price, that is $F_1 = F_2$, then the final Map (8) becomes:

$$P_{t+1} = P_t + \Big(1 - \frac{n_1}{2N}\Big)\lambda(F - P_t) + (\frac{n_1}{2N})\lambda(F - P_t)$$

and the fixed point is the fundamental price $P^* = F$. In this case it turns out that:

$$\frac{dP_{t+1}}{dP_t} = 1 - \lambda$$

and we have a situation in which the market reaches a stable equilibrium if $0 < \lambda < 2$.

Heterogeneous fundamentalists. We first show the existence of the fixed points analytically and then we study their local stability by means of graphical analysis. The condition for the existence of the steady state of the Map (8) is $P_{t+1} = P_t = P^*$. Therefore, we solve the following equation in the variable P^*:

$$T(P^*) = \Big[\frac{n_2}{2N} - \frac{(F_1 - P^*)^2 - (F_2 - P^*)^2}{(F_1 - P^*)^2 + (F_2 - P^*)^2} \Big] \lambda(F_1 - P^*) + \Big[\Big(1 - \frac{n_2}{2N}\Big) + \frac{(F_1 - P^*)^2 - (F_2 - P^*)^2}{(F_1 - P^*)^2 + (F_2 - P^*)^2} \Big] \lambda(F_2 - P^*) = 0 \tag{9}$$

The following proposition holds:

Proposition 1 *Assume $P_t > 0$ $\forall t$ and $F_1 > F_2$. Then Map (8) admits three real fixed points P_i^* with $i = 1, 2, 3$ given by:*

$$P_1^* = F_2 \qquad P_2^* = F_1 \qquad P_3^* \in (F_2, F_1) \tag{10}$$

The fixed points belong to the interval $[F_2, F_1]$.

Proof 1 *In order to compute the fixed points we have to solve Eq.(9). It may be seen that the two fundamental prices* F_1 *and* F_2 *are two of the fixed points of the model. Indeed, if we substitute* $P^* = F_1$ *we have:*

$$-\frac{n_2}{2N}\lambda(F_2 - F_1) = 0$$

Note that, assuming $P^* = F_1$ *we have a situation in which type-1 fundamentalists have performed better than type-2 fundamentalists and in this case all fundamentalists become type 1, which implies* $\frac{n_2}{2N} = 0$. *Therefore* F_1 *solves Eq.(9) and it is a solution. A similar argument holds with regards to the second fixed point,* F_2. *Assuming* $P^* = F_2$ *we have that type-2 fundamentalists have performed better than type-1 fundamentalists and in this case all fundamentalists become type 2, which implies* $\frac{n_1}{2N} = 0$. *Therefore* F_2 *solves Eq.(9) and it is a solution. For the existence of the third fixed point in the interval* (F_2, F_1) *we make use of the intermediate value theorem. In particular, consider the midpoint of the segment* $\bar{F_2F_1}$, *i.e.* $\frac{F_1 + F_2}{2}$. *Now, we take the midpoints of the two intervals* $(F_2, \frac{F_1+F_2}{2})$ *and* $(\frac{F_1+F_2}{2}, F_1)$, *namely points* I_1 *and* I_2 *and evaluate Eq.(9) in each of the points. Given that* $T(P(I_1)) < 0$ *and* $T(P(I_2)) > 0$ *and that the function* $T(P)$ *is continuous in the interval* (F_2, F_1), *it may be said that in this interval there is a solution for Eq.(9) and it is unique. This concludes the proof.*

Note that in the steady state $P_2^* = F_1$ ($P_1^* = F_2$), the share of type-2 fundamentalists (1) is zero. The presence of both types of fundamentalists is possible only in the third steady state P_3^*. Thanks to graphical analysis it will be demonstrated that the steady state P_3^* is always unstable since it delimits the basins of attraction of the other two steady states. For appropriate values of parameters, we find that it does not coincide with $(\frac{F_1+F_2}{2})$.

The possible scenarios arising in our model are described in the following Proposition (2)

Proposition 2 *Assume* $F_1 \neq F_2$. *Let* c_{min}, c_{max} *be the local minimum and the local maximum of the Map (8);* C_{min} *and* C_{max} *are their iterates respectively, that is* $C_{min} = P_{t+1}(c_{min})$ *and* $C_{max} = P_{t+1}(c_{max})$, *then there exist two disjoint invariant*[2] *intervals* $I = [P_{t+1}(c_{min}), P_{t+1}(C_{min})]$ *and* $J = [P_{t+1}(c_{max}), P_{t+1}(C_{max})]$ *such that:*

1. ***Fear or Greed scenario:*** *if* $P_{t+1}(C_{min}) < P_3^*$ *and* $P_{t+1}(C_{max}) > P_3^*$, *then Map (8) has two coexistent attractors,* P_1^* *and* P_2^*;

2. ***Fear and Greed scenario:*** *for* $P_{t+1}(C_{min}) = P_{t+1}(C_{max}) = P_3^*$ *the two attractors merge and a contact bifurcation occurs.*

In scenario (1) depicted in Figure (2), we observe the trajectory of the price converging to one of the two coexisting attractors, P_1^* or P_2^*, depending on the initial condition P_0. Indeed, if we take an initial condition P_0 close to P_1^*, then the price converges to P_1^* (Figures (2a-b) greed scenario). On the other hand, taking an initial condition P_0 close to P_2^*, then the price converges to P_2^* (Figures (2c-d) fear scenario). It may be noted that the attractor may be a fixed point, an n-period cycle, a strange attractor or the union of two coexisting attractors. Therefore in scenario (1) we have a greed or a fear scenario depending on the initial condition.

[2] A set $I \subseteq \mathbb{R}_+$ is positively (negatively) invariant if $T^i(I) \subseteq I$ ($T^i(I) \supseteq I$) $\forall i \in \mathbb{Z}_+$. Moreover, I is invariant when it is

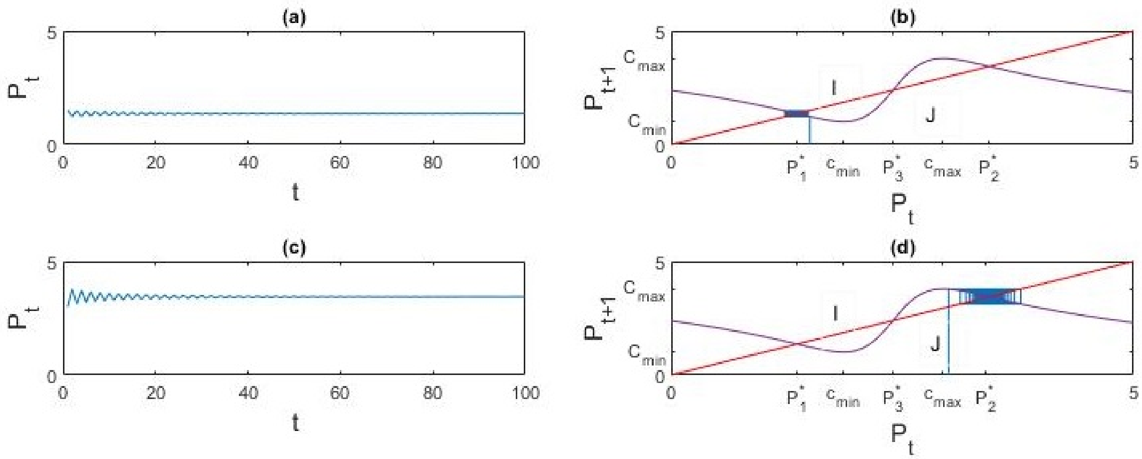

Figure 2: Coexistence of attractors for parameters $n_2 = 0.5$, $F_1 = 3.44$, $F_2 = 1.36$, $\lambda = 1.3$, $N = 0.5$ In (a) and (b)(greed scenario) an i.c. $P_0 = 1.5$ generates a trajectory converging to the fixed point P_1^*. While in (c) and (d) (fear scenario), for $P_0 = 3$ the trajectory converges to the fixed point P_2^*.

In scenario (2) shown in Figure (3) we have the occurrence of a homoclinic bifurcation. In particular, in Fig. (3a-b) the two absorbing intervals are merged. In this case the two chaotic attractors are transformed into a one-piece chaotic attractor. Fig.(3c-d) shows a more chaotic dynamic, with respect to the former, due to the merging of the two attractors which now form a q-piece chaotic attractor where complex price dynamics arise. Therefore in this scenario we have a recursive flip of the price between fear and greed regions.

Another important feature of our model is that the fixed point P_3^* could be asymmetrically distributed in the interval (F_2, F_1). In Fig.(4) we find two trajectories leading to the two coexisting attractors. This case arises for a different value of the parameter n_2 (the fraction of fundamentalists of type 2). Indeed, when we decrease the value of n_2, the basin of attraction of the fixed point F_2 enlarges with respect to that of the other fixed point F_1.

Finally, in Fig.(5), we investigate the role of the reactivity parameter λ in determining the structure of the basins of attraction. In particular, in Figure (5a) and Figure (5b) we have two disconnected basins of attraction, i.e. the basins of the two fixed points are located also in regions that do not contain the relative fixed points[3]. The greater the value of λ, the more complex the structure of the basins of attraction.

In Section (4) we describe in detail the scenarios analysed in the graphical analysis from an economic perspective in order to match our findings with the stylized facts.

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both positively and negatively invariant. Finally, a closed and positively invariant region is called trapping.

[3]see Abraham *et al.* (1997)

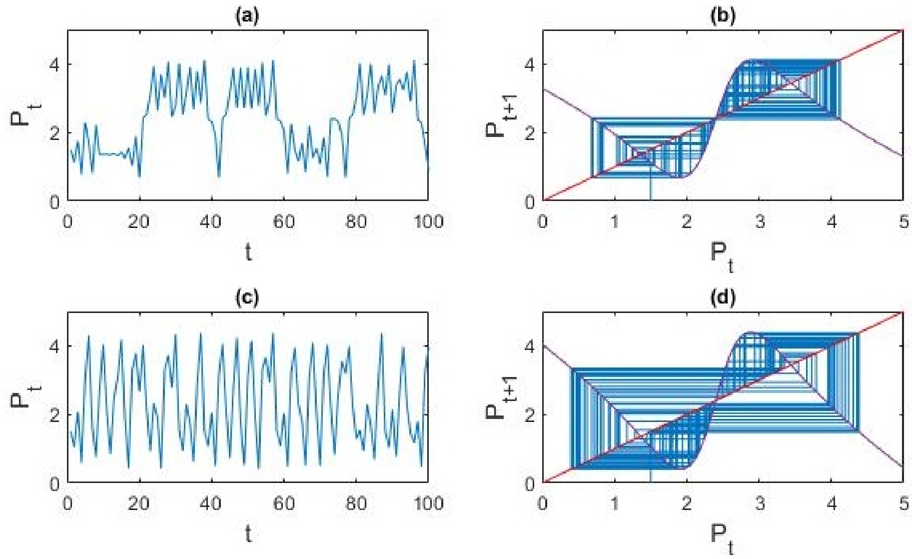

Figure 3: Homoclinic bifurcations scenario. In (a) and (b), two complex attractors merge for $n_2 = 0.5$, $F_1 = 3.44$, $F_2 = 1.36$, $\lambda = 1.79$, $N = 0.5$ and i.c. $P_0 = 1.5$. In (c) and (d) there is the occurrence of an homoclinic bifurcation for $n_2 = 0.5$, $F_1 = 3.44$, $F_2 = 1.36$, $\lambda = 2.2$, $N = 0.5$ and i.c. $P_0 = 1.5$.

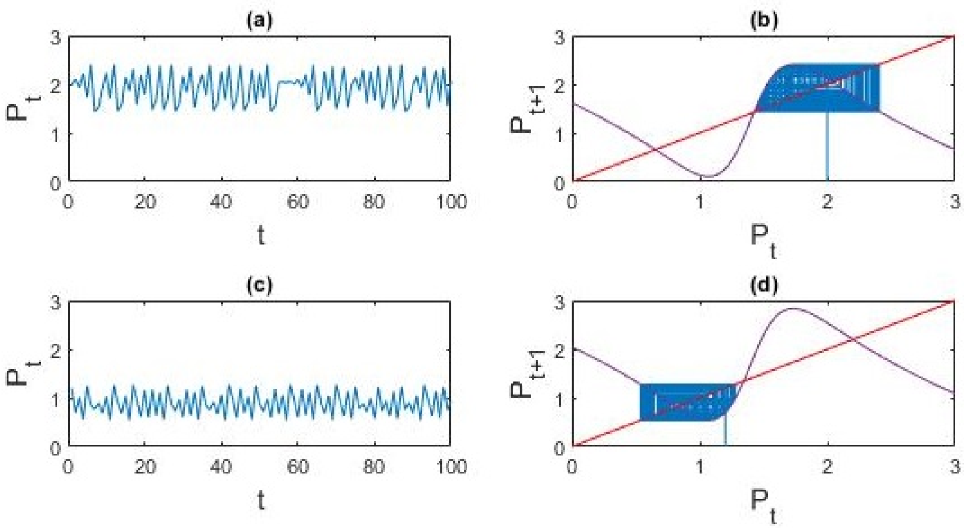

Figure 4: Asymmetric mid fixed point. In (a) and (b) the trajectory converges to the fear attractor for $n_2 = 0.7$, $F_1 = 2.04$, $F_2 = 0.65$, $\lambda = 1.8$, $N = 0.5$ and i.c. $P_0 = 2$. In (c) and (d), the trajectory leads to the greed attractor for $n_2 = 0.4$, $F_1 = 2.2$, $F_2 = 0.81$, $\lambda = 1.8$, $N = 0.5$ and i.c. $P_0 = 1.2$.

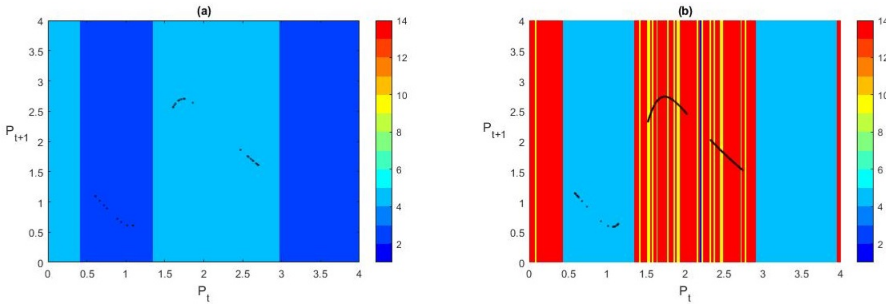

Figure 5: The role of parameter λ. Two pictures showing the basins of attraction of Map(8). In (a), $n_2 = 0.7$, $F_1 = 1.8$, $F_2 = 1$, $\lambda = 1.58$, $N = 0.5$. In (b) we use the set of parameters $n_2 = 0.7$, $F_1 = 1.8$, $F_2 = 1$, $\lambda = 1.64$, $N = 0.5$.

4 Numerical analysis

In this section we shed light on the main insights of our work. From the stability analysis we know that the three equilibria always exist. From an economic point of view, instead, our model is able to capture interesting features of financial markets. In particular, our goal is the introduction of the RAX sentiment index introduced by Elyasiani *et al.* (2018) observing how it works as a reference index of market sentiment. Instead of introducing heterogeneity into the behavioural rules of each group of agents, we highlight how agents with the same trading behavior (fundamentalists) influence market sentiment. In this sense, the RAX index is a sentiment index because it records the prevalent trend of the market (which we have referred to as fear and greed predominance). Another important element of our analysis is the role of the fundamental values perceived by the two groups of traders. In particular, the difference in the two fundamental prices matters not only in a mixed fear and greed scenario Fig.(6) but also in a fear or greed scenario Fig.(7) thanks to the signal contained in the sentiment index. Indeed, in Fig.(6a) it is apparent how the interaction of both types of fundamentalist leads to complicated dynamics and a period of volatility of the price. On the other hand, if we look to Fig.(6b) for the same level of reaction of traders, we see an amplification of price volatility when the distance between the two fundamental prices increases. Moreover, in both cases, when the fraction of fundamentalists of type i, with $i = 1, 2$, is sufficiently lower than the proportion of fundamentalists of type j, with $j = 1, 2$ and $j \neq i$, then the market reaches a stable equilibrium. Considering Fig.(6a), for $n_2 \approx 0.1$ the stable two cycle scenario disappears and a cascade of flip bifurcation becomes apparent. We note that two different attractors coexist, implying that the asymptotic dynamic of price can lead to one of the two attractors depending on the initial condition. At $n_2 \approx 0.2$ the two attractors merge into a single attractor generating a homoclinic bifurcation. In line with He and Zheng (2016), we can attribute the difference between the two fundamental prices to the uncertainty about the true value of the fundamental. Traders try to guess the fundamental price taking into account private information in addition to the public information given by the sentiment index. This gives rise to endogenous heterogeneity and switching behavior on the part of agents.

Based on our analysis, a period of stability is possible when the proportion of one type of trader is

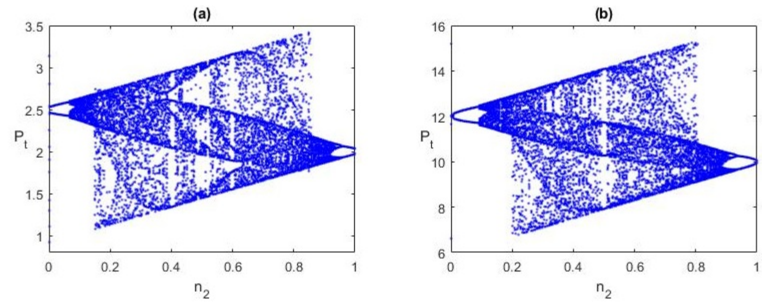

Figure 6: The role of the difference between F_1 and F_2. Two pictures showing the bifurcation diagram with respect to n_2 in the fear and greed scenario. In (a) parameters $F_1 = 2.5$, $F_2 = 2$, $\lambda = 2$, $N = 0.5$ and i.c. $P_0 = 1.5$. In (b) parameters $F_1 = 12$, $F_2 = 10$, $\lambda = 2$, $N = 0.5$ and i.c. $P_0 = 1.5$.

very low (e.g. n_2, see Fig.(8)), implying that when the market is mainly populated by one type of fundamentalist, the level of uncertainty diminishes significantly. However, also in a period of stability, the difference between the two fundamental prices determines a higher price volatility in a fear scenario (Fig.(7b)) compared to a greed scenario (Fig.(7a)).

Our framework is also able to capture the asymmetry in the return distribution of prices. In this connection, the introduction of the RAX index has the property of signaling the sentiment prevailing in the market, in order to allow traders to take their final decision on whether to buy or sell. It is self-evident that investors prefer positive returns to negative ones, and the literature provides evidence that the return distribution is negatively asymmetric which implies in turn many low positive returns and a small number of large losses. This is captured in our model, since the volatility of prices when the market faces a fear scenario is higher than the volatility occurring in the greed scenario or in the mixed fear and greed scenario. In Fig.(8) we show the three possible scenarios of our model. It is evident that, in the fear scenario, the volatility of the price is much larger than in the other two cases. These findings confirm that the RAX index is a sentiment index of fear in the sense that it is able to signal downturn periods to investors who can modify their strategies in order to avoid huge losses.

Finally, we wish to stress the role of the reactivity parameter λ and heterogeneity $(F_2 - F_1)$ in determining chaotic dynamics. This situation is described in Fig.(9) where we note the transition to more complicated dynamics when both λ and F_1 increase. Fig.(9a) shows the origin of a two-piece chaotic attractor while in Fig.(9b) we see that the two attractors give rise to a homoclinic bifurcation.

5 Stochastic model

In this section we explore the statistical properties generated by our model, showing that it is able to match a rich set of empirically observed stylized facts. The main empirical evidence that we will deal with in this section will concern heavy tails, absence of autocorrelations, volatility clustering, long memory and power law behavior characterizing the distribution of large returns (see Lux and Alfarano (2016); Lux and Marchesi (2000), He and Li (2007); He and Zheng (2016), Cont (2001) for example). In particular, we add

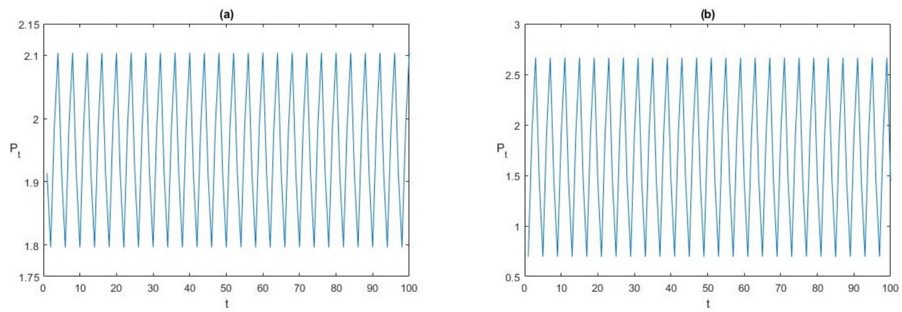

Figure 7: The role of the difference between F_1 and F_2. A stable two-cycle in (a) greed scenario for $n_2 = 0.5$, $F_1 = 2$, $F_2 = 1.9$, $\lambda = 2$, $N = 0.5$ and i.c. $P_0 = 1.5$; and (b) fear scenario for $n_2 = 0.5$, $F_1 = 2$, $F_2 = 1.36$, $\lambda = 2$, $N = 0.5$ and i.c. $P_0 = 1.5$.

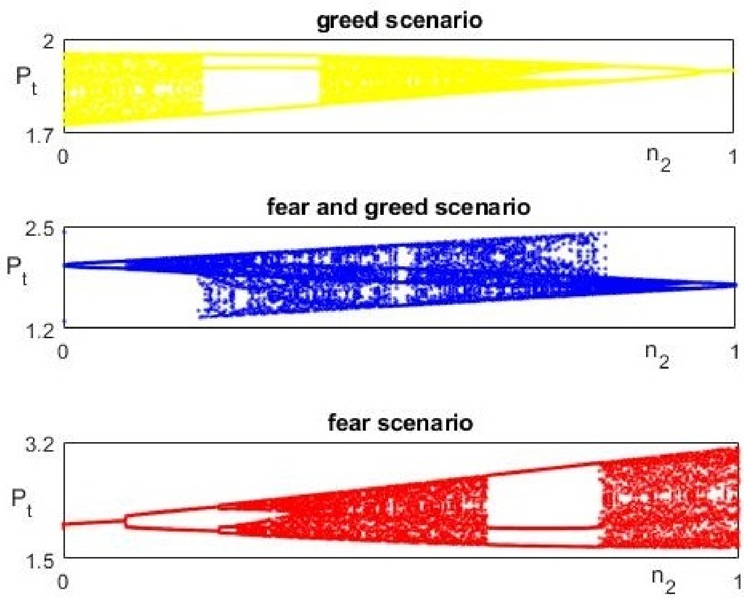

Figure 8: The three scenarios. Greed scenario (yellow diagram) with parameters $F_1 = 2$, $F_2 = 1.9$, $\lambda = 1.72$, $N = 0.5$ and i.c. $P_0 = 1.8$. Fear and greed scenario (blue diagram) with parameters $F_1 = 2$, $F_2 = 1.75$, $\lambda = 2$, $N = 0.5$ and i.c. $P_0 = 1.8$. Fear scenario (red diagram) with parameters $F_1 = 2$, $F_2 = 1.36$, $\lambda = 1.72$, $N = 0.5$ and i.c. $P_0 = 1.8$.

Table 1: Summary statistics of returns. The table reports the summary statistics including mean, standard deviation (sd), skewness, kurtosis, minimum and maximum value, Jarque-Bera test and statistic of $S\&P500$, SM.

	Mean	**sd**	**Min**	**Max**	**Skewness**	**Kurtosis**	**J-B**	**J-B statistic**
S&P500	0.0000	0.0093	-0.0001	0.0000	-0.4976	7.5827	1	2324.90
SM	0.0000	0.1175	-0.0006	0.0005	-0.1493	4.9563	1	1240.10

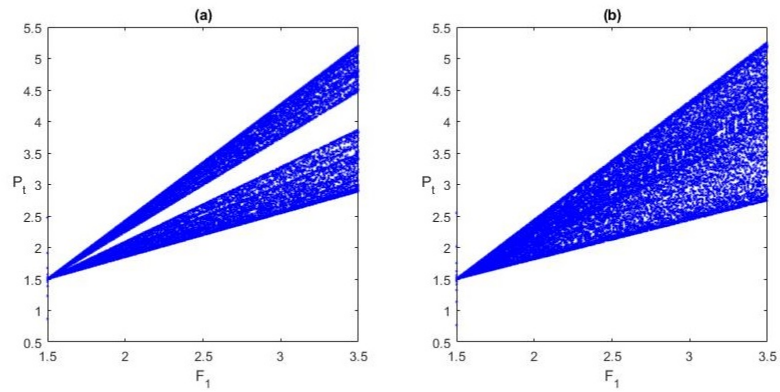

Figure 9: Bifurcation diagrams with respect to F_1 showing the transition from a 2-piece chaotic attractor to a 1-piece chaotic attractor for given values of parameters. In (a) $n_2 = 0.5$, $F_2 = 1.5$, $\lambda = 1.65$, $N = 0.5$ and i.c. $P_0 = 1.5$, in (b) $n_2 = 0.5$, $F_2 = 1.5$, $\lambda = 1.7$, $N = 0.5$ and i.c. $P_0 = 1.5$.

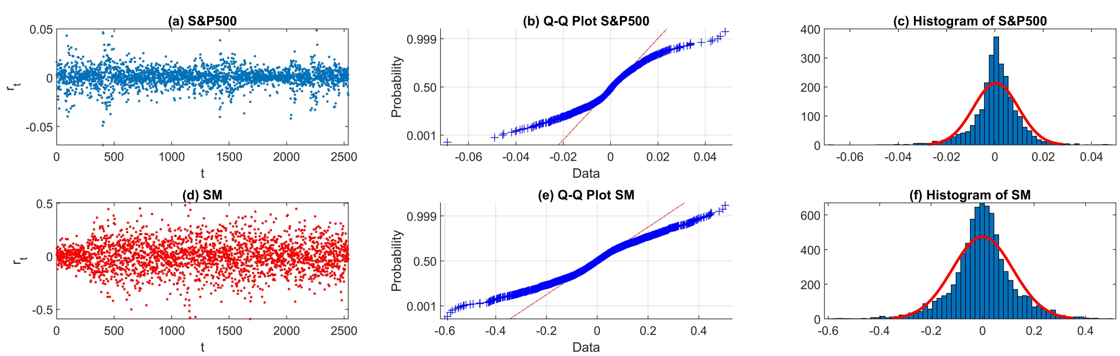

Figure 10: Returns series (panel (a)), Q-Q plot (panel (b)) and the histogram (panel (c))for $S\&P500$ index for time periods $t \in [2010, 2020]$. The same diagrams for SM are figured in panels (c)-(e) respectively.

noise to both the fundamental demand processes and examine the dynamics of the model using numerical simulations. Combining the deterministic and stochastic elements, the demand of type-1 fundamentalists is

$$D_t^{f_1} = \lambda(F_1 - P_t) + \epsilon_t^{f_1} \qquad \epsilon_t^{f_1} \sim N(0, (\sigma^{f_1})^2) \tag{11}$$

and, similarly, the demand of type-2 fundamentalists is given by

$$D_t^{f_2} = \lambda(F_2 - P_t) + \epsilon_t^{f_2} \qquad \epsilon_t^{f_2} \sim N(0, (\sigma^{f_2})^2) \tag{12}$$

where σ^{f_1} and σ^{f_2} are two positive parameter representing the standard deviations of the normal random variables. Without loss of generality, we assume that $\sigma^{f_1} = \sigma^{f_2}$. In our analysis we consider the price returns, defined as

$$r_t = ln(P_t) - ln(P_{t-1}) \tag{13}$$

focusing on the deviation from normality of their distribution. As a benchmark we consider the daily log-returns of the $S\&P500$ from January 02, 2010 to January 28, 2020. To differentiate the time series generated from the $S\&P500$ index ($S\&P500$) and the simulated stochastic model (SM) we add $S\&P500$ and SM in front of the name of each time series. In all the simulations performed we consider the following parameter set: $N = 0.5$; $n_2 = 0.5$; $F_1 = 2000$; $F_2 = 1200$; $\lambda = 0.1$, $P_0 = 1500$, $\sigma^{f_1} = \sigma^{f_2} = 0.09$. In Table(1) we provide some useful summary statistics for the returns of the $S\&P500$ and the SM and the diagrams shown in Fig. 10 will help us to interpret these statistics.

Fig.10 (a) and (d) show that the time series exhibit volatility clustering, which is characterized by intermittent and large fluctuations. Actually, the heterogeneity regarding the beliefs about the fundamental price is responsible for producing such patterns. In Fig.10 (c) and (f) we compare the shape of the distribution with a normal distribution with variance identical to the sample variance (depicted by the solid line). We note a stronger concentration around the mean, with a greater probability mass in the tails of the distribution and thinner shoulders. All these aspects are a typical deviation from the normal distribution and are to be found in the empirical literature (see Lux (1998); Lux and Marchesi (2000)).

In Fig.10 (b) and (e) we report the Q-Q plot of the returns for $S\&P500$ and SM. They display the quantile of the sample data (returns) versus the theoretical quantiles of the normal distribution. If the distribution of returns is normal, then the plot appears to be linear, which is not the case in this instance. Indeed, the tails lay below and above the 45 degree line implying that their distributions are fat-tailed, as well (see Fig.10 (b) and (e)). We also analyze the non-normality of the returns conducting the Jarque-Bera test (see Table1). In particular, we test the null hypothesis that returns follow a normal distribution at the 1% significant level. The returned value of $J - B = 1$ indicates that the Jarque-Bera test rejects the null hypothesis at the 1% significance level, additionally, the test statistic, $J - B$ statistic, is greater than the critical value, which is 5.8461, indicating a rejection of the null hypothesis.

Some of the empirical quantitative properties related to returns include a power-law behaviour, long memory and correlation to volatility. We intend to demonstrate all these stylized facts with our model.

Lux and Alfarano (2016) review a number of universal power laws characterizing financial markets. We take into account two of these in our study. The first concerns the distribution of the returns which is characterized by the presence of fat tails. It is found that fat-tailed distributed returns approximately follow an inverse cubic power law described by

$$P\Big(|\frac{r_t - \bar{r}}{sd}| > X\Big) \sim X^{-\zeta} \tag{14}$$

where $\zeta \simeq 3$ is the Pareto exponent (also called the characteristic exponent), $\bar{r}$ and sd are the mean and the standard deviation of the returns r_t, and $\frac{r_t - \bar{r}}{sd}$ is the normalized return. The pertinent literature converged on the insight of an exponent close to 3. Following He and Zheng (2016) and Gabaix *et al.* (2006), we estimate the characteristic exponent with ordinary least squares (OLS) estimating the logarithm of Eq.(14), that is

$$lnP(|\frac{r_t - \bar{r}}{sd}| > X) = -\zeta lnX + b \tag{15}$$

As shown in Table(2), for both the $S\&P500$ and the SM the value of the characteristic exponent, ζ, is close to 3, confirming the findings of empirical literature.

The second power law relates to the concept of *volatility clustering*. We can see how persistent the volatility is by estimating the following power component in the ACFs of absolute returns (see Cont (2001) and He and Zheng (2016))

$$corr(|r_t|, |r_{t+q}|) \simeq \frac{\zeta}{q^d} \tag{16}$$

where q is the number of lags, ζ is a parameter capturing the ACF of absolute returns with lag one, and d is the power exponent capturing the decay of the ACFs. We estimated Eq.(16) with non-linear least square and the results of each model are shown in Table(3). As we can see, our estimates are in line with the empirical evidence which postulates a value of d in the interval [0.2 0.4]. Fig.11 (a) and (b) plot the series of sample autocorrelation of the log-returns, absolute returns and squared returns of the simulated data and of the $S\&P500$. Both the stochastic and empirical series of returns are characterized by no linear autocorrelation but high absolute and squared autocorrelations persistent up to more than 60 periods, in line with empirical regularity observed in financial time-series referred to as *volatility clustering*.

Finally, we complete the current analysis by testing the hypothesis of long-range dependence in the volatility measured by the time-series of $|r_t|$. To this end, we compute the Lo-modified range over standard deviation or R/S statistic (also called re-scaled range) (see Lo (1991)). In Fig.11(c), we show the R/S statistic for lags ranging from 1 to 100, and it is possible to see that we reject the null hypothesis of no long-range dependence when the number is not particularly large, i.e. $q \leq 30$. Indeed, for these values of the lags, the Lo modified R/S statistic for $S\&P500$ and SM all fall out of the 95% critical interval [0.809 1.862] suggesting the presence of long memory in absolute return series.

Table 2: Power law of returns. For each return series in S&P500 and SM, we estimate $lnP(|\frac{r_t-\bar{r}}{sd}| > X) = -\zeta lnX + b$ with ordinary least squares and report ζ, the number of observation N and the R^2.

	S&P500 $lnP(\lvert\frac{r_t-\bar{r}}{sd}\rvert > X)$	SM $lnP(\lvert\frac{r_t-\bar{r}}{sd}\rvert > X)$
ζ	2.9074***	2.7087***
	(0.2564)	(0.1781)
N	46	46
R^2	0.8533	0.9117

*** significant at 1%.

Table 3: Persistence of ACFs of absolute returns. For each return series in S&P500 and SM, we estimate $corr(|r_{t+q}|, |r_t| \simeq \zeta/q^d$ with nonlinear least squares and report ζ and d.

	S&P500 $corr(\lvert r_{t+q}\rvert, \lvert r_t\rvert)$	SM $corr(\lvert r_{t+q}\rvert, \lvert r_t\rvert)$
d	0.3692***	0.2218***
	(0.0615)	(0.0163)
ζ	0.3360***	0.3219***
N	150	150
R^2	0.7359	0.8275

*** significant at 1%.

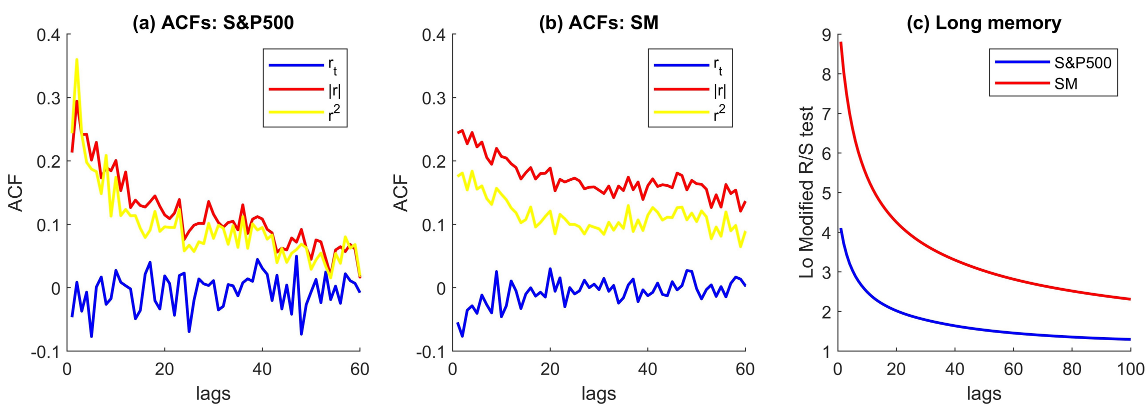

Figure 11: Volatility clustering and long-range dependence. Panels (a) and (B) plots the ACFs of returns (blue line), the absolute returns (red line) and the squared returns (yellow line) for S&P500 and SM respectively. Panel (c) plots the Lo modified R/S statistic of the absolute returns.

6 Conclusions

The aim of the paper is to model trading decisions of financial investors based on a sentiment index introduced by Elyasiani *et al.* (2018). We consider two groups of fundamentalists with heterogeneous beliefs about the fundamental value and we model an endogenous switching mechanism between the two groups of traders, relying on the sentiment index. The sentiment index is considered as a benchmark index for all agents. Depending on the value of the price, one group of agents is more aggressive than the other in buying or selling. The main finding of our work is that the introduction of the sentiment index allows the model to generate complex price dynamics. The model is able to distinguish a fear scenario, a greed scenario, or a mixed fear and greed one. The difference in the two fundamental prices perceived, thanks to the sentiment index, matters in all three scenarios.

In particular, when the proportion of the two groups of fundamentalists is similar, we observe sudden changes from fear to greed scenarios and vice versa. Based on our analysis, a stability period is possible when the proportion of one type of trader is very low. However, also in a stable period, the difference in the two fundamental prices determines greater uncertainty in the fear compared to the greed scenario.

Moreover, we observe a higher uncertainty, proxied by the range of prices attained, in a fear than in a greed scenario. As a result, our paper casts light on investor sentiment as one of the main drivers of asymmetry in stock returns (Jawadi *et al.* (2018), Verma and Soydemir (2009)).

Finally, we note the transition to more complicated dynamics when the reactivity to price changes of each group of traders increases.

The model is able to match many empirically observed stylized facts such as non-normality of the returns, heavy tails and volatility clustering.

It is possible to extend our work in several ways. First, we could introduce chartists or other types of noisy traders in order to model speculative agent behavior. Second, we could increase the heterogeneity in trading decisions by assuming that each group of agents behaves according to a different sentiment index.

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